

RECTIFIABILITY OF SPECIAL SINGULARITIES OF NON-LIPSCHITZ FUNCTIONS

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ABSTRACT. We prove rectifiability results for special singularities of non-Lipschitz functions, namely for those points where the set of Fréchet horizon supergradients contains a vector subspace.

1. INTRODUCTION

Since the celebrated result by H. Rademacher there has been a lot of interest in the study of the set of singularities of real functions. Given a function $f : \Omega \rightarrow \mathbb{R}$ defined on an open set $\Omega \subseteq \mathbb{R}^N$, a singularity of f is a point where f is not differentiable. Rademacher's theorem states that, if f is locally Lipschitz, then the set of singularities $\Sigma(f)$ of f has null Lebesgue measure. In general, however, sets with null Lebesgue measure can be very irregular and possess almost no structure. A natural question is then that of investigating the properties of the singular set for special classes of functions.

When f is convex or concave, the properties of $\Sigma(f)$ were first investigated in [18] and then developed in [30], [29], [27], [28], [2] and [3]. The basic approach in such papers is that of estimating the size of $\Sigma(f)$. We mention here a result which is essentially due to L. Zajíček and was later extended to semiconcave functions by G. Alberti, L. Ambrosio and P. Cannarsa [1]. By $\partial^F f(x)$ we denote here the Fréchet supergradient of f at x (see Definition 2.3).

Theorem ([1]). *Let f be locally semiconcave. Then, for any $k = 1, \dots, N$ the singular set $\Sigma^k(f) := \{x \in \Omega \mid \dim \partial^F f(x) = k\}$ is countably $(N - k)$ -rectifiable. In particular, $\Sigma(f)$ is countably $(N - 1)$ -rectifiable and $\Sigma^N(f)$ is at most countable.*

The importance of semiconcave/semiconvex functions arises evident in problems of optimal control, see [7], [8], [9], [10]. In particular, the minimum time function T of nonlinear smooth control dynamics with target satisfying internal sphere condition is semiconcave [8]. In the same paper, the authors also proved that T is semiconvex for the linear dynamics with convex target. In both cases, a strong controllability

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assumption on the systems (namely, T being locally Lipschitz) was demanded. However, as shown by simple examples (e.g., the well known rocket car), the Lipschitz continuity of T does not hold in general.

In order to study the regularity of non Lipschitz functions the notion of set with positive reach was used in [15]. This notion was first introduced by H. Federer [19] and then analyzed independently by several authors under different names, for example φ -convex [11], proximally smooth [12], and prox-regular sets [24]. More precisely, lower (respectively, upper) semicontinuous functions whose epigraph (resp., hypograph) has positive reach still enjoy some regularity property of semiconvex (resp. semiconcave) functions, including the rectifiability of $\Sigma^k(f)$ and almost everywhere second order differentiability. However, the Hausdorff dimension of $\Sigma^{P,\infty}(f) := \{x \in \Omega \mid \partial^{P,\infty} f \neq \emptyset\}$ can be close to N (see [15, Example 5.2]), where by $\partial^{P,\infty} f$ we denote the set of proximal horizon supergradients of f (see [15]). A natural idea for proving the positive reach property for the epigraph/hypograph of f is the representation of generalized sub/super gradient of f . By using this approach, the minimum time function T was studied in [16] and [17] under a weak controllability condition requiring T to be only continuous. In both papers, the wedgedness (see Rockafellar [26]) for normal cone to the epigraph/hypograph of T is required. It was proved in [22] that the wedgedness assumption guarantees positive reach for sets satisfying an exterior sphere condition. This result was used in [23] to investigate the relationships among functions whose hypograph satisfies the exterior sphere condition, functions with positive reach hypograph and semiconcave functions. In general, however, the exterior sphere condition is weaker than the positive reach property (see. [22] and [17]). Recently, the set of *bad points* BP_f (see Section 2 for the definition), in which wedgedness fails, was studied in [20] under the exterior sphere condition on the hypograph of the function f . More precisely, it was proved that BP_f is closed in Ω and has zero Lebesgue measure. Consequently, the hypograph of $f|_{\Omega \setminus BP_f}$ has positive reach and thus f is twice differentiable almost everywhere in Ω .

In this paper we prove some rectifiability result for the set of bad points BP_f of f . We partition the set BP_f (see (2.6)) into sets $BP_{f,k}$, $k = 1, \dots, N$, where, roughly speaking, the suffix k corresponds to the dimension of the largest vector space contained in the set $\partial^\infty f$ of Fréchet horizon supergradients of f (see Definition 2.3). We are able to prove that $BP_{f,k}$ is countably $(N - k)$ -rectifiable.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $f : \Omega \rightarrow \mathbb{R}$ be upper semi-continuous. Then the set $BP_{f,k}$ is countably $(N - k)$ -rectifiable.*

Moreover, we are able to refine the main result of [20], namely Theorem 3.1 therein.

Theorem 1.2. *Let $\Omega \subseteq \mathbb{R}^N$ be open and let $f : \Omega \rightarrow \mathbb{R}$ be continuous. If the hypograph of f satisfies the θ -exterior sphere condition for some $\theta > 0$, then the set of*

bad points BP_f is locally $(N - 1)$ -rectifiable. In particular, $\mathcal{H}^{N-1}(BP_f \cap K)$ is finite for any compact set $K \subset \mathbb{R}^N$.

Finally, in Section 4 we provide an example showing that, in general, the set $BP_{f,k}$, $k \geq 2$ may not have finite $(N - k)$ -Hausdorff measure even under the exterior sphere condition.

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2. NOTATIONS AND PRELIMINARY RESULTS

Let $\Omega \subseteq \mathbb{R}^N$ be open and let $f : \Omega \rightarrow \mathbb{R}$ be upper semi-continuous. The hypograph of f is denoted by

$$(2.1) \quad \text{hypo}(f) = \{(x, \beta) \mid x \in \Omega, \beta \leq f(x)\}.$$

The vector $(-v, \lambda) \in \mathbb{R}^N \times \mathbb{R}$ is a *Fréchet normal vector* to $\text{hypo}(f)$ at $(x, f(x))$ iff

$$(2.2) \quad \limsup_{\text{hypo}(f) \ni (y, \beta) \rightarrow (x, f(x))} \left\langle (-v, \lambda), \frac{(y, \beta) - (x, f(x))}{|y - x| + |\beta - f(x)|} \right\rangle \leq 0.$$

We denote by $N_{\text{hypo}(f)}^F(x, f(x))$ the set of Fréchet normal vectors to $\text{hypo}(f)$ at $(x, f(x))$.

Remark 2.1. If $(-v, \lambda) \in N_{\text{hypo}(f)}^F(x, f(x))$ then $\lambda \geq 0$.

We say that $(-v, \lambda)$ is a *proximal normal vector* to $\text{hypo}(f)$ at $(x, f(x))$ if there exists a constant α such that

$$\langle (-v, \lambda), (y, \beta) - (x, f(x)) \rangle \leq \alpha(\|y - x\|^2 + |\beta - f(x)|^2) \quad \forall y \in \Omega, \beta \leq f(x).$$

We denote by $N_{\text{hypo}(f)}^P(x, f(x))$ the set of proximal normal vectors to $\text{hypo}(f)$ at $(x, f(x))$. Moreover, we say that $(-v, \lambda) \in N_{\text{hypo}(f)}^P(x, f(x))$ is realized by a ball of radius $\theta > 0$ if

$$\langle (-v, \lambda), (y, \beta) - (x, f(x)) \rangle \leq \frac{\|(-v, \lambda)\|}{2\theta} (\|y - x\|^2 + |\beta - f(x)|^2) \quad \forall y \in \Omega, \beta \leq f(x).$$

We say that $\text{hypo}(f)$ satisfies the *θ -exterior sphere condition* if for any $x \in \Omega$ there exists $(-v, \lambda) \in N_{\text{hypo}(f)}^P(x, f(x))$ realized by a ball of radius θ .

Remark 2.2. It is easily seen that $N_{\text{hypo}(f)}^P(x, f(x)) \subseteq N_{\text{hypo}(f)}^F(x, f(x))$

Let us introduce some concepts of generalized differential for f at $x \in \Omega$ associated with $\text{hypo}(f)$.

Definition 2.3. Let $x \in \Omega$ and $v \in \mathbb{R}^N$. We say that v is a *Fréchet supergradient* of f at x if $(-v, 1) \in N_{\text{hypo}(f)}^F(x, f(x))$. We denote by $\partial^F f(x)$ the set of Fréchet supergradients of f at x .

We say that v is a *Fréchet horizon supergradient* of f at x if $(v, 0) \in N_{\text{hypo}(f)}^P(x, f(x))$. The set of Fréchet horizon supergradients of f at x is denoted by $\partial^\infty f(x)$.

The largest vector subspace contained in $N_{\text{hypo}(f)}^F(x, f(x))$ will be denoted by

$$(2.3) \quad NL(x) = \{ \xi \in N_{\text{hypo}(f)}^F(x, f(x)) \mid -\xi \in N_{\text{hypo}(f)}^F(x, f(x)) \}.$$

From Remark 2.1, one can see that $NL(x) \subseteq \{(v, 0) \mid -v \in \partial^\infty f(x)\}$. Let us define

$$(2.4) \quad V_x := \{v \in \mathbb{R}^N \mid (v, 0) \in NL(x)\};$$

clearly, V_x is the largest vector space contained in $\partial^\infty f(x)$ and $\dim V_x = \dim NL(x)$. We say that $v \in V_x$ is realized by a ball of radius θ if $(v, 0) \in N_{\text{hypo}(f)}^P(x, f(x))$ is realized by a ball of radius θ .

The set of *bad points* BP_f of f is defined by

$$(2.5) \quad BP_f = \{x \in \Omega \mid NL(x) \neq \{0\}\}.$$

According to the dimension of $NL(x)$, for $k = 1, \dots, N$ we introduce

$$(2.6) \quad BP_{f,k} = \{x \in BP_f \mid \dim NL(x) = k\} = \{x \in BP_f \mid \dim V_x = k\}.$$

It is clear that $BP_f = \bigcup_{k=1}^N BP_{f,k}$.

Let $k \geq 0$ and $A \subset \mathbb{R}^N$ be fixed. The k -dimensional Hausdorff measure of A is defined as

$$\mathcal{H}^k(A) := \lim_{\delta \rightarrow 0^+} \mathcal{H}_\delta^k(A) = \sup_{\delta > 0} \mathcal{H}_\delta^k(A)$$

where for any $\delta > 0$ we set

$$\mathcal{H}_\delta^k(A) := \inf \left\{ \sum_{i \in I} (\text{diam } A_i)^k \mid A \subset \bigcup_{i \in I} A_i, \text{diam } A_i < \delta \right\}.$$

The *Hausdorff dimension* of A is

$$\mathcal{H}\text{-dim}(A) := \inf\{k \geq 0 \mid \mathcal{H}^k(A) = 0\} = \sup\{k \geq 0 \mid \mathcal{H}^k(A) = \infty\}.$$

It is well known (see e.g. [19, 21]) that \mathcal{H}^k is a Borel measure on \mathbb{R}^N ; \mathcal{H}^0 is the counting measure. Moreover, if $k \in \mathbb{N}$ and S is a k -dimensional Lipschitz surface, then the surface measure of S coincides with $\frac{2^k}{\omega_k} \mathcal{H}^k \llcorner S$.

Let $k \in \mathbb{N}$; we say that $A \subset \mathbb{R}^N$ is *countably k -rectifiable* if

$$A \subset \mathcal{N} \cup \bigcup_{i=1}^{\infty} S_i$$

where S_i are suitable Lipschitz k -dimensional surfaces and \mathcal{N} is a \mathcal{H}^k -negligible set.

We say that A is *k -rectifiable* if it is countably k -rectifiable and $\mathcal{H}^k(A) < \infty$.

Any countably k -rectifiable set A satisfies $\mathcal{H}\text{-dim}(A) = k$. It is well known that, if $f : A \subset \mathbb{R}^k \rightarrow \mathbb{R}^N$ is Lipschitz continuous, then $f(A)$ is countably k -rectifiable; if A is bounded, then $f(A)$ is k -rectifiable.

In what follows, given $A \subset \mathbb{R}^N$ we define its ϵ -neighborhood $(A)_\epsilon$ by

$$(A)_\epsilon := \{x \in \mathbb{R}^N \mid \text{there exists } y \in A \text{ such that } \|x - y\| < \epsilon\}.$$

Let \mathcal{K} denote the set of closed subsets of $S^{N-1} \subset \mathbb{R}^N$; for $A, B \in \mathcal{K}$ we introduce the *Hausdorff distance* $d_H(A, B)$ by

$$d_H(A, B) = \inf\{\epsilon > 0 \mid A \subset (B)_\epsilon \text{ and } B \subset (A)_\epsilon\}.$$

It turns out (see e.g. [6]) that (\mathcal{K}, d_H) is a complete compact metric space.

We will denote by $G(N, k)$ the Grassmann manifold of all k -dimensional vector subspaces of \mathbb{R}^N ; we endow $G(N, k)$ with the distance

$$d_G(V_1, V_2) := d_H(V_1 \cap S^{N-1}, V_2 \cap S^{N-1}).$$

The metric space $(G(N, k), d_G)$ is separable and, in particular, the following property holds:

$$(2.7) \quad \forall R > 0 \exists V_1, \dots, V_m \in G(N, k) \text{ s.t. } G(N, k) \subset \bigcup_{i=1}^m B_G(V_i, R)$$

where $B_G(V_i, R)$ denote the open ball (with respect to d_G) with center V_i and radius R .

3. RECTIFIABILITY RESULTS FOR THE SET OF BAD POINTS

Let $V \in G(N, k)$ be fixed; each $z \in \mathbb{R}^N$ can be written in a unique way as $z = z_V + z_{V^\perp}$ where $z_V \in V$ and $z_{V^\perp} \in V^\perp$. For $\alpha \in (0, 1)$ we denote by $C_\alpha(V)$ the open cone along V of aperture $1/\alpha$ defined by

$$C_\alpha(V) := \{z \in \mathbb{R}^N \mid \|z_V\| > \alpha \|z\|\}.$$

If $x \in \mathbb{R}^N$ we set

$$C_\alpha(x, V) := x + C_\alpha(V) = \{z \in \mathbb{R}^N \mid \|(z - x)_V\| > \alpha \|z - x\|\};$$

It is easily seen that

$$(3.1) \quad z \in C_\alpha(x, V) \iff \exists v \in V \cap S^{N-1} \text{ such that } \langle v, z - x \rangle > \alpha \|z - x\|.$$

We also point out the following implication:

$$(3.2) \quad d_G(V_1, V_2) < R \implies C_{\alpha+R}(x, V_1) \subset C_\alpha(x, V_2)$$

which holds provided $\alpha + R < 1$. To prove (3.2) it is enough to notice that for any $z \in C_{\alpha+R}(x, V_1)$

there exists $v_1 \in V_1 \cap S^{N-1}$ such that $\langle v_1, z - x \rangle > (\alpha + R) \|z - x\|$

there exists $v_2 \in V_2 \cap S^{N-1}$ such that $\|v_1 - v_2\| \leq R$

whence

$$\langle v_2, z - x \rangle = \langle v_1, z - x \rangle - \langle v_1 - v_2, z - x \rangle > \alpha \|z - x\|,$$

i.e., $z \in C_\alpha(x, V_2)$.

For any fixed $\rho > 0$, let us introduce the sets

$$(3.3) \quad BP_{f,k}^\rho = \left\{ x \in BP_{f,k} \mid \left\langle v_x, \frac{y-x}{|y-x| + |\beta - f(x)|} \right\rangle \leq \frac{\|v_x\|}{8} \right. \\ \left. \forall v_x \in V_x, y \in B(x, \rho), \beta < f(y) \right\}.$$

Remark 3.1. If $\rho_1 > \rho_2 > 0$ then $BP_{f,k}^{\rho_1} \subseteq BP_{f,k}^{\rho_2}$.

As the following Lemma shows, the sets $BP_{f,k}^\rho$ give a partition of $BP_{f,k}$.

Lemma 3.2. *We have*

$$(3.4) \quad BP_{f,k} = \cup_{\rho > 0} BP_{f,k}^\rho.$$

In particular, from Remark 3.1 it holds

$$(3.5) \quad BP_{f,k} = \cup_{i \in \mathbb{N} \setminus \{0\}} BP_{f,k}^{1/i}.$$

Proof. Fix $x \in BP_{f,k}$ and let v_1, v_2, \dots, v_k be an orthonormal basis for V_x . By the definition of V_x we have $-v_i \in V_x$ for all $i \in \{1, 2, \dots, k\}$. Recalling (2.4), (2.3) and (2.2), there exists a constant $\rho_x > 0$ such that $B(x, \rho_x) \subset \Omega$ and for all $i \in \{1, 2, \dots, k\}$ one has

$$\left\langle v_i, \frac{y-x}{|y-x| + |\beta - f(x)|} \right\rangle \leq \frac{1}{8\sqrt{k}} \quad \text{and} \quad \left\langle -v_i, \frac{y-x}{|y-x| + |\beta - f(x)|} \right\rangle \leq \frac{1}{8\sqrt{k}}$$

for all $y \in B(x, \rho_x)$ and $\beta \leq f(y)$. Thus

$$(3.6) \quad \left| \left\langle v_i, \frac{y-x}{|y-x| + |\beta - f(x)|} \right\rangle \right| \leq \frac{1}{8\sqrt{k}}$$

for all $y \in B(x, \rho_x)$ and $\beta \leq f(y)$.

Fix $v_x \in V_x$; we have $v_x = \sum_{i=1}^k \alpha_i v_i$ for suitable $\alpha_i \in \mathbb{R}$. From (3.6), we get

$$\left\langle v_x, \frac{y-x}{|y-x| + |\beta - f(x)|} \right\rangle \leq \frac{\sum_{i=1}^k |\alpha_i|}{8\sqrt{k}}$$

for all $y \in B(x, \rho_x)$ and $\beta \leq f(y)$. On the other hand,

$$\|v_x\| = \left(\sum_{i=1}^k \alpha_i^2 \right)^{1/2} \geq \frac{\sum_{i=1}^k |\alpha_i|}{\sqrt{k}}.$$

Therefore

$$\left\langle v_x, \frac{y-x}{|y-x| + |\beta - f(x)|} \right\rangle \leq \frac{\|v_x\|}{8}$$

for all $y \in B(x, \rho_x)$ and $\beta \leq f(y)$. Thus $x \in BP_{f,k}^{\rho_x}$ and the proof is accomplished. \square

In view of a rectifiability result for the sets $BP_{f,k}$, we begin with a technical result.

Lemma 3.3. *Let $a \in \mathbb{R}^N$, $\rho > 0$ and $x, y \in BP_{f,k}^\rho \cap B(a, \frac{\rho}{2})$ be such that $d_G(V_x, V_y) < \frac{1}{8}$; then*

$$y \in \mathbb{R}^N \setminus C_{\frac{1}{4}}(x, V_x).$$

Proof. Since $x, y \in B(a, \frac{\rho}{2})$, we have $x \in B(y, \rho)$ and $y \in B(x, \rho)$. Therefore, from (3.3) if $v_x \in V_x \cap \mathcal{S}^{N-1}$ we have

$$(3.7) \quad \langle v_x, y - x \rangle \leq \frac{1}{8}(\|y - x\| + |\beta - f(x)|) \quad \text{for all } \beta \leq f(y).$$

Similarly, for any $v_y \in V_y \cap \mathcal{S}^{N-1}$ we obtain

$$(3.8) \quad \langle v_y, y - x \rangle \leq \frac{1}{8}(\|y - x\| + |\beta - f(y)|) \quad \text{for all } \beta \leq f(x).$$

We have to distinguish two cases: if $f(y) \geq f(x)$, we choose $\beta = f(x)$ in (3.7) to get

$$\langle v_x, y - x \rangle \leq \frac{1}{8}\|y - x\| \quad \forall v_x \in V_x \cap \mathcal{S}^{N-1}.$$

Recalling (3.1), this implies that $y \notin C_{\frac{1}{4}}(x, V_x)$, as desired.

If $f(y) \leq f(x)$, we choose $\beta = f(y)$ in (3.8) to get

$$\langle v_y, y - x \rangle \leq \frac{1}{8}\|y - x\| \quad \forall v_y \in V_y \cap \mathcal{S}^{N-1}.$$

Since $d_G(V_x, V_y) < \frac{1}{8}$, for any $v_x \in V_x \cap \mathcal{S}^{N-1}$ there exists $v_y = v_y(v_x) \in V_y \cap \mathcal{S}^{N-1}$ such that $\|v_x - v_y\| < \frac{1}{8}$. Therefore, for any $v_x \in V_x \cap \mathcal{S}^{N-1}$ it holds

$$(3.9) \quad \langle v_x, y - x \rangle \leq \langle v_y, y - x \rangle + |\langle v_x - v_y, y - x \rangle| \leq \frac{1}{4}\|y - x\|$$

i.e. $y \notin C_{\frac{1}{4}}(x, V_x)$, as desired. \square

We now fix $R := 1/16$ and let $V_1, \dots, V_m \in G(N, k)$ be given by (2.7). We thus divide $BP_{f,k}^\rho$ into m sets

$$(3.10) \quad BP_{f,k}^\rho = \bigcup_{j=1}^m BP_{f,k}^{\rho,j}$$

where

$$BP_{f,k}^{\rho,j} = \{x \in BP_{f,k}^\rho \mid d_G(V_x, V_j) < 1/16\}.$$

For $j = 1, \dots, m$ we denote by π_j the orthogonal projection $\mathbb{R}^n \rightarrow V_j^\perp$; clearly, $\pi_j(z) = z_{V_j^\perp} = z - z_{V_j}$.

Lemma 3.4. *The projection $\pi_j : BP_{f,k}^{\rho,j} \cap B(a, \rho/2) \rightarrow \pi_j(BP_{f,k}^{\rho,j} \cap B(a, \rho/2))$ is invertible and its inverse map is Lipschitz continuous with Lipschitz constant at most 2.*

Proof. Let $x, y \in BP_{f,k}^{\rho,j} \cap B(a, \rho/2)$ be fixed. We have $d_G(V_x, V_y) < 1/8$ and Lemma 3.3 ensures that $y \notin C_{1/4}(x, V_x)$. Since $d_G(V_x, V_j) < 1/16$, by (3.2) we deduce that $C_{1/2}(x, V_j) \subseteq C_{5/16}(x, V_j) \subseteq C_{1/4}(x, V_x)$ and, in particular, that $y \notin C_{1/2}(x, V_j)$. This implies that $\|(y-x)_{V_j}\| \leq \frac{1}{2}\|y-x\|$, whence

$$\|\pi_j(y) - \pi_j(x)\| = \|\pi_j(y-x)\| = \|(y-x) - (y-x)_{V_j}\| \geq \frac{1}{2}\|y-x\|.$$

This is enough to conclude. \square

The rectifiability of the sets $BP_{f,k}^\rho$ is now a consequence of Lemma 3.4.

Theorem 3.5. *The set $BP_{f,k}^\rho \cap K$ is $(N-k)$ -rectifiable for any $\rho > 0$ and any compact set $K \subset \mathbb{R}^N$; in particular*

$$(3.11) \quad \mathcal{H}^{N-k}(BP_{f,k}^\rho \cap K) < +\infty.$$

Proof. It will be sufficient to show that for any $j = 1, \dots, m$ the set $BP_{f,k}^{\rho,j} \cap K$ is k -rectifiable. Since K is compact, there exist $a_1, \dots, a_h \in \mathbb{R}^N$ such that

$$BP_{f,k}^{\rho,j} \cap K \subset \bigcup_{i=1}^h (BP_{f,k}^{\rho,j} \cap B(a_i, \rho/2)).$$

By Lemma 3.4, for any $i = 1, \dots, h$ the set $BP_{f,k}^{\rho,j} \cap B(a_i, \rho/2)$ is the image of

$$\pi_j^{-1} : \pi_j(BP_{f,k}^{\rho,j} \cap B(a_i, \rho/2)) \rightarrow \mathbb{R}^N,$$

i.e. of a Lipschitz map defined on a bounded subset of $V_j^\perp \equiv \mathbb{R}^{N-k}$ with Lipschitz constant at most 2. In particular, $BP_{f,k}^{\rho,j} \cap B(a_i, \rho/2)$ is $(N-k)$ -rectifiable and this allows to conclude. \square

We can finally pass to the proof of our main results.

Proof of Theorem 1.1. It is an easy consequence of Lemma 3.2 and Theorem 3.5. \square

Before passing to the proof of Theorem 1.2, we would like to discuss the relation between BP_f and the set of bad points BP_f^P considered in [20], namely,

$$BP_f^P := \{x \in \Omega \mid NL^P(x) \neq \{0\}\},$$

where $NL^P(x) = \{\xi \in N_{\text{hypo}(f)}^P(x, f(x)) \mid -\xi \in N_{\text{hypo}(f)}^P(x, f(x))\}$. From Remark 2.2 it is clear that $BP_f^P \subseteq BP_f$, but in general the two sets do not coincide.

However, the equality $BP_f = BP_f^P$ holds under the assumptions of Theorem 1.2. Indeed, from Corollary 3.1 in [20] it follows that the hypograph of $f|_{\Omega_P}$ has positive reach, where Ω_P is the open set defined by $\Omega_P := \Omega \setminus BP_f^P$. Therefore (see [14, Proposition 6.2 and 4.2] and [19, Theorem 4.8 (12)]) one has

$$N_{\text{hypo}(f|_{\Omega_P})}^P(x, f|_{\Omega_P}(x)) = N_{\text{hypo}(f|_{\Omega_P})}^F(x, f|_{\Omega_P}(x)) \quad \text{for all } x \in \Omega_P.$$

and thus

$$N_{\text{hypo}(f)}^P(x, f(x)) = N_{\text{hypo}(f)}^F(x, f(x)) \quad \text{for all } x \in \Omega_P.$$

Consequently, $NL(x) = NL^P(x)$ for all $x \in \Omega_P$. By the definition of BP_f^P , we have $NL^P(x) = \{0\}$ for all $x \in \Omega_P$. This implies that $NL(x) = \{0\}$ for all $x \in \Omega_P$, i.e. $BP_f \cap \Omega_P = \emptyset$. Thus, $BP_f \subseteq BP_f^P$, as claimed.

Proof of Theorem 1.2. Recalling 1.1, we have $\mathcal{H}^{N-1}(BP_{f,k}) = 0$ for all $k \in \{2, 3, \dots, N\}$. Since

$$BP_f = BP_{f,1} \cup \bigcup_{k=2}^N BP_{f,k},$$

the proof will be accomplished after proving that the set $BP_{f,1}$ is locally $(N-1)$ -rectifiable. From the definition (2.6), for every $x \in BP_{f,1}$ the set

$$V_x = \{tv_x \mid v_x \in \mathbb{R}^N, \|v_x\| = 1 \text{ and } t \in \mathbb{R}\}$$

is a line along v_x . Therefore by [20, Lemma 4.3], $(\pm v_x, 0) \in N_{\text{hypo}(f)}^P(x, f(x))$ is realized by a ball of radius θ , i.e.

$$\langle \pm v_x, y - x \rangle \leq \frac{1}{2\theta} (\|y - x\|^2 + |\beta - f(x)|^2) \quad \forall y \in \Omega, \beta \leq f(x).$$

From the above inequality, reasoning as in the proof of Lemma 3.3 one can obtain that the following holds. If $a \in \mathbb{R}^N$, $\rho \in (0, \theta/8]$, $x, y \in BP_{f,1} \cap B(a, \frac{\rho}{2})$ are such that $d_G(V_x, V_y) < \frac{1}{8}$, then

$$y \in \mathbb{R}^N \setminus C_{\frac{1}{4}}(x, V_x).$$

From this fact, the local $(N-1)$ -rectifiability of $BP_{f,1}$ follows (up to considering $BP_{f,1}$ instead of $BP_{f,k}^\rho$) as in the proof of Theorem 3.5. \square

4. A COUNTEREXAMPLE

By virtue of Theorem 1.2, the set of bad points BP_f is locally $(N-1)$ -rectifiable provided the θ -exterior sphere condition holds. On the contrary, an analogous $(N-k)$ -rectifiability result does not hold for $BP_{f,k}$; in other words, Theorem 1.1 cannot be refined to show that $\mathcal{H}^{N-k}(BP_{f,k} \cap K) < \infty$ for any compact set $K \subset \mathbb{R}^N$. We are going to provide an example of a continuous function $f : (-1, 1) \times (-1, 1) \rightarrow \mathbb{R}$ satisfying the θ -exterior sphere condition with $\theta = 1$ and such that $\mathcal{H}^0(BP_{f,2} \cap K) = +\infty$ for any neighbourhood K of the origin. It will be clear from the construction that what is missing is a uniform control on the radii of exterior balls (recall that, by Theorem 3.5, $BP_{f,k}^\rho$ is locally $(N-k)$ -rectifiable for any $\rho > 0$).

Let $\Omega := (-1, 1) \times (-1, 1)$; for $n \in \mathbb{N}$, $n \geq 0$ let us define $x_n^+, x_n^- \in \bar{\Omega}$ by

$$x_n^+ := (2^{-n}, 0), \quad x_n^- := (-2^{-n}, 0).$$

We also set

$$c_n^+ := \frac{x_n^+ + x_{n+1}^+}{2} = (3 \cdot 2^{-n-2}, 0) \in \Omega, \quad c_n^- := \frac{x_n^- + x_{n+1}^-}{2} = (-3 \cdot 2^{-n-2}, 0) \in \Omega$$

and

$$r_n := \frac{\|x_n^+ - x_{n+1}^+\|}{2} = \frac{\|x_n^- - x_{n+1}^-\|}{2} = 2^{-n-2}.$$

Notice that the closed balls $\overline{B(c_n^\pm, r_n)}$ are pairwise disjoint except for the case of consecutive balls, which instead are tangent, i.e., for any $n \geq 1$ one has

$$\overline{B(c_n^+, r_n)} \cap \overline{B(c_{n-1}^+, r_{n-1})} = \{x_n^+\}, \quad \overline{B(c_n^-, r_n)} \cap \overline{B(c_{n-1}^-, r_{n-1})} = \{x_n^-\}.$$

Define $f_1 : \Omega \rightarrow \mathbb{R}$ by

$$f_1(x) = \begin{cases} -\sqrt{r_n^2 - \|x - c_n^+\|^2} & \text{if } x \in B(c_n^+, r_n) \\ -\sqrt{r_n^2 - \|x - c_n^-\|^2} & \text{if } x \in B(c_n^-, r_n) \\ 0 & \text{if } x \in \Omega \setminus (\bigcup_n B(c_n^+, r_n) \cup \bigcup_n B(c_n^-, r_n)). \end{cases}$$

It is easily seen that f_1 is continuous and that $\{x_n^+, x_n^- : n \geq 1\} \subset BP_{f_1}$; more precisely

$$(4.1) \quad \begin{aligned} (1, 0) \in \partial^\infty f_1(x_n^+) & \text{ is realized by a ball of radius } r_{n-1} \\ (-1, 0) \in \partial^\infty f_1(x_n^+) & \text{ is realized by a ball of radius } r_n \\ (1, 0) \in \partial^\infty f_1(x_n^-) & \text{ is realized by a ball of radius } r_n \\ (-1, 0) \in \partial^\infty f_1(x_n^-) & \text{ is realized by a ball of radius } r_{n-1}. \end{aligned}$$

For any $x = (\xi, \eta) \in \Omega$ we also define

$$f_2(x) = -\sqrt{\eta^2 - |\eta|} = -\sqrt{1 - (1 - |\eta|)^2}.$$

One can easily check that f_2 is continuous on Ω and that $BP_{f_2} = \{(\xi, 0) : \xi \in (-1, 1)\}$; more precisely, for any $\xi \in (-1, 1)$

$$(4.2) \quad (0, 1), (-1, 0) \in \partial^\infty f_2(\xi, 0) \text{ are realized by balls of radius } 1.$$

Notice also that $f_1(x_n^\pm) = f_2(x_n^\pm) = 0$ for any $n \geq 1$. Therefore, the function $f := \inf\{f_1, f_2\}$ is continuous on Ω and $f(x_n^\pm) = f_1(x_n^\pm) = f_2(x_n^\pm) = 0$. Taking (4.1) and (4.2) into account we obtain that

$$(1, 0), (-1, 0), (0, 1), (0, -1) \in \partial^\infty f(x_n^\pm) \quad \text{for any } n \geq 1$$

whence

$$\{x_n^+, x_n^- : n \geq 1\} \subset BP_{f,2}$$

which in turn implies $\mathcal{H}^0(BP_{f,2}) = \infty$, as desired.

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