A VANISHING VISCOSITY APPROACH TO FRACTURE GROWTH IN A COHESIVE ZONE MODEL WITH PRESCRIBED CRACK PATH

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Abstract. The existence of crack evolutions based on critical points of the energy functional is proved, in the case of a cohesive zone model with prescribed crack path. It turns out that evolutions of this type satisfy a maximum stress criterion for the crack initiation. With an explicit example, it is shown that evolutions based on the absolute minimization of the energy functional do not enjoy this property.

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1. Introduction

In this paper we present a model for the study of the fracture growth in an elastic body when the cohesive forces acting between the lips of the crack are not negligible. We consider the case in which the evolution is driven by a time-dependent boundary displacement on a fixed portion of the boundary. At the moment, due to technical difficulties, we are forced to assume *a priori* that the crack is contained in a prescribed region.

In order to analyze the behaviour of the system, one can follow the time evolution of absolute minimizers of the energy. This strategy has been used by Dal Maso and Zanini in [11]. Actually, it turns out that it is not always realistic to expect the energy to be minimized at every fixed time. Indeed, it may happen that global minimization leads the system to change instantaneously in a very drastic way, jumping into a very far apart configuration. Thus, it seems reasonable to introduce a selection criterion which possibly avoids such a situation. To this aim, we will consider evolutions of critical points of the energy, taking inspiration from [7], where a vanishing viscosity approach is introduced in the context of plasticity with softening. The same technique was also used in [12], for the study of rate-independent finite-dimensional systems. In the framework of fracture mechanics, the first step in this direction has been taken by Dal Maso and Toader in [10]. Recently, Negri and Ortner (see [17]) presented an evolution based on local minimizers in the case of a connected crack with prescribed path. In [19], Toader and Zanini use a vanishing viscosity approach to handle the same problem. To the best of our knowledge, in this paper the ideas introduced in [12], [7] and [19] are applied for the first time to the case of a cohesive zone model.

We restrict our analysis to the case of *generalized antiplanar shear*. More precisely, let Ω be a bounded open set in $\mathbb{R}^N$, with Lipschitz boundary. We assume that the reference configuration is the infinite cylinder $\Omega \times \mathbb{R}$, and that the displacement $U : \Omega \times \mathbb{R} \to \mathbb{R}^{N+1}$ has the special form $U(x_1, \dots, x_N, x_{N+1}) = (0, \dots, 0, u(x_1, \dots, x_N))$, with $u : \Omega \to \mathbb{R}$. We assume also that the crack path in the reference configuration is contained in $(\Gamma \cap \overline{\Omega}) \times \mathbb{R}$, where $\Gamma \subset \mathbb{R}^N$ is a Lipschitz closed set such that $0 < \mathcal{H}^{N-1}(\Gamma \cap \overline{\Omega}) < +\infty$ and $\Omega \setminus \Gamma = \Omega^+ \cup \Omega^-$, with $\Omega^\pm$ disjoint open connected sets with Lipschitz boundary. When speaking about bulk and surface energy, we will refer to a finite portion of the cylinder, obtained by intersection with two horizontal hyperplanes separated by a unit distance. Although the case of a planar set Ω is the most interesting from the point of view of applications, no further relevant technicalities arise in considering an arbitrary $N \geq 2$.

Let us fix a time interval $[0, T]$, with $T > 0$. In the situation we consider, the evolution is driven by a time dependent displacement $w : [0, T] \to H^1(\Omega)$ imposed on a fixed portion $\partial_D \Omega$ of the boundary $\partial\Omega$. We assume that $\partial_D\Omega$ is well-separated from Γ and that its intersections with $\partial\Omega^+$ and $\partial\Omega^-$ have positive $(N-1)$-dimensional measure.

Let us now introduce the energy functional. We suppose that the unbroken part of Ω can be described in the context of linearized elasticity, so that the stored elastic energy associated to a displacement $u \in H^1(\Omega \setminus \Gamma)$ is:

$$\frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u|^2 dx.$$

In order to express the work spent to create a fracture, we need some preliminary notations. Let $u^{\pm}$ denote the trace on Γ of the restriction of u to $\Omega^{\pm}$, and let $[u]$ denote the jump $u^+ - u^-$ of u across Γ. The crack is represented by the set

$$J_u := \{x \in \Gamma : [u](x) \neq 0\}.$$

Its contribution to the energy, according to Barenblatt's cohesive zone model (see [4]), can be written as

$$\int_\Gamma g(|[u]|)\, d\mathcal{H}^{N-1},$$

where $g : [0,+\infty) \to [0,+\infty)$ is a C^1, nondecreasing, bounded, concave function with $g(0) = 0$ and $\sigma := g'(0^+) \in (0,+\infty)$. Here $g(|[u]|)$ is the energy per unit area spent to create a crack with opening $|[u]|$. Moreover, $g'(|[u]|)$ gives the force per unit area acting between the lips of the crack whose displacements are u^+ and u^-, respectively. Typically, this force decreases with the distance and hence g is concave. Since in practise the cohesive interactions have finite range, we assume g to be bounded. Therefore, the total energy associated to a displacement $u \in H^1(\Omega \setminus \Gamma)$ is given by

$$E(u) := \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u|^2 dx + \int_\Gamma g(|[u]|)\, d\mathcal{H}^{N-1}. \tag{1.1}$$

To keep the mathematical formulation as simple as possible we will neglect irreversibility. Nevertheless, this is the subject of the paper [5].

In order to give an idea of the vanishing viscosity approach we start by describing the strategy in a more general setting. Given a time-dependent functional $\mathbf{F}(u,t)$ defined for u in a Banach space Y and for $t \in [0,T]$, an evolution of critical points is a function $u : [0,T] \to Y$ which satisfies

$$0 \in \partial_u \mathbf{F}(u(t),t) \qquad \text{for a.e. } t \in [0,T], \tag{1.2}$$

where $\partial_u \mathbf{F}$ denotes the subdifferential of $\mathbf{F}$ with respect to u. The existence of such an evolution is proved by a singular perturbation method. That is, for every $\varepsilon > 0$ one considers the ε-gradient flow

$$-\varepsilon \dot{u}^\varepsilon(t) \in \partial_u \mathbf{F}(u^\varepsilon(t),t) \tag{1.3}$$

with initial datum $u^\varepsilon(0) = u_0$, where u_0 is a critical point of $\mathbf{F}(\cdot,0)$. Under suitable regularity assumptions, as $\varepsilon \to 0$ the solutions u^ε converge (in a sense to be specified) to a function u such that (1.2) holds.

Let us explain in detail this approach in our case. We apply the previous scheme to $Y = L^2(\Omega)$ and

$$\mathbf{F}(u,t) = \begin{cases} E(u) & \text{for } u \in H^1(\Omega\setminus\Gamma) \text{ and } u = w(t) \text{ on } \partial_D\Omega, \\ +\infty & \text{otherwise in } L^2(\Omega), \end{cases}$$

where E is the functional defined by (1.1). Note that in this case the functional depends on time only through the prescribed boundary conditions.

We start by observing that a minimizer $u(t)$ of (1.1) at time t is a weak solution (see Proposition 3.2) of

$$\begin{cases} \Delta u(t) = 0 & \text{in } \Omega \setminus \Gamma, \\ u(t) = w(t) & \text{on } \partial_D \Omega, \\ \partial_\nu u(t) = 0 & \text{on } \partial\Omega \setminus \partial_D \Omega, \\ \partial_\nu u^+(t) = \partial_\nu u^-(t) & \text{on } \Gamma, \\ |\partial_\nu u(t)| \leq \sigma & \text{on } \Gamma \setminus J_{u(t)}, \\ \partial_\nu u(t) = g'(|[u(t)]|)\,\mathrm{sgn}[u(t)] & \text{on } J_{u(t)}, \end{cases} \tag{1.4}$$

where with ν we denote both the inner unit normal to Ω and to Ω^+, $\mathrm{sgn}\cdot$ denotes the sign function, and $\sigma = g'(0^+)$. Any function u satisfying (1.4) will be called a *critical point* of (1.1) at time t .

Let now u_0 be a critical point of (1.1) at time $t = 0$. It turns out that a solution u^ε of (1.3) in the present situation is given by a weak solution $u^\varepsilon \in H^1((0,T); L^2(\Omega)) \cap L^\infty((0,T); H^1(\Omega \setminus \Gamma))$ of

$$\begin{cases} \Delta u^\varepsilon(t) = \varepsilon \dot{u}^\varepsilon(t) & \text{in } \Omega \setminus \Gamma, \\ u^\varepsilon(t) = w(t) & \text{on } \partial_D \Omega, \\ \partial_\nu u^\varepsilon(t) = 0 & \text{on } \partial\Omega \setminus \partial_D \Omega, \\ \partial_\nu u^\varepsilon(t)\,|_{\Omega^+} = \partial_\nu u^\varepsilon(t)\,|_{\Omega^-} & \text{on } \Gamma, \\ |\partial_\nu u^\varepsilon(t)| \leq \sigma & \text{on } \Gamma \setminus J_{u^\varepsilon(t)}, \\ \partial_\nu u^\varepsilon(t) = g'(|[u^\varepsilon(t)]|)\,\mathrm{sgn}[u^\varepsilon(t)] & \text{on } J_{u^\varepsilon(t)}, \end{cases} \tag{1.5}$$

such that $u^\varepsilon(0) = u_0$. The existence of a solution of (1.5) is proved (Theorem 4.8) by time discretization, solving suitable incremental minimum problems. Uniqueness is shown for $g \in C^{1,1}$ (Theorem 4.5), but it is not known for the general case $g \in C^1$. We will call *variational parabolic evolution* with initial datum u_0 and boundary condition w every solution of (1.5) which can be obtained by this time discretization procedure.

We show (Theorem 4.13) that, given a family $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$ of variational parabolic evolutions with initial condition u_0 and boundary datum w, parametrized by the viscosity parameter $\varepsilon \in (0,1)$, there exists a bounded measurable function $u : [0,T] \to H^1(\Omega \setminus \Gamma)$, with $u(0) = u_0$, such that the following properties hold:

- *approximability*: for every $t \in [0,T]$ there exists a sequence $\varepsilon_n(t) \to 0^+$ such that

$$u^{\varepsilon_n(t)}(t) \rightharpoonup u(t) \text{ weakly in } H^1(\Omega \setminus \Gamma); \tag{1.6}$$

- *stationarity*: for a.e. $t \in [0,T]$ the function $u(t)$ is a critical point for E at time t;
- *energy inequality*: for every $t \in [0,T]$

$$E(u(t)) \leq E(u(0)) + \int_0^t \int_{\Omega \setminus \Gamma} \nabla u(s) \cdot \nabla \dot{w}(s)\, dx\, ds. \tag{1.7}$$

We will call any such function u an *approximable quasistatic evolution* with initial condition u_0 and boundary datum w.

In the second part of the paper we study the properties of such evolutions.
In Theorem 4.14 we show that under monotone loadings, when Γ is contained in a hyperplane and Ω is symmetric with respect to Γ, the function $t \mapsto |[u(t)](x)|$ is nondecreasing for $\mathcal{H}^{N-1}$-a.e. $x \in \Gamma$. This result can be interpreted as some kind of irreversibility for the crack growth in particular situations.

The second property we consider is the fracturing time. To this aim, we introduce the *elastic evolution* $z : [0,T] \to H^1(\Omega)$, defined as the solution of

$$\begin{cases} \Delta z(t) = 0 & \text{in } \Omega \setminus \Gamma, \\ z(t) = w(t) & \text{on } \partial_D \Omega, \\ \partial_\nu z(t) = 0 & \text{on } \partial\Omega \setminus \partial_D \Omega, \end{cases}$$

for every $t \in [0,T]$. It turns out that $z(t)$ is a critical point of (1.1) at time t if and only if $\|\partial_\nu z(t)\|_{L^\infty(\Gamma)} \le \sigma$.

We are able to provide a crack initiation criterion, showing that $\sigma = g'(0^+)$ represents the *maximum sustainable stress* along Γ.

More precisely, we prove that if $t^* \in (0,T]$ is such that $\sup_{t\in[0,t^*]} \|\partial_\nu z(t)\|_{L^\infty(\Gamma)} < \sigma$, then every approximable quasistatic evolution with initial datum $u_0 = z(0)$ coincides with z for $t \in [0,t^*]$ (Theorem 4.15).

This result agrees with physical experience. Indeed, it is very well known in the engineering literature that cracks appear just when the stress reaches the value $\sigma = g'(0^+)$. The proof is obtained by studying the behaviour of absolute minimizers of the incremental minimum problems. For this reason, we use a calibration technique for free discontinuity problems (see [1, 16]).

With an explicit example (Section 9), we show that the crack initiation criterion is not satisfied by the evolution of absolute minimizers. The same example proves that in (1.7) strict inequality may occur. In particular, some energy can be dissipated passing from one branch of critical points to another one. Instead, it is known that evolutions of absolute minimizers always satisfy equality, even in the irreversible case (see [11]).

The outline of the paper is as follows. In Section 2 we fix the notations and the setting of the problem. Section 3 is devoted to the detailed study of the Euler-Lagrange conditions for the functional (1.1), while in Section 4 we state the main results. Sections 5 contains the existence and (in case $g \in C^{1,1}$) uniqueness results for variational parabolic evolutions. We show the existence of approximable quasistatic evolutions in Section 6. The irreversibility under monotone boundary conditions is the subject of Section 7, while Section 8 contains the proof of the crack initiation criterion. In Section 9 we provide an explicit example, in which approximable quasistatic evolutions and absolute minimizers evolutions are compared. The most technical part of the proof of the crack initiation criterion, in which we construct a calibration, is postponed to the Appendix.

2. Setting of the problem

In this section we give some basic definitions and we introduce the setting of the problem. We will use the following notations:

- $\mathcal{L}^k$ is the Lebesgue measure in $\mathbb{R}^k, k \in \mathbb{N}$;
- $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$.

For every set $A \subset \mathbb{R}^N$:

- 1_A is the characteristic function of A;
- A^c is the complement of A in $\mathbb{R}^N$;
- $\mathcal{D}'(A)$ is the space of distributions on A.

Through the whole chapter Ω denotes a bounded open set in $\mathbb{R}^N$, $N \ge 2$, with Lipschitz boundary. Moreover, $\Gamma \subset \mathbb{R}^N$ is a Lipschitz closed set such that $0 < \mathcal{H}^{N-1}(\Gamma\cap\overline{\Omega}) < +\infty$ and $\Omega\setminus\Gamma = \Omega^+\cup\Omega^-$, with $\Omega^\pm$ open connected sets with Lipschitz boundary and $\Omega^+\cap\Omega^- = \varnothing$. We will prescribe time dependent boundary displacements on $\partial_D\Omega \subset \partial\Omega$, where

$$\partial_D\Omega = \Lambda_D^+ \cup \Lambda_D^-,$$

with Λ_D^+ and Λ_D^- nonempty relatively open, connected, Lipschitz sets. We also assume that $\Lambda_D^\pm \subset\subset (\partial\Omega^\pm \setminus \Gamma)$, from which it follows that $\partial_D\Omega$ is well-separated from Γ. With ν we denote the *inner* unit normal vector to $\partial\Omega$, defined $\mathcal{H}^{N-1}$-a.e. in $\partial\Omega$. We will also write ν for the *inner* unit normal vector to $\partial\Omega^+$.

Let us fix a time interval $[0,T]$, with $T > 0$, and let $w \in H^1((0,T);H^1(\Omega))$ be the boundary displacement. Thus, the time derivative $\dot{w}$ of w belongs to the space $L^2((0,T);H^1(\Omega))$.
Let $B \subset \mathbb{R}^N$ be an open bounded set and let $S \subset \partial B$ be relatively open and Lipschitz. We set

$$H_0^1(B,S) := \{\psi \in H^1(B) : \psi = 0 \text{ on } S\}.$$

The symbol $\|\cdot\|$ stands for the standard norm in $L^2(\Omega)$ or $L^2(\Omega;\mathbb{R}^N)$, depending on the context. Moreover, the brackets $\langle\cdot,\cdot\rangle$ denote the dual pairing between $H^{-\frac12}(\Gamma)$ and $H^{\frac12}(\Gamma)$. For every function $u \in H^1(\Omega \setminus \Gamma)$, we will use the notation $[u] := u^+ - u^-$ and

$$J_u := \{x \in \Gamma : [u](x) \neq 0\},$$

where $u^\pm$ is the trace on Γ of the restriction of u to $\Omega^\pm$. For $t \in [0,T]$, the class $\mathcal{A}(t)$ of admissible displacements at time t is defined as

$$\mathcal{A}(t) := \{u \in H^1(\Omega \setminus \Gamma) : u = w(t) \text{ on } \partial_D\Omega\},$$

while the total energy associated to a deformation $u \in \mathcal{A}(t)$ at the same time t is

$$E(u) := \frac{1}{2}\int_{\Omega\setminus\Gamma} |\nabla u|^2 dx + \int_\Gamma g(|[u]|)\, d\mathcal{H}^{N-1}, \tag{2.1}$$

where $g : [0,+\infty) \to [0,+\infty)$ is a C^1, non decreasing, bounded, concave function with $g(0) = 0$. We will denote by $\sigma := g'(0^+) \in (0,+\infty)$ the slope of the function g at 0. For every $t \in [0,T]$, the existence of a solution to the minimum problem

$$\min_{v \in \mathcal{A}(t)} E(v) \tag{2.2}$$

is guaranteed by the direct method of the Calculus of Variations. In the next section we give the Euler-Lagrange conditions for the problem (2.2). Note that, due to the lack of convexity, the minimizer may not be unique, and there can be critical points that are not absolute minimizers.

3. Euler-Lagrange Conditions

In this section we study in detail the Euler-Lagrange conditions for a minimizer of problem (2.2), giving two equivalent formulations.

Proposition 3.1. *Let $t \in [0,T]$ be fixed and let u be a solution of (2.2). Then*

$$\int_{\Omega\setminus\Gamma} \nabla u \cdot \nabla\psi\, dx + \int_\Gamma \Big([\psi]\, g'(|[u]|)\,\mathrm{sgn}[u]\, 1_{J_u} + \sigma|[\psi]| 1_{J_u^c}\Big)\, d\mathcal{H}^{N-1} \geq 0 \tag{3.1}$$

for every $\psi \in H^1_0(\Omega \setminus \Gamma, \partial_D\Omega)$.

Proof. We start by proving that for every $\psi \in H^1_0(\Omega \setminus \Gamma, \partial_D\Omega)$

$$\begin{aligned}\lim_{\eta\to 0^+} \int_\Gamma & \frac{g(|[u] + \eta[\psi]|) - g(|[u]|)}{\eta}\, d\mathcal{H}^{N-1} \\ &= \int_\Gamma [\psi]\, g'(|[u]|)\,\mathrm{sgn}[u]\, 1_{J_u}\, d\mathcal{H}^{N-1} + \int_\Gamma \sigma|[\psi]| 1_{J_u^c}\, d\mathcal{H}^{N-1}.\end{aligned} \tag{3.2}$$

Indeed, let us fix $\psi \in H^1_0(\Omega \setminus \Gamma, \partial_D\Omega)$. Then, for $\mathcal{H}^{N-1}$-a.e. $x \in J_u^c \cap \Gamma$

$$\lim_{\eta\to 0^+} \frac{g(|[u](x) + \eta\,[\psi](x)|) - g(|[u](x)|)}{\eta} = \lim_{\eta\to 0^+} \frac{g(\eta|[\psi](x)|)}{\eta} = \sigma|[\psi](x)|, \tag{3.3}$$

while, for $\mathcal{H}^{N-1}$-a.e. $x \in J_u$

$$\lim_{\eta\to 0^+} \frac{g(|[u](x) + \eta\,[\psi](x)|) - g(|[u](x)|)}{\eta} = [\psi](x) g'(|[u](x)|)\,\mathrm{sgn}([u](x)). \tag{3.4}$$

From the fact that g is concave it follows that g' is decreasing, so that $g' \leq \sigma$. Then, we have

$$\frac{g(|[u](x) + \eta[\psi](x)|) - g(|[u](x)|)}{\eta} \leq \sigma\,|[\psi](x)| \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Gamma.$$

Thanks to (3.3) and (3.4) and applying the dominated convergence theorem we get (3.2). Being u a solution of (2.2), for every $\psi \in H^1_0(\Omega \setminus \Gamma, \partial_D\Omega)$ we have that

$$\lim_{\eta\to 0^+} \frac{E(u + \eta\psi) - E(u)}{\eta} \geq 0.$$

Using (3.2), last inequality becomes (3.1). ☐

Next proposition gives an equivalent formulation of the Euler-Lagrange conditions.

Proposition 3.2. *Let $t \in [0,T]$ be fixed and let $u \in \mathcal{A}(t)$. Then (3.1) holds if and only if the following two conditions are fulfilled:*

(a) *u satisfies*

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{D}'(\Omega \setminus \Gamma), \\ u = w(t) & \text{on } H^{\frac{1}{2}}(\partial_D \Omega), \\ \partial_\nu u = 0 & \text{on } H^{-\frac{1}{2}}(\partial\Omega \setminus \partial_D \Omega), \\ \partial_\nu u^+ = \partial_\nu u^- & \text{on } H^{-\frac{1}{2}}(\Gamma); \end{cases} \tag{3.5}$$

(b) *there exists $h \in L^\infty(\Gamma)$ such that*

$$\begin{cases} \langle \partial_\nu u, [\psi] \rangle = \int_\Gamma h[\psi]\, d\mathcal{H}^{N-1} & \forall\, \psi \in H^1_0(\Omega \setminus \Gamma, \partial_D \Omega), \\ |h| \le \sigma & \mathcal{H}^{N-1}\text{-a.e. in } \Gamma, \\ h = g'(|[u]|)\,\mathrm{sgn}([u]) & \mathcal{H}^{N-1}\text{-a.e. in } J_u. \end{cases} \tag{3.6}$$

Proof. Let us prove the two implications.

Step 1. Show that (3.1) $\Rightarrow$ (a) and (b). Specifying (3.1) for ψ and $-\psi$, with $\psi \in H^1_0(\Omega, \partial_D\Omega)$ arbitrary, we conclude that

$$\int_{\Omega\setminus\Gamma} \nabla u \cdot \nabla \psi \, dx = 0 \qquad \forall\, \psi \in H^1_0(\Omega, \partial_D \Omega),$$

that is,

$$\int_{\Omega^+} \nabla u^+ \cdot \nabla \psi^+ \, dx + \int_{\Omega^-} \nabla u^- \cdot \nabla \psi^- \, dx = 0, \tag{3.7}$$

for every $\psi^+ \in H^1_0(\Omega^+, \Lambda^+_D)$, $\psi^- \in H^1_0(\Omega^-, \Lambda^-_D)$ with $\psi^+ |_\Gamma = \psi^- |_\Gamma$. Choosing $\psi^- \equiv 0$ (hence $\psi^+ |_\Gamma \equiv 0$) we have

$$\int_{\Omega^+} \nabla u^+ \cdot \nabla \psi^+ \, dx = 0 \qquad \forall\, \psi^+ \in H^1_0(\Omega^+, \Lambda^+_D \cup \Gamma),$$

that gives

$$\begin{cases} \Delta u^+ = 0 & \text{in } \mathcal{D}'(\Omega^+), \\ \partial_\nu u^+ = 0 & \text{on } H^{-\frac{1}{2}}(\partial\Omega^+ \setminus (\Lambda^+_D \cup \Gamma)). \end{cases}$$

Analogous relations can be obtained for u^-, by choosing $\psi^+ \equiv 0$ in (3.7), so that $(3.5)_1$–$(3.5)_3$ are proved. In this way, relation (3.7) becomes (setting $\psi^+ |_\Gamma = \psi^- |_\Gamma = \psi$)

$$-\langle \partial_\nu u^+, \psi \rangle + \langle \partial_\nu u^-, \psi \rangle = 0 \qquad \forall\, \psi \in H^{\frac{1}{2}}(\Gamma),$$

that is $(3.5)_4$. Taking into account $(3.5)_1$, $(3.5)_3$ and $(3.5)_4$, (3.1) reads as

$$-\langle \partial_\nu u, [\psi] \rangle + \int_\Gamma \Big([\psi]\, g'(|[u]|)\, \mathrm{sgn}[u]\, 1_{J_u} + \sigma |[\psi]| 1_{J^c_u} \Big)\, d\mathcal{H}^{N-1} \ge 0 \tag{3.8}$$

for every $[\psi] \in Y$, where $Y := \{\, [\psi] : \psi \in H^1_0(\Omega \setminus \Gamma, \partial_D \Omega)\} \subset L^1(\Gamma)$. From (3.8), since $g' \le \sigma$, it follows that

$$\langle \partial_\nu u, z \rangle \le \sigma \, \|z\|_{L^1(\Gamma)} \qquad \forall\, z \in Y.$$

Applying the previous inequality to z and $-z$, with $z \in Y$ arbitrary, we get

$$|\langle \partial_\nu u, z \rangle| \le \sigma \, \|z\|_{L^1(\Gamma)} \qquad \forall\, z \in Y.$$

This shows that the restriction $\partial_\nu u |_Y$ of $\partial_\nu u$ to Y is linear and continuous with respect to the L^1- norm. Using the fact that $H^{\frac{1}{2}}(\Gamma)$ is dense in $L^1(\Gamma)$ we get that Y is dense in $L^1(\Gamma)$. Thus, we can extend $\partial_\nu u$ in a unique way to a linear and continuous application (also denoted with $\partial_\nu u$) $\partial_\nu u : L^1(\Gamma) \to \mathbb{R}$ with

$$|\langle \partial_\nu u, z \rangle| \le \sigma \, \|z\|_{L^1(\Gamma)} \qquad \forall\, z \in L^1(\Gamma).$$

By the representation theorem, there exists a function $h \in L^\infty(\Gamma)$ such that

$$\langle \partial_\nu u, z \rangle = \int_\Gamma h\, z \, d\mathcal{H}^{N-1} \qquad \forall\, z \in L^1(\Gamma).$$

In particular, $(3.6)_1$ holds and (3.8) becomes

$$\int_\Gamma h\,[\psi]\,d\mathcal{H}^{N-1} \le \int_\Gamma \Big([\psi]\,g'(|[u]|)\,\mathrm{sgn}[u]\,1_{J_u} + \sigma|[\psi]|1_{J_u^c}\Big)\,d\mathcal{H}^{N-1}$$

for all $[\psi]\in Y$. By density, for all $z\in L^1(\Gamma)$

$$\int_\Gamma h\,z\,d\mathcal{H}^{N-1} \le \int_\Gamma \Big(z\,g'(|[u]|)\,\mathrm{sgn}[u]\,1_{J_u} + \sigma|z|1_{J_u^c}\Big)\,d\mathcal{H}^{N-1}. \tag{3.9}$$

Using the last relation we obtain that for every $z\in L^1(\Gamma)$ with $z\ge 0$

$$\int_\Gamma z\Big(h - g'(|[u]|)\,\mathrm{sgn}[u]\,1_{J_u}\Big)\,d\mathcal{H}^{N-1} \le \int_\Gamma z\,\sigma\,1_{J_u^c} d\mathcal{H}^{N-1}.$$

From this, we conclude that for $\mathcal{H}^{N-1}$-a.e. $x\in\Gamma$

$$h(x) - g'(|[u](x)|)\,\mathrm{sgn}\,([u](x))\;1_{J_u}(x) \le \sigma\,1_{J_u^c}(x). \tag{3.10}$$

We now evaluate (3.9) in $-z$, with $z\ge 0$ arbitrary. We get that for all $z\in L^1(\Gamma)$ with $z\ge 0$

$$\int_\Gamma z\Big(h - g'(|[u]|)\,\mathrm{sgn}[u]\,1_{J_u}\Big)\,d\mathcal{H}^{N-1} \ge \int_\Gamma z(-\sigma\,1_{J_u^c})d\mathcal{H}^{N-1},$$

so that for $\mathcal{H}^{N-1}$-a.e. $x\in\Gamma$

$$h(x) - g'(|[u](x)|)\,\mathrm{sgn}\,([u](x))\;1_{J_u}(x) \ge -\sigma\,1_{J_u^c}(x). \tag{3.11}$$

Collecting (3.10) and (3.11) we have that for $\mathcal{H}^{N-1}$-a.e. $x\in\Gamma$

$$|h(x) - g'(|[u](x)|)\,\mathrm{sgn}\,([u](x))\;1_{J_u}(x)| \le \sigma\,1_{J_u^c}(x). \tag{3.12}$$

Choosing $x\in J_u$ last inequality becomes

$$h(x) - g'(|[u](x)|)\,\mathrm{sgn}\,([u](x)) = 0 \qquad \mathcal{H}^{N-1}\text{-a.e. } x\in J_u,$$

that is $(3.6)_3$. For $x\in J_u^c\cap\Gamma$ (3.12) gives

$$|h(x)|\le\sigma \qquad \mathcal{H}^{N-1}\text{-a.e. } x\in J_u^c\cap\Gamma,$$

that together with $(3.6)_3$ proves $(3.6)_2$.

Step 2. Show that (a) and (b)$\Rightarrow$ (3.1). Conversely, applying $(3.5)_1$ to $\psi\in H^1_0(\Omega\setminus\Gamma,\partial_D\Omega)$ arbitrary, integrating by parts and using relations $(3.5)_2$–$(3.6)_3$ we get (3.1). □

4. Basic Definitions and Main Results

In this section we give the basic definitions and state the main results of the chapter; all the proofs are postponed to the next sections. Proposition 3.1 motivates the following definition.

Definition 4.1. Let $w\in H^1((0,T);H^1(\Omega))$, $t\in[0,T]$, and let E be defined by (2.1). We say that a function u is a *critical point* for E at time t if $u\in\mathcal{A}(t)$ and

$$\int_{\Omega\setminus\Gamma}\nabla u\cdot\nabla\psi\,dx + \int_\Gamma \Big([\psi]\,g'(|[u]|)\,\mathrm{sgn}[u]\,1_{J_u} + \sigma|[\psi]|1_{J_u^c}\Big)\,d\mathcal{H}^{N-1} \ge 0 \tag{4.1}$$

for every $\psi\in H^1_0(\Omega\setminus\Gamma,\partial_D\Omega)$.

Throughout the whole section we will always assume that $w\in H^1((0,T);H^1(\Omega))$ and that u_0 is a critical point for E at time $t=0$. Unless otherwise stated, the hypotheses on Ω and Γ are those listed in Section 2.

4.1. **Parabolic Evolutions.** We introduce evolutions depending on a "small" viscosity parameter ε, as made precise by the following definition.

Definition 4.2. Let $\varepsilon \in (0,1)$. A *parabolic evolution* with viscosity ε, boundary datum w and initial condition u_0 is a function $u^\varepsilon : [0,T] \to H^1(\Omega \setminus \Gamma)$ such that

$(i)_\varepsilon$ $u^\varepsilon(0) = u_0$;
$(ii)_\varepsilon$ $u^\varepsilon(t) = w(t)$ on $\partial_D\Omega$ for every $t \in [0,T]$;
$(iii)_\varepsilon$ $u^\varepsilon \in H^1((0,T); L^2(\Omega)) \cap L^\infty((0,T); H^1(\Omega \setminus \Gamma))$;
$(iv)_\varepsilon$ for a.e. $t \in [0,T]$

$$\int_{\Omega\setminus\Gamma} \nabla u^\varepsilon(t) \cdot \nabla\psi \, dx + \int_{\Omega\setminus\Gamma} \varepsilon \dot{u}^\varepsilon(t)\psi \, dx \\ + \int_\Gamma \Big([\psi]\, g'(|[u^\varepsilon(t)]|) \operatorname{sgn}[u^\varepsilon(t)]\, 1_{J_{u^\varepsilon(t)}} + \sigma|[\psi]| 1_{J^c_{u^\varepsilon(t)}} \Big) \, d\mathcal{H}^{N-1} \geq 0, \tag{4.2}$$

for every $\psi \in H^1_0(\Omega \setminus \Gamma, \partial_D\Omega)$.

Remark 4.3. Arguing as in the proof of Proposition 3.2, the strong formulation of conditions $(i)_\varepsilon$–$(iv)_\varepsilon$ is easily seen to be the following: $u^\varepsilon(0) = u_0$ and for a.e. $t \in [0,T]$

$$\begin{cases} \varepsilon\dot{u}^\varepsilon(t) = \Delta u^\varepsilon(t) & \text{in } \Omega\setminus\Gamma, \\ u^\varepsilon(t) = w(t) & \text{on } \partial_D\Omega, \\ \partial_\nu u^\varepsilon(t) = 0 & \text{on } \partial\Omega\setminus\partial_D\Omega, \\ \partial_\nu u^\varepsilon(t)\,|_{\Omega^+} = \partial_\nu u^\varepsilon(t)\,|_{\Omega^-} & \text{on } \Gamma, \\ |\partial_\nu u^\varepsilon(t)| \leq \sigma & \text{on } \Gamma, \\ \partial_\nu u^\varepsilon(t) = g'(|[u^\varepsilon(t)]|)\operatorname{sgn}([u^\varepsilon(t)]) & \text{on } J_{u^\varepsilon(t)}. \end{cases}$$

As a first step, we state an existence result.

Theorem 4.4. *Let $\varepsilon \in (0,1)$. Then there exists a parabolic evolution with viscosity ε, boundary datum w and initial condition u_0.*

Next theorem shows that under slightly stronger assumptions on the function g we get also uniqueness.

Theorem 4.5. *Let $\varepsilon \in (0,1)$ and assume $g \in C^{1,1}$. Then there exists a unique parabolic evolution with viscosity ε, boundary datum w and initial condition u_0.*

For the general case $g \in C^1$ uniqueness is not known for a function u^ε satisfying $(i)_\varepsilon$–$(iv)_\varepsilon$. For this reason, in order to obtain the desired properties of the limit evolution as the viscosity parameter tends to 0, we introduce a selection criterion on the parabolic evolutions. We will select among all possible solutions of (4.2) only those obtained by a suitable approximation procedure, based on the technique of minimizing movements introduced by De Giorgi (see [2]). Given a time step $\delta \in (0,T)$, we divide the interval $[0,T]$ into subintervals $[i\delta, (i+1)\delta)$, for $i \in \mathbb{N}$ with $i\delta \leq T$. Then, at every time $i\delta$ we solve a "static" minimum problem for the energy E, adding a term which penalizes the L^2-distance between the approximate solutions at two consecutive times.

Definition 4.6. Let $\delta \in (0,T)$ and $\varepsilon \in (0,1)$. A *discrete-time evolution* with time step δ, viscosity ε, boundary datum w and initial condition u_0 is a piecewise constant function $u^{\varepsilon,\delta} : [0,T] \to H^1(\Omega\setminus\Gamma)$ such that $u^{\varepsilon,\delta}(t) := u_i^{\varepsilon,\delta}$ for $i\delta \leq t < (i+1)\delta$, where $u_0^{\varepsilon,\delta} := u_0$ and, by induction, $u_i^{\varepsilon,\delta}$ is a solution to the minimum problem

$$\min_{v\in\mathcal{A}(i\delta)} \Big\{ E(v) + \frac{\varepsilon}{2\delta} \|v - u^{\varepsilon,\delta}_{i-1}\|^2 \Big\} \tag{4.3}$$

for every $i \in \mathbb{N}$ with $i\delta \leq T$. Problem (4.3) will be also denoted by $(P)_i^{\varepsilon,\delta}$.

We now make explicit the selection criterion for parabolic evolutions.

Definition 4.7. Let $\varepsilon \in (0,1)$. A parabolic evolution is said to be a *variational parabolic evolution* with viscosity ε, boundary datum w and initial condition u_0 if there exists a family $\{u^{\varepsilon,\delta}\}_{\delta\in(0,T)}$ of discrete-time evolutions with viscosity ε, boundary datum w and initial condition u_0, such that for every $t \in [0,T]$

$$u^{\varepsilon,\delta_n}(t) \rightharpoonup u^\varepsilon(t) \text{ weakly in } H^1(\Omega \setminus \Gamma) \tag{4.4}$$

for some sequence $\delta_n \to 0^+$ as $n \to +\infty$.

Next theorem gives an existence result for variational parabolic evolutions.

Theorem 4.8. *Let $\varepsilon \in (0,1)$. Then there exists a variational parabolic evolution with viscosity ε, boundary datum w and initial condition u_0.*

Remark 4.9. In particular, the previous result implies Theorem 4.4.

The following proposition states an energy inequality for variational parabolic evolutions.

Proposition 4.10. *Let $\varepsilon \in (0,1)$ and let u^ε be a variational parabolic evolution with viscosity ε, boundary datum w and initial condition u_0. Then, for every $t \in [0,T]$*

$$\begin{aligned}\frac{1}{2}\|\nabla u^\varepsilon(t)\|^2 &+ \int_\Gamma g(|[u^\varepsilon(t)]|)\, d\mathcal{H}^{N-1} + \frac{\varepsilon}{2}\int_0^t \|\dot u^\varepsilon(s)\|^2 ds \\ &\le \frac{1}{2}\|\nabla u_0\|^2 + \int_\Gamma g(|[u_0]|)\, d\mathcal{H}^{N-1} + \int_0^t \int_{\Omega\setminus\Gamma} \nabla u^\varepsilon(s)\cdot\nabla \dot w(s)\, dx\, ds + \frac{\varepsilon}{2}\int_0^t \|\dot w(s)\|^2 ds.\end{aligned} \tag{4.5}$$

4.2. **Approximable Quasistatic Evolutions.** We give now the definition of approximable quasistatic evolution.

Definition 4.11. A bounded measurable function $u : [0,T] \to H^1(\Omega\setminus\Gamma)$ is said to be an *approximable quasistatic evolution* with boundary datum w and initial condition u_0 if there exists a family $\{u^\varepsilon\}_{\varepsilon\in(0,1)}$ of variational parabolic evolutions with boundary datum w and initial condition u_0, parametrized by the viscosity ε, such that the following three conditions are satisfied:

- *approximability*: for every $t \in [0,T]$ there exists a sequence $\varepsilon_n(t) \to 0^+$ such that
$$u^{\varepsilon_n(t)}(t) \rightharpoonup u(t) \text{ weakly in } H^1(\Omega\setminus\Gamma); \tag{4.6}$$
- *stationarity*: for a.e. $t \in [0,T]$ the function $u(t)$ is a critical point for E at time t;
- *energy inequality*: for every $t \in [0,T]$
$$E(u(t)) \le E(u(0)) + \int_0^t \int_{\Omega\setminus\Gamma} \nabla u(s)\cdot\nabla\dot w(s)\, dx\, ds. \tag{4.7}$$

Remark 4.12. It can be seen that the stationarity is a direct consequence of the approximability condition (4.6) (see the proof of Theorem 4.13).

We claim that the previous definition gives a good candidate for the description of the crack evolution. First of all, we give an existence result.

Theorem 4.13. *There exists an approximable quasistatic evolution with boundary datum w and initial condition u_0.*

We state now some results concerning the properties satisfied by the approximable quasistatic evolutions. Let us introduce some notation. For every $x \in \mathbb{R}^N$ we write $x = (x_1, x')$, where $x_1 \in \mathbb{R}$ and $x' \in \mathbb{R}^{N-1}$; we also set $X_1 := \{(x_1,x') \in \mathbb{R}^N : x_1 = 0\}$. We say that Ω is *symmetric with respect to* X_1 if $(x_1,x') \in \Omega$ implies $(-x_1,x') \in \Omega$ and Λ_D^+ can be obtained from Λ_D^- by reflection, that is $\Lambda_D^+ = \{(x_1,x') \in \mathbb{R}^N : (-x_1,x') \in \Lambda_D^-\}$. A function $v \in H^1(\Omega\setminus\Gamma)$ is said to be *odd with respect to* X_1 or simply *odd* if $v(-x_1,x') = -v(x_1,x')$ for every $(x_1,x') \in \Omega^+$. Next theorem shows that in the particular situation of monotone boundary conditions and symmetric domain, every approximable quasistatic evolution is irreversible.

Theorem 4.14. *Assume that $\Gamma \subset X_1$ and that Ω is symmetric with respect to X_1, according to the definition given above. Let $u_0 = 0$. Suppose, in addition, that the following hold:*

- *$w(t)$ is an odd function for every t;*
- *the function w has constant sign on $[0,T]\times\Lambda_D^+$;*
- *$t\mapsto |w(t)|(x_1,x')$ is nondecreasing for every $(x_1,x')\in\Lambda_D^+$.*

Then, for every approximable quasistatic evolution u with initial condition $u_0=0$ and boundary datum w the function $t\mapsto |[u(t)]|(0,x')$ is nondecreasing for $\mathcal{H}^{N-1}$-a.e. $(0,x')\in\Gamma$.

Before stating the next result we need some definitions. For every $t\in[0,T]$ we define the *elastic solution* $z(t)$ as the (unique) solution to the problem

$$\min_{v\in\mathcal{B}(t)}\left\{\frac{1}{2}\int_\Omega |\nabla v|^2 dx\right\}, \tag{4.8}$$

where

$$\mathcal{B}(t):=\{v\in H^1(\Omega): v=w(t) \text{ on } \partial_D\Omega\}\subset\mathcal{A}(t). \tag{4.9}$$

This definition comes from the fact that when there are no cracks, problem (2.2) reduces to (4.8). We will refer to the function $z:[0,T]\to H^1(\Omega)$ as *elastic evolution.*

The following crack initiation criterion proves that $\sigma:=g'(0^+)$ represents the *maximum sustainable stress* along Γ: we prove that a crack cannot appear until the elastic solution defined by (4.8) is a critical point satisfying condition $(3.6)_2$ with strict inequality.

Theorem 4.15 (crack initiation criterion)**.** *Assume that $u_0=z(0)$, $\|w\|_{L^\infty((0,T);L^\infty(\Omega))}<+\infty$ and $\|\dot w\|_{L^\infty((0,T);L^\infty(\Omega))}<+\infty$. In addition to the usual hypothesis we assume that $\partial\Omega$ is of class C^2 in a neighbourhood of $\Gamma\cap\partial\Omega$, with $\mathcal{H}^{N-2}(\overline{\partial_D\Omega}\setminus\partial_D\Omega)<+\infty$. Suppose also that there exists $t^*\in(0,T]$ such that $\sup_{t\in[0,t^*]}\|\partial_\nu z(t)\|_{L^\infty(\Gamma)}<\sigma$, with $z(t)$ defined by (4.8). Then, if u is an approximable quasistatic evolution with boundary datum w and initial condition $u_0=z(0)$, there holds $u(t)=z(t)$ for every $t\in[0,t^*]$.*

Remark 4.16. In particular, this shows the uniqueness of the approximable quasistatic evolution for $t\in[0,t^*]$ under the hypotheses of Theorem 4.15.

5. Proof of Theorem 4.8, Proposition 4.10, and Theorem 4.5

This section is devoted to the study of (variational) parabolic evolutions.

5.1. **Proof of Theorem 4.8.** Let us consider, for every $\delta\in(0,T)$, a discrete-time evolution $u^{\varepsilon,\delta}:[0,T]\to H^1(\Omega\setminus\Gamma)$ with time step δ, viscosity ε, boundary datum w and initial condition u_0. Before giving the proof of Theorem 4.8, we need two technical lemmas.

Lemma 5.1. *There exists a function $\rho:(0,T)\to[0,+\infty)$ such that $\rho(\delta)\stackrel{\delta\to0^+}{\longrightarrow}0$ and*

$$\begin{aligned}\frac{1}{2}\|\nabla u_j^{\varepsilon,\delta}\|^2&+\int_\Gamma g(|[u_j^{\varepsilon,\delta}]|)\,d\mathcal{H}^{N-1}+\frac{\varepsilon}{2\delta}\sum_{h=i}^{j-1}\|u_{h+1}^{\varepsilon,\delta}-u_h^{\varepsilon,\delta}\|^2\\ &\le\frac{1}{2}\|\nabla u_i^{\varepsilon,\delta}\|^2+\int_\Gamma g(|[u_i^{\varepsilon,\delta}]|)\,d\mathcal{H}^{N-1}+\int_{i\delta}^{j\delta}\int_{\Omega\setminus\Gamma}\nabla u^{\varepsilon,\delta}(s)\cdot\nabla\dot w(s)\,dx\,ds\\ &\qquad+\frac{\varepsilon}{2}\int_{i\delta}^{j\delta}\|\dot w(s)\|^2ds+\rho(\delta)\end{aligned} \tag{5.1}$$

for every $i,j\in\mathbb{N}$ with $0\le i<j$ and $j\delta\le T$.

Proof. Let $r\in\mathbb{N}$ be such that $i\le r<j$. Since $w\in H^1((0,T);H^1(\Omega))$, we have that

$$w_{r+1}^\delta-w_r^\delta=\int_{r\delta}^{(r+1)\delta}\dot w(s)\,ds,$$

where the integral is a Bochner integral for functions with values in $H^1(\Omega)$ and we used the notation $w(r\delta)=w_r^\delta$. This implies that

$$\nabla w_{r+1}^\delta-\nabla w_r^\delta=\int_{r\delta}^{(r+1)\delta}\nabla\dot w(s)\,ds,$$

where the integral is now a Bochner integral for functions with values in $L^2(\Omega;\mathbb{R}^N)$. Since $u_r^{\varepsilon,\delta} + w_{r+1}^\delta - w_r^\delta \in \mathcal{A}((r+1)\delta)$, by the minimality of $u_{r+1}^{\varepsilon,\delta}$ we have

$$
\begin{aligned}
&\frac{1}{2}\|\nabla u_{r+1}^{\varepsilon,\delta}\|^2 + \int_\Gamma g(|[u_{r+1}^{\varepsilon,\delta}]|)\,d\mathcal{H}^{N-1} + \frac{\varepsilon}{2\delta}\|u_{r+1}^{\varepsilon,\delta} - u_r^{\varepsilon,\delta}\|^2 \\
&\quad\le \frac{1}{2}\|\nabla u_r^{\varepsilon,\delta} + \nabla w_{r+1}^\delta - \nabla w_r^\delta\|^2 + \int_\Gamma g(|[u_r^{\varepsilon,\delta}] + [w_{r+1}^\delta] - [w_r^\delta]|)\,d\mathcal{H}^{N-1} + \frac{\varepsilon}{2\delta}\|w_{r+1}^\delta - w_r^\delta\|^2 \\
&\quad= \frac{1}{2}\|\nabla u_r^{\varepsilon,\delta}\|^2 + \int_\Gamma g(|[u_r^{\varepsilon,\delta}]|)\,d\mathcal{H}^{N-1} + \int_{r\delta}^{(r+1)\delta}\int_{\Omega\setminus\Gamma} \nabla u_r^{\varepsilon,\delta}\cdot\nabla\dot w(s)\,dx\,ds \\
&\qquad\quad + \frac{1}{2}\|\nabla w_{r+1}^\delta - \nabla w_r^\delta\|^2 + \frac{\varepsilon}{2\delta}\|w_{r+1}^\delta - w_r^\delta\|^2 \\
&\quad\le \frac{1}{2}\|\nabla u_r^{\varepsilon,\delta}\|^2 + \int_\Gamma g(|[u_r^{\varepsilon,\delta}]|)\,d\mathcal{H}^{N-1} + \int_{r\delta}^{(r+1)\delta}\int_{\Omega\setminus\Gamma} \nabla u_r^{\varepsilon,\delta}\cdot\nabla\dot w(s)\,dx\,ds \\
&\qquad\quad + \frac{1}{2}\left(\int_{r\delta}^{(r+1)\delta}\|\nabla\dot w(s)\|ds\right)^2 + \frac{\varepsilon}{2\delta}\left(\int_{r\delta}^{(r+1)\delta}\|\dot w(s)\|ds\right)^2 \\
&\quad\le \frac{1}{2}\|\nabla u_r^{\varepsilon,\delta}\|^2 + \int_\Gamma g(|[u_r^{\varepsilon,\delta}]|)\,d\mathcal{H}^{N-1} + \int_{r\delta}^{(r+1)\delta}\int_{\Omega\setminus\Gamma} \nabla u_r^{\varepsilon,\delta}\cdot\nabla\dot w(s)\,dx\,ds \\
&\qquad\quad + \frac{1}{2}\left(\max_r \int_{r\delta}^{(r+1)\delta}\|\nabla\dot w(s)\|ds\right)\int_{r\delta}^{(r+1)\delta}\|\nabla\dot w(s)\|ds + \frac{\varepsilon}{2}\int_{r\delta}^{(r+1)\delta}\|\dot w(s)\|^2 ds.
\end{aligned}
$$

Iterating last inequality for $r = j-1,\dots,i$ we get (5.1) with

$$
\rho(\delta) := \frac{1}{2}\left(\max_r \int_{r\delta}^{(r+1)\delta}\|\nabla\dot w(s)\|ds\right)\int_0^T \|\nabla\dot w(s)\|ds,
$$

that converges to 0 as $\delta\to 0^+$ by the absolute continuity of the integral. □

We define now the functions $v^{\varepsilon,\delta}\colon[0,T]\to H^1(\Omega\setminus\Gamma)$ in the following way:

$$
v^{\varepsilon,\delta}(t) := u_i^{\varepsilon,\delta} + \frac{t-i\delta}{\delta}(u_{i+1}^{\varepsilon,\delta} - u_i^{\varepsilon,\delta}) \qquad \text{for } i\delta\le t<(i+1)\delta. \tag{5.2}
$$

The second lemma gives some a priori estimates for the families $\{u^{\varepsilon,\delta}\}_{\delta\in(0,T)}$ and $\{v^{\varepsilon,\delta}\}_{\delta\in(0,T)}$.

Lemma 5.2. *There exists a positive constant $C = C(\Omega,\Gamma,u_0,w,T)$, such that*

$$
\|u_j^{\varepsilon,\delta}\|_{H^1(\Omega\setminus\Gamma)}\le C, \qquad \|v^{\varepsilon,\delta}(t)\|_{H^1(\Omega\setminus\Gamma)}\le C; \tag{5.3}
$$

$$
\frac{\varepsilon}{2}\int_0^{j\delta}\|\dot v^{\varepsilon,\delta}(s)\|^2 ds = \frac{\varepsilon}{2\delta}\sum_{h=0}^{j-1}\|u_{h+1}^{\varepsilon,\delta} - u_h^{\varepsilon,\delta}\|^2\le C; \tag{5.4}
$$

$$
\sqrt{\varepsilon}\|\dot v^{\varepsilon,\delta}\|_{L^2((0,T);L^2(\Omega))}\le C, \qquad \|v^{\varepsilon,\delta}\|_{H^1((0,T);L^2(\Omega))}\le C\sqrt{1+\frac{1}{\varepsilon}}\ ; \tag{5.5}
$$

for every $\varepsilon\in(0,1)$, $\delta\in(0,T)$, $t\in[0,T]$ and $j\in\mathbb{N}$ with $j\delta\le T$.

Proof. Let us fix $\varepsilon\in(0,1)$, $\delta\in(0,T)$, and $t\in[0,T]$. Let $j\in\mathbb{N}$ be such that $j\delta\le t<(j+1)\delta$. Consider now inequality (5.1) with $i=0$. We get

$$
\begin{aligned}
&\frac{1}{2}\|\nabla u_j^{\varepsilon,\delta}\|^2 + \int_\Gamma g(|[u_j^{\varepsilon,\delta}]|)\,d\mathcal{H}^{N-1} + \frac{\varepsilon}{2}\int_0^{j\delta}\|\dot v^{\varepsilon,\delta}(s)\|^2 ds \\
&\quad\le \frac{1}{2}\|\nabla u_0\|^2 + \int_\Gamma g(|[u_0]|)\,d\mathcal{H}^{N-1} + \int_0^{j\delta}\int_{\Omega\setminus\Gamma}\nabla u^{\varepsilon,\delta}(s)\cdot\nabla\dot w(s)\,dx\,ds \\
&\qquad + \frac{\varepsilon}{2}\int_0^{j\delta}\|\dot w(s)\|^2 ds + \rho(\delta).
\end{aligned} \tag{5.6}
$$

From (5.6), using Hölder inequality we get

$$\|\nabla u^{\varepsilon,\delta}(t)\|^2 \le c + 2\left(\int_0^t \|\nabla \dot{w}(s)\|^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \|\nabla u^{\varepsilon,\delta}(s)\|^2 ds\right)^{\frac{1}{2}}, \tag{5.7}$$

where c is a positive constant independent of δ, ε and t. By using the Gronwall Lemma [3, Lemma 4.1.8] we deduce that for every $t \in [0,T]$

$$\left(\int_0^t \|\nabla u^{\varepsilon,\delta}(s)\|^2 ds\right)^{\frac{1}{2}} \le (Tc)^{\frac{1}{2}} + 2T\|\nabla \dot{w}\|_{L^2((0,T);L^2(\Omega))}.$$

Last relation together with (5.7) implies that $\nabla u^{\varepsilon,\delta}(t)$ is bounded in $L^2(\Omega;\mathbb{R}^N)$ uniformly with respect to δ, ε and t. Then, using the Poincaré inequality we get immediately (5.3). Once (5.3) is proved, (5.4) and $(5.5)_1$ follows by (5.6). Finally, $(5.3)_2$ and $(5.5)_1$ imply $(5.5)_2$. $\square$

We are now ready to prove Theorem 4.8.

Proof of Theorem 4.8. We want to prove that there exists a function u^ε satisfying conditions $(i)_\varepsilon$–$(iv)_\varepsilon$ of Definition 4.2 and such that (4.4) holds for some sequence of time steps $\delta_n \to 0$. From $(5.3)_2$ it follows that

$$v^{\varepsilon,\delta}(t) \in B_C \quad \text{for every } \delta \in (0,T),\ t \in [0,T], \tag{5.8}$$

where

$$B_C := \{u \in H^1(\Omega \setminus \Gamma) : \|u\|_{H^1(\Omega\setminus\Gamma)} \le C\}, \tag{5.9}$$

and C is given by Lemma 5.2. Moreover, by $(5.5)_1$ and using Hölder inequality

$$\|v^{\varepsilon,\delta}(t) - v^{\varepsilon,\delta}(s)\| \le \frac{C}{\sqrt{\varepsilon}}\sqrt{|\,t-s\,|}, \quad \text{for every } s,t \in [0,T],\ \delta \in (0,T). \tag{5.10}$$

Thanks to (5.8) and (5.10), and observing that B_C is compact in $L^2(\Omega)$, we can apply a refined version of Ascoli-Arzelà Theorem (see [3, Proposition 3.3.1]). Therefore, there exist a continuous (with respect to the L^2-norm) function $u^\varepsilon : [0,T] \to B_C$ and a sequence $\delta_n \to 0^+$ such that for every $t \in [0,T]$

$$v^{\varepsilon,\delta_n}(t) \to u^\varepsilon(t) \quad \text{strongly in } L^2(\Omega). \tag{5.11}$$

From $(5.5)_2$ we get also that

$$v^{\varepsilon,\delta_n} \rightharpoonup u^\varepsilon \qquad \text{weakly in } H^1((0,T);L^2(\Omega)). \tag{5.12}$$

Let us prove (4.4). Let $t \in [0,T]$ be fixed and, for every $n \in \mathbb{N}$, let $l_n \in \mathbb{N}$ be such that $l_n\delta_n \le t < (l_n+1)\delta_n$. By (5.2) and (5.4)

$$\begin{aligned} \|v^{\varepsilon,\delta_n}(t) - u^{\varepsilon,\delta_n}(t)\|^2 &= \left(\frac{t - l_n\delta_n}{\delta_n}\right)^2 \|u^{\varepsilon,\delta_n}_{l_n+1} - u^{\varepsilon,\delta_n}_{l_n}\|^2 \\ &\le \|u^{\varepsilon,\delta_n}_{l_n+1} - u^{\varepsilon,\delta_n}_{l_n}\|^2 \le \frac{2\delta_n}{\varepsilon}C. \end{aligned} \tag{5.13}$$

When $n \to +\infty$ we get

$$\sup_{t\in[0,T]} \|v^{\varepsilon,\delta_n}(t) - u^{\varepsilon,\delta_n}(t)\| \to 0. \tag{5.14}$$

Since $u^{\varepsilon,\delta_n}(t) \in B_C$ for every $n \in \mathbb{N}$, using (5.11) and (5.14) we deduce (4.4).

We prove now conditions $(i)_\varepsilon$-$(iv)_\varepsilon$. Clearly $u^\varepsilon(0) = u_0$, so that $(i)_\varepsilon$ holds. Moreover, $(ii)_\varepsilon$ follows from the fact that, for $t \in [0,T]$ fixed, $l_n\delta_n \to t$ and consequently $w(l_n\delta_n) \to w(t)$ strongly in $H^1(\Omega)$. Using (5.12), (4.4), and the fact that $u^{\varepsilon,\delta_n}(t) \in B_C$ for every $t \in [0,T]$ we have $(iii)_\varepsilon$. It remains to prove condition $(iv)_\varepsilon$. Let us fix $t \in (0,T)$. Arguing as in the proof of Proposition 3.1, we obtain that for every $n \in \mathbb{N}$

$$\begin{aligned} &\int_{\Omega\setminus\Gamma} \nabla u^{\varepsilon,\delta_n}(t)\cdot\nabla\psi\,dx + \int_\Omega \varepsilon\dot{v}^{\varepsilon,\delta_n}(t)\psi\,dx \\ &\quad + \int_\Gamma \Big([\psi]\,g'(|[u^{\varepsilon,\delta_n}(t)]|)\,\mathrm{sgn}[u^{\varepsilon,\delta_n}(t)]\,1_{J_{u^{\varepsilon,\delta_n}(t)}} + \sigma|[\psi]|1_{J^c_{u^{\varepsilon,\delta_n}(t)}}\Big)d\mathcal{H}^{N-1} \ge 0 \end{aligned} \tag{5.15}$$

for every $\psi \in H^1_0(\Omega \setminus \Gamma, \partial_D\Omega)$. We evaluate now the mean value of inequality (5.15) from t to $t+\eta$, with $\eta > 0$ such that $t+\eta < T$:

$$\begin{aligned}\frac{1}{\eta}\int_t^{t+\eta}\int_{\Omega\setminus\Gamma} \nabla u^{\varepsilon,\delta_n}(s)\cdot\nabla\psi\,dx\,ds + \frac{1}{\eta}\int_t^{t+\eta}\int_{\Omega}\varepsilon\dot v^{\varepsilon,\delta_n}(s)\,\psi\,dx\,ds \\ \geq \frac{1}{\eta}\int_t^{t+\eta}\int_\Gamma f^{\varepsilon,n}(s)\,d\mathcal{H}^{N-1}ds,\end{aligned} \tag{5.16}$$

where

$$f^{\varepsilon,n}(s) := -[\psi]\,g'(|[u^{\varepsilon,\delta_n}(s)]|)\,\mathrm{sgn}[u^{\varepsilon,\delta_n}(s)]\,1_{J_{u^{\varepsilon,\delta_n}(s)}} - \sigma|[\psi]|1_{J^c_{u^{\varepsilon,\delta_n}(s)}}.$$

Let us consider the left-hand side of (5.16). For the first term, using Hölder inequality and $(5.3)_1$ we have that for every $s \in (t, t+\eta)$

$$\int_{\Omega\setminus\Gamma}\nabla u^{\varepsilon,\delta_n}(s)\cdot\nabla\psi\,dx \leq C\|\nabla\psi\|.$$

Hence, thanks to (4.4) and applying the Lebesgue dominated convergence Theorem

$$\lim_{n\to+\infty}\frac{1}{\eta}\int_t^{t+\eta}\int_{\Omega\setminus\Gamma}\nabla u^{\varepsilon,\delta_n}(s)\cdot\nabla\psi\,dx\,ds = \frac{1}{\eta}\int_t^{t+\eta}\int_{\Omega\setminus\Gamma}\nabla u^{\varepsilon}(s)\cdot\nabla\psi\,dx\,ds. \tag{5.17}$$

In the same way, thanks to $(5.5)_1$ and (5.12) we can apply the Lebesgue dominated convergence Theorem to the second term:

$$\lim_{n\to+\infty}\frac{1}{\eta}\int_t^{t+\eta}\int_{\Omega}\varepsilon\dot v^{\varepsilon,\delta_n}(s)\psi\,dx\,ds = \frac{1}{\eta}\int_t^{t+\eta}\int_{\Omega}\varepsilon\dot u^{\varepsilon}(s)\psi\,dx\,ds. \tag{5.18}$$

Collecting (5.17) and (5.18) we have

$$\begin{aligned}&\lim_{n\to+\infty}\left(\frac{1}{\eta}\int_t^{t+\eta}\int_{\Omega\setminus\Gamma}\nabla u^{\varepsilon,\delta_n}(s)\cdot\nabla\psi\,dx\,ds + \frac{1}{\eta}\int_t^{t+\eta}\int_{\Omega}\varepsilon\dot v^{\varepsilon,\delta_n}(s)\psi\,dx\,ds\right)\\ &= \frac{1}{\eta}\int_t^{t+\eta}\int_{\Omega\setminus\Gamma}\nabla u^{\varepsilon}(s)\cdot\nabla\psi\,dx\,ds + \frac{1}{\eta}\int_t^{t+\eta}\int_{\Omega}\varepsilon\dot u^{\varepsilon}(s)\psi\,dx\,ds.\end{aligned} \tag{5.19}$$

Let us now consider the right-hand side of (5.16). We claim that for every $s \in (t, t+\eta)$

$$\liminf_{n\to+\infty} f^{\varepsilon,n}(s) \geq -[\psi]\,g'(|[u^\varepsilon(s)]|)\,\mathrm{sgn}[u^\varepsilon(s)]\,1_{J_{u^\varepsilon(s)}} - \sigma|[\psi]|1_{J^c_{u^\varepsilon(s)}} \tag{5.20}$$

$\mathcal{H}^{N-1}$-a.e. in Γ. To prove (5.20), let us fix $s \in (t, t+\eta)$. We can extract a subsequence $(n_k)_{k\in\mathbb{N}}$, possibly depending on s, such that

$$\lim_{k\to+\infty} f^{\varepsilon,n_k}(s) = \liminf_{n\to+\infty} f^{\varepsilon,n}(s) \qquad \mathcal{H}^{N-1}\text{-a.e. in } \Gamma$$

and

$$\lim_{k\to+\infty}[u^{\varepsilon,\delta_{n_k}}(s)] = [u^\varepsilon(s)] \qquad \mathcal{H}^{N-1}\text{-a.e. in } \Gamma. \tag{5.21}$$

Now, let us fix $x \in J_{u^\varepsilon(s)}$ such that the two previous equalities hold. By (5.21) it follows that for k large enough $x \in J_{u^{\varepsilon,\delta_{n_k}}(s)}$ and $\mathrm{sgn}([u^{\varepsilon,\delta_{n_k}}(s)](x)) = \mathrm{sgn}([u^\varepsilon(s)](x))$. Hence,

$$\begin{aligned}\liminf_{n\to+\infty} f^{\varepsilon,n}(s)(x) &= \lim_{k\to+\infty} -[\psi](x)\,g'(|[u^{\varepsilon,\delta_{n_k}}(s)](x)|)\,\mathrm{sgn}[u^\varepsilon(s)](x)\\ &= -[\psi](x)\,g'(|[u^\varepsilon(s)](x)|)\,\mathrm{sgn}[u^\varepsilon(s)](x),\end{aligned}$$

for $\mathcal{H}^{N-1}$-a.e. $x \in J_{u^\varepsilon(s)}$. On the other hand, if $x \in J^c_{u^\varepsilon(s)} \cap \Gamma$ we have

$$\liminf_{n\to+\infty} f^{\varepsilon,n}(s)(x) \geq -\sigma|[\psi](x)|,$$

so that (5.20) is proved. Thus, using Fatou's Lemma and (5.20)

$$
\begin{aligned}
\liminf_{n\to+\infty} \frac{1}{\eta}\int_t^{t+\eta}\int_\Gamma f^{\varepsilon,n}(s)\, d\mathcal{H}^{N-1} ds &\geq \frac{1}{\eta}\int_t^{t+\eta}\int_\Gamma \liminf_{n\to+\infty} f^{\varepsilon,n}(s)\, d\mathcal{H}^{N-1} ds \\
&\geq \frac{1}{\eta}\int_t^{t+\eta}\int_\Gamma \Big(-[\psi]\, g'(|[u^\varepsilon(s)]|)\,\mathrm{sgn}[u^\varepsilon(s)]\, 1_{J_{u^\varepsilon(s)}} - \sigma|[\psi]| 1_{J^c_{u^\varepsilon(s)}}\Big)\, d\mathcal{H}^{N-1} ds. \qquad (5.22)
\end{aligned}
$$

Passing to the liminf as $n \to +\infty$ in (5.16) and taking into account (5.19) and (5.22) we obtain

$$
\begin{aligned}
&\frac{1}{\eta}\int_t^{t+\eta}\int_{\Omega\setminus\Gamma} \nabla u^\varepsilon(s)\cdot\nabla\psi\, dx\, ds + \frac{1}{\eta}\int_t^{t+\eta}\int_\Omega \varepsilon \dot{u}^\varepsilon(s)\psi\, dx\, ds \\
&\quad\geq \frac{1}{\eta}\int_t^{t+\eta}\int_\Gamma \Big(-[\psi]\, g'(|[u^\varepsilon(s)]|)\,\mathrm{sgn}[u^\varepsilon(s)]\, 1_{J_{u^\varepsilon(s)}} - \sigma|[\psi]| 1_{J^c_{u^\varepsilon(s)}}\Big)\, d\mathcal{H}^{N-1} ds
\end{aligned}
$$

for every $\psi \in H^1_0(\Omega\setminus\Gamma, \partial_D\Omega)$. Finally, letting η go to 0^+ we get $(iv)_\varepsilon$. □

5.2. **Proof of Proposition 4.10.** In this subsection we show the energy inequality (4.5) for variational parabolic evolutions.

Proof of Proposition 4.10. Let u^ε be a variational parabolic evolution with viscosity ε, boundary datum w and initial condition u_0. In particular, there exist a sequence of time step $\delta_n \to 0$ and a sequence of discrete time evolutions $\{u^{\varepsilon,\delta_n}\}_{n\in\mathbb{N}}$ such that (4.4) holds. By repeating the arguments used in the proof of Theorem 4.4 we obtain that relations (5.12) and (5.13) still hold true. Let now $t \in [0,T]$ be fixed. Relation (5.13) implies that

$$u^{\varepsilon,\delta_n}(t+\delta_n) - u^{\varepsilon,\delta_n}(t) \to 0 \text{ strongly in } L^2(\Omega).$$

From last relation, (4.4) and $(5.3)_1$ we deduce that

$$u^{\varepsilon,\delta_n}(t+\delta_n) \rightharpoonup u^\varepsilon(t) \text{ weakly in } H^1(\Omega\setminus\Gamma). \qquad (5.23)$$

At this point, we extract a subsequence δ_{n_k}, possibly depending on t, such that

$$[u^{\varepsilon,\delta_{n_k}}(t+\delta_{n_k})] \to [u^\varepsilon(t)] \quad \mathcal{H}^{N-1}\text{-a.e. in } \Gamma. \qquad (5.24)$$

For every $n \in \mathbb{N}$, let $l_n \in \mathbb{N}$ be such that $l_n\delta_n \leq t < (l_n+1)\delta_n$. Let us write relation (5.1) with $j = l_n + 1$ and $i = 0$. We obtain

$$
\begin{aligned}
&\frac{1}{2}\|\nabla u^{\varepsilon,\delta_n}(t+\delta_n)\|^2 + \int_\Gamma g(|[u^{\varepsilon,\delta_n}(t+\delta_n)]|)\, d\mathcal{H}^{N-1} + \frac{\varepsilon}{2}\int_0^{(l_n+1)\delta_n} \|\dot{v}^{\varepsilon,\delta_n}(s)\|^2 ds \\
&\quad\leq \frac{1}{2}\|\nabla u_0\|^2 + \int_\Gamma g(|[u_0]|) d\mathcal{H}^{N-1} + \int_0^{(l_n+1)\delta_n}\int_{\Omega\setminus\Gamma} \nabla u^{\varepsilon,\delta_n}(s)\cdot\nabla\dot{w}(s)\, dx\, ds \\
&\qquad + \frac{\varepsilon}{2}\int_0^{(l_n+1)\delta_n} \|\dot{w}(s)\|^2 ds + \rho(\delta_n). \qquad (5.25)
\end{aligned}
$$

For the left-hand side, thanks to (5.12), (5.23) and (5.24) we have

$$
\begin{aligned}
&\frac{1}{2}\|\nabla u^\varepsilon(t)\|^2 + \int_\Gamma g(|[u^\varepsilon(t)]|) d\mathcal{H}^{N-1} + \frac{\varepsilon}{2}\int_0^t \|\dot{u}^\varepsilon(s)\|^2 ds \\
&\quad\leq \liminf_{k\to+\infty}\Big(\frac{1}{2}\|\nabla u^{\varepsilon,\delta_{n_k}}(t+\delta_{n_k})\|^2 + \int_\Gamma g(|[u^{\varepsilon,\delta_{n_k}}(t+\delta_{n_k})]|)\, d\mathcal{H}^{N-1} + \frac{\varepsilon}{2}\int_0^t \|\dot{v}^{\varepsilon,\delta_{n_k}}(s)\|^2 ds\Big) \\
&\quad\leq \limsup_{n\to+\infty}\Big(\frac{1}{2}\|\nabla u^{\varepsilon,\delta_n}(t+\delta_n)\|^2 + \int_\Gamma g(|[u^{\varepsilon,\delta_n}(t+\delta_n)]|)\, d\mathcal{H}^{N-1} + \frac{\varepsilon}{2}\int_0^{(l_n+1)\delta_n} \|\dot{v}^{\varepsilon,\delta_n}(s)\|^2 ds\Big).
\end{aligned}
$$

Passing to the limsup as $n \to +\infty$ in (5.25) and taking into account last relation

$$\begin{aligned}
&\frac{1}{2}\|\nabla u^\varepsilon(t)\|^2 + \int_\Gamma g(|[u^\varepsilon(t)]|)d\mathcal{H}^{N-1} + \frac{\varepsilon}{2}\int_0^t \|\dot u^\varepsilon(s)\|^2 ds \le \frac{1}{2}\|\nabla u_0\|^2 + \int_\Gamma g(|[u_0]|)\,d\mathcal{H}^{N-1}\\
&\quad + \limsup_{n\to+\infty}\left(\int_0^{(l_n+1)\delta_n}\int_{\Omega\setminus\Gamma}\nabla u^{\varepsilon,\delta_n}(s)\cdot\nabla\dot w(s)\,dx\,ds + \frac{\varepsilon}{2}\int_0^{(l_n+1)\delta_n}\|\dot w(s)\|^2 ds + \rho(\delta_n)\right)\\
&= \frac{1}{2}\|\nabla u_0\|^2 + \int_\Gamma g(|[u_0]|)\,d\mathcal{H}^{N-1} + \int_0^t\int_{\Omega\setminus\Gamma}\nabla u^\varepsilon(s)\cdot\nabla\dot w(s)\,dx\,ds + \frac{\varepsilon}{2}\int_0^t\|\dot w(s)\|^2 ds.
\end{aligned}$$

□

We now give the proof of the uniqueness result.

5.3. **Proof of Theorem 4.5.** To start with, we prove the following auxiliary lemma.

Lemma 5.3. *For every $C_1 > 0$ there exists $C_2 > 0$ such that*

$$\|[u]\|^2_{L^2(\Gamma)} \le C_2\|u\|^2 + C_1\|\nabla u\|^2 \qquad \textit{for every } u \in H^1(\Omega\setminus\Gamma).$$

Proof. By contradiction, let us assume that the thesis does not hold. Then, there exists $C_1 > 0$ with the following property. For every $n \in \mathbb{N}$ there exists $u_n \in H^1(\Omega\setminus\Gamma)$ with $\|[u_n]\|_{L^2(\Gamma)} = 1$ such that

$$1 > n\|u_n\|^2 + C_1\|\nabla u_n\|^2.$$

Letting n go to infinity, we have that $u_n \to 0$ in $L^2(\Omega)$. Since $\|\nabla u_n\|$ is bounded, up to subsequences $u_n \rightharpoonup 0$ weakly in $H^1(\Omega\setminus\Gamma)$. This implies $[u_n] \to 0$ in $L^2(\Gamma)$, which contradicts the fact that $\|[u_n]\|_{L^2(\Gamma)} = 1$ for every $n \in \mathbb{N}$. □

We can now prove Theorem 4.5.

Proof of Theorem 4.5. By contradiction, let us assume that there exist two different parabolic evolutions u_1, u_2 with viscosity ε, boundary datum w and initial condition u_0. Specifying (4.2) for u_1 with test function $\psi = u_2 - u_1$ we obtain

$$\begin{aligned}
&\int_{\Omega\setminus\Gamma}\nabla u_1\cdot\nabla(u_2-u_1)\,dx + \int_\Omega \varepsilon\dot u_1(u_2-u_1)\,dx\\
&\quad + \int_\Gamma\Big([u_2-u_1]\,g'(|[u_1]|)\,\mathrm{sgn}[u_1]\,1_{J_{u_1}} + \sigma|[u_2]|1_{J^c_{u_1}}\Big)\,d\mathcal{H}^{N-1} \ge 0.
\end{aligned}$$

Here we omit the dependence on the time variable, fixing $t \in [0,T]$ such that (4.2) holds for both $u_1(t)$ and $u_2(t)$. Summing up last inequality with the analogous relation obtained by exchanging the role of u_1 and u_2, we get

$$\begin{aligned}
&\|\nabla(u_1-u_2)\|^2 + \frac{\varepsilon}{2}\frac{d}{dt}\left(\|u_1-u_2\|^2\right)\\
&\le \int_\Gamma\Big([u_2-u_1]\,g'(|[u_1]|)\,\mathrm{sgn}[u_1]\,1_{J_{u_1}} + \sigma|[u_2]|1_{J^c_{u_1}}\Big)\,d\mathcal{H}^{N-1}\\
&\quad + \int_\Gamma\Big([u_1-u_2]\,g'(|[u_2]|)\,\mathrm{sgn}[u_2]\,1_{J_{u_2}} + \sigma|[u_1]|1_{J^c_{u_2}}\Big)\,d\mathcal{H}^{N-1}\\
&= \int_\Gamma\Big([u_2-u_1]\Big(g'(|[u_1]|)\,\mathrm{sgn}[u_1] - g'(|[u_2]|)\,\mathrm{sgn}[u_2]\Big)\,1_{J_{u_1}\cap J_{u_2}}\,d\mathcal{H}^{N-1}\\
&\quad + \int_\Gamma |[u_2]|\Big(\sigma - g'(|[u_2]|)\Big)\,1_{J^c_{u_1}\cap J_{u_2}}\,d\mathcal{H}^{N-1}\\
&\quad + \int_\Gamma |[u_1]|\Big(\sigma - g'(|[u_1]|)\Big)\,1_{J^c_{u_2}\cap J_{u_1}}\,d\mathcal{H}^{N-1}.
\end{aligned}$$

Notice that in the right-hand side the argument of the first integral is negative if $[u_1][u_2] < 0$. Moreover, if $[u_1][u_2] > 0$ there holds $|[u_2] - [u_1]| = |\,|[u_2]| - |[u_1]|\,|$, so that

$$\begin{aligned}
&\|\nabla(u_1 - u_2)\|^2 + \frac{\varepsilon}{2}\frac{d}{dt}\left(\|u_1 - u_2\|^2\right) \\
&\quad\le \int_\Gamma |[u_2] - [u_1]|\,|\,g'(|[u_1]|) - g'(|[u_2]|)\,|\,1_{\{[u_1][u_2]>0\}}\,d\mathcal{H}^{N-1} \\
&\qquad + \int_\Gamma |[u_2]|\Big(\sigma - g'(|[u_2]|)\Big)\,1_{J^c_{u_1}\cap J_{u_2}}\,d\mathcal{H}^{N-1} \\
&\qquad + \int_\Gamma |[u_1]|\Big(\sigma - g'(|[u_1]|)\Big)\,1_{J^c_{u_2}\cap J_{u_1}}\,d\mathcal{H}^{N-1} \\
&\quad\le \int_\Gamma L|[u_1 - u_2]|^2 1_{\{[u_1][u_2]>0\}}\,d\mathcal{H}^{N-1} + \int_\Gamma L|[u_1 - u_2]|^2 1_{J^c_{u_1}\cap J_{u_2}}\,d\mathcal{H}^{N-1} \\
&\qquad + \int_\Gamma L|[u_1 - u_2]|^2 1_{J^c_{u_2}\cap J_{u_1}}\,d\mathcal{H}^{N-1} \\
&\quad\le L\|[u_1 - u_2]\|^2_{L^2(\Gamma)},
\end{aligned}$$

where $L > 0$ is the Lipschitz constant of g'. Thus, we obtained

$$\frac{\varepsilon}{2}\frac{d}{dt}\left(\|u_1 - u_2\|^2\right) \le -\|\nabla(u_1 - u_2)\|^2 + L\|[u_1 - u_2]\|^2_{L^2(\Gamma)}.$$

Applying the previous lemma with $C_1 = \frac{1}{L}$

$$\frac{\varepsilon}{2}\frac{d}{dt}\left(\|u_1 - u_2\|^2\right) \le -\|\nabla(u_1 - u_2)\|^2 + L\Big(C_2\|u_1 - u_2\|^2 + \frac{1}{L}\|\nabla(u_1 - u_2)\|^2\Big).$$

Hence, for a.e. $t \in [0, T]$

$$\frac{d}{dt}\left(\|u_1(t) - u_2(t)\|^2\right) \le \frac{2LC_2}{\varepsilon}\|u_1(t) - u_2(t)\|^2.$$

Using the version of the Gronwall Lemma stated in [3, Lemma 4.1.8] we get $u_1 = u_2$. □

6. Proof of Theorem 4.13

In order to prove the theorem, we need the following result.

Lemma 6.1. *Let X be a compact metric space. Let $p : [0, T] \to \mathbb{R}$, $p_k : [0, T] \to \mathbb{R}$ and $f_k : [0, T] \to X$ be measurable functions, for every $k \in \mathbb{N}$. For every $t \in [0, T]$ let us set*

$$\mathcal{I}(t) := \{x \in X : \exists\, k_j \to +\infty \text{ such that } x = \lim_{j\to+\infty} f_{k_j}(t) \textit{ and } p(t) = \lim_{j\to\infty} p_{k_j}(t)\}.$$

Then, the following facts hold:

- *$\mathcal{I}(t)$ is closed for all $t \in [0, T]$;*
- *for every open set $U \subseteq X$ the set $\{t \in [0, T] : \mathcal{I}(t) \cap U \neq \emptyset\}$ is measurable.*

For the proof, we refer to [8].

Proof of Theorem 4.13. We want to prove that there exists $u : [0, T] \to H^1(\Omega \setminus \Gamma)$ bounded and measurable such that the three conditions of Definition 4.11 are satisfied.

Thanks to Theorem 4.8, for every $\varepsilon \in (0, 1)$ we can consider a variational parabolic evolution u^ε with viscosity ε, boundary datum w and initial condition u_0. In particular, there exist a sequence of time steps $\delta_n \to 0^+$ and a sequence $\{u^{\varepsilon,\delta_n}\}_{n\in\mathbb{N}}$ of discrete time evolutions such that (4.4) and $(5.5)_1$ hold. This implies that for every $\varepsilon \in (0, 1)$

$$\|\dot{u}^\varepsilon\|_{L^2((0,T);L^2(\Omega))} \le \liminf_{n\to+\infty} \|\dot{v}^{\varepsilon,\delta_n}\|_{L^2((0,T);L^2(\Omega))} \le \frac{C}{\sqrt{\varepsilon}}.$$

Then, there exists a sequence $\varepsilon_n \to 0^+$ such that

$$\varepsilon_n \dot{u}^{\varepsilon_n}(t) \to 0 \text{ strongly in } L^2(\Omega) \qquad \text{for a.e. } t \in [0, T]. \tag{6.1}$$

Let $\Theta\subset[0,T]$ be such that $\mathcal{L}^1(\Theta)=0$ and $\dot w(t)$ is defined for every $t\in[0,T]\setminus\Theta$. We set for every $n\in\mathbb{N}$

$$\theta^{\varepsilon_n}(t):=\begin{cases}\displaystyle\int_{\Omega\setminus\Gamma}\nabla u^{\varepsilon_n}(t)\cdot\nabla\dot w(t)\,dx & \text{for } t\in[0,T]\setminus\Theta,\\ 0 & \text{for } t\in\Theta,\end{cases}$$

and for every $t\in[0,T]$

$$\theta(t):=\limsup_{n\to+\infty}\theta^{\varepsilon_n}(t).$$

We point out that θ, as pointwise limsup of a sequence of measurable functions, is measurable. It turns out that $\theta\in L^1(0,T)$. Indeed, by (4.4) and $(5.3)_1$ it follows that

$$u^{\varepsilon_n}(t)\in B_C\quad\forall\, n\in\mathbb{N},\ t\in[0,T],\tag{6.2}$$

where B_C is defined by (5.9). Moreover, since $w\in H^1((0,T);H^1(\Omega))$,

$$\int_0^T|\theta(t)|\,dt\le\int_0^T\limsup_{n\to+\infty}\|\nabla u^{\varepsilon_n}(t)\|\,\|\nabla\dot w(t)\|\,dt\le\overline{C}T,$$

for some constant $\overline{C}>0$. By definition of θ, for every $t\in[0,T]$ we can extract a subsequence $\varepsilon_{n_k}(t)\to0^+$, possibly depending on t, such that

$$\theta(t)=\lim_{k\to+\infty}\theta^{\varepsilon_{n_k}(t)}(t).$$

By (6.2), for every $t\in[0,T]$ we can extract a further subsequence (not relabelled) such that

$$u^{\varepsilon_{n_k}(t)}(t)\rightharpoonup u(t)\qquad\text{weakly in } H^1(\Omega\setminus\Gamma),\tag{6.3}$$

for some $u(t)\in B_C$. This shows that the set

$$\mathcal{I}(t):=\{u\in B_C:\exists\,\varepsilon_{n_k}\to0^+\text{ such that } u^{\varepsilon_{n_k}}(t)\rightharpoonup u\text{ weakly in } H^1(\Omega\setminus\Gamma)\text{ and }\theta^{\varepsilon_{n_k}}(t)\longrightarrow\theta(t)\}$$

is not empty for every $t\in[0,T]$.

In the following, we will consider B_C endowed with the metric compatible with the weak topology of $H^1(\Omega\setminus\Gamma)$. In this way, B_C becomes a compact metric space and we can apply Lemma 6.1. Hence, for every $t\in[0,T]$ the set $\mathcal{I}(t)$ is closed in B_C. Moreover, for every open set U of B_C the set $\{t\in[0,T]:\mathcal{I}(t)\cap U\neq\emptyset\}$ is measurable. Using [6, Theorem III.6], for every $t\in[0,T]$ we can select $u(t)\in\mathcal{I}(t)$ in such a way that $t\longmapsto u(t)$ is measurable from $[0,T]$ to B_C. Since $t\longmapsto u(t)$ is separably valued, we get measurability from $[0,T]$ to $H^1(\Omega\setminus\Gamma)$ endowed with the strong topology (see [21, Chapter V, Section 4]). This shows the approximability condition and the fact that u is measurable and bounded.

Let us prove the energy inequality. First, we notice that for every $t\in[0,T]\setminus\Theta$ there holds

$$\begin{aligned}\theta(t)=\limsup_{n\to\infty}\theta^{\varepsilon_n}(t)=\lim_{k\to\infty}\theta^{\varepsilon_{n_k}(t)}(t)&=\lim_{k\to\infty}\int_{\Omega\setminus\Gamma}\nabla u^{\varepsilon_{n_k}(t)}(t)\cdot\nabla\dot w(t)\,dx\\ &=\int_{\Omega\setminus\Gamma}\nabla u(t)\cdot\nabla\dot w(t)\,dx,\end{aligned}$$

since $u^{\varepsilon_{n_k}(t)}$ converges weakly to $u(t)$ in $H^1(\Omega\setminus\Gamma)$. Up to subsequences, we can assume that

$$[u^{\varepsilon_{n_k}(t)}(t)]\to[u(t)]\qquad\mathcal{H}^{N-1}\text{-a.e. in }\Gamma.$$

Consider now inequality (4.5) for the functions $u^{\varepsilon_{n_k}(t)}(t)$. Using last relation, (6.3), and Fatou's Lemma we get that for every $t \in [0,T]$

$$\begin{aligned}
&\frac{1}{2}\|\nabla u(t)\|^2 + \int_\Gamma g(|[u(t)]|)d\mathcal{H}^{N-1} \\
&\quad\le \liminf_{k\to+\infty}\left(\frac{1}{2}\|\nabla u^{\varepsilon_{n_k}(t)}(t)\|^2 + \int_\Gamma g(|[u^{\varepsilon_{n_k}(t)}(t)]|)d\mathcal{H}^{N-1}\right) \\
&\quad\le \limsup_{k\to+\infty}\left(\frac{1}{2}\|\nabla u^{\varepsilon_{n_k}(t)}(t)\|^2 + \int_\Gamma g(|[u^{\varepsilon_{n_k}(t)}(t)]|)d\mathcal{H}^{N-1}\right) \\
&\quad\le \frac{1}{2}\|\nabla u_0\|^2 + \int_\Gamma g(|[u_0]|)d\,\mathcal{H}^{N-1} + \limsup_{n\to+\infty}\left(\int_0^t\int_{\Omega\setminus\Gamma}\nabla u^{\varepsilon_n}(s)\cdot\nabla\dot w(s)\,dx\,ds\right) \\
&\quad\le \frac{1}{2}\|\nabla u_0\|^2 + \int_\Gamma g(|[u_0]|)d\,\mathcal{H}^{N-1} + \int_0^t\limsup_{n\to+\infty}\int_{\Omega\setminus\Gamma}\nabla u^{\varepsilon_n}(s)\cdot\nabla\dot w(s)\,dx\,ds \\
&\quad= \frac{1}{2}\|\nabla u_0\|^2 + \int_\Gamma g(|[u_0]|)d\,\mathcal{H}^{N-1} + \int_0^t\int_{\Omega\setminus\Gamma}\nabla u(s)\cdot\nabla\dot w(s)\,dx\,ds.
\end{aligned}$$

This shows (4.7).

It remains to prove the stationarity. For almost every $t \in [0,T]$, we can consider inequality (4.2) for the functions $u^{\varepsilon_{n_k}(t)}(t)$ with an arbitrary $\psi \in H^1_0(\Omega\setminus\Gamma, \partial_D\Omega)$:

$$\begin{aligned}
&\int_{\Omega\setminus\Gamma}\nabla u^{\varepsilon_{n_k}(t)}(t)\cdot\nabla\psi\,dx + \int_\Omega \varepsilon_{n_k}(t)\,\dot u^{\varepsilon_{n_k}(t)}(t)\,\psi\,dx \\
&\quad\ge -\int_\Gamma\left([\psi]\,g'(|[u^{\varepsilon_{n_k}(t)}(t)]|)\,\mathrm{sgn}[u^{\varepsilon_{n_k}(t)}(t)]\,1_{J_{u^{\varepsilon_{n_k}(t)}(t)}} + \sigma|[\psi]|1_{J^c_{u^{\varepsilon_{n_k}(t)}(t)}}\right)d\,\mathcal{H}^{N-1}.
\end{aligned} \tag{6.4}$$

Up to a $\mathcal{L}^1$-negligible set of times, we can also assume that (6.1) holds. Hence, passing to the limit in the left-hand side of (6.4) and using (6.3)

$$\lim_{k\to+\infty}\left(\int_{\Omega\setminus\Gamma}\nabla u^{\varepsilon_{n_k}(t)}(t)\cdot\nabla\psi\,dx + \varepsilon_{n_k}(t)\int_\Omega \dot u^{\varepsilon_{n_k}(t)}(t)\,\psi\,dx\right) = \int_{\Omega\setminus\Gamma}\nabla u(t)\cdot\nabla\psi\,dx. \tag{6.5}$$

With the same argument used to prove (5.20) one can show that

$$\liminf_{k\to+\infty} f^k(t) \ge -[\psi]\,g'(|[u(t)]|)\,\mathrm{sgn}[u(t)]\,1_{J_{u(t)}} - \sigma|[\psi]|1_{J^c_{u(t)}} \qquad \mathcal{H}^{N-1}\text{-a.e. in } \Gamma, \tag{6.6}$$

where

$$f^k(t) := -[\psi]\,g'(|[u^{\varepsilon_{n_k}(t)}(t)]|)\,\mathrm{sgn}[u^{\varepsilon_{n_k}(t)}(t)]\,1_{J_{u^{\varepsilon_{n_k}(t)}(t)}} - \sigma|[\psi]|1_{J^c_{u^{\varepsilon_{n_k}(t)}(t)}}.$$

Finally, taking the liminf of relation (6.4) and using (6.5), (6.6) and Fatou's Lemma, we get (3.1). □

7. Proof of Theorem 4.14

To fix the ideas, let us assume $w(t) \ge 0$ on Λ^+_D for every $t \in [0,T]$, the other case being analogous. We recall that for every $(\varepsilon,\delta) \in (0,1)\times(0,T)$ and for every $i \in \mathbb{N}$ with $i\delta \le T$ the functions $u^{\varepsilon,\delta}_i$ are introduced in Definition 4.6. We will divide the proof into three steps:

1. $u^{\varepsilon,\delta}_i$ is odd for every $(\varepsilon,\delta) \in (0,1)\times(0,T)$ and for every $i \in \mathbb{N}$ with $i\delta \le T$;
2. $u^{\varepsilon,\delta}_i \ge u^{\varepsilon,\delta}_{i-1}$ a.e. in Ω^+ for every $(\varepsilon,\delta) \in (0,1)\times(0,T)$ and for every $i \in \mathbb{N}$ with $i\delta \le T$.
3. Proof of Theorem 4.14.

Step 1: $u^{\varepsilon,\delta}_i$ is odd. Let us fix $(\varepsilon,\delta) \in (0,1)\times(0,T)$ and $i \in \mathbb{N}$ with $i\delta \le T$. First of all, $u^{\varepsilon,\delta}_0$ is odd since, by definition of discrete time evolution, $u^{\varepsilon,\delta}_0 = u_0 = 0$. We will then argue by induction, proving that $u_i^{\varepsilon,\delta}$ is odd under the assumption that $u_{i-1}^{\varepsilon,\delta}$ is odd. We set $v^\pm := u_i^{\varepsilon,\delta}\,|_{\Omega^\pm}$. From the results contained in Section 3 it can be inferred that v^+ satisfies the following problem in Ω^+:

$$\begin{cases} \Delta v^+ = \dfrac{\varepsilon}{\delta}(v^+ - u_{i-1}^{\varepsilon,\delta}) & \text{in } \Omega^+, \\ v^+ = w(t) & \text{on } \Lambda_D^+, \\ \partial_\nu v^+ = 0 & \text{on } \partial\Omega^+ \setminus (\Lambda_D^+ \cup \Gamma), \\ \partial_\nu v^+ = \partial_\nu v^- & \text{on } \Gamma. \end{cases}$$

Let us now define the function $\tilde{v}^- \in H^1(\Omega^-)$ such that $\tilde{v}^-(x_1, x') := -v^+(x_1, x')$ for every $(x_1, x') \in \Omega^-$. Using the definition of $\tilde{v}^-$ and the fact that $w(t)$ and $u_{i-1}^{\varepsilon,\delta}$ are odd, it follows that $v := \tilde{v}^- - v^-$ satisfies the following equation in Ω^-:

$$\begin{cases} \Delta v = \dfrac{\varepsilon}{\delta} v & \text{in } \Omega^-, \\ v = 0 & \text{on } \Lambda_D^-, \\ \partial_\nu v = 0 & \text{on } \partial\Omega^- \setminus (\Lambda_D^- \cup \Gamma), \\ \partial_\nu v = 0 & \text{on } \Gamma. \end{cases} \tag{7.1}$$

Since the unique solution of (7.1) is $v \equiv 0$, $\tilde{v}^- = v^-$ and the claim is proved.

Remark 7.1. In the same way one can show that, under the same assumptions, all the critical points of the energy functional (2.1) are odd.

Step 2: $u_i^{\varepsilon,\delta} \geq u_{i-1}^{\varepsilon,\delta}$ *a.e. in* Ω^+. First of all, notice that the thesis holds for $i = 1$, since by a truncation argument it follows that $u_j^{\varepsilon,\delta} \geq u_0^{\varepsilon,\delta} = 0$ for every $(\varepsilon, \delta) \in (0,1) \times (0,T)$ and $j \in \mathbb{N}$ with $j\delta \leq T$. The proof is then completed by the following lemma.

Lemma 7.2. *Let $i \in \mathbb{N}$ with $i \geq 2$ and $i\delta \leq T$ and assume $0 \leq u_1^{\varepsilon,\delta} \leq \ldots \leq u_{i-1}^{\varepsilon,\delta}$ a.e. in Ω^+. Then $u_{i-1}^{\varepsilon,\delta} \leq u_i^{\varepsilon,\delta}$ a.e. in Ω^+.*

Proof. Let us fix $(\varepsilon, \delta) \in (0,1) \times (0,T)$ and $i \in \mathbb{N}$ with $i\delta \leq T$. We want to show that

$$u_i^{\varepsilon,\delta} \geq u_j^{\varepsilon,\delta} \text{ a.e. in } \Omega^+ \qquad \text{for every } j = 0, 1, \ldots, i-1.$$

As already observed, there holds $u_i^{\varepsilon,\delta} \geq u_0^{\varepsilon,\delta} = 0$. By induction, let us assume $u_i^{\varepsilon,\delta} \geq u_j^{\varepsilon,\delta}$ for $j = 0, \ldots, k-1$ with k integer such that $k < i$; we shall show that this implies $u_i^{\varepsilon,\delta} \geq u_k^{\varepsilon,\delta}$. Let us define

$$Z := \{(x_1, x') \in \Omega^+ : u_i^{\varepsilon,\delta}(x_1, x') < u_k^{\varepsilon,\delta}(x_1, x')\}.$$

Assume, by contradiction, that $\mathcal{L}^N(Z) > 0$. Define now the functional $E_i : H^1(\Omega^+) \to \mathbb{R}$ as

$$E_i(v) := \frac{1}{2}\int_Z |\nabla v|^2 dx + \frac{1}{2}\int_{\Gamma \cap \overline{Z}} g(2|v|)\, d\mathcal{H}^{N-1} + \frac{\varepsilon}{2\delta}\int_Z |v - u_{i-1}^{\varepsilon,\delta}|^2 dx.$$

Let us denote with V_i^k the odd function of $H^1(\Omega \setminus \Gamma)$ coinciding a.e. with $u_k^{\varepsilon,\delta}$ in Z and such that $V_i^k = u_i^{\varepsilon,\delta}$ a.e. in $\Omega^+ \setminus Z$. Notice that $V_i^k = u_i^{\varepsilon,\delta}$ on $\partial_D\Omega$, so that V_i^k is a competitor for the problem (4.3). By the minimality of $u_i^{\varepsilon,\delta}$ we get

$$E(u_i^{\varepsilon,\delta}) + \frac{\varepsilon}{2\delta}\|u_i^{\varepsilon,\delta} - u_{i-1}^{\varepsilon,\delta}\|^2 \leq E(V_i^k) + \frac{\varepsilon}{2\delta}\|V_i^k - u_{i-1}^{\varepsilon,\delta}\|^2.$$

Since $u_i^{\varepsilon,\delta} = V_i^k$ a.e. in $\Omega^+ \setminus Z$ and using the fact that $u_{i-1}^{\varepsilon,\delta}$, $u_i^{\varepsilon,\delta}$ and V_i^k are odd, last inequality implies

$$E_i(u_i^{\varepsilon,\delta}) \leq E_i(V_i^k) = E_i(u_k^{\varepsilon,\delta}). \tag{7.2}$$

Moreover, $0 \leq u_1^{\varepsilon,\delta} \leq \ldots \leq u_{i-1}^{\varepsilon,\delta}$ a.e. in Ω^+ and $u_i^{\varepsilon,\delta} < u_k^{\varepsilon,\delta}$ a.e. in Z, so that

$$\frac{\varepsilon}{2\delta}\int_Z |u_i^{\varepsilon,\delta} - u_{i-1}^{\varepsilon,\delta}|^2 dx > \frac{\varepsilon}{2\delta}\int_Z |u_k^{\varepsilon,\delta} - u_{i-1}^{\varepsilon,\delta}|^2 dx. \tag{7.3}$$

From (7.2) and (7.3) it follows

$$\frac{1}{2}\int_Z |\nabla u_i^{\varepsilon,\delta}|^2 + \frac{1}{2}\int_{\Gamma\cap\overline{Z}} g(|[u_i^{\varepsilon,\delta}]|)\, d\mathcal{H}^{N-1} < \frac{1}{2}\int_Z |\nabla u_k^{\varepsilon,\delta}|^2 + \frac{1}{2}\int_{\Gamma\cap\overline{Z}} g(|[u_k^{\varepsilon,\delta}]|)\, d\mathcal{H}^{N-1}, \tag{7.4}$$

while from the fact that $0 \le u_{k-1}^{\varepsilon,\delta} \le u_i^{\varepsilon,\delta} < u_k^{\varepsilon,\delta}$ a.e. in Z we get

$$\frac{\varepsilon}{2\delta}\int_Z |u_i^{\varepsilon,\delta} - u_{k-1}^{\varepsilon,\delta}|^2 dx < \frac{\varepsilon}{2\delta}\int_Z |u_k^{\varepsilon,\delta} - u_{k-1}^{\varepsilon,\delta}|^2 dx. \tag{7.5}$$

Let us now denote with V_k^i the odd function of $H^1(\Omega\setminus\Gamma)$ coinciding a.e. with $u_i^{\varepsilon,\delta}$ in Z and such that $V_k^i = u_k^{\varepsilon,\delta}$ a.e. in $\Omega^+\setminus Z$. Notice that V_k^i is a competitor for the problem (4.3) with i replaced by k, since $V_k^i = u_k^{\varepsilon,\delta}$ on $\partial_D\Omega$. Collecting relations (7.4) and (7.5), we have

$$E(V_k^i) + \frac{\varepsilon}{2\delta}\,\|V_k^i - u_{k-1}^{\varepsilon,\delta}\|^2 < E(u_k^{\varepsilon,\delta}) + \frac{\varepsilon}{2\delta}\,\|u_k^{\varepsilon,\delta} - u_{k-1}^{\varepsilon,\delta}\|^2,$$

against the minimality of $u_k^{\varepsilon,\delta}$.

□

Step 3: Proof of Theorem 4.14. From the previous steps it follows that for every discrete time evolution $u^{\varepsilon,\delta}$ with time step δ, viscosity ε, boundary datum w and initial condition u_0 the function $t \mapsto |[u^{\varepsilon,\delta}(t)]|(0,x')$ is non decreasing for $\mathcal{H}^{N-1}$-a.e. $(0,x')\in\Gamma$. From this and from the definition of approximable quasistatic evolution the conclusion follows.

8. Proof of Theorem 4.15 (Crack Initiation Criterion)

We start with the following definition.

Definition 8.1. Let $\delta\in(0,T)$ and $\varepsilon\in(0,1)$. We define the *elastic discrete-time evolution* with time step δ, viscosity ε, boundary datum w and initial condition u_0 as the function $z^{\varepsilon,\delta} : [0,T] \to H^1(\Omega)$ such that $z^{\varepsilon,\delta}(t) := z_i^{\varepsilon,\delta}$ for $i\delta \le t < (i+1)\delta$, where $z_0^{\varepsilon,\delta} := z(0)$ and, by induction, $z_i^{\varepsilon,\delta}$ is the unique solution to

$$\min_{v\in\mathcal{B}(i\delta)} \left\{ \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \frac{\varepsilon}{2\delta}\|v - z_{i-1}^{\varepsilon,\delta}\|^2 \right\} \tag{8.1}$$

for $i\in\mathbb{N}$ with $i\delta\le T$. Here $\mathcal{B}(i\delta)$ is defined by (4.9).

Remark 8.2. For $j\in\mathbb{N}$ with $j\delta\le T$ one can define the functions $\alpha_j^{\varepsilon,\delta} := z_j^{\varepsilon,\delta} - z_{j-1}^{\varepsilon,\delta}$. Setting $\alpha_0^{\varepsilon,\delta} = 0$, it turns out that $\alpha_j^{\varepsilon,\delta}$ is the unique solution to the problem

$$\min_v \left\{ \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \frac{\varepsilon}{2\delta}\|v - \alpha_{j-1}^{\varepsilon,\delta}\|^2 \right\}$$

for every $j\in\mathbb{N}$ with $j\delta\le T$, where the minimum is taken over all the functions $v\in H^1(\Omega)$ such that $v = w(j\delta) - w((j-1)\delta)$ on $\partial_D\Omega$. Moreover, by a truncation argument it follows that for every $j\in\mathbb{N}$ with $j\delta\le T$

$$\|\alpha_j^{\varepsilon,\delta}\|_{L^\infty(\Omega)} \le \max_{k=1,\dots,j}\|w(k\delta) - w((k-1)\delta)\|_{L^\infty(\Omega)} \le \delta \sup_{t\in[0,T]}\|\dot{w}(t)\|_{L^\infty(\Omega)}. \tag{8.2}$$

The proof of Theorem 4.15 relies in the following two propositions. The first one shows that when ε and δ tend to zero the elastic discrete-time evolution $z^{\varepsilon,\delta}(t)$ converges strongly to $z(t)$ in $H^1(\Omega)$ uniformly with respect to $t\in[0,T]$.

Proposition 8.3. *There holds*

$$\lim_{\varepsilon\to 0^+}\lim_{\delta\to 0^+}\sup_{t\in[0,T]}\|z^{\varepsilon,\delta}(t) - z(t)\|_{H^1(\Omega)} = 0. \tag{8.3}$$

Proof. For $t \in [0,T]$ fixed, we will denote with $i_\delta = i_\delta(t)$ the only integer such that $i_\delta \delta \le t < (i_\delta + 1)\delta$. From (4.8) and (8.1) we conclude that the function $\hat{z}^{\varepsilon,\delta}(t) := z(i_\delta \delta) - z^{\varepsilon,\delta}(t)$ satisfies:

$$\int_\Omega \nabla \hat{z}^{\varepsilon,\delta}(t) \cdot \nabla \psi \, dx = \frac{\varepsilon}{\delta} \int_\Omega \alpha_{i_\delta}^{\varepsilon,\delta} \psi \, dx \qquad \forall\, \psi \in H_0^1(\Omega, \partial_D \Omega). \tag{8.4}$$

Taking $\hat{z}^{\varepsilon,\delta}(t)$ as a test function and using Hölder and Poincaré inequalities together with (8.2), we get that

$$\|\nabla \hat{z}^{\varepsilon,\delta}(t)\|^2 \le \varepsilon\, (\mathcal{L}^N(\Omega))^{\frac{1}{2}} \sup_{t \in [0,T]} \|\dot{w}(t)\|_{L^\infty(\Omega)} \, \|\hat{z}^{\varepsilon,\delta}(t)\| \le \varepsilon\, \overline{C}\, \|\nabla \hat{z}^{\varepsilon,\delta}(t)\|,$$

where $\overline{C}$ is a positive constant independent of δ, ε and t. Applying once again Poincaré inequality, from the last relation it follows that

$$\sup_{\delta \in (0,T)} \sup_{t \in [0,T]} \|\hat{z}^{\varepsilon,\delta}(t)\|_{H^1(\Omega)} \stackrel{\varepsilon \to 0^+}{\longrightarrow} 0. \tag{8.5}$$

On the other hand, the difference $z(t) - z(i_\delta \delta)$ satisfies

$$\int_\Omega \nabla (z(t) - z(i_\delta \delta)) \cdot \nabla \psi \, dx = 0 \qquad \forall\, \psi \in H_0^1(\Omega, \partial_D \Omega).$$

Considering as test function $z(t) - z(i_\delta \delta) - w(t) + w(i_\delta \delta)$ and using Hölder inequality we obtain

$$\|\nabla (z(t) - z(i_\delta \delta))\| \le \|\nabla (w(t) - w(i_\delta \delta))\|. \tag{8.6}$$

Since $w : [0,T] \to H^1(\Omega)$ is uniformly continuous, using (8.6) and Poincaré inequality we get

$$\lim_{\delta \to 0^+} \sup_{t \in [0,T]} \|z(t) - z(i_\delta \delta)\|_{H^1(\Omega)} = 0.$$

By triangular inequality the thesis follows from last relation and (8.5).

□

The following proposition, whose proof is postponed to the Appendix, shows that when ε and δ are sufficiently small, the only possible discrete-time evolution in the time interval $[0, t^*]$ is just the elastic one.

Proposition 8.4. *There exist $\overline{\varepsilon} \in (0,1)$ and a function $\hat{\delta} : (0, \overline{\varepsilon}) \to (0,T)$ with the following property. Let $\varepsilon \in (0, \overline{\varepsilon})$ and $\delta \in (0, \hat{\delta}(\varepsilon))$, and let $u^{\varepsilon,\delta} : [0,T] \to H^1(\Omega \setminus \Gamma)$ be a discrete-time evolution with time step δ, viscosity ε, boundary datum w and initial condition u_0. Then, $u^{\varepsilon,\delta}(t) = z^{\varepsilon,\delta}(t)$ for every $t \in [0, t^*]$.*

We can now give the proof of Theorem 4.15.

Proof of Theorem 4.15. Let $t \in [0, t^*]$ be fixed and let $u : [0,T] \to H^1(\Omega \setminus \Gamma)$ be an approximable quasistatic evolution with boundary datum w and initial condition u_0. Then, there exists a family $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$ of variational parabolic evolutions and a subsequence $\varepsilon_n(t) \to 0^+$ such that condition (4.6) holds:

$$u^{\varepsilon_n(t)}(t) \rightharpoonup u(t) \text{ weakly in } H^1(\Omega \setminus \Gamma). \tag{8.7}$$

Let $n \in \mathbb{N}$ be fixed and so large that $\varepsilon_n(t) \in (0, \overline{\varepsilon})$, where $\overline{\varepsilon}$ is given by Proposition 8.4. By definition of variational parabolic evolution, there exists a family of discrete-time evolutions $\{u^{\varepsilon_n(t),\delta}\}_{\delta \in (0,T)}$ and a sequence $\delta_k \to 0^+$ such that condition (4.4) is satisfied:

$$u^{\varepsilon_n(t),\delta_k}(t) \rightharpoonup u^{\varepsilon_n(t)}(t) \text{ weakly in } H^1(\Omega \setminus \Gamma). \tag{8.8}$$

For k sufficiently large we have $\varepsilon_n(t) \in (0, \overline{\varepsilon})$ and $\delta_k \in (0, \hat{\delta}(\varepsilon_n(t)))$. Hence, applying Proposition 8.4 we have $u^{\varepsilon_n(t),\delta_k}(t) = z^{\varepsilon_n(t),\delta_k}(t)$ and relations (8.7) and (8.8) become

$$z^{\varepsilon_n(t),\delta_k}(t) \rightharpoonup u^{\varepsilon_n(t)}(t) \quad \text{and} \quad u^{\varepsilon_n(t)}(t) \rightharpoonup u(t) \qquad\qquad \text{weakly in } H^1(\Omega \setminus \Gamma).$$

Thanks to Proposition 8.3 this implies $u(t) = z(t)$.

□

9. An Explicit Example

In this section we provide an explicit example to show that approximable quasistatic evolutions and evolutions based on absolute minimization of the energy functional can be quite different. In particular, we will show that they can have different fracturing times and that for approximable quasistatic evolutions strict inequality may occur in (4.7). In all the section we will refer to the following setting. The function g describing the energy needed to create a crack is

$$g(s) = \begin{cases} -\dfrac{s^2}{2R} + s & 0 \leq s < R, \\ \dfrac{R}{2} & s \geq R, \end{cases}$$

where R is a positive constant representing the range of the cohesive forces. Notice that in this case $\sigma = g'(0^+) = 1$. The set Ω is the open rectangle $\{(x,y) \in \mathbb{R}^2 : 0 \leq |x| < A,\, 0 \leq |y| < B\}$, with $\Gamma = \{0\} \times [-B, B]$, where A and B are positive constants and $\partial_D\Omega = \{-A, A\} \times (-B, B)$. We consider initial condition $u_0 = 0$. Moreover, at every time $t \in [0,T]$, in $\partial_D\Omega$ we assign the displacement

$$w(t) = \frac{t}{A}x.$$

We want to describe the evolution of the set Ω, that represents the section of an infinite cylinder subject to the diplacement w. To this aim, it will be convenient to define the functions $u_1, u_2, u_3 : [0,T] \to H^1(\Omega \setminus \Gamma)$ as:

$$u_1(t) := \frac{t}{A}x\,; \qquad u_2(t) := \frac{1}{R-2A}\begin{cases} (R-2t)x + R(t-A) & x > 0 \\ (R-2t)x - R(t-A) & x < 0 \end{cases}; \qquad u_3(t) := \begin{cases} t & x > 0 \\ -t & x < 0 \end{cases}$$

In this setting it is possible to give an explicit expression to both approximable quasistatic evolution and evolution of absolute minimizers. More precisely, we can state the following results.

Theorem 9.1. *Let $A < \frac{R}{2} < T$. Then, there exists a unique approximable quasistatic evolution u, coinciding with the evolution of the absolute minimizers of the energy, that is given by:*

$$u(t) = \begin{cases} u_1(t) & 0 \leq t < A \\ u_2(t) & A \leq t < \dfrac{R}{2} \\ u_3(t) & \dfrac{R}{2} \leq t \leq T \end{cases}$$

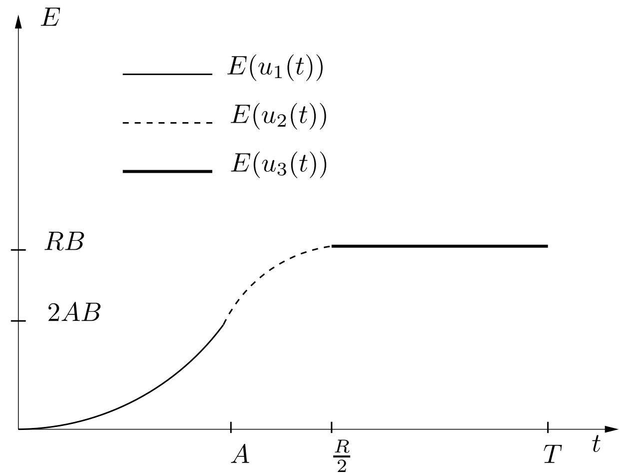

Figure 1. Energy graph for $0 < A < \frac{R}{2}$.

Remark 9.2. When the section Ω is sufficiently small, there are no differences between approximable quasistatic evolution and evolution of absolute minimizers of the energy. The section is stretched in the time interval $[0, A)$, where the only contribution to the energy comes from the elastic stored energy. For $t = A$ a crack occurs. Because of the symmetry of the problem, the crack set consists in *all* Γ. In the time interval $(A, \frac{R}{2})$ cohesive effects are observed, and the opening of the fracture grows from 0 to R. For $t > \frac{R}{2}$, cohesive forces cease to act and the opening of the crack continues to grow, without any further expense of energy. The graph of the corresponding energy is shown in Fig. 1.

Theorem 9.3. *Let $\frac{R}{2} < A < T$. Then, there exists a unique approximable quasistatic evolution u, that is given by:*

$$u(t) = \begin{cases} u_1(t) & 0 \le t < A \\ u_3(t) & A \le t \le T \end{cases}$$

Moreover, the evolution u^{am} of the absolute minimizers of the energy is uniquely determined and is given by

$$u^{am}(t) = \begin{cases} u_1(t) & 0 \le t < \bar{t} \\ u_3(t) & \bar{t} \le t \le T \end{cases}$$

where $\bar{t} := \sqrt{A\frac{R}{2}} \in (\frac{R}{2}, A)$.

Remark 9.4. When the section Ω is sufficiently large, approximable quasistatic evolution and evolution of absolute minimizers do not coincide. One can immediately see that the *maximum stress criterion* (see Theorem 4.15) is satisfied by the approximable quasistatic evolution. This is not the case for the evolution of the absolute minimizers of the energy.

Approximable quasistatic evolution. In the time interval $[0, A)$ the section Ω is stretched, there are no cracks, and the energy is a quadratic function of time. At time $t = A$ a crack occurs and the evolution continues with Ω divided into two horizontal (i.e. parallel to the plane (x, y)) pieces that become farther and farther, without any further expense of energy. No cohesive effects are observed.

Absolute minimizers evolution. In this case the section breaks "too early". Indeed, for short times the evolution coincides with the approximable quasistatic evolution. Then, a crack appears at time $t = \bar{t} < A$, which corresponds to a stress $u' = \bar{t}/A$ strictly lower than $\sigma = 1$. Hence, the crack initiation criterion is violated. Also here no cohesive effects are observed. The beaviour of the energy as a function of time is described by Fig. 2.

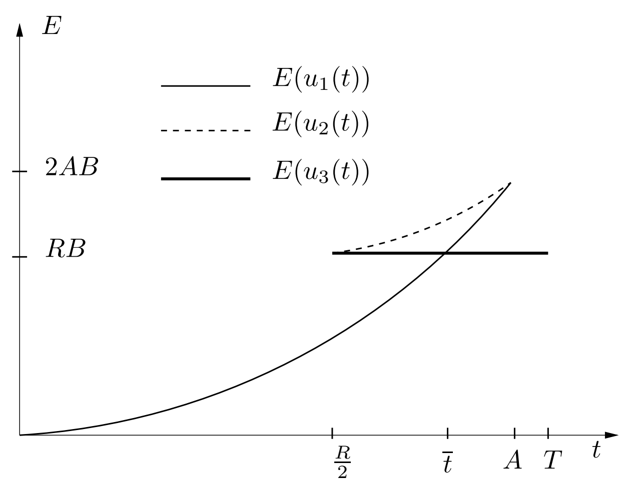

FIGURE 2. Energy graph for $A > \frac{R}{2}$.

Remark 9.5. The fact that, for Ω large, cohesive effects are not observed depends just on our choice of g. Indeed, if one considers more complicated expressions of g (e.g. g cubic for $0 \le s < R$) one can check that cohesive effects may appear.

Remark 9.6. For $A > \frac{R}{2}$ by a direct computation and using the expression of u given in Theorem 9.3 we have that for $t \in (A, T)$

$$E(u(t)) = RB < 2AB = E(u(0)) + 2B \int_0^t \int_{(-A,A)\setminus\{0\}} \nabla u(s) \cdot \nabla \dot{w}(s)\, dx\, ds,$$

so that in relation (4.7) we have the strict inequality. As expected, for the evolution of the absolute minimizers u^{am} we have equality for every $t \in [0, T]$.

Proof of Theorems 9.1 and 9.3. First of all we show that, thanks to the symmetry of the problem, the absolute minimizers of (2.1) depend only on the variable x. Indeed, let $v(x, y)$ be an admissible function for the minimization. Then, we have

$$\begin{aligned}
&\frac{1}{2} \int_{\Omega\setminus\Gamma} |\nabla v|^2\, dx\, dy + \int_\Gamma g(|[v]|)\, dy \\
&\qquad \ge \int_{-B}^{B} \Big\{ \frac{1}{2} \int_{(-A,A)\setminus\{0\}} \left| \frac{\partial v}{\partial x}(x, y) \right|^2 dx + g(|[v](y)|) \Big\} dy,
\end{aligned}$$

where the equality holds only for v such that $\frac{\partial v}{\partial y}(x, y) \equiv 0$. We define now the functional $\mathcal{F} : H^1((-A, A) \setminus \{0\}) \to \mathbb{R}$ as

$$\mathcal{F}(v) := \frac{1}{2} \int_{(-A,A)\setminus\{0\}} |v'(x)|^2 dx + g\left(|v^+(0) - v^-(0)|\right)$$

and consider the problem

$$\min_v \mathcal{F}(v),$$

where v varies in $H^1((-A, A) \setminus \{0\})$, with the boundary conditions $v(\pm A) = \pm t$. Since $\mathcal{F}$ is lower semicontinuous and coercive there exists a minimizer $\overline{v}$. Thus, if v depends on y we have

$$\frac{1}{2} \int_{\Omega\setminus\Gamma} |\nabla v|^2 dx\, dy + \int_\Gamma g(|[v]|)\, dy > \int_{-B}^{B} \mathcal{F}(\overline{v})\, dy = 2B\mathcal{F}(\overline{v}),$$

that is, minimizing F is equivalent to minimize $\mathcal{F}$. Since the same argument applies to the functional (4.3), for t fixed also the approximable quasistatic evolutions will not depend on the variable y. Notice that Ω is symmetric with respect to the coordinate plane $\{(x, y) \in \mathbb{R}^2 : x = 0\}$ and w is odd. Hence, by step 1 of the proof of Theorem 4.14 and by Remark 7.1, it follows that at every fixed $t \in [0, T]$ both the discrete time evolutions and the evolutions of absolute minimizers are odd. For this reason, we will restrict our analysis to the set $(0, A)$. Now, let $t \in [0, T]$ be fixed and let us look for the odd solutions to the Euler-Lagrange equations that do not depend on the variable y. Equations $(3.5)_1$–$(3.6)_3$ become

$$\begin{cases} u'' = 0 & \text{in } (0, A), \\ u(A) = t, & \\ |u'(0)| \le 1 & \text{if } u(0) = 0, \\ u'(0) = g'(2u(0)) & \text{if } u(0) \ne 0, \end{cases} \tag{9.1}$$

where we used the fact that u is positive in $(0, A)$. Since u is odd, for every fixed $t \in [0, T]$ the general solution can be written as

$$u(t) = \begin{cases} C_1(t)x + C_2(t) & x > 0 \\ C_1(t)x - C_2(t) & x < 0 \end{cases}$$

for some nonnegative constants $C_1(t), C_2(t)$, depending on t. We consider now three possible cases.

Solution without fracture. Let us suppose that $u(t)$ does not jump across the point $x = 0$, i.e. that $u(0) = 0$. Then we obtain the function

$$u_1(t) := \frac{t}{A}x \qquad 0 \le t \le A.$$

Notice that $u_1(t)$ is not a solution of the Euler-Lagrange equations for $t > A$, because in this case there holds $u_1'(0) = \frac{t}{A} > 1$. The energy associated to $u_1(t)$ is

$$E(u_1(t)) = \frac{2B}{A}t^2 \qquad 0 \le t \le A.$$

When there is a crack, we have to consider condition $(9.1)_4$.

Solution with small jump. Let us suppose the jump $2C_2(t)$ satisfies the relation

$$0 < 2C_2(t) = |[u(t)]| < R. \tag{9.2}$$

Then we have

$$C_1(t) = u'(t) = g'(|[u(t)]|) = -\frac{2C_2(t)}{R} + 1.$$

From this it follows that

$$C_1(t) = \frac{R - 2t}{R - 2A}, \qquad C_2(t) = \frac{R(t - A)}{R - 2A}.$$

Since in this case (9.2) must be satisfied, this choice of $C_1(t)$ and $C_2(t)$ is admissible only for $t \in (\min(A, \frac{R}{2}), \max(A, \frac{R}{2}))$, and the corresponding solution is $u_2(t)$. Notice that the behaviour in time of $u_2(t)$ changes according to the size of A. If $0 < A < \frac{R}{2}$ the solution corresponds to a cracked configuration with jump that increases from 0 (for $t = A$) to R (at time $t = \frac{R}{2}$), while the slope passes from 1 to 0. On the other side, for $A > \frac{R}{2}$ the section Ω starts divided into two horizontal pieces with jump R, and ends without crack and with slope 1. In both cases the energy is given by

$$E(u_2(t)) = \frac{2B}{2A - R}(2t^2 - 2Rt + AR).$$

Solution with big jump. If the jump of the solution is such that $2C_2(t) = |[u(t)]| \ge R$, one easily sees that the solution of (9.1) is $u_3(t)$ and is admissible only for $t \ge \frac{R}{2}$.

$A < \frac{R}{2} < T$. In this case for every $t \in [0, T]$ there is just one admissible solution to the Euler-Lagrange equations (9.1): we have $u_1(t)$ for $t \in [0, A)$, $u_2(t)$ for $t \in [A, \frac{R}{2})$ and $u_3(t)$ when $t \in [\frac{R}{2}, T]$.

$\frac{R}{2} < A < T$. In this case, the evolution of the absolute minimizers of the energy can be easily deduced from Figure 2. We have $u_1(t)$ for $t \in [0, \bar{t})$ and $u_3(t)$ when $t \in [\bar{t}, T]$. Concerning the approximable quasistatic evolution, we can apply Theorem 4.15. Then, it follows that the approximable quasistatic evolution coincides with the elastic evolution $z(t) = \frac{t}{A}x = u_1(t)$ until $\frac{t}{A} < 1$, that is in the time interval $[0, A)$. For $t \ge A$, the only possible solution to the Euler-Lagrange equations is given by $u_3(t)$. Hence, the approximable quasistatic evolution coincides with $u_1(t)$ for $t \in [0, A)$ and with $u_3(t)$ for $t \in [A, T]$. □

10. Appendix: Proof of Proposition 8.4

We will proceed by induction. To start with, we notice that the proposition holds for the initial time $t = 0$. Indeed, by definition of discrete time evolution $u_0^{\varepsilon,\delta} = z(0) = z_0^{\varepsilon,\delta}$ for every $(\varepsilon, \delta) \in (0, 1) \times (0, T)$. The proof is then completed by the following proposition. We recall that problem $(P)_i^{\varepsilon,\delta}$ is introduced in Definition 4.6.

Proposition 10.1. *There exist $\bar{\varepsilon} \in (0, 1)$ and a function $\hat{\delta} : (0, \bar{\varepsilon}) \to (0, T)$ with the following property. Let $\varepsilon \in (0, \bar{\varepsilon})$, $\delta \in (0, \hat{\delta}(\varepsilon))$ and $i \in \mathbb{N}$ with $i\delta \le t^*$. If $i \ge 2$, assume also that $u_j^{\varepsilon,\delta} = z_j^{\varepsilon,\delta}$ is the unique solution of problem $(P)_j^{\varepsilon,\delta}$ for every $j = 1, \dots, i - 1$. Then, the solution $u_i^{\varepsilon,\delta}$ of problem $(P)_i^{\varepsilon,\delta}$ is unique and there holds $u_i^{\varepsilon,\delta} = z_i^{\varepsilon,\delta}$.*

In the remaining part of the section our goal will be to show that $u_i^{\varepsilon,\delta} = z_i^{\varepsilon,\delta}$, provided that ε and δ are sufficiently small. To prove that $z_i^{\varepsilon,\delta}$ is the unique absolute minimizer of problem (4.3), we will use the technique of the calibration theory for free discontinuity problems. We remark that the solution $z_i^{\varepsilon,\delta}$ can present singularities in the part of $\partial\Omega$ where the boundary conditions change, that is in the set $G := \overline{\partial_D\Omega} \setminus \partial_D\Omega$. For this reason, we will first prove the minimality of $z_i^{\varepsilon,\delta}$ in subdomains obtained by removing from Ω a small neighbourhood of G. Then, the full minimality will be obtained by approximation. The outline of the proof is the following:

- Statement and proof of some auxiliary results;
- Proof of the minimality of $z_i^{\varepsilon,\delta}$ in a fixed set $\Omega_n \subset \Omega$;
- Full minimality: limit as $\Omega_n \nearrow \Omega$.

10.1. **Statement and proof of some auxiliary results.** In this subsection we state two lemmas that will be useful in the sequel. Let $\tilde\Omega_1$ and $\tilde\Omega_2$ be two open connected subsets of $\mathbb{R}^N$ such that $\Gamma \subset\subset \tilde\Omega_1 \subset\subset \tilde\Omega_2$ and $\tilde\Omega_k \cap \partial\Omega \subset\subset \partial\Omega \setminus \partial_D\Omega$ for $k = 1,2$. We set $\Omega_k := \tilde\Omega_k \cap \Omega$ $(k = 1,2)$, $\Omega_3 := \Omega \setminus \overline{\Omega}_2$ and choose $\tilde\Omega_2$ in such a way that Ω_2 has C^2 boundary.

The first lemma shows some properties of the function $z^{\varepsilon,\delta}(t)$ for $t \in [0,t^*]$ and ε and δ small enough. We will prove that $\|\nabla z^{\varepsilon,\delta}(t)\|_{L^\infty(\Omega_2)}$ is bounded uniformly with respect to t and that $z^{\varepsilon,\delta}(t)$ satisfies $(3.6)_2$ with strict inequality. These properties will be crucial in the construction of the calibration.

Lemma 10.2. *There exist $C_1 > 0$, $c \in (0,\sigma)$, $\overline\varepsilon \in (0,1)$ and a function $\overline\delta : (0,\overline\varepsilon) \to (0,T)$ such that*

$$\sup_{t\in[0,t^*]} \|\nabla z^{\varepsilon,\delta}(t)\|_{L^\infty(\Omega_2)} \le C_1, \tag{10.1}$$

$$\sup_{t\in[0,t^*]} \|\partial_\nu z^{\varepsilon,\delta}(t)\|_{L^\infty(\Gamma)} \le c < \sigma, \tag{10.2}$$

for every (ε,δ) with $\varepsilon \in (0,\overline\varepsilon)$ and $\delta \in (0,\overline\delta(\varepsilon))$.

Proof. We will prove (10.2), since (10.1) can be shown by using similar arguments. For every $(\varepsilon,\delta) \in (0,1)\times(0,T)$ and $t \in [0,T]$ we define $\overline z^{\varepsilon,\delta}(t) := z^{\varepsilon,\delta}(t) - z(t)$. From Definition 8.1 it follows that for every $(\varepsilon,\delta) \in (0,1)\times(0,T)$ and $t \in [0,T]$

$$\begin{cases} \Delta \overline z^{\varepsilon,\delta}(t) = \frac{\varepsilon}{\delta}\,\alpha_i^{\varepsilon,\delta} & \text{in } \Omega,\\ \overline z^{\varepsilon,\delta}(t) = w(i\delta) - w(t) & \text{on } \partial_D\Omega,\\ \partial_\nu \overline z^{\varepsilon,\delta}(t) = 0 & \text{on } \partial\Omega\setminus\partial_D\Omega, \end{cases} \tag{10.3}$$

where $i \in \mathbb{N}$ is such that $i\delta \le t < (i+1)\delta$ and $\alpha_i^{\varepsilon,\delta}$ is defined by Remark 8.2. Consider now a cut off function $\chi \in C^\infty(\overline\Omega)$ such that $0 \le \chi \le 1$, $\chi \equiv 1$ in $\overline\Omega_1$ and $\chi \equiv 0$ in Ω_3. It turns out that the function $\mu^{\varepsilon,\delta}(t) := \chi \overline z^{\varepsilon,\delta}(t)$ satisfies the following equation in Ω_2:

$$\begin{cases} \Delta\mu^{\varepsilon,\delta}(t) = \chi\Delta\overline z^{\varepsilon,\delta}(t) + \overline z^{\varepsilon,\delta}(t)\Delta\chi + 2\nabla\overline z^{\varepsilon,\delta}(t)\cdot\nabla\chi & \text{in } \Omega_2\\ \partial_\nu\mu^{\varepsilon,\delta}(t) = \overline z^{\varepsilon,\delta}(t)\partial_\nu\chi & \text{on } \partial\Omega_2. \end{cases}$$

Thanks to [20, Lemma 3.18, pag 181] we get that there exists a constant $\overline C > 0$ independent of ε, δ and t such that for every $p \in [2,+\infty)$

$$\begin{aligned} \frac{1}{\overline C}\|\mu^{\varepsilon,\delta}(t)\|_{W^{2,p}(\Omega_2)} \le{}& \varepsilon \sup_{t\in[0,T]} \|\dot w(t)\|_{L^\infty(\Omega)}\|\chi\|_{L^p(\Omega_2)} + \|\overline z^{\varepsilon,\delta}(t)\Delta\chi\|_{L^p(\Omega_2)} \\ &+ 2\|\nabla\chi\cdot\nabla\overline z^{\varepsilon,\delta}(t)\|_{L^p(\Omega_2)} + \|\overline z^{\varepsilon,\delta}(t)\partial_\nu\chi\|_{W^{\frac{1}{p'},p}(\partial\Omega_2)} + \|\mu^{\varepsilon,\delta}(t)\|_{H^1(\Omega_2)}, \end{aligned} \tag{10.4}$$

where $p' = \frac{p}{p-1}$ and we used relations (10.3) and (8.2). Notice that condition (8.3) holds also with $\overline z^{\varepsilon,\delta}(t) = z^{\varepsilon,\delta}(t) - z(t)$ replaced by $\mu^{\varepsilon,\delta}(t)$, so that applying (10.4) with $p = 2$ we get

$$\lim_{\varepsilon\to0^+}\lim_{\delta\to0^+}\sup_{t\in[0,T]} \|\mu^{\varepsilon,\delta}(t)\|_{W^{2,2}(\Omega_2)} = 0. \tag{10.5}$$

If $N=2$, by the Sobolev embedding theorem we get that

$$\lim_{\varepsilon\to0^+}\lim_{\delta\to0^+}\sup_{t\in[0,T]}\|\mu^{\varepsilon,\delta}(t)\|_{W^{1,q}(\Omega_2)}=0,\qquad q\in[2,+\infty),$$

so that, applying once again (10.4):

$$\lim_{\varepsilon\to0^+}\lim_{\delta\to0^+}\sup_{t\in[0,T]}\|\mu^{\varepsilon,\delta}(t)\|_{W^{2,q}(\Omega_2)}=0,\qquad q\in[2,+\infty).$$

Thus,

$$\lim_{\varepsilon\to0^+}\lim_{\delta\to0^+}\sup_{t\in[0,T]}\|\mu^{\varepsilon,\delta}(t)\|_{C^{1,\lambda}(\Omega_2)}=0\qquad\forall\,\lambda\in(0,1),$$

because Ω_2 is of class C^2. Since $\mu^{\varepsilon,\delta}(t)=\overline{z}^{\varepsilon,\delta}(t)$ in $\overline{\Omega}_1$, this implies

$$\lim_{\varepsilon\to0^+}\lim_{\delta\to0^+}\sup_{t\in[0,t^*]}\|\nabla z^{\varepsilon,\delta}(t)-\nabla z(t)\|_{L^\infty(\Omega_1)}=0. \tag{10.6}$$

If $N>2$, (10.6) follows applying repeatedly the Sobolev embedding theorem and estimate (10.4) starting from relation (10.5). Since $\sup_{t\in[0,t^*]}\|\partial_\nu z(t)\|_{L^\infty(\Gamma)}<\sigma$ by hypothesis, from (10.6) we get (10.2). □

The second lemma gives an existence result for an eigenfunction v_{β_0} of the laplacian operator in Ω, for which the ratio $\frac{|\nabla v_{\beta_0}|}{v_{\beta_0}}$ is nonnegative and bounded.

Lemma 10.3. *There exist two positive constants β_0 and C_2, and a strictly positive function $v_{\beta_0}\in H^1(\Omega)\cap C^1(\overline{\Omega}_2)$ such that*

$$\begin{cases}\Delta v_{\beta_0}=\beta_0 v & \text{in }\Omega,\\ \partial_\nu v_{\beta_0}=0 & \text{on }\partial\Omega\setminus\partial_D\Omega.\end{cases} \tag{10.7}$$

and

$$0\le\sup_{x\in\Omega_2}\frac{|\nabla v_{\beta_0}|}{v_{\beta_0}}\le C_2. \tag{10.8}$$

Proof. Let us fix $\beta_0>0$ and define v_{β_0} as the solution to the problem

$$\begin{cases}\Delta v_{\beta_0}=\beta_0 v & \text{in }\Omega,\\ v_{\beta_0}=1 & \text{on }\partial_D\Omega.\\ \partial_\nu v_{\beta_0}=0 & \text{on }\partial\Omega\setminus\partial_D\Omega.\end{cases}$$

It turns out that $v_{\beta_0}>0$ in $\overline{\Omega}$. Indeed, by the Strong Maximum Principle (see e.g. [18]) it follows that $v_{\beta_0}>0$ on Ω. To show that $v_{\beta_0}>0$ on $\partial\Omega$, fix $\Omega'\subset\subset\Omega$ and observe that the restriction of v_{β_0} to $\Omega\setminus\Omega'$ is the unique solution of the problem

$$\min_v\left\{\int_{\Omega\setminus\Omega'}|\nabla v|^2\,dx+\beta_0\int_{\Omega\setminus\Omega'}v^2\,dx\right\},$$

where the minimum is taken among all the $v\in H^1(\Omega\setminus\Omega')$ such that $v=v_{\beta0}>0$ on $\partial\Omega'$ and $v=1$ on $\partial_D\Omega$. By a truncation argument it follows that in $\Omega\setminus\Omega'$ $v_{\beta_0}\ge\inf_{\partial\Omega'}v_{\beta_0}>0$. One can show (10.8) with arguments similar to those used in the proof of Lemma 10.2. □

10.2. **Proof of the minimality of $z_i^{\varepsilon,\delta}$ in a fixed set $\Omega_n\subset\Omega$.** From now on we will assume $\varepsilon\in(0,\overline{\varepsilon})$ and $\delta\in(0,\overline{\delta}(\varepsilon))$, where $\overline{\varepsilon}$ and $\overline{\delta}(\varepsilon)$ are given by Lemma 10.2. The main result of this subsection is given by Proposition 10.5, where we prove that for ε and δ small enough the function $z_i^{\varepsilon,\delta}$ is the unique absolute minimizer of problem (4.3), among all competitors coinciding with $z_i^{\varepsilon,\delta}$ in a fixed neighbourhood of the set $G=\overline{\partial_D\Omega}\setminus\partial_D\Omega$. Before stating Proposition 10.5 we need some preliminary notation and we briefly introduce the notion of absolute calibration.

We consider a decreasing sequence $(G_n)_{n\in\mathbb{N}}$ of open Lipschitz sets of $\mathbb{R}^N$, such that $G_n\supset\supset G_{n+1}\supset\supset\ldots\supset\supset G$, $\mathcal{L}^N(G_n)\to0$ as $n\to+\infty$, and $\Omega_n:=\Omega\setminus\overline{G}_n$ is Lipschitz for every $n\in\mathbb{N}$. We consider also a sequence of cut off functions $\varphi_n\in H^1(\mathbb{R}^N)$ such that $0\le\varphi_n\le1$, $\varphi_n\equiv1$ in

$\mathbb{R}^N \setminus G^{n-1}$ and $\varphi_n \equiv 0$ in G^n. Since $\mathcal{H}^{N-2}(G) < +\infty$, and thus its 2-capacity is zero (see [13]), then we may choose $(G_n)_{n\in\mathbb{N}}$ and $(\varphi_n)_{n\in\mathbb{N}}$ in such a way that $\varphi_n \to 1$ strongly in $H^1(\mathbb{R}^N)$.

In the remaining part of the subsection we will assume $n \in \mathbb{N}$ fixed. Define now the sequence of functions $(r_j^{\varepsilon,\delta})_{j=0,1,\dots,i}$ in the following way. We set $r_0^{\varepsilon,\delta} := z(0)$ and, for $j = 1,\dots,i$ by induction, we choose $r_j^{\varepsilon,\delta}$ as a solution to the problem

$$\min_{v\in\mathcal{C}_n(j\delta)} \left\{ E(v) + \frac{\varepsilon}{2\delta}\|v - r_{j-1}^{\varepsilon,\delta}\|^2 \right\}, \tag{10.9}$$

where

$$\mathcal{C}_n(j\delta) := \left\{ v \in \mathcal{A}(j\delta) : v = z^{\varepsilon,\delta}(j\delta) \text{ in } \Omega \cap G_n \right\}.$$

By definition, we have $r_0^{\varepsilon,\delta} = z_0^{\varepsilon,\delta}$. In case $i \geq 2$ we also know, by the inductive hypothesis of Proposition 10.1, that for $j = 1,\dots,i-1$ the solution $r_j^{\varepsilon,\delta}$ to (10.9) is unique and coincides with $z_j^{\varepsilon,\delta}$. At this point, we want to show that the only possible choice for the function $r_i^{\varepsilon,\delta}$ is just $r_i^{\varepsilon,\delta} = z_i^{\varepsilon,\delta}$. As already mentioned, we will construct a calibration for the function $z_i^{\varepsilon,\delta}$ in $\Omega_n \times \mathbb{R}$.

Before stating next proposition, we adapt some definitions and results of [1] to the present situation. An *absolute calibration* for $z_i^{\varepsilon,\delta}$ in $\Omega_n \times \mathbb{R}$ is a bounded vector field $\phi = (\phi^x, \phi^t) : \Omega_n \times \mathbb{R} \to \mathbb{R}^N \times \mathbb{R}$ of class C^1 that satisfies the following properties (see [1, Lemma 3.7]):

(a) $\operatorname{div}\phi = \partial_t \phi^t + \operatorname{div}_x \phi^x = 0$ in $\Omega_n \times \mathbb{R}$;

(b) $\phi^t(x,t) \geq \dfrac{|\phi(x,t)|^2}{2} - \dfrac{\varepsilon}{2\delta}(t - z_{i-1}^{\varepsilon,\delta}(x))^2$ for a.e. $x \in \Omega_n$, for every $t \in \mathbb{R}$;

(c) $\begin{cases} \phi^x(x, z_i^{\varepsilon,\delta}(x)) = \nabla z_i^{\varepsilon,\delta}(x) \\ \phi^t(x, z_i^{\varepsilon,\delta}(x)) = \dfrac{|\nabla z_i^{\varepsilon,\delta}|^2}{2}(x) - \dfrac{\varepsilon}{2\delta}(z_i^{\varepsilon,\delta}(x) - z_{i-1}^{\varepsilon,\delta}(x))^2 \end{cases}$ for a.e. $x \in \Omega_n$;

(d) $\left[\displaystyle\int_{t_1}^{t_2} \phi^x(x,t)\,dt\right] \cdot \nu(x) \leq g(t_2 - t_1)$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Gamma,\ t_1 \leq t_2$.

(e) $\phi^x(x,t)\cdot\nu(x) = 0$ for $\mathcal{L}^N$-a.e. for a.e. $(x,t) \in \big(\partial\Omega_n \cap (\partial\Omega \setminus \partial_D\Omega)\big) \times \mathbb{R}$.

By a careful inspection of the proof of [1, Lemma 3.2], we get the following result.

Theorem 10.4. *Suppose that there exists an absolute calibration ϕ for $z_i^{\varepsilon,\delta}$. Assume, in addition, that condition (d) is satisfied with strict inequality for $t_1 \neq t_2$. Then $z_i^{\varepsilon,\delta}$ is the unique absolute minimizer for the problem (10.9) with the index j replaced by i and*

$$\begin{aligned}\int_{\Omega_n\setminus\Gamma} \Big(\frac{1}{2}|\nabla z_i^{\varepsilon,\delta}|^2 + \frac{\varepsilon}{2\delta}(z_i^{\varepsilon,\delta} - z_{i-1}^{\varepsilon,\delta})^2\Big)dx &= \int_{\Omega_n\setminus\Gamma}\Big(\phi^x(x,v(x))\cdot\nabla v(x) - \phi^t(x,v(x))\Big)dx \\ &\quad + \int_\Gamma \left(\int_{\min\{v^+,v^-\}}^{\max\{v^+,v^-\}} \phi^x(x,t)\,dt\right)\cdot\nu\,d\mathcal{H}^{N-1}\end{aligned} \tag{10.10}$$

for every $v \in \mathcal{C}_n(i\delta)$. Moreover, there holds

$$\int_{\Omega_n\setminus\Gamma}\Big(\phi^x(x,v(x))\cdot\nabla v(x) - \phi^t(x,v(x))\Big)dx \leq \int_{\Omega_n\setminus\Gamma}\Big(\frac{1}{2}|\nabla v|^2 + \frac{\varepsilon}{2\delta}(v - z_{i-1}^{\varepsilon,\delta})^2\Big)dx \tag{10.11}$$

and

$$\int_\Gamma \left(\int_{\min\{v^+,v^-\}}^{\max\{v^+,v^-\}} \phi^x(x,t)\,dt\right)\cdot\nu\,d\mathcal{H}^{N-1} < \int_\Gamma g(|[v]|)\,d\mathcal{H}^{N-1} \tag{10.12}$$

for every $v \in \mathcal{C}_n(i\delta)$.

We state now the main result of this subsection.

Proposition 10.5. *There exists a function* $\hat{\delta}:(0,\overline{\varepsilon})\to(0,T)$*, independent of* $n\in\mathbb{N}$*, such that the following holds. Let* $\varepsilon\in(0,\overline{\varepsilon})$ *and* $\delta\in(0,\hat{\delta}(\varepsilon))$*. Then, there exists an absolute calibration* ϕ *for* $z_i^{\varepsilon,\delta}$ *for which condition (d) is satisfied with strict inequality for* $t_1\neq t_2$ *. As a consequence,* ϕ *is such that relations (10.10), (10.11) and (10.12) are satisfied.*

We start now with the construction of the calibration ϕ, showing that conditions (a)–(e) are satisfied for δ sufficiently small. We will take inspiration from [16], where a global calibration for the Mumford-Shah functional is provided. In the quoted paper, the author considers the following calibration:

$$\begin{cases}\phi^x = \nabla z_i^{\varepsilon,\delta} + \frac{t-z_i^{\varepsilon,\delta}}{v_{\beta_0}}\nabla v_{\beta_0}\\ \phi^t = \frac{1}{2}\left|\nabla z_i^{\varepsilon,\delta} + \frac{t-z_i^{\varepsilon,\delta}}{v_{\beta_0}}\nabla v_{\beta_0}\right|^2 - \frac{\varepsilon}{2\delta}(t-z_i^{\varepsilon,\delta})^2 + \left(\frac{\varepsilon}{2\delta}-\beta_0\right)(z_i^{\varepsilon,\delta}-t)^2,\end{cases}$$

where β_0 and v_{β_0} are given by Lemma 10.3. In the present situation, we cannot use directly the previous expression. Indeed, in order (d) to be satisfied we need that when $x\in\Gamma$ and for small values of t_2-t_1 the integral $\left[\int_{t_1}^{t_2}\phi^x(x,t)\,dt\right]\cdot\nu(x)$ is sublinear as a function of the difference t_2-t_1.

For this reason, we will introduce a suitable cut off function. Let us consider a constant $\eta>0$ to be properly chosen later and a function $a:\mathbb{R}\to[0,1]$ of class C^∞, with $\operatorname{supp} a\subset(-2\eta,2\eta)$, $a\equiv 1$ in $[-\eta,\eta]$ and $|\dot{a}|\le\frac{2}{\eta}$, where with the dot we denote the derivative with respect to t. Let us consider also a function $\psi\in C^\infty(\overline{\Omega})$ such that $\psi\equiv 1$ in $\overline{\Omega}_3$, $\psi\equiv 0$ in $\overline{\Omega}_1$. Then, for $(x,t)\in\overline{\Omega}\times\mathbb{R}$ we set $\xi(x,t):=a(t)+(1-a(t))\,\psi(x)$. Our assumptions in particular imply that $\sup_{t\in\mathbb{R}}\|\nabla\xi(t,\cdot)\|_{L^\infty(\Omega)}\le C_3$ and

$$|t\dot{\xi}(x,t)|\le 4 \qquad\text{and}\qquad |t^2\dot{\xi}(x,t)|\le 8\eta \qquad \text{for every } (x,t)\in\overline{\Omega}\times\mathbb{R}, \tag{10.13}$$

where $C_3:=\|\nabla\psi\|_{L^\infty(\Omega)}$. We set now $C:=\max\{C_1,C_2,C_3\}$, where C_1 and C_2 are defined in (10.8) and (10.1). Moreover, for every $c\in(0,\sigma)$ we define $s(c)\in(0,+\infty)$ as the unique positive real number such that $g(s)=c\,s(c)$. Since g is nondecreasing, concave and has finite limit at infinity, $s(c)$ is well defined. We set now

$$\phi^x(x,t):=\xi(x,t-z_i^{\varepsilon,\delta}(x))\Big(\nabla z_i^{\varepsilon,\delta}(x)+\frac{t-z_i^{\varepsilon,\delta}(x)}{v_{\beta_0}(x)}\nabla v_{\beta_0}(x)\Big) \tag{10.14}$$

for every $(x,t)\in\Omega_n\times\mathbb{R}$. Notice that ϕ^x is bounded in $\Omega_n\times\mathbb{R}$, but not in $\Omega\times\mathbb{R}$. This is the reason why we first prove the minimality in Ω_n.

In order (c) to be satisfied, we define

$$\phi^t(x,z_i^{\varepsilon,\delta}(x)):=\frac{|\nabla z_i^{\varepsilon,\delta}(x)|^2}{2}-\frac{\varepsilon}{2\delta}(z_i^{\varepsilon,\delta}(x)-z_{i-1}^{\varepsilon,\delta}(x))^2 \tag{10.15}$$

for all $x\in\Omega_n$. To simplify the notation, in the following we will omit the dependence on variables taking into account, when deriving, that $\xi(x,\cdot)$ is always evaluated at $t-z_i^{\varepsilon,\delta}(x)$. To satisfy (a), we impose

$$\begin{aligned}\partial_t\phi^t = -\operatorname{div}_x\phi^x &= -\nabla_x\xi\cdot\nabla z_i^{\varepsilon,\delta}-(t-z_i^{\varepsilon,\delta})\,\frac{\nabla_x\xi\cdot\nabla v_{\beta_0}}{v_{\beta_0}}+\dot{\xi}\,|\nabla z_i^{\varepsilon,\delta}|^2\\ &+\dot{\xi}\,(t-z_i^{\varepsilon,\delta})\,\frac{\nabla z_i^{\varepsilon,\delta}\cdot\nabla v_{\beta_0}}{v_{\beta_0}}-\frac{\varepsilon}{\delta}\,\xi\,(z_i^{\varepsilon,\delta}-z_{i-1}^{\varepsilon,\delta})+\xi\,\frac{\nabla z_i^{\varepsilon,\delta}\cdot\nabla v_{\beta_0}}{v_{\beta_0}}\\ &+\xi\,\frac{(t-z_i^{\varepsilon,\delta})|\nabla v_{\beta_0}|^2}{v_{\beta_0}^2}-\beta_0\,\xi\,(t-z_i^{\varepsilon,\delta}).\end{aligned} \tag{10.16}$$

In this way, relations (10.14), (10.15) and (10.16) together determine ϕ^x and ϕ^t everywhere. By construction, (a), (c) and (e) hold. In the next two lemmas we show that η can be chosen in such a way that conditions (b) and (d) are satisfied.

Lemma 10.6. *Let*

$$0 < \eta < \min\left\{\frac{-c+\sqrt{c^2+c\,C s(c)}}{2C}, \frac{\sigma-c}{3C}\right\}. \tag{10.17}$$

Then, condition (d) *is satisfied.*

Proof. Let $x \in \Gamma$ and $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$. We have

$$\begin{aligned}
\left[\int_{t_1}^{t_2} \phi^x(x,t)\,dt\right]\cdot\nu(x) &= \int_{t_1}^{t_2} a(t-z_i^{\varepsilon,\delta}(x))\left(\partial_\nu z_i^{\varepsilon,\delta}(x) + \frac{t-z_i^{\varepsilon,\delta}(x)}{v_{\beta_0}(x)}\,\partial_\nu v_{\beta_0}(x)\right)dt \\
&\le c\int_{t_1-z_i^{\varepsilon,\delta}(x)}^{t_2-z_i^{\varepsilon,\delta}(x)} a(\tau)\,d\tau + C\int_{t_1-z_i^{\varepsilon,\delta}(x)}^{t_2-z_i^{\varepsilon,\delta}(x)} |\tau|\,a(\tau)\,d\tau \\
&= c\int_{\hat t_1(x)-z_i^{\varepsilon,\delta}(x)}^{\hat t_2(x)-z_i^{\varepsilon,\delta}(x)} a(\tau)\,d\tau + C\int_{\hat t_1(x)-z_i^{\varepsilon,\delta}(x)}^{\hat t_2(x)-z_i^{\varepsilon,\delta}(x)} |\tau|\,a(\tau)\,d\tau,
\end{aligned} \tag{10.18}$$

where $\hat t_1(x) \ge t_1$ and $\hat t_2(x) \le t_2$ are defined as

$$\hat t_1(x) := \begin{cases} t_1 & \text{for } t_1 - z_i^{\varepsilon,\delta}(x) \in [-2\eta, 2\eta], \\ z_i^{\varepsilon,\delta}(x) - 2\eta & \text{for } t_1 - z_i^{\varepsilon,\delta}(x) < -2\eta, \end{cases}$$

and

$$\hat t_2(x) := \begin{cases} t_2 & \text{for } t_2 - z_i^{\varepsilon,\delta}(x) \in [-2\eta, 2\eta], \\ z_i^{\varepsilon,\delta}(x) + 2\eta & \text{for } t_2 - z_i^{\varepsilon,\delta}(x) > +2\eta. \end{cases}$$

Notice that if $t_1 - z_i^{\varepsilon,\delta}(x) > 2\eta$ or $t_2 - z_i^{\varepsilon,\delta}(x) < -2\eta$ the left-hand side of (10.18) is zero and then there is nothing to prove. We remark that $[\hat t_1(x) - z_i^{\varepsilon,\delta}(x), \hat t_2(x) - z_i^{\varepsilon,\delta}(x)] \subset [-2\eta, 2\eta]$ for every $x \in \Gamma$ and $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$. We will consider two possible cases.

Step 1. $0 < t_2 - t_1 \le s(c)$.

In this case there holds

$$\int_{\hat t_1(x)-z_i^{\varepsilon,\delta}(x)}^{\hat t_2(x)-z_i^{\varepsilon,\delta}(x)} |\tau|\,d\tau \le 3\eta(\hat t_2(x) - \hat t_1(x)). \tag{10.19}$$

Indeed, if $(\hat t_1(x) - z_i^{\varepsilon,\delta}(x))(\hat t_2(x) - z_i^{\varepsilon,\delta}(x)) \ge 0$

$$\begin{aligned}
\int_{\hat t_1(x)-z_i^{\varepsilon,\delta}(x)}^{\hat t_2(x)-z_i^{\varepsilon,\delta}(x)} |\tau|\,d\tau &\le \frac{1}{2}\left|(\hat t_2(x) - z_i^{\varepsilon,\delta}(x))^2 - (\hat t_1(x) - z_i^{\varepsilon,\delta}(x))^2\right| \\
&\le \frac{1}{2}(\hat t_2(x) - \hat t_1(x))(|\hat t_1(x) - z_i^{\varepsilon,\delta}(x)| + |\hat t_2(x) - z_i^{\varepsilon,\delta}(x)|) \le 2\eta(\hat t_2(x) - \hat t_1(x)),
\end{aligned}$$

while for $(\hat t_1(x) - z_i^{\varepsilon,\delta}(x))(\hat t_2(x) - z_i^{\varepsilon,\delta}(x)) < 0$

$$\begin{aligned}
\int_{\hat t_1(x)-z_i^{\varepsilon,\delta}(x)}^{\hat t_2(x)-z_i^{\varepsilon,\delta}(x)} |\tau|\,d\tau &\le \int_{\hat t_1(x)-z_i^{\varepsilon,\delta}(x)+2\eta}^{\hat t_2(x)-z_i^{\varepsilon,\delta}(x)+2\eta} |\tau|\,d\tau \\
&= \frac{1}{2}\left((\hat t_2(x) - z_i^{\varepsilon,\delta}(x) + 2\eta)^2 - (\hat t_1(x) - z_i^{\varepsilon,\delta}(x) + 2\eta)^2\right) \\
&= \frac{1}{2}(\hat t_2(x) - \hat t_1(x))(\hat t_2(x) - z_i^{\varepsilon,\delta}(x) + \hat t_1(x) - z_i^{\varepsilon,\delta}(x) + 4\eta) \\
&\le \frac{1}{2}(\hat t_2(x) - \hat t_1(x))(\hat t_2(x) - z_i^{\varepsilon,\delta}(x) + 4\eta) \le 3\eta(\hat t_2(x) - \hat t_1(x)).
\end{aligned}$$

Using (10.18) and (10.19), since g is nondecreasing, we get

$$\begin{aligned}
\left[\int_{t_1}^{t_2} \phi^x(x,t)dt\right]\cdot\nu(x) &\le (c + 3\,\eta\,C)(\hat t_2(x) - \hat t_1(x)) < \sigma(\hat t_2(x) - \hat t_1(x)) \\
&\le g(\hat t_2(x) - \hat t_1(x)) \le g(t_2 - t_1),
\end{aligned}$$

provided

$$0<\eta<\frac{\sigma-c}{3C}. \tag{10.20}$$

Step 2. $t_1-t_2>s(c)$.

If $t_1-t_2>s(c)$ then

$$\begin{aligned}\left[\int_{t_1}^{t_2}\phi^x(x,t)dt\right]\cdot\nu(x)&\le c\int_{-2\eta}^{2\eta}a(\tau)d\tau+C\int_{-2\eta}^{2\eta}|\tau|d\tau\le 4\,c\,\eta+4\,C\,\eta^2\\ &=c\left(4\eta+\frac{4C}{c}\eta^2\right)<c\,s(c)=g(s(c))\le g(t_1-t_2),\end{aligned}$$

provided

$$4\eta+\frac{4C}{c}\eta^2<s(c).$$

Last condition is certainly satisfied for

$$0<\eta<\frac{-c+\sqrt{c^2+c\,Cs(c)}}{2C}. \tag{10.21}$$

Collecting (10.20) and (10.21) we get the thesis. □

Next lemma concludes the proof of Proposition 10.5.

Lemma 10.7. *Let $\overline{\varepsilon}$ be given by Lemma 10.2. For every $\varepsilon\in(0,\overline{\varepsilon})$ there exists $\hat{\delta}(\varepsilon)\in(0,T)$, independent of $n\in\mathbb{N}$, with the following property. If $\varepsilon\in(0,\overline{\varepsilon})$ and $\delta\in(0,\hat{\delta}(\varepsilon))$, then there exists $\eta>0$ such that conditions* (b) *and* (d) *are satisfied.*

Proof. Let $\eta>0$ be fixed and such that (10.17) holds. By Lemma 10.6 it follows that condition (d) is satisfied. We want to prove that for $\varepsilon\in(0,\overline{\varepsilon})$ and δ sufficiently small there holds

$$\phi^t\ge\frac{(\phi^x)^2}{2}-\frac{\varepsilon}{2\delta}(t-z_{i-1}^{\varepsilon,\delta})^2 \tag{10.22}$$

for every $(x,t)\in\Omega_n\times\mathbb{R}$. By construction, we already know that equality holds along the graph of $z_i^{\varepsilon,\delta}$, that is for $t=z_i^{\varepsilon,\delta}(x)$. Hence, in order to prove the previous inequality it will be sufficient to impose the following relations, obtained by deriving (10.22) with respect to t:

$$\partial_t\phi^t\ge\phi^x\cdot\partial_t\phi^x-\frac{\varepsilon}{\delta}(t-z_{i-1}^{\varepsilon,\delta})\text{ for }t>z_i^{\varepsilon,\delta}, \tag{10.23}$$

$$\partial_t\phi^t\le\phi^x\cdot\partial_t\phi^x-\frac{\varepsilon}{\delta}(t-z_{i-1}^{\varepsilon,\delta})\text{ for }t<z_i^{\varepsilon,\delta}. \tag{10.24}$$

Let us consider inequality (10.23). Thanks to (10.14) and (10.16), we get

$$\begin{aligned}&\beta_0\xi z_i^{\varepsilon,\delta}+\frac{\varepsilon}{\delta}\big((\xi-1)z_{i-1}^{\varepsilon,\delta}-\xi z_i^{\varepsilon,\delta}\big)+\left(\frac{\varepsilon}{\delta}-\beta_0\xi\right)t\\ &\quad\ge\xi\,\dot\xi\,|\nabla z_i^{\varepsilon,\delta}|^2+2\xi\,\dot\xi\,(t-z_i^{\varepsilon,\delta})\,\frac{\nabla z_i^{\varepsilon,\delta}\cdot\nabla v_{\beta_0}}{v_{\beta_0}}+\xi\,\dot\xi\,(t-z_i^{\varepsilon,\delta})^2\,\frac{|\nabla v_{\beta_0}|^2}{v_{\beta_0}^2}\\ &\qquad+\xi^2\,\frac{\nabla z_i^{\varepsilon,\delta}\cdot\nabla v_{\beta_0}}{v_{\beta_0}}+\xi^2(t-z_i^{\varepsilon,\delta})\,\frac{|\nabla v_{\beta_0}|^2}{v_{\beta_0}^2}+\nabla\xi\cdot\nabla z_i^{\varepsilon,\delta}-\dot\xi\,|\nabla z_i^{\varepsilon,\delta}|^2\\ &\qquad+(t-z_i^{\varepsilon,\delta})\,\frac{\nabla\xi\cdot\nabla v_{\beta_0}}{v_{\beta_0}}-\dot\xi\,(t-z_i^{\varepsilon,\delta})\,\frac{\nabla z_i^{\varepsilon,\delta}\cdot\nabla v_{\beta_0}}{v_{\beta_0}}\\ &\qquad-\xi\,\frac{\nabla z_i^{\varepsilon,\delta}\cdot\nabla v_{\beta_0}}{v_{\beta_0}}-\xi\,(t-z_i^{\varepsilon,\delta})\,\frac{|\nabla v_{\beta_0}|^2}{v_{\beta_0}^2}.\end{aligned} \tag{10.25}$$

When $x\in\overline{\Omega}_3$ or $t-z_i^{\varepsilon,\delta}\in(0,\eta)$ there holds $\xi\equiv1$ and (10.25) reduces to

$$\left(\frac{\varepsilon}{\delta}-\beta_0\right)(t-z_i^{\varepsilon,\delta})\ge0\Longleftrightarrow t\ge z_i^{\varepsilon,\delta},$$

that certainly holds. Suppose now $t - z_i^{\varepsilon,\delta} > \eta$ and $x \in \Omega_2$. Let us focus on the left-hand side of (10.25), that we will denote by LHS (10.25). Assuming $\delta < \frac{\varepsilon}{\beta_0 + C^2}$ and using the fact that $t > \eta + z_i^{\varepsilon,\delta}$ we obtain

$$\begin{aligned} LHS(10.25) &= \beta_0 \xi (z_i^{\varepsilon,\delta} - t) + \frac{\varepsilon}{\delta}\xi\,(z_{i-1}^{\varepsilon,\delta} - z_i^{\varepsilon,\delta}) - C^2(t - z_i^{\varepsilon,\delta}) + \frac{\varepsilon}{\delta}(t - z_{i-1}^{\varepsilon,\delta}) + C^2(t - z_i^{\varepsilon,\delta}) \\ &> \Big(\frac{\varepsilon}{\delta} - \beta_0\xi - C^2\Big)(t - z_i^{\varepsilon,\delta}) + \frac{\varepsilon}{\delta}(1-\xi)(z_i^{\varepsilon,\delta} - z_{i-1}^{\varepsilon,\delta}) + C^2(t - z_i^{\varepsilon,\delta}) \\ &> \Big(\frac{\varepsilon}{\delta} - \beta_0 - C^2\Big)\eta - \overline{\varepsilon} \sup_{t\in[0,T]} \|\dot w(t)\|_{L^\infty(\Omega)} + C^2(t - z_i^{\varepsilon,\delta}), \end{aligned} \tag{10.26}$$

where in the last inequality we used (8.2). Thanks to (10.13), for the right-hand side of (10.25) we have

$$\begin{aligned} RHS(10.25) &= \dot\xi\,(\xi-1)|\nabla z_i^{\varepsilon,\delta}|^2 + \dot\xi\,(t - z_i^{\varepsilon,\delta})(2\xi - 1)\frac{\nabla z_i^{\varepsilon,\delta}\cdot\nabla v_{\beta_0}}{v_{\beta_0}} \\ &\quad + \xi\,\dot\xi\,(t - z_i^{\varepsilon,\delta})^2\,\frac{|\nabla v_{\beta_0}|^2}{v_{\beta_0}^2} + \xi(\xi-1)\frac{\nabla z_i^{\varepsilon,\delta}\cdot\nabla v_{\beta_0}}{v_{\beta_0}} + \nabla\xi\cdot\nabla z_i^{\varepsilon,\delta} \\ &\quad + \xi(\xi-1)(t - z_i^{\varepsilon,\delta})\,\frac{|\nabla v_{\beta_0}|^2}{v_{\beta_0}^2} + (t - z_i^{\varepsilon,\delta})\,\frac{\nabla\xi\cdot\nabla v_{\beta_0}}{v_{\beta_0}} \\ &\le \frac{2C^2}{\eta} + 4C^2 + 8\eta C^2 + 2C^2 + C^2(t - z_i^{\varepsilon,\delta}). \end{aligned} \tag{10.27}$$

Collecting (10.25), (10.26) and (10.27), we have that if the following inequality holds

$$\Big(\frac{\varepsilon}{\delta} - \beta_0 - C^2\Big)\eta - \overline{\varepsilon}\|\dot w\|_{L^\infty} \ge \frac{2C^2}{\eta} + 6C^2 + 8\eta C^2, \tag{10.28}$$

then (10.23) is satisfied. In the same way one can see that the previous relation implies (10.24). Observe now that the right-hand side of (10.28) is constant, while the left-hand side tends to $+\infty$ when $\delta \to 0^+$. Hence, for δ sufficiently small relation (10.28) is satisfied and condition (b) holds. $\square$

10.3. **Limit as** $n \to +\infty$**.** Let $n \in \mathbb{N}$. By a truncation argument, it will be enough to consider competitors in the class $\mathcal{A}(i\delta) \cap L^\infty(\Omega)$. Thus, let $v \in \mathcal{A}(i\delta) \cap L^\infty(\Omega)$ with $v \neq z_i^{\varepsilon,\delta}$. We set $v_n := \varphi_n v + (1 - \varphi_n) z_i^{\varepsilon,\delta}$, recalling that $\varphi_n \equiv 1$ in $\mathbb{R}^N \setminus G^{n-1}$, $\varphi_n \equiv 0$ in G^n, and $\varphi_n \to 1$ strongly in $H^1(\mathbb{R}^N)$. From the properties of φ_n there holds $v_n \in \mathcal{C}_n(i\delta)$ for every $n \in \mathbb{N}$ and $v_n \to v$ strongly in $H^1(\Omega)$. Thanks to Proposition 10.5 and applying relations (10.10) we have

$$\begin{aligned} &\int_{\Omega_n\setminus\Gamma} \Big(\frac{1}{2}\,|\nabla z_i^{\varepsilon,\delta}|^2 + \frac{\varepsilon}{2\delta}(z_i^{\varepsilon,\delta} - z_{i-1}^{\varepsilon,\delta})^2\Big)dx \\ &\quad = \int_{\Omega_n\setminus\Gamma} \Big(\phi^x(x, v_n(x))\cdot\nabla v_n(x) - \phi^t(x, v_n(x))\Big)dx + \int_\Gamma \left(\int_{\min\{v^+,v^-\}}^{\max\{v^+,v^-\}} \phi^x(x,t)\,dt\right)\cdot\nu\,d\mathcal{H}^{N-1}, \end{aligned} \tag{10.29}$$

where we used the fact that $v_n^\pm = v^\pm$. Moreover, by (10.11) and (10.12) there holds

$$\int_{\Omega_n\setminus\Gamma} \Big(\phi^x(x, v_n(x))\cdot\nabla v_n(x) - \phi^t(x, v_n(x))\Big)dx \le \int_{\Omega_n\setminus\Gamma} \Big(\frac{1}{2}\,|\nabla v_n|^2 + \frac{\varepsilon}{2\delta}(v_n - z_{i-1}^{\varepsilon,\delta})^2\Big)dx \tag{10.30}$$

and

$$\int_\Gamma \left(\int_{\min\{v^+,v^-\}}^{\max\{v^+,v^-\}} \phi^x(x,t)\,dt\right)\cdot\nu\,d\mathcal{H}^{N-1} < \int_\Gamma g(|[v]|)\,d\mathcal{H}^{N-1}. \tag{10.31}$$

We pass now to the limit as $n \to +\infty$ in (10.29). Since $v_n \to v$ strongly in $H^1(\Omega \setminus \Gamma)$, taking into account (10.30) and (10.31) we obtain

$$\begin{aligned}\int_{\Omega\setminus\Gamma} \Big(\frac{1}{2}\,|\nabla z_i^{\varepsilon,\delta}|^2 + \frac{\varepsilon}{2\delta}(z_i^{\varepsilon,\delta} - z_{i-1}^{\varepsilon,\delta})^2\Big) dx &= \lim_{n\to+\infty} \int_{\Omega_n} \Big(\phi^x(x, v_n(x)) \cdot \nabla v_n(x) - \phi^t(x, v_n(x))\Big) dx \\ &\quad + \int_\Gamma \left(\int_{\min\{v^+,v^-\}}^{\max\{v^+,v^-\}} \phi^x(x,t)\, dt \right) \cdot \nu \, d\mathcal{H}^{N-1} \\ &< \int_{\Omega\setminus\Gamma} \Big(\frac{1}{2}\,|\nabla v|^2 + \frac{\varepsilon}{2\delta}(v - z_{i-1}^{\varepsilon,\delta})^2\Big) dx + \int_\Gamma g(|[v]|)\, d\mathcal{H}^{N-1}\end{aligned}$$

for every $v \in \mathcal{A}(i\delta) \cap L^\infty(\Omega)$ with $v \neq z_i^{\varepsilon,\delta}$.

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