Radial entire solutions for supercritical biharmonic equations*

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Abstract

We prove existence and uniqueness (up to rescaling) of positive radial entire solutions of supercritical semilinear biharmonic equations. The proof is performed with a shooting method which uses the value of the second derivative at the origin as a parameter. This method also enables us to find finite time blow up solutions. Finally, we study the convergence at infinity of regular solutions towards the explicitly known singular solution. It turns out that the convergence is different in space dimensions $n \leq 12$ and $n \geq 13$.

1 Introduction

In the present paper we investigate existence, uniqueness, asymptotic behavior and further qualitative properties of radial solutions of the supercritical biharmonic equation

$$\Delta^2 u = |u|^{p-1} u \qquad \text{in } \mathbb{R}^n, \tag{1}$$

where $n \geq 5$ and $p > \frac{n+4}{n-4}$. There are several deep motivations for the study of (1). Let us try to explain them in some detail.

We first recall that the corresponding supercritical second order equation (when $n \geq 3$ and $p > \frac{n+2}{n-2}$)

$$-\Delta u = |u|^{p-1} u \qquad \text{in } \mathbb{R}^n \tag{2}$$

was intensively studied by Xuefeng Wang in [16]. Most of the methods employed there are special for second order equations and do not apply to (1). For instance, qualitative properties of solutions require a detailed analysis of a dynamical system in the corresponding phase space which is two dimensional for (2), whereas it is four dimensional for (1). And in four dimensional spaces powerful tools such as the Poincaré-Bendixson theory are no longer available. One of our purposes is to find out which of the results in [16] continue to hold and by which new methods they can be proved.

We seek solutions u of (1) which only depend on $|x|$ so that they also solve the corresponding ordinary differential equation. Due to their homogeneity, both equations (1) and (2) are invariant under a suitable rescaling. This means that existence of a solution immediately implies the existence of infinitely many solutions, each one of them being characterized by its value at the origin. To ensure smoothness of the solution, one needs to require that $u'(0) = u'''(0) = 0$ for (1) and $u'(0) = 0$ for (2). But contrary to (2), solutions of (1) may be determined only by fixing a priori also the value of $u''(0)$. In Theorem 1 we show that positive radial solutions $u = u(|x|)$ of (1) exist and are unique,

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up to rescaling. The proof is performed with a shooting method which uses as a free parameter the "shooting concavity", namely the initial second derivative $u''(0)$. Clearly, (2) has no free parameter since one has just to fix the rescaling parameter $u(0)$.

Theorem 2 highlights further differences between (1) and (2). Firstly, it states that the shooting concavity $u''(0)$ enables to find both positive and negative finite time blow up solutions for (1). Since no free parameter is available, no such solutions exist for (2). Secondly, Theorem 2 shows that no Lyapunov functional may exist for the ordinary differential equation corresponding to (1). On the contrary, a quite helpful Lyapunov functional does exist for (2).

In Section 6 we transform (1) into an autonomous ordinary differential equation and, by exploiting the supercriticality assumption, we construct an energy functional which (of course!) is not necessarily globally monotone but which has the crucial feature of being strictly decreasing on critical points of the solution. This fact, combined with several fine estimates, enables us to prove Theorem 3, namely that positive radial entire solutions of (1) behave asymptotically as $|x| \to \infty$ like the (positive) singular solution $u_s(x) := C|x|^{-4/(p-1)}$ which solves (1) in $\mathbb{R}^n \setminus \{0\}$ for a suitable value of $C > 0$. In other words, we show that any entire positive radial solution $u = u(|x|)$ of (1) satisfies

$$\lim_{|x|\to\infty} |x|^{4/(p-1)} u(|x|) = C \tag{3}$$

for some fixed $C > 0$. Although this result is similar to that obtained in [16] for (2), its proof is completely different.

The following step is to find out whether the convergence in (3) occurs monotonically or with oscillations. To this end, we perform a stability analysis for the singular solution u_s. It turns out that for dimensions $n \geq 13$ a new critical exponent $p_c > \frac{n+4}{n-4}$ arises. The stable manifold behaves differently for $n \leq 12$ and $p > \frac{n+4}{n-4}$ or $n \geq 13$ and $\frac{n+4}{n-4} < p < p_c$ on the one hand, and for $n \geq 13$ and $p \geq p_c$ on the other hand. In Section 4 we show that strong hints give the feeling that oscillations occur in the former situation. On the other hand, in Theorem 4 we prove that monotone convergence occurs in (3) whenever $n \geq 13$ and $p > p_c$. In what follows, the notion "subcritical" and "supercritical" always refers to this new critical exponent p_c. Our results still leave open some questions, which we describe in detail in Open Problems 1–3.

Finally, let us mention that our results may also shed some light on related problems in the unit ball $B \subset \mathbb{R}^n$. For both the cases $L = -\Delta$ and $L = \Delta^2$, consider the equation

$$Lu = \lambda(1+u)^p \qquad \text{in } B \tag{4}$$

where $\lambda \geq 0$. We complement (4) with homogeneous Dirichlet boundary conditions ($u = 0$ if $L = -\Delta$ and $u = |\nabla u| = 0$ if $L = \Delta^2$). When $L = -\Delta$, it is known [10, Théorème 4] that the extremal solution u^* (corresponding to the largest value of λ for which (4) admits a positive solution) is bounded for all n and p which give rise to regular solutions of (1) oscillating around the singular solution, see [16, Proposition 3.7]. For the remaining values of n and p (when no oscillation occurs in (1)), it is known [3] that u^* is unbounded. When $L = \Delta^2$, similar results are not known due to several serious obstructions which arise. For instance, the singular solution of (4) cannot be explicitly determined, see [1, 2]. Moreover, the link with remainder terms in Hardy inequality discovered in [3] seems to fail for higher order problems [5]. Nevertheless, the results of the present paper enable us to conjecture that, when $L = \Delta^2$, extremal solutions of (4) are unbounded if and only if $n \geq 13$ and $p \geq p_c$.

This paper is organized as follows. In next section, we state our main results. In Section 3 we transform equation (1) first into an autonomous equation and subsequently into an autonomous system. In Section 4 we study the autonomous system in the "subcritical" case $(n+4)/(n-4) < p < p_c$. Finally, Sections 5, 6, 7 and 8 are devoted to the proofs of the results.

2 Results

An existence result, which covers the equation (1), was given first by Serrin and Zou [14]. In Section 5, we give a different proof which is perhaps simpler and more suitable for our purposes. Moreover, we show uniqueness and complement these results with some information on the qualitative behavior of the solution.

Theorem 1. *Let $n \geq 5$ and assume that $p > \frac{n+4}{n-4}$. Then, for any $a > 0$ the equation*

$$\Delta^2 u = u^p \qquad in\ \mathbb{R}^n \tag{5}$$

admits a unique radial positive solution $u = u(r)$ ($r = |x|$) such that $u(0) = a$ and $u(r) \to 0$ as $r \to \infty$. Moreover, u satisfies:
(i) $u'(r) < 0$ for all $r > 0$.
(ii) $\Delta u(r) < 0$ for all $r > 0$.
(iii) $(\Delta u)'(r) > 0$ for all $r > 0$.

The solutions in Theorem 1 are constructed by means of a shooting method. We keep $u(0)$ fixed, say $u(0) = 1$, and look for solutions u_γ of the initial value problem over $[0, \infty)$:

$$\begin{gathered} u_\gamma^{iv}(r) + \frac{2(n-1)}{r} u_\gamma'''(r) + \frac{(n-1)(n-3)}{r^2} u_\gamma''(r) - \frac{(n-1)(n-3)}{r^3} u_\gamma'(r) = |u_\gamma(r)|^{p-1} u_\gamma(r) \\ u_\gamma(0) = 1 \ , \qquad u_\gamma'(0) = u_\gamma'''(0) = 0 \ , \qquad u_\gamma''(0) = \gamma < 0 \ , \end{gathered} \tag{6}$$

which is the radial version of equation (1). Then, one has the following behavior with respect to the parameter γ:

Theorem 2. *There exists a unique $\overline{\gamma} < 0$ such that the solution $\overline{u}$ of (6) (for $\gamma = \overline{\gamma}$) exists on the whole interval $[0, \infty)$, is positive everywhere and vanishes at infinity.*
If $\gamma < \overline{\gamma}$, there exist $0 < R_1 < R_2 < \infty$ such that $u_\gamma(R_1) = 0$ and $\lim_{r \to R_2} u_\gamma(r) = -\infty$.
If $\gamma > \overline{\gamma}$, there exist $0 < R_1 < R_2 < \infty$ such that $u_\gamma'(r) < 0$ for $r \in (0, R_1)$, $u_\gamma'(R_1) = 0$, $u_\gamma'(r) > 0$ for $r \in (R_1, R_2)$ and $\lim_{r \to R_2} u_\gamma(r) = +\infty$.

Theorem 2 shows that entire radial solutions of (1) are necessarily of one sign so that, in what follows, we restrict our attention to positive solutions. It is a simple observation that a positive singular solution u_s of (5) is given by

$$u_s(r) = K_0^{1/(p-1)} r^{-4/(p-1)}, \tag{7}$$

where

$$K_0 = \frac{8}{(p-1)^4} \left[(n-2)(n-4)(p-1)^3 + 2(n^2 - 10n + 20)(p-1)^2 - 16(n-4)(p-1) + 32 \right] .$$

In contrast with the second order equation (2) discussed in [16], a priori the entire solutions of (5) found in Theorem 1 may have faster decay than the singular solution, see the discussion in Section 6. However, by transforming equation (6) into an autonomous 4×4 system and by means of a careful analysis of a suitable energy functional and of corresponding integrability properties, we succeed in proving the following result:

Theorem 3. *Let $n \geq 5$ and assume that $p > \frac{n+4}{n-4}$. Let $u = u(r)$ be a positive regular radial entire solution of (5) and let u_s be as in (7). Then,*

$$u(r) < \left(\frac{p+1}{2} \right)^{1/(p-1)} u_s(r) \quad for\ all\ r \geq 0 \tag{8}$$

and

$$\lim_{r\to\infty} \frac{u(r)}{u_s(r)} = 1. \tag{9}$$

We now wish to describe in which way (9) occurs. To this end, in Section 7 we perform a stability analysis of the singular solution u_s. It turns out that for dimensions $n \geq 13$ a new critical exponent $p_c > \frac{n+4}{n-4}$ becomes important:

Theorem 4. *For all $n \geq 13$ there exists $p_c > \frac{n+4}{n-4}$ such that if $p > p_c$ and if u is a regular positive radial entire solution of* (5)*, then $u(r) - K_0^{1/(p-1)} r^{-4/(p-1)}$ does not change sign infinitely many times.*

The number p_c is the unique value of $p > \frac{n+4}{n-4}$ such that

$$\begin{gathered} -(n-4)(n^3-4n^2-128n+256)(p-1)^4 + 128(3n-8)(n-6)(p-1)^3 \\ +256(n^2-18n+52)(p-1)^2 - 2048(n-6)(p-1) + 4096 = 0. \end{gathered}$$

In Proposition 1 we show that $n \mapsto p_c$ is decreasing for $n \geq 13$ and tends to 1 as $n \to \infty$.

Theorem 4 is a partial result concerning the "supercritical" case $p > p_c$, $n \geq 13$. Section 4 is devoted to the discussion of the "subcritical" case.

3 An autonomous system

In radial coordinates $r = |x|$, equation (5) reads

$$u^{iv}(r) + \frac{2(n-1)}{r}u'''(r) + \frac{(n-1)(n-3)}{r^2}u''(r) - \frac{(n-1)(n-3)}{r^3}u'(r) = u^p(r) \qquad r \in [0,\infty)\ . \tag{10}$$

Our purpose here is to transform (10) first into an autonomous equation and, subsequently, into an autonomous system. For some of the estimates which follow, it is convenient to rewrite the original assumption $p > \frac{n+4}{n-4}$ as

$$(n-4)(p-1) > 8\ . \tag{11}$$

Inspired by the proof of [16, Proposition 3.7] (see also [6, 8]) we set

$$u(r) \;=\; r^{-4/(p-1)}\, v(\log r) \quad (r>0)\ , \qquad v(t) \;=\; e^{4t/(p-1)}\, u(e^t) \quad (t \in \mathbb{R})\ . \tag{12}$$

Tedious calculations then show that

$$\frac{u'(r)}{r^3} \;=\; r^{-4p/(p-1)} \left[v'(t) - \frac{4}{p-1}v(t)\right]\ , \tag{13}$$

$$\frac{u''(r)}{r^2} \;=\; r^{-4p/(p-1)} \left[v''(t) - \frac{p+7}{p-1}v'(t) + \frac{4(p+3)}{(p-1)^2}v(t)\right]\ ,$$

$$\frac{u'''(r)}{r} \;=\; r^{-4p/(p-1)} \left[v'''(t) - \frac{3(p+3)}{p-1}v''(t) + \frac{2(p^2+10p+13)}{(p-1)^2}v'(t) - \frac{8(p+1)(p+3)}{(p-1)^3}v(t)\right]\ ,$$

$$\begin{aligned} u^{iv}(r) \;=\; r^{-4p/(p-1)} \Bigg[& v^{iv}(t) - \frac{2(3p+5)}{p-1}v'''(t) + \frac{11p^2+50p+35}{(p-1)^2}v''(t) \\ & - \frac{2(3p^3+35p^2+65p+25)}{(p-1)^3}v'(t) + \frac{8(p+1)(p+3)(3p+1)}{(p-1)^4}v(t) \Bigg]\ . \end{aligned}$$

Therefore, after the change (12), equation (10) may be rewritten as

$$v^{iv}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) + K_0 v(t) = v^p(t) \qquad t \in \mathbb{R} \;, \tag{14}$$

where the constants $K_i = K_i(n,p)$ $(i = 0, ..., 3)$ are given by

$$K_0 = \frac{8}{(p-1)^4}\left[(n-2)(n-4)(p-1)^3 + 2(n^2-10n+20)(p-1)^2 - 16(n-4)(p-1) + 32\right] ,$$

$$K_1 = -\frac{2}{(p-1)^3}\left[(n-2)(n-4)(p-1)^3 + 4(n^2-10n+20)(p-1)^2 - 48(n-4)(p-1) + 128\right] ,$$

$$K_2 = \frac{1}{(p-1)^2}\left[(n^2-10n+20)(p-1)^2 - 24(n-4)(p-1) + 96\right] ,$$

$$K_3 = \frac{2}{p-1}\left[(n-4)(p-1) - 8\right] .$$

By using (11), it is not difficult to show that $K_1 = K_3 = 0$ if $p = \frac{n+4}{n-4}$ and that

$$K_0 > 0 \;, \qquad K_1 < 0 \;, \qquad K_3 > 0 \qquad \forall n \geq 5 \;, \quad p > \frac{n+4}{n-4} \;. \tag{15}$$

On the other hand, the sign of K_2 depends on n and p. We emphasize that the sign of K_1 and K_3 is due to assumption (11) and will be essentially exploited in the proof of Theorem 3, see also the proof of Lemma 6.

Note that (14) admits the two constant solutions $v_0 \equiv 0$ and $v_s \equiv K_0^{1/(p-1)}$ which, by (12), correspond to the following solutions of (10):

$$u_0(r) \equiv 0 \;, \qquad u_s(r) = \frac{K_0^{1/(p-1)}}{r^{4/(p-1)}} \;.$$

We now write (14) as a system in $\mathbb{R}^4$. By (13) we have

$$u'(r) = 0 \quad \Longleftrightarrow \quad v'(t) = \frac{4}{p-1} v(t) \;.$$

This fact suggests us to define

$$w_1(t) = v(t) \,, \quad w_2(t) = v'(t) - \frac{4}{p-1} v(t) \,, \quad w_3(t) = v''(t) - \frac{4}{p-1} v'(t) \,, \quad w_4(t) = v'''(t) - \frac{4}{p-1} v''(t)$$

so that (14) becomes

$$\begin{cases} w_1'(t) &= \frac{4}{p-1} w_1(t) + w_2(t) \\ w_2'(t) &= w_3(t) \\ w_3'(t) &= w_4(t) \\ w_4'(t) &= C_2 w_2(t) + C_3 w_3(t) + C_4 w_4(t) + w_1^p(t) \;, \end{cases} \tag{16}$$

where

$$C_m = -\sum_{k=m-1}^{4} \frac{K_k 4^{k+1-m}}{(p-1)^{k+1-m}} \qquad \text{for } m = 1,2,3,4 \qquad \text{with } K_4 = 1 \;. \tag{17}$$

This gives first that $C_1=0$ so that the term $C_1 w_1(t)$ does not appear in the last equation of (16). Moreover, we have the explicit formulae:

$$
\begin{aligned}
C_2 &= \tfrac{2}{(p-1)^3}\left[(n-2)(n-4)(p-1)^3+2(n^2-10n+20)(p-1)^2-16(n-4)(p-1)+32\right]=\tfrac{p-1}{4}K_0,\\
C_3 &= -\tfrac{1}{(p-1)^2}\left[(n^2-10n+20)(p-1)^2-16(n-4)(p-1)+48\right],\\
C_4 &= -\tfrac{2}{p-1}\left[(n-4)(p-1)-6\right].
\end{aligned}
$$

System (16) has the two stationary points (corresponding to v_0 and v_s)

$$O\Big(0,0,0,0\Big) \qquad \text{and} \qquad P\Big(K_0^{1/(p-1)},-\frac{4}{p-1}K_0^{1/(p-1)},0,0\Big)\ .$$

Let us consider first the "regular point" O. The linearized matrix at O is

$$M_O \ = \begin{pmatrix} \frac{4}{p-1} & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & C_2 & C_3 & C_4 \end{pmatrix}$$

and the characteristic polynomial is

$$\lambda\mapsto\lambda^4+K_3\lambda^3+K_2\lambda^2+K_1\lambda+K_0\ .$$

Then, according to MAPLE, the eigenvalues are given by

$$\lambda_1=2\frac{p+1}{p-1},\quad \lambda_2=\frac{4}{p-1},\quad \lambda_3=\frac{4p}{p-1}-n,\quad \lambda_4=2\frac{p+1}{p-1}-n.$$

Since we assume that $p>\frac{n+4}{n-4}>\frac{n}{n-4}>\frac{n+2}{n-2}$, we have

$$\lambda_1>\lambda_2>0>\lambda_3>\lambda_4.$$

This means that O is a hyperbolic point and that both the stable and the unstable manifolds are two-dimensional. This is the same situation as in the exponential case (see [1]) and except for λ_2 it seems as if one could perform a formal limit $p\to\infty$.

Around the "singular point" P the linearized matrix of the system (16) is given by

$$M_P \ = \begin{pmatrix} \frac{4}{p-1} & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ pK_0 & C_2 & C_3 & C_4 \end{pmatrix}. \tag{18}$$

The corresponding characteristic polynomial is

$$\nu\mapsto\nu^4+K_3\,\nu^3+K_2\,\nu^2+K_1\nu+(1-p)K_0$$

and the eigenvalues are given by

$$
\begin{aligned}
\nu_1&=\frac{N_1+\sqrt{N_2+4\sqrt{N_3}}}{2(p-1)}, & \nu_2&=\frac{N_1-\sqrt{N_2+4\sqrt{N_3}}}{2(p-1)},\\
\nu_3&=\frac{N_1+\sqrt{N_2-4\sqrt{N_3}}}{2(p-1)}, & \nu_4&=\frac{N_1-\sqrt{N_2-4\sqrt{N_3}}}{2(p-1)},
\end{aligned}
$$

where

$$N_1 := -(n-4)(p-1)+8, \qquad N_2 := (n^2-4n+8)(p-1)^2,$$

$$\begin{aligned} N_3 \ &:= \ (9n-34)(n-2)\,(p-1)^4 + 8(3n-8)(n-6)\,(p-1)^3 \\ &\quad +(16n^2-288n+832)\,(p-1)^2 - 128(n-6)(p-1) + 256. \end{aligned}$$

The stability of the stationary point P is described by the following

Proposition 1. *Assume that* $p > \frac{n+4}{n-4}$.
(i) For any $n \geq 5$ *we have* $\nu_1, \nu_2 \in \mathbb{R}$ *and* $\nu_2 < 0 < \nu_1$.
(ii) For any $5 \leq n \leq 12$ *we have* $\nu_3, \nu_4 \notin \mathbb{R}$ *and* $\mathrm{Re}\,\nu_3 = \mathrm{Re}\,\nu_4 < 0$.
(iii) For any $n \geq 13$ *there exists* $p_c > \frac{n+4}{n-4}$ *such that:*
– *if* $p < p_c$, *then* $\nu_3, \nu_4 \notin \mathbb{R}$ *and* $\mathrm{Re}\,\nu_3 = \mathrm{Re}\,\nu_4 < 0$.
– *if* $p = p_c$, *then* $\nu_3, \nu_4 \in \mathbb{R}$ *and* $\nu_4 = \nu_3 < 0$.
– *if* $p > p_c$, *then* $\nu_3, \nu_4 \in \mathbb{R}$ *and* $\nu_4 < \nu_3 < 0$. *The number* p_c *is the unique value of* $p > \frac{n+4}{n-4}$ *such that*

$$\begin{aligned} &-(n-4)(n^3-4n^2-128n+256)(p-1)^4 + 128(3n-8)(n-6)(p-1)^3 \\ &\quad +256(n^2-18n+52)(p-1)^2 - 2048(n-6)(p-1) + 4096 = 0. \end{aligned}$$

The function $n \mapsto p_c$ *is strictly decreasing and approaches* 1 *as* $n \to \infty$.

Proof. See Section 8. □

According to Proposition 1, in any case we have

$$\nu_1 > 0, \qquad \nu_2 < 0, \qquad \mathrm{Re}\,\nu_{3/4} < 0.$$

This means that P has a three dimensional stable manifold and a one dimensional unstable manifold (as in the exponential case).

Remark 1. Consider the function

$$\phi(x) := x^4 + K_3x^3 + K_2x^2 + K_1x\ . \tag{19}$$

We have $\phi(0) = 0$ and $\phi'(0) = K_1 < 0$ for every n and p. Moreover, by the previous analysis around the point O, we know that the equation $\phi(x) = -K_0$ always has 4 real solutions, 2 positive and 2 negative. By these facts we deduce that the graphic of ϕ has the shape of W with two local minima (one positive, one negative) at level below $-K_0$ and the unique local maximum (negative) at strictly positive level. In particular, for any $-K_0 \leq \gamma \leq 0$, the equation $\phi(x) = \gamma$ has 4 real solutions. Finally, note that the level of the local maximum of ϕ coincides with $(p-1)K_0$ if and only if $p = p_c$.

4 Observations on the stable manifold of P and open problems

Let u denote a regular positive entire radial solution of (5), let v be defined according to (12) so that it solves (14), and let $\mathbf{w}(t) = \big(w_1(t), w_2(t), w_3(t), w_4(t)\big)$ be the vector solution of the corresponding first order system (16).

We first state a general result which holds for any entire regular solution:

Proposition 2. *We assume that* u *is an entire regular positive radial solution of* (5) *and that* $\mathbf{w} = (w_1, w_2, w_3, w_4)$ *is the corresponding solution of system* (16). *Then,*

$$\lim_{t\to\infty} \mathbf{w}(t) = P.$$

In particular, the trajectory $\mathbf{w}$ *is on the stable manifold of* P.

Proof. See Section 8. □

By Proposition 2 we know that $\mathbf{w}$ is on the stable manifold of the singular point P while Theorem 4 gives information on the nonoscillatory behavior of u around the singular solution u_s in the "supercritical" case. In this section, we refer to the new critical exponent p_c arising in Proposition 1. Here, we are interested in the (presumably) oscillatory behavior in the "subcritical" case, i.e. in what follows we assume:

$$n \le 12 \qquad \text{or} \qquad \Big(n \ge 13 \text{ and } \frac{n+4}{n-4} < p < p_c\Big). \tag{20}$$

We study the relative position of the hyperplane

$$H := \{\mathbf{w} \in \mathbb{R}^4 : \ w_1 = K_0^{1/(p-1)}\}$$

with respect to the tangential plane of the oscillatory component of the stable manifold

$$OS := \{s\mathbf{x} + t\mathbf{y} : \ s, t \in \mathbb{R}\}.$$

Here $\mathbf{x} \pm i\mathbf{y}$ denotes eigenvectors of the matrix M_P defined in (18) corresponding to the nonreal eigenvalues ν_3, ν_4.

Proposition 3. *The hyperplane H and the plane $P + OS$ intersect transversally, i.e.*

$$P + OS \not\subset H.$$

Proof. See Section 8. □

Open Problem 1. Since we have that $\nu_2 < \operatorname{Re}\nu_{3/4} < 0$ we know that all trajectories of system (16) which are in the stable manifold of P are eventually tangential to OS, except the trajectory being tangential to the eigenvector corresponding to ν_2. By Proposition 3 we may conclude that all these trajectories have infinitely many intersections with the hyperplane H. If the trajectory $\mathbf{w}$ corresponding to the solution u is among these, then we would have shown:

> *For $t \to \infty$, the first component $v(t) = w_1(t)$ attains infinitely many times the value $K_0^{1/(p-1)}$ so that for r near ∞, $u(r)$ oscillates infinitely often around the singular solution u_s, provided that the subcriticality assumption* (20) *is satisfied.*

In order to complete the proof of this statement, it "only" remains to show that at ∞, $t \mapsto \mathbf{w}(t)$ is *not* tangential to an eigenvector corresponding to ν_2. For this it would suffice to identify the trajectories having this property and to see that they are different from $\mathbf{w}$.

Open Problem 2. Our proof of Theorem 4 relies on a result by Elias [4] which no longer applies when $p = p_c$. Nevertheless, we believe that the statement of Theorem 4 also holds true in this limit situation. If one could show that for all $p > p_c$ the solutions u of (10) are approaching the singular solution u_s *from below*, then the same result would presumably also follow for $p = p_c$ by continuous dependence.

Open Problem 3. With the same proof of Theorem 4, one can also show that if u_α and u_β are positive radial entire solutions of (5) with shooting levels $u_\alpha(0) = \alpha$ and $u_\beta(0) = \beta$, then (under the assumptions of Theorem 4) $u_\alpha - u_\beta$ in nonoscillatory, i.e. it has at most a finite number of zeroes. A natural question arises whether all these solutions (including the singular one) are *completely ordered*, i.e. they have no crossing at all, and not only eventually.

5 Proof of Theorems 1 and 2

If $u = u(r)$ is a radial positive solution of (5) such that $u(0) = 1$ and $u(r) \to 0$ as $r \to \infty$, then

$$u_a(r) := a\, u(a^{\frac{p-1}{4}}\, r) \qquad (a > 0)$$

is a radial positive solution of (5) such that $u_a(0) = a$ and $u_a(r) \to 0$ as $r \to \infty$. Therefore, Theorem 1 follows if we prove existence and uniqueness of a solution u satisfying $u(0) = 1$.

Existence. In order to prove existence, we apply a shooting method with initial second derivative as parameter. We remark that $nu''(0) = \Delta u(0)$ and that by l'Hospital's rule

$$(\Delta u)'(0) = u'''(0) + (n-1) \lim_{r\to 0} \frac{ru''(r) - u'(r)}{r^2} = u'''(0) + \frac{n-1}{2} u'''(0) = \frac{n+1}{2} u'''(0).$$

This means that the initial conditions in (6) also read as

$$u(0) = 1 \ , \qquad u'(0) = (\Delta u)'(0) = 0 \ , \qquad \Delta u(0) = n\gamma < 0 \ . \tag{21}$$

For all $\gamma < 0$, (10)-(21) admits a unique local smooth solution u_γ defined on some right neighborhood of $r = 0$. Let

$$R_\gamma = \begin{cases} +\infty & \text{if } u(r)u'(r) < 0 \quad \forall r > 0 \\ \min\{r > 0;\ u(r)u'(r) = 0\} & \text{otherwise.} \end{cases}$$

From now on we understand that u_γ is continued on $[0, R_\gamma)$. Let

$$I^+ = \{\gamma < 0;\ R_\gamma < \infty\, ,\ u_\gamma(R_\gamma) = 0\} \ , \qquad I^- = \{\gamma < 0;\ R_\gamma < \infty\, ,\ u'_\gamma(R_\gamma) = 0\} \ .$$

We prove the following statement:

Lemma 1. *Assume $p > \frac{n+4}{n-4}$. If both I^+ and I^- are nonempty then there exists $\gamma < 0$ such that $R_\gamma = \infty$. Moreover, u_γ is defined on $[0, \infty)$ and $\lim_{r\to\infty} u_\gamma(r) = 0$.*

Proof. Since $p \geq \frac{n+4}{n-4}$, Pohozaev identity (see e.g. [12, Corollary 1]) tells us that

$$I^+ \cap I^- = \emptyset \ . \tag{22}$$

Moreover, by continuous dependence with respect to the variable initial datum γ, we have that

$$I^+ \text{ and } I^- \quad \text{are open in } (-\infty, 0) \ . \tag{23}$$

Combining (22)-(23) with the assumption, we infer that there exists $\gamma \notin I^+ \cup I^-$. Then, $R_\gamma = +\infty$ and $\lim_{r\to\infty} u_\gamma(r)$ exists (recall $u'_\gamma < 0$). Finally, this limit is necessarily 0, since u_γ solves (10). □

Remark 2. A well-known crucial difference arises when $1 < p < \frac{n+4}{n-4}$. In such case, by standard critical point theory and rescaling one has that $I^+ \cap I^- \neq \emptyset$.

Consider now the Euler equation for Sobolev minimizers (see e.g. [15]):

$$\begin{cases} v^{iv}(r) + \dfrac{2(n-1)}{r} v'''(r) + \dfrac{(n-1)(n-3)}{r^2} v''(r) - \dfrac{(n-1)(n-3)}{r^3} v'(r) = v^{\frac{n+4}{n-4}}(r), \quad r \geq 0 \\ v(0) = 1 \ , \qquad v'(0) = (\Delta v)'(0) = 0 \ , \qquad \Delta v(0) = n\delta \ , \end{cases} \tag{24}$$

where $\delta < 0$ is chosen in such a way that the unique solution of (24) is given by

$$v(r) = \frac{\big[n(n^2-4)(n-4)\big]^{\frac{n-4}{4}}}{(\sqrt{n(n^2-4)(n-4)}+r^2)^{\frac{n-4}{2}}}\ . \tag{25}$$

This explicit solution will serve as a comparison function for the initial value problem (10)-(21). For this purpose we quote a comparison principle, which has been observed by McKenna-Reichel [9] and which will turn out to be useful also in the proof of uniqueness below:

Lemma 2. *Assume that $f:\mathbb{R}\to\mathbb{R}$ is continuous and strictly increasing. Let $u,v\in C^4([0,R))$ be such that*

$$\begin{cases} \forall r\in[0,R): \qquad \Delta^2 v(r) - f(v(r)) \geq \Delta^2 u(r) - f(u(r)), \\ v(0)\geq u(0), \quad v'(0)=u'(0)=0, \quad \Delta v(0)\geq \Delta u(0), \quad (\Delta v)'(0)=(\Delta u)'(0)=0. \end{cases} \tag{26}$$

Then we have

$$\forall r\in[0,R): \qquad v(r)\geq u(r), \quad v'(r)\geq u'(r), \quad \Delta v(r)\geq \Delta u(r), \quad (\Delta v)'(r)\geq(\Delta u)'(r). \tag{27}$$

Moreover,

(i) the initial point 0 can be replaced by any initial point $\rho>0$ if all four initial data are weakly ordered.

(ii) a strict inequality in one of the initial data at $\rho\geq 0$ or in the differential inequality on (ρ,R) implies a strict ordering of $v,v',\Delta v,\Delta v'$ and $u,u',\Delta u,\Delta u'$ on (ρ,R).

With the aid of this lemma we obtain

Lemma 3. *Let $\delta<0$ be as in (24) and let v be as in (25). Let $\gamma<\delta$ and let u_γ be the local solution of (10)-(21). Then, one of the two following facts holds true:*

(i) $\gamma\in I^+$.

(ii) $0<u_\gamma(r)<v(r)$ for all $r>0$.

Proof. Since $0<v\leq 1$ we have

$$\Delta^2 v - v^p \geq \Delta^2 v - v^{(n+4)/(n-4)} = 0 = \Delta^2 u - |u|^{p-1}u,$$

as long as u exists. Hence, $v(r)>u(r)$ and $0>v'(r)>u'(r)$ for these $r>0$. Assume that $\gamma\notin I^+$. Then it is immediate from Lemma 2 that alternative (ii) holds true. □

If the alternative (ii) in Lemma 3 holds true, then the corresponding solution u_γ satisfies the requirements of Theorem 1 and existence follows.

If the alternative (i) in Lemma 3 holds true, in view of Lemma 1, existence is proved once we show that

$$I^-\neq\emptyset\ . \tag{28}$$

To this end, we consider the following Dirichlet problem

$$\begin{cases} \Delta^2 w = \lambda(1+w)^p & \text{in } B \\ w>0 & \text{in } B \\ w=\dfrac{\partial w}{\partial n}=0 & \text{on } \partial B \end{cases} \tag{29}$$

where $\lambda > 0$ and $B \subset \mathbb{R}^n$ is the unit ball. Arguing as in the proof of [2, Theorem 2.3] (see also [13]) and taking into account both Lemma 6 and Theorem 1 in [1] we infer that there exists $\overline{\lambda} > 0$ such that for all $\lambda \in (0, \overline{\lambda}]$ problem (29) admits a radial regular solution $w_\lambda = w_\lambda(r)$. So, fix one such λ and put $w(r) = \lambda^{\frac{1}{p-1}}(1 + w_\lambda(r))$. Then, w satisfies

$$\begin{cases} \Delta^2 w = w^p & \text{in } B \\ w > \lambda^{\frac{1}{p-1}} & \text{in } B \\ w = \lambda^{\frac{1}{p-1}} & \text{on } \partial B \\ \frac{\partial w}{\partial n} = 0 & \text{on } \partial B \ . \end{cases}$$

Finally, the function $u_\gamma(r) := \alpha^{\frac{4}{p-1}} w(\alpha r)$ with $\alpha = w^{\frac{1-p}{4}}(0)$ satisfies $u_\gamma(0) = 1$ and

$$\begin{cases} \Delta^2 u_\gamma = u_\gamma^p & \text{in } B_{1/\alpha} \\ u_\gamma > \alpha^{\frac{4}{p-1}} \lambda^{\frac{1}{p-1}} & \text{in } B_{1/\alpha} \\ u_\gamma = \alpha^{\frac{4}{p-1}} \lambda^{\frac{1}{p-1}} & \text{on } \partial B_{1/\alpha} \\ \frac{\partial u_\gamma}{\partial n} = 0 & \text{on } \partial B_{1/\alpha} \ . \end{cases}$$

Take $\gamma = u_\gamma''(0) < 0$. Then, $R_\gamma = \alpha^{-1}$ and $u_\gamma'(R_\gamma) = 0$. This proves that $\gamma \in I^-$ and, in turn, that (28) holds true. And this proves the existence of a positive radial solution of (5) satisfying $u(0) = 1$ and $u(x) \to 0$ as $|x| \to \infty$.

Qualitative behavior. Let $u = u(r)$ be a radial solution of (5) such that $u(0) = 1$ and $u(r) \to 0$ as $r \to \infty$. Statement (iii) follows by integrating

$$\left\{ r^{n-1} \left[\Delta u(r) \right]' \right\}' = r^{n-1} u^p(r) \tag{30}$$

over $[0, r]$ for $r > 0$.

In order to prove (i), we assume for contradiction that there exists $R_1 > 0$, the first solution of $u'(R_1) = 0$. Then, $\Delta u(R_1) = u''(R_1) \geq 0$. By using the just proved statement (iii) for $r > R_1$ we deduce that $\Delta u(r) > 0$ for all $r > R_1$ and that $u'(r) > 0$ for all $r > R_1$, against the assumption of $u(r)$ vanishing at $+\infty$. This contradiction proves (i).

Next we shall prove (ii). For contradiction, assume now that there exists $R_1 > 0$, the first solution of $\Delta u(R_1) = 0$. Then, by (iii), we know that there exist $R_2 > R_1$ and $\varepsilon > 0$ such that $\Delta u(r) \geq \varepsilon$ for all $r \geq R_2$. Multiplying by r^{n-1} this inequality yields

$$[r^{n-1} u'(r)]' \geq \varepsilon r^{n-1} \qquad \text{for all } r \geq R_2 \ .$$

Integrating this last inequality over $[R_2, r]$ for any $r > R_2$ and dividing by r^{n-1} gives

$$u'(r) \geq \frac{\varepsilon}{n} r + \frac{R_2^{n-1} u'(R_2)}{r^{n-1}} - \frac{\varepsilon R_2^n}{n r^{n-1}} \qquad \text{for all } r \geq R_2 \ .$$

Letting $r \to \infty$ we then obtain $u'(r) \to +\infty$, contradiction. Hence, also (ii) is proved.

Uniqueness. By means of the comparison principle Lemma 2, it is immediate that the family in Lemma 3 is ordered:

Lemma 4. *Let $\gamma_1 < \gamma_2$ and let u_{γ_j} be the corresponding local solution of* (10)-(21). *As long as both solutions exist, we have for $r > 0$ that*

$$u_{\gamma_1}(r) < u_{\gamma_2}(r). \tag{31}$$

Since we already proved existence, the following statement makes sense:

Lemma 5. *Let $\overline{u}$ denote a positive entire radially decreasing solution of* (5) *such that* $\overline{u}(0) = 1$ *and* $\overline{u}(r) \to 0$ *as* $r \to \infty$ *and let* $\overline{\gamma} = \overline{u}''(0)$. *For any* $\gamma < \overline{\gamma}$ *let* u_γ *be the local solution of* (10)-(21). *Then, for* $r > 0$, *as long as* u_γ *exists:*

$$u_\gamma'(r) < \overline{u}'(r). \tag{32}$$

Again, the proof follows directly from Lemma 2. In particular, Lemma 5 tells us that for any $\gamma < \overline{\gamma}$, $u_\gamma(r)$ vanishes in finite time. This proves uniqueness and completes the proof of Theorem 1. □

Proof of Theorem 2. The existence of precisely one such $\overline{\gamma}$ follows from the proof of Theorem 1.

The statement in the case $\gamma > \overline{\gamma}$ follows by arguing similarly as in Theorem 4.2 in [2]. More precisely, by Lemma 4, for $r > 0$ we have $0 < \overline{u}(r) < u_\gamma(r)$ as long as the latter exists. If there exists no $R_1 > 0$ such that $u_\gamma'(R_1) = 0$, then $u_\gamma'(r) < 0$ for all $r > 0$ so that u_γ would be a positive global solution of (6) such that $u_\gamma(r) \to 0$ as $r \to \infty$, against the uniqueness stated in Theorem 1. So, let $R_1 > 0$ be the first solution of $u_\gamma'(R_1) = 0$. Then, $\Delta u_\gamma(R_1) \geq 0$. By integrating (30) over $[0, r]$ for $r > R_1$ we deduce that $\Delta u_\gamma(r) > 0$ for all $r > R_1$ and that $u_\gamma'(r) > 0$ for all $r > R_1$. Then, $u_\gamma(r) \to +\infty$ at some (finite or infinite) $R_2 > R_1$.

In order to show that $R_2 < \infty$ we essentially refer to a reasoning, which was performed for the critical case in [7, Lemma 2]. Let $\tilde{u}(r) := u_\gamma(r) - 1$, so that it solves $\Delta^2 \tilde{u} = (1 + \tilde{u})^p$. Since $\tilde{u}(r) \nearrow \infty$ for $r \nearrow R_2$, successive integration of the differential equation shows that for some suitable $r_0 < R_2$, r_0 close enough to R_2, one has:

$$\tilde{u}(r_0) > 0, \qquad \tilde{u}'(r_0) > 0, \qquad \Delta\tilde{u}(r_0) > 0, \qquad (\Delta\tilde{u})'(r_0) > 0.$$

For any value of the rescaling parameter $\alpha > 0$,

$$u_{0,\alpha}(r) := \alpha \left(1 - \left(\frac{r}{\lambda_\alpha}\right)^2\right)^{-(n-4)/2}, \qquad \lambda_\alpha = \alpha^{-2/(n-4)} \left[(n+2)n(n-2)(n-4)\right]^{1/4},$$

solves

$$\Delta^2 u_{0,\alpha} = u_{0,\alpha}^{(n+4)/(n-4)} \leq (1 + u_{0,\alpha})^{(n+4)/(n-4)} \leq (1 + u_{0,\alpha})^p \ \text{ for } r \in [0, \lambda_\alpha).$$

Choosing $\alpha > 0$ small enough one may achieve that $\lambda_\alpha > r_0$ and that

$$\tilde{u}(r_0) > u_{0,\alpha}(r_0), \qquad \tilde{u}'(r_0) > u_{0,\alpha}(r_0), \qquad \Delta\tilde{u}(r_0) > \Delta u_{0,\alpha}(r_0), \qquad (\Delta\tilde{u})'(r_0) > (\Delta u_{0,\alpha})'(r_0).$$

That means that $u_{0,\alpha}$ is a subsolution for $\tilde{u}$ on $[r_0, \min\{R_2, \lambda_\alpha\})$. Lemma 2 yields that

$$\tilde{u}(r) \geq u_{0,\alpha}(r) \text{ on } [r_0, \min\{R_2, \lambda_\alpha\}).$$

Consequently, $R_2 \leq \lambda_\alpha < \infty$.

The statement in the case $\gamma < \overline{\gamma}$ is mostly a further consequence of Lemma 5. Indeed, for $\gamma < \overline{\gamma}$ we know that necessarily u_γ vanishes in finite time, say at $r = R_1$. Since by (32) u_γ' remains negative for all r, we necessarily have $u_\gamma(r) \to -\infty$ at some $R_2 > R_1$. By considering $-u$ for $r < R_2$ close to R_2 and observing that $-u$ solves the same differential equation, the first part of the present proof shows that also here $R_2 < \infty$. □

6 Proof of Theorem 3

In order to prove (9) we proceed in three steps. We consider the corresponding global positive solution v of (14) and show first that for $t \to +\infty$, $v \to 0$ or $v \to K_0^{1/(p-1)}$ or v oscillates infinitely many times near ∞. In a second step, we exclude the first alternative. Finally, we study solutions v being oscillatory at ∞. For this purpose, an energy functional is introduced, which helps to deduce suitable L^2-bounds on the solution v. These bounds show that the solution again and again and even faster and faster has to be in a neighbourhood of the singular point P. By local properties of the autonomous system (16), the trajectory of v is (finally) on the stable manifold of P. For these arguments it is crucial that the coefficients K_1 and K_3 have the "good" sign: $K_1 < 0$ and $K_3 > 0$.

As a first step, we prove:

Proposition 4. *Let v be a global positive solution of* (14) *and assume that there exists $L \in [0, +\infty]$ such that*

$$\lim_{t\to+\infty} v(t) \;=\; L\;.$$

Then, $L \in \{0, K_0^{1/(p-1)}\}$.

Proof. For contradiction, assume first that L is finite and $L \notin \{0, K_0^{1/(p-1)}\}$. Then, $v^p - K_0 v(t) \to \alpha := L^p - K_0 L \neq 0$ and for all $\varepsilon > 0$ there exists $T > 0$ such that

$$\alpha - \varepsilon \le v^{iv}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) \le \alpha + \varepsilon \qquad \forall t \ge T\;. \tag{33}$$

Take $\varepsilon < |\alpha|$ so that $\alpha - \varepsilon$ and $\alpha + \varepsilon$ have the same sign and let

$$\delta := \sup_{t\ge T} |v(t) - v(T)| < \infty.$$

Integrating (33) over $[T, t]$ for any $t \ge T$ yields

$$(\alpha-\varepsilon)(t-T) + C + K_1\delta \le v'''(t) + K_3 v''(t) + K_2 v'(t) \le (\alpha+\varepsilon)(t-T) + C - K_1\delta \qquad \forall t \ge T\;,$$

where $C = C(T)$ is a constant containing all the terms $v(T)$, $v'(T)$, $v''(T)$ and $v'''(T)$. Repeating twice more this procedure gives

$$\frac{\alpha-\varepsilon}{6}(t-T)^3 + O(t^2) \le v'(t) \le \frac{\alpha+\varepsilon}{6}(t-T)^3 + O(t^2) \qquad \text{as } t \to \infty\;.$$

This contradicts the assumption that v admits a finite limit as $t \to +\infty$.

Next, we exclude the case $L = +\infty$. For contradiction, assume that

$$\lim_{t\to+\infty} v(t) \;=\; +\infty. \tag{34}$$

Then, there exists $T \in \mathbb{R}$ such that

$$v^{iv}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) \ge \frac{v^p(t)}{2} \qquad \forall t \ge T\;.$$

Moreover, by integrating this inequality over $[T, t]$ (for $t \ge T$), we get

$$v'''(t) + K_3 v''(t) + K_2 v'(t) + K_1 v(t) \ge \frac{1}{2}\int_T^t v^p(s)ds + C \qquad \forall t \ge T\;, \tag{35}$$

where $C = C(T)$ is a constant containing all the terms $v(T)$, $v'(T)$, $v''(T)$ and $v'''(T)$. From (34) and (35) we deduce that there exists $T' \geq T$ such that $\alpha := v'''(T') + K_3 v''(T') + K_2 v'(T') + K_1 v(T') > 0$. Since, (14) is autonomous, we may assume that $T' = 0$. Therefore, we have

$$v^{iv}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) \geq \frac{v^p(t)}{2} \qquad \forall t \geq 0 \ , \tag{36}$$

$$v'''(0) + K_3 v''(0) + K_2 v'(0) + K_1 v(0) = \alpha > 0 \ . \tag{37}$$

We may now apply the test function method developed by Mitidieri-Pohožaev [11]. More precisely, fix $T_1 > T > 0$ and a nonnegative function $\phi \in C_c^4[0, \infty)$ such that

$$\phi(t) = \begin{cases} 1 & \text{for } t \in [0, T] \\ 0 & \text{for } t \geq T_1 \ . \end{cases}$$

In particular, these properties imply that $\phi(T_1) = \phi'(T_1) = \phi''(T_1) = \phi'''(T_1) = 0$. Hence, multiplying inequality (36) by $\phi(t)$, integrating by parts and recalling (37) yields

$$\int_0^{T_1} [\phi^{iv}(t) - K_3 \phi'''(t) + K_2 \phi''(t) - K_1 \phi'(t)] v(t) dt \geq \frac{1}{2} \int_0^{T_1} v^p(t) \phi(t) dt + \alpha \ . \tag{38}$$

We now apply Young's inequality in the following form: for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$v\phi^{(i)} = v\phi^{1/p} \frac{\phi^{(i)}}{\phi^{1/p}} \leq \varepsilon v^p \phi + C(\varepsilon) \frac{|\phi^{(i)}|^{p/(p-1)}}{\phi^{1/(p-1)}} \ , \qquad \phi^{(i)} = \frac{d^i \phi}{dt^i} \qquad (i = 1, 2, 3, 4).$$

Then, provided ε is chosen sufficiently small, (38) becomes

$$C \sum_{i=1}^{4} \int_0^{T_1} \frac{|\phi^{(i)}(t)|^{p/(p-1)}}{\phi^{1/(p-1)}(t)} dt \geq \frac{1}{4} \int_0^T v^p(t) dt + \alpha \tag{39}$$

where $C = C(\varepsilon, K_i) > 0$. We now choose $\phi(t) = \phi_0(\frac{t}{T})$, where $\phi_0 \in C_c^4([0, \infty)$, $\phi_0 \geq 0$ and

$$\phi_0(\tau) = \begin{cases} 1 & \text{for } \tau \in [0, 1] \\ 0 & \text{for } \tau \geq \tau_1 > 1 \ . \end{cases}$$

As noticed in [11], there exists a function ϕ_0 in such class satisfying moreover

$$\int_0^{\tau_1} \frac{|\phi_0^{(i)}(\tau)|^{p/(p-1)}}{\phi_0^{1/(p-1)}(\tau)} d\tau =: A_i < \infty \qquad (i = 1, 2, 3, 4).$$

Then, thanks to a change of variables in the integrals, (39) becomes

$$C \sum_{i=1}^{4} A_i T^{1 - ip/(p-1)} \geq \frac{1}{4} \int_0^T v^p(t) dt + \alpha \qquad \forall T > 0.$$

Letting $T \to \infty$, the previous inequality contradicts (34). □

In order to perform the above mentioned second step, we show that a solution v of (14) vanishes at infinity only if the corresponding vector solution $\mathbf{w}= (w_1, w_2, w_3, w_4)$ of the system (16) approaches the "regular point" O.

Proposition 5. *Assume that* $v : [T_0, \infty) \to (0, \infty)$ *exists for some* T_0*, solves* (14) *and satisfies* $\lim_{t\to\infty} v(t) = 0$*. Then for all* $k \in \mathbb{N}$*, one also has:*

$$\lim_{t\to\infty} v^{(k)}(t) = 0. \tag{40}$$

Proof. By assumption we know that for t large enough $v(t) < K_0^{1/(p-1)}$ so that by the differential equation (14) eventually $v^{iv}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t) = \left(v^{p-1}(t) - K_0\right) v(t) < 0$. This shows that

$$t \mapsto v'''(t) + K_3 v''(t) + K_2 v'(t) + K_1 v(t) \tag{41}$$

is eventually strictly decreasing. Using the assumption once more we see that there exists

$$\lim_{t\to\infty} \left(v'''(t) + K_3 v''(t) + K_2 v'(t)\right) = \lim_{t\to\infty} \left(v'''(t) + K_3 v''(t) + K_2 v'(t) + K_1 v(t)\right) \in \mathbb{R} \cup \{-\infty\}. \tag{42}$$

We distinguish several cases and start by assuming

$$\lim_{t\to\infty} \left(v'''(t) + K_3 v''(t) + K_2 v'(t)\right) = \lim_{t\to\infty} \left(v'''(t) + K_3 v''(t) + K_2 v'(t) + K_1 v(t)\right) = 0. \tag{A}$$

In this case, since (41) is strictly decreasing, one eventually has that $v'''(t) + K_3 v''(t) + K_2 v'(t) + K_1 v(t) > 0$ so that by $K_1 < 0$

$$v'''(t) + K_3 v''(t) + K_2 v'(t) > 0 \text{ for } t \text{ large enough.} \tag{43}$$

This shows that $t \mapsto v''(t) + K_3 v'(t) + K_2 v(t)$ is eventually strictly increasing so that there exists

$$\lim_{t\to\infty} \left(v''(t) + K_3 v'(t) + K_2 v(t)\right) = \lim_{t\to\infty} \left(v''(t) + K_3 v'(t)\right) \in \mathbb{R} \cup \{+\infty\}.$$

If this limit were equal to $+\infty$, then also $+\infty = \lim_{t\to\infty} (v'(t) + K_3 v(t)) = \lim_{t\to\infty} v'(t)$, which contradicts the assumption. Hence

$$\lim_{t\to\infty} \left(v''(t) + K_3 v'(t) + K_2 v(t)\right) = \lim_{t\to\infty} \left(v''(t) + K_3 v'(t)\right) \in \mathbb{R}. \tag{44}$$

We distinguish three further subcases and start with discussing

$$\lim_{t\to\infty} \left(v''(t) + K_3 v'(t)\right) = \lim_{t\to\infty} \left(v''(t) + K_3 v'(t) + K_2 v(t)\right) = 0. \tag{A1}$$

We want to show that $\lim_{t\to\infty} v'(t)$ exists and assume for contradiction that $\limsup_{t\to\infty} v'(t) > \liminf_{t\to\infty} v'(t)$. Then we have a sequence $(t_k)_{k\in\mathbb{N}}$ with $t_k \to \infty$ such that consecutively v' attains local maxima and local minima in t_k so that in particular $v''(t_k) = 0$. By (A1) we may conclude that $\lim_{k\to\infty} v'(t_k) = 0$. Since v' attains consecutively its local maxima and local minima in t_k, this would contradict $\limsup_{t\to\infty} v'(t) > \liminf_{t\to\infty} v'(t)$. Hence we have proved that $\lim_{t\to\infty} v'(t) \in \mathbb{R}$ exists. Since $\lim_{t\to\infty} v(t) = 0$, we get

$$\lim_{t\to\infty} v'(t) = 0. \tag{45}$$

From this and assumption (A1), we directly obtain that also $\lim_{t\to\infty} v''(t) = 0$. From assumption (A) we then get that also $\lim_{t\to\infty} v'''(t) = 0$. For $k \geq 4$, the differential equation (14) finally yields $\lim_{t\to\infty} v^{(k)}(t) = 0$.

Next we consider the subcase

$$\lim_{t\to\infty} \left(v''(t) + K_3 v'(t)\right) = \lim_{t\to\infty} \left(v''(t) + K_3 v'(t) + K_2 v(t)\right) = 2\alpha > 0. \tag{A2}$$

In this case, one has that eventually $v''(t)+K_3v'(t)\geq\alpha$. Multiplying this inequality by $\exp(K_3t)$ and integrating yields

$$v'(t)\geq\frac{\alpha}{K_3}+o(1)\text{ near }\infty.$$

But this is impossible in view of our assumption $\lim_{t\to\infty}v(t)=0$.

Finally we consider the subcase

$$\lim_{t\to\infty}\big(v''(t)+K_3v'(t)\big)=\lim_{t\to\infty}\big(v''(t)+K_3v'(t)+K_2v(t)\big)=2\alpha<0. \tag{A3}$$

With precisely the same reasoning as in the previous case we come up with $v'(t)\leq\frac{\alpha}{K_3}+o(1)$ for $t\to\infty$ and again, we reach a contradiction.

Now we may consider the second main case

$$\lim_{t\to\infty}\big(v'''(t)+K_3v''(t)+K_2v'(t)\big)=\lim_{t\to\infty}\big(v'''(t)+K_3v''(t)+K_2v'(t)+K_1v(t)\big)=\alpha\neq 0. \tag{B}$$

Then $t\mapsto v''(t)+K_3v'(t)+K_2v(t)$ is monotone near ∞ and admits a limit $\beta\in\mathbb{R}\cup\{\pm\infty\}$. Hence, also $\lim_{t\to\infty}(v''(t)+K_3v'(t))=\beta$. If $\beta=0$ we proceed as in Subcase (A1) and if $\beta\neq 0$ as in Subcases (A2) and (A3). □

In order to exclude the first alternative in Proposition 4, for any global regular positive solution v of (14) and any $t\in\mathbb{R}$, we define the energy function

$$E(t):=E_v(t):=\frac{1}{p+1}v^{p+1}(t)-\frac{K_0}{2}v^2(t)-\frac{K_2}{2}|v'(t)|^2+\frac{1}{2}|v''(t)|^2. \tag{46}$$

We prove first that on consecutive extrema of v, the energy is decreasing. For the proof of the following lemma, the sign of the coefficients K_1,K_3 in front of the odd order derivatives in equation (14) is absolutely crucial.

Lemma 6. *Assume that $t_0<t_1$ and that $v'(t_0)=v'(t_1)=0$. Then*

$$E(t_0)\geq E(t_1).$$

If v is not constant, then the inequality is strict.

Proof. From the differential equation (14) we find:

$$\begin{aligned}E'(s)&=v^p(s)v'(s)-K_0v(s)v'(s)-K_2v'(s)v''(s)+v''(s)v'''(s)\\&=\big(v^p(s)-K_0v(s)-K_2v''(s)\big)\,v'(s)+v''(s)v'''(s)\\&=\big(v^{iv}(s)+K_3v'''(s)+K_1v'(s)\big)\,v'(s)+v''(s)v'''(s).\end{aligned}$$

Integrating by parts, this yields:

$$\begin{aligned}E(t_1)-E(t_0)&=\int_{t_0}^{t_1}E'(s)\,ds=-\int_{t_0}^{t_1}v'''(s)\,v''(s)\,ds-K_3\int_{t_0}^{t_1}\left|v''(s)\right|^2\,ds\\&\quad+K_1\int_{t_0}^{t_1}\left|v'(s)\right|^2\,ds+\int_{t_0}^{t_1}v'''(s)\,v''(s)\,ds\\&=-K_3\int_{t_0}^{t_1}\left|v''(s)\right|^2\,ds+K_1\int_{t_0}^{t_1}\left|v'(s)\right|^2\,ds\leq 0,\end{aligned} \tag{47}$$

since $K_3>0$ and $K_1<0$. If v is not a constant, the inequality is strict. □

Lemma 6 enables us to prove:

Lemma 7. *Assume that* $v : \mathbb{R} \to (0, \infty)$ *solves* (14) *and that* $\lim_{t\to-\infty} v(t) = \lim_{t\to-\infty} v'(t) = \lim_{t\to-\infty} v''(t) = 0$. *Then it cannot happen that also* $\lim_{t\to\infty} v(t) = 0$.

Proof. Consider the energy function E defined in (46). By assumption, we have $E(-\infty) = 0$. Assume for contradiction that $\lim_{t\to\infty} v(t) = 0$. Then, by Proposition 5 we see that also $E(+\infty) = 0$. By Lemma 6, this shows that v is a constant, hence $v(t) \equiv 0$. In turn, this contradicts the assumption that $v > 0$. □

Remark 3. In terms of dynamical systems, Lemma 7 states that the regular point O does not allow for a homoclinic orbit of system (16).

We can now exclude the first alternative in Proposition 4:

Proposition 6. *Let u be a regular positive radial solution of* (5) *and let v be defined according to* (12). *Then the first alternative in Proposition 4 does not occur, i.e. it is impossible that* $\lim_{t\to\infty} v(t) = 0$.

Proof. Since u is assumed to be regular near 0 and since v is defined according to (12), we have that $\lim_{t\to-\infty} v(t) = \lim_{t\to-\infty} v'(t) = \lim_{t\to-\infty} v''(t) = 0$. If we also had $\lim_{t\to\infty} v(t) = 0$, then $v(t) \equiv 0$ by Lemma 7. A contradiction! □

As before, we assume in what follows that u is a regular positive radial solution of (5) and that v is defined according to (12) so that v solves (14). If v is eventually monotonous, then the claim of Theorem 3 follows directly from Propositions 4 and 6. So, it remains to consider solutions v, which oscillate infinitely many times near $t = \infty$, i.e. have an unbounded sequence of consecutive local maxima and minima. In the sequel we always restrict to this kind of solutions without explicit mention. We first prove the following inequalities:

Lemma 8.

$$\liminf_{t\to\infty} v(t) > 0; \tag{48}$$

$$\forall t \in \mathbb{R}: \quad 0 < v(t) < \left(\tfrac{p+1}{2}\right)^{1/(p-1)} K_0^{1/(p-1)}; \tag{49}$$

$$\forall t \in \mathbb{R}: \quad v'(t) < \tfrac{4}{p-1} \left(\tfrac{p+1}{2}\right)^{1/(p-1)} K_0^{1/(p-1)}. \tag{50}$$

Proof. Since v is defined by means of a regular solution of (5), we have that $E(-\infty) = 0$. Let $\bar{t}$ be any local maximum for v. By Lemma 6 (with $t_0 = -\infty$ and $t_1 = \bar{t}$) we immediately get (49).

Let $\{t_k\}_{k\in\mathbb{N}}$ denote the sequence of consecutive positive critical points of v, starting with the first local maximum t_1 in $[0, \infty)$ of v. In particular we have that $v'(t_k) = 0$ and $\{t_k\}_{k\in\mathbb{N}}$ is a strictly increasing sequence, diverging to $+\infty$. Since $\{E(t_k)\}_{k\in\mathbb{N}}$ is bounded from below, by Lemma 6 we see that

$$\lim_{k\to\infty} E(t_k) =: -\delta < 0$$

exists. Therefore, for k large enough we have

$$-\frac{\delta}{2} \geq \frac{1}{p+1} v^{p+1}(t_k) - \frac{K_0}{2} v^2(t_k)$$

which proves (48).

Finally, note that in view of (12), statement (i) of Theorem 1 becomes

$$v'(t) < \frac{4}{p-1} v(t) \qquad \text{for all } t \in \mathbb{R} .$$

This inequality, combined with (49), proves (50). □

By (12), we see that (49) proves (8).

In the next four lemmas we prove some summability properties over $\mathbb{R}$ of v and of its derivatives:

Lemma 9.

$$\int_{\mathbb{R}} |v'(s)|^2\, ds + \int_{\mathbb{R}} |v''(s)|^2\, ds < \infty.$$

Proof. We take the same sequence $\{t_k\}_{k\in\mathbb{N}}$ as in the proof of Lemma 8. Since $E(-\infty)=0$, we obtain from (47) that for any k :

$$-K_3 \int_{-\infty}^{t_k} \left|v''(s)\right|^2\, ds + K_1 \int_{-\infty}^{t_k} \left|v'(s)\right|^2\, ds = E(t_k) \geq \min_{\nu\in[0,\infty)} \left(\frac{1}{p+1}\nu^{p+1} - \frac{K_0}{2}\nu^2\right) > -\infty.$$

The statement follows by letting $k\to\infty$ and using again that $K_3>0$ and $K_1<0$. □

Lemma 10.

$$\int_{\mathbb{R}} |v'''(s)|^2\, ds < \infty.$$

Proof. Here the sequence $\{t_k\}_{k\in\mathbb{N}}$ from the previous lemmas is no longer adequate. Instead, we choose a monotonically increasing diverging sequence $\{\tau_k\}_{k\in\mathbb{N}}$ of flex points of v such that v is there increasing. By Lemma 8 we may achieve:

$$\tau_k > 0, \quad \tau_k \nearrow \infty, \quad 0 \leq v'(\tau_k) < \frac{4}{p-1}\left(\frac{p+1}{2}\right)^{1/(p-1)} K_0^{-1/(p-1)}, \quad v''(\tau_k)=0. \tag{51}$$

We multiply the differential equation (14) by v'' and integrate over $(-\infty,\tau_k)$:

$$\int_{-\infty}^{\tau_k} \left(v^{iv}(s) + K_3 v'''(s) + K_2 v''(s) + K_1 v'(s) + K_0 v(s)\right)\, v''(s)\, ds = \int_{-\infty}^{\tau_k} v^p(s)\, v''(s)\, ds. \tag{52}$$

We show that all the lower order terms remain bounded, when $k\to\infty$:

$$\left|\int_{-\infty}^{\tau_k} v^p(s)\, v''(s)\, ds\right| = \left|v^p(\tau_k)v'(\tau_k) - p\int_{-\infty}^{\tau_k} v^{p-1}(s)\, |v'(s)|^2\, ds\right| \leq O(1) \tag{53}$$

by (49), (51) and Lemma 9. With the same argument, one also gets

$$\left|\int_{-\infty}^{\tau_k} v(s)\, v''(s)\, ds\right| \leq O(1). \tag{54}$$

Hölder's inequality and Lemma 9 imply

$$\left|\int_{-\infty}^{\tau_k} v'(s)\, v''(s)\, ds\right| \leq O(1). \tag{55}$$

By our choice of τ_k (recall that $v''(\tau_k)=0$), we obtain:

$$\int_{-\infty}^{\tau_k} v'''(s)\, v''(s)\, ds = \left[\frac{1}{2}|v''(s)|^2\right]_{-\infty}^{\tau_k} = 0. \tag{56}$$

Finally, integrating by parts and again by our choice of τ_k, we find:

$$\int_{-\infty}^{\tau_k} v^{iv}(s)\, v''(s)\, ds = \left[v'''(s)\, v''(s)\right]_{-\infty}^{\tau_k} - \int_{-\infty}^{\tau_k} \left(v'''(s)\right)^2\, ds = -\int_{-\infty}^{\tau_k} \left(v'''(s)\right)^2\, ds. \tag{57}$$

Letting $k\to\infty$, the statement follows directly from Lemma 9 and (52)–(57). □

Lemma 11.

$$\int_{\mathbb{R}} |v^{iv}(s)|^2\, ds < \infty.$$

Proof. In view of Lemmas 8–10 we may find a sequence $\{s_k\}$ such that

$$\lim_{k\to\infty} s_k = \infty,\ v(s_k) = O(1),\ \lim_{k\to\infty} v'(s_k) = \lim_{k\to\infty} v''(s_k) = \lim_{k\to\infty} v'''(s_k) = 0.$$

We multiply the equation (14) by v^{iv} and integrate over $(-\infty, s_k]$:

$$\int_{-\infty}^{s_k} \left(v^{iv}(s)\right)^2\, ds = \int_{-\infty}^{s_k} \left(v^p(s) - K_0 v(s) - K_1 v'(s) - K_2 v''(s) - K_3 v'''(s)\right) v^{iv}(s)\, ds. \tag{58}$$

By using Lemmas 8–10 and arguing as in the previous proofs we obtain:

$$\begin{aligned}
\int_{-\infty}^{s_k} v^{iv}(s)\, v'''(s)\, ds &= \left[\frac{1}{2}|v'''(s)|^2\right]_{-\infty}^{s_k} = o(1);\\
\int_{-\infty}^{s_k} v^{iv}(s)\, v''(s)\, ds &= o(1) - \int_{-\infty}^{s_k} |v'''(s)|^2\, ds = O(1);\\
\int_{-\infty}^{s_k} v^{iv}(s)\, v'(s)\, ds &= o(1) - \int_{-\infty}^{s_k} v'''(s)\, v''(s)\, ds = o(1);\\
\int_{-\infty}^{s_k} v^{iv}(s)\, v(s)\, ds &= o(1) - \int_{-\infty}^{s_k} v'''(s)\, v'(s)\, ds = o(1) + \int_{-\infty}^{s_k} |v''(s)|^2\, ds = O(1);\\
\left|\int_{-\infty}^{s_k} v^{iv}(s)\, v^p(s)\, ds\right| &= \left|o(1) - p\int_{-\infty}^{s_k} v'''(s)\, v^{p-1}(s)\, v'(s)\, ds\right|\\
&\le o(1) + C\left(\int_{-\infty}^{s_k} |v'''(s)|^2\, ds\right)^{1/2}\left(\int_{-\infty}^{s_k} |v'(s)|^2\, ds\right)^{1/2} \le O(1).
\end{aligned}$$

Inserting all these estimates into (58), the claim follows. □

Lemma 12.

$$\int_{\mathbb{R}} v^2(s)\left(v^{p-1}(s) - K_0\right)^2\, ds < \infty.$$

Proof. From the differential equation (14), we conclude

$$\left(v^{iv}(s) + K_3 v'''(s) + K_2 v''(s) + K_1 v'(s)\right)^2 = v^2(s)\ \left(v^{p-1}(s) - K_0\right)^2.$$

The statement follows now immediately from Lemmas 9–11. □

The proof of Theorem 3 will be completed by showing:

Proposition 7. *We assume that u is an entire regular positive radial solution of* (5)*, that v is defined according to* (12) *and that $\mathbf{w} = (w_1, w_2, w_3, w_4)$ is the corresponding solution of system* (16)*. We assume further that $v = w_1$ has an unbounded sequence of consecutive local maxima and minima near $t = \infty$. Then it follows that*

$$\lim_{t\to\infty} \mathbf{w}(t) = P, \tag{59}$$

where P is the "singular" steady solution of system (16)*.*

Proof. By Lemmas 8–12, we can find a sequence $\{\sigma_k\}_{k\in\mathbb{N}}$ such that

$$\sigma_{k+1} > \sigma_k, \quad \lim_{k\to\infty}(\sigma_{k+1}-\sigma_k) = 0, \quad \lim_{k\to\infty}\sigma_k = \infty, \quad \lim_{k\to\infty}\mathbf{w}(\sigma_k) = P.$$

If (59) were not true, then there would exist a subsequence $\{k_\ell\}_{\ell\in\mathbb{N}}$ with the following properties: for any $\varepsilon > 0$ there exists ℓ_ε such that for all $\ell \geq \ell_\varepsilon$ one has that

$$|\mathbf{w}(\sigma_{k_\ell}) - P| < \varepsilon, \qquad \sigma_{k_\ell+1} - \sigma_{k_\ell} < \varepsilon^2$$

and moreover that there exists $\theta_\ell \in (\sigma_{k_\ell}, \sigma_{k_\ell+1})$ with

$$|\mathbf{w}(s) - P| < 2\varepsilon \quad \forall s \in (\sigma_{k_\ell}, \theta_\ell) \quad \text{and} \quad |\mathbf{w}(\theta_\ell) - P| = 2\varepsilon\ .$$

The triangle inequality shows that $|\mathbf{w}(\theta_\ell) - \mathbf{w}(\sigma_{k_\ell})| > \varepsilon$, hence

$$\frac{1}{\theta_\ell - \sigma_{k_\ell}}\,|\mathbf{w}(\theta_\ell) - \mathbf{w}(\sigma_{k_\ell})| > \frac{1}{\varepsilon}\ .$$

By the mean value Theorem we conclude that

$$\frac{1}{\varepsilon} < \frac{1}{\theta_\ell - \sigma_{k_\ell}}\left|\int_{\sigma_{k_\ell}}^{\theta_\ell}\mathbf{w}'(s)\,ds\right| \leq \frac{1}{\theta_\ell - \sigma_{k_\ell}}\int_{\sigma_{k_\ell}}^{\theta_\ell}\big|\mathbf{w}'(s)\big|\,ds$$

so that there exists $\tau_\ell \in [\sigma_{k_\ell}, \theta_\ell]$ with

$$\big|\mathbf{w}'(\tau_\ell)\big| > \frac{1}{\varepsilon}.$$

Since ε is arbitrarily small, $|\mathbf{w}(\sigma_{k_\ell}) - P| < \varepsilon$, $|\mathbf{w}(\tau_\ell) - P| \leq 2\varepsilon$ and since $\mathbf{w}$ solves system (16), this is impossible for large enough ℓ. A contradiction is achieved, thereby proving (59). □

7 Proof of Theorem 4

Let v be defined by (12) and let ϕ be as in (19). Assume that $n \geq 13$ and $p > p_c$. Then, by Proposition 1, there exists $\varepsilon_0 > 0$ such that the equation $\phi(x) = (p-1)K_0 + \varepsilon$ admits four real solutions for all $\varepsilon \in (0, \varepsilon_0)$. From now on, we fix $\varepsilon = \varepsilon_0/2$ so that the equation

$$\psi^{iv}(t) + K_3\psi'''(t) + K_2\psi''(t) + K_1\psi'(t) - [(p-1)K_0 + \varepsilon]\psi(t) = 0 \qquad t \in \mathbb{R}\ ,$$

is nonoscillatory in $(-\infty, 0)$ according to the definition in [4]. In other words it has four linearly independent solutions of "exponential type" $\psi_i(t) = e^{\mu_i t}$ $(i = 1, ..., 4)$ for some μ_i's being small perturbations of the ν_i's which are all real numbers. Moreover, the differential operator

$$L_0 := \left(\frac{d}{dt}\right)^4 + K_3\left(\frac{d}{dt}\right)^3 + K_2\left(\frac{d}{dt}\right)^2 + K_1\left(\frac{d}{dt}\right) + K_0$$

is disconjugate, since this is the biharmonic operator, transformed by means of (12). By differentiating (14), we obtain

$$L_0\psi(t) - pv^{p-1}(t)\psi(t) = \psi^{iv}(t) + K_3\psi'''(t) + K_2\psi''(t) + K_1\psi'(t) + p(t)\psi(t) = 0 \qquad t \in \mathbb{R}\ , \tag{60}$$

where $\psi(t) := v'(t)$ and $p(t) := K_0 - pv^{p-1}(t)$. According to Theorem 3 we know that

$$\exists\, T > 0 \text{ such that } -[(p-1)K_0 + \varepsilon] < p(t) < 0 \quad \forall t > T\ .$$

Therefore, the equation (60) is also nonoscillatory in view of [4, Corollary 1]. This shows that $v'(t) = \psi(t)$ cannot change sign infinitely many times, and therefore that $v(t) - K_0^{1/(p-1)}$ does not change sign infinitely many times. □

8 Proof of Propositions 1, 2, 3

Proof of Proposition 1. We first observe that (11) is equivalent to

$$N_1 < 0 \tag{61}$$

and that (11) implies

$$N_2 - N_1^2 = 4(n-2)(p-1)^2 + 16(n-4)(p-1) - 64 > 4(n-2)(p-1)^2 + 64 > 0. \tag{62}$$

Next, we show that

$$N_3 > \frac{(N_2 - N_1^2)^2}{16}. \tag{63}$$

Indeed, by exploiting again (11), we have:

$$N_3 - \frac{(N_2 - N_1^2)^2}{16} =$$

$$\begin{aligned}
&= 8(n-2)(n-4)(p-1)^4 + 16(n^2-10n+20)(p-1)^3 - 128(n-4)(p-1)^2 + 256(p-1) \\
&> 16(n^2-6n+12)(p-1)^3 - 128(n-4)(p-1)^2 + 256(p-1) \\
&= 64(p-1)^3 + 16(n-2)(n-4)(p-1)^3 - 128(n-4)(p-1)^2 + 256(p-1) \\
&> 64(p-1)^3 + 128(n-2)(p-1)^2 - 128(n-4)(p-1)^2 + 256(p-1) \\
&= 64(p-1)^3 + 256(p-1)^2 + 256(p-1) = 64(p+1)^2(p-1) > 0.
\end{aligned}$$

In particular, (63) implies that $N_3 > 0$. In turn, together with the fact that $N_2 > N_1^2$, this shows that $\sqrt{N_2 + 4\sqrt{N_3}} > |N_1|$ which proves statement (i) in Proposition 1.

In order to discuss the stability properties of the eigenvalues ν_3 and ν_4 we introduce the function

$$\begin{aligned}
N_4 := 16N_3 - N_2^2 = \; & -(n-4)(n^3-4n^2-128n+256)(p-1)^4 + 128(3n-8)(n-6)(p-1)^3 \\
& +256(n^2-18n+52)(p-1)^2 - 2048(n-6)(p-1) + 4096.
\end{aligned} \tag{64}$$

For $1.939447811\ldots < n < 12.56534446\ldots$, the first coefficient in (64) is positive, so that assuming

$$5 \le n \le 12,$$

we obtain with help of (11):

$$\begin{aligned}
N_4 &= -(n-4)(n^3-4n^2-128n+256)(p-1)^4 + 128(3n-8)(n-6)(p-1)^3 \\
&\quad +256(n^2-18n+52)(p-1)^2 - 2048(n-6)(p-1) + 4096 \\
&> -8(n^3-4n^2-128n+256)(p-1)^3 + 128(3n-8)(n-6)(p-1)^3 \\
&\quad +256(n^2-18n+52)(p-1)^2 - 2048(n-6)(p-1) + 4096 \\
&= 64n^2(p-1)^3 - 8(n-4)(n^2-40n+128)(p-1)^3 + 256(n^2-18n+52)(p-1)^2 \\
&\quad -2048(n-6)(p-1) + 4096 \\
&> 64n(n-4)(p-1)^3 - 64(n^2-40n+128)(p-1)^2 + 256(n^2-18n+52)(p-1)^2 \\
&\quad -2048(n-6)(p-1) + 4096 \\
&> 512n(p-1)^2 + 64(n-4)(3n-20)(p-1)^2 - 2048(n-6)(p-1) + 4096 \\
&= 2048(p-1)^2 + 192(n-4)^2(p-1)^2 - 2048(n-6)(p-1) + 4096 \\
&> 2048(p-1)^2 + 1536(n-4)(p-1) - 2048(n-6)(p-1) + 4096 \\
&= 2048(p-1)^2 - 512(n-12)(p-1) + 4096 > 0,
\end{aligned}$$

since $n \leq 12$. This, together with (61), proves statement (ii) in Proposition 1.

In order to prove statement (iii), we assume that

$$n \geq 13$$

and we study $N_4 = N_4(n,p)$ as a function of p. We compute its second derivative (with respect to p):

$$-\frac{\partial^2 N_4}{\partial p^2} = 12(n-4)(n^3-4n^2-128n+256)(p-1)^2 - 768(3n-8)(n-6)(p-1) - 512(n^2-18n+52).$$

This is a quadratic function of p which tends to $+\infty$ as $p \to +\infty$. Its minimum is smaller than the Sobolev exponent $(n+4)/(n-4)$ if and only if

$$0 < (n^3-4n^2-128n+256) - 4(3n-8)(n-6) = (n-18)(n^2+2n+12)+280.$$

This is certainly true for $n \geq 18$, while for $n = 13, \ldots, 17$, we have $\frac{\partial^2 N_4}{\partial p^2}(n, \frac{n+4}{n-4}) < 0$. Summarizing, for $p > (n+4)/(n-4)$, $\frac{\partial^2 N_4}{\partial p^2}$ has at most one zero. Therefore,

$$\text{for } p > \frac{n+4}{n-4}, \quad p \mapsto N_4(n,p) \quad \text{is either always concave or it is first convex and then concave.} \tag{65}$$

Moreover, since the first coefficient in (64) is now negative (because $n \geq 13$), we have

$$\lim_{p \to \infty} N_4(n,p) = -\infty \qquad \forall n \geq 13. \tag{66}$$

Finally, note that

$$N_4\left(n, \frac{n+4}{n-4}\right) = \frac{32768\, n^2}{(n-4)^3} > 0 \qquad \text{and} \qquad \frac{\partial N_4}{\partial p}\left(n, \frac{n+4}{n-4}\right) = \frac{20480\, n^2}{(n-4)^2} > 0. \tag{67}$$

By (65)-(66)-(67) there exists a unique $p_c > (n+4)/(n-4)$ such that

$$N_4(n,p) > 0 \text{ for all } p < p_c\ , \qquad N_4(n,p_c) = 0\ , \qquad N_4(n,p) < 0 \text{ for all } p > p_c.$$

In order to prove that $n \mapsto p_c$ is strictly decreasing we calculate $\frac{dp_c}{dn}$ by means of implicit differentiation and note first that the previous reasoning gives

$$\frac{\partial N_4}{\partial p}(n,p_c) < 0. \tag{68}$$

We proceed by calculating

$$\begin{aligned}
\frac{\partial N_4}{\partial n} &= -\left(4n^3-24n^2-224n+768\right)(p-1)^4 + 256(3n-13)(p-1)^3 \\
&\quad +512(n-9)(p-1)^2 - 2048(p-1), \\
\frac{\partial^2 N_4}{\partial n^2} &= -(12n^2-48n-224)(p-1)^4 + 768(p-1)^3 + 512(p-1)^2, \\
\frac{\partial^3 N_4}{\partial n^3} &= -24(n-2)(p-1)^4;
\end{aligned}$$

the latter being always negative for $n > 2$. Keeping $p > 1$ fixed, we consider now $n \mapsto N_4(n,p)$. First we calculate $n > 4$ such that $p = (n+4)/(n-4)$, i.e. $n = 4 + \frac{8}{p-1}$. Negativity of $\frac{\partial^3 N_4}{\partial n^3}$ shows that beyond $n = 4 + \frac{8}{p-1}$ this function is either always concave or convex first and then always concave. On the mentioned particular value we have by (67) that

$$N_4\left(4 + \frac{8}{p-1}, p\right) > 0$$

and moreover, we find that

$$\frac{\partial N_4}{\partial n}\left(4 + \frac{8}{p-1}, p\right) = 32\left(4 + \frac{8}{p-1}\right)\left(2 + \frac{8}{p-1}\right)(p-1)^4 > 0.$$

Since $N_4(n, p_c) = 0$, this shows that also

$$\frac{\partial N_4}{\partial n}(n, p_c) < 0. \tag{69}$$

By implicit differentiation we conclude from (68) and (69) that

$$\frac{dp_c}{dn} = -\frac{\frac{\partial N_4}{\partial n}(n, p_c)}{\frac{\partial N_4}{\partial p}(n, p_c)} < 0.$$

Finally one reads directly from the form of N_4 that for any $p_0 > 1$, $N_4(n, p_0)$ becomes negative, provided n is chosen large enough. This shows that $\frac{n+4}{n-4} < p_c < p_0$ for n large enough, i.e.

$$\lim_{n\to\infty} p_c = 1.$$

The proof of statement *(iii)* in Proposition 1 is so complete. □

Proof of Proposition 2. This proof is an extension of the one of Proposition 7.

It is enough to consider a solution v which converges eventually monotonically to $K_0^{1/(p-1)}$. The differential equation (14) shows that $v^{iv}(t) + K_3 v'''(t) + K_2 v''(t) + K_1 v'(t)$ eventually has a fixed sign. Let us now consider $\tilde{v}(t) := v(t) - K_0^{1/(p-1)}$. Then

$$\lim_{t\to\infty} \tilde{v}(t) = 0$$

and $\tilde{v}^{iv}(t) + K_3\tilde{v}'''(t) + K_2\tilde{v}''(t) + K_1\tilde{v}'(t)$ is also eventually of fixed sign. This shows that

$$\lim_{t\to\infty}\left(\tilde{v}'''(t) + K_3\tilde{v}''(t) + K_2\tilde{v}'(t) + K_1\tilde{v}(t)\right) = \lim_{t\to\infty}\left(\tilde{v}'''(t) + K_3\tilde{v}''(t) + K_2\tilde{v}'(t)\right) \in \mathbb{R}\cup\{\pm\infty\}$$

exists. Now we may proceed precisely as in Proposition 5. □

Proof of Proposition 3. It is enough to show that M_P has no eigenvectors with first component equal to 0. Assume for contradiction that associated to some eigenvalue ν, there exists $(a, b, c) \neq (0, 0, 0)$ such that

$$\begin{pmatrix} \frac{4}{p-1} - \nu & 1 & 0 & 0 \\ 0 & -\nu & 1 & 0 \\ 0 & 0 & -\nu & 1 \\ pK_0 & C_2 & C_3 & C_4 - \nu \end{pmatrix}\begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

This is clearly impossible. □

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