EXISTENCE OF INFINITE ENERGY SOLUTIONS OF DEGENERATE ELLIPTIC EQUATIONS

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Abstract. We establish an existence theorem for infinite energy solutions of degenerate elliptic equations whose right hand side belongs to a Orlicz Zygmund class. The function which measures the degree of degeneracy of the problem is assumed to be exponentially integrable. We also study the regularity of the solution when the right hand side belongs to a suitable Lebesgue space.

1. Introduction

The aim of this paper is to establish the existence of infinite energy solutions of degenerate elliptic equations. Let us consider the following equation

$$\operatorname{div} A(x, Du) = \operatorname{div} f \qquad \text{in} \quad \mathbb{R}^n, \qquad n \geq 2 \tag{1.1}$$

for a function $u : \mathbb{R}^n \to \mathbb{R}$. We suppose that the operator $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function satisfying the following assumptions for almost every $x \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^n$

$$|A(x, \xi) - A(x, \eta)| \leq k(x)|\xi - \eta|, \tag{1.2}$$

$$\langle A(x, \xi) - A(x, \eta), \xi - \eta \rangle \geq \frac{1}{k(x)}|\xi - \eta|^2, \tag{1.3}$$

$$A(x, 0) = 0, \tag{1.4}$$

with $k(x) \geq 1$. The above three conditions imply the following inequality

$$|\xi|^2 + |A(x, \xi)|^2 \leq \mathcal{K}(x)\langle A(x, \xi), \xi \rangle\,, \tag{1.5}$$

where $\mathcal{K}(x) = (k(x)^2 + 1)k(x)$ will be called the distortion function of the operator $A(x, \xi)$. When $k(x)$ is bounded the equation is uniformly elliptic, otherwise it is a genuine anisotropic equation. In what follows, the distortion function will belong to the exponential class $EXP(\mathbb{R}^n)$ defined via the Orlicz function $P(t) = e^t - 1$.
The model we have in mind is the operator $A(x, \xi)$ of the form $A(x, \xi) = A(x)\xi$ where the matrix $A(x)$ is given by

$$A(x) = \begin{pmatrix} \log^{-\frac{1}{3}}\left(e + \frac{1}{|x|}\right) & 0 \\ & \\ 0 & \log^{-\frac{1}{3}}\left(e + \frac{1}{|x|}\right) \end{pmatrix}$$

when $x \neq 0$ and by the zero matrix when $x = 0$.

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Definition 1.1. *A function u in the Sobolev class* $W^{1,1}_{loc}(\mathbb{R}^n)$ *such that* $A(x, Du) \in L^1_{loc}(\mathbb{R}^n;\mathbb{R}^n)$ *is a solution of equation* (1.1) *if it is a distributional solution, i.e. if the following integral identity*

$$\int_{\mathbb{R}^n} \langle A(x, Du), D\varphi\rangle \, dx = \int_{\mathbb{R}^n} \langle f, D\varphi\rangle \, dx, \tag{1.6}$$

is verified for every $\varphi \in C^\infty_0(\mathbb{R}^n)$*, whenever* $f \in L^1_{loc}(\mathbb{R}^n;\mathbb{R}^n)$*.*

Let us recall that the energy of the solution u in a measurable set $\omega \subset \mathbb{R}^n$ is

$$\mathcal{E}[u, \omega] = \int_\omega \langle A(x, Du), Du\rangle dx. \tag{1.7}$$

We say that a solution of equation (1.1) has finite energy if $\mathcal{E}[u, \omega]$ is finite for every compact set $\omega \subset \mathbb{R}^n$.

The study of the regularity properties of solutions of degenerate elliptic equations has a long history under the assumption that $\mathcal{K}(x)$ is a function exponentially integrable, since such equations naturally arise in the study of mappings with finite distortion and in non linear elasticity phenomena. A simple use of Young's inequality yields that the gradient of a finite energy solution of equation (1.1), under the assumption that the distortion function is exponentially integrable, belongs to $L^2 \log^{-1} L(\mathbb{R}^n)$. Actually, as proved in many papers (see for example [18], [13], [17], [9], [6]), it gains higher integrability in the scale of Orlicz-Zygmund classes.
A solution of equation (1.1) whose gradient belongs to $L^2 \log^{-\alpha-1} L(\mathbb{R}^n)$, for some $\alpha > 0$, clearly could be an infinite energy solution.
In recent papers ([9], [6]) also the regularity of the gradient of infinite energy solutions have been studied. More precisely, it has been shown that if the gradient of the solution belongs to a Orlicz-Zygmund class not too far from the natural one, i.e. it belongs to $L^2 \log^{-\alpha-1} L(\mathbb{R}^n)$, for α positive and sufficiently small, then the solution has finite energy. As far as we know, no existence results are available for infinite energy solutions of equations of this kind. Here we fill this gap showing that there exist infinite energy solutions of equation (1.1) if the right hand side f belongs to the Orlicz-Zygmund class $L^2 \log^{1-\alpha} L(\mathbb{R}^n)$, with $\alpha > 0$ depending on the norm of the distortion in the exponential class.

More precisely, our basic assumption is a global exponential integrability of the distortion function $\mathcal{K}(x)$ appearing in (1.5). Namely, we shall assume that

$$[\mathcal{K}] = \int_{\mathbb{R}^n} [\exp(\beta(\mathcal{K}(x) - \mathcal{K}_0)) - 1] \, dx < \infty, \tag{1.8}$$

for some $\beta > 0$ and some function $\mathcal{K}_0 \in L^\infty(\mathbb{R}^n)$ such that $1 \le \mathcal{K}_0(x) \le \mathcal{K}(x)$.
Our main result is the following

Theorem 1.2. *Assume* (1.2)-(1.4) *and suppose that* $\mathcal{K}(x)$ *satisfies* (1.8)*. There exists a constant* $\alpha_0 = \alpha_0(n, \beta, ||\mathcal{K}_0||_\infty)$*, such that if*

$$f \in L^2 \log^{1-\alpha} L(\mathbb{R}^n) \tag{1.9}$$

for $\alpha < \min\{1, \alpha_0\}$*, then the equation* (1.1) *admits a solution* u *such that* $Du \in L^2 \log^{-\alpha-1} L(\mathbb{R}^n)$*. Moreover the following estimate holds*

$$\begin{aligned} ||Du||^2_{L^2 \log^{-\alpha-1} L(\mathbb{R}^n)} &\leq c \int_{\mathbb{R}^n} |f(x)|^2 \log^{1-\alpha}(e^2 + |f|)\, dx \\ &+ c \int_{\mathbb{R}^n} [\exp(\beta(\mathcal{K}(x) - \mathcal{K}_0)) - 1]\, dx, \end{aligned} \tag{1.10}$$

for a constant $c = c(n, \beta, ||\mathcal{K}_0||_\infty)$*.*

Let us explicitly point out that we are dealing with genuine anisotropic equations, since the ratio between the eigenvalues (given by $k^2(x)$) is unbounded and we search the solution on the whole $\mathbb{R}^n$.
We will show, by mean of an example, that the regularity of the right hand side of equation (1.1) doesn't prevent us in finding infinite energy solutions. In fact, we shall construct an equation for which assumption (1.5) is satisfied for an exponentially integrable function, whose right hand side is zero and admits an infinite energy solution.
The main difficulty in dealing with equations with degenerate ellipticity is that generally we cannot use test functions proportional to solutions. We overcome this difficulty using Lipschitz test functions constructed as in the pioneering paper by Acerbi and Fusco ([1]) and a method due to Lewis ([16]) in order to establish useful a priori estimates. The desired existence result will be obtained, as usual, by an approximation procedure, obtained suitably modifying the argument of [3], since the a priori estimate is preserved in passing to the limit.
In both steps, due to the nonlinearity of the operator and since the energy of the solution could be infinite, we need suitabtle properties of the Hardy-Littlewood maximal operator in Orlicz spaces as well as non trivial techniques of functional analysis.
Finally, when $n > 2$, we study how the summability of the right hand side influences the summability of the solution u. We recall that if the right hand side belongs to $L^p_{loc}(\mathbb{R}^n)$ with $p > n$ then the solutions are locally bounded (see [6]). Here we study the complementary case, i.e. what happens when $2 < p < n$, , by using a new version of a Lemma contained in [8]. We prove that also for the problems treated here the regularity of f improves the regularity of u but, dealing with degenerate problems, the solution u earns less regularity than the non degenerate case (see [19], [2], [4], [5] and [8]). In fact, we shall prove that if $f \in L^p_{loc}(\mathbb{R}^n)$, $2 < p < n$, then $u \in L^s_{loc}(\mathbb{R}^n)$, for every $s < p^*$ (see Theorem 5.2).
The paper is organized as follows : in Section 2 we recall definitions and basic properties of Orlicz-Zygmund classes and we collect several Lemmas useful for our needs; in Section 3 we establish the a priori estimate; Section 4 is devoted to the existence result; in Section 5 we study the regularity of the solutions; in Section 6 we construct an example of degenerate equation admitting an infinite energy solution.

2. Preliminary results

In this section we recall some definitions and basic result on Orlicz spaces and maximal operator. For more details on these subjects we refer to [15] and [20].

An Orlicz function P is a continuosly increasing function such that

$$P:[0,+\infty)\to[0,+\infty), \qquad P(0)=0, \qquad \lim_{t\to\infty}P(t)=\infty.$$

The Orlicz space $L^P(\mathbb{R}^n)$ consists of those Lebesgue measurable function f defined on $\mathbb{R}^n$ for which

$$\int_{\mathbb{R}^n} P(\lambda|f|)<\infty, \quad \text{for some} \quad \lambda=\lambda(f)>0.$$

This is a complete linear metric space with respect to the following distance

$$\mathrm{dist}_P(f,g)=\inf\left\{\frac{1}{\lambda}\,:\,\int_{\mathbb{R}^n}P(\lambda|f-g|)\le 1\right\}.$$

The non linear Luxenburg functional

$$||f||_P=\inf\left\{\frac{1}{\lambda}\,:\,\int_{\mathbb{R}^n}P(\lambda|f|)\le 1\right\},$$

is homogeneous, but in general fails to satisfy the triangle inequality. In case the Orlicz function P is convex, then $||\cdot||_P$ is a norm and L^P with this norm is a Banach space.
We shall work with the Orlicz-Zygmund spaces $L^s\log^\alpha L$, $1\le s<\infty$, $\alpha\in\mathbb{R}$, which are Orlicz spaces generated by the function $P(t)=t^s\log^\alpha(e+t)$. Note that Orlicz functions that are equivalents at ∞ generate the same Orlicz space.
Let us recall that for $\alpha\ge 0$ the non-linear functional

$$[f]_{s,\alpha}=\left[\int|f|^s\log^\alpha\left(e+\frac{|f|}{\|f\|_s}\right)\right]^{\frac{1}{s}}$$

is comparable with the Luxemburg norm in the sense that

$$||f||_{L^s\log^\alpha L}\le[f]_{s,\alpha}\le 2||f||_{L^s\log^\alpha L}.$$

The following inclusions trivially hold

$$L^p\log^\beta L(\mathbb{R}^n)\subset L^p(\mathbb{R}^n)\subset L^p\log^\alpha L(\mathbb{R}^n),$$

with continuous imbeddings

$$||f||_{L^p\log^\alpha L(\mathbb{R}^n)}\le||f||_{L^p(\mathbb{R}^n)}\le||f||_{L^p\log^\beta L(\mathbb{R}^n)},$$

whenever $\alpha<0<\beta$.
We have also the following Hölder type estimates

$$||fg||_{L^c\log^\gamma L}\le C(\alpha,\beta)||f||_{L^a\log^\alpha L}||g||_{L^b\log^\beta L}, \tag{2.1}$$

whenever $a,b>1$ and $\alpha,\beta\in\mathbb{R}$ are coupled by the relations

$$\frac{1}{c}=\frac{1}{a}+\frac{1}{b}, \qquad \frac{\gamma}{c}=\frac{\alpha}{a}+\frac{\beta}{b}.$$

Moreover, the Young's inequality in Orlicz-Zygmund spaces reads as

$$st\le s^p\log^\alpha(e+s)+t^q\log^\beta(e+t), \qquad \forall s,\,t\ge 0, \tag{2.2}$$

whenever $p,q>1$ and $\alpha,\beta\in\mathbb{R}$ are coupled by the relations

$$1=\frac{1}{p}+\frac{1}{q}, \qquad \frac{\alpha}{p}+\frac{\beta}{q}=0.$$

For $\alpha > 0$, the dual Orlicz space to $L\log^\alpha L(\mathbb{R}^n)$ is the Orlicz space $EXP_{\frac{1}{\alpha}}(\mathbb{R}^n)$, generated by the function $Q(t) = \exp(t^{\frac{1}{\alpha}}) - 1$.
The well known Young's inequality tells us that

$$st \le s\log^\alpha(e+s) + \exp(t^{\frac{1}{\alpha}}) - 1, \qquad \forall s,\, t \ge 0, \tag{2.3}$$

and the Hölder's inequality reads as follows

$$||fg||_{L^1} \le c||f||_{L\log^\alpha L}||g||_{Exp_{\frac{1}{\alpha}}}. \tag{2.4}$$

Here and in what follows we will not specify the constants but only their dependence on relevant parameters. We shall also need the following elementary inequalities

$$\frac{s+t}{\log^\alpha(e^2+s+t)} \le \frac{s}{\log^\alpha(e^2+s)} + \frac{t}{\log^\alpha(e^2+t)}, \qquad \forall\ \alpha,\, s,\, t \ge 0, \tag{2.5}$$

$$s\log^\alpha(e^2+s) \le c(\alpha,\beta)[e^{\beta s} - 1], \qquad \forall\ \alpha,\, \beta,\, s \ge 0, \tag{2.6}$$

$$s^p \le cs^2\log^{-\alpha}(e^2+s), \qquad \forall\ \alpha \ge 0,\ \forall s \ge 1 \quad \forall\ 1 \le p < 2. \tag{2.7}$$

Next Lemma will be useful in what follows.

Lemma 2.1. *For a function $f \in L\log^{-\alpha-1}L(\mathbb{R}^n)$, $0 < \alpha < 1$, we have that*

$$\int_{\mathbb{R}^n} |f|\log^{-2\alpha}(e+|f|)\log^{\alpha-1}\left(e + \frac{|f|}{\log^\alpha(e+|f|)}\right) dx \le c\int_{\mathbb{R}^n} |f|\log^{-\alpha-1}(e+|f|)\, dx\,,$$

for a constant c independent of α.

Proof. Observe that

$$\begin{aligned} &\int_{\mathbb{R}^n} |f|\log^{-2\alpha}(e+|f|)\log^{\alpha-1}\left(e + \frac{|f|}{\log^\alpha(e+|f|)}\right) dx \\ =\ &\int_{\{|f|\ge e\}} |f|\log^{-2\alpha}(e+|f|)\log^{\alpha-1}\left(e + \frac{|f|}{\log^\alpha(e+|f|)}\right) dx \\ +\ &\int_{\{|f|< e\}} |f|\log^{-2\alpha}(e+|f|)\log^{\alpha-1}\left(e + \frac{|f|}{\log^\alpha(e+|f|)}\right) dx \\ =\ &I + II. \end{aligned} \tag{2.8}$$

In the set $\{|f| < e\}$ we obviously have

$$\frac{|f|}{\log^\alpha(e+|f|)} \ge \frac{1}{2}|f|.$$

Since $\alpha - 1 < 0$, we have

$$\log^{\alpha-1}\left(e + \frac{|f|}{\log^\alpha(e+|f|)}\right) \le \log^{\alpha-1}\left(e + \frac{|f|}{2}\right) \le c\log^{\alpha-1}(e+|f|),$$

for a constant c independent of α and so

$$II \le c\int_{\{|f|<e\}} |f|\log^{-2\alpha}(e+|f|)\log^{\alpha-1}(e+|f|)\, dx \le c\int_{\mathbb{R}^n} |f|\log^{-\alpha-1}(e+|f|)\, dx\,. \tag{2.9}$$

In the set $\{|f| \geq e\}$ we use that $\log(e+|f|) \leq 2|f|^{\frac{1}{2}}$, thus having

$$\log^{\alpha-1}\left(e+\frac{|f|}{\log^{\alpha}(e+|f|)}\right) \leq \log^{\alpha-1}\left(e+\frac{|f|^{\frac{1}{2}}}{2}\right) \leq c\log^{\alpha-1}(e+|f|)$$

and then

$$I \leq c\int_{\{|f|\geq e\}} |f|\log^{-2\alpha}(e+|f|)\log^{\alpha-1}(e+|f|)\,dx \leq c\int_{\mathbb{R}^n} |f|\log^{-\alpha-1}(e+|f|)\,dx. \tag{2.10}$$

The conclusion follows inserting (2.9) and (2.10) in (2.8).

□

The Hardy-Littlewood maximal function of $f \in L^1_{loc}(\Omega)$, that will be denoted by $Mf(x)$, is defined as

$$Mf(x) = \sup_{Q\ni x} \fint_Q |f(y)|dy,$$

where the supremum is taken over all cubes Q with edges parallel to the coordinate axes. Recall that the maximal function acts boundedly between Orlicz-Zygmund classes. More precisely, we have the following result.

Lemma 2.2. *For a function $f \in L^p\log^{\alpha} L(\mathbb{R}^n)$, $p > 1$, $\alpha \in \mathbb{R}$, we have that $Mf \in L^p\log^{\alpha} L(\mathbb{R}^n)$ and*

$$||Mf||_{L^p\log^{\alpha} L(\mathbb{R}^n)} \leq c(n,p,\alpha)||f||_{L^p\log^{\alpha} L(\mathbb{R}^n)}.$$

We shall use the following extension Lemma (see [1]).

Lemma 2.3. *Let $\lambda > 0$, $1 < q < \infty$, $x_0 \in \mathbb{R}^n$ and $r > 0$. Suppose that $u \in W^{1,q}(\mathbb{R}^n)$,* $\operatorname{supp} u \subset B(x_0,r)$ *and*

$$F(\lambda) = \{x : M(|Du|)(x) \leq \lambda\} \cap B(x_0, 2r) \neq \emptyset.$$

Then $u_{|F(\lambda)}$ has an extension denoted by $v = v(\cdot,\lambda)$ such that

(i) $v = u$ *on* $F(\lambda)$,

(ii) $\operatorname{supp} v \subset B(x_0, 2r)$,

(iii) $v \in W^{1,\infty}(\mathbb{R}^n)$ *and* $||Dv||_\infty \leq c(n)\lambda$.

We conclude this section stating an useful iteration lemma whose proof can be found in [10], p. 161, Lemma 3.1.

Lemma 2.4. *Let $f(\tau)$ be a non-negative bounded function defined for $0 \leq R_0 \leq t \leq R_1$. Suppose that for $R_0 \leq \tau < t \leq R_1$ we have*

$$f(\tau) \leq A(t-\tau)^{-\alpha} + B + \theta f(t), \tag{2.11}$$

where A, B, α, θ are non-negative constants, and $\theta < 1$. Then there exists a constant γ, depending only on α and θ such that for every ρ, R, R_0, $R_0 \leq \rho < R \leq R_1$, we have

$$f(\rho) \leq \gamma[A(R-\rho)^{-\alpha} + B]. \tag{2.12}$$

3. The a priori estimate

This section is devoted to establish an a priori estimate for infinite energy solutions of equation (1.1). We have the following result.

Theorem 3.1. *Let u be a solution of the equation* (1.1) *and assume* (1.2)-(1.4). *Suppose that the function $\mathcal{K}(x)$ appearing in* (1.5) *satisfies* (1.8). *Then there exists $\alpha_0 = \alpha_0(n, \beta, ||\mathcal{K}_0||_\infty)$ such that if*

$$\langle A(x, Du), Du\rangle \in L \log^{-\alpha} L(\mathbb{R}^n), \tag{3.1}$$

and

$$f \in L^2 \log^{1-\alpha} L(\mathbb{R}^n), \tag{3.2}$$

for $0 < \alpha < \min\{1, \alpha_0\}$ then the following estimate

$$\begin{aligned} ||Du||^2_{L^2 \log^{-\alpha-1} L(\mathbb{R}^n)} &\leq c \int_{\mathbb{R}^n} |f(x)|^2 \log^{1-\alpha}(e^2 + |f|)\, dx \\ &+ c \int_{\mathbb{R}^n} [\exp(\beta\mathcal{K} - \mathcal{K}_0))) - 1]\, dx \end{aligned} \tag{3.3}$$

holds true for a constant $c = c(n, \beta, ||\mathcal{K}_0||_\infty)$.

Proof. Let $T_t(u)$ be the truncation of the solution u at levels $\pm t$ defined as follows

$$T_t(s) = \min\{|s|, t\}\text{sign}(s) = \begin{cases} t & \text{if} \quad s > t, \\ s & \text{if} \quad |s| < t, \\ -t & \text{if} \quad s < -t. \end{cases} \tag{3.4}$$

Let us denote by $B_\rho = B(0, \rho)$ the ball of radius ρ centered at the origin and let us consider a family φ_ρ of cut-off functions between B_ρ and $B_{2\rho}$, that is

$$\varphi_\rho = 1, \text{ on } B_\rho \qquad |\nabla\varphi_\rho| \leq \frac{c}{\rho},$$
$$0 \leq \varphi_\rho \leq 1, \qquad \text{supp}\varphi_\rho \subset B_{2\rho},$$
$$\varphi_\rho \nearrow 1 \quad \text{and} \quad |\nabla\varphi_\rho| \to 0 \text{ uniformly as } \rho \to +\infty.$$

Next, for any ρ, let us set $u^t_\rho = T_t(u) \cdot \varphi_\rho$ and observe that, for $x \in \mathbb{R}^n \setminus B_{3\rho}$, we have

$$M(|Du^t_\rho|)(x) \leq \lambda_\rho = \frac{c}{\rho^n} \int_{B_{2\rho}} |Du^t_\rho|.$$

Then, let us consider the sets

$$E(\lambda) = \{x \in \mathbb{R}^n : \ M(|Du^t_\rho|)(x) \leq \lambda\},$$
$$F(\lambda) = E(\lambda) \cap B_{4\rho}.$$

Since $F(\lambda)$ is non empty for $\lambda > \lambda_\rho$, for such λ we consider the function v which is the Lipschitz continuous extension of $u^t_{\rho|F(\lambda)}$ to the whole $\mathbb{R}^n$, given by Lemma 2.3. Using v as test function in the equation, we have

$$\int_{F(\lambda)} \langle A(x, Du), Du^t_\rho\rangle\, dx = -\int_{\mathbb{R}^n\setminus F(\lambda)} \langle A(x, Du), Dv\rangle\, dx$$

$$
\begin{aligned}
&+ \int_{F(\lambda)} \langle f, Du^t_\rho \rangle \, dx + \int_{\mathbb{R}^n \setminus F(\lambda)} \langle f, Dv \rangle \, dx \\
&\le c(n)\lambda \int_{B_{4\rho}\setminus F(\lambda)} |A(x,Du)| \, dx \\
&+ \int_{F(\lambda)} |f||Du^t_\rho| \, dx + c(n)\lambda \int_{B_{4\rho}\setminus F(\lambda)} |f| \, dx, \qquad (3.5)
\end{aligned}
$$

where we used that supp $v \subset B_{4\rho}$. Let us introduce the function

$$
\Phi(\lambda) = \frac{1}{e^2+\lambda} \left[\log^{-\alpha-1}(e^2+\lambda) - (\alpha+1)\log^{-\alpha-2}(e^2+\lambda) \right],
$$

where $0 < \alpha < 1$ will be determined at the end of the proof. Note that the function $\Phi(\lambda)$ is positive for every $\lambda > 0$ and $\alpha < 1$. Multiplying both sides of (3.5) by $\Phi(\lambda)$ and integrating with respect to λ in the interval $(\lambda_\rho, +\infty)$, we get

$$
\begin{aligned}
&\int_{\lambda_\rho}^\infty \Phi(\lambda) \int_{F(\lambda)} \langle A(x,Du), Du^t_\rho \rangle \, dx d\lambda \\
\le\ & c(n) \int_{\lambda_\rho}^\infty \Phi(\lambda)\lambda \int_{B_{4\rho}\setminus F(\lambda)} |A(x,Du)| \, dx d\lambda \\
+\ & \int_{\lambda_\rho}^\infty \Phi(\lambda) \int_{F(\lambda)} |f||Du^t_\rho| \, dx d\lambda + c(n) \int_{\lambda_\rho}^\infty \Phi(\lambda)\lambda \int_{B_{4\rho}\setminus F(\lambda)} |f| \, dx d\lambda. \qquad (3.6)
\end{aligned}
$$

We rewrite estimate (3.6) as follows

$$
J_0 \le c(n) J_1 + J_2 + c(n) J_3, \qquad (3.7)
$$

and we estimate the integrals J_i separately.

Estimate of J_0

Using Fubini Theorem, we get

$$
\begin{aligned}
J_0 &= \int_{\lambda_\rho}^\infty \Phi(\lambda) \int_{B_{4_\rho}} \langle A(x,Du), Du^t_\rho \rangle \chi_{\{x:\, M(|Du^t_\rho|) \le \lambda\}} \, dx d\lambda \\
&= \int_{\lambda_\rho}^\infty \Phi(\lambda) \int_{E(\lambda_\rho)} \langle A(x,Du), Du^t_\rho \rangle \, dx d\lambda \\
&+ \int_{B_{4_\rho}\setminus E(\lambda_\rho)} \langle A(x,Du), Du^t_\rho \rangle \, dx \int_{\lambda_\rho}^\infty \Phi(\lambda) \, d\lambda \\
&= \left[\frac{1}{\alpha} \log^{-\alpha}(e^2+\lambda_\rho) - \log^{-\alpha-1}(e^2+\lambda_\rho) \right] \int_{E(\lambda_\rho)} \langle A(x,Du), Du^t_\rho \rangle \, dx \\
&+ \frac{1}{\alpha} \int_{B_{4_\rho}\setminus E(\lambda_\rho)} \langle A(x,Du), Du^t_\rho \rangle \log^{-\alpha}(e^2 + M(|Du^t_\rho|)) \, dx \\
&- \int_{B_{4_\rho}\setminus E(\lambda_\rho)} \langle A(x,Du), Du^t_\rho \rangle \log^{-\alpha-1}(e^2 + M(|Du^t_\rho|)) \, dx. \qquad (3.8)
\end{aligned}
$$

Estimate of J_1

Again changing the order of integration, we obtain

$$
\begin{aligned}
J_1 &= \int_{\lambda_\rho}^{\infty} \Phi(\lambda)\lambda \int_{B_{4\rho}\setminus F(\lambda)} |A(x,Du)|\,dxd\lambda \\
&\leq c(n)\int_{\lambda_\rho}^{\infty} \Phi(\lambda)\lambda \int_{B_{4\rho}} |A(x,Du)|\chi_{\{x:\,M(|Du^t_\rho|)>\lambda\}}\,dxd\lambda \\
&= c(n)\int_{B_{4\rho}} |A(x,Du)| \int_{\lambda_\rho}^{M(|Du^t_\rho|)} \Phi(\lambda)\lambda\,d\lambda\,dx \\
&\leq c(n)\int_{B_{4\rho}} |A(x,Du)| M(|Du^t_\rho|)\log^{-\alpha-1}(e^2+M(|Du^t_\rho|))\,dx. \qquad (3.9)
\end{aligned}
$$

Estimate of J_3

Arguing as in the estimate of J_1, we have

$$
J_3 \leq c(n)\int_{B_{4\rho}} |f(x)| M(|Du^t_\rho|)\log^{-\alpha-1}(e^2+M(|Du^t_\rho|))\,dx. \qquad (3.10)
$$

Estimate of J_2

Arguing as in the estimate of J_0, we get

$$
\begin{aligned}
J_2 &= \left[\frac{1}{\alpha}\log^{-\alpha}(e^2+\lambda_\rho)-\log^{-\alpha-1}(e^2+\lambda_\rho)\right]\int_{E(\lambda_\rho)} |f(x)||Du^t_\rho|\,dx \\
&+ \frac{1}{\alpha}\int_{B_{4_\rho}\setminus E(\lambda_\rho)} |f(x)||Du^t_\rho|\log^{-\alpha}(e^2+M(|Du^t_\rho|))\,dx \\
&- \int_{B_{4\rho}\setminus E(\lambda_\rho)} |f(x)||Du^t_\rho|\log^{-\alpha-1}(e^2+M(|Du^t_\rho|))\,dx \\
&\leq \frac{1}{\alpha}\log^{-\alpha}(e^2+\lambda_\rho)\int_{E(\lambda_\rho)} |f(x)||Du^t_\rho|\,dx \\
&+ \frac{1}{\alpha}\int_{B_{4_\rho}\setminus E(\lambda_\rho)} |f(x)||Du^t_\rho|\log^{-\alpha}(e^2+M(|Du^t_\rho|))\,dx. \qquad (3.11)
\end{aligned}
$$

Combining estimates (3.7)–(3.11) we obtain

$$
\begin{aligned}
&\left[\frac{1}{\alpha}\log^{-\alpha}(e+\lambda_\rho)-\log^{-\alpha-1}(e^2+\lambda_\rho)\right]\int_{E(\lambda_\rho)} \langle A(x,Du), Du^t_\rho\rangle\,dx \\
+&\ \frac{1}{\alpha}\int_{B_{4_\rho}\setminus E(\lambda_\rho)} \langle A(x,Du), Du^t_\rho\rangle\log^{-\alpha}(e^2+M(|Du^t_\rho|))\,dx \\
-&\ \int_{B_{4\rho}\setminus E(\lambda_\rho)} \langle A(x,Du), Du^t_\rho\rangle\log^{-\alpha-1}(e^2+M(|Du^t_\rho|))\,dx \\
\leq&\ c(n)\int_{B_{4\rho}} |A(x,Du)|M(|Du^t_\rho|)\log^{-\alpha-1}(e^2+M(|Du^t_\rho|))\,dx \\
+&\ c(n)\int_{B_{4\rho}} |f(x)|M(|Du^t_\rho|)\log^{-\alpha-1}(e^2+M(|Du^t_\rho|))\,dx \\
+&\ \frac{1}{\alpha}\log^{-\alpha}(e^2+\lambda_\rho)\int_{E(\lambda_\rho)} |f(x)||Du^t_\rho|\,dx
\end{aligned}
$$

$$+ \quad \frac{1}{\alpha}\int_{B_{4\rho}\setminus E(\lambda_\rho)} |f(x)||Du_\rho^t| \log^{-\alpha}(e^2+M(|Du_\rho^t|))\,dx,$$

which implies

$$\begin{aligned}
&\left[\frac{1}{\alpha}\log^{-\alpha}(e^2+\lambda_\rho)-\log^{-\alpha-1}(e^2+\lambda_\rho)\right]\int_{E(\lambda_\rho)}\langle A(x,Du),Du_\rho^t\rangle\,dx\\
+&\quad \frac{1}{\alpha}\int_{B_{4\rho}}\langle A(x,Du),Du_\rho^t\rangle\log^{-\alpha}(e^2+M(|Du_\rho^t|))\,dx\\
-&\quad \frac{1}{\alpha}\int_{E(\lambda_\rho)}\langle A(x,Du),Du_\rho^t\rangle\log^{-\alpha}(e^2+M(|Du_\rho^t|))\,dx\\
\le&\quad c(n)\int_{B_{4\rho}}|A(x,Du)|M(|Du_\rho^t|)\log^{-\alpha-1}(e^2+M(|Du_\rho^t|))\,dx\\
+&\quad c(n,\alpha)\int_{B_{4\rho}}|f(x)|M(|Du_\rho^t|)\log^{-\alpha}(e^2+M(|Du_\rho^t|))\,dx\\
+&\quad \frac{1}{\alpha}\log^{-\alpha}(e^2+\lambda_\rho)\int_{E(\lambda_\rho)}|f(x)||Du_\rho^t|\,dx. \qquad (3.12)
\end{aligned}$$

Noticing that $\log^{-\sigma}(e^2+\lambda_\rho)\le\log^{-\sigma}(e^2+M(|Du_\rho^t|))$ on the set $E(\lambda_\rho)$ for every $\sigma>0$, we can rewrite estimate (3.12) as follows

$$\begin{aligned}
&\frac{1}{\alpha}\int_{B_{4\rho}}\langle A(x,Du),Du_\rho^t\rangle\log^{-\alpha}(e^2+M(|Du_\rho^t|))\,dx\\
\le&\quad c(n)\int_{B_{4\rho}}|A(x,Du)|M(|Du_\rho^t|)\log^{-\alpha-1}(e^2+M(|Du_\rho^t|))\,dx\\
+&\quad c(n,\alpha)\int_{B_{4\rho}}|f(x)|M(|Du_\rho^t|)\log^{-\alpha}(e^2+M(|Du_\rho^t|))\,dx\\
+&\quad c(n,\alpha)\int_{E(\lambda_\rho)}(|A(x,Du)|+|f(x)|)|Du_\rho^t|\log^{-\alpha}(e^2+M(|Du_\rho^t|))\,dx. \qquad (3.13)
\end{aligned}$$

In the Appendix we will prove that the integral over $E(\lambda_\rho)$ tends to 0 as $\rho\to+\infty$, i.e.

$$\lim_{\rho\to+\infty}\int_{E(\lambda_\rho)}(|A(x,Du)|+|f(x)|)|Du_\rho^t|\log^{-\alpha}(e^2+M(|Du_\rho^t|))\,dx=0. \qquad (3.14)$$

Hence, passing to the limit as ρ goes to $+\infty$ in (3.13), we get

$$\begin{aligned}
&\frac{1}{\alpha}\int_{\mathbb{R}^n}\langle A(x,Du),DT_t(u)\rangle\log^{-\alpha}(e^2+M(|DT_t(u)|))\,dx\\
\le&\quad c(n)\int_{\mathbb{R}^n}|A(x,Du)|M(|DT_t(u)|)\log^{-\alpha-1}(e^2+M(|DT_t(u)|))\,dx\\
+&\quad c(n,\alpha)\int_{\mathbb{R}^n}|f(x)|M(|DT_t(u)|)\log^{-\alpha}(e^2+M(|DT_t(u)|))\,dx\\
\le&\quad c(n)\int_{\mathbb{R}^n}|A(x,Du)|M(|Du|)\log^{-\alpha-1}(e^2+M(|Du|))\,dx\\
+&\quad c(n,\alpha)\int_{\mathbb{R}^n}|f(x)|M(|Du|)\log^{-\alpha}(e^2+M(|Du|))\,dx, \qquad (3.15)
\end{aligned}$$

where we have used that $|DT_t(u)| \leq |Du|$ for every $t > 0$ and the monotonicity of the functions $s\log^{-\alpha-1}(e^2+s)$ and $s\log^{-\alpha}(e^2+s)$. The constant $c(n,\alpha)$ in (3.15) can be explicitly expressed by $c(n,\alpha) = c(n) + \frac{c}{\alpha}$, with c an absolute constant. Note that Young's inequality (2.2) and Lemma 2.1 imply

$$
\begin{aligned}
&c(n,\alpha)\int_{\mathbb{R}^n} |f(x)| M(|Du|)\log^{-\alpha}(e^2+M(|Du|))\,dx \leq \\
&\int_{\mathbb{R}^n} M(|Du|)^2\log^{-2\alpha}(e^2+M(|Du|))\log^{\alpha-1}\Big[e^2+M(|Du|)\log^{-\alpha}(e^2+M(|Du|))\Big]\,dx \\
&\qquad +c(n,\alpha)\int_{\mathbb{R}^n} |f(x)|^2\log^{1-\alpha}(e^2+|f|)\,dx \\
&\leq c\int_{\mathbb{R}^n} M(|Du|)^2\log^{-\alpha-1}(e^2+M(|Du|)) \\
&\qquad +c(n,\alpha)\int_{\mathbb{R}^n} |f(x)|^2\log^{1-\alpha}(e^2+|f|)\,dx, \qquad (3.16)
\end{aligned}
$$

where c is a constant independent of α. Inserting (3.16) in (3.15) and using Young's inequality we get

$$
\begin{aligned}
&\frac{1}{\alpha}\int_{\mathbb{R}^n} \langle A(x,Du), DT_t(u)\rangle \log^{-\alpha}(e^2+M(|DT_t(u)|))\,dx \\
\leq\ & c(n)\int_{\mathbb{R}^n} |A(x,Du)|^2\log^{-\alpha-1}(e^2+M(|Du|)) \\
+\ & c\int_{\mathbb{R}^n} M(|Du|)^2\log^{-\alpha-1}(e^2+M(|Du|))\,dx \\
+\ & c(n,\alpha)\int_{\mathbb{R}^n} |f(x)|^2\log^{1-\alpha}(e^2+|f|)\,dx \\
=\ & I + II + III, \qquad (3.17)
\end{aligned}
$$

where c is an absolute constant. In order to estimate the integral I in (3.17), we use (1.2), (1.4), (1.8), (1.3) and Young's inequality (2.3) as follows

$$
\begin{aligned}
I \ \leq\ & c(n)\int_{\mathbb{R}^n} k^2(x)|Du|^2\log^{-\alpha-1}(e^2+M(|Du|))\,dx \\
\leq\ & c(n)\int_{\mathbb{R}^n} \frac{k^2(x)}{\mathcal{K}}(\mathcal{K}-\mathcal{K}_0)|Du|^2\log^{-\alpha-1}(e^2+M(|Du|))\,dx \\
+\ & c(n)\int_{\mathbb{R}^n} \frac{k^2(x)}{\mathcal{K}}\mathcal{K}_0|Du|^2\log^{-\alpha-1}(e^2+M(|Du|))\,dx \\
\leq\ & c(n)\int_{\mathbb{R}^n} [\exp(\beta(\mathcal{K}-\mathcal{K}_0))-1]\,dx \\
+\ & \frac{c(n)}{\beta}\int_{\mathbb{R}^n} \frac{k^2(x)}{\mathcal{K}}|Du|^2\log^{-\alpha-1}(e^2+M(|Du|))\log\left(e^2+\frac{k^2(x)}{\mathcal{K}}|Du|^2\right)\,dx \\
+\ & c(n)||\mathcal{K}_0||_\infty\int_{\mathbb{R}^n} \frac{1}{k(x)}|Du|^2\log^{-\alpha-1}(e^2+M(|Du|))\,dx
\end{aligned}
$$

$$\begin{aligned} &\leq c(n)\int_{\mathbb{R}^n}[\exp(\beta(\mathcal{K}-\mathcal{K}_0))-1]\,dx \\ &+ c(n)\left(\frac{1}{\beta}+||\mathcal{K}_0||_\infty\right)\int_{\mathbb{R}^n}\langle A(x,Du),Du\rangle\log^{-\alpha}(e^2+M(|Du|))\,dx, \end{aligned} \tag{3.18}$$

where we have also used that $\frac{k^2(x)}{\mathcal{K}}\leq\frac{1}{k(x)}\leq 1$ and $|Du|\leq M(|Du|)$.
In order to estimate II we observe that

$$II = c\int_{\mathbb{R}^n}\left(\frac{M(|Du|)^{\frac{2}{p}}}{\log^{\frac{1}{p}}(e^2+M(|Du|))}\right)^p\log^{-\alpha}(e^2+M(|Du|))\,dx,$$

for every exponent $1<p<2$. Since the function $s^{\frac{2}{p}}\log^{-\frac{1}{p}}(e^2+s)$ is convex for $p<2$ and large s, a simple use of the Jensen's inequality yields

$$\frac{M(|Du|)^{\frac{2}{p}}}{\log^{\frac{1}{p}}(e^2+M(|Du|))}\leq M\left(\frac{|Du|^{\frac{2}{p}}}{\log^{\frac{1}{p}}(e^2+|Du|)}\right), \tag{3.19}$$

hence

$$II\leq c\int_{\mathbb{R}^n}M\left(\frac{|Du|^{\frac{2}{p}}}{\log^{\frac{1}{p}}(e^2+|Du|)}\right)^p\log^{-\alpha}(e^2+M(|Du|))\,dx.$$

Since $\log^{-\alpha}(e^2+M(|Du|))$ is an A_p weight for every $p>1$ (see [7]), by the maximal Theorem in the weighted Lebesgue spaces, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n}\left(M\left(\frac{|Du|^{\frac{2}{p}}}{\log^{\frac{1}{p}}(e^2+|Du|)}\right)\right)^p\log^{-\alpha}(e^2+M(|Du|))\,dx \\ &\leq c(n,p)\int_{\mathbb{R}^n}\left(\frac{|Du|^{\frac{2}{p}}}{\log^{\frac{1}{p}}(e^2+|Du|)}\right)^p\log^{-\alpha}(e^2+M(|Du|))\,dx \\ &= c(n,p)\int_{\mathbb{R}^n}|Du|^2\log^{-1}(e^2+|Du|)\log^{-\alpha}(e^2+M(|Du|))\,dx. \end{aligned} \tag{3.20}$$

Arguing as we did in (3.18) we get

$$\begin{aligned} II &\leq c(n)\int_{\mathbb{R}^n}[\exp(\beta(\mathcal{K}-\mathcal{K}_0))-1]\,dx \\ &+ c(n)\left(\frac{1}{\beta}+||\mathcal{K}_0||_\infty\right)\int_{\mathbb{R}^n}\langle A(x,Du),Du\rangle\log^{-\alpha}(e^2+M(|Du|))\,dx. \end{aligned} \tag{3.21}$$

Inserting (3.18) and (3.21) in (3.17), we get

$$\begin{aligned} &\frac{1}{\alpha}\int_{\mathbb{R}^n}\langle A(x,Du),DT_t(u)\rangle\log^{-\alpha}(e^2+M(|DT_t(u)|))\,dx \\ &\leq c(n)\left(\frac{1}{\beta}+||\mathcal{K}_0||_\infty\right)\int_{\mathbb{R}^n}\langle A(x,Du),Du\rangle\log^{-\alpha}(e^2+M(|Du|))\,dx \\ &+ c(n,\alpha)\int_{\mathbb{R}^n}|f(x)|^2\log^{1-\alpha}(e^2+|f|)\,dx+c(n)\int_{\mathbb{R}^n}[\exp(\beta(\mathcal{K}-\mathcal{K}_0))-1]\,dx. \end{aligned} \tag{3.22}$$

Thanks to the assumptions (3.1) and (3.2), we can pass to the limit as $t \to +\infty$ in (3.22) and by Fatou's lemma we obtain

$$\begin{aligned} &\frac{1}{\alpha}\int_{\mathbb{R}^n} \langle A(x,Du),Du\rangle \log^{-\alpha}(e^2+M(|Du|))\,dx \\ \leq\;& c(n)\left(\frac{1}{\beta}+||\mathcal{K}_0||_\infty\right)\int_{\mathbb{R}^n} \langle A(x,Du),Du\rangle \log^{-\alpha}(e^2+M(|Du|)) \\ +\;& c(n,\alpha)\int_{\mathbb{R}^n} |f(x)|^2\log^{1-\alpha}(e^2+|f|)\,dx + c(n)\int_{\mathbb{R}^n}[\exp(\beta(\mathcal{K}-\mathcal{K}_0))-1]\,dx. \end{aligned} \tag{3.23}$$

Choosing $0<\alpha<\min\{1,\alpha_0\}$, where α_0 is defined by $\frac{1}{\alpha_0}=c(n)\left(\frac{1}{\beta}+||\mathcal{K}_0||_\infty\right)$, and thank to the assumption (3.1), we can absorb the first integral in the right hand side of (3.23) by the left hand side and we arrive at

$$\begin{aligned} &\int_{\mathbb{R}^n} \langle A(x,Du),Du\rangle \log^{-\alpha}(e^2+M(|Du|))\,dx \\ \leq\;& c(n,\beta,||\mathcal{K}_0||_\infty)\int_{\mathbb{R}^n} |f(x)|^2\log^{1-\alpha}(e^2+|f|)\,dx \\ +\;& c(n,\beta,||\mathcal{K}_0||_\infty)\int_{\mathbb{R}^n}[\exp(\beta(\mathcal{K}-\mathcal{K}_0))-1]\,dx, \end{aligned} \tag{3.24}$$

which, by virtue of the assumption (1.3), implies

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{1}{k(x)}|Du|^2 \log^{-\alpha}(e^2+M(|Du|))\,dx \\ \leq\;& c(n,\beta,||\mathcal{K}_0||_\infty)\int_{\mathbb{R}^n} |f(x)|^2\log^{1-\alpha}(e^2+|f|)\,dx \\ +\;& c(n,\beta,||\mathcal{K}_0||_\infty)\int_{\mathbb{R}^n}[\exp(\beta(\mathcal{K}-\mathcal{K}_0))-1]\,dx. \end{aligned} \tag{3.25}$$

Using again Young's inequality in Orlicz spaces we get

$$\begin{aligned} &\int_{\mathbb{R}^n} |Du|^2\log^{-1}(e+|Du|)\log^{-\alpha}(e^2+M(|Du|))\,dx \\ \leq\;& \int_{\mathbb{R}^n}[\exp(\beta(\mathcal{K}-\mathcal{K}_0))-1]\,dx \\ +\;& \left(\frac{1}{\beta}+||\mathcal{K}_0||_\infty\right)\int_{\mathbb{R}^n}\frac{1}{k(x)}|Du|^2\log^{-\alpha}(e^2+M(|Du|))\,dx. \end{aligned} \tag{3.26}$$

From estimate (3.20) we deduce that

$$\begin{aligned} &\int_{\mathbb{R}^n} |Du|^2\log^{-1}(e+|Du|)\log^{-\alpha}(e^2+M(|Du|))\,dx \\ \geq\;& c\int_{\mathbb{R}^n}\left(\frac{M(|Du|)^{\frac{2}{p}}}{\log^{\frac{1}{p}}(e+M(|Du|))}\right)^p \log^{-\alpha}(e^2+M(|Du|))\,dx \\ =\;& c\int_{\mathbb{R}^n}\frac{M(|Du|)^2}{\log(e+M(|Du|))}\log^{-\alpha}(e^2+M(|Du|))\,dx \\ \geq\;& c\int_{\mathbb{R}^n}|Du|^2\log^{-\alpha-1}(e^2+|Du|)\,dx. \end{aligned} \tag{3.27}$$

Combining (3.25), (3.26) and (3.27), we get

$$\begin{aligned} & \int_{\mathbb{R}^n} |Du|^2 \log^{-\alpha-1}(e^2+|Du|)\,dx \\ \leq \;& c(n,\beta,||\mathcal{K}_0||_\infty)\int_{\mathbb{R}^n} |f(x)|^2 \log^{1-\alpha}(e^2+|f|)\,dx \\ + \;& c(n,\beta,||\mathcal{K}_0||_\infty)\int_{\mathbb{R}^n} [\exp(\beta(\mathcal{K}-\mathcal{K}_0)-1]\,dx. \end{aligned} \tag{3.28}$$

Recalling that Orlicz functions equivalent at ∞ generate the same Orlicz space and by the definition of the norm in Orlicz spaces we deduce the assertion. □

4. The main result

In this section we prove our main result concerning the existence of infinite energy solutions for the equation (1.1). The proof is achieved via an approximation procedure and it relies on the a priori estimate proved in the previous section.

Proof of Theorem 1.2. For ε a positive real number, let us define

$$A_\varepsilon(x,\xi) = \frac{A(x,\xi)+\varepsilon k(x)\xi}{1+\varepsilon k(x)}. \tag{4.1}$$

Using the structure assumptions (1.2)–(1.4), one can easily check that

$$|A_\varepsilon(x,\xi)-A_\varepsilon(x,\eta)| \leq \frac{1+\varepsilon}{\varepsilon}|\xi-\eta|, \tag{4.2}$$

$$\langle A_\varepsilon(x,\xi)-A_\varepsilon(x,\eta),\xi-\eta\rangle \geq \frac{\varepsilon}{1+\varepsilon}|\xi-\eta|^2. \tag{4.3}$$

Moreover, we have the following bounds independent of ε

$$|A_\varepsilon(x,\xi)-A_\varepsilon(x,\eta)| \leq k(x)|\xi-\eta|, \tag{4.4}$$

$$\langle A_\varepsilon(x,\xi)-A_\varepsilon(x,\eta),\xi-\eta\rangle \geq \frac{1}{k(x)}|\xi-\eta|^2. \tag{4.5}$$

The equation

$$\operatorname{div} A_\varepsilon(x,Du_\varepsilon) = \operatorname{div} f \qquad \text{in } \mathbb{R}^n, \tag{4.6}$$

is uniformly elliptic thanks to (4.2) and (4.3). Moreover assumption (1.9) implies that

$$\int_{\mathbb{R}^n} |f|^2\,dx \leq \int_{\mathbb{R}^n} |f|^2 \log^{1-\alpha}(e+|f|)\,dx,$$

and then by classical results for each $\varepsilon>0$ there exists a unique solution $u_\varepsilon \in W^{1,2}(\mathbb{R}^n)\cap L^2_{loc}(\mathbb{R}^n)$ of the equation (4.6). Therefore we can apply Theorem 3.1 to find that

$$\begin{aligned} & ||Du_\varepsilon||^2_{L^2\log^{-\alpha-1}L(\mathbb{R}^n)} \\ & \leq c\int_{\mathbb{R}^n} |f(x)|^2 \log^{1-\alpha}(e^2+|f|)\,dx + c\int_{\mathbb{R}^n} [\exp(\beta(\mathcal{K}(x)-\mathcal{K}_0))-1]\,dx. \end{aligned} \tag{4.7}$$

A standard diagonal argument gives us a subsequence, still denoted by u_ε, weakly converging to a function u. By the lower semicontinuity of the norm, the gradient of the limit map Du belongs to $L^2 \log^{-\alpha-1} L(\mathbb{R}^n)$ and

$$\begin{aligned}
&||Du||^2_{L^2 \log^{-\alpha-1} L(\mathbb{R}^n)} \\
&\leq \; c\int_{\mathbb{R}^n} |f(x)|^2 \log^{1-\alpha}(e^2 + |f|)\, dx + c\int_{\mathbb{R}^n} [\exp(\beta(\mathcal{K}(x) - \mathcal{K}_0)) - 1]\, dx. \qquad (4.8)
\end{aligned}$$

It remains to prove that u is a solution of equation (1.1). To this aim, recall that since u_ε solves equation (4.6), then

$$\int_{\mathbb{R}^n} \langle A_\varepsilon(x, Du_\varepsilon), D\varphi\rangle = \int_{\mathbb{R}^n} \langle f, D\varphi\rangle,$$

for every test function $\varphi \in C_0^\infty(\mathbb{R}^n)$. Hence, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \langle A(x, Du), D\varphi\rangle \\
= \;& \int_{\mathbb{R}^n} \langle A(x, Du) - A_\varepsilon(x, Du), D\varphi\rangle \\
+ \;& \int_{\mathbb{R}^n} \langle A_\varepsilon(x, Du) - A_\varepsilon(x, Du_\varepsilon), D\varphi\rangle \\
+ \;& \int_{\mathbb{R}^n} \langle A_\varepsilon(x, Du_\varepsilon), D\varphi\rangle \\
= \;& I^1_\varepsilon + I^2_\varepsilon + \int_{\mathbb{R}^n} \langle f, D\varphi\rangle. \qquad (4.9)
\end{aligned}$$

By (4.9) it follows

$$\left| \int_{\mathbb{R}^n} \langle A(x, Du), D\varphi\rangle - \int_{\mathbb{R}^n} \langle f, D\varphi\rangle \right| \leq |I^1_\varepsilon| + |I^2_\varepsilon|. \qquad (4.10)$$

Next step is to prove that the right hand side of (4.10) tends to zero as ε goes to zero. Using the definition of the operator $A_\varepsilon(x, \xi)$ and (4.8) we get

$$\begin{aligned}
&|I^1_\varepsilon| \leq \int_{\mathbb{R}^n} |A(x, Du) - A_\varepsilon(x, Du)||D\varphi|\, dx \\
\leq \;& ||\varphi||_{C_0^\infty} \int_{\mathrm{supp}\varphi} \varepsilon k(x)|Du - A(x, Du)|\, dx \\
\leq \;& ||\varphi||_{C_0^\infty} \int_{\mathrm{supp}\varphi} \varepsilon(k(x) + k^2(x))|Du|\, dx \\
\leq \;& \varepsilon||\varphi||_{C_0^\infty} \int_{\mathrm{supp}\varphi} (\mathcal{K} - \mathcal{K}_0)|Du|\, dx \\
+ \;& \varepsilon||\mathcal{K}_0||_\infty ||\varphi||_{C_0^\infty} \int_{\mathrm{supp}\varphi} |Du|\, dx \\
\leq \;& \varepsilon||\varphi||_{C_0^\infty} \int_{\mathrm{supp}\varphi} [\exp(\mathcal{K} - \mathcal{K}_0) - 1]\, dx + \varepsilon \frac{||\varphi||_{C_0^\infty}}{\beta} \int_{\mathrm{supp}\varphi} |Du| \log(e + |Du|) \\
+ \;& \varepsilon||\mathcal{K}_0||_\infty ||\varphi||_{C_0^\infty} \int_{\mathrm{supp}\varphi} |Du|\, dx
\end{aligned}$$

$$\leq \varepsilon c(\beta, ||\mathcal{K}_0||_\infty, ||\varphi||_{C_0^\infty}) \Bigg(\int_{\{|Du|\leq 1\}\cap \mathrm{supp}\varphi} |Du| \log(e+|Du|)\, dx$$

$$+ \int_{\{|Du|>1\}\cap \mathrm{supp}\varphi} |Du|^2 \log^{-\alpha-1}(e+|Du|)\, dx + \int_{\mathrm{supp}\varphi} [\exp(\mathcal{K}-\mathcal{K}_0)-1]\, dx \Bigg) \leq c\varepsilon \tag{4.11}$$

and then

$$\limsup_{\varepsilon\to 0} |I_\varepsilon^1| = 0. \tag{4.12}$$

Moreover using (4.4), we have

$$|I_\varepsilon^2| \leq \int_{\mathrm{supp}\varphi} |A_\varepsilon(x, Du_\varepsilon) - A_\varepsilon(x, Du)||D\varphi|\, dx$$

$$\leq ||\varphi||_{C_0^\infty} \int_{\mathrm{supp}\varphi} k(x)|Du - Du_\varepsilon|\, dx$$

$$\leq c||\varphi||_{C_0^\infty} ||\mathcal{K}-\mathcal{K}_0||_{EXP} \left(\int_{\mathrm{supp}\varphi} |Du - Du_\varepsilon|^p\, dx \right)^{\frac{1}{p}}$$

$$+ c||\varphi||_{C_0^\infty} ||\mathcal{K}_0||_\infty \left(\int_{\mathrm{supp}\varphi} |Du - Du_\varepsilon|^p\, dx \right)^{\frac{1}{p}}, \tag{4.13}$$

for every exponent p satisfying $1 < p < 2$.
Now, we remark that, since $Du_\varepsilon \rightharpoonup Du$ in $L^2 \log^{-\alpha-1} L(\mathbb{R}^n)$, then $Du_\varepsilon \rightharpoonup Du$ in $L^2 \log^{-\alpha-1} L(\mathrm{supp}\varphi)$. By (2.7) we deduce that

$$\int_{\mathrm{supp}\varphi} |Du - Du_\varepsilon|^q$$

$$= \int_{\mathrm{supp}\varphi\cap\{|Du-Du_\varepsilon|\leq 1\}} |Du - Du_\varepsilon|^q + \int_{\mathrm{supp}\varphi\cap\{|Du-Du_\varepsilon|>1\}} |Du - Du_\varepsilon|^q$$

$$\leq |\mathrm{supp}\varphi| + \int_{\mathrm{supp}\varphi} |Du - Du_\varepsilon|^2 \log^{-\alpha-1}(e + |Du - Du_\varepsilon|).$$

Hence we also have that $Du_\varepsilon \rightharpoonup Du$ in $L^q(\mathrm{supp}\varphi)$, for all $1 < q < 2$. In the Appendix (Section 7.2) , arguing as in [3], we will prove that

$$Du_\varepsilon \to Du \qquad \text{in measure.} \tag{4.14}$$

Then there exists a subsequence, still denoted by u_ε, such that $Du_\varepsilon \to Du$ strongly in $L^p(\mathrm{supp}\varphi)$, $p < q < 2$, i.e.

$$\lim_{\varepsilon\to 0} \int_{\mathrm{supp}\varphi} |Du - Du_\varepsilon|^p = 0. \tag{4.15}$$

Passing to the limit in (4.13) as $\varepsilon \to 0$ and using (4.15) we get

$$\lim_{\varepsilon\to 0} |I_\varepsilon^2| = 0. \tag{4.16}$$

Then we conclude that

$$\int_{\mathbb{R}^n} A(x, Du) D\varphi\, dx = \int_{\mathbb{R}^n} f D\varphi\, dx,$$

i.e. u is a solution of equation (1.1) such that $Du \in L^2 \log^{-\alpha-1} L(\mathbb{R}^n)$. □

5. The regularity

This section is devoted to the study of the regularity properties of solutions of equation (1.1) when the right hand side $f \in L^{\gamma}_{loc}(\mathbb{R}^n)$, $\gamma > 2$. In this case, the infinite energy solutions are actually finite energy solutions and hence their gradients belongs to the Orlicz-Zygmund class $L^2 \log^{\alpha} L_{loc}(\mathbb{R}^n)$, for a suitable $\alpha > 0$, thanks to the regularity results of [6]. Moreover, as said in the introduction, in [6] it has been proved that if $f \in L^{\gamma}_{loc}$, for some $\gamma > n$, then the solutions are locally bounded. Here we study the regularity properties of a solution in the setting of Lebesgue spaces L^{γ}_{loc} when $2 < \gamma < n$. To this aim, first of all, we prove a Lemma that we will use in the following and that can be of interest by itself.

Denote by B_t the ball of radius t centered at x_0 where $x_0 \in \Omega$ and Ω is a bounded open set of $\mathbb{R}^n$. For $\lambda > 0$ let

$$A_\lambda = \{x \in \Omega : |u(x)| > \lambda\}, \qquad A_{\lambda,t} = A_\lambda \cap B_t.$$

Moreover if $m < n$, m^* is the Sobolev embedding exponent, i.e. it results $\frac{1}{m^*} = \frac{1}{m} - \frac{1}{n}$.

Lemma 5.1. *Let $u \in W^{1,p}_{loc}(\Omega)$, $\varphi_0 \in L^r_{loc}(\Omega)$, where $1 < p < n$ and r satisfies*

$$1 < r < \frac{n}{p}. \tag{5.1}$$

Assume the following integral estimate holds

$$\int_{A_{\lambda,\tau}} |Du|^p \le c_0 \left[\int_{A_{\lambda,t}} \varphi_0 + (t-\tau)^{-\alpha} \int_{A_{\lambda,t}} |u|^q \right], \tag{5.2}$$

for every $\lambda \in \mathbb{N}$ and $R_0 \le \tau < t \le R_1$, where c_0 is a positive constant that depends only on n, p, r, q, R_0, R_1 and $|\Omega|$, α is a real positive constant and $0 < q < p^*$. Then it follows that

$$u \in L^s_{loc}(\Omega), \qquad s = (pr)^*. \tag{5.3}$$

Proof. If $q = p$ the proof of Lemma 5.1 can be found in [8]. Actually the proof in [8] works also if $q < p$. Hence it suffices to prove (5.3) when $p < q < p^*$. In the appendix, by using the same argument of [8], we will prove that for all $R_0 \le t' < t \le R_1$ with $B_{R_1} \subset \Omega$ the following estimate holds true

$$\left(\int_{B_{t'}} |T_{j+1}(u)|^{(m+1)p^*} \right)^{\frac{p}{p^*}} \le c \int_{B_t} \left[(t-t')^{-\beta}(|u|^q + 1) + \varphi_0 \right] \left(1 + |T_{j+1}(u)|^{pm}\right), \tag{5.4}$$

where $m > 0$ is a positive constant to be chosen, T_{j+1} is the truncation function at levels $\pm(j+1)$ defined in (3.4), c is a constant depending only on the data (see formula (7.13) in the appendix) and $\beta = \max\{\alpha, p\}$. Since it results

$$c(t-t')^{-\beta} \int_{B_t} (|u|^q + 1) \le c(t-t')^{-\beta} \|u\|^q_{L^{p^*}(B_t)} |B_t|^{1-\frac{q}{p^*}} + c(t-t')^{-\beta}|B_t|, \tag{5.5}$$

it remains to evaluate only the following term

$$c(t-t')^{-\beta}\int_{B_t}(|u|^q+1)|T_{j+1}(u)|^{pm}. \tag{5.6}$$

We proceed by steps.
Step 1: If $1<r\le \frac{p^*}{q}$ then it results $pm\frac{p^*}{p^*-q}\le pm\frac{r}{r-1}$ and Hölder's and Young's inequalities give

$$\begin{aligned}
&c(t-t')^{-\beta}\int_{B_t}|u|^q|T_{j+1}(u)|^{pm}\le\\
&c(t-t')^{-\beta}\left(\int_{B_t}|u|^{p^*}\right)^{\frac{q}{p^*}}\left(\int_{B_t}|T_{j+1}(u)|^{pm\frac{p^*}{p^*-q}}\right)^{1-\frac{q}{p^*}}\le\\
&c(t-t')^{-\beta}c(\|u\|^q_{L^{p^*}(B_t)},|B_t|)\left(\int_{B_t}|T_{j+1}(u)|^{pm\frac{r}{r-1}}\right)^{1-\frac{1}{r}},
\end{aligned}$$

where as before $c(\|u\|^q_{p^*},|B_t|)$ denotes a positive constant that depends only on the quantities $\|u\|^q_{L^{p^*}(B_t)}$ and $|B_t|$ that can vary from line to line. Moreover, since by assumption (5.1) we have $1-1/r<p/p^*$, from the previous inequality we deduce

$$\begin{aligned}
&c(t-t')^{-\beta}\int_{B_t}|u|^q|T_{j+1}(u)|^{pm}\le\\
&\le\frac{\varepsilon}{2}\left(\int_{B_t}|T_{j+1}(u)|^{pm\frac{r}{r-1}}\right)^{\frac{p}{p^*}}+C(\varepsilon,c,\|u\|^q_{L^{p^*}(B_t)},|B_t|)(t-t')^{-\beta\frac{(n-p)r}{n-pr}},
\end{aligned}\tag{5.7}$$

for every fixed $\varepsilon\in(0,1)$. Similarly we have

$$\begin{aligned}
&c(t-t')^{-\beta}\int_{B_t}|T_{j+1}(u)|^{pm}\le\\
&\le\frac{\varepsilon}{2}\left(\int_{B_t}|T_{j+1}(u)|^{pm\frac{r}{r-1}}\right)^{\frac{p}{p^*}}+C(\varepsilon,c,|B_t|)(t-t')^{-\beta\frac{(n-p)r}{n-pr}}.
\end{aligned}\tag{5.8}$$

Using (5.5), (5.7) and (5.8) in (5.4) we deduce the following inequality

$$\begin{aligned}
&\left(\int_{B_{t'}}|T_{j+1}(u)|^{(m+1)p^*}\right)^{\frac{p}{p^*}}\le c\int_{B_t}\varphi_0(1+|T_{j+1}(u)|^{mp})+\\
&\varepsilon\left(\int_{B_t}|T_{j+1}(u)|^{pm\frac{r}{r-1}}\right)^{\frac{p}{p^*}}+C(\varepsilon,c,\|u\|^q_{L^{p^*}(B_t)},|B_t|)\left[(t-t')^{-\beta}+(t-t')^{-\beta\frac{(n-p)r}{n-pr}}\right],
\end{aligned}\tag{5.9}$$

for every fixed $\varepsilon\in(0,1)$.

We prove now that the previous inequality implies the assertion. In what follows we denote again by c a constant that depends only on the data c_0, α, p, n and r which can vary from line to line. Let us choose m such that

$$\frac{pmr}{r-1}=(m+1)p^*=(pr)^*, \tag{5.10}$$

that is $m = n(r-1)/(n-rp) > 0$ by assumption (5.1). Thanks to this choice we have

$$c \int_{B_t} \varphi_0 (1 + |T_{j+1}(u)|^{mp}) \leq \tag{5.11}$$
$$c\|\varphi_0\|_{L^r(B_t)} \left\{ |B_t|^{1-\frac{1}{r}} + \left[\int_{B_t} |T_{j+1}(u)|^{(m+1)p^*} \right]^{1-\frac{1}{r}} \right\}.$$

We observe also that using again assumption (5.1), we have $1 - 1/r < p/p^*$ and so the righ-hand side of (5.11) can be controlled by

$$c\|\varphi_0\|_{L^r(B_t)} \left\{ |B_t|^{1-\frac{1}{r}} + 1 + \left[\int_{B_t} |T_{j+1}(u)|^{(m+1)p^*} \right]^{\frac{p}{p^*}} \right\}. \tag{5.12}$$

Choosing $\varepsilon = 1/4$ and R_1 as follows

$$R_1 \leq 1 : \qquad c\|\varphi_0\|_{L^r(B_{R_1})} \leq 1/4, \tag{5.13}$$

and combining the previous estimates, we deduce

$$\left(\int_{B_{t'}} |T_{j+1}(u)|^{(m+1)p^*} \right)^{\frac{p}{p^*}} \leq \frac{1}{2} \left(\int_{B_t} |T_{j+1}(u)|^{(m+1)p^*} \right)^{\frac{p}{p^*}} +$$
$$c\|\varphi_0\|_{L^r(B_t)} \left[|B_t|^{1-\frac{1}{r}} + 1 \right] + C(c, \|u\|^q_{L^{p^*}(B_t)}, |B_t|)(t-t')^{-\beta(m+1)}. \tag{5.14}$$

Let $R_0 \leq \rho < R \leq R_1$ be arbitrarily fixed. Thus by the previous inequality, for every t and t' such that $\rho \leq t' < t \leq R$ we obtain

$$\left(\int_{B_{t'}} |T_{j+1}(u)|^{(m+1)p^*} \right)^{\frac{p}{p^*}} \leq \frac{1}{2} \left(\int_{B_t} |T_{j+1}(u)|^{(m+1)p^*} \right)^{\frac{p}{p^*}} +$$
$$c\|\varphi_0\|_{L^r(B_R)} \left[|B_R|^{1-\frac{1}{r}} + 1 \right] + C(c, \|u\|^q_{L^{p^*}(B_R)}, |B_R|)(t-t')^{-\beta(m+1)}, \tag{5.15}$$

since the constant C in (5.14) depends on $\|u\|^q_{L^{p^*}(B_t)}$ and on $|B_t|$ in an increasing way. Applying Lemma 2.4 we conclude that

$$\left(\int_{B_\rho} |T_{j+1}(u)|^{(m+1)p^*} \right)^{\frac{p}{p^*}} \leq \tag{5.16}$$
$$C(c, \beta(m+1)) \left\{ \|\varphi_0\|_{L^r(B_R)} \left[|B_R|^{1-\frac{1}{r}} + 1 \right] + C(c, \|u\|^q_{L^{p^*}(B_R)}, |B_R|, \beta(m+1))(R-\rho)^{-\beta(m+1)} \right\},$$

where $C(c, \beta(m+1)) = c\gamma$ with γ as in (2.12). Letting $j \to +\infty$ in (5.16), and recalling (5.10), we obtain

$$\left(\int_{B_\rho} |u|^{(pr)^*} \right)^{\frac{p}{p^*}} \leq \tag{5.17}$$
$$C(c, \beta(m+1)) \left\{ \|\varphi_0\|_{L^r(B_R)} \left[|B_R|^{1-\frac{1}{r}} + 1 \right] + C(c, \|u\|^q_{L^{p^*}(B_R)}, |B_R|, \beta(m+1))(R-\rho)^{-\beta(pr)^*/p^*} \right\},$$

from which the assertion follows.

Step 2: If $\frac{p^*}{q} \geq \frac{n}{p}$ there is nothing to prove. If $r_0 \equiv \frac{p^*}{q} < \frac{n}{p}$ we have to consider the remaining case $\frac{n}{p} > r > \frac{p^*}{q}$. Since $\varphi_0 \in L^r_{loc}(\Omega)$ we also have that $\varphi_0 \in L^{\frac{p^*}{q}}_{loc}(\Omega)$. Thus by the result of step 1 we deduce that $u \in L^{(p\frac{p^*}{q})^*}_{loc}(\Omega)$ and we can replace estimate (5.7) by the following

$$\int_{B_t} |u|^q |T_{j+1}(u)|^{pm} \leq \left(\int_{B_t} |u|^{(p\frac{p^*}{q})^*} \right)^{\frac{q}{(p\frac{p^*}{q})^*}} \left(\int_{B_t} |T_{j+1}(u)|^{pm \frac{(p\frac{p^*}{q})^*}{(p\frac{p^*}{q})^* - q}} \right)^{1 - \frac{q}{(p\frac{p^*}{q})^*}} . \tag{5.18}$$

We note that if $r \leq \frac{(p\frac{p^*}{q})^*}{q}$ then $pm\frac{(p\frac{p^*}{q})^*}{(p\frac{p^*}{q})^*-q} \leq pm\frac{r}{r-1}$. Hence we have

$$\left(\int_{B_t} |T_{j+1}(u)|^{pm \frac{(p\frac{p^*}{q})^*}{(p\frac{p^*}{q})^* - q}} \right)^{1 - \frac{q}{(p\frac{p^*}{q})^*}} \leq C(|B_t|) \left(\int_{B_t} |T_{j+1}(u)|^{pm\frac{r}{r-1}} \right)^{1-\frac{1}{r}} , \tag{5.19}$$

and thus proceeding as in step 1 we deduce the assertion.
Now if $\frac{(p\frac{p^*}{q})^*}{q} \geq \frac{n}{p}$ the proof is concluded, otherwise it remains to prove the theorem when $\frac{n}{p} > r > r_1 \equiv \frac{(p\frac{p^*}{q})^*}{q}$. Under such a condition on r we can proceed exactly as before (i.e. since $\varphi_0 \in L^r_{loc}(\Omega)$ implies $\varphi_0 \in L^{r_1}_{loc}(\Omega)$ and hence $u \in L^{(pr_1)^*}_{loc}(\Omega)$ ...) and conclude that if $r \leq r_2 = \frac{(r_1 p)^*}{q}$ then $u \in L^{(pr)^*}_{loc}(\Omega)$. Again the proof is concluded if $r_2 \geq \frac{n}{p}$, otherwise we need to consider the case $\frac{n}{p} > r > r_2$ which can be treated exactly as before. Notice that the sequence $r_{i+1} = \frac{(r_i p)^*}{q}$, $i \in \mathbb{N}$ is strictly increasing. As a matter of fact we have

$$r_{i+1} > r_i \qquad \Leftrightarrow \qquad r_i > \frac{n(q-p)}{pq}, \tag{5.20}$$

where the second inequality can be easily proved by induction. We complete the proof showing that there exists a value $\iota \in \mathbb{N}$ such that $r_\iota \geq \frac{n}{p}$ (and hence the procedure ends after no more than ι steps). To this aim we observe that by the monotonicity of this sequence it follows that

$$\exists \lim_{i \to +\infty} r_i = l, \qquad l \in \left(\frac{n(q-p)}{pq}, +\infty \right] . \tag{5.21}$$

The assertion follows proving that $l = +\infty$. As a matter of fact, if $l \in \mathbb{R}^+$ it follows that

$$l = \frac{(lp)^*}{q} \qquad \Leftrightarrow \qquad l = \frac{n(q-p)}{pq}$$

and this contradicts (5.21). □

Now we can prove the following regularity result

Theorem 5.2. *Assume* (1.2) *and* (1.3) *and let* u *be a finite energy solution of the equation* (1.1). *Suppose that* $\mathcal{K}$ *satisfies* (1.8) *and that*

$$f \in L^\gamma_{loc}(\mathbb{R}^n) \qquad 2 < \gamma < n. \tag{5.22}$$

Then we have

$$u \in L^s_{loc}(\mathbb{R}^n), \qquad \forall\ s < \gamma^* = \frac{n\gamma}{n-\gamma}. \tag{5.23}$$

Proof. Let us fix a ball B_{R_1} and consider $R_0 \le \tau < t \le R_1$. Define the function

$$\varphi = \eta^2[u - T_\lambda(u)], \quad \lambda > 0,$$

where T_λ denotes the truncation at levels $\pm\lambda$ defined in (3.4) and $\eta \in C_0^\infty(B_{R_1})$ is a cut-off function such that

$$\operatorname{supp} \eta \subset B_t, \qquad 0 \le \eta \le 1, \qquad \eta = 1 \text{ in } B_\tau, \qquad |\nabla \eta| \le \frac{c}{t-\tau}.$$

Using φ as test function in (1.1), we get

$$\begin{aligned} & \int_{B_{R_1}} \langle A(x,Du), D(u - T_\lambda(u))\eta^2\rangle \, dx + 2\int_{B_{R_1}} \langle A(x,Du), \eta D\eta(u - T_\lambda(u))\rangle \, dx \\ = & \int_{B_{R_1}} \langle f(x), D(u - T_\lambda(u))\eta^2\rangle \, dx + 2\int_{B_{R_1}} \langle f(x), \eta D\eta(u - T_\lambda(u))\rangle \, dx. \end{aligned}$$

Setting

$$A_{\lambda,r} = \{x \in B_r : \ |u(x)| > \lambda\},$$

we can rewrite previous equality as follows

$$\begin{aligned} & \int_{A_{\lambda,R_1}} \eta^2 \langle A(x,Du), Du\rangle \, dx \\ = & -2\int_{A_{\lambda,R_1}} \langle A(x,Du), \eta D\eta(u - T_\lambda(u))\rangle \, dx \\ + & \int_{A_{\lambda,R_1}} \eta^2 \langle f(x), Du\rangle \, dx \\ + & \ 2\int_{A_{\lambda,R_1}} \langle f(x), \eta D\eta(u - T_\lambda(u))\rangle \, dx. \end{aligned} \tag{5.24}$$

Thanks to the assumptions (1.2)-(1.4) and using the properties of η, we obtain

$$\begin{aligned} & \int_{A_{\lambda,R_1}} \frac{1}{k(x)}|Du|^2\eta^2 \, dx \le \frac{c}{t-\tau}\int_{A_{\lambda,R_1}} k(x)|Du||\eta||u| \, dx \\ + & \int_{A_{\lambda,R_1}} |f(x)||Du|\eta^2 \, dx + \frac{c}{t-\tau}\int_{A_{\lambda,R_1}} |f(x)||\eta||u| \, dx. \end{aligned} \tag{5.25}$$

A simple use of Young's inequality yields

$$\begin{aligned} & \int_{A_{\lambda,R_1}} \frac{1}{k(x)}|Du|^2\eta^2 \, dx \le \frac{c}{(t-\tau)^2}\int_{A_{\lambda,R_1}} k^3(x)|u|^2 \, dx \\ + & \ \varepsilon\int_{A_{\lambda,R_1}} \frac{1}{k(x)}|Du|^2|\eta|^2 \, dx + \frac{c}{(t-\tau)^2}\int_{A_{\lambda,R_1}} |u|^2 \, dx \\ + & \int_{A_{\lambda,R_1}} k(x)|f(x)|^2 \, dx + \int_{A_{\lambda,R_1}} |f(x)|^2 \, dx. \end{aligned} \tag{5.26}$$

Choosing ε sufficiently small, we get

$$\begin{aligned} & \int_{A_{\lambda,R_0}} \frac{1}{k(x)}|Du|^2 \, dx \le \frac{c}{(t-\tau)^2}\int_{A_{\lambda,R_1}} k^3(x)|u|^2 \, dx \\ + & \ c\int_{A_{\lambda,R_1}} k(x)|f(x)|^2 \, dx, \end{aligned} \tag{5.27}$$

where we used that $k(x) \geq 1$.
Setting $\varphi_0 = k(x)|f(x)|^2$, since $k^3(x) \in EXP_{loc}(\mathbb{R}^n)$ and $|f|^2 \in L^{\frac{\gamma}{2}}$, we observe that $\varphi_0 \in L^{\frac{\gamma}{2}} \log^{-\frac{1}{3}} L(\mathbb{R}^n)$ and hence $\varphi_0 \in L^r(\mathbb{R}^n)$ for every $r < \frac{\gamma}{2}$. Moreover, using Holder's inequality in Orlicz spaces, from (5.27) we deduce that

$$\int_{A_{\lambda,R_0}} \frac{|Du|^2}{\log(e+|Du|)}\,dx \leq \frac{c}{(t-\tau)^2}\int_{A_{\lambda,R_1}} |u|^2\log(e+|u|)\,dx + c\int_{A_{\lambda,R_1}} \varphi_0\,dx, \tag{5.28}$$

and therefore

$$\int_{A_{\lambda,R_0}} |Du|^p\,dx \leq \frac{c}{(t-\tau)^2}\int_{A_{\lambda,R_1}} |u|^q\,dx + c\int_{A_{\lambda,R_1}} \varphi_0\,dx, \tag{5.29}$$

for every p and q satisfying $\frac{2n}{n+2} < p < 2 < q < p^*$. Notice that by assumption (5.22) and being $p < 2$, it follows that $1 < r < \frac{\gamma}{2} < \frac{n}{p}$. Hence we can apply Lemma 5.1 and we deduce that $u \in L^s_{loc}(\mathbb{R}^n)$, $s = (pr)^*$, for every $p < 2$ and $r < \frac{\gamma}{2}$ and hence (5.23) follows. □

6. An Example

For $x \in B(0,e^{-1}) \setminus \{0\} \subset \mathbb{R}^2$, let us introduce the function

$$u(x_1,x_2) = \frac{x_1}{|x|}\exp\log^{\frac{2}{3}}\frac{1}{|x|}\,.$$

Elementary calculations yield that

$$\frac{\partial u}{\partial x_1} = \frac{1}{|x|}\rho(|x|) - \frac{x_1^2}{|x|^3}\rho(|x|)\left(1+\frac{2}{3\log^{\frac{1}{3}}\frac{1}{|x|}}\right),$$

$$\frac{\partial u}{\partial x_2} = -\frac{x_1x_2}{|x|^3}\rho(|x|)\left(1+\frac{2}{3\log^{\frac{1}{3}}\frac{1}{|x|}}\right),$$

where in order to simplify the notations we set $\rho(|x|) = \exp\log^{\frac{2}{3}}\frac{1}{|x|}$. Next, let us consider the function

$$v(x_1,x_2) = -\frac{x_2}{|x|}\exp\log^{\frac{2}{3}}\frac{1}{|x|}, \qquad x \neq 0,$$

and introduce the matrix

$$A(x) = \frac{1}{u_{x_1}v_{x_2} - u_{x_2}v_{x_1}}\begin{pmatrix} u_{x_2}^2 + v_{x_2}^2 & -u_{x_1}u_{x_2} - v_{x_1}v_{x_2} \\ -u_{x_1}u_{x_2} - v_{x_1}v_{x_2} & u_{x_1}^2 + v_{x_1}^2 \end{pmatrix}.$$

It is well known that (see for example [14])

$$\mathrm{div}A(x)\nabla u = \mathrm{div}(-v_{x_2}, v_{x_1}) = 0.$$

Moreover the eigenvalues of the matrix $A(x)$ are given by $\frac{1}{k(x)}$ and $k(x)$ where

$$k(x) = 3\log^{\frac{1}{3}}\frac{1}{|x|}\,.$$

Since

$$\langle A(x)\nabla u, \nabla u\rangle = \frac{2\rho^2(|x|)}{3|x|^2 \log^{\frac{1}{3}} \frac{1}{|x|}},$$

we have that

$$\int_{B(0,e^{-1})} \langle A(x)\nabla u, \nabla u\rangle \, dx = c\int_0^{e^{-1}} \frac{\rho^2(r)}{2r \log^{\frac{1}{3}} \frac{1}{r}} \, dr$$

$$= \quad -c\int_0^{e^{-1}} \rho(r)\rho'(r)\, dr = -c[\rho^2(r)]_0^{e^{-1}} = +\infty\,.$$

One can easily check that

$$\int_{B(0,e^{-1})\cap \mathrm{supp}\varphi} \langle A(x)\nabla u, \nabla \varphi\rangle \, dx = e^2 \int_{\partial B(0,e^{-1})\cap \mathrm{supp}\varphi} \varphi \, dx_2 \qquad \forall \varphi \in C_0^\infty(\mathbb{R}^2)$$

Hence setting

$$\tilde{u}(x) = \begin{cases} u(x) & \text{in } B(0,e^{-1})\setminus\{0\} \\ \\ e^2 x_1 & \text{in } \mathbb{R}^2 \setminus B(0,e^{-1}) \end{cases}$$

$$\tilde{A}(x) = \begin{cases} A(x) & \text{in } B(0,e^{-1})\setminus\{0\} \\ \\ \mathbf{I} & \text{in } \mathbb{R}^2 \setminus B(0,e^{-1}) \end{cases}$$

and

$$\tilde{k}(x) = \begin{cases} k(x) & \text{in } B(0,e^{-1})\setminus\{0\} \\ \\ 1 & \text{in } \mathbb{R}^2 \setminus B(0,e^{-1}) \end{cases}$$

we have that $\tilde{u}$ is an infinite energy solution of the divergence type equation

$$\mathrm{div}(\tilde{A}(x)\nabla \tilde{u}) = 0 \qquad\qquad \text{in } \mathbb{R}^2\,.$$

Note that $\tilde{A}$ satisfies the assumptions (1.2)-(1.5), for $\mathcal{K} = (\tilde{k}^2+1)\tilde{k}$. Moreover there exists $\beta > 0$ such that $\exp(\beta\mathcal{K}) \in L^1(B(0,e^{-1}))$, hence also (1.8) is satisfied with $\mathcal{K}_0 = 2$ in $\mathbb{R}^2$.

7. Appendix

In Subsection 7.1 we prove the estimate (3.14) while in Subsection 7.2 we prove the statement (4.14). Finally in Subsection 7.3 we prove the estimate (5.4).

7.1. **Proof of estimate** (3.14)**.** Observe that

$$\lim_{\rho\to+\infty}\lambda_\rho = 0,$$

$$E(\lambda_\rho)\subset\left\{x\in B_{4\rho}:\ M(|\varphi_\rho DT_t(u)|)(x)\le\frac{ct}{\rho}+\lambda_\rho\right\}=A_\rho,$$

and that there exists a sequence $\rho_h\to\infty$ such that

$$\chi_{A_{\rho_h}}\to 0 \qquad\qquad a.e.$$

Now, using the sublinearity of the maximal operator, the monotonicity of the function $\frac{t}{\log^\alpha(e+t)}$, the elementary inequality (2.5) the property of the function φ_ρ and the definition of the A_ρ, we obtain

$$\begin{aligned}
&\int_{E(\lambda_\rho)}[|A(x,Du)|+|f(x)|]M(|Du_\rho^t|)\log^{-\alpha}(e^2+M(|Du_\rho^t|))\,dx\\
\le\ &\int_{E(\lambda_\rho)}[|A(x,Du)|+|f(x)|]\left(\frac{M(|\varphi_\rho DT_t(u)|)}{\log^\alpha(e^2+M(|\varphi_\rho DT_t(u)|)}+\frac{M(|T_t(u)D\varphi_\rho|)}{\log^\alpha(e^2+M(|T_t(u)D\varphi_\rho|)}\right)dx\\
\le\ &\frac{c}{\rho}\int_{\mathbb{R}^n}[|A(x,Du)|+|f(x)|]M(|T_t(u)|)\,dx\\
+\ &\int_{A_\rho}[|A(x,Du)|+|f(x)|]\frac{M(|\varphi_\rho DT_t(u)|)}{\log^\alpha(e^2+M(|\varphi_\rho DT_t(u)|)}\,dx\\
=\ &\frac{c}{\rho}\int_{\mathbb{R}^n}[|A(x,Du)|+|f(x)|]M(|T_t(u)|)\,dx\\
+\ &\int_{\mathbb{R}^n}[|A(x,Du)|+|f(x)|]\frac{M(|\varphi_\rho DT_t(u)|)}{\log^\alpha(e^2+M(|\varphi_\rho DT_t(u)|)}\chi_{A_\rho}\,dx\\
\le\ &\frac{ct}{\rho}\int_{\mathbb{R}^n}[|A(x,Du)|+|f(x)|]\,dx+\frac{\frac{ct}{\rho}+\lambda_\rho}{\log^\alpha(e^2+\frac{ct}{\rho}+\lambda_\rho)}\int_{\mathbb{R}^n}(|A(x,Du)|+|f(x)|)\,dx.
\end{aligned}$$

Thanks to the assumption (3.1) we have that $|A(x,Du)|\in L^1(\mathbb{R}^n)$. Hence passing to the limit as $\rho\to+\infty$ we get the conclusion.

□

7.2. **Proof of the statement** (4.14)**.** Let us denote by ω the set supp φ. Our aim is to prove that for every η, $\lambda>0$, there exists $\nu=\nu(\eta,\lambda)$ such that

$$|\{x\in\omega:\ |Du_\varepsilon-Du_{\varepsilon'}|>\lambda\}|<\eta, \tag{7.1}$$

for every ε and ε' in $(0,\nu)$. For some $B>1$, let us define

$$E_1=\{x\in\omega:\ |Du_\varepsilon|>B\}\cup\{x\in\omega:\ |Du_{\varepsilon'}|>B\},$$

$$E_2=\{x\in\omega:\ |Du_\varepsilon|\le B,\ |Du_{\varepsilon'}|\le B,\ |Du_\varepsilon-Du_{\varepsilon'}|>\lambda\}.$$

First, let us observe that

$$\{x\in\omega:\ |Du_\varepsilon-Du_{\varepsilon'}|>\lambda\}\subset E_1\cup E_2.$$

Since $||Du_\varepsilon||^q_{L^q(\omega)}\le c$, we have that

$$|E_1|<\frac{\eta}{2},$$

for $B = \max\{1, \frac{8c}{\eta}\}$, independently of $\varepsilon, \varepsilon'$. We may always suppose that η is sufficiently small to have that $B = \frac{8c}{\eta}$. Now the definition of A_ε at (4.1) and elementary calculations yield

$$
\begin{aligned}
&\langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle \\
\geq \quad & \frac{1}{k(x)}|Du_\varepsilon - Du_{\varepsilon'}|^2 - \frac{|\varepsilon - \varepsilon'|k}{(1+\varepsilon k)(1+\varepsilon' k)}|\langle A(x, Du_{\varepsilon'}) - Du_{\varepsilon'}, Du_\varepsilon - Du_{\varepsilon'}\rangle|
\end{aligned}
$$

and then

$$
\begin{aligned}
&\int_{E_2} \frac{1}{k(x)}|Du_\varepsilon - Du_{\varepsilon'}|^2 dx \leq \int_{E_2} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle dx \\
+ \quad & |\varepsilon - \varepsilon'| \int_{E_2} \frac{k}{(1+\varepsilon k)(1+\varepsilon' k)}|\langle A(x, Du_{\varepsilon'}) - Du_{\varepsilon'}, Du_\varepsilon - Du_{\varepsilon'}\rangle| dx \\
\leq \quad & \int_{E_2} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle dx + \int_{E_2} \frac{1}{2k(x)}|Du_\varepsilon - Du_{\varepsilon'}|^2 dx \\
+ \quad & |\varepsilon - \varepsilon'|^2 \int_{E_2} \frac{k^3}{(1+\varepsilon k)^2(1+\varepsilon' k)^2}|A(x, Du_{\varepsilon'}) - Du_{\varepsilon'}|^2 dx. \qquad (7.2)
\end{aligned}
$$

From (7.2), using the definition of E_2, we deduce that

$$
\begin{aligned}
&\int_{E_2} \frac{1}{2k(x)}|Du_\varepsilon - Du_{\varepsilon'}|^2 dx \\
\leq \quad & \int_{E_2} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle dx \\
+ \quad & |\varepsilon - \varepsilon'|^2 \int_{E_2} k^3(k^2B^2 + B^2) dx \\
\leq \quad & \int_{E_2} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle dx \\
+ \quad & c(\beta, [\mathcal{K}], ||\mathcal{K}_0||_\infty, |\omega|)\frac{|\varepsilon - \varepsilon'|^2}{\eta^2}, \qquad (7.3)
\end{aligned}
$$

where we used that $B = \frac{8c}{\eta}$. We can verify, as in [3], that E_2 is a compact set. In order to estimate the first integral in the right hand side of (7.3), let us denote by $E_{2,t}$, for every $t > 0$, the set

$$
E_{2,t} = \Big\{x \in E_2 : \operatorname{dist}(x, \partial E_2) > t\Big\}.
$$

Consider the subset $L_t = E_{2,\frac{t}{2}} \setminus \overline{E_{2,t}}$ and a smooth cut-off function $\psi_t \in C_0^\infty(E_{2,\frac{t}{2}}; [0,1])$ such that $\psi_t = 1$ on $E_{2,t}$. As the thickness of the strip L_t is of order t, we have an upper bound of the form $||\nabla\psi_t||_\infty \leq \frac{c}{t}$. Using $\psi_t(u_\varepsilon - u_{\varepsilon'})$ as test function in equation (4.6), we get

$$
\begin{aligned}
&\int_{E_2} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle dx \\
= \quad & \int_{E_{2,t}} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), \psi_t(Du_\varepsilon - Du_{\varepsilon'})\rangle dx \\
+ \quad & \int_{E_2 \setminus \overline{E_{2,t}}} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle dx
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{E_{2,\frac{t}{2}}} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), \psi_t(Du_\varepsilon - Du_{\varepsilon'})\rangle dx \\
&+ \int_{E_2\setminus\overline{E_{2,t}}} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle dx \\
&- \int_{E_{2,\frac{t}{2}}\setminus\overline{E_{2,t}}} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), \psi_t(Du_\varepsilon - Du_{\varepsilon'})\rangle dx \\
&= -\int_{E_{2,\frac{t}{2}}} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), \nabla\psi_t(u_\varepsilon - u_{\varepsilon'})\rangle dx \\
&+ \int_{E_2\setminus\overline{E_{2,t}}} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), Du_\varepsilon - Du_{\varepsilon'}\rangle dx \\
&- \int_{E_{2,\frac{t}{2}}\setminus\overline{E_{2,t}}} \langle A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'}), \psi_t(Du_\varepsilon - Du_{\varepsilon'})\rangle dx \\
&\le \frac{c}{t}\int_{E_2} |A_\varepsilon(x, Du_\varepsilon) - A_{\varepsilon'}(x, Du_{\varepsilon'})||u_\varepsilon - u_{\varepsilon'}| dx \\
&+ cB^2 \int_{E_2\setminus\overline{E_{2,t}}} k(x)dx + cB^2 \int_{E_{2,\frac{t}{2}}\setminus\overline{E_{2,t}}} k(x)dx \\
&\le B\frac{c}{t}\left(\int_\omega k^3(x)\,dx\right)^{\frac{1}{3}}\left(\int_\omega |u_\varepsilon - u_{\varepsilon'}|^{\frac{3}{2}}\,dx\right)^{\frac{2}{3}} dx + cB^2\left(\int_\omega k^3(x)\,dx\right)^{\frac{1}{3}} |E_2\setminus\overline{E_{2,t}}|^{\frac{2}{3}} \\
&\le c(\beta, [\mathcal{K}], ||\mathcal{K}_0||_\infty, |\omega|)\left(\frac{B}{t}||u_\varepsilon - u_{\varepsilon'}||_{\frac{3}{2}} + B^2 t^{\frac{2}{3}}\right) \\
(7.4)&\le c(\beta, [\mathcal{K}], ||\mathcal{K}_0||_\infty, |\omega|)\left(\frac{||u_\varepsilon - u_{\varepsilon'}||_{\frac{3}{2}}}{\eta t} + \frac{t^{\frac{2}{3}}}{\eta^2}\right),
\end{aligned}
$$

where, in the last lines, we used (1.2) and that $B = \frac{8c}{\eta}$. Choosing $t = \eta^6$ and inserting estimate (7.4) in (7.3), we finally obtain

$$
\begin{aligned}
&\int_{E_2} \frac{1}{2k(x)}|Du_\varepsilon - Du_{\varepsilon'}|^2 dx \\
&\le \frac{c}{\eta^7}||u_\varepsilon - u_{\varepsilon'}||_{\frac{3}{2}} + c\eta^2 + c\frac{|\varepsilon - \varepsilon'|^2}{\eta^2}.
\end{aligned}
\tag{7.5}
$$

The strong convergence of the sequence u_ε in $L^{\frac{3}{2}}$ allows us to choose $\varepsilon, \varepsilon'$ such that

$$
\int_{E_2} \frac{1}{2k(x)}|Du_\varepsilon - Du_{\varepsilon'}|^2 dx \le c\eta,
\tag{7.6}
$$

and hence the conclusion follows.

□

7.3. **Proof of estimate** (5.4)**.** Let $m > 0$, $j \ge 1$ and $B_{R_1} \subset \Omega$ arbitrarily fixed. Multiplying (5.2) by

$$
(1+k)^{pm-1}\delta_k, \qquad \delta_k = \begin{cases} 1 & \text{if } k \le j, \\ 0 & \text{if } k > j, \end{cases}
$$

and summing on k we obtain

$$\sum_{k=0}^{+\infty}(1+k)^{pm-1}\delta_k\sum_{n=k}^{+\infty}\int_{B_\tau\cap B(n)}|Du|^p \leq$$
$$\sum_{k=0}^{+\infty}(1+k)^{pm-1}\delta_k\sum_{n=k}^{+\infty}\int_{B_\tau\cap B(n)}(C_0|u|^q+c_0\varphi_0),$$

where $C_0=c_0(t-\tau)^{-\alpha}$ and

$$B(n)=\{x\in\Omega : n\leq|u|<n+1\}.$$

Using the equality

$$\sum_{k=0}^{+\infty}(1+k)^{\lambda}\delta_k\sum_{n=k}^{+\infty}\int_{B(n)}|\psi| = \sum_{n=0}^{+\infty}\int_{B(n)}|\psi|\sum_{k=0}^{n}(1+k)^{\lambda}\delta_k,$$

the previous inequality becomes

$$\sum_{n=0}^{+\infty}\int_{B_\tau\cap B(n)}|Du|^p\sum_{k=0}^{n}(1+k)^{pm-1}\delta_k \leq \tag{7.7}$$
$$\sum_{n=0}^{+\infty}\int_{B_t\cap B(n)}(C_0|u|^q+c_0\varphi_0)\sum_{k=0}^{n}(1+k)^{pm-1}\delta_k.$$

The left-hand side of (7.7) can be estimated as follows

$$\sum_{n=0}^{+\infty}\int_{B_\tau\cap B(n)}|Du|^p\sum_{k=0}^{n}(1+k)^{pm-1}\delta_k =$$
$$\sum_{n=0}^{j}\int_{B_\tau\cap B(n)}|Du|^p\sum_{k=0}^{n}(1+k)^{pm-1}\delta_k + \sum_{n=j+1}^{+\infty}\int_{B_\tau\cap B(n)}|Du|^p\sum_{k=0}^{n}(1+k)^{pm-1}\delta_k \geq$$
$$c_1\sum_{n=0}^{j}\int_{B_\tau\cap B(n)}|Du|^p(1+n)^{pm}\geq c_1\sum_{n=0}^{j}\int_{B_\tau\cap B(n)}|Du|^p|u|^{pm} =$$
$$c_2\int_{B_\tau\cap\{|u|\leq j+1\}}|D\left[|T_{j+1}(u)|^m T_{j+1}(u)\right)]|^p = c_2\int_{B_\tau}|D\left[|T_{j+1}(u)|^m T_{j+1}(u)\right)]|^p, \tag{7.8}$$

where $c_1 = (\max\{1, pm\})^{-1}$, $c_2 = c_1(m+1)^{-p}$ and T_{j+1} is the truncation function defined in (3.4). Analogously we can estimate the right-hand side of (7.7) as follows

$$
\begin{aligned}
&\sum_{n=0}^{+\infty}\int_{B_t\cap B(n)}(C_0|u|^q+c_0\varphi_0)\sum_{k=0}^{n}(1+k)^{pm-1}\delta_k =\\
&\sum_{n=0}^{j}\int_{B_t\cap B(n)}(C_0|u|^q+c_0\varphi_0)\sum_{k=0}^{n}(1+k)^{pm-1}\delta_k +\\
&\sum_{n=j+1}^{+\infty}\int_{B_t\cap B(n)}(C_0|u|^q+c_0\varphi_0)\sum_{k=0}^{n}(1+k)^{pm-1}\delta_k \le\\
&\sum_{n=0}^{j}\int_{B_t\cap B(n)}c_3(C_0|u|^q+c_0\varphi_0)(1+n)^{pm}+\sum_{n=j+1}^{+\infty}\int_{B_t\cap B(n)}c_3(C_0|u|^q+c_0\varphi_0)(1+j)^{pm}\le\\
&\int_{B_t\setminus A(j+1)}c_3(C_0|u|^q+c_0\varphi_0)(1+|u|)^{pm}+\\
&\int_{B_t\cap A(j+1)}c_3(C_0|u|^q+c_0\varphi_0)(1+|T_{j+1}(u)|)^{pm}\le\\
&\int_{B_t}c_3(C_0|u|^q+c_0\varphi_0)(1+|T_{j+1}(u)|)^{pm}, \qquad (7.9)
\end{aligned}
$$

where $c_3 = (\min\{1, pm\})^{-1}$. Using (7.8) and (7.9) in (7.7) we obtain

$$
\int_{B_t}|D\left[|T_{j+1}(u)|^m T_{j+1}(u)\right]|^p \le c_4\int_{B_t}(C_0|u|^q+\varphi_0)(1+|T_{j+1}(u)|)^{pm}, \qquad (7.10)
$$

where $c_4 = c_2^{-1}\max\{1, c_0\}c_3$. We estimate now the term in the left-hand side of (7.10). We recall that (7.10) is verified for all τ and t such that $R_0 \le \tau < t \le R_1$. Let t' and τ arbitrarily fixed satisfying $R_0 \le t' < \tau < t$ and let ν be a cut-off function such that

$$
\mathrm{supp}\nu \subset B_\tau, \quad 0\le\nu\le1, \quad \nu=1 \text{ in } B_{t'}, \quad |D\nu|\le 2(\tau-t')^{-1}.
$$

Then using Sobolev inequality we obtain

$$
\begin{aligned}
&\int_{B_\tau}|D\left[|T_{j+1}(u)|^m T_{j+1}(u)\right]|^p \ge \int_{B_\tau}|D\left[|T_{j+1}(u)|^m T_{j+1}(u)\right]|^p\,\nu^p \ge\\
&\frac{1}{2^p}\int_{B_\tau}|D\left[|T_{j+1}(u)|^m T_{j+1}(u)\nu\right]|^p - \frac{2^p}{(\tau-t')^p}\int_{B_\tau}|T_{j+1}(u)|^{(m+1)p}\ge\\
&c_5\left(\int_{B_\tau}|T_{j+1}(u)|^{(m+1)p^*}\nu^{p^*}\right)^{\frac{p}{p^*}} - \frac{2^p}{(\tau-t')^p}\int_{B_\tau}|u|^p|T_{j+1}(u)|^{mp}\ge\\
&c_5\left(\int_{B_{t'}}|T_{j+1}(u)|^{(m+1)p^*}\right)^{\frac{p}{p^*}} - \frac{2^p}{(\tau-t')^p}\int_{B_t}|u|^p|T_{j+1}(u)|^{mp}, \qquad (7.11)
\end{aligned}
$$

where c_5 is a constant that depends only on p and N ($c_5 = 2^{-p}S$ where S is the constant of Sobolev for $W_0^{1,p}(B_\tau)$). Using (7.11) in (7.10) and recalling the value of C_0 we have, for every $R_0 \leq t' < \tau < t \leq R_1$, that the following estimate holds

$$\left(\int_{B_{t'}} |T_{j+1}(u)|^{(m+1)p^*}\right)^{\frac{p}{p^*}} \leq \tag{7.12}$$
$$\frac{1}{c_5}\int_{B_t}\left[\frac{c_4c_0|u|^q}{(t-\tau)^\alpha}+\frac{2^p|u|^p}{(\tau-t')^p}\right](1+|T_{j+1}(u)|^{mp})+\frac{c_4}{c_5}\int_{B_t}\varphi_0(1+|T_{j+1}(u)|^{mp}).$$

Choose $\tau=(t+t')/2$ that is such that $t-\tau=\tau-t'=(t-t')/2$. Then by equation (7.12), since obviously it results $|u|^p \leq |u|^q+1$, it follows that for every $R_0 \leq t' < t \leq R_1 \leq 1$ with $B_{R_1} \subset \Omega$ the following estimate holds true

$$\left(\int_{B_{t'}} |T_{j+1}(u)|^{(m+1)p^*}\right)^{\frac{p}{p^*}} \leq c\int_{B_t}\left[C_1(|u|^q+1)+\varphi_0\right](1+|T_{j+1}(u)|^{mp}),$$

where

$$c=[(c_4+1)/c_5](2^p+2^\alpha)(c_0c_4+2^p), \quad C_1=1/(t-t')^\beta, \quad \beta=\max\{\alpha,p\}, \tag{7.13}$$

that is the assertion.

□

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