Dimension reduction of a crack evolution problem in a linearly elastic plate

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Abstract

A two dimensional model which describes the evolution of a crack in a plate is deduced from a three dimensional linearly elastic Griffith's type model. The result is achieved by adopting the framework of energetic solutions for rate-independent processes, to model three dimensional fracture evolution, in conjunction with a variational dimension reduction procedure.

Keywords: Dimension reduction, rate-independent processes, crack evolution, linear elasticity

1 Introduction

In this paper we consider the time evolution of a single crack in a body occupying a cylindrical region whose height is much smaller than the diameter. In particular, our main interest is to approximate this three dimensional

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problem with an appropriate two dimensional evolution model. Hence, the framework of energetic solutions for rate-independent processes, to model three dimensional fracture evolution, will be used in conjunction with a variational dimension reduction procedure.

Energetic solutions for rate-independent processes have been extensively studied in recent years; see for instance [15] and references therein. For a crack evolution problem the notion of energetic solution is based on a global energetic stability principle related to Griffith's criterion [13] which asserts that a crack grows if the energy release rate is bigger than the fracture toughness and it is stationary otherwise. These concepts have been developed along a fruitful line of research generated by the revision of fracture mechanics by Francfort and Marigo [11], see [10] and references therein. In this context crack growth is studied, differently from Griffith's approach, without prescribing the crack path position. Nonetheless in several contexts it is still convenient to start the study within the framework in which the crack path is known in advance, see e.g. [7, 18, 19] and references therein.

While variational convergence for static problems have been well studied, by using the variational property of Γ-limits which ensures convergence of minima and minimizers (for an account of those we refers to the books of Braides [6] and Dal Maso [9]), the related questions for evolutionary systems have been addressed only recently (see [17] and references therein). Actually, we apply, in this paper, a Γ-convergence based scheme for rate-independent evolutionary problems developed by Mielke, Roubíček and Stefanelli in [17].

In particular, we consider a sequence of cylinders whose height is scaled by an adimensional parameter ε. Each of these regions shall denote the reference configuration of a body in which a single crack may advance along a fixed path. The mechanical response of the bodies is characterized by an energy consisting of the sum of a linearly elastic bulk part and, following Barenblatt, a cohesive crack surface energy. Also the surface part of the energy is split into the sum of two parts so to appropriately describe the behavior parallel and orthogonal to the axes of the thin cylinders. A vertical crack may evolve, along a fixed path, but only dissipating energy. Following Griffith, the dissipation functional is assumed to be essentially proportional to the area swept by the advancement of the crack: thus no dissipation is associated to the "interatomic" forces which generate the cohesive surface energy. We believe that our model may well represent, for instance, the evolution of a crack in an elastic body reinforced by soft and hyperelastic fibers (whose bulk contribution is negligible), that is, in bodies in which the presence of the fibers is detected only after the opening of the crack. Our assumptions allow also to set the cohesive surface energy equal to zero and hence to obtain the classical fracture evolution described by Griffith's model.

The bodies are not subject to body or contact forces, and the evolution in time is driven only by a time dependent Dirichlet condition on part of the lateral boundary of the bodies.

The two dimensional evolution problem is obtained, under appropriate scalings, by letting ε go to zero within the framework of Mielke, Roubíček and Stefanelli in [17]. The theory put forward by these authors turns out to be very robust and can be easily adapted to our problem.

Dimension reduction problems for rate-independent processes have been considered only very recently, and to our best knowledge the only related result is due to Babadjian [2] who obtained, by using a different scaling, a 2-dimensional evolutionary model of a free crack in a nonlinear elastic membrane.

The paper is organized as follows: in Section 2 the sequence of three-dimensional problems is described and the time evolution problem is precisely defined. In Section 3 the domains and consequently some fields are rescaled following classical ideas introduced by Ciarlet and Destuynder [8]. Section 4 is devoted to the computation of some Γ-limits of the energy which are relevant to the asymptotic analysis of the evolutionary system. The two dimensional rate-independent evolutionary limit problem is stated in Section 5, whose energetic solutions turn out to be those associated with the classical energy functional of plates. The convergence of energetic solutions is proven by checking the assumptions needed to apply the abstract convergence scheme of [17]. Section 6 is devoted to write down the equilibrium equations for the limit problem under some additional regularity assumptions.

Notation. We use the following convention for indexing vector and tensor components: Greek indices α and β take their values in the set $\{1, 2\}$ and Latin indices i, j and k in the set $\{1, 2, 3\}$. With a little abuse of notation, and because this is a common practice and does not give rise to any mistake, we use to call "sequences" even those families indicized by a continuous parameter $\varepsilon \in (0, 1)$. The component k of a vector v will be denoted either with $(v)_k$ or v_k and an analogous notation will be used to denote tensor components.

2 The $3D$ - problem

We consider a body occupying in the reference configuration the region

$$\Omega^\varepsilon := \omega \times (-\frac{h}{2}\varepsilon, \frac{h}{2}\varepsilon) \subset \mathbb{R}^3,$$

where ω is a bounded open Lipschitz subset of $\mathbb{R}^2$ and h, $\varepsilon > 0$. The constant parameter h is kept to clarify the role played by the thickness of the body in the limit problem.

A crack path is prescribed by

$$\Gamma^\varepsilon_L := \gamma_L \times (-\frac{h}{2}\varepsilon, \frac{h}{2}\varepsilon),$$

where $\gamma_L \subset \overline{\omega}$ is a regular oriented simple curve with length L and intersecting $\partial\omega$ only in its first endpoint denoted by P_0, see Fig. 1.

We suppose that the boundary $\partial\omega$ is partitioned into a relatively open Neumann region, $\partial_N\omega$, containing P_0 and a Dirichlet region, $\partial_D\omega$, such that $\mathscr{H}^1(\partial_D\omega) > 0$, where $\mathscr{H}^d$ denotes the d-dimensional Hausdorff measure. We further set $\partial_N\Omega^\varepsilon := \partial_N\omega \times (-\frac{h}{2}\varepsilon, \frac{h}{2}\varepsilon)$ and $\partial_D\Omega^\varepsilon := \partial_D\omega \times (-\frac{h}{2}\varepsilon, \frac{h}{2}\varepsilon)$; of course we have $\mathscr{H}^2(\partial_D\Omega^\varepsilon) > 0$.

For a chosen parametrization $\gamma\colon [0, L] \to \gamma_L$ and an initial path length $\sigma_{in} \in (0, L)$, we consider the following family of admissible cracks

$$\Gamma^\varepsilon_\sigma := \gamma_\sigma \times (-\frac{h}{2}\varepsilon, \frac{h}{2}\varepsilon) \quad \text{for } \sigma \in [\sigma_{in}, L],$$

where $\gamma_\sigma = \{\, \gamma(s) \mid 0 \leq s \leq \sigma \,\}$. The admissible crack $\Gamma^\varepsilon_\sigma$ is uniquely determined by its length σ.

For $\sigma \in [\sigma_{in}, L]$ we set $\Omega^\varepsilon_\sigma := \Omega^\varepsilon \setminus \Gamma^\varepsilon_\sigma$, and for $\psi \in H^1(\Omega^\varepsilon; \mathbb{R}^3)$ we denote by

$$\mathcal{A}_\varepsilon(\psi, \sigma) := \{\, u \in H^1(\Omega^\varepsilon_\sigma; \mathbb{R}^3) \mid u = \psi \text{ on } \partial_D\Omega^\varepsilon \,\}$$

the set of admissible displacements with boundary value ψ on $\partial_D\Omega^\varepsilon$ and crack $\Gamma^\varepsilon_\sigma$. Note that $\sigma_1 \leq \sigma_2$ implies $H^1(\Omega^\varepsilon_{\sigma_1}; \mathbb{R}^3) \subseteq H^1(\Omega^\varepsilon_{\sigma_2}; \mathbb{R}^3)$.

Time-dependent boundary displacements will be prescribed by

$$\hat{\psi}^\varepsilon \in C^1([0,T]; H^1(\Omega^\varepsilon; \mathbb{R}^3)), \tag{2.1}$$

so that the time derivative $t \mapsto \dot{\hat{\psi}}^\varepsilon(t)$ belongs to $C([0,T]; H^1(\Omega^\varepsilon; \mathbb{R}^3))$ and its spatial gradient $t \mapsto D\dot{\hat{\psi}}^\varepsilon(t)$ belongs to $C([0,T]; L^2(\Omega^\varepsilon; \mathbb{R}^{3\times 3}))$.

For $w \in H^1(\Omega^\varepsilon; \mathbb{R}^3)$ we denote the strain by Ew, i.e.,

$$Ew = \frac{Dw + Dw^T}{2}.$$

For every $t \in [0,T]$, $u \in H^1(\Omega^\varepsilon_L; \mathbb{R}^3)$ and $\sigma \in [\sigma_{in}, L]$, we assume that the bulk elastic energy is of the form

$$\frac{1}{2}\int_{\Omega^\varepsilon_\sigma} \mathbb{C}Eu \cdot Eu \, dx \qquad \text{if } u \in \mathcal{A}_\varepsilon(\hat{\psi}^\varepsilon(t), \sigma), \tag{2.2}$$

and $+\infty$ else. The elasticity tensor $\mathbb{C}$ is assumed to be positive definite, i.e., there exists $\mu > 0$ such that

$$\mathbb{C}A \cdot A \geq \mu |A|^2 \;\text{ for every } A \in \mathbb{R}^{3\times 3}_{\text{sym}}. \tag{2.3}$$

Moreover, we assume the following usual symmetry properties

$$\mathbb{C}_{ijkl} = \mathbb{C}_{ijlk} = \mathbb{C}_{klij}, \qquad i,j,k,l = 1,2,3, \tag{2.4}$$

and that the material has monoclinic symmetry with respect to the (x_1, x_2)-plane, which implies

$$\mathbb{C}_{\alpha\beta\gamma 3} = \mathbb{C}_{\alpha 333} = 0, \qquad \alpha, \beta, \gamma = 1, 2. \tag{2.5}$$

Let $\hat{g}^\varepsilon_v : \mathbb{R} \to [0, +\infty)$, $\hat{g}^\varepsilon_p : \mathbb{R}^2 \to [0, +\infty)$ be the surface energy densities; the regularity of these functions will be made precise in the next section. For given $t \in [0,T]$, $u \in H^1(\Omega^\varepsilon_L; \mathbb{R}^3)$, and $\sigma \in [\sigma_{in}, L]$ we assume that the energy spent to open the crack $\Gamma^\varepsilon_\sigma$ is given by

$$\int_{\Gamma^\varepsilon_\sigma} \Big(\hat{g}^\varepsilon_v([u]_v) + \hat{g}^\varepsilon_p([u]_p) \Big) \, d\mathscr{H}^2 \qquad \text{if } u \in \mathcal{A}_\varepsilon(\hat{\psi}^\varepsilon(t), \sigma), \tag{2.6}$$

and $+\infty$ else. In the formula above $[u]$ denotes the jump of u across the oriented crack surface $\Gamma^\varepsilon_\sigma$ and

$$[u]_v := [u] \cdot \mathrm{e}_3, \qquad [u]_p := \sum_{\alpha=1}^{2} \big([u] \cdot \mathrm{e}_\alpha\big)\, \mathrm{e}_\alpha$$

are the projections of the jump along the x_3-axis and on the (x_1, x_2)-plane, respectively.

Remark 2.1 For most materials it would be natural to assume $\hat{g^\varepsilon_v}$ to be even. Our assumptions are more general and embrace also this case. A similar remark may be stated for $\hat{g^\varepsilon_p}$. Instead, we do not impose that $[u]\cdot\nu \geq 0$, where ν denotes the normal to Γ^ε_L, which would avoid the possibility of material interpenetration.

In the sequel we found it convenient to adopt an additive splitting (cfr. [12]) for the displacement and for this reason we introduce the set of admissible displacements $\mathcal{A}_\varepsilon(\sigma) := \mathcal{A}_\varepsilon(0, \sigma) = \{\, u \in H^1(\Omega^\varepsilon_\sigma; \mathbb{R}^3) \mid u = 0 \text{ on } \partial_D \Omega^\varepsilon \,\}$. Recalling (2.2) and (2.6), we define the bulk energy

$$\widehat{\mathscr{E}}^b_\varepsilon(t, u, \sigma) := \begin{cases} \dfrac{1}{2} \displaystyle\int_{\Omega^\varepsilon_\sigma} \mathbb{C} E(u + \hat{\psi}^\varepsilon(t)) \cdot E(u + \hat{\psi}^\varepsilon(t))\, dx & \text{if } u \in \mathcal{A}_\varepsilon(\sigma), \\ +\infty & \text{else,} \end{cases}$$

and the surface energy

$$\widehat{\mathscr{E}}^s_\varepsilon(t, u, \sigma) := \begin{cases} \displaystyle\int_{\Gamma^\varepsilon_\sigma} \Big(\hat{g^\varepsilon_v}([u]_v) + \hat{g^\varepsilon_p}([u]_p)\Big)\, d\mathscr{H}^2 & \text{if } u \in \mathcal{A}_\varepsilon(\sigma), \\ +\infty & \text{else,} \end{cases}$$

since $[\hat{\psi}^\varepsilon(t)] = 0$ $\mathscr{H}^2$-a.e. on Γ^ε_L.

Finally, we define the functional

$$\widehat{\mathscr{E}}_\varepsilon : [0, T] \times H^1(\Omega^\varepsilon_L; \mathbb{R}^3) \times [\sigma_{in}, L] \to (-\infty, +\infty]$$

by

$$\widehat{\mathscr{E}}_\varepsilon(t, u, \sigma) := \widehat{\mathscr{E}}^b_\varepsilon(t, u, \sigma) + \widehat{\mathscr{E}}^s_\varepsilon(t, u, \sigma).$$

We interpret $\widehat{\mathscr{E}}_\varepsilon(\cdot,\cdot,\sigma)$ as the total elastic energy for a crack of length σ. We allow the length σ to increase in time but to such a process we associate a dissipation of energy. For this reason we define the dissipation distance

$$\widehat{\mathscr{D}}_\varepsilon(\sigma,\tilde\sigma) := \begin{cases} \displaystyle\int_{\Gamma^\varepsilon_{\tilde\sigma}\setminus\Gamma^\varepsilon_\sigma} \hat\kappa^\varepsilon(x)\,d\mathscr{H}^2(x) & \text{if } \tilde\sigma \geq \sigma, \\ +\infty & \text{else,} \end{cases}$$

where $\hat\kappa^\varepsilon : \Gamma^\varepsilon_L \to (0,+\infty)$ is taken to be a continuous function.

The $\widehat{\mathscr{D}}_\varepsilon$-dissipation of a function $\hat\sigma : [0,T] \to [\sigma_{in}, L]$ during a time interval $[t_1,t_2] \subseteq [0,T]$ is defined by

$$\mathrm{Diss}_{\widehat{\mathscr{D}}_\varepsilon}(\hat\sigma;[t_1,t_2]) := \sup\Big\{ \sum_{j=1}^M \widehat{\mathscr{D}}_\varepsilon(\hat\sigma(r_{j-1}),\hat\sigma(r_j)) : M \in \mathbb{N}, \\ t_1 \leq r_0 < \cdots < r_M \leq t_2 \Big\}.$$

We observe that if $\mathrm{Diss}_{\widehat{\mathscr{D}}_\varepsilon}(\hat\sigma;[t_1,t_2]) < +\infty$ then $\hat\sigma$ is non-decreasing in $[t_1,t_2]$ and

$$\mathrm{Diss}_{\widehat{\mathscr{D}}_\varepsilon}(\hat\sigma;[t_1,t_2]) = \widehat{\mathscr{D}}_\varepsilon(\hat\sigma(t_1),\hat\sigma(t_2)).$$

Following Mielke (see e.g. [15] and references therein) we specify the meaning of energetic solution of the rate-independent process associated with the functionals $\widehat{\mathscr{E}}_\varepsilon$ and $\widehat{\mathscr{D}}_\varepsilon$.

Definition 2.2 *An* energetic solution *associated with the functionals $\widehat{\mathscr{E}}_\varepsilon$ and $\widehat{\mathscr{D}}_\varepsilon$, is a function $(\hat u_\varepsilon,\hat\sigma_\varepsilon) : [0,T] \to H^1(\Omega^\varepsilon_L;\mathbb{R}^3) \times [\sigma_{in}, L]$ such that:*

(i) the map $t \mapsto \partial_t\widehat{\mathscr{E}}_\varepsilon(t,\hat u_\varepsilon(t),\hat\sigma_\varepsilon(t)) \in L^1(0,T)$;

(ii) for every $t \in [0,T]$, $(\hat u_\varepsilon(t),\hat\sigma_\varepsilon(t))$ satisfies the stability condition

$$(\hat S)_\varepsilon \qquad \widehat{\mathscr{E}}_\varepsilon(t,\hat u_\varepsilon(t),\hat\sigma_\varepsilon(t)) < +\infty, \text{ and,}$$

$$\widehat{\mathscr{E}}_\varepsilon(t,\hat u_\varepsilon(t),\hat\sigma_\varepsilon(t)) \leq \widehat{\mathscr{E}}_\varepsilon(t,\tilde u,\tilde\sigma) + \widehat{\mathscr{D}}_\varepsilon(\hat\sigma_\varepsilon(t),\tilde\sigma)$$

$$\text{for every } (\tilde u,\tilde\sigma) \in H^1(\Omega^\varepsilon_L;\mathbb{R}^3) \times [\sigma_{in}, L],$$

and the energy balance

$$(\hat E)_\varepsilon \quad \widehat{\mathscr{E}}_\varepsilon(t,\hat u_\varepsilon(t),\hat\sigma_\varepsilon(t)) + \mathrm{Diss}_{\widehat{\mathscr{D}}_\varepsilon}(\hat\sigma_\varepsilon;[0,t]) = \widehat{\mathscr{E}}_\varepsilon(0,\hat u_\varepsilon(0),\hat\sigma_\varepsilon(0)) + \\ + \int_0^t \partial_t\widehat{\mathscr{E}}_\varepsilon(s,\hat u_\varepsilon(s),\hat\sigma_\varepsilon(s))\,ds.$$

In the definition above, by $\partial_t\widehat{\mathscr{E}}_\varepsilon$ we mean the partial derivative of $\widehat{\mathscr{E}}_\varepsilon$ with respect to the first variable; in our case, for a given triplet (t,u,σ) with $u \in \mathcal{A}_\varepsilon(\hat{\psi}^\varepsilon(t),\sigma)$ we have

$$\partial_t\widehat{\mathscr{E}}_\varepsilon(t,u,\sigma) = \int_{\Omega^\varepsilon_\sigma} \mathbb{C}E\dot{\hat{\psi}}^\varepsilon(t)\cdot E(u+\psi^\varepsilon(t))\,dx,$$

which is the power produced by the time dependent displacement imposed on the Dirichlet part of the boundary.

A pair $(\hat{u}_\varepsilon(t),\hat{\sigma}_\varepsilon(t))$ satisfying $(\hat{S})_\varepsilon$ at time t is called stable since the energy in any other state plus the dissipation involved in the change of states is greater than the energy of the stable state.

Also, $(\hat{S})_\varepsilon$ implies that, for fixed t and $\hat{\sigma}_\varepsilon(t)$, the displacement $\hat{u}_\varepsilon(t)$ is a global minimizer of the energy $\widehat{\mathscr{E}}_\varepsilon(t,\cdot,\hat{\sigma}_\varepsilon(t))$.

The condition $(\hat{E})_\varepsilon$ is a balance equation, indeed, on the left–hand side we find the sum of the energy at time t plus the dissipation involved in the time interval $[0,t]$, while on the right–hand side we have the energy at time 0 plus the work done by the external loads in the time interval $[0,t]$.

3 The rescaled problem

Here we reformulate the problem stated in the previous section on a cylinder whose height is independent of ε. To this end, we set $\Omega := \Omega^1$, $\partial_N\Omega := \partial_N\Omega^1$, $\partial_D\Omega := \partial_D\Omega^1$, $\Gamma_\sigma := \Gamma^1_\sigma$, and $\Omega_\sigma := \Omega^1_\sigma$.

For any $\varepsilon > 0$, let $p_\varepsilon : \Omega \to \Omega^\varepsilon$ be defined by $p_\varepsilon(x_1,x_2,x_3) := (x_1,x_2,\varepsilon x_3)$.

For a generic $\hat{v} \in H^1(\Omega^\varepsilon_\sigma;\mathbb{R}^3)$ we let $v \in H^1(\Omega_\sigma;\mathbb{R}^3)$ be defined by $v_\alpha := \frac{1}{\varepsilon}\hat{v}_\alpha \circ p_\varepsilon$ and $v_3 := \hat{v}_3 \circ p_\varepsilon$. With this notation we have

$$\frac{1}{\varepsilon}(E\hat{v})\circ p_\varepsilon = \begin{pmatrix} (Ev)_{\alpha\beta} & \frac{1}{\varepsilon}(Ev)_{\alpha 3} \\ \frac{1}{\varepsilon}(Ev)_{3\beta} & \frac{1}{\varepsilon^2}(Ev)_{33} \end{pmatrix} =: E^\varepsilon v. \tag{3.1}$$

Similarly, concerning the boundary displacement, we assume that

> there exists a function $\psi \in C^1([0,T];H^1(\Omega;\mathbb{R}^3))$ such that $\psi_\alpha := \frac{1}{\varepsilon}\hat{\psi}^\varepsilon_\alpha \circ p_\varepsilon$ and $\psi_3 := \hat{\psi}^\varepsilon_3 \circ p_\varepsilon$ and $(E\psi)_{i3} = 0$. (3.2)

Remark 3.1 The function ψ denotes the boundary displacement. The assumption $(E\psi)_{33} = 0$ means that the boundary is not stretched along the direction x_3 while $(E\psi)_{\alpha 3} = 0$ implies that $(E\psi)_{\alpha 3}\tau_\alpha = 0$ and $(E\psi)_{\alpha 3}\nu_\alpha = 0$ where τ_α are the components of the tangent vector to $\partial\omega$ and ν_α those of the normal to $\partial_D\Omega$. The former means that segments on the boundary in the x_3-direction will be rigidly deformed and will remain, after deformation, orthogonal to the deformed mean-line of the boundary, while the latter implies that ψ_α depends linearly on x_3 on the Dirichlet part of the boundary. We believe, since the height of the cylinder is small, that these are quite natural assumptions.

Let $\mathcal{A}(\sigma) := \{\, u \in H^1(\Omega_\sigma; \mathbb{R}^3) \mid u = 0 \text{ on } \partial_D\Omega \,\}$ be the set of admissible displacements and define the rescaled bulk energy

$$\mathscr{E}^b_\varepsilon(t, u, \sigma) := \begin{cases} \dfrac{1}{2}\displaystyle\int_{\Omega_\sigma} \mathbb{C}E^\varepsilon(u + \psi(t)) \cdot E^\varepsilon(u + \psi(t))\, dx & \text{if } u \in \mathcal{A}(\sigma), \\ +\infty & \text{else.} \end{cases} \tag{3.3}$$

Note that for $\hat{u} \in \mathcal{A}_\varepsilon(\sigma)$ we have

$$\begin{aligned} \frac{1}{\varepsilon^3}\widehat{\mathscr{E}^b_\varepsilon}(t, \hat{u}, \sigma) &= \frac{1}{2\varepsilon}\int_{\Omega^\varepsilon_\sigma} \mathbb{C}\frac{E(\hat{u} + \hat{\psi}^\varepsilon(t))}{\varepsilon} \cdot \frac{E(\hat{u} + \hat{\psi}^\varepsilon(t))}{\varepsilon}\, dy \\ &= \frac{1}{2}\int_{\Omega_\sigma} \mathbb{C}E^\varepsilon(u + \psi(t)) \cdot E^\varepsilon(u + \psi(t))\, dx = \mathscr{E}^b_\varepsilon(t, u, \sigma) \end{aligned}$$

where $u \in \mathcal{A}(\sigma)$.

We assume, for all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^2$

$$\hat{g^\varepsilon_v}(s) := \varepsilon^2 g_v(s) \quad \text{and} \quad \hat{g^\varepsilon_p}(\xi) := \varepsilon g_p(\xi) \tag{3.4}$$

with $g_v : \mathbb{R} \to [0, +\infty)$, $g_p : \mathbb{R}^2 \to [0, +\infty)$ satisfying the following assumptions

(G1) $g_v(0) = 0$;

(G2) g_v is lower semicontinuous;

(G3) $g_p(0) = 0$;

(G4) there exists $C > 0$ such that $g_p(\xi) \le C|\xi|$ for every $\xi \in \mathbb{R}^2$;

(G5) setting $g_p^0(\xi) := \limsup_{s\to 0^+} \frac{g_p(s\xi)}{s}$, there exist constants $C, \alpha, \ell > 0$ such that

$$\left| g_p^0(\xi) - \frac{g_p(s\xi)}{s} \right| \leq C s^\alpha$$

for every $\xi \in \mathbb{R}^2$ with $|\xi| = 1$ and for every $s \in (0, \ell)$;

(G6) g_p is lower semicontinuous.

From these assumptions it follows that g_p^0 is positively one-homogeneous, lower semicontinuous and satisfies conditions analogous to $(G3)$ and $(G4)$. Assumption $(G5)$ corresponds to a similar hypothesis made in [4]. All these conditions will be used in the next section.

We define the rescaled surface energy by

$$\mathscr{E}_\varepsilon^s(u,\sigma) := \begin{cases} \displaystyle\int_{\Gamma_\sigma} \left(g_v([u]_v) + \frac{g_p(\varepsilon [u]_p)}{\varepsilon} \right) d\mathscr{H}^2 & \text{if } u \in \mathcal{A}(\sigma), \\ +\infty & \text{else.} \end{cases} \tag{3.5}$$

Note that for $\hat{u} \in \mathcal{A}_\varepsilon(\sigma)$ we have

$$\begin{aligned} \frac{1}{\varepsilon^3} \widehat{\mathscr{E}}_\varepsilon^s(t, \hat{u}, \sigma) &= \frac{1}{\varepsilon^3} \int_{\Gamma_\sigma^\varepsilon} \left(\hat{g}_v^\varepsilon([\hat{u}]_v) + \hat{g}_p^\varepsilon([\hat{u}]_p) \right) d\mathscr{H}^2 \\ &= \frac{1}{\varepsilon^3} \int_{\Gamma_\sigma} \varepsilon \Big(\varepsilon^2 g_v([u]_v) + \varepsilon g_p(\varepsilon [u]_p) \Big) \, d\mathscr{H}^2 \\ &= \mathscr{E}_\varepsilon^s(u,\sigma). \end{aligned}$$

The rescaled total energy is then defined by

$$\mathscr{E}_\varepsilon(t,u,\sigma) := \mathscr{E}_\varepsilon^b(t,u,\sigma) + \mathscr{E}_\varepsilon^s(u,\sigma). \tag{3.6}$$

About the rescaling of the fracture toughness, we assume

$$\hat{\kappa}^\varepsilon = \varepsilon^2 \kappa \tag{3.7}$$

with $\kappa : \Gamma_L \to (\kappa_m, +\infty)$ continuous and $\kappa_m > 0$. The rescaled dissipation distance is defined by

$$\mathscr{D}(\sigma, \tilde{\sigma}) := \begin{cases} \int_{\Gamma_{\tilde{\sigma}} \setminus \Gamma_\sigma} \kappa(x) \, d\mathscr{H}^2(x) & \text{if } \tilde{\sigma} \geq \sigma, \\ +\infty & \text{else.} \end{cases} \tag{3.8}$$

We note that

$$\frac{\widehat{\mathscr{D}}_\varepsilon(\sigma,\tilde\sigma)}{\varepsilon^3} = \mathscr{D}(\sigma,\tilde\sigma).$$

We conclude this section by rewriting Definition 2.2 on the fixed domain. The map $\mathrm{Diss}_{\mathscr{D}}$ is defined analogously to $\mathrm{Diss}_{\widehat{\mathscr{D}}_\varepsilon}$, and in the same way we observe that if $\mathrm{Diss}_{\mathscr{D}}(\sigma;[t_1,t_2]) < +\infty$ then $\sigma : [t_1,t_2] \to [\sigma_{in}, L]$ is non-decreasing and $\mathrm{Diss}_{\mathscr{D}}(\sigma;[t_1,t_2]) = \mathscr{D}(\sigma(t_1),\sigma(t_2))$.

Definition 3.2 *An* energetic solution *associated with the functionals $\mathscr{E}_\varepsilon$ and $\mathscr{D}$, is a function $(u_\varepsilon,\sigma_\varepsilon) : [0,T] \to H^1(\Omega_L;\mathbb{R}^3) \times [\sigma_{in}, L]$ such that*

(i) the map $t \mapsto \partial_t \mathscr{E}_\varepsilon(t, u_\varepsilon(t), \sigma_\varepsilon(t)) \in L^1(0,T)$;

(ii) for every $t \in [0,T]$, $(u_\varepsilon(t),\sigma_\varepsilon(t))$ satisfies the stability condition

$$(S)_\varepsilon \qquad \mathscr{E}_\varepsilon(t,u_\varepsilon(t),\sigma_\varepsilon(t)) < +\infty, \text{ and,}$$

$$\mathscr{E}_\varepsilon(t,u_\varepsilon(t),\sigma_\varepsilon(t)) \le \mathscr{E}_\varepsilon(t,\tilde u,\tilde\sigma) + \mathscr{D}(\sigma_\varepsilon(t),\tilde\sigma)$$

for every $(\tilde u,\tilde\sigma) \in H^1(\Omega_L;\mathbb{R}^3) \times [\sigma_{in}, L]$,

and the energy balance

$$(E)_\varepsilon \quad \mathscr{E}_\varepsilon(t,u_\varepsilon(t),\sigma_\varepsilon(t)) + \mathrm{Diss}_{\mathscr{D}}(\sigma_\varepsilon;[0,t]) = \mathscr{E}_\varepsilon(0,u_\varepsilon(0),\sigma_\varepsilon(0)) + \\ + \int_0^t \partial_t \mathscr{E}_\varepsilon(s,u_\varepsilon(s),\sigma_\varepsilon(s))\,ds.$$

Whenever $\mathscr{E}_\varepsilon(t,u,\sigma)$ is finite, the power $\partial_t\mathscr{E}_\varepsilon(t,u,\sigma)$ takes the form

$$\partial_t\mathscr{E}_\varepsilon(t,u,\sigma) = \partial_t\mathscr{E}^b_\varepsilon(t,u,\sigma) = \int_{\Omega_\sigma} \mathbb{C}E^\varepsilon\dot\psi(t)\cdot E^\varepsilon(u+\psi(t))\,dx. \tag{3.9}$$

Remark 3.3 Let $t \mapsto (u_\varepsilon(t),\sigma_\varepsilon(t))$ be an energetic solution associated with the functionals $\mathscr{E}_\varepsilon$ and $\mathscr{D}$. From the stability condition $(S)_\varepsilon$, choosing $(\tilde u,\tilde\sigma) = (0,\sigma_\varepsilon(t))$ and recalling that $(E\psi)_{i3} = 0$, it follows that $\mathscr{E}_\varepsilon(t,u_\varepsilon(t),\sigma_\varepsilon(t))$ is uniformly bounded in ε and t. From the balance of energy $(E)_\varepsilon$ and the fact that $t \mapsto \partial_t\mathscr{E}_\varepsilon(t,u_\varepsilon(t),\sigma_\varepsilon(t)) \in L^1(0,T)$ it follows that $\mathrm{Diss}_{\mathscr{D}}(\sigma_\varepsilon;[0,t])$ is finite and therefore $t \mapsto \sigma_\varepsilon(t)$ is non-decreasing and $\mathrm{Diss}_{\mathscr{D}}(\sigma_\varepsilon;[0,t]) = \mathscr{D}(\sigma_\varepsilon(0),\sigma_\varepsilon(t))$.

Throughout the rest of the paper we shall tacitly assume all the requirements made in this and the previous sections.

4 Upper and lower bounds

In this section we state and prove two lemmas concerning upper and lower bounds for the limit of the energies as ε goes to 0 which shall be useful in identifying the limit problem.

To state our results it is useful to introduce the following tensor components

$$\mathbb{C}^0_{\alpha\beta\gamma\delta} := \mathbb{C}_{\alpha\beta\gamma\delta} - \frac{\mathbb{C}_{\alpha\beta 33}\mathbb{C}_{\gamma\delta 33}}{\mathbb{C}_{3333}}, \qquad \alpha, \beta, \gamma, \delta = 1, 2, \tag{4.1}$$

and remark that, setting

$$f(A) := \mathbb{C}A \cdot A \qquad \text{for every } A \in \mathbb{R}^{3\times 3}_{\text{sym}}$$

and taking into account (2.4) and (2.5) we have (see also [1])

$$\mathbb{C}^0 \tilde{A} \cdot \tilde{A} = \min_{\tilde{z}\in\mathbb{R}^2, z_3\in\mathbb{R}} f\begin{pmatrix} \tilde{A} & \tilde{z} \\ \tilde{z}^\top & z_3 \end{pmatrix} \qquad \text{for every } \tilde{A} \in \mathbb{R}^{2\times 2}_{\text{sym}}, \tag{4.2}$$

where the minimum is achieved for $\tilde{z} = 0$ and

$$z_3 = -\sum_{\alpha,\beta=1}^{2} \frac{\mathbb{C}_{33\alpha\beta}\tilde{A}_{\alpha\beta}}{\mathbb{C}_{3333}}.$$

For $\sigma \in [\sigma_{in}, L]$, let

$$KL(\Omega_\sigma; \mathbb{R}^3) := \{\, u \in H^1(\Omega_\sigma; \mathbb{R}^3) \mid (Eu)_{i3} = 0 \text{ for } i = 1, 2, 3 \,\}$$

be the set of Kirchhoff-Love displacements. Hereafter following the notation above we denote by $\tilde{E}(u + \psi(t))$ the 2×2-matrix with components

$$\tilde{E}(u + \psi(t))_{\alpha\beta} := E(u + \psi(t))_{\alpha\beta}$$

and we define

$$\begin{aligned} \mathscr{E}_0(t, u, \sigma) := \ & \frac{1}{2}\int_{\Omega_\sigma} \mathbb{C}^0 \tilde{E}(u + \psi(t)) \cdot \tilde{E}(u + \psi(t))\, dx + \\ & + \int_{\Gamma_\sigma} \Big(g_v([u]_v) + g_p^0([u]_p) \Big)\, d\mathscr{H}^2 \end{aligned} \tag{4.3}$$

if $u \in KL(\Omega_\sigma; \mathbb{R}^3)$ and $+\infty$ otherwise.

Lemma 4.1 *If $t_\varepsilon \to t$, $\sigma_\varepsilon \to \sigma$ and $u_\varepsilon \rightharpoonup u$ in $H^1(\Omega_L;\mathbb{R}^3)$ as $\varepsilon \to 0^+$, then*

$$\liminf_{\varepsilon\to 0^+} \mathscr{E}_\varepsilon(t_\varepsilon, u_\varepsilon, \sigma_\varepsilon) \geq \mathscr{E}_0(t,u,\sigma). \tag{4.4}$$

Proof: Since

$$\liminf_{\varepsilon\to 0^+} \mathscr{E}_\varepsilon(t_\varepsilon, u_\varepsilon, \sigma_\varepsilon) \geq \liminf_{\varepsilon\to 0^+} \mathscr{E}^b_\varepsilon(t_\varepsilon, u_\varepsilon, \sigma_\varepsilon) + \liminf_{\varepsilon\to 0^+} \mathscr{E}^s_\varepsilon(u_\varepsilon, \sigma_\varepsilon)$$

it sufficies to prove that

$$\liminf_{\varepsilon\to 0^+} \mathscr{E}^b_\varepsilon(t_\varepsilon, u_\varepsilon, \sigma_\varepsilon) \geq \frac{1}{2}\int_{\Omega_\sigma} \mathbb{C}^0 \tilde{E}(u+\psi(t))\cdot \tilde{E}(u+\psi(t))\,dx \tag{4.5}$$

and

$$\liminf_{\varepsilon\to 0^+} \mathscr{E}^s_\varepsilon(u_\varepsilon, \sigma_\varepsilon) \geq \int_{\Gamma_\sigma} \Big(g_v([u]_v) + g^0_p([u]_p)\Big)\,d\mathscr{H}^2 \tag{4.6}$$

when the liminfs are finite limits.

We start by proving (4.6). Since the energy $\mathscr{E}^s_\varepsilon(u_\varepsilon,\sigma_\varepsilon)$ is finite and $g_v(0) = g_p(0) = 0$, it follows that $u_\varepsilon \in \mathcal{A}(\sigma_\varepsilon)$ and

$$\int_{\Gamma_{\sigma_\varepsilon}} \left(g_v([u_\varepsilon]_v) + \frac{g_p(\varepsilon[u_\varepsilon]_p)}{\varepsilon}\right) d\mathscr{H}^2 = \int_{\Gamma_L} \left(g_v([u_\varepsilon]_v) + \frac{g_p(\varepsilon[u_\varepsilon]_p)}{\varepsilon}\right) d\mathscr{H}^2.$$

Since $u_\varepsilon \rightharpoonup u$ in $H^1(\Omega_L;\mathbb{R}^3)$ we have that $[u_\varepsilon]_v \to [u]_v$ in $L^2(\Gamma_L)$ and $[u_\varepsilon]_p \to [u]_p$ in $L^2(\Gamma_L;\mathbb{R}^2)$, and hence, up to subsequences,

$$[u_\varepsilon]_v \to [u]_v, \quad [u_\varepsilon]_p \to [u]_p \quad \mathscr{H}^2 - \text{a.e.}.$$

Moreover, since $\sigma_\varepsilon \to \sigma$, we get $u \in H^1(\Omega_\sigma;\mathbb{R}^3)$.

By Severini-Egorov theorem we have that for every $\eta > 0$ there exists a closed subset F_η of Γ_L such that $\mathscr{H}^2(\Gamma_L \setminus F_\eta) < \eta$ and

$$[u_\varepsilon]_p \to [u]_p \text{ uniformly in } F_\eta.$$

Let

$$B_\eta := \{x \in \Gamma_L \;:\; |[u]_p(x)| \leq 1/\eta\}.$$

Then, for any ε small enough we have

$$|[u_\varepsilon]_p(x)| \leq \frac{1}{\eta} + 1 \quad \text{for every } x \in B_\eta \cap F_\eta,$$

hence $\varepsilon|[u_\varepsilon]_p(x)| < \ell$, where ℓ is the constant of assumption $(G5)$. From $g_p \geq 0$ and $(G5)$ we deduce

$$
\begin{aligned}
\int_{\Gamma_L} \frac{g_p(\varepsilon[u_\varepsilon]_p)}{\varepsilon}\, d\mathscr{H}^2 &\geq \int_{B_\eta\cap F_\eta} \frac{g_p(\varepsilon[u_\varepsilon]_p)}{\varepsilon}\, d\mathscr{H}^2 \\
&\geq \int_{B_\eta\cap F_\eta} \big(g_p^0([u_\varepsilon]_p) - C\varepsilon^\alpha |[u_\varepsilon]_p|^{\alpha+1}\big)\, d\mathscr{H}^2 \\
&\geq \int_{\Gamma_L} g_p^0([u_\varepsilon]_p)\, d\mathscr{H}^2 - \int_{\Gamma_L\setminus(B_\eta\cap F_\eta)} g_p^0([u_\varepsilon]_p)\, d\mathscr{H}^2 - C\varepsilon^\alpha (\frac{1}{\eta}+1)^{\alpha+1},
\end{aligned}
$$

but, by $(G4)$,

$$
\begin{aligned}
\int_{\Gamma_L\setminus(B_\eta\cap F_\eta)} g_p^0([u_\varepsilon]_p)\, d\mathscr{H}^2 &\leq C\int_{\Gamma_L\setminus(B_\eta\cap F_\eta)} |[u_\varepsilon]_p|\, d\mathscr{H}^2 \\
&\leq C\mathscr{H}^2(\Gamma_L\setminus(B_\eta\cap F_\eta))^{1/2}\Big(\int_{\Gamma_L} |[u_\varepsilon]_p|^2\, d\mathscr{H}^2\Big)^{1/2} \\
&\leq C\mathscr{H}^2(\Gamma_L\setminus(B_\eta\cap F_\eta))^{1/2}.
\end{aligned}
$$

Hence, by Fatou's lemma, $(G2)$ and $(G6)$ we obtain

$$
\begin{aligned}
&\liminf_{\varepsilon\to 0^+} \int_{\Gamma_L} \big(g_v([u_\varepsilon]_v) + \frac{g_p(\varepsilon[u_\varepsilon]_p)}{\varepsilon}\big)\, d\mathscr{H}^2 \geq \\
&\quad\geq \liminf_{\varepsilon\to 0^+} \int_{\Gamma_L} g_v([u_\varepsilon]_v) + g_p^0([u_\varepsilon]_p)\, d\mathscr{H}^2 - C\mathscr{H}^2(\Gamma_L\setminus(B_\eta\cap F_\eta))^{1/2} \\
&\quad\geq \int_{\Gamma_L} g_v([u]_v) + g_p^0([u]_p)\, d\mathscr{H}^2 - C\mathscr{H}^2(\Gamma_L\setminus(B_\eta\cap F_\eta))^{1/2}
\end{aligned}
$$

and letting $\eta \to 0$ we finally deduce (4.6).

We now prove (4.5). From (2.3) and (3.3) it follows that

$$\sup_\varepsilon \|E^\varepsilon(u_\varepsilon + \psi(t_\varepsilon))\|_{L^2} < +\infty$$

and hence

$$\|E(u_\varepsilon + \psi(t_\varepsilon))_{i3}\|_{L^2} \leq C\varepsilon.$$

Thus, passing to the limit,

$$E(u + \psi(t))_{i3} = 0,$$

which implies $u + \psi(t) \in KL(\Omega_\sigma; \mathbb{R}^3)$, hence $u \in KL(\Omega_\sigma; \mathbb{R}^3)$ since by assumption $E(\psi(t))_{i3} = 0$ for every t.

Noticing that, for $\alpha, \beta = 1, 2$,

$$E^\varepsilon(u_\varepsilon + \psi(t_\varepsilon))_{\alpha\beta} = E(u_\varepsilon + \psi(t_\varepsilon))_{\alpha\beta} \rightharpoonup E(u + \psi(t))_{\alpha\beta} \ \text{ in } L^2(\Omega_L)$$

and using property (4.2) we find

$$\begin{aligned}
\liminf_{\varepsilon\to 0^+} \mathscr{E}^b_\varepsilon(t_\varepsilon, u_\varepsilon, \sigma_\varepsilon) &= \frac{1}{2} \liminf_{\varepsilon\to 0^+} \int_\Omega f\big(E^\varepsilon(u_\varepsilon + \psi(t_\varepsilon))\big)\, dx \\
&\geq \frac{1}{2} \liminf_{\varepsilon\to 0^+} \int_\Omega \mathbb{C}^0 \tilde{E}(u_\varepsilon + \psi(t_\varepsilon)) \cdot \tilde{E}(u_\varepsilon + \psi(t_\varepsilon))\, dx \\
&\geq \frac{1}{2} \int_{\Omega_\sigma} \mathbb{C}^0 \tilde{E}(u + \psi(t)) \cdot \tilde{E}(u + \psi(t))\, dx,
\end{aligned}$$

which concludes the proof. ■

Lemma 4.2 *Let $t \in [0, T]$, $\sigma \in [\sigma_{in}, L]$. For every $(t_\varepsilon, \sigma_\varepsilon) \to (t, \sigma)$, $\breve{\sigma} \in [\sigma_{in}, L]$ and $\breve{u} \in H^1(\Omega_{\breve{\sigma}}; \mathbb{R}^3)$ there exist $\breve{\sigma}_\varepsilon \in [\sigma_{in}, L]$ and $\breve{u}_\varepsilon \in H^1(\Omega_{\breve{\sigma}_\varepsilon}; \mathbb{R}^3)$ such that $\breve{u}_\varepsilon \rightharpoonup \breve{u}$ in $H^1(\Omega_L; \mathbb{R}^3)$, $\breve{\sigma}_\varepsilon \to \breve{\sigma}$ and*

$$\limsup_{\varepsilon\to 0^+} \big[\mathscr{E}_\varepsilon(t_\varepsilon, \breve{u}_\varepsilon, \breve{\sigma}_\varepsilon) + \mathscr{D}(\sigma_\varepsilon, \breve{\sigma}_\varepsilon)\big] \leq \mathscr{E}_0(t, \breve{u}, \breve{\sigma}) + \mathscr{D}(\sigma, \breve{\sigma}).$$

Proof: We may assume that $\breve{\sigma} \geq \sigma$ and $\breve{u} \in KL(\Omega_{\breve{\sigma}}; \mathbb{R}^3)$, otherwise there is nothing to prove.

Let

$$\breve{\sigma}_\varepsilon := \begin{cases} \sigma_\varepsilon & \text{if } \sigma = \breve{\sigma} \text{ and } \sigma_\varepsilon \geq \sigma, \\ \breve{\sigma} & \text{else.} \end{cases}$$

Note that $\breve{\sigma}_\varepsilon \to \breve{\sigma}$ and $\breve{\sigma}_\varepsilon \geq \breve{\sigma}$.

If $\sigma = \breve{\sigma}$, then $\mathscr{D}(\sigma, \breve{\sigma}) = 0$ and $\mathscr{D}(\sigma_\varepsilon, \breve{\sigma}_\varepsilon)$ is equal to 0 if $\sigma_\varepsilon \geq \sigma$, while is finite if $\sigma_\varepsilon < \sigma$. In both cases, $\mathscr{D}(\sigma_\varepsilon, \breve{\sigma}_\varepsilon) \to 0 = \mathscr{D}(\sigma, \breve{\sigma})$. While if $\sigma < \breve{\sigma}$, then $\mathscr{D}(\sigma_\varepsilon, \breve{\sigma}_\varepsilon) = \mathscr{D}(\sigma_\varepsilon, \breve{\sigma})$ which, for small ε, is finite and converges to $\mathscr{D}(\sigma, \breve{\sigma})$.

Thus, to prove the claim it suffices to deal with the convergence of the energy. Let

$$z_3 := -\sum_{\alpha,\beta=1}^{2} \frac{\mathbb{C}_{33\alpha\beta} \widetilde{E}(\breve{u} + \psi(t))_{\alpha\beta}}{\mathbb{C}_{3333}},$$

and choose $\varphi_\varepsilon \in C_0^\infty(\Omega)$ such that $\varphi_\varepsilon \to z_3$ and $\varepsilon\partial\varphi_\varepsilon/\partial x_\alpha \to 0$ in $L^2(\Omega)$. Set

$$\eta_\varepsilon(x_1, x_2, x_3) := \int_0^{x_3} \varphi_\varepsilon(x_1, x_2, s)\, ds$$

and

$$\begin{cases} \check{u}^\varepsilon_\alpha := \check{u}_\alpha, \\ \check{u}^\varepsilon_3 := \check{u}_3 + \varepsilon^2 \eta_\varepsilon, \end{cases}$$

so to have

$$\check{u}^\varepsilon \in H^1(\Omega_{\check{\sigma}}; \mathbb{R}^3) \subseteq H^1(\Omega_{\check{\sigma}_\varepsilon}; \mathbb{R}^3)$$

and, by (3.1),

$$E^\varepsilon(\check{u}^\varepsilon + \psi(t_\varepsilon)) = \begin{pmatrix} E(\check{u} + \psi(t_\varepsilon))_{\alpha\beta} & \dfrac{\varepsilon}{2}\dfrac{\partial \eta_\varepsilon}{\partial x_\alpha} \\ \text{sym} & \varphi_\varepsilon \end{pmatrix},$$

since $\check{u}$, $\psi(t_\varepsilon) \in KL(\Omega_L; \mathbb{R}^3)$. Taking the limit we have

$$\lim_{\varepsilon\to 0^+} \left\| E^\varepsilon(\check{u}^\varepsilon + \psi(t_\varepsilon)) - \begin{pmatrix} \widetilde{E}(\check{u} + \psi(t)) & 0 \\ 0 & -\displaystyle\sum_{\alpha,\beta=1}^{2} \frac{\mathbb{C}_{33\alpha\beta}\widetilde{E}(\check{u} + \psi(t))_{\alpha\beta}}{\mathbb{C}_{3333}} \end{pmatrix} \right\|_{L^2} = 0,$$

and therefore, concerning the bulk part,

$$\lim_{\varepsilon\to 0^+} \int_\Omega \mathbb{C}E^\varepsilon(\check{u}^\varepsilon + \psi(t_\varepsilon)) \cdot E^\varepsilon(\check{u}^\varepsilon + \psi(t_\varepsilon))\, dx = \int_\Omega \mathbb{C}^0 \tilde{E}(\check{u} + \psi(t)) \cdot \tilde{E}(\check{u} + \psi(t))\, dx.$$

Passing to the surface part of the energy, we first notice that $[\check{u}^\varepsilon]_p = [\check{u}]_p$, $[\check{u}^\varepsilon]_v = [\check{u}]_v$. Then, from $(G4)$ we have

$$\frac{g_p(\varepsilon[\check{u}^\varepsilon]_p)}{\varepsilon} \le C\frac{|\varepsilon[\check{u}^\varepsilon]_p|}{\varepsilon} = C|[\check{u}]_p|,$$

and by the dominated convergence theorem we get

$$\lim_{\varepsilon\to 0^+} \int_{\Gamma_{\check{\sigma}_\varepsilon}} g_v([\check{u}^\varepsilon]_v) + \frac{g_p(\varepsilon[\check{u}^\varepsilon]_p)}{\varepsilon}\, d\mathscr{H}^2 = \int_{\Gamma_{\check{\sigma}}} g_v([\check{u}]_v) + g_p^0([\check{u}]_p)\, d\mathscr{H}^2$$

and the proof is concluded. ■

5 Convergence of solutions

Let us begin with the following definition.

Definition 5.1 *An* energetic solution *associated with the functionals* $\mathscr{E}_0$ *and* $\mathscr{D}$*, is a function* $(u, \sigma) : [0, T] \to H^1(\Omega_L; \mathbb{R}^3) \times [\sigma_{in}, L]$*, with the map* $t \mapsto \partial_t \mathscr{E}_0(t, u(t), \sigma(t)) \in L^1(0, T)$ *and satisfying for every* $t \in [0, T]$ *a stability condition* $(S)_0$ *and an energy balance condition* $(E)_0$*:*

$(S)_0$ $\mathscr{E}_0(t, u(t), \sigma(t)) < +\infty$*, and, for every* $(\tilde{u}, \tilde{\sigma}) \in H^1(\Omega_L; \mathbb{R}^3) \times [\sigma_{in}, L]$

$$\mathscr{E}_0(t, u(t), \sigma(t)) \le \mathscr{E}_0(t, \tilde{u}, \tilde{\sigma}) + \mathscr{D}(\sigma(t), \tilde{\sigma}),$$

$(E)_0$ $\mathscr{E}_0(t, u(t), \sigma(t)) + \mathrm{Diss}_{\mathscr{D}}(\sigma; [0, t]) = \mathscr{E}_0(0, u(0), \sigma(0)) +$

$$+ \int_0^t \partial_t \mathscr{E}_0(s, u(s), \sigma(s))\, ds.$$

Similarly to (3.9), whenever $\mathscr{E}_0(t, u, \sigma)$ is finite, we have

$$\partial_t \mathscr{E}_0(t, u, \sigma) = \int_{\Omega_\sigma} \mathbb{C}^0 \tilde{E} \dot{\psi}(t) \cdot \tilde{E}(u + \psi(t))\, dx. \tag{5.1}$$

Remark 5.2 From the definition of $\mathscr{E}_0$ and the stability condition $(S)_0$, we deduce that if $t \mapsto (u(t), \sigma(t))$ is an energetic solution associated with the functionals $\mathscr{E}_0$ and $\mathscr{D}$, then $u(t) \in KL(\Omega_{\sigma(t)}; \mathbb{R}^3)$ for any $t \in [0, T]$.

The aim of this section is to prove, for every fixed ε, the existence of an energetic solution associated with $\mathscr{E}_\varepsilon$ and $\mathscr{D}$, and to show that a sequence of energetic solutions converges, as ε goes to 0, to an energetic solution associated with $\mathscr{E}_0$ and $\mathscr{D}$. We shall achieve both results by applying general abstract theorems proven in [12, 14, 17]. In what follows we write and check the assumptions needed to apply those results. For convenience of the reader, following the notation of [17], we let $\mathcal{Q} := H^1(\Omega_L; \mathbb{R}^3) \times [\sigma_{in}, L]$. Accordingly, we set $q = (u, \sigma)$, and therefore we write, for instance, $\mathscr{E}_\varepsilon(t, q)$ in place of $\mathscr{E}_\varepsilon(t, u, \sigma)$; also, if we find it convenient, we write $\mathscr{D}(q, \tilde{q})$ instead of $\mathscr{D}(\sigma, \tilde{\sigma})$.

The *set of stable states* $\mathcal{S}_\varepsilon(t)$, for $t \in [0, T]$ and $\varepsilon \ge 0$, is

$$\mathcal{S}_\varepsilon(t) := \{q \in \mathcal{Q} \,:\, \mathscr{E}_\varepsilon(t, q) < +\infty,\, \mathscr{E}_\varepsilon(t, q) \le \mathscr{E}_\varepsilon(t, \tilde{q}) + \mathscr{D}(q, \tilde{q})\ \forall\, \tilde{q} \in \mathcal{Q}\}, \tag{5.2}$$

and a sequence $(t_l, q_{\varepsilon_l})_{l\in\mathbb{N}}$ is called a *stable sequence* with respect to $(\mathscr{E}_{\varepsilon_l})$ and $(\mathcal{S}_{\varepsilon_l})$ if $\varepsilon_l \to 0$,

$$q_{\varepsilon_l} \in \mathcal{S}_{\varepsilon_l}(t_l) \quad \text{and} \quad \sup_{l\in\mathbb{N}} \mathscr{E}_{\varepsilon_l}(t_l, q_{\varepsilon_l}) < +\infty. \tag{5.3}$$

The dissipation $\mathscr{D}$ satisfies the following properties, corresponding to (2.2)–(2.4) of [17]:

Pseudo distance:

$$\begin{aligned}&\mathscr{D}(\sigma_1,\sigma_1) = 0 \text{ and } \mathscr{D}(\sigma_1,\sigma_3) \le \mathscr{D}(\sigma_1,\sigma_2) + \mathscr{D}(\sigma_2,\sigma_3)\\ &\text{for every } \sigma_1,\sigma_2,\sigma_3 \in [\sigma_{in}, L].\end{aligned} \tag{5.4}$$

Lower semi-continuity of $\mathscr{D}$:

$$\mathscr{D} : [\sigma_{in}, L] \times [\sigma_{in}, L] \to [0, +\infty] \text{ is lower semi-continuous.} \tag{5.5}$$

Positivity of $\mathscr{D}$:

$$\begin{aligned}&\text{if a sequence } (\sigma_\varepsilon) \text{ in } [\sigma_{in}, L] \text{ and } \sigma \in [\sigma_{in}, L] \text{ are such that}\\ &\min\{\mathscr{D}(\sigma_\varepsilon,\sigma), \mathscr{D}(\sigma,\sigma_\varepsilon)\} \to 0, \text{ then } \sigma_\varepsilon \to \sigma.\end{aligned} \tag{5.6}$$

From (5.5) it follows (2.5) of [17], that is

Lower Γ-limit for $\mathscr{D}$:

$$\begin{aligned}&\text{for any pair of stable sequences } (t_l, q_{\varepsilon_l}),\ (\tilde{t}_l, \tilde{q}_{\varepsilon_l}) \text{ such that}\\ &(t_l, q_{\varepsilon_l}) \rightharpoonup (t, q),\ (\tilde{t}_l, \tilde{q}_{\varepsilon_l}) \rightharpoonup (\tilde{t}, \tilde{q}) \text{ in } [0,T]\times\mathcal{Q}, \text{ we have}\\ &\mathscr{D}(q,\tilde{q}) \le \liminf\nolimits_{l\to\infty} \mathscr{D}(q_{\varepsilon_l}, \tilde{q}_{\varepsilon_l}).\end{aligned} \tag{5.7}$$

From Korn's inequality and (2.3) we find

$$\begin{aligned}\mathscr{E}_\varepsilon(t,u,\sigma) \ge \mathscr{E}^b_\varepsilon(t,u,\sigma) &\ge \frac{1}{2}\int_{\Omega_\sigma} \mathbb{C}E^\varepsilon(u+\psi(t))\cdot E^\varepsilon(u+\psi(t))\,dx\\ &\ge \mu\|E^\varepsilon(u+\psi(t))\|^2_{L^2} \ge \mu\|E(u+\psi(t))\|^2_{L^2}\\ &\ge C_1\|Eu\|^2_{L^2} - C_2\|E\psi(t)\|^2_{L^2} \ge C_1\|u\|^2_{H^1} - C_2\|E\psi(t)\|^2_{L^2}.\end{aligned} \tag{5.8}$$

This chain of inequalities together with a similar computation for $\mathscr{E}_0$ shows that the set $\bigcup_{\varepsilon\geq 0}\{\, q\in\mathcal{Q} \,|\, \mathscr{E}_\varepsilon(t,q)\leq E\,\}$ is relatively compact. Moreover, for any $\varepsilon\geq 0$ the functionals $\mathscr{E}_\varepsilon(t,\cdot)$ are weakly lower semicontinuous in $\mathcal{Q}$ due to the convexity of the bulk part of the energy, the lower semicontinuity of the integrands g_v, g_p and g_p^0 and the convergence in L^2 of the jumps. Thus the sublevels are also closed and the following property (corresponding to (2.6) of [17]) holds

Compactness of energy sublevels:

$$
\begin{aligned}
&\text{for all } t\in[0,T] \text{ and all } E\in\mathbb{R} \text{ we have}\\
&(i)\ \{q\in\mathcal{Q}\,:\,\mathscr{E}_\varepsilon(t,q)\leq E\} \text{ is compact for any } \varepsilon\geq 0;\\
&(ii)\ \textstyle\bigcup_{\varepsilon\geq 0}\{q\in\mathcal{Q}\,:\,\mathscr{E}_\varepsilon(t,q)\leq E\} \text{ is relatively compact.}
\end{aligned}
\tag{5.9}
$$

Since $\psi\in C^1([0,T];H^1(\Omega;\mathbb{R}^3))$ then $\mathscr{E}_\varepsilon(\cdot,q)\in C^1([0,T])$ for all $\varepsilon\geq 0$ and all $q\in\mathcal{Q}$ for which $\mathscr{E}_\varepsilon(\cdot,q)<+\infty$.

If $\mathscr{E}_\varepsilon(s,u,\sigma)<+\infty$ for $s\in[0,T]$ and since $(E\dot{\psi})_{i3}=0$ (see (3.2)), from (3.9) we have

$$
\begin{aligned}
|\partial_t\mathscr{E}_\varepsilon(s,u,\sigma)| &\leq c\Big(\|E^\varepsilon\dot{\psi}(s)\|_{L^2}^2+\|E^\varepsilon(u+\psi(s))\|_{L^2}^2\Big)\\
&\leq c\Big(\|E\dot{\psi}(s)\|_{L^2}^2+\mathscr{E}_\varepsilon(s,u,\sigma)\Big),
\end{aligned}
$$

which, together with a similar computation for $\mathscr{E}_0$, leads to (see (2.7) of [17])

Uniform control of the power $\partial_t\mathscr{E}_\varepsilon$:

$$
\begin{aligned}
&\text{there exist } c_0^E\in\mathbb{R} \text{ and } c_1^E>0 \text{ such that}\\
&\text{for any } \varepsilon\geq 0,\ t\in[0,T] \text{ and } q\in\mathcal{Q},\\
&\text{if } \mathscr{E}_\varepsilon(t,q)<+\infty \text{ then } \mathscr{E}_\varepsilon(\cdot,q)\in C^1([0,T]) \text{ and}\\
&|\partial_t\mathscr{E}_\varepsilon(s,q)|\leq c_1^E(c_0^E+\mathscr{E}_\varepsilon(s,q)) \text{ for all } s\in[0,T].
\end{aligned}
\tag{5.10}
$$

From the definition of $\mathscr{E}_0$ (given in (4.3)) it follows that if $\mathscr{E}_0(0,u,\sigma)$ is finite then $u\in KL(\Omega_\sigma;\mathbb{R}^3)$ and thus $\mathscr{E}_0(t,u,\sigma)$ is finite for every $t\in[0,T]$. Since ψ is C^1, condition (2.8) of [17] is satisfied, namely

Uniform time-continuity of the power $\partial_t\mathscr{E}_0$:

$$\begin{aligned}&\text{for every } \eta > 0 \text{ and } E \in \mathbb{R} \text{ there exists } \delta > 0 \text{ such that}\\ &\mathscr{E}_0(0, q) \le E,\ |t_1 - t_2| < \delta \Rightarrow |\partial_t\mathscr{E}_0(t_1, q) - \partial_t\mathscr{E}_0(t_2, q)| < \eta.\end{aligned} \tag{5.11}$$

By Lemma 4.1 we get the following property (2.10) of [17]

Lower Γ-limit for $\mathscr{E}_\varepsilon$:

$$\begin{aligned}&\text{for any stable sequence } (t_l, q_{\varepsilon_l}) \text{ such that}\\ &(t_l, q_{\varepsilon_l}) \rightharpoonup (t, q) \text{ in } [0, T] \times \mathcal{Q}, \text{ we have}\\ &\mathscr{E}_0(t, q) \le \liminf_{l\to+\infty} \mathscr{E}_{\varepsilon_l}(t_l, q_{\varepsilon_l}).\end{aligned} \tag{5.12}$$

The following proposition essentially corresponds to property (2.9) of [17]; the only difference is that we establish the convergence in the open interval $(0, T)$ instead of its closure. This fact will not affect the arguments used in the sequel.

Proposition 5.3 *[Conditioned continuous convergence of the power] If $(t, q) \in (0, T) \times \mathcal{Q}$ and $(t_{\varepsilon_l}, q_{\varepsilon_l})$ is a stable sequence such that $(t_{\varepsilon_l}, q_{\varepsilon_l}) \rightharpoonup (t, q)$ in $[0, T] \times \mathcal{Q}$, then*

$$\partial_t\mathscr{E}_{\varepsilon_l}(t_{\varepsilon_l}, q_{\varepsilon_l}) \to \partial_t\mathscr{E}_0(t, q).$$

Proof: For simplicity we write ε in place of ε_l. We combine the argument of the proof of Proposition 3.3 of [12], where a single energy functional is considered, with the upper and lower bound Lemma 4.1 and Lemma 4.2.

We start by showing that

$$\mathscr{E}_\varepsilon(t_\varepsilon, u_\varepsilon, \sigma_\varepsilon) \to \mathscr{E}_0(t, u, \sigma). \tag{5.13}$$

Indeed, from the stability of the sequence $(t_\varepsilon, u_\varepsilon, \sigma_\varepsilon)$ and by Lemma 4.2, there exist $\check{\sigma}_\varepsilon \in [\sigma_{in}, L]$ and $\check{u}_\varepsilon \in H^1(\Omega_L; \mathbb{R}^3)$ such that $\check{u}_\varepsilon \rightharpoonup u$ in $H^1(\Omega_L; \mathbb{R}^3)$, $\check{\sigma}_\varepsilon \to \sigma$ such that

$$\limsup_{\varepsilon\to0^+} \mathscr{E}_\varepsilon(t_\varepsilon, u_\varepsilon, \sigma_\varepsilon) \le \limsup_{\varepsilon\to0^+} \left[\mathscr{E}_\varepsilon(t_\varepsilon, \check{u}_\varepsilon, \check{\sigma}_\varepsilon) + \mathscr{D}(\sigma_\varepsilon, \check{\sigma}_\varepsilon)\right] \le \mathscr{E}_0(t, u, \sigma).$$

Hence (5.13) follows from the inequality above and Lemma 4.1.

Let now $h > 0$ be such that $t \pm 2h \in [0,T]$. Then for any ε small enough $t_\varepsilon \in (t-h, t+h)$ and hence $t_\varepsilon \pm h \in [0,T]$, and let $K_0 > 0$ be such that $\mathscr{E}_\varepsilon(t_\varepsilon, q_\varepsilon), \mathscr{E}_0(t,q) \leq K_0$. We have

$$\begin{aligned}
&\Big|\frac{1}{h}\big[\mathscr{E}_\varepsilon(t_\varepsilon \pm h, u_\varepsilon, \sigma_\varepsilon) - \mathscr{E}_\varepsilon(t_\varepsilon, u_\varepsilon, \sigma_\varepsilon)\big] \mp \partial_t \mathscr{E}_\varepsilon(t_\varepsilon, u_\varepsilon, \sigma_\varepsilon)\Big| = \\
&\quad = |\partial_t \mathscr{E}_\varepsilon(\bar t_\varepsilon, u_\varepsilon, \sigma_\varepsilon) - \partial_t \mathscr{E}_\varepsilon(t_\varepsilon, u_\varepsilon, \sigma_\varepsilon)| \\
&\quad = \Big| \int_{\Omega_{\sigma_\varepsilon}} \Big[\mathbb{C} E^\varepsilon \dot\psi(\bar t_\varepsilon) \cdot E^\varepsilon\big(\psi(\bar t_\varepsilon) - \psi(t_\varepsilon)\big) + \\
&\qquad\qquad + \mathbb{C} E^\varepsilon\big(\dot\psi(\bar t_\varepsilon) - \dot\psi(t_\varepsilon)\big) \cdot E^\varepsilon(u_\varepsilon + \psi(t_\varepsilon))\Big]\, dx\Big| \\
&\quad \leq C\Big(\|E\dot\psi(\bar t_\varepsilon)\|_{L^2} \|E\big(\psi(\bar t_\varepsilon) - \psi(t_\varepsilon)\big)\|_{L^2} + \|E\big(\dot\psi(\bar t_\varepsilon) - \dot\psi(t_\varepsilon)\big)\|_{L^2} K_0\Big) \\
&\quad =: \omega_{K_0}(h)
\end{aligned} \tag{5.14}$$

where $\bar t_\varepsilon$ is between t_ε and $t_\varepsilon \pm h$, and where ω_{K_0} is independent of ε because of the continuity at time t of $D\psi$ and $D\dot\psi$ stated in assumption (3.2).

By applying Lemma 4.1 and (5.13) we have

$$\liminf_{\varepsilon\to 0} \frac{1}{h}\big[\mathscr{E}_\varepsilon(t_\varepsilon \pm h, u_\varepsilon, \sigma_\varepsilon) - \mathscr{E}_\varepsilon(t_\varepsilon, u_\varepsilon, \sigma_\varepsilon)\big] \geq \frac{1}{h}\big[\mathscr{E}_0(t \pm h, u, \sigma) - \mathscr{E}_0(t, u, \sigma)\big]. \tag{5.15}$$

By using (5.14), then the inequality above and finally (5.11), we find

$$\begin{aligned}
&\limsup_{\varepsilon\to 0^+} \partial_t \mathscr{E}_\varepsilon(t_\varepsilon, u_\varepsilon, \sigma_\varepsilon) \leq \\
&\quad \leq \limsup_{\varepsilon\to 0^+} \frac{1}{h}\big[\mathscr{E}_\varepsilon(t_\varepsilon, u_\varepsilon, \sigma_\varepsilon) - \mathscr{E}_\varepsilon(t_\varepsilon - h, u_\varepsilon, \sigma_\varepsilon)\big] + \omega_{K_0}(h) \\
&\quad = \omega_{K_0}(h) - \liminf_{\varepsilon\to 0^+} \frac{1}{h}\big[\mathscr{E}_\varepsilon(t_\varepsilon - h, u_\varepsilon, \sigma_\varepsilon) - \mathscr{E}_\varepsilon(t_\varepsilon, u_\varepsilon, \sigma_\varepsilon)\big] \\
&\quad \leq \omega_{K_0}(h) - \frac{1}{h}\big[\mathscr{E}_0(t-h, u, \sigma) - \mathscr{E}_0(t, u, \sigma)\big] \leq \partial_t \mathscr{E}_0(t, u, \sigma) + 2\omega_{K_0}(h).
\end{aligned}$$

In the same way, but appropriately choosing the signs in (5.14) and (5.15), we get

$$\liminf_{\varepsilon\to 0^+} \partial_t \mathscr{E}_\varepsilon(t_\varepsilon, u_\varepsilon, \sigma_\varepsilon) \geq \partial_t \mathscr{E}_0(t, u, \sigma) - 2\omega_{K_0}(h),$$

and the conclusion follows by letting h go to zero. ■

By Proposition 2.2 of [17] we get that Lemma 4.2 implies property (2.11) of [17], namely

Conditioned upper-semicontinuity of stable sets:

$$\begin{aligned}&\text{for any stable sequence } (t_l, q_{\varepsilon_l}) \text{ such that}\\ &(t_l, q_{\varepsilon_l}) \rightharpoonup (t, q) \text{ in } [0,T]\times\mathcal{Q} \text{ we have that } q \in \mathcal{S}_0(t).\end{aligned} \tag{5.16}$$

We now prove the existence of energetic solutions associated with the functionals $\mathscr{E}_\varepsilon$ and $\mathscr{D}$, for fixed $\varepsilon > 0$.

Theorem 5.4 *Let $q_\varepsilon^0 = (u_\varepsilon^0, \sigma_\varepsilon^0) \in \mathcal{S}_\varepsilon(0)$. There exists an energetic solution $(u_\varepsilon, \sigma_\varepsilon) : [0,T] \to \mathcal{Q}$ associated with $\mathscr{E}_\varepsilon$ and $\mathscr{D}$, such that $(u_\varepsilon(0), \sigma_\varepsilon(0)) = (u_\varepsilon^0, \sigma_\varepsilon^0)$.*

Proof: The proof follows by applying Theorem 3.4 of Mielke [16] (see also [12, 14, 17]). Since we already know that the assumptions on the energy functional (5.9), (5.10), (5.11) with $\mathscr{E}_0$ replaced by $\mathscr{E}_\varepsilon$, and the conditions on the dissipation distance (5.4), (5.5), (5.6) hold true, to apply Theorem 3.4 of [16] we only need to check the following compatibility conditions:

if (t_k, q_k), $k \in \mathbb{N}$, is a stable sequence in the sense of [16] such that $(t_k, q_k) \rightharpoonup (t, q)$ in $[0,T]\times\mathcal{Q}$ then

$$\begin{aligned}&\partial_t \mathscr{E}_\varepsilon(t_k, q_k) \to \partial_t \mathscr{E}_\varepsilon(t, q),\\ &q \in \mathcal{S}_\varepsilon(t).\end{aligned}$$

The convergence of the powers follows from (3.9), using the time continuity of $\dot{\psi}$. Passing to the stability condition $q \in \mathcal{S}_\varepsilon(t)$, setting $q = (u, \sigma)$, we have to prove that

$$\mathscr{E}_\varepsilon(t, u, \sigma) \le \mathscr{E}_\varepsilon(t, \tilde{u}, \tilde{\sigma}) + \mathscr{D}(\sigma, \tilde{\sigma}) \quad \text{for every } (\tilde{u}, \tilde{\sigma}) \in \mathcal{Q}.$$

Let $(\tilde{u}, \tilde{\sigma}) \in \mathcal{Q}$. Without loss of generality we may assume $\tilde{u} \in \mathcal{A}(\sigma)$ and $\sigma \le \tilde{\sigma}$. Let $\delta > 0$ and $q_k = (u_k, \sigma_k) \in \mathcal{S}_\varepsilon(t_k)$. Hence

$$\mathscr{E}_\varepsilon(t_k, u_k, \sigma_k) \le \mathscr{E}_\varepsilon(t_k, \tilde{u}, \tilde{\sigma} + \delta) + \mathscr{D}(\sigma_k, \tilde{\sigma} + \delta),$$

and using the lower semicontinuity of $\mathscr{E}_\varepsilon(t,\cdot)$ in $\mathcal{Q}$ together with the time regularity of ψ we deduce

$$\mathscr{E}_\varepsilon(t,u,\sigma) \le \liminf_{k\to+\infty} \mathscr{E}_\varepsilon(t_k,u_k,\sigma_k) \le \mathscr{E}_\varepsilon(t,\tilde{u},\tilde{\sigma}+\delta) + \mathscr{D}(\sigma,\tilde{\sigma}+\delta).$$

The arbitrariness of δ concludes the proof. ∎

We are now in a position to state our main dimension reduction result. Indeed the next theorem collects the existence of an energetic solution associated with $\mathscr{E}_0$ and $\mathscr{D}$, and the convergence properties of the sequence of solutions associated with $\mathscr{E}_\varepsilon$ and $\mathscr{D}$.

Theorem 5.5 *Let, for any $\varepsilon > 0$, $q_\varepsilon = (u_\varepsilon, \sigma_\varepsilon) : [0,T] \to \mathcal{Q}$ be an energetic solution associated with $\mathscr{E}_\varepsilon$ and $\mathscr{D}$. Then there exist an energetic solution $q = (u,\sigma) : [0,T] \to \mathcal{Q}$ associated with $\mathscr{E}_0$ and $\mathscr{D}$ and a sequence (ε_n) converging to zero such that, if $\mathscr{E}_{\varepsilon_n}(0, q_{\varepsilon_n}(0)) \to \mathscr{E}_0(0, q(0))$, then*

(1) $\mathscr{E}_{\varepsilon_n}(t, q_{\varepsilon_n}(t)) \to \mathscr{E}_0(t, q(t))$ for every $t \in [0,T]$;

(2) $\mathscr{D}(\sigma_{\varepsilon_n}(0), \sigma_{\varepsilon_n}(t)) \to \mathscr{D}(\sigma(0), \sigma(t))$ for every $t \in [0,T]$;

(3) $\partial_t \mathscr{E}_{\varepsilon_n}(\cdot, q_{\varepsilon_n}(\cdot)) \to \partial_t \mathscr{E}_0(\cdot, q(\cdot))$ in $L^1(0,T)$;

(4) $\sigma_{\varepsilon_n}(t) \to \sigma(t)$ for every $t \in [0,T]$;

(5) for every $t \in [0,T]$ there exists a t-dependent subsequence (ε_{n^t}) of (ε_n) such that $u_{\varepsilon_{n^t}}(t) \rightharpoonup u(t)$ in $H^1(\Omega_L;\mathbb{R}^3)$.

Proof: By Remark 3.3, the function $t \mapsto \sigma_\varepsilon(t)$ is non-decreasing for any ε. Hence we can apply Helly's theorem (see e.g. [3], Theorem 3.5 Chapter 1) obtaining that there exist a sequence (ε_n) and a non-decreasing limit function $\sigma : [0,T] \to [\sigma_{in}, L]$ such that

$$\sigma_{\varepsilon_n}(t) \to \sigma(t) \quad \text{for every } t \in [0,T]. \tag{5.17}$$

We have thus proved *(4)*.

Since $t \mapsto (u_{\varepsilon_n}(t), \sigma_{\varepsilon_n}(t))$ is an energetic solution it follows, from the stability condition $(S)_{\varepsilon_n}$, that $\mathscr{E}_{\varepsilon_n}(t, u_{\varepsilon_n}(t), \sigma_{\varepsilon_n}(t))$ is bounded uniformly in ε_n and hence, from Korn's inequality (see (5.8)), $u_{\varepsilon_n}(t)$ is uniformly bounded in H^1 for every $t \in [0,T]$. Thus, for every $t \in [0,T]$ there exists a subsequence (ε_{n^t}) such that

$$u_{\varepsilon_{n^t}}(t) \rightharpoonup: u(t) \quad \text{in } H^1(\Omega_L;\mathbb{R}^3), \tag{5.18}$$

that is *(5)*, and

$$\theta(t) := \limsup_{n\to\infty} \partial_t \mathscr{E}_{\varepsilon_n}(t, q_{\varepsilon_n}(t)) = \lim_{n\to\infty} \partial_t \mathscr{E}_{\varepsilon_{n^t}}(t, q_{\varepsilon_{n^t}}(t)). \tag{5.19}$$

From (5.17), (5.18), (5.19) and Proposition 5.3 it follows that

$$\theta(t) = \partial_t \mathscr{E}_0(t, u(t), \sigma(t)). \tag{5.20}$$

By Theorem 3.1 of [17] and its proof we deduce that $t \mapsto (u(t), \sigma(t))$ is an energetic solution associated with $\mathscr{E}_0$ and $\mathscr{D}$.

Let us prove *(1)*. By Lemma 4.2 there exist $\check{q}_{\varepsilon_n} := (\check{u}_{\varepsilon_n}, \check{\sigma}_{\varepsilon_n}) \in \mathcal{Q}$ such that

$$\check{q}_{\varepsilon_n} \rightharpoonup q(t) = (u(t), \sigma(t)) \text{ in } \mathcal{Q}$$

and

$$\limsup_{n\to\infty} \left[\mathscr{E}_{\varepsilon_n}(t, \check{u}_{\varepsilon_n}, \check{\sigma}_{\varepsilon_n}) + \mathscr{D}(\sigma_{\varepsilon_n}(t), \check{\sigma}_{\varepsilon_n})\right] \le \mathscr{E}_0(t, u(t), \sigma(t)).$$

This inequality, together with the stability condition $(S)_{\varepsilon_n}$, implies

$$\limsup_{n\to\infty} \mathscr{E}_{\varepsilon_n}(t, u_{\varepsilon_n}(t), \sigma_{\varepsilon_n}(t)) \le \mathscr{E}_0(t, u(t), \sigma(t)). \tag{5.21}$$

For every t we may find a subsequence ε'_{n^t} of (ε_n) such that

$$\liminf_{n\to\infty} \mathscr{E}_{\varepsilon_n}(t, u_{\varepsilon_n}(t), \sigma_{\varepsilon_n}(t)) = \lim_{n\to\infty} \mathscr{E}_{\varepsilon'_{n^t}}(t, u_{\varepsilon'_{n^t}}(t), \sigma_{\varepsilon'_{n^t}}(t)). \tag{5.22}$$

From Korn's inequality (5.8) we deduce that there exists a subsequence (not relabeled) and a limit function $u^*(t) \in H^1(\Omega_L;\mathbb{R}^3)$ such that

$$u_{\varepsilon'_{n^t}}(t) \rightharpoonup u^*(t) \text{ in } H^1(\Omega_L;\mathbb{R}^3).$$

We note that

$$\sigma_{\varepsilon'_{n^t}}(t) \to \sigma(t) \quad \text{for every } t \in [0,T].$$

Let $\bar{u} \in H^1(\Omega_{\sigma(t)};\mathbb{R}^3)$. By Lemma 4.2 there exist $\tilde{u}_{\varepsilon'_{n^t}} \rightharpoonup \bar{u}$ and $\tilde{\sigma}_{\varepsilon'_{n^t}}(t) \to \sigma(t)$ such that

$$\mathscr{E}_0(t, \bar{u}, \sigma(t)) \ge \limsup_{n\to+\infty} \left[\mathscr{E}_{\varepsilon'_{n^t}}(t, \tilde{u}_{\varepsilon'_{n^t}}, \tilde{\sigma}_{\varepsilon'_{n^t}}(t)) + \mathscr{D}(\sigma_{\varepsilon'_{n^t}}(t), \tilde{\sigma}_{\varepsilon'_{n^t}}(t))\right]$$

$$\geq \liminf_{n\to+\infty} \mathscr{E}_{\varepsilon'_{n}t}(t, u_{\varepsilon'_{n}t}, \sigma_{\varepsilon'_{n}t}(t)) \geq \mathscr{E}_0(t, u^*(t), \sigma(t)), \qquad (5.23)$$

where the second inequality holds thanks to the stability condition $(S)_{\varepsilon'_{n}t}$, while the last one follows by Proposition 4.1. Therefore $u^*(t)$ minimizes $\mathscr{E}_0(t, \cdot, \sigma(t))$ and hence

$$\mathscr{E}_0(t, u^*(t), \sigma(t)) = \mathscr{E}_0(t, u(t), \sigma(t)),$$

for every $t \in [0, T]$. From this equality, (5.22) and (5.23) it follows

$$\liminf_{n\to\infty} \mathscr{E}_{\varepsilon_n}(t, u_{\varepsilon_n}(t), \sigma_{\varepsilon_n}(t)) \geq \mathscr{E}_0(t, u(t), \sigma(t)),$$

and, from (5.21), we obtain *(1)*. Since $t \mapsto \sigma_{\varepsilon_n}(t)$ is non-decreasing and $\mathscr{D}$ is continuous on its effective domain, *(2)* follows from (5.17).

Let us prove *(3)*. By Remark 3.3 and (5.10) we derive that the functions $\partial_t \mathscr{E}_{\varepsilon_n}(\cdot, q_{\varepsilon_n}(\cdot))$ form a bounded sequence in $L^\infty(0, T)$. Passing to a subsequence (not relabeled) we have $\partial_t \mathscr{E}_{\varepsilon_n}(\cdot, q_{\varepsilon_n}(\cdot)) \overset{*}{\rightharpoonup} \theta_*$ in $L^\infty(0, T)$. By taking the limit in the balance of energy $(E)_{\varepsilon_n}$ we find

$$\mathscr{E}_0(t, u(t), \sigma(t)) + \mathscr{D}(\sigma(0), \sigma(t)) = \mathscr{E}_0(0, u(0), \sigma(0)) + \int_0^t \theta_*(s)\, ds,$$

and, using the fact that $t \mapsto (u(t), \sigma(t))$ is an energetic solution associated with $\mathscr{E}_0$ and $\mathscr{D}$ and recalling (5.20), we get

$$\mathscr{E}_0(0, u(0), \sigma(0)) + \int_0^t \theta(s)\, ds = \mathscr{E}_0(0, u(0), \sigma(0)) + \int_0^t \theta_*(s)\, ds.$$

Hence $\theta = \theta_*$ a.e. in $(0, T)$ and by Lemma 3.5 of [12] we get condition *(3)*. ■

Remark 5.6 If the rate-independent problem associated with $\mathscr{E}_0$ and $\mathscr{D}$ has a unique energetic solution, the property stated in point *(5)* of Theorem 5.5 holds for the whole sequence u_{ε_n}.

6 Euler-Lagrange equations

The main purpose of this section is to rewrite the definition of energetic solution in terms of Kirchhoff-Love displacements. Moreover we deduce the

Euler-Lagrange equations associated with the limit problem. Our deduction is quite formal, in the sense that we suppose that the fields involved are regular enough to justify the computations.

Let $t \mapsto (u(t), \sigma(t))$ be an energetic solution associated with the functionals $\mathscr{E}_0$ and $\mathscr{D}$. Setting $w(t) := u(t) + \psi(t)$, we have $w(t) \in KL(\Omega_{\sigma(t)}, \mathbb{R}^3)$ and $w(t) = \psi(t)$ on $\partial_D\Omega$ for every $t \in [0, T]$.

In the sequel we fix the time t and, to simplify the notation, we omit to write the dependence on this variable. Since $w \in KL(\Omega_\sigma, \mathbb{R}^3)$ then, denoting $\omega_\sigma := \omega \setminus \gamma_\sigma$, there exist $\rho_\alpha \in H^1(\omega_\sigma)$ and $\xi \in H^2(\omega_\sigma)$ such that $w_\alpha = \rho_\alpha - x_3\xi_{,\alpha}$ and $w_3 = \xi$. Therefore

$$\tilde{E}w = \tilde{E}\rho - x_3 D^2\xi.$$

With

$$M(\xi) := \frac{h^3}{12}\mathbb{C}^0 D^2\xi, \quad N(\rho) = h\mathbb{C}^0\tilde{E}\rho,$$

$$\bar{g}_p([\rho], [D\xi]) := \int_{-h/2}^{h/2} g_p^0([\rho_\alpha - x_3\xi_{,\alpha}]\mathrm{e}_\alpha)\, dx_3,$$

the limit energy rewrites as

$$\begin{aligned}\mathscr{E}_0(t, u, \sigma) &= \frac{1}{2}\int_{\omega_\sigma} \Big(N(\rho)\cdot\tilde{E}\rho + M(\xi)\cdot D^2\xi\Big)\, d\mathscr{H}^2 + \\ &\quad + \int_{\gamma_\sigma} \Big(hg_v([\xi]) + \bar{g}_p([\rho], [D\xi])\Big)\, d\mathscr{H}^1 =: \mathscr{E}_{KL}(t, \rho, \xi, \sigma).\end{aligned}$$

In terms of ρ and ξ we can rewrite Definition 5.1 of energetic solution associated with $\mathscr{E}_{KL}$ and $\mathscr{D}$.

Definition 6.1 *An* energetic solution *associated with the functionals $\mathscr{E}_{KL}$ and $\mathscr{D}$, is a function $(\rho, \xi, \sigma) : [0, T] \to H^1(\omega_L; \mathbb{R}^2) \times H^2(\omega_L) \times [\sigma_{in}, L]$ such that the map $t \mapsto \partial_t\mathscr{E}_{KL}(t, \rho(t), \xi(t), \sigma(t)) \in L^1(0, T)$ and satisfying for every*

$t \in [0,T]$ *a stability condition* $(S)_{KL}$ *and an energy balance condition* $(E)_{KL}$*:*

$(S)_{KL}$ $\mathscr{E}_{KL}(t,\rho(t),\xi(t),\sigma(t)) < +\infty$*, and,*

$$\mathscr{E}_{KL}(t,\rho(t),\xi(t),\sigma(t)) \leq \mathscr{E}_{KL}(t,\tilde{\rho},\tilde{\xi},\tilde{\sigma}) + \mathscr{D}(\sigma(t),\tilde{\sigma})$$

$$\text{for every } (\tilde{\rho},\tilde{\xi},\tilde{\sigma}) \in H^1(\omega_L;\mathbb{R}^2) \times H^2(\omega_L) \times [\sigma_{in}, L],$$

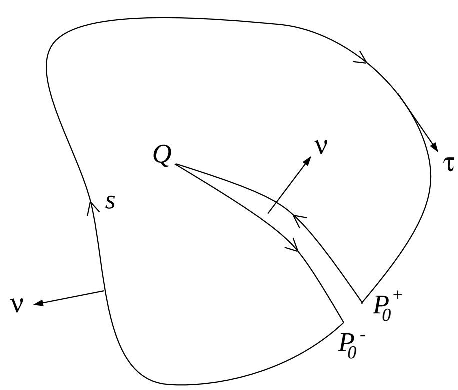
$(E)_{KL}$ $$\mathscr{E}_{KL}(t,\rho(t),\xi(t),\sigma(t)) + \mathrm{Diss}_{\mathscr{D}}(\sigma;[0,t]) = \mathscr{E}_{KL}(0,\rho(0),\xi(0),\sigma(0)) + \\ + \int_0^t \partial_t \mathscr{E}_{KL}(s,\rho(s),\xi(s),\sigma(s))\,ds.$$

Let (ρ,ξ,σ) be an energetic solution associated with $\mathscr{E}_{KL}$ and $\mathscr{D}$. Following Bourdin, Francfort and Marigo [5], from the stability condition $(S)_{KL}$ we derive the following stationarity necessary conditions

$$\int_{\omega_\sigma} \Big(N(\rho)\cdot \tilde{E}\hat{\rho} + M(\xi)\cdot D^2\hat{\xi} \Big)\, d\mathscr{H}^2 + \\ + \int_{\gamma_\sigma} \Big(h g_v'([\xi])[\hat{\xi}] + D_1\bar{g}_p([\rho],[D\xi])\cdot[\hat{\rho}] + D_2\bar{g}_p([\rho],[D\xi])\cdot[D\hat{\xi}] \Big)\, d\mathscr{H}^1 = 0$$

for all $\hat{\rho} \in H^1(\omega_\sigma;\mathbb{R}^2)$ and $\hat{\xi} \in H^2(\omega_\sigma)$, and

$$\partial_\sigma \mathscr{E}_{KL}(t,\rho,\xi,\sigma) + \kappa(\gamma(\sigma)) \geq 0.$$



The domain ω_σ and the notation used

For a region ω with smooth boundary $\partial\omega$ the conditions above yield the following equilibrium equations which hold for a.e. $t \in (0,T)$

$$
\left.\begin{array}{l}
\mathrm{div} N(\rho) = 0,\\
\mathrm{div}\,\mathrm{div} M(\xi) = 0,
\end{array}\right]\ \text{in } \omega_\sigma
$$

$$
\left.\begin{array}{l}
N(\rho)\nu = 0,\\
(M(\xi)\nu\cdot\tau)_{,s} + \mathrm{div} M(\xi)\cdot\nu = 0,\\
M(\xi)\nu\cdot\nu = 0,
\end{array}\right]\ \text{on } \partial_N\omega_\sigma
$$

$$
\left.\begin{array}{l}
[N(\rho)]\nu = 0,\\
-(N(\rho)\nu)^+ + D_1\bar{g}_p([\rho],[D\xi]) = 0,\\
[(M(\xi)\nu\cdot\tau)_{,s} + \mathrm{div} M(\xi)\cdot\nu] = 0,\\
\big((M(\xi)\nu\cdot\tau)_{,s} + \mathrm{div} M(\xi)\cdot\nu\big)^+ + hg_v'([\xi]) - (D_2\bar{g}_p([\rho],[D\xi])\cdot\tau)_{,s} = 0,\\
[M(\xi)\nu\cdot\nu] = 0,\\
\big(M(\xi)\nu\cdot\nu\big)^+ - D_2\bar{g}_p([\rho],[D\xi])\cdot\nu = 0,
\end{array}\right]\ \text{on } \gamma_\sigma
$$

$$
\begin{array}{l}
[\![M(\xi)\nu\cdot\tau]\!](P_0^+) + D_2\bar{g}_p([\rho](P_0),[D\xi](P_0))\cdot\tau = 0,\\
-[\![M(\xi)\nu\cdot\tau]\!](P_0^-) + D_2\bar{g}_p([\rho](P_0),[D\xi](P_0))\cdot\tau = 0,\\
[\![M(\xi)\nu\cdot\tau]\!](Q) = 0,\\
\partial_\sigma\mathscr{E}_0(t,\rho,\xi,\sigma) + \kappa(\gamma(\sigma)) \geq 0,\\
\dot{\sigma} \geq 0,\\
\big(\partial_\sigma\mathscr{E}_0(t,\rho,\xi,\sigma) + \kappa(\gamma(\sigma))\big)\dot{\sigma} = 0,
\end{array}
$$

where ν and τ denote the unit normal and tangent vectors to $\partial\omega_\sigma$ as represented in figure, s is the arclength, $f_{,s}$ denotes the derivative of f with respect to s and, finally, for the jump points $P = P_0^-$, P_0^+, Q of ν, $[\![M(\xi)\nu\cdot\tau]\!](P)$ denotes the jump in P along the boundary $\partial\omega_\sigma$ of the trace of $M(\xi)\nu\cdot\tau$.

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