ORLICZ CAPACITIES AND APPLICATIONS TO SOME EXISTENCE QUESTIONS FOR ELLIPTIC PDEs HAVING MEASURE DATA

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Abstract

We study the sequence u_n, which is solution of $-\text{div}(a(x, \nabla u_n)) + \Phi''(|u_n|)\, u_n = f_n + g_n$ in Ω an open bounded set of $\mathbf{R}^N$ and $u_n = 0$ on $\partial\Omega$, when f_n tends to a measure concentrated on a set of null Orlicz-capacity. We considere the relation between this capacity and the N-function Φ, and prove a non-existence result.

1 Introduction

Let Ω be a bounded open subset of $\mathbf{R}^N$, $N > 2$, we study the non-existence of a solution for the following nonlinear elliptic problem (that is our model problem)

$$\begin{cases} -\Delta u + |u|^{q-1}\, u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

in the following sense : let f_n be a sequence of smooth functions that tends to a measure μ in a sense that we will precise. Let u_n be the sequence of solutions of

$$\begin{cases} -\Delta u_n + |u_n|^{q-1}\, u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

we will consider the case, with respect to the measure μ and the value of q, where u_n converge to a function u that does not satisfy (1.1).

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Let us, first, recall the following result due to H. Brezis (see [9]). Let Ω be a bounded open subset of $\mathbf{R}^N$, $N > 2$, with $0 \in \Omega$, let f be a function in $L^1(\Omega)$, and let f_n be a sequence of $L^\infty(\Omega)$ functions such that

$$\lim_{n\to+\infty} \int_{\Omega\setminus B_\rho(0)} |f_n - f|\, dx = 0\,, \qquad \forall \rho > 0\,. \tag{1.3}$$

Let u_n be the sequence of solutions of (1.2) with $q \geq \frac{N}{N-2}$. Then u_n converges to the unique solution u of the equation $-\Delta u + |u|^{q-1}\,u = f$.

If $f = 0$, an example of functions f_n satisfying condition (1.3)

is that of a sequence of nonnegative $L^\infty(\Omega)$ functions converging in the weak$*$ topology of measures to δ_0, the Dirac mass concentrated at the origin. In this case, u_n converges to zero. The result of [9] is strongly connected with a theorem by P. Bénilan and H. Brezis (see [9]), which states that the problem $-\Delta u + |u|^{q-1}\,u = \delta_0$ has no distributional solution if $q \geq \frac{N}{N-2}$. On the other hand (see [7] and [9]), if $q < \frac{N}{N-2}$, then there exists a unique solution of

$$\begin{cases} -\Delta u + |u|^{q-1}\,u = \delta_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Thus the preceding theorem can be seen as a nonexistence result for this problem, in the sense that if one looks for solutions obtained for approximation of (1.4), then one does not find a "reasonable" solution.

The "dividing range" $\frac{N}{N-2}$ basically depends on two facts: the linearity of the laplacian operator (i.e., the dependence of order 1 with respect to the gradient of u), and the fact that the Dirac δ_0 is a measure which is concentrated on a point: a set of zero N-capacity. In the case $q \geq \frac{N}{N-2}$, which is equivalent to $2q' \leq N$, δ_0 is not "absolutely continuous" with respect to the N-capacity and hence also to the $2q'$-capacity and there is no solution of (1.4). If $q < \frac{N}{N-2}$, which is equivalent to $2q' > N$, δ_0 is "absolutely continuous" with respect to the $2q'$-capacity and there is a solution of (1.4).

This fact is strictly related to the result of [14], where a necessary and sufficient condition for the existence of a solution is given. More precisely, the equation

$$\begin{cases} -\Delta u + |u|^{q-1}\,u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

has a solution if and only if μ belongs to $L^1(\Omega)+W^{-2,q}(\Omega)$. If μ is a measure that is "absolutely continuous" with respect to the $(2,q')$-capacity, which is defined in Definition 2.4, then μ belongs to $L^1(\Omega)+W^{-2,q}(\Omega)$ and (1.5) has a solution. Moreover in [5], the singularities for (1.5) are removable if and only if μ is "absolutely continuous" with respect to the $(2,q')$-capacity.

In order to point out the relations between these results and capacities, we recall that we have (according to Gagliardo-Nirenberg inequalities)

$$\mathrm{cap}_{2,q'}(E)=0 \quad\Longrightarrow\quad \mathrm{cap}_{1,2q'}(E)=0\,,$$

and that, by [1], Theorem 5.5.1, we have, for every set E,

$$\mathrm{cap}_{1,2q'+\varepsilon}(E)=0 \quad\Longrightarrow\quad \mathrm{cap}_{2,q'}(E)=0\,, \quad \forall\varepsilon>0\,.$$

The result of [9] has been extended to nonlinear operators of Leray-Lions type and measure concentrated on sets of null r-capacities in [20] :

Theorem 1.1 *Let $p<r\le N$, and let $\lambda=\lambda^+-\lambda^-$ be a bounded Radon measure concentrated on a set E of zero r-capacity. Let $f_n=f_n^\oplus-f_n^\ominus$ (with $f_n^\oplus$ and $f_n^\ominus$ nonnegative funtions) be a sequence of $L^\infty(\Omega)$ functions that converges to λ in the sense*

$$\lim_{n\to+\infty}\int_\Omega f_n^\oplus\,\varphi\,dx=\int_\Omega\varphi\,d\lambda^+\,,\qquad \lim_{n\to+\infty}\int_\Omega f_n^\ominus\,\varphi\,dx=\int_\Omega\varphi\,d\lambda^-\,,$$

for every function φ which is continuous and bounded on Ω. Let g be a function in $L^1(\Omega)$, and let g_n be a sequence of $L^\infty(\Omega)$ functions which converges to g weakly in $L^1(\Omega)$. Let

$$q>\frac{r(p-1)}{r-p}\,,$$

and let u_n be the solution in $W_0^{1,p}(\Omega)$ of the problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u_n))+|u_n|^{q-1}\,u_n=f_n+g_n & \text{in } \Omega,\\ u_n=0 & \text{on } \partial\Omega. \end{cases}$$

Then, as n tends to infinity, $|\nabla u_n|^{p-1}$ converges strongly to $|\nabla u|^{p-1}$ in $L^\sigma(\Omega)$, for every $\sigma < \frac{pq}{(q+1)(p-1)}$, where u is the unique entropy solution of

$$\text{(1.6)} \qquad \begin{cases} -\operatorname{div}(a(x,\nabla u)) + |u|^{q-1}\, u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover,

$$\lim_{n\to+\infty} \int_\Omega |u_n|^{q-1}\, u_n\, \varphi\, dx = \int_\Omega |u|^{q-1}\, u\, \varphi\, dx + \int_\Omega \varphi\, d\lambda\,,\ \forall \varphi \in C^0_{\mathrm{c}}(\Omega)\,.$$

Remark 1.2 *Since this theorem deals with rather general operators and measures, the concept of solution in the sense of distributions of problems like (1.6) may not be convenient in order to have uniqueness of solutions. Hence the notion of entropy solutions (see Definition 2.8) has been used.*

In order to avoid the loss between $q \geq \frac{N}{N-2}$ (see [9]) and $q > \frac{N}{N-2}$ (see Theorem 1.1 with $p = 2$ and $r = N$) and between the $2q'$-capacity and the $(2, q')$-capacity, we will extend the result of [20] to low order terms more general that $|u|^{q-1}u$ in the context of Orlicz spaces. The best approach for this new context will involve also the notion of Orlicz capacity. Such a notion has been already introduced in literature (see [4]). In spite of this, we will adopt a new equivalent definition (see Definition 2.5), which is closer to the classical one used with Sobolev spaces.

2 The main results

2.1 Definitions

First let be the various definitions useful for understanding the results.

Definition 2.1 *An N-function is a function Φ continuous on $[0,\infty[$, strictly increasing, convex, and such that $\lim_{x\to 0} \Phi(x)/x = 0$, $\lim_{x\to\infty} \Phi(x)/x = +\infty$. For our purposes we will assume also that $\Phi \in C^1([0,\infty[)$, Φ' strictly increasing, and*

$$\text{(2.1)} \qquad c_1 \min(s^{q_1-1}, s^{q_2-1})\Phi'(t) \leq \Phi'(st) \leq c_2 \max(s^{q_1-1}, s^{q_2-1})\Phi'(t)$$

Condition (2.1) means that the growth of Φ "lies between" that one of the powers t^{q_1}, t^{q_2} (see Section 4).

Definition 2.2 *The N-function Φ belongs to Δ_2 if there exist $c > 0$ and $t_0 \geq 0$ such that*

$$\Phi(2t) \leq c\,\Phi(t) \qquad \forall t \geq t_0.$$

Definition 2.3 *The complementary function of Φ, denoted by $\widetilde{\Phi}$, is defined by*

$$\widetilde{\Phi}(t) = \sup_{s\geq 0}[st - \Phi(t)] \qquad \forall t \geq 0$$

It can be proved that if Φ is an N-function, also $\widetilde{\Phi}$ is an N-function. If Φ' is strictly increasing, $(\widetilde{\Phi})'(t) = (\Phi')^{-1}(t)\,\forall t \geq 0$.

Let us recall that Ω is an open bounded set of $\mathbf{R}^N$. Let Φ satisfying Definition 2.1. The Orlicz class $L^\Phi(\Omega)$ is defined by

$$L^\Phi(\Omega) = \left\{ f \in L^1_{loc}(\Omega) : \int_\Omega \Phi(|f|)dx < +\infty \right\}$$

The Orlicz class $L^\Phi(\Omega)$, equipped with the norm

$$\|f\|_\Phi = \inf\left\{ k > 0 : \int_\Omega \Phi\left(\frac{|f|}{k}\right) dx \leq 1 \right\}$$

becomes the so-called Orlicz space, which is a reflexive Banach space whose dual is $L^{\widetilde{\Phi}}(\Omega)$. In the following we will assume that the reader is familiar with the Orlicz space theory, deeply studied (for instance) in [17, 19, 22].

One can also define the Orlicz-Marcinkiewicz spaces by

$$M^\Phi(\Omega) = \left\{ f \in L^1_{loc}(\Omega) : \text{ meas}\,\{f > t\}\Phi(t) \text{ is bounded} \right\}$$

Definition 2.4 *Let $0 < \alpha < N$ and let r be a real number, with $r > 1$. Let K be a compact subset of Ω. The (α, r)-capacity of K with respect to Ω is defined as:*

$$\text{cap}_{\alpha,r}(K) = \inf\left\{ \|u\|^r_{W_0^{\alpha,r}(\Omega)} : u \in C^\infty_{\text{c}}(\Omega),\ u \geq \chi_K \right\},$$

where χ_K is the characteristic function of K; we will use the convention that $\inf\emptyset = +\infty$. The (α, r)-capacity of any open subset U of Ω is then defined by:

$$\text{cap}_{\alpha,r}(U) = \sup\left\{ \text{cap}_{\alpha,r}(K),\ K \text{ compact},\ K \subset U \right\},$$

and the (α, r)-capacity of any set $E \subset \Omega$ by

$$\mathrm{cap}_{\alpha,r}(E) = \inf \left\{ \mathrm{cap}_{\alpha,r}(U),\ U \text{ open},\ E \subset U \right\}.$$

We introduce now the following definition, which represents a generalization of the previous one. We will see in Section 4 that this formulation is equivalent to that one appearing in [4].

Definition 2.5 *Let K be a compact subset of Ω. Let A satisfying Definition 2.1. The A-capacity of K with respect to Ω is defined as:*

$$\mathrm{cap}_{1,A}(K) = \inf \left\{ \|\nabla u\|_A : u \in C_c^\infty(\Omega),\ u \geq \chi_K \right\},$$

where χ_K is the characteristic function of K; we will use the convention that $\inf \emptyset = +\infty$. *The A-capacity of any open subset U of Ω is then defined by:*

$$\mathrm{cap}_{1,A}(U) = \sup \left\{ \mathrm{cap}_{1,A}(K),\ K \text{ compact},\ K \subset U \right\},$$

and the A-capacity of any set $E \subset \Omega$ by

$$\mathrm{cap}_{1,A}(B) = \inf \left\{ \mathrm{cap}_{1,A}(U),\ U \text{ open},\ E \subset U \right\}.$$

Let p be a real number, with $1 < p < N$, and let p' be its conjugate Hölder exponent (i.e., $1/p + 1/p' = 1$). Let $a : \Omega \times \mathbf{R}^N \to \mathbf{R}^N$ be a Carathéodory function (i.e., $a(\cdot, \xi)$ is measurable on Ω for every ξ in $\mathbf{R}^N$, and $a(x, \cdot)$ is continuous on $\mathbf{R}^N$ for almost every x in Ω), such that the following holds:

$$a(x,\xi) \cdot \xi \geq \alpha \, |\xi|^p, \tag{2.2}$$

$$|a(x,\xi)| \leq \beta \, [b(x) + |\xi|^{p-1}], \tag{2.3}$$

$$[a(x,\xi) - a(x,\eta)] \cdot (\xi - \eta) > 0, \tag{2.4}$$

for almost every x in Ω, for every ξ, η in $\mathbf{R}^N$, with $\xi \neq \eta$, where α and β are two positive constants, and b is a nonnegative function in $L^{p'}(\Omega)$.

Under assumptions (2.2), (2.3) and (2.4), $u \mapsto -\mathrm{div}(a(x, \nabla u))$ is a uniformly elliptic, coercive and pseudomonotone operator acting from $W_0^{1,p}(\Omega)$ to its dual $W^{-1,p'}(\Omega)$, and so it is surjective (see [18]).

Let us denote $C(\Omega)$ the space of the real valued continuous functions on Ω, equipped with the topology of uniform convergence on compact subsets of Ω. If K is compact, $C(K)$ is usually normed with the supremum norm, $\|\cdot\|_{L^\infty(K)}$. $C_c(\Omega)$ is the subset of $C(\Omega)$ consisting of functions with compact support contained in Ω. The dual of the space $C_c(\Omega)$ is denoted by $\mathcal{M}(\Omega)$, the bounded measures on Ω. The set of positive measures on Ω, is denoted by $\mathcal{M}^+(\Omega)$. For K compact, the symbol $\mathcal{M}^+(K)$ has analogous meaning; such space will be used mainly in some intermediate auxiliary statements in Section 4.

Let λ be a bounded measure on Ω. We say that λ is *concentrated* on a set E if $\lambda(B) = \lambda(B \cap E)$ for every Borelian subset B of Ω. Thanks to the Hahn decomposition theorem, given a signed Radon measure λ on Ω, we can decompose it as the difference of two nonnegative, mutually singular, measures:

$$\lambda = \lambda^+ - \lambda^- .$$

If λ is concentrated on a set E, as a consequence of the fact that λ^+ and λ^- are mutually singular, we have that λ^+ is concentrated on a set E^+, λ^- is concentrated on a set E^-, and $E^+ \cap E^- = \emptyset$.

Definition 2.6 *Let λ be a measure, decomposed as $\lambda^+ - \lambda^-$, and let be approximations f_n of λ made in the following way: $f_n = f_n^\oplus - f_n^\ominus$, where $\{f_n^\oplus\}$ and $\{f_n^\ominus\}$ are sequences of nonnegative $L^\infty(\Omega)$ functions such that*

$$(2.5)\quad \lim_{n\to+\infty} \int_\Omega f_n^\oplus \varphi\, dx = \int_\Omega \varphi\, d\lambda^+ , \qquad \lim_{n\to+\infty} \int_\Omega f_n^\ominus \varphi\, dx = \int_\Omega \varphi\, d\lambda^- ,$$

for every function φ which is continuous and bounded on Ω.

We explicitly remark that $f_n^\oplus$ and $f_n^\ominus$ may not be the positive and negative parts of f_n (that is to say, their supports may not be disjoint). Observe that choosing $\varphi \equiv 1$ in (2.5) we obtain

$$(2.6)\qquad \|f_n^\oplus\|_{L^1(\Omega)} \le c , \qquad \|f_n^\ominus\|_{L^1(\Omega)} \le c .$$

Since we will deal with right hand side which are some measures, the solution may not be in $L^1_{\mathrm{loc}}(\Omega)$, thus there distributional gradient may not be defined. Thus we will use the following definition of "gradient".

Before this, we define, for $k > 0$,

$$T_k(s) = \max(-k, \min(k, s)) , \qquad \forall s \in \mathbf{R} ,$$

the truncature at levels $\pm k$.

Definition 2.7 *Let u be a measurable function on Ω such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$. Then (see [6], Lemma 2.1) there exists a unique measurable function $v : \Omega \to \mathbf{R}^N$ such that*

$$\nabla T_k(u) = v\,\chi_{\{|u|\le k\}}\,, \quad \textit{almost everywhere in } \Omega, \textit{ for every } k > 0.$$

We will define the gradient of u as the function v, and we will denote it by $v = \nabla u$. If u belongs to $W_0^{1,1}(\Omega)$, then this gradient coincides with the usual gradient in distributional sense.

For right hand side which are some measures or $L^1(\Omega)$ function, there is no uniqueness of distributional solutions of nonlinear elliptic equations thus we will use the following notion of entropy solution (see [6]).

Definition 2.8 *Let g be a function in $L^1(\Omega)$ and let Φ be an N-function. A measurable function u such that $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$ is an entropy solution of the equation*

$$(2.7) \qquad \begin{cases} -\mathrm{div}(a(x,\nabla u)) + \Phi''(|u|)\,u = g & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases}$$

if $\Phi''(|u|)|u|$ belongs to $L^1(\Omega)$, and

$$\int_\Omega a(x,\nabla u)\cdot\nabla T_k(u-\varphi)\,dx + \int_\Omega \Phi''(|u|)\,u\,T_k(u-\varphi)\,dx \le \int_\Omega g\,T_k(u-\varphi)\,dx\,,$$

for every φ in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, and for every $k > 0$.

We recall the following result (see [6], Theorem 6.1, Theorem 5.1 and Corollary 4.3).

Theorem 2.9 *Let g be a function in $L^1(\Omega)$. Then there exists a unique entropy solution of (2.7). Moreover this solution is also a solution of (2.7) in the sense of distribution.*

2.2 Nonlinear problem

The first main result (proved in Section 3) of the paper is the following

Theorem 2.10 *Let A be an N-function satisying the assumptions of Definition 2.1, with $q_1 = p$, $q_2 = N$, and let λ be a bounded measure concentrated on a set E of zero A-capacity. Let f_n be a sequence of functions converging to λ in the sense of Definition 2.6. Let g be a function in $L^1(\Omega)$, and let g_n be a sequence of $L^\infty(\Omega)$ functions which converges to g weakly in $L^1(\Omega)$. Let $\Phi \in C^2([0,\infty[)$ be an N-function such that*

$$\Phi'(t) \le t\,\Phi''(t) \qquad \forall t \ge 0 \tag{2.8}$$

$$\int^{+\infty} \frac{(\widetilde{A})'(t)\,\Phi^{-1}(t^{p'})}{t^{p'}}\,dt < +\infty\,, \tag{2.9}$$

and let u_n be the solution in $W_0^{1,p}(\Omega)$ of the problem

$$\begin{cases} -\mathrm{div}(a(x,\nabla u_n)) + \Phi''(|u_n|)\,u_n = f_n + g_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

Then, as n tends to infinity, $|\nabla u_n|^{p-1}$ converges strongly to $|\nabla u|^{p-1}$ in $L^\Theta(\Omega)$, for every N-function $\Theta \in \Delta_2$ such that

$$\int^{+\infty} \frac{\Theta'(t)\,\Phi^{-1}(t^{p'})}{t^{p'}}\,dt < +\infty\,, \tag{2.11}$$

where u is the unique entropy solution of

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) + \Phi''(|u|)\,u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

Moreover,

$$\lim_{n\to+\infty} \int_\Omega \Phi''(u_n)\,u_n\,\varphi\,dx = \int_\Omega \Phi''(u)\,u\,\varphi\,dx + \int_\Omega \varphi\,d\lambda\,,\ \forall \varphi \in C_c^0(\Omega)\,. \tag{2.13}$$

Remark 2.11 *According to Theorem 2.9, Theorem 2.10 is also true with solutions in the distributional sense (but there is no uniqueness result).*

Remark 2.12 *Assumption (2.8) implies that the growth of the nonlinear part of the equation is superlinear (recall that $\Phi'(t)/t$ is increasing because of the convexity of Φ).*

Remark 2.13 *Let us now consider the Theorem 2.10 in the case $\Phi(t) = t^{q+1}$. The condition (2.9) can be reformulated as follows:*

$$\int^{+\infty} \frac{(\widetilde{A})'(s)}{s^{\frac{p'q}{q+1}}}\, ds < +\infty\,. \tag{2.14}$$

Remark 2.14 *Set $\Phi(t) = t^{q+1}$ and $A(t) = t^r$ (thus $(\widetilde{A})'(t) = c\, t^{1/(r-1)}$), then (2.9) (or (2.14)) becomes :*

$$\int^{+\infty} t^{\frac{1}{r-1}+\frac{p'}{q+1}-p'} dt < +\infty\,.$$

This condition can be true only if $r > p$. In this case, this condition is equivalent to

$$q > \frac{r(p-1)}{r-p}.$$

Remark 2.15 *Set $\Phi(t) = t^{q+1}$ and $A(t) = t^r \log^{r-1+\varepsilon}(e+t)$ for some $\varepsilon > 0$. There exist some constants $c_1, \dots, c_3'$, such that one has*

$$c_1\, t^{r-1} \log^{r-1+\varepsilon}(e+t) \le A'(t) \le c_2\, t^{r-1} \log^{r-1+\varepsilon}(e+t), \qquad \forall t > c_3$$

and therefore (see Definition 2.3)

$$c_1'\, t^{\frac{1}{r-1}} \log^{-\frac{r-1+\varepsilon}{r-1}}(e+t) \le (\widetilde{A})'(t) \le c_2'\, t^{\frac{1}{r-1}} \log^{-\frac{r-1+\varepsilon}{r-1}}(e+t), \qquad \forall t > c_3'$$

thus (2.9) (or (2.14)) becomes

$$\int^{+\infty} t^{\frac{1}{r-1}+\frac{p'}{q+1}-p'} \log^{-\frac{r-1+\varepsilon}{r-1}}(e+t) dt < +\infty\,.$$

If $r > p$, this is equivalent to

$$q \ge \frac{r(p-1)}{r-p}.$$

Remark 2.16 *Let $p < r \leq N$, and $A(t) = t^r$. Then (2.10) has not solutions (in the sense of Theorem 2.10) for any Φ such that*

$$\int^{+\infty} \frac{\Phi^{-1}(t^{p'})}{t^{p'-r'+1}}\, dt < +\infty\,. \tag{2.15}$$

We give here some examples of functions Φ for which (2.9) or (2.15) apply:

$$\Phi_1(t) = t^{q+1} \quad \forall q > \frac{r(p-1)}{r-p};$$

$$\Phi_2(t) = t^{\frac{r(p-1)}{r-p}+1}[\log(e+t)]^{k+1} \quad \forall k > \frac{r(p-1)}{r-p};$$

$$\Phi_3(t) = t^{\frac{r(p-1)}{r-p}+1}[\log(e+t)]^{\frac{r(p-1)}{r-p}+1}[\log(e+\log(e+t))]^{k+1} \quad \forall k > \frac{r(p-1)}{r-p};$$

$$\Phi_4(t) = e^t - t - 1;$$

$$\Phi_5(t) = e^{e^t-1} - t - 1\,.$$

Remark 2.17 *If $A(t)/t^p$ is non-increasing, then (2.9) has no solution in Φ. On the other hand, when $A(t) = t^r$, the above condition means $r \leq p$, and in this case we have existence for 2.16, see Remark 1.10 of [20]. This motivates the bound $q_1 = p$ in Theorem 2.10. Notice also that q_2 cannot be bigger than N because there is not set of r-capacity null for $r > N$.*

According to Remark 2.14, one can compare Theorem 2.10 and the result of [20].

Remark 2.18 *Set $A(t) = t^r$ and $\Phi(t) = t^{q+1}$, then the problem studied is*

$$\begin{cases} -\mathrm{div}(a(x,\nabla u)) + |u|^{q-1}\, u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.16}$$

and the condition (2.9) (or (2.14)) is equivalent to

$$q > \frac{r(p-1)}{r-p}, \tag{2.17}$$

i.e. the same condition found in [20], therefore our Theorem 2.10 is a generalization of Theorem 1.6 of [20].

2.3 Linear problem

Let now study the linear case (where $p = 2$) with still $\Phi(t) = t^{q+1}$:

$$
\left\{ \begin{array}{ll} -\Delta u + |u|^{q-1}\, u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{2.18}
$$

If $A(t) = t^r$, according to Remark 2.14, (2.9) becomes $q > \frac{r}{r-2}$ (or $r > 2q'$). When $r = N$, one can see that Theorem 2.10 is thus weaker than the one of [9], where the condition is $q \geq \frac{N}{N-2}$. However if we set $A_r(t) = t^r \log^{r-1+\varepsilon}(e+t)$, according to Remark 2.15, (2.9) becomes $q \geq \frac{r}{r-2}$. Therefore the capacities cap_{1,A_r} give us the possibility to allow also the case $q = \frac{r}{r-2}$ in Theorem 2.10.

The second main result of the paper is the following one, obtained by extending in the Orlicz setting the nonlinear potential techniques of [1] (see Section 4):

Theorem 2.19 *Let $s > 1$, $0 < \beta s < N$, $\beta \in \mathbf{N}$. If*

$$
\int_0^1 (\widetilde{A})'(t^{1-\beta s})\, dt < \infty \tag{2.19}
$$

then

$$
\text{cap}_{1,A}(E) = 0 \Rightarrow \text{cap}_{\beta,s}(E) = 0
$$

Setting $\beta = 2$ and $s = q'$ in Theorem 2.19, we get the following : If

$$
\int_0 (\widetilde{A})'(t^{1-2q'})dt < \infty \tag{2.20}
$$

then

$$
\text{cap}_{1,A}(E) = 0 \Rightarrow \text{cap}_{2,q'}(E) = 0\,.
$$

As a consequence of the previous Theorem, taking particular cases of the parameters involved, we get the following (already known) remark (see [20]):

Remark 2.20 *Set* $A(t) = t^r$. *If* $q > \frac{r}{r-2}$ *(or equivalently* $r > 2q'$*, (2.9), (2.14) or (2.20)) in the Theorem 2.19, then we have*

$$\mathrm{cap}_{1,r}(E) = 0 \Rightarrow \mathrm{cap}_{2,q'}(E) = 0.$$

Moreover if λ *is a measure concentrated on a set* E *of null* $(1, r)$*-capacity then* λ *is not absolutely continuous with respect to the* $(2, q')$*-capacity.*

Let us go further studying the relation between these capacities :

Lemma 2.21 *Let* $A(t) = t^r \log^{r-1+\varepsilon}(e + t)$ *for some* $\varepsilon > 0$ *and* q *such that* $2q' < N$. *If* $q \geq \frac{r}{r-2}$ *that is (2.9) or (2.20), one has according to Theorem 2.19,*

$$\mathrm{cap}_{1,t^r \log^{r-1+\varepsilon}(e+t)}(E) = 0 \Rightarrow \mathrm{cap}_{2,q'}(E) = 0\,. \tag{2.21}$$

If $q < \frac{r}{r-2}$ *one has*

$$\mathrm{cap}_{2,q'}(E) = 0 \Rightarrow \mathrm{cap}_{1,2q'}(E) = 0 \Rightarrow \mathrm{cap}_{1,t^r \log^{r-1+\varepsilon}(e+t)}(E) = 0\,. \tag{2.22}$$

Proof. If $q \geq \frac{r}{r-2}$, using Theorem 2.19 whith $\beta = 2$ and $s = q'$, and $A(t) = t^r \log^{r-1+\varepsilon}(e + t)$, one get (2.21). If $q < \frac{r}{r-2}$ that is equivalent to $r < 2q'$, one has $A(t) \leq t^{2q'}$ near infinity thus

$$\mathrm{cap}_{1,t^r \log^{r-1+\varepsilon}(e+t)}(E) \leq c\,\mathrm{cap}_{1,2q'}(E)\,,$$

and according to Adams-Hedberg [1], Theorem 5.5.1, p. 148, one has

$$\mathrm{cap}_{1,2q'}(E) \leq c\,\mathrm{cap}_{2,q'}(E)\,.$$

■

The relation between Theorem 2.10 and our results about capacities is given by the following

Remark 2.22 *Let us consider (2.18),* $A(t) = t^r \log^{r-1+\varepsilon}(e + t)$ *for some* $\varepsilon > 0$. *If* q *is such that (2.9) is true, that is* $q \geq r/(r-2)$*, and* λ *concentrated on a set of null* $(1, t^r \log^{r-1+\varepsilon}(e + t))$*-capacity (and moreover of null* $(2, q')$*-capacity according to Lemma 2.21), then (2.18) has no solution in the sense* *of Theorem 2.10. If now q is such that (2.9) is false, that is $q < r/(r-2)$, and that λ is concentrated on a set of null $(1, t^r \log^{r-1+\varepsilon}(e+t))$-capacity but absolutely continuous with respect to the $(1, t^{2q'})$-capacity (this is possible since $r < 2q'$) and thus also to the $(2, q')$-capacity (according to Lemma 2.21), then there exist solutions of (2.18) (see Gallouët and Morel [14]).*

3 Proof of the nonexistence result

Before giving the proof of Theorem 2.10, we need to construct, as in [11], a sequence of suitable cut-off functions, built after λ and E (the proof of [20] works also for Sobolev-Orlicz spaces).

Lemma 3.1 *Let $\lambda = \lambda^+ - \lambda^-$ be a Radon measure concentrated on a set E of zero r-capacity, with $1 < r \le N$. Then for every $\delta > 0$ there exist two $C_c^\infty(\Omega)$ function ψ_δ^+ and ψ_δ^- such that*

$$0 \le \psi_\delta^+ \le 1\,,\ 0 \le \psi_\delta^- \le 1\,,\quad \|\nabla \psi_\delta^+\|_A \le \delta\,,\ \|\nabla \psi_\delta^-\|_A \le \delta\,, \tag{3.1}$$

$$0 \le \int_\Omega (1 - \psi_\delta^+)\, d\lambda^+ \le \delta\,,\quad 0 \le \int_\Omega (1 - \psi_\delta^-)\, d\lambda^- \le \delta\,, \tag{3.2}$$

$$0 \le \int_\Omega \psi_\delta^-\, d\lambda^+ \le \delta\,,\quad 0 \le \int_\Omega \psi_\delta^+\, d\lambda^- \le \delta\,. \tag{3.3}$$

Lemma 3.2 *Let $\rho > 0$, and let $\{v_n\}$ be a sequence of functions bounded in $M^{\Phi'}(\Omega)$. Suppose that, for every $k > 0$, we have*

$$\int_\Omega |\nabla T_k(v_n)|^p\, dx \le c\, k\,,$$

for some positive constant c. Then $\{|\nabla v_n|^{p-1}\}$ is bounded in $M^\Psi(\Omega)$, with

$$\Psi(s) = \frac{s^{p'}}{\Phi^{-1}(s^{p'})}\,.$$

Proof. We follow the lines of the proof of [6], Lemma 4.2. Let σ be a fixed positive real number. We have, for every $k > 0$,

$$\begin{aligned} \text{meas}\,\{|\nabla v_n| > \sigma\} &= \text{meas} \left\{ \begin{array}{c} |\nabla v_n| > \sigma \\ |v_n| \le k \end{array} \right\} + \text{meas} \left\{ \begin{array}{c} |\nabla v_n| > \sigma \\ |v_n| > k \end{array} \right\} \\ &\le \text{meas} \left\{ \begin{array}{c} |\nabla v_n| > \sigma \\ |v_n| \le k \end{array} \right\} + \text{meas}\,\{|v_n| > k\}\,. \end{aligned} \tag{3.4}$$

Moreover,

$$\text{meas}\left\{\begin{matrix} |\nabla v_n| > \sigma \\ |v_n| \le k \end{matrix}\right\} \le \frac{1}{\sigma^p}\int_\Omega |\nabla T_k(v_n)|^p\,dx \le c\,\frac{k}{\sigma^p}\,.$$

Since by the assumptions on v_n there exists a positive constant c such that

$$\text{meas}\,\{|v_n| > k\} \le \frac{c}{\Phi'(k)}\,,$$

(3.4) then implies

$$\text{meas}\,\{|\nabla v_n| > \sigma\} \le c\,\frac{k}{\sigma^p} + \frac{c}{\Phi'(k)}\,,$$

and this latter inequality holds for every $k > 0$. Minimizing on k, we get $c\,k\,\Phi'(k) = \sigma^p$ (recall that $k\,\Phi'(k) \ge \Phi(k)$ and $\Phi^{-1}(ck) \le c\,\Phi^{-1}(k)$, for all $c > 1$, thanks to the convexity of Φ)

$$\text{meas}\,\{|\nabla v_n| > \sigma\} \le \frac{c\,\Phi^{-1}(\sigma^p)}{\sigma^p}\,,$$

thus

$$\text{meas}\,\{|\nabla v_n|^{p-1} > \sigma\} \le \frac{c\,\Phi^{-1}(\sigma^{p'})}{\sigma^{p'}}\,,$$

which is the desired result. ■

Lemma 3.3 *Let Ψ and Θ be N-functions. If moreover*

$$\int^{+\infty} \frac{\Theta'(t)}{\Psi(t)}dt < +\infty$$

then one has

$$M^\Psi(\Omega) \subset L^\Theta(\Omega).$$

and for any $s > 0$ the following inequality hold:

$$\int_\Omega \Theta(|v|)\,dx \le \text{meas}\,(\Omega)\,\Theta(s) + \left(\sup_{t>0}\Psi(t)\,\text{meas}\,\{|v| > t\}\right)\int_s^{+\infty}\frac{\Theta'(t)}{\Psi(t)}dt$$

Proof. Let v be a function in $M^{\Psi}(\Omega)$. One has, for all $s > 0$,

$$\int_{\Omega} \Theta(|v|)\,dx = \int_0^{+\infty} \Theta'(t)\,\mathrm{meas}\,\{|v| > t\}dt$$
$$\leq \mathrm{meas}\,(\Omega)\,\Theta(s) + \int_s^{+\infty} \frac{\Theta'(t)}{\Psi(t)}\Psi(t)\,\mathrm{meas}\,\{|v| > t\}dt$$

from which the assertion follows. ■

Lemma 3.4 *Let $\{v_n\}$ be a sequence of $W_0^{1,p}(\Omega)$ functions such that*

$$\int_{\Omega} |\nabla T_k(v_n)|^p\,dx \leq c\,k\,,$$

for some positive constant c. Then there exists a subsequence, still denoted by v_n, and a measurable function v, such that v_n converges to v almost everywhere in Ω.

Proof. See [6], Proof of Theorem 6.1, Step 2. ■

Proof of Theorem 2.10.

We will follow [20] which has used some of the ideas contained in [11] when dealing with nonlinear elliptic equations with measure data.

Then, since the operator is monotone, there exists a unique solution u in $W_0^{1,p}(\Omega)$ of the following nonlinear elliptic problem (this result is well known and is a consequence of [18] ; it is, for example, proved in Theorem 2.9)

$$\text{(3.5)} \qquad \begin{cases} -\mathrm{div}(a(x,\nabla u)) + \Phi''(|u|)\,u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the sense that

$$\text{(3.6)} \qquad \int_{\Omega} a(x,\nabla u)\cdot\nabla\varphi\,dx + \int_{\Omega} \Phi''(|u|)\,u\,\varphi\,dx = \int_{\Omega} f\,\varphi\,dx\,,$$

for every φ in $W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ and for $\varphi = u$, so that $\Phi(|u|)$ (and $\Phi''(|u|)u^2$) belongs to $L^1(\Omega)$.

We define $\omega(n,m,\delta)$ any quantity (depending on n, m and δ) such that

$$\lim_{\delta\to 0^+}\ \lim_{m\to+\infty}\ \lim_{n\to+\infty} |\omega(n,m,\delta)| = 0\,.$$

Similarly, if the quantity we are considering does not depend one or more of the three parameters n, m and δ, we will omit the dependence from it in ω. For example, $\omega(n,\delta)$ is any quantity such that

$$\lim_{\delta\to 0^+}\lim_{n\to+\infty}|\omega(n,\delta)|=0\,.$$

Step 1: *A priori* estimates.

Since $T_k(u_n)$ is in $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, we can choose it as test function in the weak formulation of (2.10). We get, using (2.2), (2.6), and the boundedness of $\{g_n\}$ in $L^1(\Omega)$,

$$\alpha\int_\Omega|\nabla T_k(u_n)|^p\,dx+\int_\Omega\Phi''(|u_n|)|u_n|\,|T_k(u_n)|\,dx\le c\,k\,, \tag{3.7}$$

for some positive constant c. Dropping the first, nonnegative term of the left hand side of the preceding inequality, we have

$$k\int_{\{|u_n|\ge k\}}\Phi''(|u_n|)|u_n|\,dx\le\int_\Omega\Phi''(|u_n|)|u_n|\,|T_k(u_n)|\,dx\le c\,k\,,$$

so that

$$\int_{\{|u_n|\ge k\}}\Phi''(|u_n|)|u_n|\,dx\le c\,. \tag{3.8}$$

By (2.8) this implies

$$\Phi'(k)\,\mathrm{meas}\,\{|u_n|\ge k\}\le k\,\Phi''(k)\,\mathrm{meas}\,\{|u_n|\ge k\}\le c\,,$$

and so $\{u_n\}$ is bounded in $M^{\Phi'}(\Omega)$. Furthermore,

$$\int_{\{|u_n|<k\}}\Phi''(|u_n|)|u_n|\,dx\le k\,\Phi''(k)\,\mathrm{meas}\,(\Omega)\,,$$

and so, using (3.8),

$$\Phi''(|u_n|)\,u_n \text{ is bounded in } L^1(\Omega). \tag{3.9}$$

The boundedness of u_n in $M^{\Phi'}(\Omega)$, and Lemma 3.2, which can be applied since (3.7) also implies that

$$\int_\Omega|\nabla T_k(u_n)|^p\,dx\le c\,k\,, \tag{3.10}$$

yields

(3.11) $\{|\nabla u_n|^{p-1}\}$ is bounded in $M^\Psi(\Omega)$, with $\Psi(s) = \frac{s^{p'}}{\Phi^{-1}(s^{p'})}$.

Now let $\Theta_1 \in \Delta_2$ be an N-function verifying the assumption (2.11), and let $\Theta \in \Delta_2$ be any N-function "well dominated" by Θ_1. We formalize this domination writing

$$\Theta_1(t) = \varphi(\Theta(t)) \qquad \forall t \geq 0$$

for some φ increasing, continuous, such that $\lim_{t\to\infty} \varphi(t)/t = +\infty$. By (3.11) and Lemma 3.3 the set $\{\varphi(\Theta(|\nabla u_n|^{p-1}))\}$ is bounded in $L^1(\Omega)$, therefore the sequence

(3.12) $\Theta(|\nabla u_n|^{p-1})$ is equiintegrable.

On the other hand, using again (3.10), by Lemma 3.4, and up to some subsequence still denoted by u_n, u_n converges almost everywhere to a measurable function u, and so $T_k(u_n)$ converges almost everywhere to $T_k(u)$. Using (3.9) and Fatou lemma, one has $\Phi''(|u|)u \in L^1(\Omega)$.

Moreover, (3.10) implies that $\{T_k(u_n)\}$ is bounded in $W_0^{1,p}(\Omega)$, so that, by the weak lower semicontinuity of the norm, $T_k(u)$ belongs to $W_0^{1,p}(\Omega)$ for every $k > 0$, and thus u has a gradient ∇u in the sense of Definition 2.7.

As for the gradients of u_n, we remark that u_n is the solution of the equation $-\mathrm{div}(a(x, \nabla u_n)) = f_n^\oplus - f_n^\ominus + g_n - \Phi''(|u_n|)\, u_n$, and that the right hand side is bounded in $L^1(\Omega)$ by (2.6) and (3.9). By a result in [8], this implies that, up to subsequences,

(3.13) ∇u_n converges almost everywhere to ∇u.

From now on, we will suppose to have already extracted from u_n a subsequence (which we still denote by u_n), with the properties we have proved before. By (3.13) we have also

(3.14) $\Theta(|\nabla u_n|^{p-1})$ converges almost everywhere to $\Theta(|\nabla u|^{p-1})$.

By (3.12) and (3.14), we can apply Vitali's theorem, and we get $|\nabla u_n|^{p-1} \in L^\Theta(\Omega)$ and

$$\int_\Omega \Theta(|\nabla u_n|^{p-1})\, dx \to \int_\Omega \Theta(|\nabla u|^{p-1})\, dx. \tag{3.15}$$

By (3.13) and (3.15), applying the Fatou lemma to the sequence of nonnegative functions $c_\Theta(\Theta(|\nabla u|^{p-1}) + \Theta(|\nabla u_n|^{p-1})) - \Theta(||\nabla u|^{p-1} - |\nabla u_n|^{p-1}|)$, where

c_Θ is the constant appearing in the Δ_2 condition for Θ, we get

$$\int_\Omega \Theta(||\nabla u|^{p-1} - |\nabla u_n|^{p-1}|)\, dx \to 0 \tag{3.16}$$

from which, since $\Theta \in \Delta_2$, we get (see e.g. Theorem 1.3 p.8 of [19])

$$|\nabla u_n|^{p-1} \to |\nabla u|^{p-1} \quad \text{strongly in } L^\Theta(\Omega). \tag{3.17}$$

Notice that we obtained (3.17) for all $\Theta \in \Delta_2$ *well dominated* by some Θ_1 such that (2.11) holds. Such functions $\Theta \in \Delta_2$ verify condition (2.11), and, on the other hand, arguing as in [17] (see Chapter II, Section 8, n.1, p.60), it is easy to show that any Δ_2 N-function satisfying the condition (2.11) is *well dominated* by an N-function of the same type. The conclusion is that we have (3.17) for all Θ verifying (2.11).

Observe that, by the assumption (2.3) on a, the argument above shows also that

$$a(x, \nabla u_n) \to a(x, \nabla u) \quad \text{strongly in } (L^\Theta(\Omega))^N, \tag{3.18}$$

for every function $\Theta \in \Delta_2$ such that $\int^{+\infty} \frac{\Theta'(t)}{\Psi(t)} dt < \infty$. In particular, one can choose $\Theta = \widetilde{A}$ thanks to (2.9). Thus the last convergence is also in $L^1(\Omega)$.

Step 2: Energy estimates.

Let $\Psi_\delta = \psi_\delta^+ + \psi_\delta^-$, where ψ_δ^+ and ψ_δ^- are as in Lemma 3.1. Then

$$\int_{\{u_n > 2m\}} \Phi''(u_n)\, u_n\, (1 - \Psi_\delta)\, dx = \omega(n, m, \delta)\,, \tag{3.19}$$

and

$$\int_{\{u_n < -2m\}} \Phi''(|u_n|)|u_n|\, (1 - \Psi_\delta)\, dx = \omega(n, m, \delta)\,, \tag{3.20}$$

We will only prove (3.19), since the proof of (3.20) is identical. We choose $\beta_m(u_n)\,(1 - \Psi_\delta)$ as test function in the weak formulation of (2.10), where $\beta_m(s)$ is defined as

$$\beta_m(s) = \begin{cases} 0 & \text{if } s \le m, \\ \dfrac{s}{m} - 1 & \text{if } m < s \le 2m, \\ 1 & \text{if } s > 2m. \end{cases} \tag{3.21}$$

We obtain, using the fact that the derivative of $\beta_m(s)$ is different from zero only where $m < s < 2m$,

$$
\begin{aligned}
&\text{(A)} && \frac{1}{m}\int_{\{m<u_n<2m\}} a(x,\nabla u_n)\cdot\nabla u_n\,(1-\Psi_\delta)\,dx \\
&\text{(B)} && -\int_\Omega a(x,\nabla u_n)\cdot\nabla\Psi_\delta\,\beta_m(u_n)\,dx \\
&\text{(C)} && +\int_\Omega \Phi''(|u_n|)\,u_n\,\beta_m(u_n)\,(1-\Psi_\delta)\,dx \\
&\text{(D)} && =\int_\Omega f_n^\oplus\,\beta_m(u_n)\,(1-\Psi_\delta)\,dx \\
&\text{(E)} && -\int_\Omega f_n^\ominus\,\beta_m(u_n)\,(1-\Psi_\delta)\,dx \\
&\text{(F)} && +\int_\Omega g_n\,\beta_m(u_n)\,(1-\Psi_\delta)\,dx\,.
\end{aligned}
$$

We have, by (3.18), by Egorov theorem, and since $\beta_m(u_n)$ converges to $\beta_m(u)$ almost everywhere in Ω and in the weak$*$ topology of $L^\infty(\Omega)$,

$$-(B)=\int_\Omega a(x,\nabla u)\cdot\nabla\Psi_\delta\,\beta_m(u)\,dx+\omega(n)=\omega(n,m)\,,$$

and the last passage is due to the fact that $\beta_m(u)$ converges to zero in the weak$*$ topology of $L^\infty(\Omega)$ as m tends to infinity. For the same reason, we have

$$(F)=\omega(n,m)\,.$$

Finally, by (3.2) and (3.3),

$$
\begin{aligned}
(D)&\le\int_\Omega f_n^\oplus\,(1-\Psi_\delta)\,dx=\int_\Omega f_n^\oplus\,(1-\psi_\delta^+)\,dx+\int_\Omega f_n^\oplus\,\psi_\delta^-\,dx \\
&=\int_\Omega(1-\psi_\delta^+)\,d\lambda^+ +\int_\Omega\psi_\delta^-\,d\lambda^+ +\omega(n) \\
&=\omega(n,\delta)\,.
\end{aligned}
$$

Since (A) and $-$(E) are nonnegative, and since

$$(C)\ge\int_{\{u_n>2m\}}\Phi''(u_n)\,u_n\,(1-\Psi_\delta)\,dx\,,$$

we get (3.19).

Step 3: Passing to the limit.

We are now ready to conclude the proof of Theorem 2.10, showing that u is the entropy solution of (2.12) with datum g.

Let φ be a function in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, let $M = \|\varphi\|_{L^\infty(\Omega)}$, let $k > 0$, and choose $T_k(u_n - \varphi)(1 - \Psi_\delta)$ as test function in the weak formulation of (2.10). We get

$$
\begin{aligned}
&\text{(A)} && \int_\Omega a(x, \nabla T_k(u_n)) \cdot \nabla T_k(u_n - \varphi)\,(1 - \Psi_\delta)\,dx \\
&\text{(B)} && \quad - \int_\Omega a(x, \nabla u_n) \cdot \nabla \Psi_\delta\, T_k(u_n - \varphi)\,dx \\
&\text{(C)} && \quad + \int_\Omega \Phi''(|u_n|)\, u_n\, T_k(u_n - \varphi)\,(1 - \Psi_\delta)\,dx \\
&\text{(D)} && = \int_\Omega f_n^\oplus\, T_k(u_n - \varphi)\,(1 - \Psi_\delta)\,dx \\
&\text{(E)} && \quad - \int_\Omega f_n^\ominus\, T_k(u_n - \varphi)\,(1 - \Psi_\delta)\,dx \\
&\text{(F)} && \quad + \int_\Omega g_n\, T_k(u_n - \varphi)\,(1 - \Psi_\delta)\,dx\,.
\end{aligned}
$$

Using (3.18), (2.9), Lemma 3.3, one has the convergence of $a(x, \nabla u_n)$ to $a(x, \nabla u)$ in $L^{\widetilde{A}}(\Omega)$. Thus using (3.1), we get

$$
-(B) = \int_\Omega a(x, \nabla u) \cdot \nabla \Psi_\delta\, T_k(u - \varphi)\,dx + \omega(n) = \omega(n, \delta)\,.
$$

Using (3.2) and (3.3), we obtain

$$
|(D)| + |(E)| \le k \int_\Omega (f_n^\oplus + f_n^\ominus)\,(1 - \Psi_\delta)\,dx = \omega(n, \delta)\,.
$$

It is then easy to see that

$$
(F) = \int_\Omega g\, T_k(u - \varphi)\,dx + \omega(n, \delta)\,,
$$

so that we only have to deal with (A) and (B). Let $m > k + M$ be fixed. We then have

$$
\begin{aligned}
&\text{(G)} && \text{(C)} = \int_{\{-2m \le u_n \le 2m\}} \Phi''(|u_n|)\, u_n\, T_k(u_n - \varphi)\,(1 - \Psi_\delta)\,dx \\
&\text{(H)} && \quad + \int_{\{u_n > 2m\}} \Phi''(u_n)\, u_n\, k\,(1 - \Psi_\delta)\,dx \\
&\text{(I)} && \quad + \int_{\{u_n < -2m\}} \Phi''(|u_n|)\, |u_n|\, k\,(1 - \Psi_\delta)\,dx\,.
\end{aligned}
$$

It is easily seen that (recall that $\Phi''(|u|)\,u \in L^1(\Omega)$)

$$
\begin{aligned}
(G) &= \int_{\{-2m\le u\le 2m\}} \Phi''(|u|)\,u\,T_k(u-\varphi)\,(1-\Psi_\delta)\,dx + \omega(n) \\
&= \int_\Omega \Phi''(|u|)\,u\,T_k(u-\varphi)\,(1-\Psi_\delta)\,dx + \omega(n,m) \\
&= \int_\Omega \Phi''(|u|)\,u\,T_k(u-\varphi)\,dx + \omega(n,m,\delta)\,.
\end{aligned}
$$

We then have, by (3.19),

$$
(H) = k\int_{\{u_n>2m\}} \Phi''(u_n)\,u_n\,(1-\Psi_\delta)\,dx = \omega(n,m,\delta)\,,
$$

and, by (3.20),

$$
(I) = k\int_{\{u_n<-2m\}} \Phi''(|u_n|)\,|u_n|\,(1-\Psi_\delta)\,dx = \omega(n,m,\delta)\,,
$$

so that

$$
(C) = \int_\Omega \Phi''(|u|)\,u\,T_k(u-\varphi)\,dx + \omega(n,\delta)\,.
$$

Finally, we have

$$
\begin{aligned}
&\text{(J)} \qquad (A) = \int_\Omega [a(x,\nabla u_n) - a(x,\nabla\varphi)]\cdot\nabla T_k(u_n-\varphi)\,(1-\Psi_\delta)\,dx \\
&\text{(K)} \qquad\qquad + \int_\Omega a(x,\nabla\varphi)\cdot\nabla T_k(u_n-\varphi)\,(1-\Psi_\delta)\,dx\,.
\end{aligned}
$$

Since the integrand function in (J) is nonnegative, and converges almost everywhere in Ω to $[a(x,\nabla u) - a(x,\nabla\varphi)]\cdot\nabla T_k(u-\varphi)$, as n tends to infinity and then δ tends to zero, Fatou lemma implies

$$
\int_\Omega [a(x,\nabla u) - a(x,\nabla\varphi)]\cdot\nabla T_k(u-\varphi)\,dx \le \liminf_{\delta\to 0^+}\ \liminf_{n\to+\infty}\,(J)\,.
$$

Moreover, since $a(x,\nabla\varphi)$ belongs to $(L^{p'}(\Omega))^N$, we have

$$
(K) = \int_\Omega a(x,\nabla\varphi)\cdot\nabla T_k(u-\varphi)\,dx + \omega(n,\delta)\,,
$$

so that, putting together the results for (J) and (K), we have

$$
\int_\Omega a(x,\nabla u)\cdot\nabla T_k(u-\varphi)\,dx \le \liminf_{\delta\to 0^+}\ \liminf_{n\to+\infty}\,(A)\,.
$$

Summing up the results we have obtained so far, we have

$$\int_\Omega a(x,\nabla u)\cdot\nabla T_k(u-\varphi)\,dx + \int_\Omega \Phi''(|u|)\,u\,T_k(u-\varphi)\,dx \le \int_\Omega g\,T_k(u-\varphi)\,dx\,,$$

and so u is the entropy solution of (2.12). Observe that, since the solution u does not depend on the subsequences we have extracted, then the whole sequence u_n converges to u.

To conclude the proof of the theorem, it only remains to prove (2.13). In order to do this, we choose a test function $\varphi \in C_c^\infty(\Omega)$ in the weak formulation of (2.10). We get

$$\int_\Omega a(x,\nabla u_n)\cdot\nabla\varphi\,dx + \int_\Omega \Phi''(|u_n|)\,u_n\,\varphi\,dx = \int_\Omega (f_n+g_n)\,\varphi\,dx\,.$$

Thanks to (3.18), and to the assumptions on f_n and g_n, we have

$$\int_\Omega \Phi''(|u_n|)\,u_n\,\varphi\,dx = -\int_\Omega a(x,\nabla u)\cdot\nabla\varphi\,dx + \int_\Omega g\,\varphi\,dx + \int_\Omega \varphi\,d\lambda + \omega(n)\,.$$

Since the entropy solution of (2.12) is also a distributional solution of the same problem, we have for the same φ,

$$\int_\Omega a(x,\nabla u)\cdot\nabla\varphi\,dx + \int_\Omega \Phi''(|u|)\,u\,\varphi\,dx = \int_\Omega g\,\varphi\,dx\,,$$

and so we have proved that (2.13) holds for every φ in $C_c^\infty(\Omega)$. Since $\Phi''(|u_n|)\,u_n$ is bounded in $L^1(\Omega)$, (2.13) can then be extended by density to the functions in $C_c^0(\Omega)$. ∎

4 Comparison between Orlicz-type capacities

We need to make some preliminary considerations about the growth of N-functions. Let us begin with the following

The assumption (2.1) made in Definition 2.1 is a way to describe that a certain growth is between two powers. Let us recall it for A a N-function

$$c_1 \min(s^{q_1-1}, s^{q_2-1})A'(t) \le A'(st) \le c_2 \max(s^{q_1-1}, s^{q_2-1})A'(t)$$

for all $s, t > 0$. It implies all the following inequalities:

$$c_1 \min(s^{q_1}, s^{q_2})A(t) \le A(st) \le c_2 \max(s^{q_1}, s^{q_2})A(t) \tag{4.1}$$

(4.2) $$c_1 \min(s^{\frac{q_1}{q_1-1}}, s^{\frac{q_2}{q_2-1}})\widetilde{A}(t) \le \widetilde{A}(st) \le c_2 \max(s^{\frac{q_1}{q_1-1}}, s^{\frac{q_2}{q_2-1}})\widetilde{A}(t)$$

(4.3) $$c_1 \min(s^{\frac{1}{q_1}}, s^{\frac{1}{q_2}})A^{-1}(t) \le A^{-1}(st) \le c_2 \max(s^{\frac{1}{q_1}}, s^{\frac{1}{q_2}})A^{-1}(t)$$

(4.4) $$c_1 \min(s^{\frac{q_1-1}{q_1}}, s^{\frac{q_2-1}{q_2}})\widetilde{A}^{-1}(t) \le \widetilde{A}^{-1}(st) \le c_2 \max(s^{\frac{q_1-1}{q_1}}, s^{\frac{q_2-1}{q_2}})\widetilde{A}^{-1}(t)$$

(4.5) $$c_1 \min(s^{\frac{1}{q_1-1}}, s^{\frac{1}{q_2-1}})\widetilde{A}'(t) \le \widetilde{A}'(st) \le c_2 \max(s^{\frac{1}{q_1-1}}, s^{\frac{1}{q_2-1}})\widetilde{A}'(t)$$

for all $s, t > 0$. The constants c_1, c_2 need not to be the same in each line. We remark that the inequalities stated above are not equivalent; however, with the help of the arguments given in [21], it can be proved that 2.1 implies them all. Remark that (4.1) implies that $A \in \Delta_2$.

We give now some basic definitions, and fix some notation, borrowed mainly from the book by Adams and Hedberg [1].

If $f \in L^1$, its Fourier transform is the bounded and continuous function

$$\mathcal{F}f(\xi) = \int_{\mathbf{R}^N} f(x)e^{-ix\xi}dx$$

$\mathcal{F}$ can be extended by continuity to a bijection on L^2 (Plancherel's theorem). The Bessel kernel is defined by

$$G_\alpha = \mathcal{F}^{-1}\left((1+|\xi|^2)^{\frac{\alpha}{2}}\right) \qquad (\alpha \in \mathbf{R})$$

It can be shown that the following integral formula holds:

$$G_\alpha(x) = \frac{1}{(4\pi)^{\alpha/2}\Gamma(\alpha/2)}\int_0^\infty t^{(\alpha-N)/2}e^{-\pi|x|^2/t-t/(4\pi)}\frac{dt}{t} \qquad (\alpha > 0)$$

Moreover, G_α is positive and integrable over $\mathbf{R}^N$.

Let $\mu \in \mathcal{M}^+(K)$, and let g be an nonnegative mesurable function. The convolution $g * \mu$ is defined by

$$g * \mu(x) = \int_K g(x-y)d\mu(y)$$

and the following equality holds (where $\check{g}(-x) = g(x)$) :

(4.6) $$\int_{\mathbf{R}^N}(g*\mu)f dx = \int_K(\check{g}*f)d\mu = \int_{\mathbf{R}^N}(\check{g}*f)d\mu$$

The Hardy-Littlewood fractional maximal function of a measure μ for $0 \le \alpha < N$, $\delta > 0$, is defined by

$$M_{\alpha,\delta}\mu(x) = \sup_{0<r\le\delta}\frac{\mu(B(x,r))}{|B(x,r)|^{(N-\alpha)/N}}.$$

In the sequel we will use the following inequality, trivial in the context of Lebesgue spaces:

Lemma 4.1 *The following inequality hold:*

$$A(\|f\|_A) \le \Psi_A\left(\int_\Omega A(|f|)dx\right) \tag{4.7}$$

where

$$\Psi_A(s) = \sup_{t>0} A\left(\frac{t}{A^{-1}\left(\frac{A(t)}{s}\right)}\right).$$

Moreover, the following bounds for Ψ_A *hold:*

$$c_1 \min\left(A(s^{1/q_1}), A(s^{1/q_2})\right) \le \Psi_A(s) \le c_2 \max\left(A(s^{1/q_1}), A(s^{1/q_2})\right). \tag{4.8}$$

Proof. By definition of Ψ_A we have

$$A\left(\frac{t}{A^{-1}\left(\frac{A(t)}{s}\right)}\right) \le \Psi_A(s) \qquad \forall s,t>0$$

or, equivalently,

$$\frac{t}{A^{-1}\left(\frac{A(t)}{s}\right)} \le A^{-1}(\Psi_A(s)) \qquad \forall s,t>0$$

$$\frac{t}{A^{-1}(\Psi_A(s))} \le A^{-1}\left(\frac{A(t)}{s}\right) \qquad \forall s,t>0$$

$$A\left(\frac{t}{A^{-1}(\Psi_A(s))}\right) \le \frac{A(t)}{s} \qquad \forall s,t>0$$

and therefore, replacing t by $|f(x)|$ and s by $\int_\Omega A(|f(x)|)dx$ and integrating over Ω

$$\int_\Omega A\left(\frac{|f(x)|}{A^{-1}\left(\Psi_A\left(\int_\Omega A(|f(x)|)dx\right)\right)}\right) \le 1$$

By definition of Orlicz norm we deduce

$$\|f\|_A \le A^{-1}\left(\Psi_A\left(\int_\Omega A(|f(x)|)dx\right)\right)$$

from which the first part of the assertion follows. The two bounds for Ψ_A can be proved in the same way, we will show only the upper one. By (4.1) we have

$$\sigma t \leq c_2 A^{-1}(\max(\sigma^{q_1}, \sigma^{q_2})A(t)) \qquad \forall \sigma, t > 0$$

Setting

$$\max(\sigma^{q_1}, \sigma^{q_2}) = 1/s \Leftrightarrow \sigma = \min(s^{-1/q_1}, s^{-1/q_2})$$

we have

$$\min(s^{-1/q_1}, s^{-1/q_2})t \leq c_2 A^{-1}(\tfrac{A(t)}{s}) \qquad \forall s, t > 0$$

$$\frac{t}{A^{-1}(\frac{A(t)}{s})} \leq c_2 \max(s^{1/q_1}, s^{1/q_2}) \qquad \forall s, t > 0$$

thus, using (4.1),

$$A\left(\frac{t}{A^{-1}(\frac{A(t)}{s})}\right) \leq \tilde{c}_2 \max(A(s^{1/q_1}), A(s^{1/q_2})) \qquad \forall s, t > 0$$

from which the assertion follows. ■

Remark 4.2 Note that Ψ_A in (4.7) is an increasing function, such that $\Psi(0+) = 0$. In the following we will use its natural extension in 0, by setting $\Psi(0) = 0$.

Remark 4.3 Inequality (4.7) can be related to the inequality for Jensen means proved in [13].

We will use in the sequel also the following Theorem, which is part of Theorem 1 of [3]:

Theorem 4.4 *Let $0 < \alpha < N$, $\delta > 0$ and let K be compact in $\mathbf{R}^N$. The there exists a positive constant c such that*

$$\|G_\alpha * \mu\|_A \leq c \|M_{\alpha,\delta}\mu\|_A \qquad \forall \mu \in \mathcal{M}^+(K) \tag{4.9}$$

Let us now denote by $B_n(x)$, $n \in \mathbf{Z}$, the open ball with radius 2^{-n} centered at x, and by B_n the ball $B_n(0)$. We will call η the characteristic function for B_0: $\eta = \chi_{B_0}$ so that Supp $\eta = \overline{B_0}$, η is nonnegative, bounded, lower semicontinuous and $\eta(rx)$ is a decreasing function of $r > 0$ for any $x \in \mathbf{R}^N$.

We define η_n, $n \in \mathbf{Z}$, by setting $\eta_n(x) = 2^{nN}\eta(2^n x)$ so that Supp $\eta_n = \overline{B}_n$, $\int \eta_n dx = \int \eta dx$. Notice that

$$
\begin{aligned}
\eta_n * \mu(x) &= \int_{\mathbf{R}^N} \eta_n(x-y)d\mu(y) = \int_{\mathbf{R}^N} 2^{nN}\eta(2^n(x-y))d\mu(y) \\
&= \int_{B_n(x)} 2^{nN}\eta(2^n(x-y))d\mu(y) = 2^{nN}\int_{B_n(x)} d\mu(y) \\
&= 2^{nN}\mu(B_n(x))
\end{aligned}
\tag{4.10}
$$

The following lemmas state some inequalities, which can be shown by using the standard techniques used in [1]. We will prove them by completeness.

Lemma 4.5 *Let $0 < \alpha < N$ and $\mu \in \mathcal{M}^+(K)$, then there exists a constant c such that*

$$
c^{-1}M_{\alpha,1}\mu(x) \le \sup_{n\ge 0}(2^{-n\alpha}\eta_n * \mu(x)) \le c\, M_{\alpha,2}\mu(x). \tag{4.11}
$$

Proof. We have

$$
\begin{aligned}
M_{\alpha,1}\mu(x) &= \sup_{0<r\le 1}\frac{\mu(B(x,r))}{|B(x,r)|^{(N-\alpha)/N}} = \sup_{n\ge 0}\sup_{2^{-n-1}\le r\le 2^{-n}}\frac{\mu(B(x,r))}{|B(x,r)|^{(N-\alpha)/N}} \\
&\le \sup_{n\ge 0}\frac{\mu(B(x,2^{-n}))}{|B(x,2^{-n-1})|^{(N-\alpha)/N}} = \sup_{n\ge 0}\frac{\mu(B_n(x))}{[c_N(2^{-n-1})^N]^{(N-\alpha)/N}} \\
&= c\sup_{n\ge 0}2^{n(N-\alpha)}\mu(B_n(x)) = c\sup_{n\ge 0}(2^{-n\alpha}\eta_n * \mu(x))\,.
\end{aligned}
$$

The other inequality can be proved similarly. ∎

Lemma 4.6 *If $k > 0$ and $\mu \in \mathcal{M}^+(K)$, then the following inequality holds:*

$$
\eta_n * A(k\eta_n * \mu)(x) \le c_N A(2^{nN}k\mu(B_{n-1}(x)) \tag{4.12}
$$

Proof. We have

$$
\eta_n * A(k\eta_n * \mu)(x) = \int_{\mathbf{R}^N}\eta_n(x-y)A(k\eta_n * \mu)(y)dy
$$

and by (4.10)

$$
\eta_n * A(k\eta_n * \mu)(x) = \int_{\mathbf{R}^N}\eta_n(x-y)A(k2^{nN}\mu(B_n(y)))dy\,;
$$

since the last integral is in fact over $B_n(x)$ and $B_n(y) \subset B_{n-1}(x)\, \forall y \in B_n(x)$, we get

$$\begin{aligned} \eta_n * A(k\eta_n * \mu)(x) &\leq \int_{\mathbf{R}^N} \eta_n(x-y) A(k 2^{nN} \mu(B_{n-1}(x)))dy \\ &= A(k2^{nN}\mu(B_{n-1}(x))) \int_{\mathbf{R}^N} \eta_n(x-y)dy \end{aligned}$$

and the lemma is therefore proved. ∎

The main tool that will be used in the following is the generalization of the so-called Wolff's inequality (see [1], Theorem 4.5.2 p. 109) in the framework of Orlicz spaces. Even if in fact the proof is a generalization of that one given in [1], we show extensively the argument, because we think that in this case the refinement of the estimations is not completely standard.

Theorem 4.7 *Let $0 < \alpha < N$ and $\mu \in \mathcal{M}^+(K)$. The following inequality holds:*

$$A(\|G_\alpha * \mu\|_A) \leq c\, \Psi_A \left(\int_{\mathbf{R}^N} W^\mu_{\alpha,\tilde{A}}(x) d\mu \right) \tag{4.13}$$

where

$$W^\mu_{\alpha,\tilde{A}}(x) = \int_0^4 t^\alpha A' \left(\frac{\mu(B(x,t))}{t^{N-\alpha}} \right) \frac{dt}{t}$$

for some constant c depending on A, α, N but independent of μ.

Proof. By (4.9) we have

$$A(\|G_\alpha * \mu\|_A) \leq c\, A(\|M_{\alpha,1}\mu\|_A)$$

and therefore, by (4.7),

$$A(\|G_\alpha * \mu\|_A) \leq c\, \Psi_A \left(\int_{\mathbf{R}^N} A(|M_{\alpha,1}\mu|)dx \right).$$

Applying inequality (4.11) we get

$$A(\|G_\alpha * \mu\|_A) \leq c\, \Psi_A \left(\int_{\mathbf{R}^N} A\left(|\sup_{n\geq 0}(2^{-n\alpha}(\eta_n * \mu)(x))| \right) dx \right)$$

from which

$$\begin{aligned}
\Psi_A^{-1}\left(c^{-1}A(\|G_\alpha * \mu\|_A)\right) &\le \int_{\mathbf{R}^N} A\left(|\sup_{n\ge 0}(2^{-n\alpha}(\eta_n * \mu)(x))|\right) dx \\
&\le \int_{\mathbf{R}^N} \sum_{n=0}^{\infty} A(2^{-n\alpha}(\eta_n * \mu)(x))\, dx \\
&\le c\int_{\mathbf{R}^N} \sum_{n=0}^{\infty} A'(2^{-n\alpha}(\eta_n * \mu)(x))2^{-n\alpha}(\eta_n * \mu)(x)\, dx.
\end{aligned}$$

By (4.6) and (4.12) we get

$$\begin{aligned}
\Psi_A^{-1}\left(c^{-1}A(\|G_\alpha * \mu\|_A)\right) &\le c\int_{\mathbf{R}^N} \sum_{n=0}^{\infty} 2^{-n\alpha}\eta_n * A'(2^{-n\alpha}(\eta_n * \mu)(x))\, d\mu \\
&\le c\int_{\mathbf{R}^N} \sum_{n=0}^{\infty} 2^{-n\alpha} A'(2^{n(N-\alpha)}\mu(B_{n-1}(x)))\, d\mu.
\end{aligned}$$

On the other hand, setting

$$W^{\mu}_{\alpha,\tilde{A}}(x) = \int_0^4 t^\alpha A'\left(\frac{\mu(B(x,t))}{t^{N-\alpha}}\right)\frac{dt}{t}$$

we have

$$\begin{aligned}
W^{\mu}_{\alpha,\tilde{A}}(x) &= \sum_{n=0}^{\infty}\int_{2^{-n+1}}^{2^{-n+2}} t^\alpha A'\left(\frac{\mu(B(x,t))}{t^{N-\alpha}}\right)\frac{dt}{t} \\
&\ge \sum_{n=0}^{\infty}\int_{2^{-n+1}}^{2^{-n+2}} 2^{-n\alpha+\alpha} A'\left(\frac{\mu(B(x,2^{-(n-1)}))}{2^{-n(N-\alpha)+2(N-\alpha)}}\right)\frac{dt}{2^{-n+2}} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}\int_{2^{-n+1}}^{2^{-n+2}} 2^\alpha\cdot 2^{-n\alpha} A'\left(2^{-2(N-\alpha)}\cdot 2^{n(N-\alpha)}\mu(B_{n-1}(x))\right)\frac{dt}{2^{-n+1}} \\
&= 2^{\alpha-1}\sum_{n=0}^{\infty} 2^{-n\alpha} A'\left(2^{-2(N-\alpha)}\cdot 2^{n(N-\alpha)}\mu(B_{n-1}(x))\right) \\
&\ge 2^{\alpha-1}c_{A',N,\alpha}\sum_{n=0}^{\infty} 2^{-n\alpha} A'\left(2^{n(N-\alpha)}\mu(B_{n-1}(x))\right).
\end{aligned}$$

From the relations obtained, the assertion follows. ■

In order to fix some more notation, let us recall the definition of Hausdorff measure. Let $h(r)$ be an increasing function, defined ($\le +\infty$) for $r \ge 0$, and satisfying $h(0) = 0$. Let $E \subset \mathbf{R}^N$, and consider coverings of E by countable unions of (open or closed) balls $\{B(x_i, r_i)\}_{i=1}^{\infty}$ with radii $\{r_i\}_{i=1}^{\infty}$. Then for any ρ, $0 < \rho \leq \infty$, a set function $\Lambda_h^{(\rho)}$ is defined by

$$\Lambda_h^{(\rho)}(E) = \inf \sum_{i=1}^{\infty} h(r_i)$$

where the infimum is taken over all such coverings with $\sup_{i=1} r_i \leq \rho$. Clearly $\Lambda_h^{(\rho)}(E)$ is a decreasing function of ρ, so $\lim_{\rho \to 0} \Lambda_h^{(\rho)}(E)$ exists ($\leq +\infty$), and we can define

$$\Lambda_h(E) = \lim_{\rho \to 0} \Lambda_h^{(\rho)}(E).$$

This is the Hausdorff measure of E with respect to the function h. If $h(r) = r^\alpha$, we write Λ_α for Λ_{r^α}. The set function $\Lambda_h^{(\infty)}$ is called the Hausdorff capacity. The following results are known from [1].

Lemma 4.8 $\Lambda_h^{(\infty)}(E) = 0$ *if and only if* $\Lambda_h(E) = 0$

Theorem 4.9 *Let h be an increasing function on $[0, \infty[$ such that $h(0) = 0$, and let $K \subset \mathbf{R}^N$ be a compact set. Then*

$$\mu(K) \leq \Lambda_h^{(\infty)}(K)$$

for all $\mu \in \mathcal{M}^+(K)$ such that $\mu(B(x,r)) \leq h(r)$ for all balls $B(x,r)$. Furthermore, there is a constant $c > 0$, depending only on N, and a $\mu \in \mathcal{M}^+(K)$, satisfying $\mu(B(x,r)) \leq h(r)$ for all $B(x,r)$, such that

$$\Lambda_h^{(\infty)}(K) \leq c\, \mu(K)$$

Following [4], let us now give these definitions.

Definition 4.10 *Let E be any measurable subset of $\mathbf{R}^N$. The $(\alpha, A)-$capacity of E is defined as:*

$$\mathrm{Cap}_{\alpha,A}(E) = \inf \left\{ \|f\|_A : f \in L^A(\mathbf{R}^N), G_\alpha * f \geq \chi_E \right\}$$

Definition 4.11 *Let $E \subset K$ be a set measurable for all $\mu \in \mathcal{M}^+(K)$. Let us set*

$$D_{\alpha,A}(E) = \sup \left\{ \mu(E) : \mu \in \mathcal{M}^+(K), \mu \text{ concentrated on } E, \| G_\alpha * \mu\|_{\widetilde{A}} \leq 1 \right\}$$

The following theorem (which is a particular case of Theorem 11, part 2 of [4], see also [2]) holds:

Theorem 4.12 *Let K be a compact set in $\mathbf{R}^N$. Then*

$$A^{-1}(\mathrm{Cap}_{\alpha,A}(K)) = D_{\alpha,A}(K) = \sup_{\mu\in\mathcal{M}^+(K)} \frac{\mu(K)}{\|G_\alpha * \mu\|_{\widetilde{A}}}$$

This theorem represents a dual definition of the capacity $\mathrm{Cap}_{\alpha,A}$, which generalizes an analogous result for the $\mathrm{Cap}_{\alpha,r}$ capacity (see Theorem 2.2.7 of [1], p. 21).

We have now all the background in order to prove the following

Theorem 4.13 *Let $0 < \alpha < N$, h be an increasing function on $[0,\infty[$ such that $h(0) = 0$,*

$$\int_0^4 t^{\alpha-1}(\widetilde{A})'\left(\frac{h(t)}{t^{N-\alpha}}\right) dt = H < \infty$$

and let $E \subset \mathbf{R}^N$ be a set satisfying $\Lambda_h^{(\infty)}(E) > 0$. Then there exists a constant $c_A > 0$, independent of h and E, such that

$$\Lambda_h^{(\infty)}(E) \le \Theta(c_A A^{-1}(\mathrm{Cap}_{\alpha,A}(E)))$$

where $\Theta(t)$ is an increasing function such that $\Theta(0+) = 0$. In particular,

$$\mathrm{Cap}_{\alpha,A}(E) = 0 \Rightarrow \Lambda_h(E) = 0\,.$$

Proof. Let K be compact with $\Lambda_h^{(\infty)}(K) > 0$, and let $\mu \in \mathcal{M}^+(K)$ be given by Theorem 4.9, such that

$$\mu(B(x,t)) \le h(t) \text{ for all balls } B(x,t) \tag{4.14}$$

and

$$c^{-1}\Lambda_h^{(\infty)}(K) \le \mu(K) \le \Lambda_h^{(\infty)}(K) \tag{4.15}$$

By Wolff's inequality (Theorem 4.7)

$$\begin{aligned}\widetilde{A}(\|G_\alpha * \mu\|_{\widetilde{A}}) &\le c\,\Psi_{\widetilde{A}}\left(\int_{\mathbf{R}^N} W^\mu_{\alpha,A}(x)d\mu\right)\\ &= c\,\Psi_{\widetilde{A}}\left(\int_{\mathbf{R}^N}\left(\int_0^4 t^{\alpha-1}(\widetilde{A})'\left(\frac{\mu(B(x,t))}{t^{N-\alpha}}\right)dt\right)d\mu\right)\end{aligned}$$

and therefore, by (4.14),

$$\begin{aligned}\widetilde{A}(\|G_\alpha * \mu\|_{\widetilde{A}}) &\le c\,\Psi_{\widetilde{A}}\left(\int_{\mathbf{R}^N}\left(\int_0^4 t^{\alpha-1}(\widetilde{A})'\left(\frac{h(t)}{t^{N-\alpha}}\right)dt\right)d\mu\right)\\ &= c\,\Psi_{\widetilde{A}}\left(\int_{\mathbf{R}^N} H d\mu\right)\\ &= c\,\Psi_{\widetilde{A}}(H\mu(K))\end{aligned}$$

thus

$$\|G_\alpha * \mu\|_{\widetilde{A}} \le \widetilde{A}^{-1}(c\,\Psi_{\widetilde{A}}(H\mu(K)))\,. \tag{4.16}$$

On the other hand, by Theorem 4.12

$$A^{-1}(\mathrm{Cap}_{\alpha,A}(K)) \ge \frac{\mu(K)}{\|G_\alpha * \mu\|_{\widetilde{A}}}$$

and therefore, by (4.16),

$$A^{-1}(\mathrm{Cap}_{\alpha,A}(K)) \ge \frac{\mu(K)}{\widetilde{A}^{-1}(c\,\Psi_{\widetilde{A}}(H\mu(K)))}$$

and by (4.15) and Remark 4.2,

$$A^{-1}(\mathrm{Cap}_{\alpha,A}(K)) \ge c_{\widetilde{A}^{-1}}\frac{\Lambda_h^{(\infty)}(K)}{\widetilde{A}^{-1}(\Psi_{\widetilde{A}}(H\Lambda_h^{(\infty)}(K)))} = c_{\widetilde{A}^{-1}}\Theta^{-1}(\Lambda_h^{(\infty)}(K))$$

from which the assertion follows. The property $\Theta(0+) = 0$ is true because $\Theta^{-1}(t) = t/\widetilde{A}^{-1}(\Psi_{\widetilde{A}}(Ht))$ and therefore, from (4.8), we get (for t small)

$$\frac{t}{(Ht)^{1/q_2}} \le \Theta^{-1}(t) \le \frac{t}{(Ht)^{1/q_1}}$$

from which

$$H^{1/(q_1-1)}t^{q_1/(q_1-1)} \le \Theta(t) \le H^{1/(q_2-1)}t^{q_2/(q_2-1)}.$$

The last part follows from Lemma 4.8. Let us now extend the results to a general set E, not necessarily compact. There exists E' countable intersection of open sets such that $E \subset E'$ and $\Lambda_h^{(\infty)}(E) = \Lambda_h^{(\infty)}(E')$ and $\mathrm{Cap}_{\alpha,A}(E) = \mathrm{Cap}_{\alpha,A}(E')$ ($\mathrm{Cap}_{\alpha,A}$ is an outer capacity according to [4], and

$\Lambda_h^{(\infty)}$ is an outer measure according to [23]). Using the fact that $\Lambda_h^{(\infty)}$ satisfies the assumptions of Choquet's theorem (see [23] Chapter 2.7), one has

$$\Lambda_h^{(\infty)}(E') = \sup\left\{\Lambda_h^{(\infty)}(K),\ K \text{ compact},\ K \subset E'\right\} .$$

Moreover, using Theorem 9 of [4], one has also

$$\mathrm{Cap}_{\alpha,A}(E') = \sup\left\{\mathrm{Cap}_{\alpha,A}(K),\ K \text{ compact},\ K \subset E'\right\} .$$

Hence the results obtained for compact sets can be extend to general sets.

■

Remark 4.14 The proof of Theorem 4.13 follows the ideas used to prove the Theorem 5.1.13 of [1], page 137. We remark that, with respect to the original proof, our constant is rougher, but simpler, and the proof is slightly shorter. In our context this (small) simplification is possible because we don't need finer constants.

Let us recall now the following Theorem, proved in [1] (see Theorem 5.1.9, p. 134)

Theorem 4.15 *Let $s > 1$ and $0 < \beta s < N$, and let $E \subset \mathbf{R}^N$. Set $h(t) = t^{N-\beta s}$. Then there is c independent of E such that*

$$\mathrm{Cap}_{\beta,s}(E)^s \le c\,\Lambda_h^{(1)}(E)$$

and moreover

$$\Lambda_h(E) < \infty \Rightarrow \mathrm{Cap}_{\beta,s}(E) = 0$$

Let us now consider some relations between Definition 2.4, Definition 2.5 and Definition 4.10. Let us first consider the case $A(t) = t^r$, $r > 1$: Definition 4.10 reduces to

$$\mathrm{Cap}_{\alpha,r}(E) = \inf\left\{\|f\|_r : f \in L^r(\mathbf{R}^N),\ G_\alpha * f \ge \chi_E\right\}.$$

Denoting by $L^{\alpha,r}(\mathbf{R}^N)$ the Bessel potential spaces

$$L^{\alpha,r}(\mathbf{R}^N) = \left\{h : h = G_\alpha * f,\ f \in L^r(\mathbf{R}^N)\right\},$$

whose norm is given by $\|h\|_{\alpha,r} = \|f\|_r$, we can write also

$$\mathrm{Cap}_{\alpha,r}(E) = \inf\left\{\|h\|_{\alpha,r} : h \in L^{\alpha,r}(\mathbf{R}^N),\ h \geq \chi_E\right\}.$$

At this point we use the result of Calderon: for $\alpha \in \mathbf{N}$, $W^{\alpha,r}(\mathbf{R}^N) = L^{\alpha,r}(\mathbf{R}^N)$, $1 < r < \infty$, with equivalence of norms, i.e. there is a constant c such that for all f

$$c^{-1}\|f\|_{\alpha,r} \leq \|f\|_{W^{\alpha,r}} \leq c\|f\|_{\alpha,r} \tag{4.17}$$

Hence it is clear that $\mathrm{Cap}_{\alpha,\mathrm{r}}$ is equivalent to

$$\mathrm{Cap}'_{\alpha,r}(E, \mathbf{R}^N) = \inf\left\{\|h\|_{W^{\alpha,r}} : h \in C_c^\infty(\mathbf{R}^N),\ h \geq \chi_E\right\}.$$

Similarly, one can prove, for $\alpha = 1$, that the capacity defined in Definition 4.10 is equivalent to

$$\mathrm{Cap}'_{1,A}(E, \mathbf{R}^N) = \inf\left\{\|\nabla h\|_A + \|h\|_A : h \in C_c^\infty(\mathbf{R}^N),\ h \geq \chi_E\right\}.$$

This extension is possible because the proof of Calderon's theorem can be obtained in the context of Orlicz spaces by straightforward generalization of that one given in [24], see Chapter V, Section 3, Theorem 3. All the properties of Orlicz spaces needed in the proof are heredited from those ones true for Lebesgue spaces. We briefly list them and give references for their proofs. For separability properties (density of $C_0^\infty(\mathbf{R}^N)$ in $L^A(\mathbf{R}^N)$ and in $W^{1,A}(\mathbf{R}^N)$) we refer to Section 2 of [12]. For properties obtained by interpolation, we refer to the paper [15]. Finally, for the boundedness of the Riesz transforms R_j in Orlicz spaces we refer to the book by Kokilashvili and Krbec [16], Theorem 3.1.1 page 97.

Proof of Theorem 2.19: Let E be such that $\mathrm{cap}_{1,A}(E) = 0$, we will first prove that $\mathrm{Cap}_{1,A}(E) \leq c\,\mathrm{cap}_{1,A}(E)$ and thus $\mathrm{Cap}_{1,A}(E) = 0$. Let us consider K a compact set, let $\varphi \in C_c^\infty(\Omega)$ such that $\varphi \geq \chi_K$ thus $\|\nabla\varphi\|_A \leq \mathrm{cap}_{1,A}(K)$ such φ can be use in the definition of $\mathrm{Cap}'_{1,A}(K, \mathbf{R}^N)$ thus $\mathrm{Cap}'_{1,A}(K, \mathbf{R}^N) \leq c\,\mathrm{cap}_{1,A}(K)$ so $\mathrm{Cap}_{1,A}(K) \leq c\,\mathrm{cap}_{1,A}(K)$. Using now Theorem 2 and Theorem 9 of [4], one has for all open set U

$$\mathrm{Cap}_{1,A}(U) = \sup\left\{\mathrm{Cap}_{1,A}(K),\ K \text{ compact},\ K \subset U\right\},$$

and for all set B

$$\mathrm{Cap}_{1,A}(B) = \inf\left\{\mathrm{Cap}_{1,A}(U),\ U \text{ open},\ B \subset U\right\}.$$

Hence $\mathrm{Cap}_{1,A}(B) \leq c\,\mathrm{cap}_{1,A}(B)$, for all borelian B, thus $\mathrm{Cap}_{1,A}(E) = 0$.

According to (2.19), $h(t) = t^{N-\beta s}$ satisfies the hypotheses of Theorem 4.13 with $\alpha = 1$, thus $\Lambda_h(E) = 0$. Using now Theorem 4.15, we get $\mathrm{Cap}_{\beta,s}(E) = 0$. It remains now to prove that $\mathrm{cap}_{\beta,s}(E) = 0$.

Since $\mathrm{Cap}_{\beta,s}(E) = 0$, there exists a sequence U_n of open sets in $\mathbf{R}^N$ such that $E \subset U_n$ and $\mathrm{Cap}_{\beta,s}(U_n) \leq \frac{1}{n}$. Since the capacity is nondecreasing, we can suppose that $U_n \subset \Omega$ (by replacing U_n with $U_n \cap \Omega$, recall that $E \subset \Omega$). Hence there exists $h_n \in C_c^\infty(\mathbf{R}^N)$ such that $h \geq \chi_{U_n}$ and $\|h_n\|_{L^{\beta,s}} \leq \frac{2}{n}$. Moreover using the Calderon result, one has $\|h_n\|_{W^{\beta,s}} \leq \frac{c}{n}$. Let now be $\widetilde{K}$ a compact set in Ω, and $\widetilde{U}$ an open set such that $\widetilde{K} \subset \widetilde{U} \subset\subset \Omega$. There exists $\xi \in C_c^\infty(\Omega)$ such that $\xi \geq \chi_{\widetilde{U}}$. Hence $\xi h_n \in C_c^\infty(\Omega)$, $\xi h_n \geq \chi_{\widetilde{U} \cap U_n}$ and $\|\xi h_n\|_{W^{\beta,s}} \leq c\,\|h_n\|_{W^{\beta,s}} \leq \frac{c}{n}$. Thus for all $K \subset \widetilde{U} \cap U_n$, $\mathrm{cap}_{\beta,s}(K) \leq \frac{c}{n}$, hence $\mathrm{cap}_{\beta,s}(\widetilde{U} \cap U_n) \leq \frac{c}{n}$. Finally, since $E \cap \widetilde{K} \subset \widetilde{U} \cap U_n$, $\mathrm{cap}_{\beta,s}(E \cap \widetilde{K}) \leq \frac{c}{n}$ for all $n \in N$ so

$\mathrm{cap}_{\beta,s}(E \cap \widetilde{K}) = 0$. Since Ω is the union of increasing compact, one has $\mathrm{cap}_{\beta,s}(E) = 0$.

■

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References

[1] D.R. Adams, L.I. Hedberg, *Function spaces and potential theory*, Grundlehren der matematischen wissenschaften, **314**, Springer-Verlag, Berlin, 1996.

[2] N. Aissaoui, Bessel potentials in Orlicz spaces, *Rev. Mat. Univ. Complut. Madrid*, **10** (1997), 1, 55-79.

[3] N. Aissaoui, Some developments of Strongly Nonlinear Potential Theory, *Libertas Math.*, **19** (1999), 155-170.

[4] N. Aissaoui, A. Benkirane, Capacités dans les espaces d'Orlicz, *Ann. Sci. Math. Québec*, **18** (1994), 1, 1-23.

[5] P. Baras, M. Pierre, Singularités éliminables pour des équations semi-linéaires, *Ann. Inst. Fourier (Grenoble)*, **34** (1984), 185–206.

[6] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J.L. Vazquez, An L^1 theory of existence and uniqueness of nonlinear elliptic equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **22** (1995), 240–273.

[7] P. Bénilan, H. Brezis, M. Crandall, A semilinear ellitptic equation in $L^1(\mathbf{R}^N)$, *Ann. Sc. Norm. Sup. Pisa Cl. Sci.*, **2** (1975), 523–555.

[8] L. Boccardo, T. Gallouët, Nonlinear elliptic equations with right-hand side measures, *Comm. Partial Differential Equations*, **17** (1992), 641–655.

[9] H. Brezis, Nonlinear elliptic equations involving measures, in *Contributions to nonlinear partial differential equations (Madrid, 1981)*, 82–89, Res. Notes in Math., 89, Pitman, Boston, Mass.-London, 1983.

[10] G. Choquet, Theory of Capacities, Ann. Inst. Fourier (Grenoble), **5** (1953-1954), 131-295, Ch. 1, Thm 4.1, p. 142

[11] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, Renormalized solutions for elliptic equations with general measure data, *Ann. Scuola Norm. Sup. Pisa CL. Sci.*, **28** (1999), 741–808.

[12] T.K. Donaldson, N.S. Trudinger, Orlicz-Sobolev spaces and imbedding theorems, *J. Funct. Anal.*, **8** (1971), 52-75.

[13] A. Fiorenza, An inequality for Jensen Means, *Nonlinear Anal.*, **16** (1991), 2, 191-198.

[14] T. Gallouët, J.M. Morel, Resolution of a semilinear equation in L^1, *Proc. Roy. Soc. Edinburgh*, **96** (1984), 275–288.

[15] J. Gustavsson, J. Peetre, Interpolation of Orlicz spaces,
Studia Math., **60** (1977), 33-59.

[16] V. Kokilashvili, M. Krbec, *Weighted inequalities in Lorentz and Orlicz spaces*, World Scientific, 1991

[17] M.A. Krasnosel'skii, Ya. B. Rutickii, *Convex functions and Orlicz Spaces*, Noordhoff Ltd., 1961.

[18] J. Leray, J.-L. Lions, Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder, *Bull. Soc. Math. France*, **93** (1965), 97–107.

[19] L. Maligranda, *Orlicz Spaces and Interpolation*, Dep. de Matematica Univ. Estadual de Campinas, Campinas, Brasil, 1989

[20] L. Orsina, A. Prignet, Nonexistence of solutions for some nonlinear elliptic equations involving measures, *Proc. Roy. Soc. Edinburgh*, **130A** (2000), 167-187.

[21] L.E. Persson, Interpolation with a parameter function, *Math. Scand.*, **59** (1986), 199-222.

[22] M.M.Rao, Z.D.Ren *Theory of Orlicz Spaces*, Marcel Dekker, 1991.

[23] C.A. Rogers *Hausdorff Measures*, Cambridge University Press, 1970.

[24] E.M. Stein *Singular Integrals and Differentiability properties of functions*, Princeton University Press, 1970.