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Ph.D. Thesis in Mathematics

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# Lower semicontinuity and relaxation for integral and supremal functionals

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# Contents

# Chapter 1

# Introduction

In this Ph.D. Thesis we shall be concerned with some aspects of the central problem of the Calculus of Variations which is to find the minimum points of functionals. In our study we will consider, in particular, the integral functional

$$I(u,\Omega) = \int_\Omega f(x,u(x),\nabla u(x))dx, \tag{1.1}$$

and the supremal functional

$$S(u,\Omega) = \operatorname*{ess\,sup}_{x\in\Omega} g(x,u(x),\nabla u(x)), \tag{1.2}$$

where $\Omega \subseteq \mathbb{R}^n$ is an open set, $f, g$ are two Borel functions from $\Omega \times \mathbb{R} \times \mathbb{R}^n$ with values in $[0,\infty]$ and $u$ belongs to a suitable space of functions. In the following we will always denote by $x, s$ and $\xi$ the three variables of $f$ and $g$, called geometric, function and gradient variable respectively.

Some of the most powerful tools to prove the existence of minimum points for such functionals are the so called *Direct Methods*. These methods move by the fact that, given a topological space $(X,\tau)$, we have that a function $F : X \to [0,\infty]$ admits a minimum point every time $F$ is *lower semicontinuous* (briefly *l.s.c.*)[1], that is,

$$x_h, x \in X, x_h \to x \text{ in } \tau \quad \Longrightarrow \quad F(x) \leq \liminf_{h\to\infty} F(x_h),$$

and there exists a *compact*[2] *minimizing sequence*, that is a sequence $\{y_h\}_{h=1}^\infty \subseteq X$ which admits a converging subsequence and such that

$$\inf_{x\in X} F(x) = \lim_{h\to\infty} F(y_h).$$

Indeed, in these hypotheses, we have that every limit point $y_0$ of $\{y_h\}_{h=1}^\infty$ satisfies the inequality

$$F(y_0) \leq \liminf_{h\to\infty} F(y_h) = \inf_{x\in X} F(x),$$

so that $y_0$ is just a minimum point of $F$.

Clearly the two conditions considered above work in opposite directions: indeed, roughly speaking, the lower semicontinuity of $F$ is simpler to prove when the topology $\tau$ has *many* open sets,

[1] Actually this definition is the one of *sequential lower semicontinuity*: however, since we will always deal with this notion, which may be different from the topological notion of lower semicontinuity given by means of open sets, we simplify our notations with this little abuse. Note also that when the space $(X,\tau)$ is a metric space, as for instance $\mathbb{R}^n$ endowed with the Euclidean distance, the two notions agree.

[2] Even in this case, since we will always deal with the notion of *sequential compactness*, we can simplify our notations.

while, on the contrary, we have more opportunities to prove the compactness of some minimizing sequence when $\tau$ has *few* open sets.

Now, if we consider the functionals $I$ and $S$ given by (1.1) and (1.2), in order to apply the Direct Methods, we have to find a suitable topological space on which they can be defined and such that it allows to prove the lower semicontinuity and the compactness. It is worth noting that the notion of lower semicontinuity was introduced for the first time in the framework of the Calculus of Variations by Tonelli in 1913 just to treat functionals like (1.1) (see the monographs [70] and [71]; for further details see also [62]).

Since our aim is to find regular minimum points of $I$ and $S$, certainly the first attempt to try is to define them on $C^1(\Omega)$, that is, the set of the derivable functions on $\Omega$ with continuous first derivatives, endowed with its natural topology. However, it is easy to see that this space is not suitable to apply the Direct Methods since its topology, even if it makes easy the proof of the lower semicontinuity of the considered functionals, requires too much conditions to prove the compactness of the minimizing sequences. Thus, since we want $I$ and $S$ to be defined at least on $C^1(\Omega)$, we need to find more proper larger functional spaces on which they can be defined and in which, this time, some minimum points can be found by more powerful compactness theorems.

The Sobolev spaces, whose theory has been developed to solve this particular kind of problems, provide a suitable domain of definition for $I$ and $S$, because, by the properties of their weak (weak*) topologies, the compactness of the minimizing sequences can be obtained by requiring simple conditions on the functionals, even if the lower semicontinuity could be more difficult to prove.

The lower semicontinuity of $I$ and $S$ defined on Sobolev spaces has been deeply studied and necessary and sufficient conditions on $f$ and $g$ to obtain it have been found (see for instance De Giorgi [35], Ioffe [56] and Olech [65] for the integral setting and Barron and Jensen [14], Barron, Jensen and Wang [15] and Barron and Liu [16] for the supremal setting). In particular, the fundamental role of the convexity of $f$ and of the level convexity of $g$ in the gradient variable has been understood[3].

Assuming now that $f$ and $g$ guarantee the lower semicontinuity of $I : W^{1,p}_{\text{loc}}(\Omega) \to [0,\infty]$ (with $1 < p < \infty$) and $S : W^{1,\infty}_{\text{loc}}(\Omega) \to [0,\infty]$, we can simply prove the existence of a minimum by letting, for instance,

$$f(x,s,\xi) \geq c(|s|^p + |\xi|^p), \tag{1.3}$$

where $c > 0$, and

$$g(x,s,\xi) \geq \theta_\infty(|s| + |\xi|), \tag{1.4}$$

where $\theta_\infty : [0,\infty) \to [0,\infty)$ and $\lim_{t\to\infty} \theta_\infty(t) = \infty$. Indeed, (1.3) and (1.4) guarantee that every minimizing sequence of $I$ and $S$ is locally bounded in $W^{1,p}_{\text{loc}}(\Omega)$ and $W^{1,\infty}_{\text{loc}}(\Omega)$ respectively and then compact with respect to their weak (weak*) topologies.

Obviously, the minimum points found in this way can be considered only as *generalized minimum points* of the starting problem; the question of knowing if they really belong to $C^1(\Omega)$ is proper to the so called *regularity* theory[4].

Nevertheless a large class of remarkable functionals, as the class of integral functionals in which the integrand grows only linearly, does not satisfy the coercivity conditions (1.3) and (1.4) so that

[3] As far as possible, from now on, $f$ will denote a convex (in the gradient variable) function and $g$ will denote a level convex (in the gradient variable) one. However, we will always state all the properties of every function considered in the statements and proofs.

[4] We point out that, at least in the integral setting, it may happen that not only such a generalized minimum point does not belong to $C^1(\Omega)$ but also

$$\inf\left\{I(u,\Omega) : u \in W^{1,p}_{\text{loc}}(\Omega)\right\} < \inf\quad I(u,\Omega) : u \in C^1(\Omega) \quad .$$

When this particular situation occurs we say we have a *Lavrentiev* (or *Gap*) *phenomenon* (see [57], [60] and [73]). Nevertheless the Sobolev spaces have such an important role that often a minimum point in these spaces is already considered satisfactory.

the compactness theorems on the Sobolev spaces cannot be used anymore. Therefore, we would like to extend again the domain of definition of our functionals on larger topological spaces as, for instance, $L^1_{\mathrm{loc}}(\Omega)$, $C(\Omega)$ or $BV_{\mathrm{loc}}(\Omega)$, endowed with their natural topologies, in order to look for the minimum points in these new spaces: clearly this strategy requires to prove more difficult lower semicontinuity theorems, but it allows to have better compactness theorems (we will consider in particular the case of $BV_{\mathrm{loc}}(\Omega)$ endowed with the $w^*$-$BV_{\mathrm{loc}}(\Omega)$ topology).

However, functions in the spaces quoted above don't have a gradient representable as a function, contrarily to the ones in Sobolev spaces. Thus, it is not clear what could be the meaning of $I$ and $S$ when computed, for instance, on a function belonging to $BV_{\mathrm{loc}}(\Omega)$. As a consequence, it must be understood how we can build up extensions of $I$ and $S$ on these spaces in such a way that they are l.s.c. with respect to their own topologies.

Before presenting one of the possible answers to this question, let us give a short background on the area of the nonparametric surfaces. We follow Serrin [68].

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded open set with Lipschitz boundary. If $u \in \mathrm{Aff}(\Omega)$, the set of the piecewise affine functions on $\Omega$ (that is, the functions such that their graph is a polyhedral surface; see Chapter 2) the notion of area is elementary:

$$A(u,\Omega) = \mathrm{Area}(\mathrm{graph}(u)) = \sum \text{area of the plane faces} = \int_\Omega \sqrt{1+|\nabla u(x)|^2}dx.$$

However, it might be of interest a definition of the area that can be meaningful even for a continuous curved surface. To this purpose, we could use the integral expression defined above but it is not so clear which class of functions it can be really applied to. Then, as for the definition of the arc length, we could define the area of a continuous surface as the supremum of the areas of the approximating polyhedra, but, as Schwartz's counterexample shows, this method leads to contradictions (see for instance [27] page 540).

Lebesgue had the brilliant idea to define the area of a continuous surface obtained as the graph of $u \in C(\Omega)$, as

$$\mathcal{A}(u,\Omega) = \inf\left\{\liminf_{h\to\infty} A(u_h,\Omega) : u_h \in \mathrm{Aff}(\Omega),\ u_h \to u \text{ in } L^\infty(\Omega)\right\}.$$

Subsequently Serrin proved that not only, as obvious, $\mathcal{A}$ is l.s.c. on $C(\Omega)$ with respect to the uniform convergence, but also that, for every $u \in \mathrm{Aff}(\Omega)$, $A(u,\Omega) = \mathcal{A}(u,\Omega)$, that is $\mathcal{A}$ is really an extension of A (see [68] Theorem 1): these ideas are the basis of the modern concept of relaxation.

For our purposes, since we focus our attention on the problem of the extension of functionals, we introduce the notion of relaxation as follows[5].

Let us consider a topological space $(X,\tau)$, $Y \subseteq X$ be a dense subset with respect to $\tau$ and $F : Y \to [0,\infty]$. We can always define the following trivial extension of $F$ on $X$, given by

$$\tilde{F}(x) = \begin{cases} F(x) & \text{if } x \in Y, \\ \infty & \text{if } x \notin X \setminus Y, \end{cases}$$

but in general this functional is not l.s.c. on $X$ with respect to $\tau$. Nevertheless, by means of $\tilde{F}$, we can define the functional $R[\tau](F) : X \to [0,\infty]$, called the *relaxed functional* of $F$ on X with respect to $\tau$, as

$$R[\tau](F)(x) = \sup\left\{G(x) : G \le \tilde{F}, G \text{ is l.s.c. with respect to } \tau\right\};$$

note that the supremum of any family of l.s.c. functionals is l.s.c. too. Clearly, $R[\tau](F)$ is the greatest functional less or equal to $\tilde{F}$ which is l.s.c. on $X$ with respect to $\tau$ and it can be easily

[5]As for the notion of lower semicontinuity, even in this case the definition that follows is the one of *sequential relaxation*.

proved that it can be characterized, for every $x \in X$, as

$$R[\tau](F)(x) = \inf\left\{\liminf_{h\to\infty} F(x_h) : x_h \in Y, x_h \to x \text{ in } \tau\right\}. \tag{1.5}$$

It also satisfies the equality

$$\inf_{x\in X} R[\tau](F)(x) = \inf_{y\in Y} F(y), \tag{1.6}$$

that provides important information on the original minimum problem on $Y$.

However, even if the lower semicontinuity is now satisfied, it may happen that the functional $R[\tau](F)$ just built up is not an extension of $F$ on $X$ since it could exist $x \in Y$ such that $R[\tau](F)(x) < F(x)$. In order to really have an extension we must have $R[\tau](F) = F$ on $Y$ and this happens if and only if $F$ is l.s.c. on $Y$ with respect to $\tau$, that is

$$x_h, x \in Y, x_h \to x \text{ in } \tau \quad \Longrightarrow \quad F(x) \leq \liminf_{h\to\infty} F(x_h),$$

For these reasons, in order to obtain further extensions via relaxation of $I$ and $S$ on certain topological spaces larger than the Sobolev ones, first of all we have to understand the lower semicontinuity properties of these functionals defined on Sobolev spaces with respect to the new considered topologies. This argument is developed in Chapters 4 and 5.

In Chapter 4 we look for conditions on $f$ in order to obtain the lower semicontinuity of $I$ on $W^{1,1}_{\mathrm{loc}}(\Omega)$ with respect to the $L^1_{\mathrm{loc}}(\Omega)$ convergence. This is a classical problem in the Calculus of Variations, first studied by Serrin in [69]. Here we propose slightly different versions of the results contained in the papers of Gori and Marcellini [55], Gori, Maggi and Marcellini [54] and Gori and Maggi [53], in which new and very mild conditions on $f$ that guarantee the lower semicontinuity of $I$ are given.

The proofs of the main results of Chapter 4 are based on certain approximation theorems for convex functions (depending continuously on a parameter), collected in Chapter 3. In particular, we propose the proofs of two very recent theorems contained in [53] which provide new methods of approximations by means of convex cones and of strictly convex functions.

In Chapter 5 we consider the functional $S$ defined on $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and we study its lower semicontinuity with respect to the $L^\infty_{\mathrm{loc}}(\Omega)$ topology. Some conditions on $g$ in this contest have been found first by Gori and Maggi [52]: in this chapter we improve these results, considering the problem of the necessary conditions too.

As already stated, the theorems of Chapters 4 and 5 can be read as results that provide hypotheses on $f$ and $g$, that guarantee the equalities $R\left[L^1_{\mathrm{loc}}\right](I) = I$ and $R\left[L^\infty_{\mathrm{loc}}\right](S) = S$ on $W^{1,1}_{\mathrm{loc}}(\Omega)$ and $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ respectively. Nevertheless these results give no information about the possibility to represent (and to easily compute) $R\left[L^1_{\mathrm{loc}}\right](I)$ on $L^1_{\mathrm{loc}}(\Omega) \setminus W^{1,1}_{\mathrm{loc}}(\Omega)$ as an integral either $R\left[L^\infty_{\mathrm{loc}}\right](S)$ on $C(\Omega) \setminus W^{1,\infty}_{\mathrm{loc}}(\Omega)$ as a supremal[6].

Moving from these remarks, in Chapter 7 we consider $I$ and $S$ defined on $W^{1,1}_{\mathrm{loc}}(\Omega)$ and we approach the particular problem to understand if it is possible to find one of their extensions on $BV_{\mathrm{loc}}(\Omega)$ that is l.s.c. with respect to the $w^*$-$BV_{\mathrm{loc}}(\Omega)$ topology and such that could be explicitly represented on this space as an integral and as a supremal respectively.

We stress that the relaxation is not the unique strategy to extend the functionals $I$ and $S$ on $BV_{\mathrm{loc}}(\Omega)$ in a lower semicontinuous way: indeed, another appropriate way to extend $I$ and $S$, different from the relaxation, is suggested by Serrin in [68] and [69] and described below.

First of all let us consider the following notion of convergence: given an open set $\Omega$ and a function $u \in BV_{\mathrm{loc}}(\Omega)$, we say that a sequence $\{u_h\}_{h=1}^\infty$ converges to $u$ in $w^*$-$BV_{\mathrm{loc}}(\updownarrow \Omega)$ if there exists a sequence $\{\Omega_h\}_{h=1}^\infty$ of open sets such that $u_h \in BV_{\mathrm{loc}}(\Omega_h)$ and, for every open set $\Omega' \subset\subset \Omega$,

[6]For what concerns the integral setting, the problem of the representation of $R\ \ L^1_{\mathrm{loc}}\ \ (I)$ has been deeply studied, and several general integral formulas to represent $R\ \ L^1_{\mathrm{loc}}\ \ (I)$ at least on $BV_{\mathrm{loc}}(\Omega)$ have been found (see for instance Dal Maso [30]).

we have $\Omega' \subset\subset \Omega_h$ if $h$ is large enough and $u_h \to u$ in $w^*$-$BV(\Omega')$ as $h \to \infty$. Then, for every $u \in BV_{\mathrm{loc}}(\Omega)$, we define

$$I^*(u,\Omega) = \inf\left\{\liminf_{h\to\infty} I(u_h,\Omega_h) : u_h \in W^{1,1}_{\mathrm{loc}}(\Omega_h),\, u_h \to u \text{ in } w^*\text{-}BV_{\mathrm{loc}}(\updownarrow \Omega)\right\}, \tag{1.7}$$

and

$$S^*(u,\Omega) = \inf\left\{\liminf_{h\to\infty} S(u_h,\Omega_h) : u_h \in W^{1,1}_{\mathrm{loc}}(\Omega_h),\, u_h \to u \text{ in } w^*\text{-}BV_{\mathrm{loc}}(\updownarrow \Omega)\right\}. \tag{1.8}$$

The analogy between $I^*$ and $S^*$ and the functionals $R\left[w^*\text{-}BV_{\mathrm{loc}}\right](I)$ and $R\left[w^*\text{-}BV_{\mathrm{loc}}\right](S)$ is clear. Moreover, for every $u \in BV_{\mathrm{loc}}(\Omega)$, it holds

$$I^*(u,\Omega) \le R\left[w^*\text{-}BV_{\mathrm{loc}}\right](I)(u,\Omega) \quad \text{and} \quad S^*(u,\Omega) \le R\left[w^*\text{-}BV_{\mathrm{loc}}\right](S)(u,\Omega),$$

and, as we will see, in some cases even the equality holds.

In Chapter 7 we present the main results of the theory involving $I^*$, developed by Serrin in [69] and by Goffman and Serrin in [49], and of the one about $S^*$, developed by Gori in [51]. In particular, when $f$ and $g$ depend only on the gradient variable (and of course they are convex and level convex respectively), we prove not only that $I^*$ and $S^*$ extend $I$ and $S$ on $BV_{\mathrm{loc}}(\Omega)$ and that they are l.s.c. with respect to the $w^*$-$BV_{\mathrm{loc}}(\Omega)$ convergence, but also, that they can be also represented on $BV_{\mathrm{loc}}(\Omega)$ as an integral and a supremal given by [7]

$$\int_\Omega f(\nabla u(x))dx + \int_\Omega f^\infty\left(\frac{dD^s u}{d|D^s u|}(x)\right) d|D^s u|(x),$$

and

$$\left[\operatorname*{ess\,sup}_{x\in\Omega} g(\nabla u(x))\right] \vee \left[|D^s u|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} g^\natural\left(\frac{dD^s u}{d|D^s u|}(x)\right)\right].$$

In the same chapter, by means of the representation formula found for $S^*$ and following the analogous theory developed for the integral functionals by Anzellotti, Buttazzo and Dal Maso [8], we define also a generalized Dirichlet problem for supremal functionals defined on $BV(\Omega)$ and we prove the existence of a minimum on $BV(\Omega)$ even if $g$ is not coercive (that is, (1.4) does not hold): in some sense, this existence result justifies the extension of $S$ on $BV(\Omega)$.

It is worth noting that the lower semicontinuity theorems and the representations formulas on $BV_{\mathrm{loc}}(\Omega)$ about the functionals (1.7) and (1.8), presented in Chapter 7, are obtained as corollaries of more general results proved for analogous functionals defined on Radon measures: in fact in Chapter 6 the well known theory for the integral functional (see [49], [69]) is presented together with several new results about the supremal setting, which generalize the paper of Gori [51].

We stress that, even if these results about the integral functionals are classical, we present their proofs together with the ones for the supremal functionals just to show the complete analogy that there exists, even under a technical point of view, between integral and supremal settings.

Moreover in the same chapter, starting by the lower semicontinuity results obtained by Bouchitté and Buttazzo for non convex integral functionals defined on Radon measures (see [18]), we approach the analogous problem for the non level convex supremal ones, finding necessary and sufficient conditions. When these results are applied to the setting of the functions of bounded variation, they lead to the existence of a minimum point for a particular non level convex and one dimensional problem, stated and solved in Chapter 7, which presents some similar aspects to the celebrated Mumford-Shah image segmentation problem (see also Alicandro, Braides and Cicalese [3]).

We conclude saying that Chapter 2 contains some brief notes on measure theory, functions of bounded variation and convex analysis, in which the main definitions and theorems on these topics

[7] See Chapter 2 for notations.

are recalled. However, in this chapter, we prove also a certain number of new propositions, most of them about the fine properties of level convex functions, that are fundamental for the proofs of the remaining chapters.

# Chapter 2

# Preliminary results

In this chapter we introduce most of the definitions and notations we will need, and we collect several propositions and theorems that will be useful tools to prove the main results presented in this thesis. In the following we will implicitly refer to this chapter for every (non trivial) symbol used.

## 2.1 Notes on measure theory

Let us consider the measurable space $(\Omega, \mathcal{B}(\Omega))$ where $\Omega \subseteq \mathbb{R}^n$ is an open set and $\mathcal{B}(\Omega)$ is the $\sigma$-algebra of the Borel subsets of $\Omega$.

We say that $K$ is *well contained* in $\Omega$ (briefly $K \subset\subset \Omega$) if $\mathrm{cl}(K) \subseteq \Omega$, that is, the *closure* of $K$ is a subset of $\Omega$. Moreover, for every $x \in \mathbb{R}^n$ and $\rho > 0$, we set

$$B(x,\rho) = \{y \in \mathbb{R}^n : |x-y| < \rho\} \quad \text{and} \quad \overline{B(x,\rho)} = \mathrm{cl}\left(B(x,\rho)\right).$$

We set also $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$.

If $\mu : \mathcal{B}(\Omega) \to [0,\infty]$ is a positive measure then it is called a *positive Borel measure* on $\Omega$. Moreover $\mu$ is called a *positive Radon measure* (resp. *finite positive Radon measure*) on $\Omega$, if, for every $K \in \mathcal{B}(\Omega)$, $K \subset\subset \Omega$, it is also $\mu(K) < \infty$ (resp. $\mu(\Omega) < \infty$): the set of the positive Radon measures on $\Omega$ is denoted by $\mathcal{M}^+(\Omega)$. Obviously the Lebesgue measure on $\mathbb{R}^n$, denoted with $\mathcal{L}^n$, belongs to $\mathcal{M}^+(\mathbb{R}^n)$ so as the standard Dirac measure centered on $x \in \mathbb{R}^n$ denoted with $\delta_x$.

Let us fix $m \in \mathbb{N}$: if $\lambda : \mathcal{B}(\Omega) \to \mathbb{R}^m$ is a vector measure then it is called a *finite Radon measure* on $\Omega$ while if $\lambda : \{K \in \mathcal{B}(\Omega) : K \subset\subset \Omega\} \to \mathbb{R}^m$ and, for every $K \in \mathcal{B}(\Omega)$, $K \subset\subset \Omega$, $\lambda$ is a vector measure on $\mathcal{B}(K)$ then it is called a *Radon measure* on $\Omega$. We denote the set of Radon measures (resp. finite Radon measures) with $\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ (resp. $\mathcal{M}(\Omega, \mathbb{R}^m)$)[1].

Let us consider now $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m) \cup \mathcal{M}^+(\Omega)$. For every $B \in \mathcal{B}(\Omega)$ the *total variation*[2] of $\lambda$ on $B$ is defined by[3]

$$|\lambda|(B) = \sup\left\{\sum_{j=1}^{r} |\lambda(B_j)| : B_j \in \mathcal{B}(\Omega), B_j \subset\subset \Omega, \bigcup_{j=1}^{r} B_j \subseteq B, B_i \cap B_j = \emptyset, \forall i \neq j\right\}. \tag{2.1}$$

It can be proved that $|\lambda| \in \mathcal{M}^+(\Omega)$, that $\lambda \in \mathcal{M}(\Omega, \mathbb{R}^m)$ implies $|\lambda|(\Omega) < \infty$ and that if $\lambda \in \mathcal{M}^+(\Omega)$ then $|\lambda| = \lambda$.

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[1] Clearly $\mathcal{M}^+(\Omega) \subseteq \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R})$, $\mu \in \mathcal{M}^+(\Omega) : \mu(\Omega) < \infty \subseteq \mathcal{M}(\Omega, \mathbb{R})$ and $\mathcal{M}(\Omega, \mathbb{R}^m) \subseteq \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$.

[2] We present here the definition of total variation directly for Radon measures: it is simple to verify that, when $\lambda$ is a finite Radon measure, it is equivalent to the usual one (see for instance [7] page 3).

[3] When $x \in \mathbb{R}^n$, $|x|$ means $\|x\|_{\mathbb{R}^n}$ the standard Euclidean norm on $\mathbb{R}^n$.

Given now $\lambda \in \mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m) \cup \mathcal{M}^+(\Omega)$ we say that $\lambda$ is *concentrated* on a set $B \in \mathcal{B}(\Omega)$ if $|\lambda|(\Omega \setminus B) = 0$. If $\lambda_1, \lambda_2 \in \mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m) \cup \mathcal{M}^+(\Omega)$ we say that $\lambda_1$ is *singular* with respect to $\lambda_2$ (briefly $\lambda_1 \perp \lambda_2$) if there exist $B_1, B_2 \in \mathcal{B}(\Omega)$ such that $\lambda_1$ is concentrated in $B_1$, $\lambda_2$ is concentrated in $B_2$ and $B_1 \cap B_2 = \emptyset$.

If now we consider $\lambda \in \mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m) \cup \mathcal{M}^+(\Omega)$ and $\mu \in \mathcal{M}^+(\Omega)$ we say that $\lambda$ is *absolutely continuous* with respect to $\mu$ (briefly $\lambda << \mu$) if $\mu(B) = 0$ implies $|\lambda|(B) = 0$. Moreover, fixed $E \in \mathcal{B}(\Omega)$, we denote with $\lambda \llcorner E$ the *restriction* of $\lambda$ to E, that is, the element of $\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m) \cup \mathcal{M}^+(\Omega)$ defined, for every $B \in \mathcal{B}(\Omega)$ (when $\lambda \notin \mathcal{M}(\Omega,\mathbb{R}^m) \cup \mathcal{M}^+(\Omega)$, we need $B \subset\subset \Omega$ too), as $(\lambda \llcorner E)(B) = \lambda(E \cap B)$. For every $E \in \mathcal{B}(\mathbb{R}^n)$ we will always denote with $\mathcal{L}^n$ the measure $\mathcal{L}^n \llcorner E$.

We write for short $\lambda(x)$ instead of $\lambda(\{x\})$ and we denote

$$A_\lambda = \{x \in \Omega : \lambda(x) \neq 0\},$$

the *set of the atoms* of $\lambda$. Finally, if $\mu \in \mathcal{M}^+(\Omega)$, we define the *support* of $\mu$ as the set

$$\operatorname{spt}(\mu) = \operatorname{cl}\left(\left\{x \in \Omega : \forall \rho > 0, \mu(B(x,\rho)) > 0\right\}\right),$$

while, if $\lambda \in \mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$, we set $\operatorname{spt}(\lambda) = \operatorname{spt}(|\lambda|)$: note that $\lambda$ is concentrated on $\operatorname{spt}(\lambda)$.

Let us consider now a Borel function $u : \Omega \to \mathbb{R}$, that is a function which is measurable with respect to the $\sigma$-algebra $\mathcal{B}(\Omega)$ (the same definition holds even if $u : \Omega \to \mathbb{R}^m$), and $\mu \in \mathcal{M}^+(\Omega)$. It is surely well known the notion of integral on $B \in \mathcal{B}(\Omega)$ of $u$ with respect to $\mu$. We prefer instead to remember that the *essential supremum* on $B \in \mathcal{B}(\Omega)$ of $u$ with respect to $\mu$ is defined as

$$\mu\text{-}\operatorname*{ess\,sup}_{x \in B} u(x) = \begin{cases} \inf\left\{\sup_{x \in B \setminus A} u(x) : A \in \mathcal{B}(\Omega), A \subseteq B, \mu(A) = 0\right\} & \text{if } \mu(B) \neq 0, \\ -\infty & \text{if } \mu(B) = 0, \end{cases}$$

pointing out that, if $\mu << \mathcal{L}^n$, then[4]

$$\mu\text{-}\operatorname*{ess\,sup}_{x \in B} u(x) \leq \operatorname*{ess\,sup}_{x \in B} u(x). \tag{2.2}$$

We set also, for every $a, b \in \mathbb{R}$, the notations $a \vee b = \sup\{a,b\}$ and $a \wedge b = \inf\{a,b\}$.

Let $\mu \in \mathcal{M}^+(\Omega)$, $m \geq 1$ and $u : \Omega \to \mathbb{R}^m$ be a Borel function: we say $u \in L^1_\mu(\Omega,\mathbb{R}^m)$ (resp. $L^\infty_\mu(\Omega,\mathbb{R}^m)$) if it is

$$\int_\Omega |u(x)| d\mu(x) < \infty, \quad \left(\text{resp. } \mu\text{-}\operatorname*{ess\,sup}_{x \in \Omega} |u(x)| < \infty\right),$$

and, in this case, we set

$$\int_\Omega u(x) d\mu(x) = \left(\int_\Omega u_1(x) d\mu(x), \ldots, \int_\Omega u_m(x) d\mu(x)\right) \in \mathbb{R}^m,$$

where $u = (u_1, \ldots, u_m)$. When, for every $K \in \mathcal{B}(\Omega)$, $K \subset\subset \Omega$ and $\mu(K) > 0$[5], $u \in L^1_\mu(K,\mathbb{R}^m)$ (resp. $L^\infty_\mu(K,\mathbb{R}^m)$), we say $u \in L^1_{{\rm loc},\mu}(\Omega,\mathbb{R}^m)$ (resp. $L^\infty_{{\rm loc},\mu}(\Omega,\mathbb{R}^m)$)[6].

Fixed $\lambda \in \mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$, we say that a scalar valued (resp. $\mathbb{R}^m$-valued) Borel function $u$ belongs to $L^1_\lambda(\Omega)$ (resp. $L^1_\lambda(\Omega,\mathbb{R}^m)$) if

$$\int_\Omega |u(x)| d|\lambda|(x) < \infty,$$

[4] When the Lebesgue's measure is considered, we write for short $\operatorname{ess\,sup}$ instead of $\mathcal{L}^n$-$\operatorname{ess\,sup}$.

[5] This last condition on $K$ is clearly relevant only to define $L^\infty_{{\rm loc},\mu}(\Omega,\mathbb{R}^m)$.

[6] When $m = 1$ we write for short $L^1_\mu(\Omega)$, $L^\infty_\mu(\Omega)$, $L^1_{{\rm loc},\mu}(\Omega)$ and $L^\infty_{{\rm loc},\mu}(\Omega)$ and when $\mu = \mathcal{L}^n$, $\mu$ is omitted.

and, in this case, we set

$$\int_\Omega u(x)d\lambda(x) = \left(\int_\Omega u(x)d\lambda_i(x), \dots, \int_\Omega u(x)d\lambda_m(x)\right) \in \mathbb{R}^m,$$

$$\left(\text{resp. } \int_\Omega u(x)d\lambda(x) = \sum_{i=1}^m \int_\Omega u_i(x)d\lambda_i(x) \in \mathbb{R}\right),$$

where $\lambda = (\lambda_1, \dots, \lambda_m)$ and, for every $i \in \{1, \dots, m\}$,

$$\int_\Omega u(x)d\lambda_i(x) = \int_\Omega u(x)d\lambda_i^+(x) - \int_\Omega u(x)d\lambda_i^-(x),$$

$$\left(\text{resp. } \int_\Omega u_i(x)d\lambda_i(x) = \int_\Omega u_i(x)d\lambda_i^+(x) - \int_\Omega u_i(x)d\lambda_i^-(x)\right),$$

where $\lambda_i^+ = \frac{|\lambda_i|+\lambda_i}{2}$ and $\lambda_i^- = \frac{|\lambda_i|-\lambda_i}{2}$ belong to $\mathcal{M}^+(\Omega)$. When, for every $K \in \mathcal{B}(\Omega)$, $K \subset\subset \Omega$, $u \in L^1_\lambda(K)$ (resp. $L^1_\lambda(K, \mathbb{R}^m)$), we say $u \in L^1_{\text{loc},\lambda}(\Omega)$ (resp. $L^1_{\text{loc},\lambda}(\Omega, \mathbb{R}^m)$).

If we consider now $\mu \in \mathcal{M}^+(\Omega)$ and a function $u \in L^1_\mu(\Omega, \mathbb{R}^m)$ (resp. $L^1_{\text{loc},\mu}(\Omega, \mathbb{R}^m)$) we denote with $u \cdot \mu$ the element of $\mathcal{M}(\Omega, \mathbb{R}^m)$ (resp. $\mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$) defined, for every $B \in \mathcal{B}(\Omega)$ (resp. $B \in \mathcal{B}(\Omega)$, $B \subset\subset \Omega$), as

$$(u \cdot \mu)(B) = \int_B u(x)d\mu(x);$$

it is well known that $|u \cdot \mu| = |u| \cdot \mu$ and $u \cdot \mu << \mu$. A quite standard result of measure theory is that, given $\mu \in \mathcal{M}^+(\Omega)$, we can decompose in a unique way[7] a measure $\lambda \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$ as

$$\lambda = \lambda^a \cdot \mu + \lambda^s, \tag{2.3}$$

where $\lambda^a \in L^1_{\text{loc},\mu}(\Omega, \mathbb{R}^m)$ and $\lambda^s \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$ with $\lambda^s \perp \mu$: we call $\lambda^a \cdot \mu$ the *absolutely continuous part* of $\lambda$ while $\lambda^s$ its *singular part*. With the notation quoted above, we can also write

$$\lambda = \lambda^a \cdot \mu + \lambda^c + \lambda^\#, \tag{2.4}$$

where $\lambda^c = \lambda^s \llcorner (\Omega \setminus A_\lambda)$ is said the *Cantor part* of $\lambda$ while $\lambda^\# = \lambda^s \llcorner A_\lambda$ is said its *atomic part*. These decompositions of $\lambda$ obviously depend on $\mu$ even if in the notations $\lambda^a$, $\lambda^s$, $\lambda^c$ and $\lambda^\#$ the measure $\mu$ is not expressly named: however, every time one of these decompositions is used, the measure $\mu$ will be clear by the context. Finally note that, if $\lambda \in \mathcal{M}(\Omega, \mathbb{R}^m)$, then $\lambda^a \in L^1_\mu(\Omega, \mathbb{R}^m)$ and $\lambda^s, \lambda^c, \lambda^\# \in \mathcal{M}(\Omega, \mathbb{R}^m)$.

The linear space $\mathcal{M}(\Omega, \mathbb{R}^m)$, endowed with the norm $\|\lambda\|_\mathcal{M} = |\lambda|(\Omega)$, is a Banach space identifiable with the dual of the linear space $C_0(\Omega, \mathbb{R}^m)$, (that is, the set of the continuous functions $\varphi$ defined on $\Omega$ with values on $\mathbb{R}^m$ with the property that, for every $\varepsilon > 0$, there exists $K_\varepsilon \subset\subset \Omega$ such that, for every $x \in \Omega \setminus K_\varepsilon$, $|\varphi(x)| < \varepsilon$), by the duality

$$\langle \lambda, \varphi \rangle = \int_\Omega \varphi(x)d\lambda(x) = \sum_{i=1}^m \int_\Omega \varphi_i(x)d\lambda_i(x),$$

while the linear space $\mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$ can be identified with the dual of the locally convex linear space $C_c(\Omega, \mathbb{R}^m)$, (that is, the set of the continuous functions on $\Omega$ with values on $\mathbb{R}^m$ such that every component has compact support), with the same duality. We remember that, given a function

[7] This is the Radon-Nikodým Theorem (see for instance [7] Theorem 1.28). The uniqueness of the decomposition has to be interpreted in this way: if $\lambda = \lambda_1^a \cdot \mu + \lambda_1^s = \lambda_2^a \cdot \mu + \lambda_2^s$, then, for $\mu$-a.e. $x \in \Omega$, $\lambda_1^a(x) = \lambda_2^a(x)$ and, for every $B \in \mathcal{B}(\Omega)$, $B \subset\subset \Omega$, $\lambda_1^s(B) = \lambda_2^s(B)$.

$\varphi \in C(\Omega, \mathbb{R}^m)$ (that is, the set of the continuous functions on $\Omega$ with values on $\mathbb{R}^m$)[8], its *support* is defined as

$$\mathrm{spt}(\varphi) = \mathrm{cl}\left(\{x \in \Omega : \varphi(x) \neq 0\}\right).$$

For this, given $\lambda_h, \lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ (resp. $\mathcal{M}(\Omega, \mathbb{R}^m)$), we say $\lambda_h \to \lambda$ in $w^*$-$\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ (resp. $w^*$-$\mathcal{M}(\Omega, \mathbb{R}^m)$) if, for every $\varphi \in C_c(\Omega)$ (resp. $C_0(\Omega)$), we have

$$\lim_{h\to\infty} \int_\Omega \varphi(x) d\lambda_h(x) = \int_\Omega \varphi(x) d\lambda(x).$$

Let us note that, for technical reasons, in the previous definition we have considered scalar valued test functions $\varphi$: however, the definitions given agree with the ones obtained considering vector value test functions and the duality before described.

It is suitable at this point to introduce also the following notion of convergence. Let $\{\Omega_h\}_{h=1}^\infty$ be a family of open sets and let $\lambda_h \in \mathcal{M}_{\mathrm{loc}}(\Omega_h, \mathbb{R}^m)$, $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$: we say $\lambda_h \to \lambda$ in $w^*$-$\mathcal{M}_{\mathrm{loc}}(\updownarrow \Omega, \mathbb{R}^m)$ if, for every $K \subset\subset \Omega$, we have $K \subseteq \Omega_h$ if $h$ is large enough and, for every $\varphi \in C_c(\Omega)$,

$$\lim_{h\to\infty} \int_{\Omega_h} \varphi(x) d\lambda_h(x) = \int_\Omega \varphi(x) d\lambda(x).$$

The following compactness theorem for Radon measures holds (see for instance [7] Theorem 1.59 and Corollary 1.60).

**Theorem 2.1.** *Let $\{\lambda_h\}_{h=1}^\infty \subseteq \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ (resp. $\mathcal{M}(\Omega, \mathbb{R}^m)$) be a sequence such that, for every $K \subset\subset \Omega$,*

$$\sup\Big\{|\lambda_h|(K) : h \in \mathbb{N}\Big\} < \infty \qquad \Big(\text{resp. } \sup\Big\{|\lambda_h|(\Omega) : h \in \mathbb{N}\Big\} < \infty\Big).$$

*Then there exists a subsequence $\{\lambda_{h_k}\}_{k=1}^\infty$ and $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ (resp. $\mathcal{M}(\Omega, \mathbb{R}^m)$) such that $\lambda_{h_k} \to \lambda$ in $w^*$-$\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ (resp. $w^*$-$\mathcal{M}(\Omega, \mathbb{R}^m)$).*

Given a measure $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and $\rho > 0$, we define the *convolution* of $\lambda$ with step $\rho$ as

$$\lambda_\rho(x) = \int_{B(x,\rho)} \rho^{-n} k\left(\frac{x-y}{\rho}\right) d\lambda(y) : \Omega_\rho \to \mathbb{R}^m. \tag{2.5}$$

In the previous definition $k : \mathbb{R}^n \to [0,1]$ is a *convolution kernel*, that is $k \in C_c^\infty(\mathbb{R}^n)$, for every $x \in \mathbb{R}^n$, $k(x) = k(-x)$, $\mathrm{spt}(k) \subseteq B(0,1)$ and $\int_{\mathbb{R}^n} k(x)dx = 1$ (we require also that $k(0) \neq 0$), and

$$\Omega_\rho = \{x \in \Omega : d(x, \partial\Omega) > \rho\},$$

where $d(x, \partial\Omega) = \inf\{|x-y| : y \in \partial\Omega\}$ and $\partial\Omega$ is the topological boundary of $\Omega$.

It is well known that $\lambda_\rho \in C^\infty(\Omega_\rho, \mathbb{R}^m)$ and it can be simply proved using Fubini's Theorem that $\lambda_\rho \cdot \mathcal{L}^n \to \lambda$ in $w^*$-$\mathcal{M}_{\mathrm{loc}}(\updownarrow \Omega, \mathbb{R}^m)$ as $\rho \to 0$ (see [7] Theorem 2.2). In the following, referring to a convolution kernel $k$, we will set $k_\rho(x) = \rho^{-n} k\left(\rho^{-1}x\right)$.

Now we propose a theorem on measures that describes the point-wise behavior of the convolutions of a measure and that will be fundamental in proving some results of the following chapters. The principal tools used in its proof are the Besicovich's Derivation Theorem for Radon measures and the Lebesgue's points Theorem (see for instance [7] Theorem 2.22 and Corollary 2.23). This theorem generalizes Theorem 3 in [51].

**Theorem 2.2.** *Let $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$. Then, referring to the decomposition* (2.3) *with respect to $\mathcal{L}^n$, we have*

*(i) for $\mathcal{L}^n$-a.e. $x \in \Omega$, $\lim_{\rho\to 0} |\lambda_\rho(x) - \lambda^a(x)| = 0$;*

[8] When $m = 1$ we write for short $C(\Omega)$, $C_0(\Omega)$ and $C_c(\Omega)$.

*(ii) for $|\lambda^s|$-a.e. $x \in \operatorname{spt}(\lambda^s)$, there exists a sequence of positive numbers $\{\rho_h\}_{h=1}^{\infty}$, depending on $x$ and decreasing to zero, such that*[9]

$$\lim_{h\to\infty} \left| \frac{d\lambda^s}{d|\lambda^s|}(x) - \frac{\lambda_{\rho_h}(x)}{\int_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y)} \right| = 0. \tag{2.6}$$

For simplicity we prove Theorem 2.2 by means of two lemmas that, in our opinion, are interesting on their own. The first lemma is a simple fact from the measure theory and, in this form, can be found in [6], Theorem 2.3.

**Lemma 2.3.** *Let $\mu \in \mathcal{M}^+(\Omega)$ such that $\mu \perp \mathcal{L}^n$. Then, for $\mu$-a.e. $x \in \operatorname{spt}(\mu)$ and, for every $\sigma \in (0,1)$, we have*

$$\sigma^n \le \limsup_{\rho\to 0} \frac{\mu(B(x,\sigma\rho))}{\mu(B(x,\rho))} \le 1.$$

*Proof.* Let us set

$$\Omega(\mu) = \left\{ x \in \operatorname{spt}(\mu) : \lim_{\rho\to 0} \frac{\mu(B(x,\rho))}{\rho^n} = \infty \right\} : \tag{2.7}$$

since $\mu(\Omega \setminus \Omega(\mu)) = 0$ and $\mathcal{L}^n(\Omega(\mu)) = 0$ (see [7] Theorem 2.22), we achieve the proof showing the wanted relation for every $x \in \Omega(\mu)$. Let us suppose, by contradiction, there exist $x_0 \in \Omega(\mu)$ and $\sigma_0 \in (0,1)$ such that

$$\sigma_0^n > \limsup_{\rho\to 0} \frac{\mu(B(x_0,\sigma_0\rho))}{\mu(B(x_0,\rho))}.$$

Then there exists $\rho_0$ such that, for every $0 < \rho \le \rho_0$, $\mu(B(x_0,\sigma_0\rho)) \le \sigma_0^n \mu(B(x_0,\rho))$. If we call $\omega(\rho) = \mu(B(x_0,\rho))$ we have that, for every $0 < \rho \le \rho_0$, $\omega(\sigma_0\rho) \le \sigma_0^n\omega(\rho)$. Then, for every $h \in \mathbb{N}$, $\sigma_0^{-nh}\omega(\sigma_0^h\rho_0) \le \omega(\rho_0)$, thus

$$\frac{\mu(B(x_0,\sigma_0^h\rho_0))}{(\sigma_0^h\rho_0)^n}\rho_0^n \le \omega(\rho_0) = \mu(B(x_0,\rho_0)) < \infty.$$

If now $h \to \infty$, then $\sigma_0^h\rho_0 \to 0$ and the left hand side of the previous inequality tends to infinity: thus the contradiction is found. □

In particular, fixed $\sigma \in (0,1)$, for $\mu$-a.e. $x \in \operatorname{spt}(\mu)$, there exists a sequence $\{\rho_h\}_{h=1}^{\infty}$, depending on $x$ and decreasing to zero, such that, for every $h \in \mathbb{N}$,

$$\mu(B(x,\sigma\rho_h)) \ge \sigma^{n+1}\mu(B(x,\rho_h)). \tag{2.8}$$

**Lemma 2.4.** *Let $\lambda^s \in \mathcal{M}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$, $\mu \in \mathcal{M}^+(\Omega)$ such that $\lambda^s \perp \mathcal{L}^n$ and $\mu << \mathcal{L}^n$. Then, for $|\lambda^s|$-a.e. $x \in \operatorname{spt}(\lambda^s)$, there exists a sequence of positive numbers $\{\rho_h\}_{h=1}^{\infty}$, depending on $x$ and decreasing to zero, such that*

$$\lim_{h\to\infty} \frac{\int_{B(x,\rho_h)} k_{\rho_h}(x-y)d\mu(y)}{\int_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y)} = 0 \tag{2.9}$$

*and*

$$\lim_{h\to\infty} \frac{\int_{B(x,\rho_h)} k_{\rho_h}(x-y)\left|\frac{d\lambda^s}{d|\lambda^s|}(y) - \frac{d\lambda^s}{d|\lambda^s|}(x)\right| d|\lambda^s|(y)}{\int_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y)} = 0. \tag{2.10}$$

[9]We point out that, for $|\lambda^s|$-a.e. $x \in \operatorname{spt}(\lambda^s)$, $\frac{d\lambda^s}{d|\lambda^s|}(x) \in S^{m-1}$ and $\int_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y) \neq 0$.

*Proof.* Obviously $\lambda^s \perp \mu$. Thus, let us consider $\Omega(|\lambda^s|)$ defined as in (2.7), and the set

$$\Omega_0 = \Omega(|\lambda^s|) \cap \left\{ x \in \operatorname{spt}(\lambda^s) : \lim_{\rho\to 0} \frac{\mu(B(x,\rho))}{|\lambda^s|(B(x,\rho))} = 0,\, x \text{ is a Lebesgue's point for } \frac{d\lambda^s}{d|\lambda^s|} \right\}.$$

Clearly $|\lambda^s|(\Omega \setminus \Omega_0) = 0$: then the proof is achieved if, for every $x \in \Omega_0$, (2.9) and (2.10) hold. By the properties of the convolution kernel $k$, we can find $\sigma \in (0,1)$ and $c, M > 0$ such that, for every $x \in \mathbb{R}^n$, $k(x) \leq M$ and, for every $x \in B(0,\sigma)$, $k(x) \geq c$ (remember that $k(0) \neq 0$). Thus, let us fix $x \in \Omega_0$ and, since $\Omega_0 \subseteq \Omega(|\lambda^s|)$, let us consider, with respect to $|\lambda^s|$, the sequence $\{\rho_h\}_{h=1}^{\infty}$ ($\rho_h < \sigma$) given by (2.8). Then we have

$$0 \leq \lim_{h\to\infty} \frac{\int_{B(x,\rho_h)} k_{\rho_h}(x-y)d\mu(y)}{\int_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y)} = \lim_{h\to\infty} \frac{\int_{B(x,\rho_h)} k\left(\frac{x-y}{\rho_h}\right) d\mu(y)}{\int_{B(x,\rho_h)} k\left(\frac{x-y}{\rho_h}\right) d|\lambda^s|(y)} \leq \lim_{h\to\infty} \frac{M}{c} \frac{\mu(B(x,\rho_h))}{|\lambda^s|(B(x,\sigma\rho_h))}$$

$$= \lim_{h\to\infty} \frac{M}{c} \frac{\mu(B(x,\rho_h))}{|\lambda^s|(B(x,\rho_h))} \cdot \frac{|\lambda^s|(B(x,\rho_h))}{|\lambda^s|(B(x,\sigma\rho_h))} \leq \lim_{h\to\infty} \frac{M}{c\sigma^{n+1}} \frac{\mu(B(x,\rho_h))}{|\lambda^s|(B(x,\rho_h))} = 0,$$

and

$$0 \leq \lim_{h\to\infty} \frac{\int_{B(x,\rho_h)} k_{\rho_h}(x-y) \left| \frac{d\lambda^s}{d|\lambda^s|}(y) - \frac{d\lambda^s}{d|\lambda^s|}(x) \right| d|\lambda^s|(y)}{\int_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y)}$$

$$= \lim_{h\to\infty} \frac{\int_{B(x,\rho_h)} k\left(\frac{x-y}{\rho_h}\right) \left| \frac{d\lambda^s}{d|\lambda^s|}(y) - \frac{d\lambda^s}{d|\lambda^s|}(x) \right| d|\lambda^s|(y)}{\int_{B(x,\rho_h)} k\left(\frac{x-y}{\rho_h}\right) d|\lambda^s|(y)} \leq \lim_{h\to\infty} \frac{M}{c} \frac{\int_{B(x,\rho_h)} \left| \frac{d\lambda^s}{d|\lambda^s|}(y) - \frac{d\lambda^s}{d|\lambda^s|}(x) \right| d|\lambda^s|(y)}{|\lambda^s|(B(x,\sigma\rho_h))}$$

$$= \lim_{h\to\infty} \frac{M}{c} \frac{\int_{B(x,\rho_h)} \left| \frac{d\lambda^s}{d|\lambda^s|}(y) - \frac{d\lambda^s}{d|\lambda^s|}(x) \right| d|\lambda^s|(y)}{|\lambda^s|(B(x,\rho_h))} \cdot \frac{|\lambda^s|(B(x,\rho_h))}{|\lambda^s|(B(x,\sigma\rho_h))}$$

$$\leq \lim_{h\to\infty} \frac{M}{c\sigma^{n+1}} \frac{\int_{B(x,\rho_h)} \left| \frac{d\lambda^s}{d|\lambda^s|}(y) - \frac{d\lambda^s}{d|\lambda^s|}(x) \right| d|\lambda^s|(y)}{|\lambda^s|(B(x,\rho_h))} = 0,$$

that prove (2.9) and (2.10). □

*Proof of Theorem 2.2. (i).* Let us fix $\lambda \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$ and $M > 0$ such that, for every $x \in \mathbb{R}^n$, $k(x) \leq M$, then

$$|\lambda_\rho(x) - \lambda^a(x)| = \left| \int_{B(x,\rho)} k_\rho(x-y)d\lambda(y) - \lambda^a(x) \right|$$

$$\leq \int_{B(x,\rho)} k_\rho(x-y)|\lambda^a(y) - \lambda^a(x)|dy + \int_{B(x,\rho)} k_\rho(x-y)d|\lambda^s|(y)$$

$$\leq M\rho^{-n} \int_{B(x,\rho)} |\lambda^a(y) - \lambda^a(x)|dy + M\rho^{-n} \int_{B(x,\rho)} d|\lambda^s|(y)$$

$$= M\rho^{-n} \int_{B(x,\rho)} |\lambda^a(y) - \lambda^a(x)|dy + M \frac{|\lambda^s|(B(x,\rho))}{\rho^n},$$

that, for $\mathcal{L}^n$-a.e. $x \in \Omega$, goes to zero when $\rho \to 0$.

*(ii).* Let us fix $\lambda \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$ and consider $\lambda^s$ and $\mu = |\lambda^a(x)| \cdot \mathcal{L}^n$: since $\lambda^s \perp \mathcal{L}^n$ and $\mu << \mathcal{L}^n$ we can apply Lemma 2.4 to $\lambda^s$ and $\mu$. Thus, we can find $M \subseteq \Omega$ with $|\lambda^s|(M) = 0$

such that, for every $x \in \Omega \setminus M$, there exists a sequence $\{\rho_h\}_{h=1}^{\infty}$ satisfying the conditions (2.9) and (2.10) of Lemma 2.4. Then we simply end since, for every $x \in \Omega \setminus M$,

$$\left| \frac{\lambda_{\rho_h}(x)}{\int_{B(x,\rho_h)} k_{\rho_h}(x-y) d|\lambda^s|(y)} - \frac{d\lambda^s}{d|\lambda^s|}(x) \right|$$

$$\leq \frac{\int_{B(x,\rho_h)} k_{\rho_h}(x-y)|\lambda^a(y)|dy}{\int_{B(x,\rho_h)} k_{\rho_h}(x-y) d|\lambda^s|(y)} + \frac{\int_{B(x,\rho_h)} k_{\rho_h}(x-y) \left| \frac{d\lambda^s}{d|\lambda^s|}(y) - \frac{d\lambda^s}{d|\lambda^s|}(x) \right| d|\lambda^s|(y)}{\int_{B(x,\rho_h)} k_{\rho_h}(x-y) d|\lambda^s|(y)},$$

that goes to zero as $h \to \infty$ by the conditions (2.9) and (2.10). □

We end this section with two useful propositions.

**Proposition 2.5.** *Let $\lambda_1, \lambda_2 \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ such that $\lambda_1 \perp \lambda_2$. Then, for $|\lambda_1|$-a.e. $x \in \Omega$, $\frac{d\lambda_1}{d|\lambda_1|}(x) = \frac{d(\lambda_1+\lambda_2)}{d|\lambda_1+\lambda_2|}(x)$.*

*Proof.* Let $A \in \mathcal{B}(\Omega)$ such that $|\lambda_1|(\Omega \setminus A) = |\lambda_2|(A) = 0$: clearly we can prove the wanted equality only for $|\lambda_1|$-a.e. $x \in A$. Since $|\lambda_1 + \lambda_2| = |\lambda_1| + |\lambda_2|$ (see for instance Lemma 6.10), $\lambda_1 << |\lambda_1|$ and $\lambda_1, \lambda_2 << |\lambda_1 + \lambda_2|$, we have that, for every $B \in \mathcal{B}(A)$,

$$\lambda_1(B) = \int_B \frac{d\lambda_1}{d|\lambda_1|}(x) d|\lambda_1|(x),$$

and

$$\lambda_1(B) = \int_B \frac{d\lambda_1}{d|\lambda_1+\lambda_2|}(x) d|\lambda_1+\lambda_2|(x) = \int_B \frac{d\lambda_1}{d|\lambda_1+\lambda_2|}(x) d|\lambda_1|(x).$$

Thus, for $|\lambda_1|$-a.e. $x \in A$, $\frac{d\lambda_1}{d|\lambda_1|}(x) = \frac{d\lambda_1}{d|\lambda_1+\lambda_2|}(x)$[10]. We end noting that, with a similar argument, for $|\lambda_1|$-a.e. $x \in A$, $\frac{d\lambda_2}{d|\lambda_1+\lambda_2|}(x) = 0$. □

**Proposition 2.6.** *Let $\lambda_1, \lambda_2 \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ such that $\lambda_1 \perp \lambda_2$ and let $\varphi : \Omega \to [0, \infty]$ be a Borel function. Then*

$$\int_\Omega \varphi(x) d|\lambda_1 + \lambda_2|(x) = \int_\Omega \varphi(x) d|\lambda_1|(x) + \int_\Omega \varphi(x) d|\lambda_2|(x),$$

*and*

$$|\lambda_1+\lambda_2|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} \varphi(x) = \left[ |\lambda_1|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} \varphi(x) \right] \vee \left[ |\lambda_2|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} \varphi(x) \right].$$

*Proof.* Let $A \in \mathcal{B}(\Omega)$ such that $|\lambda_1|(\Omega \setminus A) = |\lambda_2|(A) = 0$. Since $|\lambda_1 + \lambda_2| = |\lambda_1| + |\lambda_2|$, we have

$$\int_\Omega \varphi(x) d|\lambda_1+\lambda_2|(x) = \int_A \varphi(x) d|\lambda_1+\lambda_2|(x) + \int_{\Omega\setminus A} \varphi(x) d|\lambda_1+\lambda_2|(x)$$

$$= \int_A \varphi(x) d|\lambda_1|(x) + \int_{\Omega\setminus A} \varphi(x) d|\lambda_2|(x) = \int_\Omega \varphi(x) d|\lambda_1|(x) + \int_\Omega \varphi(x) d|\lambda_2|(x),$$

and

$$|\lambda_1+\lambda_2|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} \varphi(x) = \left[ |\lambda_1+\lambda_2|\text{-}\operatorname*{ess\,sup}_{x\in A} \varphi(x) \right] \vee \left[ |\lambda_1+\lambda_2|\text{-}\operatorname*{ess\,sup}_{x\in\Omega\setminus A} \varphi(x) \right]$$

$$= \left[ |\lambda_1|\text{-}\operatorname*{ess\,sup}_{x\in A} \varphi(x) \right] \vee \left[ |\lambda_2|\text{-}\operatorname*{ess\,sup}_{x\in\Omega\setminus A} \varphi(x) \right] = \left[ |\lambda_1|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} \varphi(x) \right] \vee \left[ |\lambda_2|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} \varphi(x) \right].$$

□

[10] Note that if $\mu \in \mathcal{M}^+(\Omega)$, $\varphi : \Omega \to \mathbb{R}^m$ is a Borel function such that, for every $B \in \mathcal{B}(\Omega)$, $\int_B \varphi(x) d\mu = 0$ then, for $\mu$-a.e. $x \in \Omega$, $\varphi(x) = 0$.

## 2.2 Notes on $BV$, Sobolev and Lipschitz functions

Let us introduce now the notion of function of (locally) bounded variation (for more details see [7] Chapter 3 or [40] Chapter 5). A Borel function $u : \Omega \to \mathbb{R}$ is said to have *bounded variation* on $\Omega$, or briefly $u \in BV(\Omega)$, if $u \in L^1(\Omega)$ and there exists $Du \in \mathcal{M}(\Omega, \mathbb{R}^n)$ such that, for every $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \varphi(x) dDu(x) = -\int_\Omega u(x)\nabla\varphi(x)dx,$$

or, in other words, if $u$ has a *distributional gradient* belonging to $\mathcal{M}(\Omega, \mathbb{R}^n)$.

Since $Du \in \mathcal{M}(\Omega, \mathbb{R}^n)$, following (2.3), it can be decomposed with respect to $\mathcal{L}^n$ as

$$Du = \nabla u \cdot \mathcal{L}^n + D^s u,$$

where $\nabla u \in L^1(\Omega, \mathbb{R}^n)$, $D^s u \in \mathcal{M}(\Omega, \mathbb{R}^n)$ and $D^s u \perp \mathcal{L}^n$. When $D^s u = 0$ we say $u \in W^{1,1}(\Omega)$ and if $u \in L^\infty(\Omega)$ and $\nabla u \in L^\infty(\Omega, \mathbb{R}^n)$ too, we say $u \in W^{1,\infty}(\Omega)$.

When, for every open set $\Omega' \subset\subset \Omega$, $u \in BV(\Omega')$ (resp. $u \in W^{1,1}(\Omega')$, $u \in W^{1,\infty}(\Omega')$), we say $u \in BV_{\text{loc}}(\Omega)$ (resp. $u \in W^{1,1}_{\text{loc}}(\Omega)$, $u \in W^{1,\infty}_{\text{loc}}(\Omega)$): in this case $u \in L^1_{\text{loc}}(\Omega)$ (resp. $u \in L^1_{\text{loc}}(\Omega)$, $u \in L^\infty_{\text{loc}}(\Omega)$) and $Du \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^n)$ (resp. $Du = \nabla u \cdot \mathcal{L}^n$ with $\nabla u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^n)$, $Du = \nabla u \cdot \mathcal{L}^n$ with $\nabla u \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^n)$). Finally we will consider sometimes functions belonging to $W^{1,1}(\Omega, \mathbb{R}^m)$ or $W^{1,\infty}(\Omega, \mathbb{R}^m)$: the definitions of these spaces are clear and then omitted.

Given $m \in \mathbb{N}$ and $u_h, u \in L^1_{\text{loc}}(\Omega, \mathbb{R}^m)$ (resp. $L^\infty_{\text{loc}}(\Omega, \mathbb{R}^m)$), we say $u_h \to u$ in $L^1_{\text{loc}}(\Omega, \mathbb{R}^m)$ (resp. $L^\infty_{\text{loc}}(\Omega, \mathbb{R}^m)$) if, for every $K \in \mathcal{B}(\Omega)$, $K \subset\subset \Omega$ and $\mathcal{L}^n(K) > 0$,

$$\lim_{h\to\infty} \int_K |u_h(x) - u(x)|dx = 0, \quad \left(\lim_{h\to\infty} \operatorname*{ess\,sup}_{x\in K} |u_h(x) - u(x)| = 0\right),$$

while considering $u_h, u \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^m)$, we say $u_h \to u$ in $w^*$-$L^\infty_{\text{loc}}(\Omega, \mathbb{R}^m)$ if, for every $K \in \mathcal{B}(\Omega)$, $K \subset\subset \Omega$ and $v \in L^1(K)$,

$$\lim_{h\to\infty} \int_K v(x)u_h(x)dx = \int_K v(x)u(x)dx.$$

The definitions of convergence in $L^1(\Omega, \mathbb{R}^m)$, $L^\infty(\Omega, \mathbb{R}^m)$ and $w^*$-$L^\infty(\Omega, \mathbb{R}^m)$ are obvious.

It is well known the validity of the proposition below.

**Proposition 2.7.** *Let $\{u_h\}_{h=1}^\infty \subseteq L^\infty_{\text{loc}}(\Omega, \mathbb{R}^m)$ (resp. $L^\infty(\Omega, \mathbb{R}^m)$) be a sequence such that, for every $K \in \mathcal{B}(\Omega)$, $K \subset\subset \Omega$,*

$$\sup\left\{\operatorname*{ess\,sup}_{x\in K} |u_h(x)| : h \in \mathbb{N}\right\} < \infty \quad \left(\text{resp. } \sup\left\{\operatorname*{ess\,sup}_{x\in \Omega} |u_h(x)| : h \in \mathbb{N}\right\} < \infty\right).$$

*Then there exists a subsequence $\{u_{h_k}\}_{k=1}^\infty$ and $u \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^m)$ (resp. $u \in L^\infty(\Omega, \mathbb{R}^m)$) such that $u_{h_k} \to u$ in $w^*$-$L^\infty_{\text{loc}}(\Omega, \mathbb{R}^m)$ (resp. $w^*$-$L^\infty(\Omega, \mathbb{R}^m)$).*

If now $u_h, u \in BV_{\text{loc}}(\Omega)$ (resp. $W^{1,1}_{\text{loc}}(\Omega)$), we say $u_h \to u$ in $w^*$-$BV_{\text{loc}}(\Omega)$ (resp. $W^{1,1}_{\text{loc}}(\Omega)$) if $u_h \to u$ in $L^1_{\text{loc}}(\Omega)$ and $Du_h \to Du$ in $w^*$-$\mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^n)$ (resp. $\nabla u_h \to \nabla u$ in $L^1_{\text{loc}}(\Omega, \mathbb{R}^n)$). Also in this case it is clear what we mean for convergence in $w^*$-$BV(\Omega)$ and $W^{1,1}(\Omega)$.

The following compactness theorem for $BV_{\text{loc}}(\Omega)$ functions holds (see [7] Theorem 3.23). For the notion of set with *Lipschitz boundary* see [40] page 127.

**Theorem 2.8.** *Let $\{u_h\}_{h=1}^\infty \subseteq BV_{\text{loc}}(\Omega)$ be a sequence such that, for every $K \subset\subset \Omega$,*

$$\sup\left\{\int_K |u(x)|dx + |Du_h|(K) : h \in \mathbb{N}\right\} < \infty.$$

*Then there exists a subsequence $\{u_{h_k}\}_{k=1}^{\infty}$ and $u \in BV_{\rm loc}(\Omega)$ such that $u_{h_k} \to u$ in $w^*$-$BV_{\rm loc}(\Omega)$. Moreover if $\Omega$ is a bounded open set with Lipschitz boundary and*

$$\sup\left\{\int_{\Omega}|u(x)|dx + |Du_h|(\Omega) : h \in \mathbb{N}\right\} < \infty,$$

*then $\{u_h\}_{h=1}^{\infty} \subseteq BV(\Omega)$, $u \in BV(\Omega)$ and $u_{h_k} \to u$ in $w^*$-$BV(\Omega)$.*

When $\Omega = (a, b) \subseteq \mathbb{R}$ and $u \in BV(a, b)$ it is meaningful, following (2.4), to decompose $Du \in \mathcal{M}(a, b)$ with respect to $\mathcal{L}^1$ as $Du = u' \cdot \mathcal{L}^1 + D^c u + D^{\#} u$ where the notations are clear. If $D^c u = 0$ we say $u \in SBV(a, b)$. For this space the following simple compactness theorem holds (see [7] Theorem 4.8: here we present a very special case).

**Theorem 2.9.** *Let $(a, b) \subseteq \mathbb{R}$ be an open bounded interval and let $\{u_h\}_{h=1}^{\infty} \subseteq SBV(a, b) \cap L^{\infty}(a, b)$ be a sequence such that*[11]

$$\sup_{h\in\mathbb{N}}\left\{\operatorname*{ess\,sup}_{x\in(a,b)}|u_h(x)| + \operatorname*{ess\,sup}_{x\in(a,b)}|u_h'(x)| + \#\left(A_{Du}\right)\right\} < \infty.$$

*Then there exists a subsequence $\{u_{h_k}\}_{k=1}^{\infty}$ and $u \in SBV(a, b) \cap L^{\infty}(a, b)$ such that $u_{h_k} \to u$ in $w^*$-$BV(a, b)$ and $u_{h_k} \to u$ in $w^*$-$L^{\infty}(a, b)$.*

As for Radon measures, let $\{\Omega_h\}_{h=1}^{\infty}$ be a family of open sets and let $u_h \in L^1_{\rm loc}(\Omega_h, \mathbb{R}^m)$, $u \in L^1_{\rm loc}(\Omega, \mathbb{R}^m)$: we say $u_h \to u$ in $L^1_{\rm loc}(\updownarrow \Omega, \mathbb{R}^m)$ if, for every $K \subset\subset \Omega$, we have $K \subseteq \Omega_h$ if $h$ is large enough and

$$\lim_{h\to\infty}\int_K |u_h(x) - u(x)|dx = 0.$$

If instead $u_h \in BV_{\rm loc}(\Omega_h)$ (resp. $W^{1,1}_{\rm loc}(\Omega_h)$), $u \in BV_{\rm loc}(\Omega)$ (resp. $W^{1,1}_{\rm loc}(\Omega)$), we say $u_h \to u$ in $w^*$-$BV_{\rm loc}(\updownarrow \Omega)$ (resp. $W^{1,1}_{\rm loc}(\updownarrow \Omega)$) if $u_h \to u$ in $L^1_{\rm loc}(\updownarrow \Omega)$ and $Du_h \to Du$ in $w^*$-$\mathcal{M}_{\rm loc}(\updownarrow \Omega, \mathbb{R}^n)$ (resp. $\nabla u_h \to \nabla u$ in $L^1_{\rm loc}(\updownarrow \Omega, \mathbb{R}^n)$).

Also in this case we suppose known the fundamental properties of the convolutions of a function $u$ belonging to $L^1_{\rm loc}(\Omega)$, $BV_{\rm loc}(\Omega)$ or $W^{1,1}_{\rm loc}(\Omega)$, and in particular the fact that, using the same notations than for the measures,

$$u_\rho(x) = \int_{B(x,\rho)} \rho^{-n} k\left(\frac{x-y}{\rho}\right) u(y)dy : \Omega_\rho \to \mathbb{R},$$

belongs to $C^{\infty}(\Omega_\rho)$ and $u_\rho \to u$ in $L^1_{\rm loc}(\updownarrow \Omega)$.

Moreover, when $u \in BV_{\rm loc}(\Omega)$, $\nabla u_\rho = (Du)_\rho$ and $(Du)_\rho \cdot \mathcal{L}^n \to Du$ in $w^*$-$\mathcal{M}_{\rm loc}(\updownarrow \Omega, \mathbb{R}^n)$, that is, by definition, $u_\rho \to u$ in $w^*$-$BV_{\rm loc}(\updownarrow \Omega)$, while if $u \in W^{1,1}_{\rm loc}(\Omega)$, $\nabla u_\rho = (\nabla u)_\rho$ and $(\nabla u)_\rho \to \nabla u$ in $L^1_{\rm loc}(\updownarrow \Omega, \mathbb{R}^n)$, that is, by definition, $u_\rho \to u$ in $W^{1,1}_{\rm loc}(\updownarrow \Omega)$.

Let now $\Omega$ be a bounded open set in $\mathbb{R}^n$. We say that a function $v \in W^{1,\infty}(\Omega)$ is *piecewise affine* in $\Omega$ if there exists a family $\{\Omega_j\}_{j=1}^{N}$ of open disjoint subsets of $\Omega$ such that

$$\mathcal{L}^n\left(\Omega \setminus \bigcup_{j=1}^{N}\Omega_j\right) = 0,$$

and such that, for every $j \in \{1, ..., N\}$, $x \in \Omega_j$,

$$v(x) = \langle \xi_j, x\rangle + q_j,$$

where $\xi_j \in \mathbb{R}^n$ and $q_j \in \mathbb{R}$. We shall denote the space of such functions with $\mathrm{Aff}(\Omega)$.

The next theorem is a particular case of Theorem 1.8, Chapter 2 in [29].

[11] With $\#(A)$ we mean the number of element of $A$. Remember that $A_{Du}$ is the set of the atoms of $Du$.

**Theorem 2.10.** *Let $\Omega$ be a bounded open set with Lipschitz boundary and let $u \in W^{1,1}(\Omega)$. Then, for every $\varepsilon > 0$, there exists $v_\varepsilon \in \mathrm{Aff}(\Omega)$ such that*

$$\int_\Omega |\nabla u(x) - \nabla v_\varepsilon(x)| dx < \varepsilon.$$

It is known that $\mathrm{Lip}_{\mathrm{loc}}(\Omega)$ that is the space of the locally Lipschitz functions defined on $\Omega$, can be identified with $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ (see [40] Theorem 5, page 131), thus, in our arguments, we will always identify these two spaces. The following celebrated theorem due to Rademacher holds (see [40] Theorem 2, page 81).

**Theorem 2.11.** *Let $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and let $\Omega_d \subseteq \Omega$ be the set of the points in which $u$ is differentiable. Then $\mathcal{L}^n(\Omega \setminus \Omega_d) = 0$. In particular $\Omega_d$ is dense in $\Omega$.*

Let $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and let $\Omega_d$ be the set of the points in which $u$ is differentiable: for every $x \in \Omega$, we set[12],

$$\overline{\nabla} u(x) = \{\xi : \exists x_k \to x, x_k \in \Omega_d, \nabla u(x_k) \to \xi\}, \tag{2.11}$$

and

$$\partial_c u(x) = \mathrm{co}\left(\overline{\nabla} u(x)\right), \tag{2.12}$$

and we call $\partial_c u(x)$ the *Clarke's gradient* of $u$ in $x$.

Some results about Clarke's gradient are presented here: the proof of Proposition 2.12 can be found in Clarke [26] (see also Lebourg [58]) while Theorem 2.13 follows from Corollary 8.47, Proposition 7.15 and Theorem 9.61 in [67].

**Proposition 2.12.** *Let $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$. Then the following properties hold for $\partial_c u$:*

*(i) $\partial_c u$ does not change if, in its definition, we consider any $\Omega_1 \subseteq \Omega_d$ such that $\mathcal{L}^n(\Omega_d \setminus \Omega_1) = 0$;*

*(ii) for every $x \in \Omega$, $\partial_c u(x)$ is nonempty, convex and compact;*

*(iii) the multi-valued map $\partial_c u$ from $\Omega$ to the nonempty, convex and compact subsets of $\mathbb{R}^n$ is outer semicontinuous, that is, if $x_k \to x$, $\xi_k \in \partial_c u(x_k)$ and $\xi_k \to \xi$ then $\xi \in \partial_c u(x)$;*

*(iv) if $u \in C^1(\Omega)$ then, for every $x \in \Omega$, $\partial_c u(x) = \nabla u(x)$.*

**Theorem 2.13.** *Let $u, u_h \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ such that $u_h \to u$ with respect to the $L^\infty_{\mathrm{loc}}(\Omega)$ convergence and let $x_0 \in \Omega$ be such that $\nabla u(x_0)$ exists. Then there exists a subsequence $\{u_{h_k}\}_{k=1}^\infty$ and two sequences $\{x_k\}_{k=1}^\infty$, $\{\xi_k\}_{k=1}^\infty$ such that, for every $k \in \mathbb{N}$, $\xi_k \in \partial_c u_{h_k}(x_k)$ and it holds $x_k \to x_0$, $u_{h_k}(x_k) \to u(x_0)$ and $\xi_k \to \nabla u(x_0)$.*

## 2.3 Some tools from convex analysis

### 2.3.1 Principal definitions and properties

A set $C \subset \mathbb{R}^m$ is said *convex* if, for every $\xi, \eta \in C$, $t \in (0,1)$, we have $t\xi + (1-t)\eta \in C$ while $C$ is said a *cone* if $0 \in C$ and, for every $\xi \in C$, $t > 0$, it is $t\xi \in C$.

If we consider any family of convex sets (resp. cones) $\{C_i\}_{i \in I}$ then $\bigcap_{i \in I} C_i$ is convex (resp. a cone) too. Thus, given a set $A \subset \mathbb{R}^m$, it is well defined the *convex envelope* of $A$ as the set

$$\mathrm{co}(A) = \bigcap \Big\{ C \subseteq \mathbb{R}^m : A \subset C, C \,\text{is convex} \Big\},$$

[12] See the next section for the definition of convex set and for the notation $\mathrm{co}(A)$, that means the convex envelope of the set $A$.

so as the *positive conic envelope* of $A$ given by the set

$$\text{pos}(A) = \{0\} \cup \bigcap \Big\{ C \subseteq \mathbb{R}^m : A \subset C, C \text{ is a cone} \Big\}.$$

The next important theorem about the convex envelopes is due to Carathéodory (see [67] Theorem 2.27).

**Theorem 2.14.** *Let $A \subseteq \mathbb{R}^m$, $A \neq \emptyset$. Then*

$$\text{co}(A) = \left\{ \sum_{j=1}^{m+1} \lambda_j \xi_j : \xi_j \in A, \lambda_j \geq 0, \sum_{j=1}^{m+1} \lambda_j = 1 \right\}.$$

Given now $A \subseteq \mathbb{R}^m$ and $\xi_0 \in \mathbb{R}^m$ it is possible to define also the positive conic envelope of $A$ with respect to $\xi_0$ as

$$\text{pos}_{\xi_0}(A) = \xi_0 + \text{pos}(A - \xi_0),$$

and this set can be easily characterized as

$$\text{pos}_{\xi_0}(A) = \{\xi_0\} \cup \{\lambda(\xi - \xi_0) + \xi_0 : \lambda > 0, \xi \in A\}.$$

Let us note that if $A$ is convex, then $\text{pos}_{\xi_0}(A)$ is convex too and that in general $\text{pos}_{\xi_0}(A)$ may fail to be closed even if $A$ it is (see for example [67] Chapter 3, Section G).

Let us give now some fundamental definitions: for our purposes we can confine ourselves to consider only non negative functions defined on the whole space $\mathbb{R}^m$. We remember that $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$.

Let $\phi : \mathbb{R}^m \to [0, \infty]$:

- $\phi$ is *convex* if, for every $\xi, \eta \in \mathbb{R}^m$, $t \in (0, 1)$, $\phi(t\xi + (1-t)\eta) \leq t\phi(\xi) + (1-t)\phi(\eta)$. If, whenever $\xi \neq \eta$, the previous inequality is strict we say that $\phi$ is *strictly convex*;
- $\phi$ is *level convex* if, for every $\xi, \eta \in \mathbb{R}^m$, $t \in (0, 1)$, $\phi(t\xi + (1-t)\eta) \leq \phi(\xi) \vee \phi(\eta)$;
- $\phi$ is *sub-linear* if, for every $\xi, \eta \in \mathbb{R}^m$, $\phi(\xi + \eta) \leq \phi(\xi) + \phi(\eta)$[13];
- $\phi$ is *sub-maximal* if, for every $\xi, \eta \in \mathbb{R}^m$, $\phi(\xi + \eta) \leq \phi(\xi) \vee \phi(\eta)$;
- $\phi$ is *positively homogeneous* of degree $r \in [0, \infty)$ if, for every $\xi \in \mathbb{R}^m$, $t > 0$, it is $\phi(t\xi) = t^r \phi(\xi)$;
- $\phi$ is *demi-coercive* if there exist $a > 0$, $b \geq 0$ and $\eta \in \mathbb{R}^m$ such that, for every $\xi \in \mathbb{R}^m$, $a|\xi| \leq \phi(\xi) + \langle \eta, \xi \rangle + b$;
- $\phi$ is *non constant on straight lines*, briefly *n.c.s.l.*, if its restriction to any straight line is a non constant function.

We remember that a convex and finite function is locally Lipschitzian (see De Giorgi [35] Theorem 3, Section 2 or [40] Theorem 1, page 236), while this is trivially false for a level convex function. However, it can be proved that both convex and level convex functions are differentiable $\mathcal{L}^n$ almost everywhere (see Theorem 2.11 above for the convex case and Crouzeix [28] Theorem 1 for the level convex case).

Given a function $\phi : \mathbb{R}^m \to [0, \infty]$ we define the *domain* of $\phi$ the set

$$\text{dom}(\phi) = \{\xi \in \mathbb{R}^m : \phi(\xi) < \infty\},$$

[13] Note that some authors (see for instance [49] or [67]) call sub-linear a function that satisfies the inequality just introduced together with the positively homogeneity of degree 1 (see the definition below).

and when $\mathrm{dom}(\phi) \neq \emptyset$ we say that $\phi$ is *proper*: we will always consider such functions in order to avoid triviality. The *epigraph* of $\phi$ is the set

$$\mathrm{epi}(\phi) = \{(\xi, r) \in \mathbb{R}^m \times [0, \infty) : r \geq \phi(\xi)\} \subseteq \mathbb{R}^{m+1},$$

while the *sub-level set* of $\phi$ to level $r \in [0, \infty)$ is the set

$$E_\phi(r) = \{\xi \in \mathbb{R}^m : \phi(\xi) \leq r\} \subseteq \mathbb{R}^m.$$

The following three propositions will be used in several situations even if sometimes they will not be expressly quoted. The proofs of Propositions 2.15 and 2.16 are trivial and then omitted (for the first part of Proposition 2.15 see Proposition 2.4 in [67]).

**Proposition 2.15.** *A function $f : \mathbb{R}^m \to [0, \infty]$ is convex if and only if $\mathrm{epi}(f) \subseteq \mathbb{R}^{m+1}$ is convex. A function $g : \mathbb{R}^m \to [0, \infty]$ is level convex if and only if, for every $r \in [0, \infty)$, $E_g(r) \subseteq \mathbb{R}^m$ is convex.*

**Proposition 2.16.** *Let $f_h : \mathbb{R}^m \to [0, \infty]$ be a sequence functions and $f = \sup\{f_h : h \in \mathbb{N}\}$. Then*

*(i) $f$ is l.s.c. if, for every $h \in \mathbb{N}$, $f_h$ is l.s.c.;*

*(ii) $f$ is convex if, for every $h \in \mathbb{N}$, $f_h$ is convex;*

*(iii) $f$ is level convex if, for every $h \in \mathbb{N}$, $f_h$ is level convex;*

*(iv) $f$ is sub-linear if, for every $h \in \mathbb{N}$, $f_h$ is sub-linear;*

*(v) $f$ is sub-maximal if, for every $h \in \mathbb{N}$, $f_h$ is sub-maximal.*

**Proposition 2.17.** *Let $f : \mathbb{R}^m \to [0, +\infty]$ be a proper and positively homogeneous of degree 1 function: then $f$ is convex if and only if $f$ is sub-linear. Let $g : \mathbb{R}^m \to [0, +\infty]$ be a proper and positively homogeneous of degree 0 function: then $g$ is level convex if and only if $g$ is sub-maximal.*

*Proof.* In order to prove the *if* part, let $\xi, \eta \in \mathbb{R}^m$ and $t \in (0, 1)$. We have

$$f(t\xi + (1-t)\eta) \leq f(t\xi) + f((1-t)\eta) = tf(\xi) + (1-t)f(\eta),$$

and

$$g(t\xi + (1-t)\eta) \leq g(t\xi) \vee g((1-t)\eta) = g(\xi) \vee g(\eta).$$

In order to prove the *only if* part, let $\xi, \eta \in \mathbb{R}^m$. Then

$$f(\xi + \eta) = 2f\left(\frac{\xi + \eta}{2}\right) \leq f(\xi) + f(\eta) \quad \text{and} \quad g(\xi + \eta) = g\left(\frac{\xi + \eta}{2}\right) \leq g(\xi) \vee g(\eta),$$

and we end the proof. □

We state now the celebrated Jensen's inequality for convex functions and another Jensen's type inequality involving level convex functions whose simple proof can be found for instance in [15] Theorem 1.2.

**Theorem 2.18.** *Let $f : \mathbb{R}^m \to [0, +\infty]$ be proper, l.s.c and convex function, $\mu \in \mathcal{M}^+(\Omega)$ with $\mu(\Omega) = 1$ and $\varphi \in L^1_\mu(\Omega, \mathbb{R}^m)$. Then*

$$f\left(\int_\Omega \varphi(x) d\mu(x)\right) \leq \int_\Omega f(\varphi(x)) d\mu(x).$$

**Theorem 2.19.** *Let $g : \mathbb{R}^m \to [0, +\infty]$ be a proper, l.s.c. and level convex function, $\mu \in \mathcal{M}^+(\Omega)$ with $\mu(\Omega) = 1$ and $\varphi \in L^1_\mu(\Omega, \mathbb{R}^m)$. Then*

$$g\left(\int_\Omega \varphi(x)d\mu(x)\right) \leq \mu\text{-}\operatorname*{ess\,sup}_{x\in\Omega} g(\varphi(x)).$$

Finally let $f : \mathbb{R}^m \to [0, \infty)$ be a convex function and $\xi_0 \in \mathbb{R}^m$: the set

$$\partial f(\xi_0) = \Big\{ v \in \mathbb{R}^m : \langle v, \xi - \xi_0 \rangle + f(\xi_0) \leq f(\xi), \quad \forall \xi \in \mathbb{R}^m \Big\},$$

is called the *sub-differential* of $f$ at $\xi_0$ and it is always a non empty, closed and convex set (see [66] Section 23). The sub-differential of $f$ can be seen also as a multi-valued map from $\mathbb{R}^m$ to the class of non empty, closed and convex subset of $\mathbb{R}^m$ and, from this point of view, $\partial f$ is *outer semicontinuous*, that is, if $\xi_h \to \xi$ and $v_h \in \partial f(\xi_h)$ is such that $v_h \to v$ then $v \in \partial f(\xi)$ (see [66] Theorem 24.5).

### 2.3.2 Recession functions and related topics

Given a function $\phi : \mathbb{R}^m \to [0, \infty]$, the *recession function* of $\phi$ is the function defined, for every $\xi \in \mathbb{R}^m$, as[14]

$$\phi^\infty(\xi) = \inf \left\{ \liminf_{h\to\infty} \frac{\phi(t_h \xi_h)}{t_h} : \xi_h \to \xi, t_h \uparrow \infty \right\},$$

and similarly, we define also, for every $\xi \in \mathbb{R}^m$,

$$\phi^\natural(\xi) = \inf \left\{ \liminf_{h\to\infty} \phi(t_h \xi_h) : \xi_h \to \xi, t_h \uparrow \infty \right\}.$$

Strictly related to $\phi^\infty$ and $\phi^\natural$ respectively are the functions defined, for every $\xi \in \mathbb{R}^m$, as[15]

$$\phi^0(\xi) = \sup \left\{ \limsup_{h\to\infty} \frac{\phi(t_h \xi)}{t_h} : t_h \downarrow 0 \right\},$$

and

$$\phi^\flat(\xi) = \sup \left\{ \limsup_{h\to\infty} \phi(t_h \xi) : t_h \downarrow 0 \right\}.$$

We underline that the definition of $\phi^\natural$ has been already proposed in the work of Gori [51], while the one of $\phi^\flat$ appears here for the first time.

Note also that, for every $\xi \in \mathbb{R}^m$, we can find suitable sequences $\xi_h \to \xi$ and $t_h \uparrow \infty$ (resp. $t_h \downarrow 0$) such that $t_h^{-1}\phi(t_h \xi_h) \to \phi^\infty(\xi)$ and $\phi(t_h \xi_h) \to \phi^\natural(\xi)$ (resp. $t_h^{-1}\phi(t_h \xi) \to \phi^0(\xi)$ and $\phi(t_h \xi) \to \phi^\flat(\xi)$).

We list now several propositions, involving the functions just introduced, which will be very useful to prove, in particular, the theorems of Chapters 6 and 7. We remark that most of them are new.

**Proposition 2.20.** *Let $f : \mathbb{R}^m \to [0, \infty]$ be a proper function. Then $f^\infty$ is l.s.c., positively homogeneous of degree 1 and $f^\infty(0) = 0$. Moreover, if $f$ is convex then $f^\infty$ is convex too.*

*Let $g : \mathbb{R}^n \to [0, \infty]$ be a proper function. Then $g^\natural$ is l.s.c., positively homogeneous of degree 0 but not necessarily proper. Moreover, if $g$ is level convex then $g^\natural$ is level convex too.*

[14] Note that the definition of recession function here presented agrees with the standard one given, for instance, in [67] (see [67] Definition 3.17 and Theorem 3.21).

[15] The definition of $\phi^0$ follows [18] equation (2.7).

*Proof.* The proof of the part involving $f^\infty$ can be found in [67] Theorem 3.21.

Thus let us study $g^\natural$. Let us fix $\xi_h, \xi_0 \in \mathbb{R}^m$ such that $\xi_h \to \xi_0$. Then, for every $h \in \mathbb{N}$, there exist two sequences $\{t_h^j\}_{j=1}^\infty \subseteq \mathbb{R}$, and $\{\xi_h^j\}_{j=1}^\infty \subseteq \mathbb{R}^m$ such that $t_h^j \uparrow \infty$, $\xi_h^j \to \xi_h$ and $g(t_h^j \xi_h^j) \to g^\natural(\xi_h)$ as $j \to \infty$. Thus, for every $h \in \mathbb{N}$, there exists $j_h$ such that, $t_h^{j_h} \geq h$, $\left|\xi_h^{j_h} - \xi_h\right| \leq \frac{1}{h}$ and $g\left(t_h^{j_h}\xi_h^{j_h}\right) \leq g^\natural(\xi_h) + \frac{1}{h}$ and, unless to extract a subsequence, we can suppose that $t_h^{j_h} \uparrow \infty$ and $\xi_h^{j_h} \to \xi_0$ as $h \to \infty$. Then

$$g^\natural(\xi_0) \leq \liminf_{h\to\infty} g\left(t_h^{j_h}\xi_h^{j_h}\right) \leq \liminf_{h\to\infty}\left(g^\natural(\xi_h) + \frac{1}{h}\right) = \liminf_{h\to\infty} g^\natural(\xi_h),$$

and the lower semicontinuity is proved.

The proof of the positive homogeneity of degree 0 is very simple and it can be omitted.

Let us prove now the level convexity, that is, that for every $r \in [0,\infty)$ the set $\{\xi : g^\natural(\xi) \leq r\}$ is convex. If $r < \inf\{g(\xi) : \xi \in \mathbb{R}^m\}$ there is nothing to prove. Thus, fixed $r \geq \inf\{g(\xi) : \xi \in \mathbb{R}^m\}$ and $E_g(r) = \{\xi : g(\xi) \leq r\}$ we have

$$\{\xi : g^\natural(\xi) \leq r\} = \left\{\xi : \exists t_h \uparrow \infty, \xi_h \to \xi, \text{ such that } \lim_{h\to\infty} g(t_h\xi_h) \leq r\right\}$$

$$= \bigcap_{\varepsilon>0}\left\{\xi : \exists t_h \uparrow \infty, \xi_h \to \xi \text{ such that } \forall h \in \mathbb{N}, g(t_h\xi_h) \leq r + \varepsilon\right\}$$

$$= \bigcap_{\varepsilon>0}\left\{\xi : \exists \tau_h \downarrow 0, \zeta_h \in E_g(r+\varepsilon) \text{ such that } \tau_h\zeta_h \to \xi\right\} = \bigcap_{\varepsilon>0} E_g^\infty(r+\varepsilon).$$

For every $\varepsilon > 0$, $E_g(r+\varepsilon) \neq \emptyset$ and by definition we have that $E_g^\infty(r+\varepsilon)$ is the so called *horizon cone* of $E_g(r+\varepsilon)$ (see [67] Definition 3.3): since the horizon cone of a convex set is convex (see [67] Theorem 3.6) we end the proof. □

**Proposition 2.21.** *Let $f : \mathbb{R}^m \to [0,\infty]$ be a proper, l.s.c. and positively homogeneous of degree 1 function. Then $f = f^\infty$. Let $g : \mathbb{R}^m \to [0,\infty]$ be a proper, l.s.c. and positively homogeneous of degree 0 function. Then $g = g^\natural$.*

*Proof.* We make the proof only for $g$ and $g^\natural$ since the other case is completely analogous. Let $\xi \in \mathbb{R}^m$: considering $\xi_h = \xi$ and $t_h \uparrow \infty$ we obtain $g^\natural(\xi) \leq g(\xi)$. In order to prove the converse inequality we use the lower semicontinuity of $g$: indeed, if we consider $\xi_h \to \xi$ and $t_h \uparrow \infty$ such that $g(t_h\xi_h) \to g^\natural(\xi)$, then

$$g(\xi) \leq \liminf_{h\to\infty} g(\xi_h) = \liminf_{h\to\infty} g(t_h\xi_h) = g^\natural(\xi),$$

and we achieve the proof. □

**Proposition 2.22.** *Let $f : \mathbb{R}^m \to [0,\infty]$ be a proper, l.s.c. and sub-linear function. Then $f^0$ is l.s.c., positively homogeneous of degree 1, convex and, for every $\xi \in \mathbb{R}^m$,*

$$f^0(\xi) = \sup_{t>0}\frac{f(t\xi)}{t}.$$

*Let $g : \mathbb{R}^m \to [0,\infty]$ be a proper, l.s.c. and sub-maximal function. Then $g^\flat$ is l.s.c., positively homogeneous of degree 0, level convex and, for every $\xi \in \mathbb{R}^m$,*

$$g^\flat(\xi) = \sup_{t>0} g(t\xi).$$

*Proof.* Following Proposition 2.2 in [18], which proves the part of the proposition involving $f$, we can easily prove the part about $g$.

Indeed, $g^\flat$ is clearly positively homogeneous of degree 0. Fixed now $\xi \in \mathbb{R}^m$, for every $r > 0$, we set

$$\alpha(r) = \sup\{g(t\xi) : 0 < t \leq r\}.$$

We have that $\alpha$ is increasing and $g^\flat(\xi) = \lim_{r\to 0} \alpha(r)$. However, by the sub-maximality of $g$ we have, for every $r > 0$,

$$\alpha(2r) = \sup_{0<t\leq 2r} g(t\xi) = \sup_{0<t\leq r} g(2t\xi) \leq \sup_{0<t\leq r} g(t\xi) = \alpha(r),$$

that implies that $\alpha$ is constant on $(0, \infty)$. Therefore

$$g^\flat(\xi) = \sup_{t>0} g(t\xi),$$

thus, in particular, being $g^\flat$ the supremum of a family of l.s.c. and sub-maximal functions, it is also l.s.c. and sub-maximal. Finally, using Proposition 2.17, the level convexity of $g^\flat$ follows too. □

**Proposition 2.23.** *Let $g : \mathbb{R}^m \to [0, \infty]$ be a proper, l.s.c. and level convex function such that, for every $\xi \in \mathbb{R}^m \setminus \{0\}$, $g^\natural(\xi) = \infty$. Then there exists a function $\theta_\infty : [0, \infty) \to [0, \infty)$ such that, for every $\xi \in \mathbb{R}^m$,*

$$g(\xi) \geq \theta_\infty(|\xi|) \quad \text{and} \quad \lim_{t\to\infty} \theta_\infty(t) = \infty. \tag{2.13}$$

*Proof.* It is sufficient to prove that, for every $h \in \mathbb{N}$, there exists $r_h > 0$ such that, for every $\xi \in \mathbb{R}^m \setminus \{0\}$, $|\xi| > r_h$, we have $g(\xi) > h$. Let us suppose by contradiction that there exists $h_0 \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$, we can find $\xi_k \in \mathbb{R}^m$, $|\xi_k| > k$, with $f(\xi_k) \leq h_0$. Then $|\xi_k| \to \infty$ and, unless to extract a (not relabelled) subsequence, $\frac{\xi_k}{|\xi_k|} \to \nu \in S^{m-1}$. Moreover, for every $M \in \mathbb{N}$, there exists $k_M$ such that, for every $k \geq k_M$, we have $|\xi_k| > M$. Then, considered $\eta_0 \in \mathbb{R}^m$ such that $g(\eta_0) < \infty$,

$$g\left(\eta_0 - \frac{M}{|\xi_k|}\eta_0 + M\frac{\xi_k}{|\xi_k|}\right) = g\left(\left(1 - \frac{M}{|\xi_k|}\right)\eta_0 + \frac{M}{|\xi_k|}\xi_k\right) \leq g(\eta_0) \vee g(\xi_k) \leq g(\eta_0) \vee h_0 < \infty,$$

and, by the lower semicontinuity of $g$, we find, for every $M \in \mathbb{N}$, the relation

$$g(\eta_0 + M\nu) \leq \liminf_{k\to\infty} g\left(\eta_0 - \frac{M}{|\xi_k|}\eta_0 + M\frac{\xi_k}{|\xi_k|}\right) \leq g(\eta_0) \vee h_0 < \infty.$$

However

$$\infty > g(\eta_0) \vee h_0 \geq \liminf_{M\to\infty} g\left(M\left(\frac{\eta_0}{M} + \nu\right)\right) \geq g^\natural(\nu) = \infty,$$

and the contradiction is found. □

**Proposition 2.24.** *Let $\gamma : \mathbb{R}^m \to [0, \infty]$ be a proper, l.s.c. and sub-maximal function*[16] *such that, for every $\xi \in \mathbb{R}^m \setminus \{0\}$, $\gamma^\flat(\xi) = \infty$. Then there exists a function $\theta_0 : (0, \infty) \to [0, \infty)$ such that, for every $\xi \in \mathbb{R}^m \setminus \{0\}$,*

$$\gamma(\xi) \geq \theta_0(|\xi|) \quad \text{and} \quad \lim_{t\to 0} \theta_0(t) = \infty. \tag{2.14}$$

*Proof.* It suffices to prove that, for every $h \in \mathbb{N}$, there exists $\varepsilon_h > 0$ such that, for every $\xi \in \mathbb{R}^m \setminus \{0\}$, $|\xi| < \varepsilon_h$, we have $\gamma(\xi) > h$. Let us suppose by contradiction that there exists $h_0 \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$, we can find $\xi_k \in \mathbb{R}^m \setminus \{0\}$, $|\xi_k| \leq \frac{1}{k}$, with $\gamma(\xi_k) \leq h_0$. Then $|\xi_k| \to 0$ and unless

[16] In the following we will call $\gamma$ every function that is sub-maximal but not level convex.

to extract a (not relabelled) subsequence, $\frac{\xi_k}{|\xi_k|} \to \nu \in S^{m-1}$. Let $\{t_h\}_{h=1}^{\infty} \subseteq (0,\infty)$ such that $t_h \downarrow 0$ and

$$\lim_{h\to\infty} \gamma(t_h \nu) = \infty.$$

For every $h, k \in \mathbb{N}$, there exists $j_{h,k} \in \mathbb{N}$ such that $\frac{t_h}{|\xi_k|} \leq j_{h,k} < \frac{t_h}{|\xi_k|} + 1$ and since $t_h \frac{\xi_k}{|\xi_k|} \to t_h \nu$ as $k \to \infty$, then also $j_{h,k}\xi_k \to t_h\nu$ as $k \to \infty$. By the lower semicontinuity and the sub-maximality of $\gamma$, we have

$$\gamma(t_h\nu) \leq \liminf_{k\to\infty} \gamma(j_{h,k}\xi_k) = \liminf_{k\to\infty} \gamma\left(\sum_{i=1}^{j_{h,k}} \xi_k\right) \leq \liminf_{k\to\infty} \gamma(\xi_k).$$

Then, for every $h \in \mathbb{N}$,

$$\gamma(t_h\nu) \leq \liminf_{k\to\infty} \gamma(\xi_k) \leq h_0 < \infty.$$

Taking the limit as $h \to \infty$, we find a contradiction and the proof is achieved. □

**Proposition 2.25.** *Let $f : \mathbb{R}^m \to [0,\infty]$ be a proper, l.s.c. and convex function. Let us define the function $\hat{f} : \mathbb{R}^m \times \mathbb{R} \to [0,\infty]$ in this way:*

$$\hat{f}(\xi,\tau) = \begin{cases} \tau f(\frac{\xi}{\tau}) & \text{if } \tau > 0, \\ f^\infty(\xi) & \text{if } \tau = 0, \\ \infty & \text{if } \tau < 0. \end{cases}$$

*Then $\hat{f}$ is proper, l.s.c., positively homogeneous of degree 1 and convex (in particular sub-linear) on $\mathbb{R}^{m+1}$.*

*Proof.* See [30] Theorem 3.1. □

**Proposition 2.26.** *Let $g : \mathbb{R}^m \to [0,\infty]$ be a proper, l.s.c. and level convex function. Let us define the function $\hat{g} : \mathbb{R}^m \times \mathbb{R} \to [0,\infty]$ in this way:*

$$\hat{g}(\xi,\tau) = \begin{cases} g(\frac{\xi}{\tau}) & \text{if } \tau > 0, \\ g^\natural(\xi) & \text{if } \tau = 0, \\ \infty & \text{if } \tau < 0. \end{cases} \tag{2.15}$$

*Then $\hat{g}$ is proper, l.s.c., positively homogeneous of degree 0 and level convex (in particular sub-maximal) on $\mathbb{R}^{m+1}$.*

*Proof.* The functional $\hat{g}$ is clearly proper and positive homogeneous of degree 0. In order to prove the lower semicontinuity we work in the following way. Let us fix $(\xi_h, \tau_h) \to (\xi_0, \tau_0)$: since $\hat{g}$ is lower semicontinuous both on $\mathbb{R}^m \times (0,\infty)$ (because of the lower semicontinuity of $g$) and on $\mathbb{R}^m \times (-\infty, 0)$, the lower semicontinuity inequality has to be proved only in the case in which $\tau_0 = 0$. If this is the case, we can find $I_1, I_2, I_3$ disjoint subsets of $\mathbb{N}$ such that $I_1 \cup I_2 \cup I_3 = \mathbb{N}$ and such that, if $h \in I_1$ then $\tau_h > 0$, if $h \in I_2$ then $\tau_h = 0$ and if $h \in I_3$ then $\tau_h < 0$. Since

$$\liminf_{h\to\infty} \hat{g}(\xi_h,\tau_h) = \inf\left\{\liminf_{h\to\infty, h\in I_i} \hat{g}(\xi_h,\tau_h) : i \in \{1,2,3\}, \#(I_1) = \infty\right\},$$

it suffices to prove the lower semicontinuity inequality of $\hat{g}$ only in the three cases in which $I_i = \mathbb{N}$, $i \in \{1,2,3\}$. However if, for every $h \in \mathbb{N}$, $\tau_h > 0$, by the definition of $g^\natural$,

$$\hat{g}(\xi_0, 0) = g^\natural(\xi_0) \leq \liminf_{h\to\infty} g\left(\frac{\xi_h}{\tau_h}\right) = \liminf_{h\to\infty} \hat{g}\left(\xi_h, \tau_h\right),$$

if, for every $h \in \mathbb{N}$, $\tau_h = 0$, by the lower semicontinuity of $g^\natural$ (see Proposition 2.20),

$$\hat{g}(\xi_0, 0) = g^\natural(\xi_0) \leq \liminf_{h\to\infty} g^\natural(\xi_h) = \liminf_{h\to\infty} \hat{g}\left(\xi_h, \tau_h\right),$$

and finally if, for every $h \in \mathbb{N}$, $\tau_h < 0$ the inequality is trivially satisfied: thus the lower semicontinuity of $\hat{g}$ is achieved.

Let us show now that $\hat{g}$ is level convex, that is, for every $r \in [0, \infty)$, the set

$$\Big\{(\xi,\tau) : \hat{g}(\xi,\tau) \le r\Big\} = \Big\{(\xi,\tau) : \tau > 0, g(\xi\tau^{-1}) \le r\Big\} \cup \Big\{(\xi,0) : g^{\natural}(\xi) \le r\Big\} = A \cup B$$

is convex. We know that $A \cup B$ is closed and, since $g^{\natural}$ is l.s.c. and level convex, that $B$ is a convex and closed set. We note also that $A$ is convex since $A = \emptyset$ or, if $A \neq \emptyset$, we have that the convex set $E_g(r) = \{\xi : g(\xi) \le r\} \neq \emptyset$ and $A = \{(t\xi, t) : \xi \in E_g(r), t > 0\}$ that is convex[17].

If $A = \emptyset$ then $A \cup B = B$ that is convex. If $A \neq \emptyset$ we are going to prove that $A \cup B = \mathrm{cl}(A)$ that is convex since $A$ is convex. Clearly $\mathrm{cl}(A) \subseteq A \cup B$: to prove the converse we only need to prove that $B \subseteq \mathrm{cl}(A)$.

In order to prove this let us fix $(\xi_0, 0) \in B$ and show that there exist two sequences $\zeta_h \to \xi_0$ and $\tau_h \downarrow 0$ such that, for every $h \in \mathbb{N}$, $(\zeta_h, \tau_h) \in A$, that is, $g(\zeta_h\tau_h^{-1}) \le r$. Since $A \neq \emptyset$ there exists $\xi_1 \in \mathbb{R}^m$ such that $g(\xi_1) \le r$: we claim that $\{\xi_1 + t\xi_0 : t \ge 0\} \subseteq E_g(r)$.

If $\xi_0 = 0$ there is nothing to prove; instead, supposing $\xi_0 \neq 0$, let fix $t \ge 0$ and consider $\xi_h \to \xi_0$ ($\xi_h \neq 0$), $t_h \uparrow \infty$ ($|t_h\xi_h| > t|\xi_0|$) such that $g(t_h\xi_h) \to g^{\natural}(\xi_0)$: then, for every $h \in \mathbb{N}$, by the level convexity of $g$,

$$g\left(\xi_1 + t|\xi_0|\frac{t_h\xi_h}{|t_h\xi_h|}\right) \le g(t_h\xi_h) \vee g(\xi_1),$$

since the point $\xi_1 + t|\xi_0|\frac{t_h\xi_h}{|t_h\xi_h|}$ belongs to the segment joining $\xi_1$ and $t_h\xi_h$. However

$$\lim_{h\to\infty}\left(\xi_1 + t|\xi_0|\frac{t_h\xi_h}{|t_h\xi_h|}\right) = \xi_1 + t\xi_0,$$

and then, by the lower semicontinuity of $g$,

$$g(\xi_1 + t\xi_0) \le \liminf_{h\to\infty}\left(g(t_h\xi_h) \vee g(\xi_1)\right) \le r,$$

that proves the claim.

At last setting, for every $h \in \mathbb{N}$, $\zeta_h = \frac{\xi_1 + h\xi_0}{h}$ and $\tau_h = \frac{1}{h}$, we have $\hat{g}(\zeta_h, \tau_h) = g(\zeta_h\tau_h^{-1}) \le r$, that is $(\zeta_h, \tau_h) \in A$, and moreover

$$\lim_{h\to\infty}(\zeta_h, \tau_h) = (\xi_0, 0) \in \mathrm{cl}(A),$$

that ends the proof. □

**Proposition 2.27.** *Let $\gamma : \mathbb{R} \to [0, \infty]$ be a continuous and sub-maximal function and let $m = \sup\{\gamma(\xi) : \xi \in \mathbb{R}\}$. Then, for every $\xi \in [0, \infty)$, $\gamma(\xi) = m$ or, for every $\xi \in (-\infty, 0]$, $\gamma(\xi) = m$.*

*Proof.* First of all we prove that $\gamma(0) = m$. Indeed, if this is not true then $\gamma(0)$ is finite, there exists $\varepsilon > 0$ such that $\gamma(0) + 2\varepsilon \le m$ and $\xi_0 \in \mathbb{R}$ such that $\gamma(\xi_0) \ge \gamma(0) + \varepsilon$. Then, for every $h \in \mathbb{N}$, by the sub-maximality of $\gamma$, also $\gamma\left(\frac{\xi_0}{h}\right) \ge \gamma(0) + \varepsilon$. But $\frac{\xi_0}{h} \to 0$ as $h \to \infty$ thus, by continuity of $\gamma$, $\gamma(0) = \lim_{h\to\infty}\gamma\left(\frac{\xi_0}{h}\right) \ge \gamma(0) + \varepsilon$ that is a contradiction.

Let us suppose now, again by contradiction, that there exist $\xi_1, \xi_2 > 0$ such that $\gamma(\xi_1) \vee \gamma(-\xi_2) < m$: by continuity of $\gamma$, we can suppose also that $\xi_1, \xi_2 \in \mathbb{Q}$. Writing $\frac{\xi_1}{\xi_2} = \frac{k}{h}$, where $h, k \in \mathbb{N}$, we have $h\xi_1 - k\xi_2 = 0$ and

$$m = \gamma(0) = \gamma(h\xi_1 - k\xi_2) \le \gamma(\xi_1) \vee \gamma(-\xi_2) < m :$$

having found a contradiction, we achieve the proof. □

[17]Indeed, since $A = \mathrm{pos}\,(E_g(r) \times \{1\}) \setminus \{(0,0)\} \subseteq \mathbb{R}^m \times \mathbb{R}$ and $(\xi, \tau) \in A$ implies $\tau > 0$, the convexity simply follows.

Let us point out that if $\gamma$ is only l.s.c. and sub-maximal on $\mathbb{R}$, then the thesis of Proposition 2.27 is false: to verify this fact one can consider for instance the function $\gamma$ defined, for every $\xi \in \mathbb{R} \setminus \mathbb{Z}$, as $\gamma(\xi) = M > 0$ and, for every $\xi \in \mathbb{Z}$, as $\gamma(\xi) = 0$.

The proofs of the following two propositions are simple and for this reason they are omitted (Proposition 2.29 can be find also in [28]).

**Proposition 2.28.** *Let $f : \mathbb{R}^m \to [0, \infty]$ be a proper, l.s.c. and demi-coercive function, that is, there exist $a > 0$, $b \geq 0$ and $\eta \in \mathbb{R}^m$ such that, for every $\xi \in \mathbb{R}^m$, $f(\xi) \geq a|\xi| - \langle \eta, \xi \rangle - b$. Then, for every $\xi \in \mathbb{R}^m$, the following properties hold:*

*(i)* $f^\infty(\xi) \leq f^\natural(\xi)$*;*

*(ii) if* $f^\natural(\xi) < \infty$ *then* $f^\infty(\xi) = 0$*;*

*(iii)* $f^\infty(\xi) \geq a|\xi| - \langle \eta, \xi \rangle$*.*

**Proposition 2.29.** *Let $g : \mathbb{R} \to [0, \infty]$ be a proper and l.s.c. function. Then $g$ is level convex if and only if $g$ belongs to one of the three following classes:*

*(i) $g$ is not decreasing on $\mathbb{R}$;*

*(ii) $g$ is not increasing on $\mathbb{R}$;*

*(iii) there exists $x_0 \in \mathbb{R}$ such that $g$ is not increasing on $(-\infty, x_0]$ and not decreasing on $[x_0, \infty)$.*

*In particular if $g : \mathbb{R}^m \to [0, \infty]$ is a proper, l.s.c. and level convex function, then $g(0) = \inf\{g(\xi) : \xi \in \mathbb{R}^m\}$ if and only if, for every $\nu \in S^{m-1}$, the function $t \mapsto g(\nu t)$ is not decreasing on $(0, \infty)$.*

The two propositions below describe some properties of the composition of a level convex function with a strictly increasing one.

**Proposition 2.30.** *Let $g : \mathbb{R}^m \to [0, \infty]$ be a proper, l.s.c and level convex function, $s = \sup\{g(\xi) : \xi \in \mathbb{R}^m\} \in [0, \infty]$ and $\Theta : [0, s] \to [0, \bar{s}]$ be a continuous, strictly increasing function with $\bar{s} = \sup\{\Theta(t) : t \in [0, s]\} \in [0, \infty]$. Then the composition $\Theta \circ g : \mathbb{R}^m \to [0, \bar{s}]$ is proper, l.s.c. and level convex. Moreover $(\Theta \circ g)^\natural = \Theta \circ g^\natural$.*

*Proof.* Clearly $\Theta \circ g$ is proper and l.s.c.. Let $\xi, \eta \in \mathbb{R}^m$ and $t \in (0, 1)$, then

$$(\Theta \circ g)(t\xi + (1 - t)\eta) = \Theta(g(t\xi + (1 - t)\eta)) \leq \Theta(g(\xi) \vee g(\eta)) = \Theta(g(\xi)) \vee \Theta(g(\eta)),$$

that is $\Theta \circ g$ is level convex too. In order to prove that $(\Theta \circ g)^\natural = \Theta \circ g^\natural$, we point out that $\Theta$ is bijective and $\Theta^{-1} : [0, \bar{s}] \to [0, s]$ is still continuous and strictly increasing. Let us fix $\xi_0 \in \mathbb{R}^m$ and let $t_h \uparrow \infty$, $\xi_h \to \xi_0$ such that $(\Theta \circ g)(t_h \xi_h) \to (\Theta \circ g)^\natural(\xi_0)$. Then, using the continuity of $\Theta^{-1}$,

$$(\Theta \circ g)^\natural(\xi_0) = \lim_{h \to \infty} \Theta(g(t_h \xi_h)) = \Theta \circ \Theta^{-1} \left( \lim_{h \to \infty} \Theta(g(t_h \xi_h)) \right) = \Theta \left( \lim_{h \to \infty} g(t_h \xi_h) \right) \geq \Theta \circ g^\natural(\xi_0).$$

Conversely let $\bar{t}_h \uparrow \infty$, $\bar{\xi}_h \to \xi_0$ such that $g\left(\bar{t}_h \bar{\xi}_h\right) \to g^\natural(\xi_0)$. Then, using the continuity of $\Theta$,

$$(\Theta \circ g^\natural)(\xi_0) = \Theta\left(g^\natural(\xi_0)\right) = \Theta\left(\lim_{h \to \infty} g\left(\bar{t}_h \bar{\xi}_h\right)\right) = \lim_{h \to \infty} \Theta\left(g\left(\bar{t}_h \bar{\xi}_h\right)\right) \geq (\Theta \circ g)^\natural(\xi_0)$$

and the equality is finally achieved. □

**Proposition 2.31.** *Let $g : \mathbb{R}^m \to [0, \infty]$ be a proper, l.s.c and level convex function, $s = \sup\{g(\xi) : \xi \in \mathbb{R}^m\} \in [0, \infty]$ and $K_g = \{\xi \in \mathbb{R}^m : g^\natural(\xi) < s\}$. Let us suppose that $\mathrm{cl}(K_g)$ doesn't contains any straight line. Then there exists a continuous and strictly increasing function $\Theta : [0, s] \to [0, \infty]$ such that the composition $\Theta \circ g : \mathbb{R}^m \to [0, \infty]$ is demi-coercive.*

*Proof.* First of all let us note that, by the properties of $g^\natural$, $\mathrm{cl}(K_g)$ is a closed cone of $\mathbb{R}^m$. Moreover, it is quite simple to prove that there exists a closed and convex cone $C$ with non empty interior such that it does not contain any straight line and $\mathrm{cl}(K_f) \subseteq C$.

We claim that there exists $\eta_0 \in \mathbb{R}^m$, such that

$$\mathrm{cl}(K_f) \setminus \{0\} \subseteq C \setminus \{0\} \subseteq \{\xi \in \mathbb{R}^m : \langle \eta_0, \xi \rangle > 0\}. \tag{2.16}$$

Following Proposition 1 in [53], let us define $C \cap B^m(0,1) = C^m$ and

$$\eta_0 = \int_{C^m} \xi d\xi.$$

Moreover, for every $\zeta \in S^{m-1}$, let us define the transformation $r_\zeta$ as the reflection of $\mathbb{R}^m$ with respect to the hyperplane[18] $\zeta^\perp$, that is in formulas, for every $\xi \in \mathbb{R}^m$,

$$r_\zeta(\xi) = \xi - 2\langle \xi, \zeta \rangle \zeta.$$

Now let us suppose by contradiction that there exists $\zeta \in S^{m-1} \cap C^m \cap \eta_0^\perp$. This implies that[19]

$$0 = \langle \eta_0, \zeta \rangle = \int_{C^m} \langle \xi, \zeta \rangle d\xi = \int_{\mathbb{R}^m} 1_{C^m}(\xi) \langle \xi, \zeta \rangle d\xi.$$

Setting $H_\zeta^+ = \{\xi : \langle \xi, \zeta \rangle > 0\}$ and $H_\zeta^- = \{\xi : \langle \xi, \zeta \rangle < 0\}$ it follows,

$$0 = \int_{H_\zeta^+} 1_{C^m}(\xi) \langle \xi, \zeta \rangle d\xi + \int_{H_\zeta^-} 1_{C^m}(\xi) \langle \xi, \zeta \rangle d\xi.$$

We apply the change of variables $\xi = r_\zeta(\eta)$ to the second integral (note that $r_\zeta(H_\zeta^+) = H_\zeta^-$, that $r_\zeta$ preserves $m$-dimensional volumes, so that its Jacobian is equal to 1, and that $r_\zeta$ is norm preserving). We have,

$$\int_{H_\zeta^-} 1_{C^m}(\xi) \langle \xi, \zeta \rangle d\xi = \int_{H_\zeta^+} 1_{C^m}(r_\zeta(\eta)) \langle r_\zeta(\eta), \zeta \rangle d\eta.$$

Since, for every $\eta \in \mathbb{R}^m$, $\langle r_\zeta(\eta), \zeta \rangle = -\langle \eta, \zeta \rangle$ we conclude

$$0 = \int_{H_\zeta^+} \Big\{ 1_{C^m}(\xi) - 1_{C^m}(r_\zeta(\xi)) \Big\} \langle \xi, \zeta \rangle d\xi. \tag{2.17}$$

Let us remark that, for every $\xi \in H_\zeta^+$,

$$1_{C^m}(\xi) - 1_{C^m}(r_\zeta(\xi)) \geq 0. \tag{2.18}$$

Indeed, this is trivial if $\xi \in C^m$ or $|\xi| > 1$ (since $r_\zeta$ is norm preserving). If $\xi \in (H_\zeta^+ \cap B^m(0,1)) \setminus C^m$ then $r_\zeta(\xi) \in \mathbb{R}^m \setminus C^m$, since, if otherwise $r_\zeta(\xi) \in C^m$, from $\zeta \in C^m$ and $\xi = 2\langle \zeta, \xi \rangle \zeta + r_\zeta(\xi)$, it would be $\xi \in C$ (note that the convex cones are closed under summation with non negative coefficients) and then in particular, by $|\xi| \leq 1$, $\xi \in C^m$.

From (2.17), (2.18) and the definition of $H_\zeta^+$, for $\mathcal{L}^m$-a.e. $\xi \in H_\zeta^+$, it is

$$1_{C^m}(\xi) - 1_{C^m}(r_\zeta(\xi)) = 0.$$

Now $\zeta \in H_\zeta^+ \cap C^m$ and then, for the properties of the convex set with non empty interior, there exists a sequence $\{\zeta_k\}_{k=1}^\infty \subseteq C^m$, converging to $\zeta$ and such that $r_\zeta(\zeta_k) \in C^m$. By the continuity of

[18] Given a vector $\zeta \in \mathbb{R}^m$, then $\zeta^\perp = \{\xi \in \mathbb{R}^m : \langle \zeta, \xi \rangle = 0\}$.

[19] In the following $1_K$ denotes the characteristic function of the set $K$ that is the function defined as $1_K(x) = 1$ if $x \in K$, $1_K(x) = 0$ if $x \notin K$.

$r_\zeta$ and since $C^m$ is closed, it is $r_\zeta(\zeta) = -\zeta \in C^m$. Then $\zeta, -\zeta \in C^m$ that is $\{t\zeta : t \in \mathbb{R}\} \subseteq C$: thus a contradiction is found and (2.16) is proved. Clearly (2.16) implies that there exists $\varepsilon > 0$ such that

$$\mathrm{cl}(K_f) \setminus \{0\} \subseteq \left\{\xi \in \mathbb{R}^m \setminus \{0\} : \left\langle \frac{\xi}{|\xi|}, \eta_0 \right\rangle > \varepsilon \right\},$$

where, without loss of generality, we can also assume $|\eta_0| = 1$.

Let us suppose at first $s < \infty$. For every $h \in \mathbb{N}$, we set

$$C_h = \left\{\xi : g(\xi) \leq s\left(1 - \frac{1}{h}\right)\right\} \quad \text{and} \quad D_h = C_h \cap \left\{\xi \in \mathbb{R}^m \setminus \{0\} : \left\langle \frac{\xi}{|\xi|}, \eta_0 \right\rangle \leq \varepsilon \right\}.$$

Obviously $\{C_h\}_{h=1}^\infty$ and $\{D_h\}_{h=1}^\infty$ are two increasing sequences of sets. Moreover we have that every $D_h$ is bounded. Indeed, if by contradiction there exists $h_0 \in \mathbb{N}$ such that $D_{h_0}$ is unbounded, we can find a sequence $\{\xi_j\}_{j=1}^\infty \subseteq D_{h_0}$ such that $|\xi_j| \to \infty$ and $\frac{\xi_j}{|\xi_j|} \to \xi_0 \in S^{m-1}$. Then

$$s\left(1 - \frac{1}{h_0}\right) \geq \liminf_{j\to\infty} g\left(|\xi_j| \frac{\xi_j}{|\xi_j|}\right) \geq g^\natural(\xi_0),$$

but since $\xi_0 \in \left\{\xi \in \mathbb{R}^m \setminus \{0\} : \left\langle \frac{\xi}{|\xi|}, \eta_0 \right\rangle \leq \varepsilon \right\}$ it should be $g^\natural(\xi_0) = s$ and the contradiction is found.

Let us define now

$$\theta(0) = 2\sup\{|\xi| : \xi \in D_1\}, \quad (\theta(t) = 0 \text{ if } D_1 = \emptyset),$$

and, for every $h \in \mathbb{N}$, $t \in \left(s\left(1 - \frac{1}{h}\right), s\left(1 - \frac{1}{h+1}\right)\right]$,

$$\theta(t) = 2\sup\{|\xi| : \xi \in D_{h+1}\}, \quad (\theta(t) = 0 \text{ if } D_{h+1} = \emptyset).$$

Then $\theta : [0, s) \to [0, \infty)$ is clearly increasing and $\sup\{\theta(t) : t \in [0, s)\} = \infty$: thus we can define also $\theta(s) = \infty$.

Let us fix now $\xi_0 \in \left\{\xi \in \mathbb{R}^m \setminus \{0\} : \left\langle \frac{\xi}{|\xi|}, \eta_0 \right\rangle \leq \varepsilon \right\}$: if $g(\xi_0) = 0$ then $\xi_0 \in D_1$ and

$$(\theta \circ g)(\xi_0) = \theta(0) = 2\sup\{|\xi| : \xi \in D_1\} \geq 2|\xi_0|;$$

if $0 < g(\xi_0) < s$ then there exists $h_0 \in \mathbb{N}$ such that $g(\xi_0) \in \left(s\left(1 - \frac{1}{h_0}\right), s\left(1 - \frac{1}{h_0+1}\right)\right]$ and then $\xi_0 \in D_{h_0+1}$ and

$$(\theta \circ g)(\xi_0) = 2\sup\{|\xi| : \xi \in D_{h_0+1}\} \geq 2|\xi_0|;$$

if at last $g(\xi_0) = s$ then

$$(\theta \circ g)(\xi_0) = \infty \geq 2|\xi_0|.$$

Now we can easily prove that $(\theta \circ g)$ is demi-coercive. Indeed, fixed $\xi \in \mathbb{R}^m \setminus \{0\}$, if $\left\langle \frac{\xi}{|\xi|}, \eta_0 \right\rangle \leq \varepsilon$ then

$$(\theta \circ g)(\xi) + \langle \xi, \eta_0 \rangle \geq 2|\xi| - |\xi| = |\xi|,$$

while, if $\left\langle \frac{\xi}{|\xi|}, \eta_0 \right\rangle > \varepsilon$, then

$$(\theta \circ g)(\xi) + \langle \xi, \eta_0 \rangle \geq \left\langle \frac{\xi}{|\xi|}, \eta_0 \right\rangle |\xi| > \varepsilon|\xi|.$$

Thus, since trivially $(\theta \circ g)(0) \geq 0$, it holds that, for every $\xi \in \mathbb{R}^m$, $(\theta \circ g)(\xi) + \langle \xi, \eta_0 \rangle \geq \varepsilon|\xi|$.

We achieve the proof of the case $s < \infty$ choosing any function $\Theta : [0, s] \to [0, \infty]$ which is continuous, strictly increasing and such that, for every $t \in [0, s]$, $\theta(t) \leq \Theta(t)$: in this way $\Theta \circ g$ is proper, l.s.c. and demi-coercive. The construction of the function $\Theta$ is simple and it can be omitted.

In order to treat the case $s = \infty$, we can use the same argument once $s\left(1 - \frac{1}{h}\right)$ is changed with $h$ and the definition given for $\theta(0)$ is used to define $\theta$ on $[0, 1]$. □

The two propositions below are based on the following simple equalities: if $t_h, l \in \mathbb{R}$, then

$$\liminf_{h\to\infty}(t_h \vee l) = \left(\liminf_{h\to\infty} t_h\right) \vee l \quad \text{and} \quad \limsup_{h\to\infty}(t_h \vee l) = \left(\limsup_{h\to\infty} t_h\right) \vee l.$$

**Proposition 2.32.** *Let $\gamma : \mathbb{R}^m \to [0,\infty]$ be a proper and Borel function and let $l_j, l \in \mathbb{R}$ such that $l_j \downarrow l$. If, for every $j \in \mathbb{N}$, $\gamma \vee l_j$ is l.s.c. then $\gamma \vee l$ is l.s.c. too.*

*Proof.* Let $\xi_h, \xi_0 \in \mathbb{R}^m$ such that $\xi_h \to \xi_0$. Then, for every $j \in \mathbb{N}$,

$$(\gamma \vee l)(\xi_0) \le (\gamma \vee l_j)(\xi_0) \le \liminf_{h\to\infty}(\gamma \vee l_j)(\xi_h) = \left(\liminf_{h\to\infty} \gamma(\xi_h)\right) \vee l_j,$$

and, letting $j \to \infty$, we obtain

$$(\gamma \vee l)(\xi_0) \le \left(\liminf_{h\to\infty} \gamma(\xi_h)\right) \vee l = \liminf_{h\to\infty}(\gamma(\xi_h) \vee l),$$

that completes the proof. □

**Proposition 2.33.** *Let $\gamma : \mathbb{R}^m \to [0,\infty]$ be a proper and Borel function and let $l \in \mathbb{R}$. Then $(\gamma \vee l)^\flat = \gamma^\flat \vee l$.*

*Proof.* Let us fix $\xi_0 \in \mathbb{R}^m$. If we consider $t_h \downarrow 0$ such that $\gamma(t_h\xi_0) \to \gamma^\flat(\xi_0)$, then

$$(\gamma \vee l)^\flat(\xi_0) \ge \limsup_{h\to\infty}(\gamma \vee l)(t_h\xi_0) = \left(\limsup_{h\to\infty} \gamma(t_h\xi_0)\right) \vee l = \gamma^\flat(\xi_0) \vee l.$$

Conversely let $\bar{t}_h \downarrow 0$ such that $(\gamma \vee l)(\bar{t}_h\xi_0) \to (\gamma \vee l)^\flat(\xi_0)$. Then

$$\gamma^\flat(\xi_0) \vee l \ge \left(\limsup_{h\to\infty} \gamma(\bar{t}_h\xi_0)\right) \vee l = \limsup_{h\to\infty}(\gamma \vee l)(\bar{t}_h\xi_0) = (\gamma \vee l)^\flat(\xi_0),$$

and the proof is achieved. □

Finally we state the following theorem involving demi-coercive and convex functions (see Anzellotti, Buttazzo and Dal Maso [8] Theorem 2.4) in which, in particular, the equivalence between demi-coercivity and the property to be n.c.s.l. is proved.

**Theorem 2.34.** *Let $f : \mathbb{R}^m \to [0,\infty]$ be a proper, l.s.c. and convex function and let $\xi_0 \in \mathbb{R}^m$. Then the following conditions are equivalent:*

*(a) $f$ is demi-coercive;*

*(b) $f$ is n.c.s.l.;*

*(c) there are no straight line containing $\xi_0$ along which $f$ is constant;*

*(d) the set $\{\xi \in \mathbb{R}^m : f^\infty(\xi) = 0\}$ contains no straight line;*

*(e) the set $\{\xi \in \mathbb{R}^m : 2f(\xi_0) = f(\xi_0 + \xi) + f(\xi_0 - \xi)\}$ is bounded;*

*(f) for every $\xi \in \mathbb{R}^m$, $\xi \ne 0$, there exists $t > 0$ such that $2f(\xi_0) < f(\xi_0 + t\xi) + f(\xi_0 - t\xi)$;*

*(g) for every $\xi \in \mathbb{R}^m$, $\xi \ne 0$, we have $f^\infty(\xi) + f^\infty(-\xi) > 0$;*

*(h) there exist $a, b \in \mathbb{R}$, $a > 0$, such that, for every $\xi \in \mathbb{R}^m$, $f(\xi_0 + \xi) + f(\xi_0 - \xi) \ge a|\xi| - b$;*

*(i) there exists $\eta \in \mathbb{R}^m$ such that, for every $\xi \in \mathbb{R}^m$, $\xi \ne 0$, $f^\infty(\xi) - \langle \eta, \xi \rangle > 0$.*

# Chapter 3

# Approximation of convex functions

In order to study the lower semicontinuity of the integral functional $I$ given by (1.1) it is fundamental to be able to approximate from below the integrand $f$ with more regular functions. For this purpose many efforts has been spent to find different strategies to build such approximation. In this chapter we present several theorems on this topic: while in the first part we state only the classical results by De Giorgi and Serrin, the second part contains some recent theorems obtained by Gori and Maggi in [53].

## 3.1 De Giorgi's and Serrin's approximation methods

Let $\Sigma \subseteq \mathbb{R}^d$ be an open set and let $f : \Sigma \times \mathbb{R}^n \to [0, \infty]$: we say that $f$ has *compact support* on $\Sigma$ if there exists $\Sigma' \subset\subset \Sigma$ such that, for every $(t, \xi) \in (\Sigma \setminus \Sigma') \times \mathbb{R}^n$, we have $f(t, \xi) = 0$.

The following approximation result was proved by De Giorgi (see [35] Theorem 3, Section 3).

**Theorem 3.1.** *Let $\Sigma \subseteq \mathbb{R}^d$ be an open set and $f : \Sigma \times \mathbb{R}^n \to [0, \infty)$ be a continuous function with compact support in $\Sigma$ such that, for every $t \in \Sigma$, $f(t, \cdot)$ is convex on $\mathbb{R}^n$. Then there exists a sequence $\{\alpha_q\}_{q=1}^{\infty} \subseteq C_c^{\infty}(\mathbb{R}^n)$, $\alpha_q \geq 0$ such that, setting*

$$a_{q,0}(t) = \int_{\mathbb{R}^n} f(t, \eta) \left\{ (n+1)\alpha_q(\eta) + \sum_{h=1}^{n} \eta_h \frac{\partial \alpha_q}{\partial \xi_h}(\eta) \right\} d\eta, \tag{3.1}$$

*and, for every $h \in \{1, \dots, n\}$,*

$$a_{q,h}(t) = -\int_{\mathbb{R}^n} f(t, \eta) \frac{\partial \alpha_q}{\partial \xi_h}(\eta) d\eta, \tag{3.2}$$

*the sequence of functions given, for every $j \in \mathbb{N}$, by*

$$f_j(t, \xi) = \max_{1 \leq j \leq q} \left\{ 0, \ \ a_{q,0}(t) + \sum_{h=1}^{n} a_{q,h}(t) \xi_h \right\},$$

*satisfies the following conditions:*

*(i) for every $j \in \mathbb{N}$, $f_j : \Sigma \times \mathbb{R}^n \to [0, \infty)$ is a continuous function with compact support in $\Sigma$ such that, for every $t \in \Sigma$, $f_j(t, \cdot)$ is convex on $\mathbb{R}^n$. Moreover, for every $(t, \xi) \in \Sigma \times \mathbb{R}^n$, $f_j(t, \xi) \leq f_{j+1}(t, \xi)$ and*

$$f(t, \xi) = \sup_{j \in \mathbb{N}} f_j(t, \xi);$$

*(ii) for every $j \in \mathbb{N}$, there exists a constant $M_j > 0$ such that, for every $(t,\xi) \in \Sigma \times \mathbb{R}^n$,*

$$|f_j(t,\xi)| \leq M_j\,(1+|\xi|)\,, \tag{3.3}$$

*and, for every $t \in \Sigma$, $\xi_1, \xi_2 \in \mathbb{R}^n$,*

$$|f_j(t,\xi_1) - f_j(t,\xi_2)| \leq M_j\,|\xi_1 - \xi_2|\,. \tag{3.4}$$

The next theorem is instead due to Serrin (see [69] Lemmas 5 and 8).

**Theorem 3.2.** *Let $\Sigma \subseteq \mathbb{R}^d$ be an open set and $f : \Sigma \times \mathbb{R}^n \to [0,\infty)$ be a continuous function such that, for every $t \in \Sigma$, $f(t,\cdot)$ strictly convex on $\mathbb{R}^n$. Then, for every $\varepsilon > 0$ and $\Sigma' \subset\subset \Sigma$, there exists a continuous function $\bar{f} : \Sigma \times \mathbb{R}^n \to [0,\infty)$ with compact support on $\Sigma'$ such that*[1]*:*

*(i) for every $t \in \Sigma$, $\bar{f}(t,\cdot)$ is convex;*

*(ii) for every $(t,\xi) \in \Sigma \times \mathbb{R}^n$, $\bar{f}(t,\xi) \leq f(t,\xi) + \varepsilon$;*

*(iii) for every $t \in \Sigma', \xi \in B\left(0,\frac{1}{\varepsilon}\right)$, $|\bar{f}(t,\xi) - f(t,\xi)| \leq \varepsilon$;*

*(iv) $\nabla_\xi \bar{f} : \Sigma \times \mathbb{R}^n \to \mathbb{R}^n$ exists, is continuous and there exists a constant $M > 0$ such that, for every $(t,\xi) \in \Sigma \times \mathbb{R}^n$,*

$$\left|\nabla_\xi \bar{f}(t,\xi)\right| \leq M,$$

*and, for every $t_1, t_2 \in \Sigma, \xi \in \mathbb{R}^n$,*

$$\left|\nabla_\xi \bar{f}(t_1,\xi) - \nabla_\xi \bar{f}(t_2,\xi)\right| \leq M|t_1 - t_2|(1+|\xi|).$$

## 3.2 Approximation of convex and demi-coercive functions

Let us introduce now two approximation results that will be instrumental in the future and that can be found in [53] Theorem 4 and Theorem 5 respectively.

### 3.2.1 Approximation by means of convex cones

The following theorem says that a convex, demi-coercive function can be approximated by demi-coercive *cones*. The advantage of this approximation is given by the fact that these cones, under a certain point of view, behave better than the supporting hyper-planes.

**Theorem 3.3.** *Let $\Sigma \subseteq \mathbb{R}^d$ be an open set and $f : \Sigma \times \mathbb{R}^n \to [0,\infty)$ be a l.s.c. (resp. continuous) function such that, for every $t \in \Sigma$, $f(t,\cdot)$ is convex and demi-coercive on $\mathbb{R}^n$. For every $\xi_0 \in \mathbb{R}^n$, let us define $\mathrm{P}_{\xi_0} f : \Sigma \times \mathbb{R}^n \to [-1,\infty)$ as*

$$\mathrm{P}_{\xi_0} f(t,\xi) = \inf\left\{\alpha : (\xi,\alpha) \in \mathrm{pos}_{(\xi_0,-1)}\left(\mathrm{epi}\, f(t,\cdot)\right)\right\}.$$

*Then we have that:*

*(i) $\mathrm{P}_{\xi_0} f$ is l.s.c. (resp. continuous) and, for every $t \in \Sigma$, $\mathrm{P}_{\xi_0} f(t,\cdot)$ is convex, demi-coercive and can be characterized as the greatest function less than or equal to $f$ such that, for every $t \in \Sigma$, the map*

$$\xi \mapsto (1 + \mathrm{P}_{\xi_0} f(t,\xi+\xi_0))$$

*is positively homogeneous of degree 1;*

[1] With $\nabla_\xi f(x,s,\xi)$ we mean the gradient of $f(x,s,\cdot)$.

*(ii) there exists a sequence* $\{\xi_k\}_{k=1}^{\infty} \subseteq \mathbb{R}^n$ *such that, for every* $(t,\xi) \in \Sigma \times \mathbb{R}^n$,

$$f(t,\xi) = \sup_{k\in\mathbb{N}} \mathrm{P}_{\xi_k}\, f(t,\xi).$$

*Proof.* Let us prove (i). By the definition of $\mathrm{P}_{\xi_0} f$ we have, for every $t \in \Sigma$,

$$\mathrm{epi}\left(\mathrm{P}_{\xi_0}\, f(t,\cdot)\right) = \mathrm{cl}\left(\mathrm{pos}_{(\xi_0,-1)}\left(\mathrm{epi}\, f(t,\cdot)\right)\right).$$

Since, for every $t \in \Sigma$, $\mathrm{epi}\,(f(t,\cdot)) \subseteq \mathrm{epi}\,(\mathrm{P}_{\xi_0}\, f(t,\cdot))$ we obtain $\mathrm{P}_{\xi_0} f \le f$ on $\Sigma \times \mathbb{R}^n$ and, since the positive conic envelope of a convex set is convex too[2], we have also that, for every $t \in \Sigma$, $\mathrm{P}_{\xi_0}\, f(t,\cdot)$ is convex. The map $\xi \mapsto (1 + \mathrm{P}_{\xi_0}\, f(t,\xi+\xi_0))$ clearly is positively homogeneous of degree 1.

In order to show that $\mathrm{P}_{\xi_0} f$ is the greatest function less than or equal to $f$ such that it satisfies the property about the positive homogeneity of degree 1 asked in (i), we note that, if $g$ is such that $g \le f$ and, for every $t \in \Sigma$, $\xi \mapsto (1 + g(t,\xi+\xi_0))$ is positively homogeneous of degree 1, then, for every $t \in \Sigma$,

$$\mathrm{epi}(g(t,\cdot)) \supseteq \mathrm{cl}\left(\mathrm{pos}_{(\xi_0,-1)}\left(\mathrm{epi}\, f(t,\cdot)\right)\right) = \mathrm{epi}\left(\mathrm{P}_{\xi_0}\, f(t,\cdot)\right),$$

so that $g \le \mathrm{P}_{\xi_0} f$.

It worth noting also that, for every $(t,\xi) \in \Sigma \times \mathbb{R}^n$, $1 + \mathrm{P}_{\xi_0}\, f(t,\xi+\xi_0) \le f^{\infty}(t,\xi)$[3]. Indeed, by means of Definitions 3.3 and 3.17 and Theorem 3.21 in [67], for every $t \in \Sigma$, it holds

$$\mathrm{epi}\left(f^{\infty}(t,\cdot)\right) = \left\{(\xi,\alpha) : \exists (\xi_h,\alpha_h) \in \mathrm{epi}(f(t,\cdot)), t_h \to \infty \text{ such that } \frac{(\xi_h,\alpha_h)}{t_h} \to (\xi,\alpha)\right\}.$$

Thus let $(\xi,\alpha)$, $(\xi_h,\alpha_h)$ and $t_h$ as above: since $(\xi_h,\alpha_h) \in \mathrm{epi}(f(t,\cdot))$ we have

$$\frac{(\xi_h - \xi_0, \alpha_h + 1)}{t_h} + (\xi_0,-1) \in \mathrm{pos}_{(\xi_0,-1)}\left(\mathrm{epi}\, f(t,\cdot)\right)$$

and then, passing to the limit as $h \to \infty$, we obtain

$$(\xi+\xi_0, \alpha-1) \in \mathrm{cl}\left(\mathrm{pos}_{(\xi_0,-1)}\left(\mathrm{epi}\, f(t,\cdot)\right)\right) = \mathrm{epi}\left(\mathrm{P}_{\xi_0}\, f(t,\cdot)\right)$$

if and only if

$$(\xi,\alpha) \in \mathrm{epi}\left(1 + \mathrm{P}_{\xi_0}\, f(t,\cdot+\xi_0)\right),$$

that proves the wanted inequality.

Now let us point out that, by definition, we have

$$\mathrm{P}_{\xi_0}\, f(t,\xi) = \inf\left\{\alpha : (\xi,\alpha) \in \mathrm{pos}_{(\xi_0,-1)}\left(\mathrm{epi}\, f(t,\cdot)\right)\right\}$$

$$= \inf\left\{ -1 + \tau(\beta+1) : \tau \ge 0, \beta \ge f(t,\eta), \xi = \xi_0 + \tau(\eta - \xi_0)\right\}$$

$$= -1 + \inf\left\{\tau + \tau f\left(t, \xi_0 + \frac{\xi-\xi_0}{\tau}\right) : \tau > 0\right\}. \tag{3.5}$$

Using this formula, by a simple computation, we can prove that, for every $t \in \Sigma$, $\mathrm{P}_{\xi_0}\, f(t,\cdot)$ is demi-coercive.

[2]See [67] Chapter 3, Section G.

[3]With $f^{\infty}(t,\xi)$ we mean $(f(t,\cdot))^{\infty}(\xi)$.

Moreover (3.5) allows to achieve also the lower semicontinuity (resp. continuity) of $\mathrm{P}_{\xi_0} f$. Indeed, let us consider the function $\hat{f} : \Sigma \times \mathbb{R}^n \times \mathbb{R} \to [0, \infty)$ given, as in Proposition 2.25, by

$$\hat{f}(t,\xi,\tau) = \begin{cases} \tau f\left(t, \frac{\xi}{\tau}\right) & \text{if } \tau > 0, \\ f^\infty(t,\xi) & \text{if } \tau = 0, \\ +\infty & \text{if } \tau < 0. \end{cases} \tag{3.6}$$

By Theorem 3.1 of [30], $\hat{f}$ is l.s.c. on $\Sigma \times \mathbb{R}^n \times \mathbb{R}$ and then the same property is had also by $\overline{f}(t,\xi,\tau) = \tau + \hat{f}(t, \xi - \xi_0, \tau)$. By (3.5), the definition of $\overline{f}$ and the remark made above about $f^\infty$ we have

$$\mathrm{P}_{\xi_0} f(t,\xi) = -1 + \inf\left\{\overline{f}(t,\xi,\tau) : \tau \geq 0\right\},$$

but, since $\tau \leq \overline{f}(t,\xi,\tau)$ and $\overline{f}(t,\xi,1) = 1 + f(t,\xi)$, it is in fact

$$\mathrm{P}_{\xi_0} f(t,\xi) = -1 + \inf\left\{\overline{f}(t,\xi,\tau) : 0 \leq \tau \leq 1 + f(t,\xi)\right\}.$$

In particular, by the lower semicontinuity of $\overline{f}$, for every $(t,\xi)$ there exists $0 \leq \tau_{(t,\xi)} \leq 1 + f(t,\xi)$ such that

$$\mathrm{P}_{\xi_0} f(t,\xi) = -1 + \overline{f}\left(t,\xi,\tau_{(t,\xi)}\right).$$

Let us verify now the lower semicontinuity of $\mathrm{P}_{\xi_0} f$. Let $(t_h,\xi_h), (t_1,\xi_1) \in \Sigma \times \mathbb{R}^n$ such that $(t_h,\xi_h) \to (t_1,\xi_1)$ and suppose, without loss of generality, that

$$\liminf_{h\to\infty} \mathrm{P}_{\xi_0} f(t_h,\xi_h) = \lim_{h\to\infty} \mathrm{P}_{\xi_0} f(t_h,\xi_h).$$

Let us define also $\tau_h = \tau_{(t_h,\xi_h)}, \tau_1 = \tau_{(t_1,\xi_1)}$. If it is $\tau_h \to \infty$, since

$$\tau_h \leq \overline{f}(t_h,\xi_h,\tau_h) = 1 + \mathrm{P}_{\xi_0} f(t_h,\xi_h),$$

the lower semicontinuity inequality is trivially verified. If otherwise $\tau_h$ is bounded we can suppose $\tau_h \to \tau_0$ and then

$$1 + \mathrm{P}_{\xi_0} f(t_1,\xi_1) = \overline{f}(t_1,\xi_1,\tau_1) \leq \overline{f}(t_1,\xi_1,\tau_0) \leq \liminf_{h\to\infty} \overline{f}(t_h,\xi_h,\tau_h) = \liminf_{h\to\infty} \left(1 + \mathrm{P}_{\xi_0} f(t_h,\xi_h)\right),$$

that completes the proof of the lower semicontinuity of $\mathrm{P}_{\xi_0} f$.

Finally, again by (3.5), we have also that the continuity of $f$ implies the upper semicontinuity (and then continuity) of $\mathrm{P}_{\xi_0} f$.

In order to prove (ii) we show at first that

$$f(t,\xi) = \sup_{\xi_0\in\mathbb{R}^n} \mathrm{P}_{\xi_0} f(t,\xi). \tag{3.7}$$

Without loss of generality, we can drop the dependence on $t$. Moreover we can reduce us to consider the one dimensional case, as we can see by the following argument. Fixed $\nu \in S^{n-1}$, we can define, for every $\rho \in \mathbb{R}$,

$$f_\nu(\rho) = f(\rho\nu),$$

then, by the maximality property of $\mathrm{P}_{\xi_0} f$, we have that, for every $\rho_0, \rho \in \mathbb{R}$,

$$\mathrm{P}_{\rho_0\nu} f(\rho\nu) = \mathrm{P}_{\rho_0} f_\nu(\rho).$$

If the approximation holds in dimension one we have, for every $\nu \in S^{n-1}$ and $\rho \in \mathbb{R}$,

$$f_\nu(\rho) = \sup_{\rho_0\in\mathbb{R}} \mathrm{P}_{\rho_0} f_\nu(\rho)$$

so that,

$$f(\rho\nu) = f_\nu(\rho) = \sup_{\rho_0\in\mathbb{R}} \mathrm{P}_{\rho_0} f_\nu(\rho) = \sup_{\rho_0\in\mathbb{R}} \mathrm{P}_{\rho_0\nu} f(\rho\nu) \leq \sup_{\xi_0\in\mathbb{R}^n} \mathrm{P}_{\xi_0} f(\rho\nu) \leq f(\rho\nu),$$

that implies (3.7).

Therefore we prove (3.7) for $n = 1$. For this we consider a function $f : \mathbb{R} \to [0, \infty)$ which is convex (and then continuous) and non constant (that is demi-coercive) and $\xi_0 \in \mathbb{R}$, and define the following set

$$\Lambda_f(\xi_0) = \Big\{ v \in \mathbb{R} : v(\xi - \xi_0) - 1 \leq f(\xi), \quad \forall \xi \in \mathbb{R} \Big\}.$$

Let us note that, for every $\xi_0 \in \mathbb{R}$, $0 \in \Lambda_f(\xi_0)$. Moreover, for every $\xi, \xi_0, v \in \mathbb{R}$, $v \neq 0$,

$$v(\xi - \xi_0) + f(\xi_0) = v\left(\xi - \xi_0 - \frac{1 + f(\xi_0)}{v}\right) - 1, \tag{3.8}$$

and then

$$v \in \partial f(\xi_0) \quad \text{if and only if} \quad v \in \Lambda_f\left(\xi_0 - \frac{1 + f(\xi_0)}{v}\right).$$

Let us define $\xi^*(\xi_0, v) = \xi_0 - \frac{1+f(\xi_0)}{v}$. For every $\xi_0 \in \mathbb{R}$, we set $\alpha_f(\xi_0) = \inf \Lambda_f(\xi_0) \leq 0$ and $\beta_f(\xi_0) = \sup \Lambda_f(\xi_0) \geq 0$. By the maximality property of $\mathrm{P}_{\xi_0} f$, we have

$$\mathrm{P}_{\xi_0} f(\xi) = \max\Big\{\alpha_f(\xi_0)(\xi - \xi_0) - 1, \beta_f(\xi_0)(\xi - \xi_0) - 1\Big\}. \tag{3.9}$$

Since $f$ is not constant and convex it is either $\lim_{\xi\to\infty} f(\xi) = \infty$ or $\lim_{\xi\to-\infty} f(\xi) = \infty$: we consider the case in which only $\lim_{\xi\to\infty} f(\xi) = \infty$ proving in this hypothesis that, for every $\xi \in \mathbb{R}$,

$$f(\xi) = \sup_{\xi_0\in\mathbb{R}} \Big\{\beta_f(\xi_0)(\xi - \xi_0) - 1\Big\} : \tag{3.10}$$

the other two cases ($\lim_{\xi\to-\infty} f(\xi) = \infty$ and $\lim_{|\xi|\to\infty} f(\xi) = \infty$) follow immediately.

Let us consider the following (possibly empty) set,

$$A = \{\xi \in \mathbb{R} : \partial f(\xi) = \{0\}\},$$

and claim that if $A$ is not empty then it is connected and $f$ is constant on $A$. If $A = \{\xi_0\}$ the claim is obvious. If instead there are $\xi_1, \xi_2 \in A$, $\xi_1 < \xi_2$, let us consider $\xi_1 < \eta < \xi_2$: by definition of sub-differential, $f(\xi_1) = f(\xi_2) = m$ and $f(\eta) \geq m$. However, by the convexity of $f$, $f(\eta) \leq m$ too. Then, for every $\xi_1 < \eta < \xi_2$, we have $f(\eta) = m$ and this implies $\partial f(\eta) = \{0\}$, that is $\eta \in A$: thus the claim is proved.

By (3.8) we immediately see that (3.10) holds on every $\xi \in \mathbb{R}\backslash A$ with the supremum attained on $\xi_0 = \xi^*(\xi, v)$, for every choice of $v \neq 0$, $v \in \partial f(\xi)$: if $A = \emptyset$ the proof is achieved.

Thus, let us suppose that $A$ is non empty. Since $f(\xi) \to \infty$ as $\xi \to \infty$, we have that $A$ is bounded from above, so that we can consider $\xi_A = \sup A < \infty$. By continuity of $f$, it is $f(\xi_A) = m$. By choosing $\xi_h \in A$ with $\xi_h \to \xi_A$, we have also

$$\{0\} = \limsup_{h\to\infty} \partial f(\xi_h) \subseteq \partial f(\xi_A).$$

Therefore $0 \in \partial f(\xi_A)$ and, for every $\xi \in A$, we have $\xi \leq \xi_A$ and $f(\xi) = f(\xi_A) = m$. Let us fix $\xi \in A$, and show the validity of (3.10). If $\xi_A \in \mathbb{R}\backslash A$, there exists $v_A \in \partial f(\xi_A)$ with $v_A > 0$, such that $[0, v_A] \subseteq \partial f(\xi_A)$ (remember that $\partial f(\xi_A)$ is convex). Then, considering $v_h = h^{-1} v_A \in \partial f(\xi_A)$, we have

$$v_h(\xi - \xi_A) + f(\xi_A) = v_h(\xi - \xi^*(\xi_A, v_h)) - 1 \leq \beta_f(\xi^*(\xi_A, v_h))(\xi - \xi^*(\xi_A, v_h)) - 1 \leq f(\xi) = m,$$

and since the left hand side tends to $f(\xi_A) = m$ as $h \to \infty$, we conclude. If otherwise $\xi_A \in A$, let us consider $\xi_h \to \xi_A$, with $\xi_h > \xi_A$. We know that

$$\limsup_{h\to\infty} \partial f(\xi_h) \subseteq \partial f(\xi_A) = \{0\},$$

then we can choose $v_h \downarrow 0$ with $v_h \in \partial f(\xi_h)$. In particular it is

$$v_h(\xi - \xi_h) + f(\xi_h) \leq \beta_f(\xi^*(\xi_h, v_h))(\xi - \xi^*(\xi_h, v_h)) - 1 \leq f(\xi) = m,$$

and since the left hand side tends to $f(\xi_A) = m$ as $h \to \infty$, we complete the proof of (3.7).

It remains to show the validity of (ii) once it is known (3.7). This can be simply proved by means of the following lemma.

**Lemma 3.4.** *Let $A \subseteq \mathbb{R}^n$ and let $\mathcal{G}$ be a set of l.s.c. functions from $A$ to $\mathbb{R}$. Let us define, for every $x \in A$, $f(x) = \sup_{g\in\mathcal{G}} g(x)$. Then there exists a sequence $\{g_h\}_{h=1}^{\infty} \subseteq \mathcal{G}$, such that, for every $x \in A$, $f(x) = \sup_{h\in\mathbb{N}} g_h(x)$.*

Since its proof consists only in a slight modification of the argument used in the proof of Lemma 9.2 in [43], we omit the details. □

### 3.2.2 Approximation by means of strictly convex functions

As a consequence of Theorem 3.3 we show that the class of functions that can be represented as a countable supremum of strictly convex ones is characterized by the demi-coercivity.

**Theorem 3.5.** *Let $\Sigma \subseteq \mathbb{R}^d$ be an open set and let $f : \Sigma \times \mathbb{R}^n \to [0, \infty)$ be a continuous function. Then the two following conditions are equivalent:*

*(i) for every $t \in \Sigma$, $f(t, \cdot)$ is convex and demi-coercive.*

*(ii) there exists a sequence $\{f_j\}_{j=1}^{\infty}$ such that, for every $(t, \xi) \in \Sigma \times \mathbb{R}^n$,*

$$f(t, \xi) = \sup_{j\in\mathbb{N}} f_j(t, \xi), \tag{3.11}$$

*where, for every $j \in \mathbb{N}$, $f_j : \Sigma \times \mathbb{R}^n \to [-2, \infty)$ is a continuous function such that, for every $t \in \Sigma$, $f_j(t, \cdot)$ is strictly convex in $\mathbb{R}^n$ and, for every $\Sigma' \subset\subset \Sigma$, there exists a constant $C_{j,\Sigma'} > 0$ such that, for every $(t, \xi) \in \Sigma' \times \mathbb{R}^n$,*

$$f_j(t, \xi) \leq C_{j,\Sigma'}(1 + |\xi|). \tag{3.12}$$

*Proof.* The proof of $(ii) \Rightarrow (i)$ is obvious and can be omitted.

In order to prove the converse it is sufficient to show that every conic type function given in Theorem 3.3 can be approximated from below by a sequence of continuous functions, strictly convex in the variable $\xi$. Thus, we can suppose also that $f(t, \cdot)$ is positively homogeneous of degree 1.

We start proving that there exists a continuous function $N : \Sigma \to \mathbb{R}^{n+1}$, such that, for every $t \in \Sigma$,

$$\text{epi}(f(t, \cdot))\backslash\{0\} \subset \{M \in \mathbb{R}^{n+1} : \langle M, N(t)\rangle > 0\}. \tag{3.13}$$

Following exactly the first part of Proposition 2.31, we can prove that the vector valued function

$$N(t) = \int_{K(t)} N dN,$$

where $K(t) = \text{epi}\,(f(t, \cdot)) \cap B^{n+1}(0, 1)$, satisfies just the above condition.

Then we only have to show that $N(t)$ is continuous on $\Sigma$. To this end let us fix $t_h, t_0 \in \Sigma$, $t_h \to t_0$, and consider

$$C = \sup\Big\{f(t,\xi) : t \in \{t_0\} \cup \{t_h\}_{h=1}^{\infty}, |\xi| \leq 1\Big\} \vee 1 < \infty,$$

since it is the supremum of a continuous function on a compact set. For

$$S(t) = \Big\{(\xi,\alpha) \in \mathbb{R}^{n+1} : |\xi| \leq 1, \alpha \in [0,C]\Big\} \cap \operatorname{epi}(f(t,\cdot)),$$

we have

$$\begin{aligned}|N(t_h) - N(t_0)| &\leq \mathcal{L}^{n+1}(K(t_h) \triangle K(t_0)) \\ &\leq \mathcal{L}^{n+1}(S(t_h) \triangle S(t_0)) = \int_{B^n(0,1)} |f(t_h,\xi) - f(t_0,\xi)| d\xi \to 0,\end{aligned}$$

as $h \to \infty$, where we used the notation $A \triangle B = (A \setminus B) \cup (B \setminus A)$. Thus also the continuity of $N(t)$ is proved.

Since it must be $N(t) = (\nu(t), a(t))$ with $\nu(t) \in \mathbb{R}^n, a(t) > 0$, then the inclusion (2.16) implies that, for every $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$g(t,\xi) = f(t,\xi) + \frac{\langle \nu(t), \xi \rangle}{a(t)} > 0. \tag{3.14}$$

The function $c(t) = \min\{g(t,v) : v \in S^{n-1}\} > 0$ is continuous on $\Sigma$ and $c(t)|\xi| \leq g(t,\xi)$.

For every $\delta \in (0,1)$, let us consider

$$h_\delta(t,\xi) = \sqrt{\delta^2 + c(t)^2|\xi|^2} - \delta, \quad (t,\xi) \in \Sigma \times \mathbb{R}^n.$$

Obviously, for every $\delta \in (0,1)$, $0 \leq h_\delta(t,\xi) \leq c(t)|\xi|$ and $h_\delta(t,\xi) \to c(t)|\xi|$ as $\delta \downarrow 0$. Moreover for every $(t,\xi) \in \Sigma \times \mathbb{R}^n$, $\delta \in (0,1)$,

$$\frac{dh_\delta}{d\delta}(t,\xi) = -1 + \frac{\delta}{\sqrt{\delta^2 + c^2(t)|\xi|^2}} \leq 0,$$

so that it follows $h_\delta(t,\xi) \uparrow c(t)|\xi|$ as $\delta \downarrow 0$. Let us note also that, for every $t \in \Sigma$, $\delta \in (0,1)$ fixed, $h_\delta(t,\cdot)$ is strictly convex. In order to prove this, we consider $\xi, \eta \in \mathbb{R}^n$, $\xi \neq \eta$, $\lambda \in (0,1)$; if $|\xi| \neq |\eta|$ then

$$\begin{aligned}\sqrt{\delta^2 + c(t)^2|\lambda\xi + (1-\lambda)\eta|^2} &\leq \sqrt{\delta^2 + c(t)^2(\lambda|\xi| + (1-\lambda)|\eta|)^2} \\ &< \lambda\sqrt{\delta^2 + c(t)^2|\xi|^2} + (1-\lambda)\sqrt{\delta^2 + c(t)^2|\eta|^2};\end{aligned}$$

if $|\xi| = |\eta|$ then $|\lambda\xi + (1-\lambda)\eta| < \lambda|\xi| + (1-\lambda)|\eta|$, thus

$$\begin{aligned}\sqrt{\delta^2 + c(t)^2|\lambda\xi + (1-\lambda)\eta|^2} &< \sqrt{\delta^2 + c(t)^2(\lambda|\xi| + (1-\lambda)|\eta|)^2} \\ &\leq \lambda\sqrt{\delta^2 + c(t)^2|\xi|^2} + (1-\lambda)\sqrt{\delta^2 + c(t)^2|\eta|^2}.\end{aligned}$$

Now we define, for every $\delta \in (0,1)$, $f_\delta : \Sigma \times \mathbb{R}^n \to \mathbb{R}$ as

$$f_\delta(t,\xi) = (1-\delta)g(t,\xi) + \delta h_\delta(t,\xi) - \frac{\langle \nu(t), \xi \rangle}{a(t)}, \quad (t,\xi) \in \Sigma \times \mathbb{R}^n.$$

This is a continuous function, strictly convex in the variable $\xi$ (as it is a sum of a convex function and a strictly convex one). Clearly $f_\delta(t,\xi) \to f(t,\xi)$ as $\delta \downarrow 0$. Since $h_\delta(t,\xi) \leq c(t)|\xi| \leq g(t,\xi)$ we have also

$$f_\delta(t,\xi) \leq g(t,\xi) - \frac{\langle \nu(t), \xi \rangle}{a(t)} = f(t,\xi).$$

For every $(t,\xi) \in \Sigma \times \mathbb{R}^n$, $\delta \in (0,1)$,

$$\frac{df_\delta}{d\delta}(t,\xi) = -g(t,\xi) + h_\delta(t,\xi) + \delta \frac{dh_\delta}{d\delta}(t,\xi) \leq 0,$$

then $f_\delta(t,\xi) \uparrow f(t,\xi)$ as $\delta \downarrow 0$. However, note that $f_\delta$ is not necessarily bounded from below, as required in the statement: indeed, defining the open set $\Sigma_+ = \{t \in \Sigma : \nu(t) \neq 0\}$, we have that, for every $t \in \Sigma \setminus \Sigma_+$, surely $f_\delta(t,\cdot) \geq 0$, while, when $t \in \Sigma_+$, $f_\delta(t,\cdot)$ could tend to $-\infty$ as $|\xi| \to \infty$.

In order to solve this problem is sufficient to prove the existence of a continuous function $g^* : \Sigma \times \mathbb{R}^n \to [-1,\infty)$ such that, for every $t \in \Sigma_+$, $g^*(t,\cdot)$ is strictly convex, for every $t \in \Sigma \setminus \Sigma_+$, $g^*(t,\cdot) = 0$, and $g^* \leq f$. Indeed, in this case, we can define, for every $(t,\xi) \in \Sigma \times \mathbb{R}^n$, the function

$$f^*_\delta(t,\xi) = f_\delta(t,\xi) \vee g^*(t,\xi),$$

that is continuous, as the maximum between two continuous functions, $f \geq f^*_\delta \geq g^* \geq -1$, and for every $t \in \Sigma$, $f^*_\delta(t,\cdot)$ is strictly convex, since, for every $t \in \Sigma_+$, it is the maximum of two strictly convex functions while, for every $t \in \Sigma \setminus \Sigma_+$, $f^*_\delta(t,\cdot) = f_\delta(t,\cdot)$ that is strictly convex. Moreover $f^*_\delta(t,\xi) \uparrow f(t,\xi)$ as $\delta \downarrow 0$ since $f_\delta(t,\xi) \uparrow f(t,\xi)$ as $\delta \downarrow 0$ and $g^* \leq f$.

For what concerns the existence of $g^*$ we can argue as follows. Let us define, for every $s \in \mathbb{R}$,

$$\varphi(s) = (s \vee 0) - 1 + \exp\left(-\frac{|s|}{2}\right),$$

which is strictly convex, belongs to $C^2(\mathbb{R})$ and $-1 \leq \varphi(s) \leq (s \vee 0)$. Moreover, for every $t \in \Sigma_+$, let us define also $b(t) = \left|\frac{\nu(t)}{a(t)}\right| > 0$ and $\mu(t) = -\frac{\nu(t)}{a(t)b(t)}$. Clearly

$$f(t,\xi) \geq c(t)|\xi| + b(t)\langle \mu(t), \xi \rangle$$

and, setting $d(t) = (b(t) \wedge 1)(c(t) \wedge 1) > 0$, we have, for every $v \in S^{n-1}$,

$$f(t,\xi) \geq d(t)|\xi| + b(t)\langle \mu(t), \xi \rangle \geq d(t)\langle v, \xi \rangle + b(t)\langle \mu(t), \xi \rangle.$$

Then it follows that[4], for every $t \in \Sigma_+$ and $v \in S^{n-1}$,

$$f(t,\xi) \geq \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \Big(\langle e(t,v), \xi \rangle \vee 0\Big) d\mathcal{H}^{n-1}(v)$$

$$\geq \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \varphi\Big(\langle e(t,v), \xi \rangle\Big) d\mathcal{H}^{n-1}(v) =: g^*(t,\xi),$$

where $e(t,v) = d(t)v + b(t)\mu(t)$. Thus, let us define $g^* : \Sigma \times \mathbb{R}^n \to \mathbb{R}$ as

$$g^*(t,\xi) = \begin{cases} \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \varphi\Big(\langle e(t,v), \xi \rangle\Big) d\mathcal{H}^{n-1}(v) & \text{if } t \in \Sigma_+, \\ 0 & \text{if } t \in \Sigma \setminus \Sigma_+. \end{cases} \tag{3.15}$$

Clearly $-1 \leq g^* \leq f$ on $\Sigma \times \mathbb{R}^n$ and, by Lebesgue dominated convergence Theorem, it can be verified that $g^*$ is continuous on $\Sigma_+ \times \mathbb{R}^n$. If now $(t_0,\xi_0) \in (\Sigma \setminus \Sigma_+) \times \mathbb{R}^n$ then $\lim_{t \to t_0} b(t) = \lim_{t \to t_0} d(t) = 0$ (we remark that $a(t)$ is locally uniformly positive on $\Sigma$), thus $\lim_{t \to t_0} e(t,v) = 0$ uniformly as $v \in S^{n-1}$. As a consequence $\lim\limits_{(t,\xi) \to (t_0,\xi_0)} g^*(t,\xi) = 0$ that implies that $g^*$ is continuous on $\Sigma \times \mathbb{R}^n$.

[4] Here and in the following $\mathcal{H}^n$ denotes the usual $n$-dimensional Haussdorf measure and $\omega_n = \mathcal{H}^n(S^n)$.

It remains to check that, for every $t \in \Sigma_+$, $g^*(t,\cdot)$ is strictly convex. By the fact that $\varphi \in C^2(\mathbb{R})$ and by Lebesgue dominated convergence Theorem, we have, for every $t \in \Sigma_+$, $g^*(t,\cdot) \in C^2(\mathbb{R}^n)$ and[5]

$$\nabla^2_\xi g^*(t,\xi) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \varphi''\Big(\langle e(t,v),\xi\rangle\Big) e(t,v) \otimes e(t,v) d\mathcal{H}^{n-1}(v).$$

Let us fix $(t,\xi) \in \Sigma_+ \times \mathbb{R}^n$ and $w \in S^{n-1}$: to achieve the proof it suffices to show that

$$\big\langle \nabla^2_\xi g^*(t,\xi) w, w \big\rangle > 0.$$

Since

$$\big\langle \nabla^2_\xi g^*(t,\xi) w, w \big\rangle = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \varphi''\Big(\langle e(t,v),\xi\rangle\Big) \langle e(t,v), w\rangle^2 d\mathcal{H}^{n-1}(v),$$

we only have to prove that, for $\mathcal{H}^{n-1}$-a.e. $v \in S^{n-1}$, it cannot be

$$\langle e(t,v), w\rangle = \langle d(t)v + b(t)\mu(t), w\rangle = 0.$$

But if this holds, by continuity of $e(t,\cdot)$, for every $v \in S^{n-1}$, we also have

$$\langle d(t)v + b(t)\mu(t), w\rangle = 0.$$

For $v \in w^\perp$ we deduce that $w \in \mu(t)^\perp$ and then, for $v = w$, we get $w = 0 \notin S^{n-1}$.

The proof of the last property stated in (ii) is simple and can be omitted. □

[5] With $\nabla^2_\xi g(t,\xi)$ we mean the hessian matrix of $g(t,\cdot)$. Moreover $\otimes$ denotes the standard tensorial product between two vectors of $\mathbb{R}^n$.

# Chapter 4

# Lower semicontinuity for integral functionals

## 4.1 A brief historical background

As announced in the introduction, this chapter is devoted to the problem of determining some new conditions sufficient to guarantee the lower semicontinuity, with respect to the $L^1_{\text{loc}}(\Omega)$ convergence, of functionals of the type

$$I(u,\Omega)=\int_\Omega f(x,u(x),\nabla u(x))dx,$$

defined on the Sobolev space $W^{1,1}_{\text{loc}}(\Omega)$. This means to find out conditions on $f$ such that, for every $u_h, u\in W^{1,1}_{\text{loc}}(\Omega)$, $u_h\to u$ in $L^1_{\text{loc}}(\Omega)$, it follows

$$I(u,\Omega)\le\liminf_{h\to\infty} I(u_h,\Omega).$$

In the following the function $f$ will satisfy the usual conditions

$$\begin{cases} f:\Omega\times\mathbb{R}\times\mathbb{R}^n\to[0,\infty], \\ \text{for every } (x,s)\in\Omega\times\mathbb{R},\ f(x,s,\cdot) \text{ is convex in } \mathbb{R}^n. \end{cases} \tag{4.1}$$

One of the first results on this argument is an example due to Aronszajn (see [64] page 54) which shows that conditions (4.1), together with the continuity of $f$, are not sufficient for the lower semicontinuity of $I$. Several years later, Serrin was able to prove the following fundamental theorem in which some sufficient conditions for the lower semicontinuity are presented (see [69] Theorem 12).

**Theorem 4.1.** *Let $f$ be a continuous function satisfying (4.1) and one of the following conditions:*

*(a) for every $(x,s)\in\Omega\times\mathbb{R}$, $\lim_{|\xi|\to\infty} f(x,s,\xi)=\infty$;*

*(b) for every $(x,s)\in\Omega\times\mathbb{R}$, $f(x,s,\cdot)$ is strictly convex in $\mathbb{R}^n$;*

*(c) for every $i,j\in\{1,\dots,n\}$, the derivatives $\frac{\partial f}{\partial x_i}$, $\frac{\partial f}{\partial \xi_j}$ and $\frac{\partial^2 f}{\partial x_i\partial\xi_j}$ exist and are continuous.*

*Then the functional $I$ is l.s.c. on $W^{1,1}_{\text{loc}}(\Omega)$ with respect to the $L^1_{\text{loc}}(\Omega)$ convergence.*

The conditions (a), (b) and (c) quoted above are clearly independent, in the sense that we can find a continuous function $f$ satisfying just one of them, but none of the other ones. However, the proof of Theorem 4.1 is essentially the same for every condition considered; indeed, the proof is based on an approximation theorem for convex functions depending continuously on parameters (see [69] Lemma 5) that can be applied, in particular, when $f$ satisfies (a), (b) or (c). This fact suggests the possibility to find a suitable condition, implied independently by (a), (b) and (c), which is still sufficient for the lower semicontinuity of $I$. This problem, that is still open and seems to be very difficult, is the argument of this chapter.

The program to unify in a unique condition (a), (b) and (c) has to start, first of all, from the analysis of the lower semicontinuity theorems and counterexamples we can find in literature: in this way it could be understood what kind of difficulties we should expect.

The next theorem, again due to Serrin (see [69] Theorem 11), is relevant also because many authors tried to improve it, dealing, in particular, with condition (i).

**Theorem 4.2.** *Let $f$ be a continuous function satisfying (4.1) and one of the following conditions:*

*(i) there exists a modulus of continuity[1] $\omega$ such that, for every $(x_1, s_1), (x_2, s_2) \in \Omega \times \mathbb{R}$, $\xi \in \mathbb{R}^n$,*

$$|f(x_1, s_1, \xi) - f(x_2, s_2, \xi)| \leq \omega(|x_1 - x_2| + |s_1 - s_2|)(1 + f(x_1, s_1, \xi)); \tag{4.2}$$

*(ii) there exist two moduli of continuity $\omega$ and $\sigma$ and a constant $C > 0$ such that, for every $t > t_0$ big enough, we have $\sigma(t) \leq Ct$ and, for every $(x_1, s_1), (x_2, s_2) \in \Omega \times \mathbb{R}$, $\xi \in \mathbb{R}^n$,*

$$|f(x_1, s_1, \xi) - f(x_2, s_2, \xi)| \leq \omega(|x_1 - x_2|)(1 + f(x_1, s_1, \xi)) + \sigma(|s_1 - s_2|). \tag{4.3}$$

*Then, for every $u_h, u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ such that $u_h \to u$ in $L^1_{\mathrm{loc}}(\Omega)$, we have*

$$I(u, \Omega) \leq \liminf_{h \to \infty} I(u_h, \Omega),$$

*assuming in addition, when the case (i) is considered, that $u \in C(\Omega)$.*

The following lower semicontinuity result was proved by Dal Maso (see [30] Theorem 3.2) and just moves in the direction described above: note that the assumption of continuity on the limit function $u$ of Theorem 4.2 is removed[2], while some *coercivity* and *growth conditions* are now introduced.

**Theorem 4.3.** *Let $\Omega$ be a bounded open set and $f$ be a Borel function satisfying (4.1) such that, for $\mathcal{H}^n$-a.e. $(x_0, s_0) \in \Omega \times \mathbb{R}^n$, $f(x_0, s_0, \cdot)$ is continuous on $\mathbb{R}^n$ and, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $(x, s) \in B(x_0, \delta) \times B(s_0, \delta)$, $\xi \in \mathbb{R}^n$,*

$$|f(x, s, \xi) - f(x_0, s_0, \xi)| < \varepsilon(1 + |\xi|). \tag{4.4}$$

*Let us suppose that, for every $r > 0$, there exist $M > 0$, $m, A \in C^0(\Omega)$, with, for every $x \in \Omega$, $m(x) > 0, A(x) \geq 0$, and $a \in L^1(\Omega)$, such that, for every $(x, s, \xi) \in \Omega \times [-r, r] \times \mathbb{R}^n$,*

$$m(x)|\xi| - a(x) \leq f(x, s, \xi) \leq M|\xi| + A(x).$$

*Then, for every $u_h, u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ such that $u_h \to u$ in $L^1_{\mathrm{loc}}(\Omega)$,*

$$I(u, \Omega) \leq \liminf_{h \to \infty} I(u_h, \Omega),$$

*assuming in addition that $u \in L^\infty(\Omega)$.*

[1] A function $\omega$ is said a modulus of continuity if $\omega : (0, \infty) \to (0, \infty)$ and $\lim_{t \to 0} \omega(t) = 0$

[2] In fact Dal Maso was able to extend his analysis to $u \in BV_{\mathrm{loc}}(\Omega)$, also considering sequences of integral functionals which $\Gamma$-*converge*

In the same paper, Dal Maso proposed also a revisited form of the example by Aronszajn quoted above.

A recent extension of Theorems 4.2(i) and 4.3 is due to Fonseca and Leoni (see [43] Theorem 1.1)[3]:

**Theorem 4.4.** *Let $f$ be a Borel function satisfying (4.1) and such that, for every $(x_0, s_0) \in \Omega \times \mathbb{R}$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $(x, s) \in B(x_0, \delta) \times B(s_0, \delta)$ and $\xi \in \mathbb{R}^n$,*

$$f(x_0, s_0, \xi) - f(x, s, \xi) \leq \varepsilon \left(1 + f(x, s, \xi)\right). \tag{4.5}$$

*Then the functional $I$ is l.s.c. on $W^{1,1}_{\text{loc}}(\Omega)$ with respect to the $L^1_{\text{loc}}(\Omega)$ convergence.*

Note that the assumption (4.5) is a kind of lower semicontinuity of $f$ with respect to $(x, s) \in \Omega \times \mathbb{R}$, *uniform* with respect to $\xi \in \mathbb{R}^n$. This condition seems more natural than the continuity conditions (4.2) and (4.4) required in Theorems 4.2(i) and 4.3, because of the following theorem due to Fusco (see [45] Proposition 3.1) in which, dealing with the case $f(x, s, \xi) = a(x)\,|\xi|$, the author points out that a necessary condition for the lower semicontinuity of the functional $I$ is the lower semicontinuity of the function $a$.

**Theorem 4.5.** *Let $\Omega = (0, 1)$. Let $f : \Omega \times \mathbb{R} \to [0, \infty)$ defined, for every $(x, \xi) \in \Omega \times \mathbb{R}$, as $f(x, \xi) = a(x)\,|\xi|$, where $a : \Omega \to [0, \infty)$ is a bounded Borel function. Then, for every $u \in W^{1,1}_{\text{loc}}(\Omega)$,*

$$R\left[L^1_{\text{loc}}\right](I)\,(u, \Omega) = \int_\Omega \overline{a}(x)\,|u'(x)|\;dx\,, \tag{4.6}$$

*where $\overline{a}(x) = \sup\Big\{b(x) : b(x) \leq a(x)$ for $\mathcal{L}^1$-a.e. $x \in \Omega,\ b$ is l.s.c. on $\Omega\Big\}$. Thus in particular*

$$I\,(u, \Omega) = \int_\Omega a(x)\,|u'(x)|\,dx$$

*is l.s.c on $W^{1,1}_{\text{loc}}(\Omega)$ with respect to the $L^1_{\text{loc}}(\Omega)$ convergence if and only if $a$ is l.s.c. on $\Omega$ (that is, there exists a l.s.c. function $\bar{a} : \Omega \to [0, \infty)$ such that, for $\mathcal{L}^1$-a.e. $x \in \Omega$, $a(x) = \bar{a}(x)$).*

Some researches had the aim to weaken the assumptions on $f$ related to the dependence on $s$. The next result, due to De Giorgi, Buttazzo and Dal Maso (see [37] Theorem 1), was the starting point of this kind of results: here, roughly speaking, it is proved that, when the dependence on $x$ is dropped, the lower semicontinuity of $I$ is *always* verified every time $f$ is convex on the gradient variable and it is regular enough to allow the composition $f(u(x), \nabla u(x))$ to be measurable.

**Theorem 4.6.** *Let $f : \mathbb{R} \times \mathbb{R}^n \to [0, \infty]$ be a Borel function such that, for every $s \in \mathbb{R}$, $f(s, \cdot)$ is convex on $\mathbb{R}^n$, for every $\xi \in \mathbb{R}^n$, $f(\cdot, \xi)$ is measurable on $\mathbb{R}$ and, in particular, $f(\cdot, 0)$ is l.s.c. on $\mathbb{R}$. If*

$$\limsup_{|\xi| \to 0} \frac{\left(f(s, 0) - f(s, \xi)\right)^+}{|\xi|} \in L^1_{\text{loc}}(\mathbb{R}), \tag{4.7}$$

*then the functional $I$ is l.s.c. on $W^{1,1}_{\text{loc}}(\Omega)$ with respect to the $L^1_{\text{loc}}(\Omega)$ convergence.*

Theorem 4.6 was generalized by Ambrosio [4], and subsequently by De Cicco [31], [32] to the $BV$ setting. In 1999 Fonseca and Leoni (see [43] Theorem 1.5) obtained the same conclusion of Theorem 4.6 for integrands $f$, depending explicitly on the $x$ variable too, under an assumption of *continuity* of $f$ on $x \in \Omega$ uniform with respect to $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$ (similar to (4.5)), together with a condition like (4.7) in which the variable $x$ is also considered.

[3] Also in this case the authors are able to consider $u \in BV_{\text{loc}}(\Omega)$: we quoted again the particular case when $u_h, u \in W^{1,1}_{\text{loc}}(\Omega)$, for a better comparison with the other results presented.

At this point it should be clear that the dependence of $f$ on $x$ must be treated carefully in studying the lower semicontinuity of $I$ with respect to the $L^1_{\rm loc}(\Omega)$ convergence. Of course the dependence on $x$ gives difficulties not only technical because of the existence of Aronszajn's counterexample (in which $f$ just depends on $x$). In particular, *measurability* of $f$ with respect to $x$ is not appropriate thanks to Theorem 4.5.

In the paper of Gori and Marcellini [55] (see also Gori [50]) it was specifically considered the dependence of $f$ with respect to $x \in \Omega$. In the results quoted above some qualified assumptions of continuity, or lower semicontinuity, of $f$ with respect to $x$ *uniformly* on $\xi \in \mathbb{R}^n$ were supposed: in [55] (see Theorem 1.6) it was proposed a new *local* condition that, in addition to the continuity of $f$ and (4.1), it is sufficient for the lower semicontinuity.

**Theorem 4.7.** *Let $f$ be a continuous function satisfying (4.1) and such that, for every open set $\Omega' \times H \times K \subset\subset \Omega \times \mathbb{R} \times \mathbb{R}^n$ there exists a constant $L = L_{\Omega' \times H \times K}$ such that, for every $x_1, x_2 \in \Omega'$, $s \in H$ and $\xi \in K$,*

$$|f(x_1, s, \xi) - f(x_2, s, \xi)| \leq L\,|x_1 - x_2|\,. \tag{4.8}$$

*Then the functional $I$ is l.s.c. on $W^{1,1}_{\rm loc}(\Omega)$ with respect to the $L^1_{\rm loc}(\Omega)$ convergence.*

Condition (4.8) means that $f$ is locally Lipschitz continuous with respect to $x$, *locally* with respect to $(s, \xi)$ and not necessarily *globally*, that is, the Lipschitz constant is not uniform for $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$. This allows us to obtain, as a corollary, an improvement of Serrin's Theorem 4.1(c) since, when only the gradient[4] $\nabla_x f$ exists and is continuous, this implies the Lipschitz continuity of $f$ with respect to $x$ on the compact subsets of $\Omega \times \mathbb{R} \times \mathbb{R}^n$ (even if not necessarily Lipschitz continuity of $f$ on the full set $\Omega \times \mathbb{R} \times \mathbb{R}^n$): Theorem 4.7 can be considered the first strict generalization of Theorem 4.1.

Moreover Theorem 4.7 underlines that hypothesis (c) in Theorem 4.1 seems to have a different nature than hypotheses (a) and (b): indeed, roughly speaking, what assumption (c) really contains is a regularity requirement in the geometric variable $x$, while (a) and (b) carry on some geometric constraints on the convexity of $f$ in the gradient variable $\xi$.

Thus, in order to unify all the conditions (a), (b) and (c), we could try to understand if condition (4.8) of Theorem 4.7 can be further improved and if hypotheses (a) and (b) come from the same source: we give an answer to these questions in the following section.

## 4.2 Statement of the main theorem

Before stating the main result of this chapter, it is suitable to make some remarks. Let us note that the condition (4.8) of Theorem 4.7 can be formulated in the following equivalent way: for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$,

$$f(\cdot, s, \xi) \in W^{1,\infty}_{\rm loc}(\Omega),$$

and, for every $\Omega' \times H \times K \subset\subset \Omega \times \mathbb{R} \times \mathbb{R}^n$ there exists a constant $L = L_{\Omega' \times H \times K}$ such that, for every $(s, \xi) \in H \times K$,

$$\operatorname*{ess\,sup}_{x \in \Omega'} |\nabla_x f(x, s, \xi)| \leq L.$$

This way to read the condition (4.8) provides the right point of view to find a generalization of Theorem 4.7. Indeed, a new sufficient condition can be found looking for a suitable *summability condition* on the weak derivatives $\nabla_x f$, rather than a qualified continuity assumption as, for instance, Hölder continuity in the $x$ variable, that, as we will see in the following, it is not a sufficient condition for the lower semicontinuity (see Theorem 4.13).

Always referring to Theorem 4.1, it is worth noting that one of the main advantages of (c) (and its generalizations) with respect to (a) and (b) is that this condition allows also to treat integrands

[4] With $\nabla_x f(x, s, \xi)$ we mean the (in case weak) gradient of $f(\cdot, s, \xi)$. As already said, an analogous definition holds for $\nabla_\xi f(x, s, \xi)$ too.

$f$ that are constant in the gradient variable for some pair $(x, s)$. The common root of (a) and (b) is just this one: they are hypotheses assumed to exclude the cases in which $f(x, s, \cdot)$ can be constant on straight lines, that is (a) and (b) assure that $f(x, s, \cdot)$ satisfies what we called the n.c.s.l. property in the gradient variable, condition that we know to be equivalent to the notion of demi-coercivity (see Theorem 2.34).

Theorem 4.8, that of course generalizes Theorems 4.1 and 4.7, is the main result of this chapter and it follows by the researches carried on by Gori, Maggi and Marcellini (see [54] Theorem 1.2) and Gori and Maggi (see [53] Theorem 6): here the remarks just made take the form of a precise statement.

**Theorem 4.8.** *Let $f$ be a continuous function satisfying (4.1) and one of the following conditions:*

*(1) for every $(s, \xi)$, $f(\cdot, s, \xi) \in W^{1,1}_{\mathrm{loc}}(\Omega)$, and, for every open set $\Omega' \times H \times K \subset\subset \Omega \times \mathbb{R} \times \mathbb{R}^n$, there exists a constant $L = L_{\Omega' \times H \times K}$ such that, for every $(s, \xi) \in H \times K$,*

$$\int_{\Omega'} |\nabla_x f(x, s, \xi)| dx \le L;$$

*(2) for every $(x_0, s_0) \in \Omega \times \mathbb{R}$, it is that either, for every $\xi \in \mathbb{R}^n$,*

$$f(x_0, s_0, \xi) \equiv \inf \{ f(x, s, \xi) : (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \},$$

*or there exists $\delta > 0$ such that, for every $(x, s) \in B(x_0, \delta) \times B(s_0, \delta)$,*

$$f(x, s, \cdot) \text{ is demi-coercive on } \mathbb{R}^n.$$

*Then the functional $I$ is l.s.c. on $W^{1,1}_{\mathrm{loc}}(\Omega)$ with respect to the $L^1_{\mathrm{loc}}(\Omega)$ convergence.*

We point out that Theorem 4.8(1) is a less general version of Theorem 1.2 proved in [54]: we prefer to present here this simplified result since, assuming $f$ continuous instead of Carathéodory and locally bounded function, many technical difficulties can be avoided and the fundamental ideas of the proof can be easier understood. Clearly we refer to the original paper [54] for further details.

Finally, the following simple and useful corollary holds.

**Corollary 4.9.** *Let $f$ be a continuous function satisfying (4.1) and one of the following conditions:*

*(1) $\nabla_x f$ exists and is continuous,*

*(2) for every $(x, s) \in \Omega \times \mathbb{R}$, $f(x, s, \cdot)$ is demi-coercive on $\mathbb{R}^n$.*

*Then the functional $I$ is l.s.c. on $W^{1,1}_{\mathrm{loc}}(\Omega)$ with respect to the $L^1_{\mathrm{loc}}(\Omega)$ convergence.*

## 4.3 Proof of the main theorem

We start presenting a simple proposition that is useful to treat lower semicontinuity problems for integral functionals. We state it in a very particular form that can be easily generalized to different contests.

**Proposition 4.10.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty]$ be a Borel function. Let us consider, for every $B \in \mathcal{B}(\Omega)$, $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, the functional*

$$I(u, B) = \int_B f(x, u(x), \nabla u(x)) dx,$$

*and suppose that, for every $B \subset\subset \Omega$ open ball, $I(\cdot, B)$ is l.s.c. on $W^{1,1}(B)$ with respect to the $L^1(B)$ convergence. Then $I(\cdot, \Omega)$ is l.s.c. on $W^{1,1}_{\mathrm{loc}}(\Omega)$ with respect to the $L^1_{\mathrm{loc}}(\Omega)$ convergence.*

*Proof.* By a simple modification of Corollary 2 page 28 in [40] we can find a sequence $\{B_i\}_{i=1}^{\infty}$ of disjoint open balls well contained in $\Omega$, such that

$$\mathcal{L}^n\left(\Omega \setminus \bigcup_{i=1}^{\infty} B_i\right) = 0.$$

Let us consider now $u_h, u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, $u_h \to u$ in $L^1_{\mathrm{loc}}(\Omega)$. Clearly, for every $i \in \mathbb{N}$, we have also $u_h, u \in W^{1,1}(B_i)$, $u_h \to u$ in $L^1(B_i)$, and then, fixed $k \in \mathbb{N}$,

$$I\left(u, \bigcup_{i=1}^{k} B_i\right) = \sum_{i=1}^{k} I(u, B_i) \le \sum_{i=1}^{k} \liminf_{h\to\infty} I(u_h, B_i)$$

$$\le \liminf_{h\to\infty} \sum_{i=1}^{k} I(u_h, B_i) \le \liminf_{h\to\infty} \sum_{i=1}^{\infty} I(u_h, B_i) = \liminf_{h\to\infty} I(u_h, \Omega),$$

where we have used the fact that $f$ is non negative. Letting now $k \to \infty$ the lower semicontinuity inequality is achieved. □

In particular this proposition shows that, without loss of generality, in the following it will be always allowed to consider $\Omega$ bounded with Lipschitz boundary and prove the lower semicontinuity of $I$ on $W^{1,1}(\Omega)$ with respect to the $L^1(\Omega)$ convergence.

### 4.3.1 The condition on the geometric variable

The proof of Theorem 4.8(1) is structured as follows. We first prove a lower semicontinuity result, namely Lemma 4.11, under certain technical hypotheses: in particular we show the validity of a chain rule involving Sobolev functions (see equation (4.18)) that represents the core of the lemma. Subsequently, combining Lemma 4.11 and Theorem 3.1, we achieve the proof of Theorem 4.8(1). As already stated, the proofs provided here are a slightly different and simplified version of the original ones presented in [54].

Before proving the theorem, in order to complete the description of the lower semicontinuity results involving conditions on the geometric variable, we have to remember that recently Fusco, Giannetti and Verde [46], De Cicco, Fusco and Verde [33] and De Cicco and Leoni [34] proposed several generalizations of Theorem 4.8(1) (or, better, of the more general Theorem 1.2 in [54]).

In [46] and [33] the authors proved in particular that the same set of hypotheses on $f$ given in Theorem 4.8(1) is still sufficient to prove also the lower semicontinuity, with respect to the $L^1_{\mathrm{loc}}(\Omega)$ convergence, of the standard integral functional, proposed by Dal Maso [30], that extends $I$ on $BV_{\mathrm{loc}}(\Omega)$. In [34] instead, by means of a very sophisticated chain rule that is the main result of the paper, more general conditions are found in order to obtain the lower semicontinuity of $I$ on $W^{1,1}_{\mathrm{loc}}(\Omega)$.

However, it is worth noting that all these results, as Theorem 4.7 and of course Theorem 4.8(1) too, are proved by using, as a crucial step, the possibility to approximate $f$ with the affine functions built up by means of Theorem 3.1.

This fact underlines the power of Theorem 3.1 in studying the lower semicontinuity with respect to the $L^1_{\mathrm{loc}}(\Omega)$ convergence of functionals in which the integrand $f$ satisfies some regularity conditions on the geometric variable.

Let us prove now the first step of the proof of Theorem 4.8(1) (compare it with Lemma 4.1 in [54]).

**Lemma 4.11.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty)$ be a continuous function satisfying (4.1) and such that:*

*(i) there exists an open set* $\Omega' \times H \subset\subset \Omega \times \mathbb{R}$ *such that, for every* $(x, s, \xi) \in (\Omega \setminus \Omega') \times (\mathbb{R} \setminus H) \times \mathbb{R}^n$, $f(x, s, \xi) = 0$*,*

*(ii)* $\nabla_\xi f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *exists and is continuous. Moreover, for every* $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$, $\nabla_\xi f(\cdot, s, \xi) \in W^{1,1}(\Omega, \mathbb{R}^n)$ *and, for every* $K \subset\subset \mathbb{R}^n$*, there exists* $L = L_K$ *such that, for every* $(s, \xi) \in \mathbb{R} \times K$*,*

$$\sum_{i=1}^{n} \int_\Omega \left| \frac{\partial \nabla_\xi f}{\partial x_i}(x, s, \xi) \right| dx \leq L, \tag{4.9}$$

*(iii) there exists a constant* $M$ *such that, for every* $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$*,*

$$|\nabla_\xi f(x, s, \xi)| \leq M, \tag{4.10}$$

*and, for every* $(x, s) \in \Omega \times \mathbb{R}$*,* $\xi_1, \xi_2 \in \mathbb{R}^n$*,*

$$|\nabla_\xi f(x, s, \xi_1) - \nabla_\xi f(x, s, \xi_2)| \leq M \, |\xi_1 - \xi_2| \,. \tag{4.11}$$

*Then the functional* $I$ *is l.s.c. on* $W^{1,1}(\Omega)$ *with respect to the* $L^1(\Omega)$ *convergence.*

*Proof.* Let $u_h, u \in W^{1,1}(\Omega)$ such that $u_h \to u$ in $L^1(\Omega)$. We will prove that

$$\liminf_{h \to \infty} I(u_h, \Omega) \geq I(u, \Omega). \tag{4.12}$$

Without loss of generality, we can assume that

$$\liminf_{h \to \infty} I(u_h, \Omega) = \lim_{h \to \infty} I(u_h, \Omega) < \infty,$$

and that $u_h$ converges almost everywhere to $u$ in $\Omega$. Moreover, by (4.10) and (4.11), there exists a constant $M'$ such that, for every $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$,

$$|f(x, s, \xi)| \leq M'(1 + |\xi|),$$

thus, in particular, we have $I(u, \Omega) < \infty$.

Since $u \in W^{1,1}(\Omega)$ and since $\partial\Omega$ is supposed Lipschitz, fixed $\varepsilon > 0$, by Theorem 2.10, there exists $v_\varepsilon \in \mathrm{Aff}(\Omega)$ such that

$$\int_\Omega |\nabla u(x) - \nabla v_\varepsilon(x)| dx \leq \varepsilon, \tag{4.13}$$

and by Fatou's Lemma and the finiteness of $I(u, \Omega)$, we can also choose $v_\varepsilon$ such that

$$\int_\Omega f(x, u(x), \nabla v_\varepsilon(x)) dx \geq \int_\Omega f(x, u(x), \nabla u(x)) dx - \varepsilon. \tag{4.14}$$

Since $v_\varepsilon \in \mathrm{Aff}(\Omega)$, we can find $\{\Omega_j\}_{j=1}^N$ disjoint open subsets of $\Omega$ such that $\mathcal{L}^n \left( \Omega \setminus \bigcup_{j=1}^N \Omega_j \right) = 0$ and $\{\xi_j\}_{j=1}^N \subseteq \mathbb{R}^n$ such that $\nabla v_\varepsilon(x) = \xi_j$ when $x \in \Omega_j$.

Now let us take a sequence of non negative functions $\{\beta_k^\varepsilon\}_{k=1}^\infty \subseteq C_c^\infty(\Omega)$ such that, for every $k \in \mathbb{N}$, $j \in \{1, \ldots, N\}$, $\beta_k^\varepsilon \in C_c^\infty(\Omega_j)$ and, for $\mathcal{L}^n$-a.e. $x \in \Omega$, $\beta_k^\varepsilon(x) \uparrow 1$. By Beppo Levi's Theorem, there exists an index $k_\varepsilon$ such that, for every $k \geq k_\varepsilon$, we have

$$\int_\Omega \beta_k^\varepsilon(x) f(x, u(x), \nabla v_\varepsilon(x)) dx \geq \int_\Omega \beta_k^\varepsilon(x) f(x, u(x), \nabla u(x)) dx - 2\varepsilon. \tag{4.15}$$

We write the difference of the integrands in (4.12) in this way

$$f(x, u_h(x), \nabla u_h(x)) - f(x, u(x), \nabla u(x)) = f(x, u_h(x), \nabla u_h(x)) - f(x, u_h(x), \nabla v_\varepsilon(x)) \tag{4.16}$$

$$+f(x,u_h(x),\nabla v_\varepsilon(x)) - f(x,u(x),\nabla v_\varepsilon(x)) + f(x,u(x),\nabla v_\varepsilon(x)) - f(x,u(x),\nabla u(x)).$$

By the convexity of $f(x,s,\xi)$ with respect to $\xi$ we have

$$f(x,u_h(x),\nabla u_h(x)) - f(x,u_h(x),\nabla v_\varepsilon(x)) \geq \langle \nabla_\xi f(x,u_h(x),\nabla v_\varepsilon(x)), \nabla u_h(x) - \nabla v_\varepsilon(x)\rangle$$

$$= \langle \nabla_\xi f(x,u_h(x),\nabla v_\varepsilon(x)), \nabla u_h(x)\rangle - \langle \nabla_\xi f(x,u(x),\nabla v_\varepsilon(x)), \nabla u(x)\rangle$$

$$+\langle \nabla_\xi f(x,u(x),\nabla v_\varepsilon(x)), \nabla u(x) - \nabla v_\varepsilon(x)\rangle + \langle \nabla_\xi f(x,u(x),\nabla v_\varepsilon(x)) - \nabla_\xi f(x,u_h(x),\nabla v_\varepsilon(x)), \nabla v_\varepsilon(x)\rangle.$$

Using the inequality just found together with (4.16), multiplying for $\beta_k^\varepsilon$ and integrating over $\Omega$, we obtain

$$\int_\Omega \beta_k^\varepsilon(x)\Big\{ f(x,u_h(x),\nabla u_h(x)) - f(x,u(x),\nabla u(x))\Big\}dx$$

$$\geq \int_\Omega \beta_k^\varepsilon(x)\Big\{ \langle \nabla_\xi f(x,u_h(x),\nabla v_\varepsilon(x)), \nabla u_h(x)\rangle - \langle \nabla_\xi f(x,u(x),\nabla v_\varepsilon(x)), \nabla u(x)\rangle\Big\}dx$$

$$+\int_\Omega \beta_k^\varepsilon(x)\langle \nabla_\xi f(x,u(x),\nabla v_\varepsilon(x)), \nabla u(x) - \nabla v_\varepsilon(x)\rangle dx$$

$$+\int_\Omega \beta_k^\varepsilon(x)\langle \nabla_\xi f(x,u(x),\nabla v_\varepsilon(x)) - \nabla_\xi f(x,u_h(x),\nabla v_\varepsilon(x)), \nabla v_\varepsilon(x)\rangle dx$$

$$+\int_\Omega \beta_k^\varepsilon(x)\Big\{ f(x,u_h(x),\nabla v_\varepsilon(x)) - f(x,u(x),\nabla v_\varepsilon(x))\Big\}dx$$

$$+\int_\Omega \beta_k^\varepsilon(x)\Big\{ f(x,u(x),\nabla v_\varepsilon(x)) - f(x,u(x),\nabla u(x))\Big\}dx.$$

We remember that, by (4.10), for every $(x,s)\in\Omega\times\mathbb{R}$ and for every $v_\varepsilon$,

$$|\nabla_\xi f(x,s,\nabla v_\varepsilon(x))| \leq M;$$

then, by (4.13), we have

$$\int_\Omega \beta_k^\varepsilon(x)\langle \nabla_\xi f(x,u(x),\nabla v_\varepsilon(x)), \nabla u(x) - \nabla v_\varepsilon(x)\rangle dx \geq -M\int_\Omega |\nabla u(x) - \nabla v_\varepsilon(x)|\, dx \geq -M\varepsilon.$$

Moreover, being $(x,s)\to\beta_k^\varepsilon(x)f(x,s,\nabla v_\varepsilon(x))$ and $(x,s)\to\beta_k^\varepsilon(x)\nabla_\xi f(x,s,\nabla v_\varepsilon(x))$ continuous and bounded functions, by the Lebesgue's dominated convergence theorem we have,

$$\lim_{h\to\infty}\int_\Omega \beta_k^\varepsilon(x)\Big\{ f(x,u_h,\nabla v_\varepsilon) - f(x,u,\nabla v_\varepsilon)\Big\}dx = 0,$$

$$\lim_{h\to\infty}\int_\Omega \beta_k^\varepsilon(x)\langle \nabla_\xi f(x,u,\nabla v_\varepsilon) - \nabla_\xi f(x,u_h,\nabla v_\varepsilon), \nabla v_\varepsilon\rangle dx = 0,$$

and in conclusion, by means of (4.15), we obtain that, for every $\varepsilon > 0$ and for every $v_\varepsilon$, $k \geq k_\varepsilon$,

$$\liminf_{h\to\infty}\int_\Omega \beta_k^\varepsilon(x)\Big\{ f(x,u_h(x),\nabla u_h(x)) - f(x,u(x),\nabla u(x))\Big\}dx$$

$$\geq \liminf_{h\to\infty}\int_\Omega \beta_k^\varepsilon(x)\Big\{ \langle \nabla_\xi f(x,u_h(x),\nabla v_\varepsilon(x)), \nabla u_h(x)\rangle - \langle \nabla_\xi f(x,u(x),\nabla v_\varepsilon(x)), \nabla u(x)\rangle\Big\}dx - M\varepsilon - 2\varepsilon.$$

Then we claim that the proof is complete if we show that, for every fixed $v_\varepsilon$ and $k \geq k_\varepsilon$, we have

$$\lim_{h\to\infty}\int_\Omega \beta_k^\varepsilon(x)\Big\{ \langle \nabla_\xi f(x,u_h(x),\nabla v_\varepsilon(x)), \nabla u_h(x)\rangle - \langle \nabla_\xi f(x,u(x),\nabla v_\varepsilon(x)), \nabla u(x)\rangle\Big\}dx = 0. \quad (4.17)$$

Indeed, since $0 \le \beta_k^\varepsilon(x) \le 1$, we shall have that, for every $k \ge k_\varepsilon$,

$$\liminf_{h\to\infty} \int_\Omega f(x, u_h(x), \nabla u_h(x))dx \ge \int_\Omega \beta_k^\varepsilon(x) f(x, u(x), \nabla u(x))dx - M\varepsilon - 2\varepsilon,$$

and letting $k \to \infty$, by Beppo Levi's Theorem, we obtain

$$\liminf_{h\to\infty} \int_\Omega f(x, u_h(x), \nabla u_h(x))dx \ge \int_\Omega f(x, u(x), \nabla u(x))dx - M\varepsilon - 2\varepsilon.$$

Now the dependence from $v_\varepsilon$ is vanished so that letting $\varepsilon \to 0$ we gain (4.12).

Thus it remains to prove (4.17): we stress that it suffices to prove this relation when $v_\varepsilon$ and $k$ are fixed. In order to achieve this we prove that[5]

$$\int_\Omega \beta_k^\varepsilon(x)\Big\{\langle \nabla_\xi f(x, u_h(x), \nabla v_\varepsilon(x)), \nabla u_h(x)\rangle - \langle \nabla_\xi f(x, u(x), \nabla v_\varepsilon(x)), \nabla u(x)\rangle\Big\}dx$$

$$= -\int_\Omega \sum_{i=1}^n \left\{ \int_{u(x)}^{u_h(x)} \frac{\partial}{\partial x_i}\Big(\beta_k^\varepsilon(x)\nabla_\xi f(x, s, \nabla v_\varepsilon(x))\Big)^{(i)} ds \right\} dx. \tag{4.18}$$

In order to prove this, fixed $j \in \{1, \dots, N\}$, let us consider the continuous function

$$g_j(x, s) = \beta_k^\varepsilon(x)\nabla_\xi f(x, s, \xi_j) : \Omega_j \times \mathbb{R} \to \mathbb{R}^n,$$

with $\mathrm{spt}(g_j) \subseteq \Omega_j \times H$, that we can considered defined on the whole space $\mathbb{R}^n \times \mathbb{R}$ (extending $g_j$ to zero out of $\Omega_j \times H$). For $\rho > 0$ we define[6]

$$g_{j,\rho}(x, s) = \int_{B(x,\rho)} k_\rho(x - y) g_j(y, s)dy : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n.$$

Clearly, for every $\rho > 0$, $g_{j,\rho} \in C_c(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ and, if $\rho < \rho_j$ small enough, then $\mathrm{spt}(g_{j,\rho}) \subseteq \Omega_j \times H$ too. Moreover, by the properties of convolutions we have also that, for every $s \in \mathbb{R}$, $g_{j,\rho}(\cdot, s) \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

We claim now that there exists $C_{j,\rho} > 0$ such that, for every $x, y \in \mathbb{R}^n$, $s \in \mathbb{R}$,

$$|g_{j,\rho}(x, s) - g_{j,\rho}(y, s)| \le C_{j,\rho}|x - y|. \tag{4.19}$$

Indeed, for every $i \in \{1, \dots, n\}$, $(x, s) \in \mathbb{R}^n \times \mathbb{R}$, we have

$$\left|\frac{\partial g_{j,\rho}}{\partial x_i}(x, s)\right| = \left|\int_{B(x,\rho)} k_\rho(x - y)\frac{\partial g_j}{\partial x_i}(y, s)dy\right| \le \rho^{-n} \int_{\mathbb{R}^n} \left|\frac{\partial g_j}{\partial x_i}(x, s)\right| dx$$

$$\le \rho^{-n}\left(\sup_{x\in\Omega} |\nabla\beta_k^\varepsilon(x)|\right) \int_{\mathbb{R}^n} |\nabla_\xi f(x, s, \xi_j)|\, dx + \rho^{-n} \int_{\mathbb{R}^n} \left|\frac{\partial \nabla_\xi f}{\partial x_i}(x, s, \xi_j)\right| dx$$

$$\le \rho^{-n} M \left(\sup_{x\in\Omega} |\nabla\beta_k^\varepsilon(x)|\right) \mathcal{L}^n(\Omega_j') + \rho^{-n} L_{K_j} = \rho^{-n} L_j, \tag{4.20}$$

where we have used the properties (i), (4.9) with constant $L_{K_j}$ where $K_j = \{\xi_j\}$ and (4.10): this obviously implies (4.19).

It is worth noting that by the previous inequality we can deduce that, for every $s \in \mathbb{R}$,

$$\int_{\mathbb{R}^n} \left|\frac{\partial g_j}{\partial x_i}(x, s)\right| dx \le L_j, \tag{4.21}$$

[5]Till the end of the proof with the notation $\xi^{(i)}$ we will denote the $i$-th component of the vector $\xi \in \mathbb{R}^n$.

[6]We refer to the notations for the convolutions introduced in Chapter 2.

and then, by means of the properties of the convolutions with respect to the $L^1$ norm, for every $\rho > 0$, $s \in \mathbb{R}$, we have also

$$\int_{\mathbb{R}^n} \left| \frac{\partial g_{j,\rho}}{\partial x_i}(x,s) \right| dx \leq L_j. \tag{4.22}$$

The validity of (4.19) implies, by standard arguments[7], for every $\rho < \rho_j$, $u, v \in W^{1,1}(\Omega_j)$, the equality

$$\int_{\Omega_j} \langle g_{j,\rho}(x, v(x)), \nabla v(x)\rangle - \langle g_{j,\rho}(x, u(x)), \nabla u(x)\rangle dx = -\int_{\Omega_j} \left\{ \sum_{i=1}^n \int_{u(x)}^{v(x)} \frac{\partial g^{(i)}_{j,\rho}}{\partial x_i}(x,s) ds \right\} dx.$$

Now we prove that, passing to the limit as $\rho \to 0$ in the previous formula we can obtain the same relation for $g_j$ too. In order to prove the equality

$$\lim_{\rho\to 0} \int_{\Omega_j} \langle g_{j,\rho}(x, v(x)), \nabla v(x)\rangle dx = \int_{\Omega_j} \langle g_j(x, v(x)), \nabla v(x)\rangle dx,$$

since the boundedness of $g$ and the summability of $\nabla v$, we only need to prove that

$$\lim_{\rho\to 0} g_{j,\rho}(x, v(x)) = g_j(x, v(x)), \tag{4.23}$$

for $\mathcal{L}^n$-a.e. $x \in \Omega_j$: but this easily follows by the property of the convolutions together with the continuity of $g_j$. Thus it remains to prove that, for every $i \in \{1, \dots, n\}$,

$$\lim_{\rho\to 0} \int_{\Omega_j} \left\{ \int_{u(x)}^{v(x)} \frac{\partial g^{(i)}_{j,\rho}}{\partial x_i}(x,s) ds \right\} dx = \int_{\Omega_j} \left\{ \int_{u(x)}^{v(x)} \frac{\partial g^{(i)}_{j}}{\partial x_i}(x,s) ds \right\} dx.$$

Indeed

$$\int_{\Omega_j} \left| \int_{u(x)}^{v(x)} \frac{\partial g^{(i)}_{j,\rho}}{\partial x_i}(x,s) ds - \int_{u(x)}^{v(x)} \frac{\partial g^{(i)}_{j}}{\partial x_i}(x,s) ds \right| dx \leq \int_{\Omega_j} \left\{ \int_H \left| \frac{\partial g^{(i)}_{j,\rho}}{\partial x_i}(x,s) - \frac{\partial g^{(i)}_{j}}{\partial x_i}(x,s) \right| ds \right\} dx$$

$$= \int_H \left\{ \int_{\Omega_j} \left| \frac{\partial g^{(i)}_{j,\rho}}{\partial x_i}(x,s) - \frac{\partial g^{(i)}_{j}}{\partial x_i}(x,s) \right| dx \right\} ds,$$

that goes to zero as $\rho \to 0$ since, by (ii) and the standard properties of the convolutions, we have, for every fixed $s \in H$,

$$\lim_{\rho\to 0} \int_{\Omega_j} \left| \frac{\partial g^{(i)}_{j,\rho}}{\partial x_i}(x,s) - \frac{\partial g^{(i)}_{j}}{\partial x_i}(x,s) \right| dx = 0,$$

and moreover, using (4.21) and (4.22), it is, for every $\rho > 0$, $s \in H$,

$$\int_{\Omega_j} \left| \frac{\partial g^{(i)}_{j,\rho}}{\partial x_i}(x,s) - \frac{\partial g^{(i)}_{j}}{\partial x_i}(x,s) \right| dx \leq \int_{\Omega_j} \left| \frac{\partial g^{(i)}_{j,\rho}}{\partial x_i}(x,s) \right| dx + \int_{\Omega_j} \left| \frac{\partial g^{(i)}_{j}}{\partial x_i}(x,s) \right| dx \leq 2L_j,$$

that is, it is bounded on $H$ uniformly on $\rho$: then we are in the position to apply the Lebesgue's dominated convergence Theorem. We conclude that, for every $j \in \{1, \dots, N\}$, $u, v \in W^{1,1}(\Omega_j)$, also

$$\int_{\Omega_j} \langle g_j(x, v(x)), \nabla v(x)\rangle - \langle g_j(x, u(x)), \nabla u(x)\rangle dx = -\int_{\Omega_j} \left\{ \sum_{i=1}^n \int_{u(x)}^{v(x)} \frac{\partial g^{(i)}_{j}}{\partial x_i}(x,s) ds \right\} dx.$$

[7] The following relation can be proved considering at first $u, v \in C^\infty(\Omega_j) \cap W^{1,1}(\Omega_j)$ and then arguing by approximation.

Hence (4.18) immediately follows since

$$\int_\Omega \beta_k^\varepsilon(x)\Big\{\langle\nabla_\xi f(x,u_h(x),\nabla v_\varepsilon(x)),\nabla u_h(x)\rangle-\langle\nabla_\xi f(x,u(x),\nabla v_\varepsilon(x)),\nabla u(x)\rangle\Big\}dx$$

$$=\sum_{j=1}^N\int_{\Omega_j}\langle g_j(x,v(x)),\nabla v(x)\rangle-\langle g_j(x,u(x)),\nabla u(x)\rangle dx=-\sum_{j=1}^N\int_{\Omega_j}\left\{\sum_{i=1}^n\int_{u(x)}^{v(x)}\frac{\partial g_j^{(i)}}{\partial x_i}(x,s)ds\right\}dx$$

$$=-\int_\Omega\sum_{i=1}^n\left\{\int_{u(x)}^{u_h(x)}\frac{\partial}{\partial x_i}\Big(\beta_k^\varepsilon(x)\nabla_\xi f(x,s,\nabla v_\varepsilon(x))\Big)^{(i)}ds\right\}dx.$$

Now by (4.18) we have can prove (4.17): indeed

$$\left|\int_\Omega \beta_k^\varepsilon(x)\Big\{\langle\nabla_\xi f(x,u_h(x),\nabla v_\varepsilon(x)),\nabla u_h(x)\rangle-\langle\nabla_\xi f(x,u(x),\nabla v_\varepsilon(x)),\nabla u(x)\rangle\Big\}dx\right|$$

$$\leq\sum_{i=1}^n\sum_{j=1}^N\int_{\Omega_j}\left|\int_{u(x)}^{u_h(x)}\left|\frac{\partial\beta_k^\varepsilon}{\partial x_i}(x)\right||\nabla_\xi f(x,s,\xi_j)|\,ds\right|dx+\sum_{i=1}^n\sum_{j=1}^N\int_{\Omega_j}\left|\int_{u(x)}^{u_h(x)}\left|\frac{\partial\nabla_\xi f}{\partial x_i}(x,s,\xi_j)\right|ds\right|dx.$$

Using (4.10), we have

$$\sum_{i=1}^n\sum_{j=1}^N\int_{\Omega_j}\left|\int_{u(x)}^{u_h(x)}\left|\frac{\partial\beta_k^\varepsilon}{\partial x_i}(x)\right||\nabla_\xi f(x,s,\xi_j)|\,ds\right|dx\leq nM\left(\sup_{x\in\Omega}|\nabla\beta_k^\varepsilon(x)|\right)\int_\Omega|u_h(x)-u(x)|\,dx,$$

which goes to zero as $h\to\infty$. On the other hand,

$$\sum_{i=1}^n\sum_{j=1}^N\int_{\Omega_j}\left|\int_{u(x)}^{u_h(x)}\left|\frac{\partial\nabla_\xi f}{\partial x_i}(x,s,\xi_j)\right|ds\right|dx\leq\sum_{i=1}^n\sum_{j=1}^N\int_{D_{j,h}}\left|\frac{\partial\nabla_\xi f}{\partial x_i}(x,s,\xi_j)\right|dxds,$$

where

$$D_{j,h}=\Big\{(x,s)\in\Omega_j\times H:\min\{u_h(x),u(x)\}\leq s\leq\max\{u_h(x),u(x)\}\Big\}.$$

But now

$$\lim_{h\to\infty}|D_{j,h}|\leq\lim_{h\to\infty}\int_{\Omega_j}|u_h(x)-u(x)|\,dx=0,$$

and, if we define $K=\{\xi_j\}_{j=1}^N$ and consider $L_K$ as in hypothesis (4.9) we obtain, for every $j\in\{1,\dots,N\}$, $i\in\{1,\dots,n\}$,

$$\int_{\Omega_j\times\mathbb{R}}\left|\frac{\partial\nabla_\xi f}{\partial x_i}(x,s,\xi_j)\right|dxds=\int_H\left(\int_{\Omega_j}\left|\frac{\partial\nabla_\xi f}{\partial x_i}(x,s,\xi_j)\right|dx\right)ds\leq|H|\,L_K.$$

Hence, for every $j\in\{1,\dots,N\}$, $i\in\{1,\dots,n\}$,

$$\lim_{h\to\infty}\int_{D_{j,h}}\left|\frac{\partial\nabla_\xi f}{\partial x_i}(x,s,\xi_j)\right|dx\,ds=0,$$

from which we conclude that (4.17) holds and this ends the proof. □

Now we are ready for the proof of Theorem 4.8(1). We start modifying the integrand $f$ in order to satisfy the hypotheses of Theorem 3.1, then we manipulate De Giorgi's approximating functions in order to apply on them Lemma 4.11.

*Proof of Theorem 4.8(1).* By Proposition 4.10 we can suppose $\Omega$ bounded and study the lower semicontinuity on $W^{1,1}(\Omega)$ with respect to the $L^1(\Omega)$ convergence. Let $\{\gamma_i\}_{i=1}^{\infty}$ be an increasing sequence of smooth functions with compact support in $\Omega \times \mathbb{R}$, converging point-wise to 1 in $\Omega \times \mathbb{R}$. Since $\gamma_i(x,s)f(x,s,\xi)$ is an increasing sequence of functions which point-wise converges to $f(x,s,\xi)$, by a standard argument, it is sufficient to prove the stated lower semicontinuity assuming directly that there exists $\Omega' \times H \subset\subset \Omega \times \mathbb{R}$ such that, for every $(x,s,\xi) \in (\Omega \backslash \Omega') \times (\mathbb{R} \backslash H) \times \mathbb{R}^n$, $f(x,s,\xi) = 0$. In this way, by the hypotheses of Theorem 4.8(1), we have that, for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$,

$$f(\cdot, s, \xi) \in W^{1,1}(\Omega), \tag{4.24}$$

and, for every $K \subset\subset \mathbb{R}^n$ there exists a constant $L = L_K$ such that, for every $(s,\xi) \in \mathbb{R} \times K$,

$$\int_\Omega |\nabla_x f(x,s,\xi)| \, dx \le L,$$

or, equivalently,

$$\sum_{i=1}^{n} \int_\Omega \left| \frac{\partial f}{\partial x_i}(x,s,\xi) \right| dx \le L. \tag{4.25}$$

Since $f$ has compact support in $(x,s)$, it is possible to approximate it with an increasing sequence $\{f_j(x,s,\xi)\}_{j=1}^{\infty}$ as in Theorem 3.1. We would like to apply Lemma 4.11 to such functions, but this is not possible. Thus we denote by $k$ a convolution kernel in $\mathbb{R}^n$ and we consider the functions

$$g_j(x,s,\xi) = \int_{\mathbb{R}^n} k_{\rho_j}(\eta) f_j(x,s,\xi - \eta) d\eta,$$

where, setting $M_j$ the constant related to $f_j$ in the statement of Theorem 3.1, we choose $\rho_j = (jM_j)^{-1}$. By the Lipschitz continuity (3.4) of $f_j$ with respect to $\xi \in \mathbb{R}^n$, we have that, for every $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$,

$$\begin{aligned} |g_j(x,s,\xi) - f_j(x,s,\xi)| &\le \int_{\mathbb{R}^n} |f_j(x,s,\xi-\eta) - f_j(x,s,\xi)| \, k_{\rho_j}(\eta) d\eta \\ &\le M_j \int_{B(0,\rho_j)} |\eta| \, k_{\rho_j}(\eta) d\eta \le \frac{1}{j}, \end{aligned}$$

so that

$$f_j(x,s,\xi) - \frac{2}{j} \le g_j(x,s,\xi) - \frac{1}{j} \le f_j(x,s,\xi) \le f(x,s,\xi). \tag{4.26}$$

By the Beppo Levi's Theorem, for every $u \in W^{1,1}(\Omega)$ we have

$$\lim_{j\to\infty} \int_\Omega f_j\left(x, u(x), \nabla u(x)\right) dx = \int_\Omega f\left(x, u(x), \nabla u(x)\right) dx,$$

and thus, if we consider the sequence of integrals

$$I_j(u,\Omega) = \int_\Omega \left\{ g_j\left(x, u(x), \nabla u(x)\right) - \frac{1}{j} \right\} dx, \tag{4.27}$$

by (4.26) we obtain that $I_j(u,\Omega)$ converges, as $j \to \infty$, to the main integral $I(u,\Omega)$, and that, at the same time, $I_j(u,\Omega) \le I(u,\Omega)$ for every $j \in \mathbb{N}$. Therefore

$$I(u,\Omega) = \sup_{j\in\mathbb{N}} I_j(u,\Omega),$$

and then, since the supremum of a family of l.s.c. functionals is l.s.c. too, it suffices to show that every $I_j(u,\Omega)$ is lower semicontinuous on $W^{1,1}(\Omega)$ with respect to the $L^1(\Omega)$ convergence: since $\Omega$ is bounded, this is the same to prove this property for the functional

$$\int_\Omega g_j\left(x,u(x),\nabla u(x)\right)dx.$$

In order to achieve this we shall invoke Lemma 4.11: let us verify that, for every $j\in\mathbb{N}$, $g_j$ satisfies its hypotheses.

Clearly every $g_j$ is non negative, continuous, convex in the gradient variable and has compact support in $(x,s)$, being a convolution in $\xi$ of $f_j\le f$. Furthermore $\nabla_\xi g_j$ exists and is continuous.

Next we verify (4.10) and (4.11). By the Lipschitz continuity (3.4) of $f_j$, we have, for every $(x,s)\in\Omega\times\mathbb{R}$, $\xi\in\mathbb{R}^n$

$$|g_j(x,s,\xi_1)-g_j(x,s,\xi_2)|\le\int_{\mathbb{R}^n}|f_j(x,s,\xi_1-\eta)-f_j(x,s,\xi_2-\eta)|\,k_{\rho_j}(\eta)d\eta\le M_j\left|\xi_1-\xi_2\right|,$$

that implies $|\nabla_\xi g_j(x,s,\xi)|\le M_j$. By the definition of convolution

$$|\nabla_\xi g_j(x,s,\xi_1)-\nabla_\xi g_j(x,s,\xi_2)|$$

$$\le\int_{\mathbb{R}^n}|f_j(x,s,\xi_1-\eta)-f_j(x,s,\xi_2-\eta)|\left|\nabla k_{\rho_j}(\eta)\right|d\eta\le M_jT_j\left|\xi_1-\xi_2\right|,$$

where

$$T_j=\int_{\mathbb{R}^n}\left|\nabla k_{\rho_j}(\eta)\right|d\eta.$$

Therefore assumptions (4.10) and (4.11) are satisfied with $M'_j=\max\{M_j,M_jT_j\}$.

It remains to prove the weak derivability of $g_j$ in the $x$ variable and to verify (4.9). We shall start examining such properties for the coefficients $a_{q,h}$ in Theorem 3.1. From (3.2) we have that $a_{q,h}$ is continuous. For every $\psi\in C_c^\infty(\Omega)$, by (4.24) we have

$$\int_\Omega a_{q,h}(x,s)\frac{\partial\psi}{\partial x_i}(x)dx=\int_\Omega\left(-\int_{\mathbb{R}^n}f(x,s,\eta)\frac{\partial\alpha_q}{\partial\xi_h}(\eta)d\eta\right)\frac{\partial\psi}{\partial x_i}(x)dx$$

$$=-\int_{\mathbb{R}^n}\left(\int_\Omega f(x,s,\eta)\frac{\partial\psi}{\partial x_i}(x)dx\right)\frac{\partial\alpha_q}{\partial\xi_h}(\eta)d\eta=\int_{\mathbb{R}^n}\left(\int_\Omega\frac{\partial f}{\partial x_i}(x,s,\eta)\psi(x)dx\right)\frac{\partial\alpha_q}{\partial\xi_h}(\eta)d\eta$$

$$=\int_\Omega\left(\int_{\mathbb{R}^n}\frac{\partial f}{\partial x_i}(x,s,\eta)\frac{\partial\alpha_q}{\partial\xi_h}(\eta)d\eta\right)\psi(x)dx;$$

furthermore, thanks to the main assumption (4.25), we obtain

$$\int_\Omega\left|\frac{\partial a_{q,h}}{\partial x_i}(x,s)\right|dx=\int_\Omega\left|\int_{\mathbb{R}^n}\frac{\partial f}{\partial x_i}(x,s,\eta)\frac{\partial\alpha_q}{\partial\xi_h}(\eta)d\eta\right|dx$$

$$\le\left(\sup_{\xi\in\mathbb{R}^n}|\nabla\alpha_q(\xi)|\right)\int_{\mathrm{spt}(\alpha)}\left(\int_\Omega\left|\frac{\partial f}{\partial x_i}(x,s,\eta)\right|dx\right)d\eta\le\left(\sup_{\xi\in\mathbb{R}^n}|\nabla\alpha_q(\xi)|\right)\mathcal{L}^n\left(\mathrm{spt}(\alpha_q)\right)L_{\mathrm{spt}(\alpha_q)}.$$

Hence $a_{q,h}(\cdot,s)\in W^{1,1}(\Omega)$ and there exists a constant $R_{q,h}$ such that, for every $s\in\mathbb{R}$,

$$\int_\Omega\left|\frac{\partial a_{q,h}}{\partial x_i}(x,s)\right|dx\le R_{q,h}.$$

The same analysis carried on for $a_{q,h}$ applies unchanged to $a_{q,0}$, so that $a_{q,0}(x,s)$ is a continuous function such that, for every $s\in\mathbb{R}$, $a_{q,0}(\cdot,s)\in W^{1,1}(\Omega)$, and there exists a constant $R_{q,0}$ such that

$$\int_\Omega\left|\frac{\partial a_{q,0}}{\partial x_i}(x,s)\right|dx\le R_{q,0}.$$

In particular $a_{q,0}(\cdot,s)+\sum_{h=1}^n a_{q,h}(\cdot,s)\xi_h \in W^{1,1}(\Omega)$, from which,

$$f_j(\cdot,s,\xi)=\max_{q\in\{1,\dots,j\}}\left\{0,a_{q,0}(\cdot,s)+\sum_{h=1}^n a_{q,h}(\cdot,s)\xi_h\right\}\in W^{1,1}(\Omega),$$

and in the end we have also that $\nabla_\xi g_j(\cdot,s,\xi)\in W^{1,1}(\Omega,\mathbb{R}^n)$, with

$$\frac{\partial \nabla_\xi g_j}{\partial x_i}(x,s,\xi)=\int_{\mathbb{R}^n}\frac{\partial f_j}{\partial x_i}(x,s,\xi-\eta)\nabla k_{\rho_j}(\eta)d\eta.$$

Since,

$$\int_\Omega\left|\frac{\partial f_j}{\partial x_i}(x,s,\xi)\right|dx\le\sum_{q=1}^j\left\{\int_\Omega\left|\frac{\partial}{\partial x_i}\left(a_{q,0}(x,s)+\sum_{h=1}^n a_{q,h}(x,s)\xi_h\right)\right|dx\right\}$$

$$\le\sum_{q=1}^j\int_\Omega\left|\frac{\partial a_{q,0}}{\partial x_i}(x,s)\right|dx+|\xi|\sum_{q=1}^j\sum_{h=1}^n\int_\Omega\left|\frac{\partial a_{q,h}}{\partial x_i}(x,s)\right|dx\le S_j\left(1+|\xi|\right),$$

for a suitable constant $S_j$ (depending on $R_{q,0}$, $R_{q,h}$, $j$), we deduce

$$\int_\Omega\left|\frac{\partial\nabla_\xi g_j}{\partial x_i}(x,s,\xi)\right|dx\le\int_\Omega\left(\int_{\mathbb{R}^n}\left|\nabla k_{\rho_j}(\eta)\right|\left|\frac{\partial f_j}{\partial x_i}(x,s,\xi-\eta)\right|d\eta\right)dx$$

$$\le\int_{B_{\rho_j}(0)}\left|\nabla k_{\rho_j}(\eta)\right|\left(\int_\Omega\left|\frac{\partial f_j}{\partial x_i}(x,s,\xi-\eta)\right|dx\right)d\eta$$

$$\le\int_{B_{\rho_j}(0)}S_j\left(1+|\xi-\eta|\right)\left|\nabla k_{\rho_j}(\eta)\right|d\eta\le S_jT_j(1+|\xi|+\rho_j).$$

Hence, fixed $K\subset\subset\mathbb{R}^n$, if $L_{j,K}=nS_jT_j(1+T+\rho_j)$, (where $T$ is equal to the radius of a ball centered at the origin and containing $K$), we have, for every $(s,\xi)\in\mathbb{R}\times K$,

$$\sum_{i=1}^n\int_\Omega\left|\frac{\partial\nabla_\xi g_j}{\partial x_i}(x,s,\xi)\right|dx\le L_{j,K},$$

that is $g_j$ satisfies the condition (4.9) of Lemma 4.11 too. Thus Lemma 4.11 applies to every $g_j$, providing the lower semicontinuity of each $I_j$. By the previous arguments this ends the proof of Theorem 4.8(1). □

### 4.3.2 The condition on the gradient variable

In this section we use Theorem 3.5, together with the blow up method by Fonseca and Müller [44] and the original argument of Serrin [69], in order to prove Theorem 4.8(2): as already said the following proof can be found in [53] Theorem 6.

It is worth noting that Theorem 4.8(2) has been recently generalized by Maggi [59] who proved in particular that, under the same set of hypotheses of Theorem 4.8(2), the standard extension of the integral functional $I$ on $BV_{\rm loc}(\Omega)$ proposed by Dal Maso [30] is lower semicontinuous on $BV_{\rm loc}(\Omega)$ with respect to the $L^1_{\rm loc}(\Omega)$ convergence.

*Proof of Theorem 4.8(2).* By Proposition 4.10 we can assume $\Omega$ bounded and then also $f\ge 2$. By the usual blow up method (see for example the first part of the proof of Theorem 1.1 of Fonseca and Leoni [43]), the problem is reduced to the following one: given $(x_0,s_0,\xi_0)\in\Omega\times\mathbb{R}\times\mathbb{R}^n$, $\varepsilon_h\downarrow 0$

and $w_h \in W^{1,1}(Q^n)$ such that $w_h \to w_0$ in $L^1(Q^n)$, where $Q^n = (0,1)^n$ and $w_0(y) = \langle \xi_0, y\rangle$, we have to prove

$$\lim_{h\to\infty} \int_{Q^n} f(x_0 + \varepsilon_h y, s_0 + \varepsilon_h w_h(y), \nabla w_h(y))dy \geq f(x_0, s_0, \xi_0). \tag{4.28}$$

If it is $f(x_0, s_0, \cdot) \equiv \inf\{f(x, s, \xi) : (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n\}$, then (4.28) is trivially verified.

Otherwise there exists $\delta > 0$ such that $f$ satisfies the hypotheses of Theorem 3.5 with $\Sigma = B(x_0, \delta) \times B(s_0, \delta)$: clearly we can suppose, for every $h \in \mathbb{N}$, $\varepsilon_h < \delta$ and

$$\operatorname*{ess\,sup}_{y\in Q^n} |w_0(y)| \leq \frac{\delta}{\varepsilon_h}.$$

In particular we have, for every $(x, s, \xi) \in B(x_0, \delta) \times B(s_0, \delta) \times \mathbb{R}^n$,

$$f(x, s, \xi) = \sup_{j\in\mathbb{N}} f_j(x, s, \xi), \tag{4.29}$$

where every $f_j$ is continuous, with values in $[0, \infty)$, and such that $f_j(x, s, \cdot)$ is strictly convex on $\mathbb{R}^n$. We know also that, taking $\delta$ small enough, we can find $C_j > 0$ such that, for every $(x, s, \xi) \in B(x_0, \delta) \times B(s_0, \delta) \times \mathbb{R}^n$,

$$f_j(x, s, \xi) \leq C_j(1 + |\xi|). \tag{4.30}$$

Let us fix now $\varepsilon > 0$ and $\delta' \in (0, \delta)$. We can apply Theorem 3.2 to each $f_j$, with $\Sigma' = B(x_0, \delta') \times B(s_0, \delta')$, to find $\bar{f}_j : B(x_0, \delta) \times B(s_0, \delta) \times \mathbb{R}^n \to [0, \infty)$ continuous, convex in the gradient variable and satisfying (i), (ii), (iii), (iv) and (v) of Theorem 3.2 with constant $M_j$.

We need to truncate the functions $w_h$ in order apply the described approximation. Let us consider the sets

$$E_h = \left\{y \in Q^n : |w_h(y)| \leq \frac{\delta}{\varepsilon_h}\right\},\ E_h^+ = \left\{y \in Q^n : w_h(y) > \frac{\delta}{\varepsilon_h}\right\},\ E_h^- = \left\{y \in Q^n : w_h(y) < -\frac{\delta}{\varepsilon_h}\right\},$$

and define the sequence

$$v_h(y) = \begin{cases} w_h(y), & y \in E_h,\\ \frac{\delta}{\varepsilon_h}, & y \in E_h^+,\\ -\frac{\delta}{\varepsilon_h}, & y \in E_h^-. \end{cases} \tag{4.31}$$

It is $v_h \in W^{1,1}(Q^n)$, $v_h \to w_0$ in $L^1(Q^n)$ and $\lim_{h\to\infty} \mathcal{L}^n(Q^n \setminus E_h) = 0$.

By the definition of $E_h$ and $v_h$, by (4.29), (4.30) and property (ii) of $\bar{f}_j$, we have

$$\begin{aligned}
\int_{Q^n} f(x_0 + \varepsilon_h y, s_0 + \varepsilon_h w_h(y), \nabla w_h(y))dy &\geq \int_{E_h} f_j(x_0 + \varepsilon_h y, s_0 + \varepsilon_h w_h(y), \nabla w_h(y))dy \\
&= \int_{E_h} f_j(x_0 + \varepsilon_h y, s_0 + \varepsilon_h v_h(y), \nabla v_h(y))dy \\
&= \int_{Q^n} f_j(x_0 + \varepsilon_h y, s_0 + \varepsilon_h v_h(y), \nabla v_h(y))dy - \int_{Q^n\setminus E_h} f_j(x_0 + \varepsilon_h y, s_0 + \varepsilon_h v_h(y), 0)dy \\
&\geq \int_{Q^n} f_j(x_0 + \varepsilon_h y, s_0 + \varepsilon_h v_h(y), \nabla v_h(y))dy - C_j\mathcal{L}^n(Q^n\setminus E_h) \\
&\geq \int_{Q^n} \bar{f}_j(x_0 + \varepsilon_h y, s_0 + \varepsilon_h v_h(y), \nabla v_h(y))dy - C_j\mathcal{L}^n(Q^n\setminus E_h) - \varepsilon.
\end{aligned} \tag{4.32}$$

Let us consider the function $g_{j,h} : Q^n \times \mathbb{R} \to \mathbb{R}^n$ defined, for $(y, s) \in Q^n \times \mathbb{R}$, as

$$g_{j,h}(y, s) = \nabla_\xi \bar{f}_j(x_0 + \varepsilon_h y, s_0 + \varepsilon_h s, \xi_0).$$

By Theorem 3.2 it is a continuous and bounded function with compact support, and it satisfies an uniform Lipschitz condition, that is, for every $y, z \in Q^n$, $s \in \mathbb{R}$,

$$|g_{j,h}(y,s) - g_{j,h}(z,s)| \leq M_j \varepsilon_h (1 + |\xi_0|)|y - z|.$$

Now, for every $y \in Q^n$, the convexity of $\bar{f}_j$ in the $\xi$ variable implies

$$\bar{f}_j(x_0 + \varepsilon_h y, s_0 + \varepsilon_h v_h(y), \nabla v_h(y)) \geq \bar{f}_j(x_0 + \varepsilon_h y, s_0 + \varepsilon_h v_h(y), \xi_0)$$
$$+\langle g_{j,h}(y, w_0(y)) - g_{j,h}(y, v_h(y)), \xi_0\rangle + \langle g_{j,h}(y, v_h(y)), \nabla v_h(y)\rangle - \langle g_{j,h}(y, w_0(y)), \xi_0\rangle,$$

and then

$$\int_{Q^n} \bar{f}_j(x_0 + \varepsilon_h y, s_0 + \varepsilon_h v_h(y), \nabla v_h(y))dy \geq \int_{Q^n} \bar{f}_j(x_0 + \varepsilon_h y, s_0 + \varepsilon_h v_h(y), \xi_0)dy$$
$$+ \int_{Q^n} \langle g_{j,h}(y, w_0(y)) - g_{j,h}(y, v_h(y)), \xi_0\rangle dy + \int_{Q^n} \langle g_{j,h}(y, v_h(y)), \nabla v_h(y)\rangle - \langle g_{j,h}(y, w_0(y)), \xi_0\rangle dy.$$

Let us evaluate the limit as $h \to \infty$ of the right hand side of the previous inequality. Since, for $\mathcal{L}^n$-a.e. $x \in Q^n$, $v_h(x) \to w_0(x)$ (and then $\varepsilon_h v_h(x) \to 0$), by Fatou's Lemma we have

$$\liminf_{h\to\infty} \int_{Q^n} \bar{f}_j(x_0 + \varepsilon_h y, s_0 + \varepsilon_h v_h(y), \xi_0)dy \geq \bar{f}_j(x_0, s_0, \xi_0),$$

while, by the continuity and boundedness of $g_{j,h}$, it follows

$$\lim_{h\to\infty} \int_{Q^n} \langle g_{j,h}(y, w_0(y)) - g_{j,h}(y, v_h(y)), \xi_0\rangle dy = 0.$$

Let us show now that also

$$\lim_{h\to\infty} \int_{Q^n} \langle g_{j,h}(y, v_h(y)), \nabla v_h(y)\rangle - \langle g_{j,h}(y, w_0(y)), \xi_0\rangle dy = 0.$$

In order to see this let us note that, by a mollification argument similar to the one used to prove (4.18) in the proof of Theorem 4.8(1), the following formula holds [8]

$$\int_{Q^n} \langle g_{j,h}(y, v_h(y)), \nabla v_h(y)\rangle - \langle g_{j,h}(y, w_0(y)), \xi_0\rangle dy = -\int_{Q^n} \left\{ \sum_{i=1}^{n} \int_{w_0(y)}^{v_h(y)} \frac{\partial g_{j,h}^{(i)}}{\partial y_i}(y,s)ds \right\} dy,$$

from which follows

$$\left| \int_{Q^n} \langle g_{j,h}(y, v_h(y)), \nabla v_h\rangle - \langle g_{j,h}(y, w_0(y)), \xi_0\rangle dy \right|$$
$$\leq \sum_{i=1}^{n} \int_{Q^n} \left| \int_{w_0(y)}^{v_h(y)} \left| \frac{\partial g_{j,h}^{(i)}}{\partial y_i}(y,s) \right| ds \right| dy \leq n M_j \varepsilon_h (1 + |\xi_0|) \int_{Q^n} |w_0 - v_h| dy,$$

that goes to zero as $h \to \infty$.

If now $\varepsilon$ is so small that $|\xi_0| \leq 1/\varepsilon$, property (iii) of $\bar{f}_j$ implies $\bar{f}_j(x_0, s_0, \xi_0) \geq f_j(x_0, s_0, \xi_0) - \varepsilon$, and hence, by (4.32), it is that

$$\liminf_{h\to\infty} \int_{Q^n} f(x_0 + \varepsilon_h y, s_0 + \varepsilon_h w_h(y), \nabla w_h(y))dy \geq f_j(x_0, s_0, \xi_0) - 2\varepsilon.$$

Thus, it suffices to let $\varepsilon$ tend to zero and then take the supremum on $j \in \mathbb{N}$ to conclude the proof. □

**Remark 4.12.** Note that the same proof allows also to consider the limit function $u \in BV_{\text{loc}}(\Omega)$, and then to give a direct lower bound of the relaxed functional $R[L^1_{\text{loc}}](I)$ on $BV_{\text{loc}}(\Omega)$.

[8] In the following $(g_{j,h})^{(i)}$ denotes the $i$-th component of $g_{j,h}$.

## 4.4 A counterexample to the lower semicontinuity

As explained in the first section of this chapter, Aronszajn's example shows that conditions (4.1), together with the continuity of the integrand $f$, are not sufficient to guarantee the lower semicontinuity on $W^{1,1}(\Omega)$ with respect to the $L^1(\Omega)$ convergence of the integral functional $I$. However, Theorem 4.7 shows that if $f$ satisfies (4.8) too, that is $f$ is locally Lipschitz continuous in $x$ uniformly for $(x, \xi)$ belonging to a compact set, then the lower semicontinuity is achieved.

Theorem 4.8(1), in which suitable summability conditions on $\nabla_x f$ are required, improves condition (4.8): nevertheless it is interesting to understand what we can expect supposing, instead of the Lipschitz continuity of $f$, its *Hölder continuity* with respect to $x$, that is, there exists $\alpha \in (0,1)$ such that, for every compact set $K \subset \Omega \times \mathbb{R} \times \mathbb{R}^n$, we can find a constant $L = L_K$ such that, for every $(x_1, s, \xi), (x_2, s, \xi) \in K$,

$$|f(x_1, s, \xi) - f(x_2, s, \xi)| \leq L|x_1 - x_2|^\alpha. \tag{4.33}$$

By modifying a one dimensional (that is n=1) version of Aronszajn's Example proposed by Dal Maso[9], Gori, Maggi and Marcellini solved this problem showing that, for every $\alpha \in (0,1)$, there exists a one dimensional integral functional for which the lower semicontinuity fails and such that its integrand satisfies (4.1) and (4.33) (see [54] Example 4.2, see also [55] Theorem 2.1 in which this fact was proved using multiple integrals, that is $n \geq 2$.).

Here we present a simplified version just of Example 4.2 in [54].

**Theorem 4.13.** *Let $\varepsilon > 0$ and $\Omega = (-\varepsilon, 1+\varepsilon)$. For every $\alpha \in (0,1)$, there exist a sequence $\{u_h\}_{h=1}^\infty \subseteq W^{1,\infty}(\Omega)$, such that $u_h \to 0$ in $L^\infty(\Omega)$, and a function[10] $b : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ (both of them depending on $\alpha$) such that:*

*(i) $b$ is bounded and uniformly continuous in $\overline{\Omega} \times \mathbb{R}$;*

*(ii) there exists a constant $L$ such that, for every $x_1, x_2 \in \overline{\Omega}$ and $s \in \mathbb{R}$,*

$$|b(x_1, s) - b(x_2, s)| \leq L\,|x_1 - x_2|^\alpha\,; \tag{4.34}$$

*(iii) setting, for every $(x, s, \xi) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}$,*

$$f(x, s, \xi) = |b(x, s)\xi - 1|\,,$$

*we have that $f(x, s, \xi)$ is continuous, satisfies (4.1) and the Hölder continuity property (4.33) and*

$$\lim_{h\to\infty} I(u_h, \Omega) = \lim_{h\to\infty} \int_\Omega f(x, u_h(x), u_h'(x))dx < \int_\Omega f(x, 0, 0)dx = I(0, \Omega). \tag{4.35}$$

*In particular the functional $I$ is not lower semicontinuous on $W^{1,\infty}(\Omega)$ with respect to the $L^\infty(\Omega)$ convergence.*

*Proof.* Let us fix $\alpha \in (0,1)$ and choose $k \in \mathbb{N}$ such that $k(1-\alpha) \geq 2$ (then $k > 2$).

[9] Note that the original example by Aronszajn is related to a *double integral*, that is, $n = 2$, and Aronszajn's integrand $f(x, \xi)$ does not explicitly depends on $s$. A *one-dimensional* version of Aronszajn's example was known to Dal Maso, who gave us some handwritten notes on the subject: however, in this case, the integrand $f$ depends on all the three variables $x$, $s$ and $\xi$.

[10] In this section $\overline{\Omega}$ denotes $\mathrm{cl}(\Omega)$.

First of all let us build up the sequence $u_h$. Let us fix $h \in 2\mathbb{N} = \{2i : i \in \mathbb{N}\}$ with $h \geq h_0$, where[11] $h_0 \geq \frac{1}{2^{\frac{1}{\alpha}} 4\varepsilon}$, and define $u_h$ on $\Omega$ in this way: for every $r \in \{0, 1, ..., h^k - 1\}$,

$$u_h(x) = \begin{cases} \frac{4h^k}{h(h+1)}x + \frac{1}{h+1} - \frac{4r}{h(h+1)} & \text{if } x \in \left[\frac{r}{h^k}, \frac{r}{h^k} + \frac{1}{4h^k}\right], \\ \frac{1}{h} & \text{if } x \in \left[\frac{r}{h^k} + \frac{1}{4h^k}, \frac{r}{h^k} + \frac{2}{4h^k}\right], \\ -\frac{4h^k}{h(h+1)}x + \frac{1}{h} + \frac{4r+2}{h(h+1)} & \text{if } x \in \left[\frac{r}{h^k} + \frac{2}{4h^k}, \frac{r}{h^k} + \frac{3}{4h^k}\right], \\ \frac{1}{h+1} & \text{if } x \in \left[\frac{r}{h^k} + \frac{3}{4h^k}, \frac{r+1}{h^k}\right], \end{cases}$$

and, for every $x \in (-\varepsilon, 0] \cup [1, 1+\varepsilon)$, $u_h(x) = \frac{1}{h+1}$. Trivially we have $u_h \in W^{1,\infty}(\Omega)$, $u_h \to 0$ in $L^\infty(\Omega)$ and, since we are considering $h \in 2\mathbb{N}$, the graphs of the functions $u_h$ are disjoint. Moreover an easy computation gives

$$\lim_{h\to\infty} \int_\Omega |u_h'(x)|\, dx = \lim_{h\to\infty} \frac{2h^k}{h(h+1)} = \infty.$$

In order to define the function $b$ let us consider, for every $h \in 2\mathbb{N}$, $h \geq h_0$ and $r \in \{0, 1, ..., h^k - 1\}$, the subsets of $\overline{\Omega} \times \mathbb{R}$ given by

$$A_h^r = \left\{\left(x, \frac{4h^k}{h(h+1)}x + \frac{1}{h+1} - \frac{4r}{h(h+1)}\right) : x \in \left[\frac{r}{h^k}, \frac{r}{h^k} + \frac{1}{4h^k}\right]\right\},$$

$$B_h^r = \left\{\left(x, -\frac{4h^k}{h(h+1)}x + \frac{1}{h} + \frac{4r+2}{h(h+1)}\right) : x \in \left[\frac{r}{h^k} + \frac{2}{4h^k}, \frac{r}{h^k} + \frac{3}{4h^k}\right]\right\},$$

$$C_h = \overline{\Omega} \times \left\{\frac{1}{2h} + \frac{1}{2(h-1)}\right\},$$

$$D_1 = \{-\varepsilon, 1+\varepsilon\} \times \mathbb{R}, \quad \text{and} \quad D_2 = \overline{\Omega} \times \left[(-\infty, 0] \cup \left[\frac{1}{2h_0} + \frac{1}{2(h_0-1)}, +\infty\right)\right],$$

whose geometric meaning, with respect to the sequence $\{u_h : h \in 2\mathbb{N}, h \geq h_0\}$, is clear. Let us consider now the closed set

$$S = \left(\bigcup_{h,r} A_h^r\right) \cup \left(\bigcup_{h,r} B_h^r\right) \cup \left(\bigcup_h C_h\right) \cup D_1 \cup D_2,$$

and define the function $b$ on $S$ as

$$b(x,s) = \begin{cases} \frac{h(h+1)}{4h^k} & \text{if } (x,s) \in A_h^r, \\ -\frac{h(h+1)}{4h^k} & \text{if } (x,s) \in B_h^r, \\ 0 & \text{if } (x,s) \in \left(\bigcup_h C_h\right) \cup D_1 \cup D_2. \end{cases}$$

Since $\lim_{h\to\infty} \frac{h(h+1)}{4h^k} = 0$, we have that $b$ is continuous on $S$; moreover note that, on $A_h^r$ and on $B_h^r$, $b$ assumes the value of the inverse of $u_h'$ on the points in which the graph of $u_h$ intersects these sets.

Now we extend $b$ on $\overline{\Omega} \times \mathbb{R}$. Let us consider $s_0 \in \bigcup_h \left[\frac{1}{h+1}, \frac{1}{h}\right]$ and observe that $S \cap (\overline{\Omega} \times \{s_0\})$ contains only a finite number of points among which there are $(-\varepsilon, s_0)$ and $(1+\varepsilon, s_0)$: thus, in

[11] This value is chosen only for technical reasons, in order to compute the value (4.36) in a simpler way.

the segment $\overline{\Omega} \times \{s_0\}$, we define $b$ by a linear interpolation between every pair of adjacent points in which the value of $b$ is known.

For every $h \in 2\mathbb{N}$, $h \geq h_0$, the described extension allows us to build up $b$ continuous in the strip $R_h = \overline{\Omega} \times \left[\frac{1}{h+1}, \frac{1}{h}\right]$. We have, in particular, that $b$ is defined in every set of the form $\overline{\Omega} \times \left\{\frac{1}{j}\right\}$, where $j \in \mathbb{N}$ and $j \geq h_0$, if $h_0$ is even, or $j \geq h_0 + 1$ if $h_0$ is odd.

Now we complete the extension of $b$ setting, for every $h \in 2\mathbb{N}$, $h \geq h_0$

$$b(x, s) = \begin{cases} b\left(x, \frac{1}{h}\right)\left[\left(\frac{1}{h} - s\right) 2h(h+1) + 1\right] & \text{if } (x, s) \in E_h, \\ b\left(x, \frac{1}{h+1}\right)\left[\left(\frac{1}{h+1} - s\right) 2(h+1)(h+2) + 1\right] & \text{if } (x, s) \in F_h, \end{cases}$$

where

$$E_h = \overline{\Omega} \times \left[\frac{1}{h}, \frac{1}{2h} + \frac{1}{2(h-1)}\right] \quad \text{and} \quad F_h = \overline{\Omega} \times \left[\frac{1}{2(h+2)} + \frac{1}{2(h+1)}, \frac{1}{h+1}\right].$$

Since this extension makes $b$ continuous on $\overline{\Omega} \times \mathbb{R}$ and since, for every $(x, s) \in D_1 \cup D_2$, $b(x, s) = 0$, we have that $b$ is uniformly continuous on $\overline{\Omega} \times \mathbb{R}$ and then condition (i) of the theorem is verified.

We want now to prove that $b$ satisfies condition (ii). Fixed $h \in 2\mathbb{N}$, $h \geq h_0$ we have that, by construction, the value

$$M_h = \sup\left\{\frac{|b(x, s) - b(y, s)|}{|x - y|^\alpha} : (x, s), (y, s) \in R_h \cup E_h \cup F_h; x \neq y\right\} \tag{4.36}$$

is in fact a maximum which is obtained by the pairs of points of the type

$$\left(\frac{r}{h^k} + \frac{1}{4h^k}, \frac{1}{h}\right), \left(\frac{r}{h^k} + \frac{2}{4h^k}, \frac{1}{h}\right), \quad \text{or} \quad \left(\frac{r}{h^k} + \frac{3}{4h^k}, \frac{1}{h+1}\right), \left(\frac{r+1}{h^k}, \frac{1}{h+1}\right), \tag{4.37}$$

for every $r \in \left\{0, 1, ..., h^k - 1\right\}$[12]. Computing (4.36), using any of the previous pair of points, we have

$$M_h = \frac{2h(h+1)}{4h^k} \cdot 4^\alpha h^{\alpha k} = \frac{4^\alpha}{2} \cdot \frac{h(h+1)}{h^{k-\alpha k}} \leq \frac{4^\alpha}{2} \cdot \frac{2h^2}{h^{k-\alpha k}} \leq 4^\alpha \cdot \frac{h^2}{h^{k-\alpha k}},$$

and since, by definition $k(1-\alpha) \geq 2$, we have

$$M_h \leq 4^\alpha \cdot \frac{h^2}{h^{k-\alpha k}} \leq 4^\alpha$$

that doesn't depend on $h$. Thus, since outside $\bigcup_h (R_h \cup E_h \cup F_h)$, that is in $D_1 \cup D_2$, we have $b \equiv 0$ we conclude that surely $b$ satisfies (4.34) for $L = 4^\alpha$.

Finally let us prove that the function $f(x, s, \xi) = |b(x, s)\xi - 1|$ satisfies condition (iii). Clearly $f$ is continuous and satisfies (4.1) and (4.33). Moreover

$$I(0, \Omega) = \int_\Omega |b(x, 0) \cdot 0 - 1|\, dx = 1 + 2\varepsilon$$

while, for every $h \in 2\mathbb{N}$, $h \geq h_0$,

$$I(u_h, \Omega) = \int_\Omega |b(x, u_h(x)) \cdot u_h'(x) - 1|\, dx = \left(\sum_{j=1}^{h^k} 2 \cdot \frac{1}{4h^k}\right) + 2\varepsilon = \frac{1}{2} + 2\varepsilon.$$

Thus

$$\lim_{h \to \infty} I(u_h, \Omega) = \frac{1}{2} + 2\varepsilon < 1 + 2\varepsilon = I(u, \Omega)$$

and (4.35) is also verified. □

---

[12] For the pair of points $-\varepsilon, \frac{1}{h+1}$ , $0, \frac{1}{h+1}$ and $1 - \frac{1}{4h^k}, \frac{1}{h+1}$ , $1 + \varepsilon, \frac{1}{h+1}$ , because of the choice of $h_0$, the value of the ratio in (4.36) is surely less then the one of the pairs in (4.37).

The theorem just proved provides also a useful information about Theorem 4.8(2). Indeed, it is simple to verify that the function $f$ of Theorem 4.13 is such that, for every $(x_0, s_0) \in \Omega \times \mathbb{R}$, either, for every $\xi \in \mathbb{R}$,

$$f(x_0, s_0, \xi) \equiv 1,$$

or there exists $\delta > 0$ such that, for every $(x, s) \in B(x_0, \delta) \times B(s_0, \delta)$,

$$f(x, s, \cdot) \text{ is demi-coercive on } \mathbb{R}.$$

There is no contradiction with Theorem 4.8(2) because it is $\inf f = 0$: however, this shows how the hypotheses of Theorem 4.8(2) are close to be sharp.

We note also that, when $n = 1$, the demi-coercivity reduces to ask that

$$f(x, s, \cdot) \quad \text{is not constant,} \tag{4.38}$$

for every fixed pair $(x, s)$. However, if in the hypotheses of Theorem 4.8(2) we replace the demi-coercivity condition with (4.38) and assume $n \geq 2$, then the lower semicontinuity will fail again. This fact can be deduced by the original Aronszajn's counterexample [64] in which a continuous function $\nu : (0,1)^2 \to S^1$ is built up in such a way that the integral functional generated by $f(x, \xi) = |\langle \nu(x), \xi \rangle|$, where $(x, \xi) \in (0,1)^2 \times \mathbb{R}^2$, it is not lower semicontinuous (see also [30] Example 4.1). Clearly $f(x, \cdot)$ satisfies (4.38) but fails to be demi-coercive.

## 4.5 Appendix: the vectorial case

In this section we deal with a function $f$ satisfying the conditions

$$\begin{cases} f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{mn} \to [0, \infty], \\ \text{for every } (x, s) \in \Omega \times \mathbb{R}^m, \ f(x, s, \cdot) \text{ is convex in } \mathbb{R}^{mn}, \end{cases} \tag{4.39}$$

where $\Omega$ is an open set of $\mathbb{R}^n$, $n, m \geq 1$. As in the scalar case we want to find conditions on $f$ such that, for every $u_h, u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ such that $u_h \to u$ in $L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$,

$$I(u, \Omega) \leq \liminf_{h \to \infty} I(u_h, \Omega). \tag{4.40}$$

The lower semicontinuity results by Serrin [69] (see Theorems 4.1 and 4.2) were proved for the scalar case $m = 1$: some efforts have been spent to understand the question of their validity in the vectorial setting[13], that is, when $m \geq 2$.

### 4.5.1 Some counterexamples

Eisen [38] showed with an example that Theorem 4.1(c) is false in the vectorial case: obviously the same example shows that the same holds for Theorems 4.7 and 4.8(1) and Corollary 4.9(1) too. We propose here a simplified version of the example quoted above (see Gori and Maggi [52] Example 2.1) that, as we will explain in the next chapter, it is meaningful even in the context of the supremal functionals.

**Example 4.14.** Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ be a function defined, for every $(s_1, s_2, \xi_1, \xi_2) = (s, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2$, as

$$f(s_1, s_2, \xi_1, \xi_2) = (\xi_1 s_2 - \xi_2 s_1 - 1)^2 .$$

[13] However, in studying the lower semicontinuity problems in the vectorial setting, the notion of convexity in the gradient variable of $f$ is replaced by the more general and natural one of quasiconvexity (see Morrey [63] and Dacorogna [29]).

Clearly $f$ is non negative, smooth and convex in the gradient variable $\xi = (\xi_1, \xi_2)$. Let us consider the sequence of functions in $C^\infty((0,1), \mathbb{R}^2)$ given, for every $h \in \mathbb{N}$, by

$$u_h(x) = \left( \frac{1}{2\pi h} \sin(2\pi h^2 x), \frac{1}{2\pi h} \cos(2\pi h^2 x) \right).$$

An easy computation gives $u_h \to 0$ in $L^\infty((0,1), \mathbb{R}^2)$ as $h \to \infty$. However, for every $h \in \mathbb{N}$, $I(u_h, (0,1)) = 0$, while $I(0, (0,1)) = 1$ so that the lower semicontinuity fails. Note in particular that, for *every* $x \in (0,1)$, $f(u_h(x), \nabla u_h(x)) = 0$.

Later Cerný and Malý [25] proved also that Theorem 4.1(b) (and of course Theorem 4.8(2) and Corollary 4.9(2) too) is false in the vectorial setting.

**Example 4.15.** Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ be a function defined, for every $(s_1, s_2, \xi_1, \xi_2) = (s, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2$, as

$$f(s_1, s_2, \xi_1, \xi_2) = \frac{\xi_2^2 + \exp(\xi_1)}{\exp\left((s_1^2 + s_2^2)^2\right)}.$$

Clearly $f$ is non negative, smooth and convex in the gradient variable $\xi = (\xi_1, \xi_2)$. Let us consider the sequence of functions in $W^{1,\infty}((0,1), \mathbb{R}^2)$ given, for every $h \in \mathbb{N}$, by $u_h(x) = (u_{h,1}(x), u_{h,2}(x))$, where, for every $x \in [0, \frac{1}{h}]$,

$$u_{h,1}(x) = \begin{cases} -h^4 x & \text{if } x \in \left[0, \frac{1}{h^3}\right], \\ -h \cos\left(\frac{\pi h^3}{4}\left(x - \frac{1}{h^3}\right)\right) & \text{if } x \in \left[\frac{1}{h^3}, \frac{5}{h^3}\right], \\ h - h^3\left(h - \frac{1}{h}\right)\left(x - \frac{5}{h^3}\right) & \text{if } x \in \left[\frac{5}{h^3}, \frac{6}{h^3}\right], \\ \frac{h^2}{h^2 - 6}\left(\frac{1}{h} - x\right) & \text{if } x \in \left[\frac{6}{h^3}, \frac{1}{h}\right], \end{cases}$$

and

$$u_{h,2}(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{h^3}\right], \\ -h \sin\left(\frac{\pi h^3}{4}\left(x - \frac{1}{h^3}\right)\right) & \text{if } x \in \left[\frac{1}{h^3}, \frac{5}{h^3}\right], \\ 0 & \text{if } x \in \left[\frac{5}{h^3}, \frac{1}{h}\right], \end{cases}$$

and extended by periodicity on $(0,1)$. A computation gives $u_h \to 0$ in $L^\infty((0,1), \mathbb{R}^2)$ as $h \to \infty$ and

$$\lim_{h \to \infty} I(u_h, (0,1)) = \frac{1}{e}.$$

Since $I(0, (0,1)) = 1$ the lower semicontinuity fails.

Following Theorem 4.1, it remains only to understand the situation about condition (a). The following example given by Cerný and Malý [24] shows that, when $f$ is lower semicontinuous (but not continuous), even if it is coercive, the lower semicontinuity may fail. However, they are not able to consider $f$ continuous as in Theorem 4.1(a): as we will see this is due to the fact that in these hypotheses a lower semicontinuity theorem can be proved (see Theorem 4.19).

**Example 4.16.** Let $f : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ be a function defined, for every $(s_1, s_2, \xi_1, \xi_2) = (s, \xi) \in \mathbb{R}^2 \times \mathbb{R}^2$, as

$$f(s_1, s_2, \xi_1, \xi_2) = \begin{cases} |\xi_1 - 1| + |\xi_2| + \frac{1}{|s_2|} + |s_1| + |s_2| & \text{if } s_2 \neq 0, \\ 2|\xi_1 - 1| + |\xi_2| + |s_1| + |s_2| & \text{if } s_2 = 0. \end{cases}$$

Clearly $f$ is non negative, lower semicontinuous and convex in the gradient variable $\xi = (\xi_1, \xi_2)$. Moreover, for every $(s_1, s_2, \xi_1, \xi_2) \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$f(s_1, s_2, \xi_1, \xi_2) \geq |s_1| + |s_2| + |\xi_1| + |\xi_2| - 1,$$

that is $f$ is coercive. Let us consider the sequence of functions in $W^{1,1}((0,1), \mathbb{R}^2)$ given, for every $h \in \mathbb{N}$, by $u_h(x) = (u_{h,1}(x), u_{h,2}(x))$, where, for every $x \in [0, \frac{1}{h}]$,

$$u_{h,1}(x) = \begin{cases} x & \text{if } x \in \left[0, \frac{1}{h} - \frac{1}{h^4}\right], \\ \left(h^3 - 1\right)\left(\frac{1}{h} - x\right) & \text{if } x \in \left[\frac{1}{h} - \frac{1}{h^4}, \frac{1}{h}\right], \end{cases}$$

and

$$u_{h,2}(x) = \begin{cases} 0 & \text{if } x \in \left[0, \frac{1}{h} - \frac{1}{h^4}\right], \\ \sqrt{h^4\left(x - \frac{1}{h} + \frac{1}{h^4}\right)\left(\frac{1}{h} - x\right)} & \text{if } x \in \left[\frac{1}{h} - \frac{1}{h^4}, \frac{1}{h}\right], \end{cases}$$

and extended by periodicity on $(0,1)$. We can see that $u_h \to 0$ in $L^\infty((0,1), \mathbb{R}^2)$ as $h \to \infty$ and

$$\lim_{h\to\infty} I(u_h, (0,1)) = 1.$$

Since $I(0, (0,1)) = 2$ the lower semicontinuity fails.

### 4.5.2 Lower semicontinuity theorems

Examples 4.14 and 4.15 seem to indicate that, when the vectorial case is considered, we need a coercivity assumption on the integrand in order to obtain the lower semicontinuity. Moreover, as Example 4.16 says, even if this condition is verified, $f$ must be more regular than lower semicontinuous. Nevertheless under the investigation of the scalar case made in the previous sections, we became aware of the fact that, in particular, two results without coercivity assumptions hold in the vector-valued case too, once provided the dependence on the $s$ variable is dropped (note that the dependence on $s$ is fundamental in the counterexamples proposed). The first one is a vector-valued version of Theorem 4.7 (see Gori, Maggi and Marcellini [54] Theorem 5.1).

**Theorem 4.17.** *Let $f$ be a continuous function satisfying (4.39) such that $f = f(x, \xi)$ and, for every open set $\Omega' \times K \subset\subset \Omega \times \mathbb{R}^{mn}$ there exists a constant $L = L_{\Omega' \times K}$ such that, for every $x_1, x_2 \in \Omega'$, $\xi \in K$,*

$$|f(x_1, s, \xi) - f(x_2, s, \xi)| \leq L\,|x_1 - x_2|. \tag{4.41}$$

*Then the functional $I$ is l.s.c. on $W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^m)$ with respect to the $L^1_{\text{loc}}(\Omega, \mathbb{R}^m)$ convergence.*

*Proof.* By Proposition 4.10, that can be applied also to the vectorial setting, we can suppose $\Omega$ bounded and with Lipschitz boundary. Moreover let us assume at first that $f$ satisfies the following conditions too:

(i) there exists an open set $\Omega' \subset\subset \Omega$, such that, for every $x \in \Omega \setminus \Omega'$ and for every $\xi \in \mathbb{R}^{mn}$, $f(x, \xi) = 0$;

(ii) $\nabla_\xi f(x, \xi)$ exists, is continuous on $\Omega \times \mathbb{R}^{mn}$ and such that, for every $\xi \in \mathbb{R}^{mn}$ $\nabla_\xi f(\cdot, \xi) \in W^{1,\infty}(\Omega, \mathbb{R}^{mn})$;

(iii) there exists a constant $M$ such that, for every $(x, \xi) \in \Omega \times \mathbb{R}^{mn}$,

$$|\nabla_\xi f(x, \xi)| \leq M, \tag{4.42}$$

and, for every $x \in \Omega$, $\xi_1, \xi_2 \in \mathbb{R}^{mn}$,

$$|\nabla_\xi f(x, \xi_1) - \nabla_\xi f(x, \xi_2)| \leq M\,|\xi_1 - \xi_2|. \tag{4.43}$$

Let $u_h, u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ such that $u_h \to u$ in $L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$. Without loss of generality, we can assume that

$$\liminf_{h\to\infty} I(u_h, \Omega) = \lim_{h\to\infty} I(u_h, \Omega) < \infty,$$

and that $u_h$ converges almost everywhere to $u$ in $\Omega$. Moreover, from (4.42) and (4.43), we have $I(u, \Omega) < \infty$.

As in the proof of Lemma 4.11, for every $\varepsilon > 0$, there exists $v_\varepsilon \in \mathrm{Aff}(\Omega, \mathbb{R}^m)$ such that,

$$\int_\Omega |\nabla u(x) - \nabla v_\varepsilon(x)|\, dx \le \varepsilon, \quad \text{and} \quad \int_\Omega f(x, \nabla v_\varepsilon(x))\, dx \ge \int_\Omega f(x, \nabla u(x)) dx - \varepsilon. \tag{4.44}$$

Since $v_\varepsilon \in \mathrm{Aff}(\Omega, \mathbb{R}^m)$, we consider $\{\Omega_j\}_{j=1}^N$ such that, for every $j \in \{1, \ldots, N\}$, $x \in \Omega_j$, $\nabla v_\varepsilon(x) = \xi_j \in \mathbb{R}^{mn}$ in $\Omega_j$. Now let us consider the sequence of functions $\{\beta_k^\varepsilon\}_{k=1}^\infty \subset C_c^\infty(\Omega)$ as in Lemma 4.11 and, using the same argument, we obtain the lower semicontinuity if we prove that, for every $v_\varepsilon$ and $k$ fixed,

$$\lim_{h\to\infty} \int_\Omega \beta_k^\varepsilon(x) \Big\{ \nabla_\xi f(x, \nabla v_\varepsilon(x)) \cdot \nabla u_h(x) - \nabla_\xi f(x, \nabla v_\varepsilon(x)) \cdot \nabla u(x) \Big\} dx = 0. \tag{4.45}$$

Observing that, for every $j \in \{1, \ldots, N\}$, $\beta_k^\varepsilon(x) \nabla_\xi f(x, \xi_j) \in W^{1,\infty}(\Omega_j, \mathbb{R}^{mn}) \cap C_c(\Omega_j, \mathbb{R}^{mn})$, we have

$$\begin{aligned}
&\left| \int_\Omega \beta_k^\varepsilon(x) \Big\{ \nabla_\xi f(x, \nabla v_\varepsilon(x)) \cdot \nabla u_h(x) - \nabla_\xi f(x, \nabla v_\varepsilon(x)) \cdot \nabla u(x) \Big\} dx \right| \\
&\le \sum_{j=1}^N \left| \int_{\Omega_j} \beta_k^\varepsilon(x) \nabla_\xi f(x, \xi_j) \cdot (\nabla u_h(x) - \nabla u(x)) dx \right| \\
&\le \sum_{j=1}^N \sum_{i=1}^n \sum_{q=1}^m \left| \int_{\Omega_j} \beta_k^\varepsilon(x) \frac{\partial f}{\partial \xi_{q,i}}(x, \xi_j) \cdot \frac{\partial}{\partial x_i} \left( u_h^{(q)}(x) - u^{(q)}(x) \right) dx \right| \\
&= \sum_{j=1}^N \sum_{i=1}^n \sum_{q=1}^m \left| \int_{\Omega_j} \frac{\partial}{\partial x_i} \left( \beta_k^\varepsilon(x) \frac{\partial f}{\partial \xi_{q,i}}(x, \xi_j) \right) \cdot \left( u_h^{(q)}(x) - u^{(q)}(x) \right) dx \right| \\
&\le \sum_{j=1}^N \sum_{i=1}^n \sum_{q=1}^m C_{i,j} \int_{\Omega_j} \left| u_h^{(q)}(x) - u^{(q)}(x) \right| dx,
\end{aligned}$$

which goes to zero as $h \to \infty$, since, for every $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, N\}$,

$$C_{i,j} = \sup \left\{ \left| \frac{\partial}{\partial x_i} \left( \beta_k^\varepsilon(x) \nabla_\xi f(x, \xi_j) \right) \right| : x \in \Omega'_j \right\} < \infty.$$

In order to treat the general case, we only have to use the same (in fact simpler) argument used in the proof of Theorem 4.8(1). □

The next proposition states that Corollary 4.9(2) holds in the vectorial case if $f = f(x, \xi)$ (see Gori, Maggi [53] Proposition 3).

**Proposition 4.18.** *Let $f$ be a continuous function satisfying (4.39) such that $f = f(x, \xi)$ and, for every $x \in \Omega$, $f(x, \cdot)$ is convex and demi-coercive on $\mathbb{R}^{mn}$. Then the functional $I$ is l.s.c. on $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ with respect to the $L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ convergence.*

*Proof.* We consider at first the case in which $f(x,\cdot)$ is strictly convex. The proof of Theorem 4.1(b), valid in the scalar case, is based on Lemmas 4, 5, 7, 8 of [69]. However, Lemmas 4, 5, 8 are also true in the vectorial setting. Thus, Lemma 7 is the only one which makes Serrin's Theorem 4.1(b) work only in the scalar setting: however, if $f$ does not depend on $s$, we can easily extend Lemma 7 to the vectorial setting and find that Theorem 4.1(b) is true also in this more general contest.

Once known the lower semicontinuity for the strictly convex case we end in a standard way applying Theorem 3.5. □

Clearly Examples 4.14 and 4.15 show that, if we allow the presence of the $s$ variable, Proposition 4.18 and Theorem 4.17 are false.

The following result, proved by Fonseca and Leoni in [42] (see Theorem 1.1 therein), shows that if $f$ is coercive and satisfies some regularity conditions then the lower semicontinuity can be proved. In particular this theorem implies the vectorial version of Theorem 4.1(a).

**Theorem 4.19.** *Let $f$ be a l.s.c. function satisfying (4.39). Let us suppose that, for every $(x_0,s_0)\in\Omega\times\mathbb{R}^m$ either, for every $\xi\in\mathbb{R}^{mn}$, $f(x_0,s_0,\xi)=0$, or there exists $c_0,\delta_0>0$ and a continuous function $g:B((x_0,s_0),\delta_0)\to\mathbb{R}^{mn}$ such that the composition*

$$f(x,s,g(x,s))\in L^\infty\left(B\left((x_0,s_0),\delta_0\right)\right), \tag{4.46}$$

*and, for every $(x,s)\in B((x_0,s_0),\delta_0)$,*

$$f(x,s,\xi)\geq c_0\,|\xi|-\frac{1}{c_0}.$$

*Then the functional $I$ is l.s.c. on $W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^m)$ with respect to the $L^1_{\rm loc}(\Omega,\mathbb{R}^m)$ convergence.*

A question posed in [42] is about the necessity of assumption (4.46). This hypothesis comes out by the use of the approximation theorem by Ambrosio (see [4] Lemma 1.5, Statement A). Example 4.16 shows that such assumption in general cannot be dropped. However, it may be interesting to note that, if we assume a local coerciveness with superlinear growth, then Theorem 4.19 holds without (4.46), as stated in the next proposition (see Gori, Maggi and Marcellini [54] Proposition 5.4).

**Proposition 4.20.** *Let $f$ be a l.s.c. function satisfying (4.39). Let us suppose that, for every $(x_0,s_0)\in\Omega\times\mathbb{R}^m$ either, for every $\xi\in\mathbb{R}^{mn}$, $f(x_0,s_0,\xi)=0$, or there exists $c_0,\delta_0>0$, $p_0>1$ such that, for every $(x,s)\in B((x_0,s_0),\delta_0)$,*

$$f(x,s,\xi)\geq c_0\,|\xi|^{p_0}-\frac{1}{c_0}. \tag{4.47}$$

*Then the functional $I$ is l.s.c. on $W^{1,1}_{\rm loc}(\Omega,\mathbb{R}^m)$ with respect to the $L^1_{\rm loc}(\Omega,\mathbb{R}^m)$ convergence.*

*Proof.* The usual blow up method reduce the problem of the lower semicontinuity to prove the inequality

$$\liminf_{h\to\infty}\int_{Q^n} f(x_0+\varepsilon_k y,s_0+\varepsilon_h w_h(y),\nabla w_h(y))dy\geq f(x_0,s_0,\xi_0),$$

where $(x_0,s_0,\xi_0)\in\Omega\times\mathbb{R}\times\mathbb{R}^n$, $\varepsilon_h\downarrow 0$ and $\{w_h\}_{h=1}^\infty\subseteq W^{1,1}(Q^n)$ is a sequence such that $w_h\to w_0$ in $L^1(Q^n)$, where $Q^n=(0,1)^n$ and $w_0(y)=\langle\xi_0,y\rangle$.

This trivially holds if, for the pair $(x_0,s_0)$ it is $f(x_0,s_0,\cdot)=0$. Otherwise there exist $\delta_0,c_0>0$, $p_0>1$ such that, for every $(x,s,\xi)\in\overline{B((x_0,s_0),\delta_0)}\times\mathbb{R}^{mn}$,

$$f(x,s,\xi)\geq c_0\,|\xi|^{p_0}-\frac{1}{c_0}.$$

If we set $M = \overline{B\left((x_0, s_0), \delta_0\right)}$ we are in the hypotheses of Lemma 1.5, Statement B by Ambrosio [4] and thus there exist continuous functions $a_h : M \to \mathbb{R}$ and $b_h : M \to \mathbb{R}^{mn}$ such that

$$f(x,s,\xi) = \sup_{h\in\mathbb{N}} \left\{ a_h(x,s) + b_h(x,s) \cdot \xi \right\},$$

for every $(x,s,\xi) \in \overline{B_{\delta_0}(x_0,s_0)} \times \mathbb{R}^{mn}$. Now we can conclude the proof working as in Theorem 1.1 of [42] quoted above. □

Finally we note that, under the assumption (4.47), if the coerciveness constant doesn't depend on $s$, we can consider the case in which $f$ is only Borel measurable with respect to $x$. The following result follows Proposition 5.6 in Gori and Marcellini [55].

**Proposition 4.21.** *Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{nm}$ be a Borel function such that, for $\mathcal{L}^n$-a.e. $x \in \Omega$, $f(x,\cdot,\cdot)$ is l.s.c., for every $s \in \mathbb{R}$, $f(x,s,\cdot)$ is convex and, for every $(s,\xi) \in \mathbb{R}^m \times \mathbb{R}^{mn}$,*

$$f(x,s,\xi) \geq a(x)\,|\xi|^p\,, \tag{4.48}$$

*where $p > 1$ and $a : \Omega \to [0,\infty]$ is a Borel function such that*

$$a^{-\frac{1}{p-1}} \in L^1_{\mathrm{loc}}(\Omega). \tag{4.49}$$

*Then the functional $I$ is l.s.c. on $W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$ with respect to the $L^1_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$ convergence.*

*Proof.* Let $u_h, u \in W^{1,1}_{\mathrm{loc}}\left(\Omega,\mathbb{R}^m\right)$ such that $u_h \to u$ in $L^1_{\mathrm{loc}}\left(\Omega,\mathbb{R}^m\right)$. We have to prove that

$$\liminf_{h\to\infty} I\left(u_h,\Omega\right) \geq I\left(u,\Omega\right). \tag{4.50}$$

To this aim we can assume that

$$\liminf_{h\to\infty} I\left(u_h,\Omega\right) = \lim_{h\to\infty} I\left(u_h,\Omega\right) = C < +\infty.$$

By Hölder inequality and by the coercivity assumption (4.48), for every $B \in \mathcal{B}(\Omega)$, $h \in \mathbb{N}$, we have

$$\int_B |\nabla u_h(x)|\,dx = \int_B a(x)^{\frac{1}{p}}\,|\nabla u_h(x)| \cdot a(x)^{-\frac{1}{p}}\,dx$$

$$\leq \left\{\int_B a(x)\,|\nabla u_h(x)|^p\,dx\right\}^{\frac{1}{p}} \cdot \left\{\int_B a(x)^{-\frac{1}{p}\cdot\frac{p}{p-1}}\,dx\right\}^{\frac{(p-1)}{p}}$$

$$\leq \left\{\int_B f\left(x,u_h(x),\nabla u_h(x)\right)dx\right\}^{\frac{1}{p}} \cdot \left\{\int_B a(x)^{-\frac{1}{p-1}}\,dx\right\}^{\frac{p-1}{p}} \leq C^{\frac{1}{p}}\left(\int_B a(x)^{-\frac{1}{p-1}}\right)^{\frac{p-1}{p}}.$$

Then $\{\nabla u_h\}_{h=1}^{\infty}$ is a sequence locally equi-integrable on $\Omega$ and in fact $u_h \to u$ in $w$-$W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$: therefore we can apply the lower semicontinuity theorem by Ioffe (see [56]). □

# Chapter 5

# Lower semicontinuity for supremal functionals

As stated in the introduction, in this chapter we consider the problem, first approached by Gori and Maggi in [52], to understand in which cases the functional

$$S(u,\Omega) = \operatorname*{ess\,sup}_{x\in\Omega} g(x,u(x),\nabla u(x)),$$

is l.s.c. on $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ with respect to the $L^{\infty}_{\mathrm{loc}}(\Omega)$ convergence.

The more natural problem of the lower semicontinuity of $S$ with respect to the $w^*$-$W^{1,\infty}_{\mathrm{loc}}(\Omega)$ convergence has been deeply studied by several authors (see Barron and Jensen [14], Barron and Liu [16] and Barron, Jensen and Wang (see [15]) and it has been immediately pointed out the fundamental role of the level convexity of $g$ in the gradient variable.

One of the main results on this topic is due to Barron, Jensen and Wang (see [15] Theorem 3.4), who provide for supremal functionals a result similar to the classical one of Ioffe [56] in the integral setting.

**Theorem 5.1.** *Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0,\infty]$ be a proper and Borel function such that, for $\mathcal{L}^n$-a.e. $x \in \Omega$, $g(x,\cdot,\cdot)$ is l.s.c. and, for every $s \in \mathbb{R}$, $g(x,s,\cdot)$ is level convex. Then the functional $S$ is l.s.c. on $W^{1,\infty}_{\mathrm{loc}}(\Omega)$ with respect to the $w^*$-$W^{1,\infty}_{\mathrm{loc}}(\Omega)$ convergence.*

In order to prove this theorem the authors show first the fact that the lower semicontinuity of a supremal functional can be always reduced to the lower semicontinuity of the elements of a suitable family of integrals whose integrands can also be infinite. Then they conclude proving that the hypotheses of Theorem 5.1 allow to apply to every functional of the considered family just Ioffe's theorem quoted above.

Considering now the lower semicontinuity of $S$ with respect to the $L^{\infty}_{\mathrm{loc}}(\Omega)$ convergence, it is quite simple to understand that the *integral reduction* approach by Barron, Jensen and Wang is not suitable anymore. Indeed, the theory of lower semicontinuity for integrals with respect to this kind of convergence requires the integrand to be regular enough in the lower order variables and, in particular, to be finite (as seen in Chapter 4).

Thus the problem must be approached in a different way. Following [52], by means of suitable tools from the convex analysis, we are able to find simple conditions on $g$ sufficient for the lower semicontinuity, that generalize Theorem 1.3 in [52]; moreover we carry on also an analysis on the necessary conditions. Example 5.2, Theorem 5.5 and Theorem 5.8 below are the main results of this approach.

Finally we stress the fact that in the whole chapter we are dealing with scalar valued functions $u : \Omega \to \mathbb{R}$. However, the situation in the vectorial setting, at least about the sufficient conditions, is clarified by Example 4.14. Indeed, it shows that Theorems 5.4 and 5.5 don't hold in the vectorial case, that is, when $g : \Omega \times \mathbb{R}^m \times \mathbb{R}^{mn} \to [0,\infty]$ and $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$.

## 5.1 A counterexample

An important remark about Theorem 5.1 is that it holds even if we consider the lower semicontinuity of $S$ on $W^{1,1}_{\mathrm{loc}}(\Omega)$ with respect to the $w$-$W^{1,1}_{\mathrm{loc}}(\Omega)$ convergence.

In Example 5.2 below we consider an upper semicontinuous function $g = g(x,\xi) : (0,1)\times\mathbb{R} \to [0,\infty)$ such that, for every $x \in (0,1)$, $g(x,\cdot)$ is level convex and such that the supremal functional $S$, related to $g$, is not l.s.c. on a suitable sequence of functions in $C^\infty([0,1])$ converging with respect to the $w^*$-$BV(0,1)$ convergence. This means that if in Theorem 5.1 we want to consider weaker convergences than the $w$-$W^{1,1}_{\mathrm{loc}}(\Omega)$, like the $w^*$-$BV_{\mathrm{loc}}(\Omega)$ convergence, we have to strengthen the measurability assumption of $g$ on the geometric variable.

This example, that can be found in [52] Example 2.2, suggests that a suitable regularity condition on $g$, in order to obtain the lower semicontinuity of $S$, is its global lower semicontinuity on $\Omega\times\mathbb{R}\times\mathbb{R}^n$ (obviously together with the level convexity on the gradient variable).

**Example 5.2.** Let us show that there exist a closed set $K \subset [0,1]$ of $\mathcal{L}^1$-positive measure, a sequence $\{u_h\}_{h=1}^\infty \subseteq C^\infty([0,1])$ such that $u_h \to 0$ in $L^\infty(0,1)$ and with

$$\sup_{h\in\mathbb{N}}\left\{\int_0^1 |u_h'(x)|dx\right\} < \infty, \tag{5.1}$$

and $0 < c < d$, $0 < \varepsilon < \frac{d-c}{2}$ such that, for every $x \in K$, $h \in \mathbb{N}$,

$$0 < c+\varepsilon < u_h'(x) < d-\varepsilon. \tag{5.2}$$

In this way, setting $f(\xi) = \left(\xi - \frac{(d-c)}{2}\right)^2$ and $g(x,\xi) = 1_K(x)f(\xi)$, we obtain

$$S(0,(0,1)) = \left(\frac{d-c}{2}\right)^2 > \left(\frac{d-c}{2}-\varepsilon\right)^2 \geq \liminf_{h\to\infty} S(u_h,(0,1)).$$

In order to do this, let us fix $t > 3$ and, for every $0 < a < b < 1$ with $b - a > t^{-1}$, we set

$$T_h([a,b]) = \left[a, \frac{b-a}{2} - \frac{1}{2t^h}\right] \cup \left[\frac{b-a}{2} + \frac{1}{2t^h}, b\right].$$

Then we define, by induction on $h \in \mathbb{N}$,

$$\mathcal{E}_1 = T_1([0,1]) = \left\{I_i^1\right\}_{i=1}^2, \quad \mathcal{E}_h = \bigcup_{i=1}^{2^{h-1}} T_h\left(I_i^{h-1}\right) = \left\{I_i^h\right\}_{i=1}^{2^h},$$

where, for every $h \in \mathbb{N}$, the intervals $I_i^h$ are ordered in such a way that $\sup I_i^h < \inf I_{i+1}^h$. Finally we set

$$K_h = \bigcup_{i=1}^{2^h} I_i^h, \quad \text{and} \quad K = \bigcap_{h=1}^\infty K_h.$$

Clearly $K$ is a closed set, and since

$$\mathcal{L}^1\left(I_i^h\right) = \frac{1}{2^h}\left\{1 - \frac{1}{t}\sum_{k=0}^{h-1}\left(\frac{2}{t}\right)^k\right\},$$

we have $\mathcal{L}^1(K) = (t-3)/(t-2)$ ($K$ is simply one of the standard variants of the Cantor's Set).

Fixed now $h \in \mathbb{N}$, for every $i \in \{1, \dots, 2^h\}$, let us define $u_h$ on $I_i^h$ as the affine function that takes the value 0 on the left extreme of $I_i^h$ and the value $2^{-h-1}$ on the right extreme. Then we extend $u_h$ on $[0,1]$ in a smooth way, under the constraint that, for every $i \in \{1, \dots, 2^h\}$ the total variation of $u_h$ on the interval between $I_i^h$ and $I_{i+1}^h$ is controlled by $3 \cdot 2^{-h-1}$.

Clearly, for every $x \in (0,1)$, it is $|u_h(x)| \leq 3 \cdot 2^{-h-1}$, and then $u_h \to 0$ in $L^\infty(0,1)$. Moreover the total variation can be estimated, looking carefully at the construction, by

$$\int_0^1 |u_h'(x)|\, dx \leq \left(2^h + (2^h - 1)\right) \frac{3}{2^{h+1}},$$

that implies (5.1). Finally, for every $h \in \mathbb{N}$ and $x \in K_h$, it is

$$u_h'(x) = \frac{1}{2^{h+1}} \cdot \frac{1}{\mathcal{L}^1\left(I_i^h\right)} = \frac{1}{2 \cdot \left\{1 - \frac{1}{t}\sum_{k=0}^{h-1}\left(\frac{2}{t}\right)^k\right\}},$$

thus, in particular, since we have

$$\lim_{h\to\infty} \left\{1 - \frac{1}{t}\sum_{k=0}^{h-1}\left(\frac{2}{t}\right)^k\right\} = \frac{t-3}{t-2} \in (0,1),$$

we can choose $c$, $d$ and $\varepsilon$ satisfying (5.2).

**Remark 5.3.** Example 5.2 provides a counterexample also in the integral setting: indeed, if we consider the integral functional $I$ in (1.1) built up starting from the function $g$ of Example 5.2 it says that Ioffe's lower semicontinuity Theorem (see [56]) doesn't hold with respect to $w^*$-$BV(\Omega)$ convergence, at least with upper semicontinuity with respect to the geometric variable $x$. This fact is classically shown with a counterexample by Carbone and Sbordone [23] (see also [22]).

## 5.2 Sufficient conditions

As Example 5.2 suggests, we will consider the case in which $g$ is l.s.c. on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and, of course, level convex in the gradient variable.

By means of Theorem 2.13 we can prove our lower semicontinuity results. The first one is quite surprising: it shows that in the scalar case we still have lower semicontinuity with respect to the $L^\infty_{\mathrm{loc}}(\Omega)$ convergence on $C^1$ sequences, even if level convexity is dropped (see [52] Theorem 1.1).

**Theorem 5.4.** *Let $g : \Omega\times\mathbb{R}\times\mathbb{R}^n \to [0,\infty]$ be a proper and l.s.c. function and let $\{u_h\}_{h=1}^\infty \subseteq C^1(\Omega)$, $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ such that $u_h \to u$ in $L^\infty_{\mathrm{loc}}(\Omega)$. Then*

$$S(u,\Omega) \leq \liminf_{h\to\infty} S(u_h, \Omega).$$

*Proof.* Let us take $t > \liminf_{h\to\infty} S(u_h,\Omega)$ (we can suppose $t < \infty$). By the continuity of $u_h$ and $\nabla u_h$, the function $g(x, u_h(x), \nabla u_h(x))$ of the $x$ variable is l.s.c. on $\Omega$, thus the essential supremum in the definition of $S$ is a point-wise supremum, that is,

$$S(u_h,\Omega) = \sup_{x\in\Omega} g(x, u_h(x), \nabla u_h(x)).$$

Then there exists $h_t \in \mathbb{N}$ such that, for every $h \geq h_t$, $x \in \Omega$, we have $g(x, u_h(x), \nabla u_h(x)) \leq t$.

Let now $\Omega_d \subseteq \Omega$ the set in which $u$ is differentiable and fix $x_0 \in \Omega_d$: we know that $\mathcal{L}^n(\Omega \setminus \Omega_d) = 0$ and that, applying Theorem 2.13 and Proposition 2.12(iv), there exists a subsequence $\{u_{h_k}\}_{k=1}^\infty$

and a sequence $\{x_k\}_{k=1}^{\infty}$, such that $x_k \to x_0$, $u_{h_k}(x_k) \to u(x_0)$ and $\nabla u_{h_k}(x_k) \to \nabla u(x_0)$. Hence, by the lower semicontinuity of $g$, we have, for every $x_0 \in \Omega_d$,

$$g(x_0, u(x_0), \nabla u(x_0)) \leq \liminf_{h\to\infty} g(x_h, u_h(x_h), \nabla u_h(x_h)) \leq t,$$

that is $S(u, \Omega) \leq t$: then the thesis follows. □

In order to prove a general lower semicontinuity theorem we need to assume the level convexity of $g$ in the gradient variable. We prove that if $g$ is l.s.c. and level convex in $\xi$ then $S$ is lower semicontinuous on $W^{1,\infty}_{\text{loc}}(\Omega)$ with respect to the $L^{\infty}_{\text{loc}}(\Omega)$ topology. Let us note that this theorem generalizes Theorem 1.3 in [52].

**Theorem 5.5.** *Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty]$ be a proper, l.s.c. function such that, for every $(x, s) \in \Omega \times \mathbb{R}$, $g(x, s, \cdot)$ is level convex. Then the functional $S$ is l.s.c. in $W^{1,\infty}_{\text{loc}}(\Omega)$ with respect to the $L^{\infty}_{\text{loc}}(\Omega)$ convergence.*

*Proof.* Let $u, u_h \in W^{1,\infty}_{\text{loc}}(\Omega)$ such that $u_h \to u$ in $L^{\infty}_{\text{loc}}(\Omega)$. We can suppose that

$$\liminf_{h\to\infty} S(u_h, \Omega) = \lim_{h\to\infty} S(u_h, \Omega) = L < +\infty.$$

Let now $t > L$: then there exists $h_t \in \mathbb{N}$ such that, for every $h \geq h_t$, $S(u_h, \Omega) \leq t$. In particular we can find $\Omega_1 \subseteq \Omega$, with $\mathcal{L}^n(\Omega \setminus \Omega_1) = 0$, such that, for every $h \geq h_t$, $x \in \Omega_1$, we have $g(x, u_h(x), \nabla u_h(x)) \leq t$ and $\nabla u(x), \nabla u_h(x)$ exist.

We claim now that the l.s.c. and the level convexity of $g$ with respect to the gradient variable imply that, for every $h \geq h_t$, $x \in \Omega$, $\xi \in \partial_c u_h(x)$, we have $g(x, u_h(x), \xi) \leq t$.

Indeed, let us consider $h \geq h_t$, $x \in \Omega$ and the set $\overline{\nabla} u_h(x)$ as in (2.12) built up with respect to $\Omega_1$. By the l.s.c. of $g$ we trivially have that, for every $x \in \Omega$, $\xi \in \overline{\nabla} u_h(x)$, $g(x, u_h(x), \xi) \leq t$.

Now, by Proposition 2.12(i), we still have $\partial_c u_h(x) = \text{co}\left(\overline{\nabla} u_h(x)\right)$ and then, by Theorem 2.14, for every $\xi \in \partial_c u_h(x)$, there exist $\{\xi_j\}_{j=1}^{n+1} \subseteq \overline{\nabla} u_h(x)$ and $\{\lambda_j\}_{j=1}^{n+1}$, with $0 \leq \lambda_j \leq 1$ and $\sum_{j=1}^{n+1} \lambda_j = 1$, such that $\xi = \sum_{j=1}^{n+1} \lambda_j \xi_j$. Thus, by the definition of level convexity,

$$g(x, u_h(x), \xi) = g\left(x, u_h(x), \sum_{j=1}^{n+1} \lambda_j \xi_j\right) \leq \bigvee_{j=1}^{n+1} g(x, u_h(x), \xi_j) \leq t,$$

and the claim is proved.

Fixed now $x_0 \in \Omega_1$ we have that, by Theorem 2.13, there exists a subsequence $\{u_{h_k}\}_{k=1}^{\infty}$ and two sequences $\{x_k\}_{k=1}^{\infty}$, $\{\xi_k\}_{k=1}^{\infty}$ such that, for every $k \in \mathbb{N}$, $\xi_k \in \partial_c u_{h_k}(x_k)$ and it holds $x_k \to x_0$, $u_{h_k}(x_k) \to u(x_0)$ and $\xi_k \to \nabla u(x_0)$. Then

$$(x_k, u_{h_k}(x_k), \xi_k) \to (x_0, u(x_0), \nabla u(x_0))$$

and by the lower semicontinuity of $g$ it is $g(x_0, u(x_0), \nabla u(x_0)) \leq t$. Thus $S(u, \Omega_1) = S(u, \Omega) \leq t$. Since this holds for every $t > L$ it follows $S(u, \Omega) \leq L$ and we get the proof. □

## 5.3 Necessary conditions

We end this chapter trying to prove that the level convexity of $g$ in the gradient variable is a necessary condition for the lower semicontinuity of $S$. What we would like it happens, in order to find a theorem stating necessary and sufficient conditions, is that supposing $g$ l.s.c. on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ and that, for every open set $\Omega' \subseteq \Omega$, $S(u, \Omega')$ is l.s.c. on $W^{1,\infty}_{\text{loc}}(\Omega')$ with respect to the $L^{\infty}_{\text{loc}}(\Omega')$ convergence, then $g$ is level convex in the $\xi$ variable.

However, the situation is more complicated it seems. We propose here an example that shows a particular situation we can find.

**Example 5.6.** Let $\Omega = (0,1)$ and $g : \Omega \times \mathbb{R} \times \mathbb{R} \to [2, \infty)$ be a continuous function such that, for every $(x,s) \in \Omega \times \mathbb{R}$, $g(x,s,\cdot)$ is level convex. By Theorem 5.5, for every $\Omega' \subseteq \Omega$ open set, $S(u,\Omega')$ is l.s.c on $W^{1,\infty}_{\text{loc}}(\Omega')$ with respect to the $L^{\infty}_{\text{loc}}(\Omega')$ convergence. Let us define the following closed set

$$A = \{(x,x,\xi) : x \in \Omega, \xi \in (-\infty, 0]\} \subseteq \Omega \times \mathbb{R} \times \mathbb{R},$$

and let us consider a continuous function $h : A \to [0,1]$ such that, for every $x \in \Omega$, $h(x,x,\cdot)$ is not level convex in the $\xi$ variable. We set now

$$g'(x,s,\xi) = \begin{cases} g(x,s,\xi) & \text{if} \quad (x,s,\xi) \in (\Omega \times \mathbb{R} \times \mathbb{R}) \setminus A, \\ h(x,s,\xi) & \text{if} \quad (x,s,\xi) \in A; \end{cases}$$

by construction we have that $g'$ is a l.s.c. function that doesn't satisfy the hypotheses of Theorem 5.5 about the level convexity. Note also that $g'$ cannot be obtain by $g$ by means of a trivial modification of $g$ on a set of the type $M \times \mathbb{R} \times \mathbb{R}^n$, where $M \subseteq \Omega$ and $\mathcal{L}^1(M) = 0$.

Let us denote with $S'$ the supremal functional associated to $g'$ and prove that, for every $\Omega' \subseteq \Omega$ open set, $S'(u,\Omega')$ is l.s.c. on $W^{1,\infty}_{\text{loc}}(\Omega')$ with respect to the $L^{\infty}_{\text{loc}}(\Omega')$ convergence.

In order to do this we claim at first that, for every function $u \in W^{1,\infty}_{\text{loc}}(\Omega)$, defined $\Omega_d = \{x \in \Omega : u'(x) \,\text{exists}\}$, the set

$$Q(u) = \{x : (x, u(x), u'(x)) \in A\} \cap \Omega_d$$

has $\mathcal{L}^1$-null measure: let us note at first that every point of $Q(u)$ is isolated. Indeed, for every $x \in Q(u)$ we have that $u'(x)$ exists, $u(x) = x$ and $u'(x) \le 0$: then, in particular, we have

$$\lim_{y \to x} \frac{u(y) - u(x)}{y - x} \le 0.$$

It follows that we can find a neighbor $V(x)$ of $x$ such that, for every $y \in V(x) \setminus \{x\}$, $u(y) \neq y$, else there exists $y_n \to x$ such that $u(y_n) = y_n$ that implies

$$\lim_{n \to \infty} \frac{u(y_n) - u(x)}{y_n - x} = \lim_{n \to \infty} \frac{y_n - x}{y_n - x} = 1.$$

Now let us define, for every $i \in \mathbb{N}$,

$$Q_i(u) = \left\{ x \in Q(u) : B_{\frac{1}{i}}(x) \cap Q(u) = \{x\} \right\} \subseteq \Omega.$$

Trivially there are only a finite number of elements in every $Q_i(u)$ and since $Q(u) = \bigcup_{i=1}^{\infty} Q_i(u)$ we have that $Q(u)$ is countable thus, in particular, of $\mathcal{L}^1$-null measure and the claim is proved.

From this fact it follows that, for every open set $\Omega' \subseteq \Omega$ and for every function $u \in W^{1,\infty}_{\text{loc}}(\Omega')$, $S(u,\Omega') = S'(u,\Omega')$. Indeed, we have

$$S'(u,\Omega') = \operatorname*{ess\,sup}_{x \in \Omega'} g'(x,u(x),u'(x)) = \operatorname*{ess\,sup}_{x \in (\Omega_d \setminus Q(u)) \cap \Omega'} g(x,u(x),u'(x)) =$$

$$\operatorname*{ess\,sup}_{x \in (\Omega_d \setminus Q(u)) \cap \Omega'} g(x,u(x),u'(x)) = S(u,\Omega').$$

Thus $S'(u,\Omega')$ is l.s.c. on $W^{1,\infty}_{\text{loc}}(\Omega')$ with respect to the $L^{\infty}_{\text{loc}}(\Omega')$ convergence even if $g'$ is not level convex in the gradient variable.

Example 5.6 show that, starting from a l.s.c. function $g$ such that its associated supremal functional is l.s.c., without further information on the regularity of $g$, we have no hope to prove nothing about its convexity property.

However, at the same time, the example shows that it is natural to identify two l.s.c. functions $g, g'$ with the property that, for every $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ and for $\mathcal{L}^n$-a.e. $x \in \Omega$,

$$g(x, u(x), \nabla u(x)) = g'(x, u(x), \nabla u(x)),$$

since their supremal functionals agree. This identification clearly defines an equivalence relationship that we denote by $\sim$.

From this new point of view, given for instance a l.s.c. function $g$, the level convexity as a necessary condition to the lower semicontinuity have to be interpreted in the following way: there exists a l.s.c. function $g'$ such that $g \sim g'$ and, for every $(x, s) \in \Omega \times \mathbb{R}$, $g'(x, s, \cdot)$ is level convex.

In the following we will study this relationship restricted to a particular class of functions satisfying same regularities and we try to understand the form of the equivalence classes. Our purpose, in order to prove necessary conditions, is to find a particular class of functions in which every equivalence class for the relationship $\sim$ is a singleton. Evidently, by Example 5.6, we cannot consider only l.s.c. functions but we need much more regularity.

The following simple proposition identify suitable regularity conditions.

**Proposition 5.7.** *Let $g, g' : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty]$ be proper and l.s.c. functions such that, for every $\xi \in \mathbb{R}^n$, $g(\cdot, \cdot, \xi)$ and $g'(\cdot, \cdot, \xi)$ are continuous on $\Omega \times \mathbb{R}$ and let us suppose that $g \sim g'$. Then $g = g'$.*

*Proof.* Let us fix $(x_0, s_0, \xi_0) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$ and $u(x) = s_0 + \langle \xi_0, x - x_0 \rangle$. Then, since $g \sim g'$, we have that for $\mathcal{L}^n$-a.e. $x \in \Omega$, $g(x, u(x), \xi_0) = g'(x, u(x), \xi_0)$. In particular this holds for a dense subset of $\Omega$ thus, by continuity, we conclude that $g(x_0, s_0, \xi_0) = g'(x_0, s_0, \xi_0)$. □

The following theorem shows that, considering the set of functions of the proposition above, the level convexity is just a necessary condition for lower semicontinuity.

**Theorem 5.8.** *Let $g : \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0, \infty]$ be a proper and l.s.c. function such that, for every $\xi \in \mathbb{R}^n$, $g(\cdot, \cdot, \xi)$ is continuous on $\Omega \times \mathbb{R}$ and such that, for every $\Omega' \subseteq \Omega$ open set, $S(\cdot, \Omega')$ is l.s.c. on $W^{1,\infty}_{\text{loc}}(\Omega')$ with respect to the $w^*$-$W^{1,\infty}_{\text{loc}}(\Omega)$ convergence. Then, for every $(x, s) \in \Omega \times \mathbb{R}$, $g(x, s, \cdot)$ is level convex.*

*Proof.* We argue by contradiction. If there exists $(x_0, s_0) \in \Omega \times \mathbb{R}$ such that $g(x_0, s_0, \cdot)$ is not level convex, then there exists $r_0 \in \mathbb{R}$ such that $E_{g(x_0,s_0,\cdot)}(r_0)$ is not convex in $\mathbb{R}^n$. Thus there are $\xi_1, \xi_2 \in E_{g(x_0,s_0,\cdot)}(r_0)$ and $t \in (0, 1)$ such that

$$\xi_0 = t\xi_1 + (1 - t)\xi_2 \notin E_{g(x_0,s_0,\cdot)}(r_0).$$

Let us consider now the function $u(x) = s_0 + \langle \xi_0, x - x_0 \rangle \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^n)$. It's quite simple to find a sequence $u_h \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^n)$ such that $u_h \to u$ in $w^*$-$W^{1,\infty}_{\text{loc}}(\mathbb{R}^n)$ and such that, for every $h \in \mathbb{N}$, when $\nabla u_h$ exists, its value is $\xi_1$ or $\xi_2$. Such a sequence can be built up working in this way. Let us consider the function $v : \mathbb{R} \to \mathbb{R}^n$ defined componentwise as

$$v_i(y) = \begin{cases} \xi_{1,i} & y \in (x_{0,i}, x_{0,i} + t), \\ \xi_{2,i} & y \in (x_{0,i} + t, x_{0,i} + 1), \end{cases}$$

where $i \in \{1, \ldots, n\}$, and extended on $\mathbb{R}$ by periodicity. Defined now $v_h(y) = v(hy) : \mathbb{R} \to \mathbb{R}^n$, we can consider the functions

$$u_h(x) = s_0 + \sum_{i=1}^{n} \int_{x_{0,i}}^{x_i} v_{h,i}(y)dy \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^n).$$

It's simple to verify that $u_h \to u$ in $w^*$-$W^{1,\infty}_{\text{loc}}(\mathbb{R}^n)$. Then, using this sequence, for every $\delta > 0$, we have

$$r_0 < g(x_0, s_0, \xi_0) \leq S(u, B(x_0, \delta)) \leq \liminf_{h \to \infty} S(u_h, B(x_0, \delta)).$$

Defining $2\varepsilon = g(x_0, s_0, \xi_0) - r_0 > 0$, by the continuity of $g(\cdot, \cdot, \xi)$ and the hypotheses on $u_h$, if $h$ is big enough and $\delta$ is small enough, we have, for every $x \in B(x_0, \delta)$,

$$g(x, u_h(x), \xi_1) \leq r_0 + \frac{\varepsilon}{2}, \quad g(x, u_h(x), \xi_2) \leq r_0 + \frac{\varepsilon}{2}.$$

Thus it follows

$$r_0 + \varepsilon \leq \liminf_{h \to \infty} S\left(u_h, B(x_0, \delta)\right) \leq r_0 + \frac{\varepsilon}{2}$$

and we have found a contradiction. □

# Chapter 6

# Functionals defined on measures

In the first part of this chapter we study some particular integral and supremal functionals defined on the space of Radon measures. Let $f, g : \mathbb{R}^m \to [0, \infty]$ be proper and Borel functions and let us consider the following functionals defined, for every $\lambda \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$, as

$$\mathbb{I}_m(\lambda, \Omega) = \int_\Omega f(\lambda^a(x))dx + \int_\Omega f^\infty \left( \frac{d\lambda^s}{d|\lambda^s|}(x) \right) d|\lambda^s|(x), \tag{6.1}$$

and

$$\mathbb{S}_m(\lambda, \Omega) = \left[ \operatorname*{ess\,sup}_{x \in \Omega} g(\lambda^a(x)) \right] \vee \left[ |\lambda^s|\text{-}\operatorname*{ess\,sup}_{x \in \Omega} g^\natural \left( \frac{d\lambda^s}{d|\lambda^s|}(x) \right) \right], \tag{6.2}$$

where the measure $\lambda$ has been decomposed as in (2.3) with respect to the Lebesgue's measure $\mathcal{L}^n$.

The main purpose of the first part of this chapter is to prove that, if we suppose $f$ l.s.c. and convex and $g$ l.s.c. and level convex, then $\mathbb{I}_m$ and $\mathbb{S}_m$ are l.s.c. on $\mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$ with respect to the $w^*$-$\mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$ convergence.

The functional $\mathbb{I}_m$ has been deeply studied by several authors and in particular by Goffman and Serrin [49], Ambrosio and Buttazzo [5], Bouchitté [17], Bouchitté and Valadier [21] and De Giorgi, Ambrosio and Buttazzo [36] (see also the book of Buttazzo [22]). In these papers necessary and sufficient conditions on $f$ for the lower semicontinuity of $\mathbb{I}_m$ have been found even when $f$ depends on the geometric variable $x$.

On the contrary, the functional $\mathbb{S}_m$ defined above seems to be new.

The strategy we follow in order to prove the lower semicontinuity of $\mathbb{I}_m$ and $\mathbb{S}_m$ is essentially the same of Serrin [69] and, in particular, of Goffman and Serrin [49], where the study of the functional $\mathbb{I}_m$ is approached for the first time. Indeed, as already shown in [51], the arguments of these papers can be adapted to the supremal setting too[1].

Before stating the main theorem concerning $\mathbb{I}_m$ and $\mathbb{S}_m$, let us define, for every $\lambda \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$ and with respect to the decomposition (2.3) of $\lambda$ with respect to $\mathcal{L}^n$, the following functionals

$$I_m(\lambda, \Omega) = \int_\Omega f(\lambda^a(x))dx, \tag{6.3}$$

$$S_m(\lambda, \Omega) = \operatorname*{ess\,sup}_{x \in \Omega} g(\lambda^a(x)), \tag{6.4}$$

$$I^*_m(\lambda, \Omega) = \inf \left\{ \liminf_{h \to \infty} I_m(\lambda_h, \Omega_h) : \lambda_h \in \mathcal{M}_{\text{loc}}(\Omega_h, \mathbb{R}^m), \lambda_h << \mathcal{L}^n, \lambda_h \to \lambda\ w^*\text{-}\mathcal{M}_{\text{loc}}(\updownarrow \Omega, \mathbb{R}^m) \right\}, \tag{6.5}$$

[1]Note that in [51] it is considered the less general case in which $\mathbb{S}_m$ is defined on $BV_{\text{loc}}(\Omega)$, that is $\lambda = Du$ (see Chapter 7).

and

$$S_m^*(\lambda,\Omega) = \inf\left\{\liminf_{h\to\infty} S_m(\lambda_h,\Omega_h) : \lambda_h \in \mathcal{M}_{\rm loc}(\Omega_h,\mathbb{R}^m), \lambda_h << \mathcal{L}^n, \lambda_h \to \lambda\ w^*\text{-}\mathcal{M}_{\rm loc}(\updownarrow \Omega,\mathbb{R}^m)\right\}. \tag{6.6}$$

The following theorem shows the complete analogy between the functionals (6.1) and (6.2).

**Theorem 6.1.** *Let $f, g : \mathbb{R}^m \to [0,\infty]$ be proper and Borel functions and consider the two following pairs of conditions:*

*(a) $f$ is l.s.c. and convex on $\mathbb{R}^m$,*

*(b) for every open set $\Omega \subseteq \mathbb{R}^n$, the functional $\mathbb{I}_m$ is l.s.c. on $\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ with respect to the $w^*$-$\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ convergence,*

*and*

*(i) $g$ is l.s.c. and level convex on $\mathbb{R}^m$,*

*(ii) for every open set $\Omega \subseteq \mathbb{R}^n$, the functional $\mathbb{S}_m$ is l.s.c. on $\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ with respect to the $w^*$-$\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ convergence.*

*Then (a) is equivalent to (b) so as (i) to (ii). Moreover when the previous conditions are satisfied, for every $\lambda \in \mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$, we have*

$$\mathbb{I}_m(\lambda,\Omega) = I_m^*(\lambda,\Omega) = \lim_{\rho\to 0} I_m(\lambda_\rho\cdot\mathcal{L}^n,\Omega_\rho), \tag{6.7}$$

$$\mathbb{S}_m(\lambda,\Omega) = S_m^*(\lambda,\Omega) = \lim_{\rho\to 0} S_m(\lambda_\rho\cdot\mathcal{L}^n,\Omega_\rho), \tag{6.8}$$

*where $\lambda_\rho$ denotes the convolution of $\lambda$, and $\mathbb{I}_m$ and $\mathbb{S}_m$ are convex and level convex respectively on $\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$.*

The second part of the chapter is devoted to the study of the lower semicontinuity of other two classes of functionals defined, for every $\lambda \in \mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$, as

$$\mathbf{I}_m(\lambda,\Omega) = \int_\Omega f(\lambda^a(x))dx + \int_\Omega f^\infty\left(\frac{d\lambda^c}{d|\lambda^c|}(x)\right) d|\lambda^c|(x) + \sum_{x\in A_\lambda} \sigma(\lambda^\#(x)), \tag{6.9}$$

and

$$\mathbf{S}_m(\lambda,\Omega) = \left[\operatorname*{ess\,sup}_{x\in\Omega} g(\lambda^a(x))\right] \vee \left[|\lambda^c|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} g^\natural\left(\frac{d\lambda^c}{d|\lambda^c|}(x)\right)\right] \vee \left[\bigvee_{x\in A_\lambda} \gamma(\lambda^\#(x))\right], \tag{6.10}$$

where $f, g, \gamma, \sigma : \mathbb{R}^m \to [0,\infty]$ are Borel functions.

First of all we remark that $\mathbf{I}_m$ and $\mathbf{S}_m$ generalize $\mathbb{I}_m$ and $\mathbb{S}_m$ respectively: in fact, when $\sigma$ is positively homogeneous of degree 1 and $\gamma$ is positively homogeneous of degree 0, we have

$$\sum_{x\in A_\lambda} \sigma(\lambda^\#(x)) = \int_{A_\lambda} \sigma\left(\frac{d\lambda^\#}{d|\lambda^\#|}(x)\right) d|\lambda^\#|(x) \quad \text{and} \quad \bigvee_{x\in A_\lambda} \gamma(\lambda^\#(x)) = |\lambda^\#|\text{-}\operatorname*{ess\,sup}_{x\in A_\lambda} \gamma\left(\frac{d\lambda^\#}{d|\lambda^\#|}(x)\right).$$

If we consider now $\sigma = f^\infty$ and $\gamma = g^\natural$, applying Propositions 2.5 and 2.6, we obtain, for every $\lambda \in \mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$, $\mathbb{I}_m(\lambda,\Omega) = \mathbf{I}_m(\lambda,\Omega)$ and $\mathbb{S}_m(\lambda,\Omega) = \mathbf{S}_m(\lambda,\Omega)$: then, when this particular case is considered, we have that Theorem 6.1 already solves the problem of the lower semicontinuity of $\mathbf{I}_m$ and $\mathbf{S}_m$.

It is also clear that, even if we suppose that $f$ and $g$ are convex and level convex, $\mathbf{I}_m$ and $\mathbf{S}_m$ could fail to be convex and level convex on $\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ because of the presence of the functions $\sigma$ and $\gamma$: for this reason, with a little abuse, we will call $\mathbf{I}_m$ the *non convex* functional and $\mathbf{S}_m$ the *non level convex* one, contrarily to $\mathbb{I}_m$ and $\mathbb{S}_m$.

Finally let us note also that the value of $\sigma$ and $\gamma$ for $\xi = 0$ does not enter in the computation of $\mathbf{I}_m$ and $\mathbf{S}_m$, thus these two functions could be defined only on $\mathbb{R}^m\setminus\{0\}$: however, to simplify several arguments, we prefer to define $\sigma$ and $\gamma$ on the whole space $\mathbb{R}^m$ adding the condition $\sigma(0) = \gamma(0) = 0$ that, as noted, doesn't reduce the generality of the statements.

The lower semicontinuity properties of $\mathbf{I}_m$ have been studied for the first time by Bouchitté and Buttazzo in [18] and other aspects, as the problem of the relaxation or the one of the integral representation, have been analyzed by the same authors in subsequent works (see [19], [20]). In the supremal case instead, this kind of study has not been developed, even if a functional very similar to $\mathbf{S}_m$ but defined on $BV_{\rm loc}(a,b)$ has been recently studied by Alicandro, Braides and Cicalese in [3] (see also Chapter 7).

Among the results proved for the functional $\mathbf{I}_m$ it is interesting to quote the following one in which we can find both necessary and sufficient conditions (it follows from Theorem 3.3 [18] and Theorem 2.3 [19]).

**Theorem 6.2.** *Let $f : \mathbb{R}^m \to [0,\infty]$ and $\sigma : \mathbb{R}^m \to [0,\infty]$ be proper and Borel functions with $\sigma(0) = 0$. Then the following conditions are equivalent:*

*(a) $f$ is l.s.c. and convex on $\mathbb{R}^m$, $\sigma$ is l.s.c and sub-linear on $\mathbb{R}^m$ with $\sigma(0) = 0$ and, for every $\xi \in \mathbb{R}^m \setminus \{0\}$, $f^\infty(\xi) = \sigma^0(\xi)$,*

*(b) for every open set $\Omega \subseteq \mathbb{R}^n$, the functional $\mathbf{I}_m$ is l.s.c. on $\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ with respect to the $w^*$-$\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ convergence.*

Here, keeping in mind the previous theorem proved for $\mathbf{I}_m$, we propose some analogous results on $\mathbf{S}_m$. We are able to prove in particular the two following theorems where we give some necessary and sufficient conditions for the lower semicontinuity: even in this case it is clear the analogy between the integral and the supremal setting.

**Theorem 6.3.** *Let $g : \mathbb{R}^m \to [0,\infty]$ and $\gamma : \mathbb{R}^m \to [0,\infty]$ be proper and Borel functions with $\gamma(0) = 0$. Let us suppose that, for every open set $\Omega \subseteq \mathbb{R}^n$, the functional $\mathbf{S}_m$ is l.s.c. on $\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ with respect to the $w^*$-$\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ convergence. Then $g$ is l.s.c. and level convex on $\mathbb{R}^m$ and, defined $l = \inf\{g(\xi) : \xi \in \mathbb{R}^m\}$, $\gamma \vee l$ is l.s.c. and sub-maximal on $\mathbb{R}^m$. Moreover, for every $\xi \in \mathbb{R}^m \setminus \{0\}$, $g^\natural(\xi) = (\gamma \vee l)^\flat(\xi)$.*

**Theorem 6.4.** *Let $g : \mathbb{R}^m \to [0,\infty]$ be a l.s.c. and level convex function, $l = \inf\{g(\xi) : \xi \in \mathbb{R}^m\}$ and $\gamma : \mathbb{R}^m \to [0,\infty]$ such that $\gamma \vee l$ is a l.s.c and sub-maximal function with $\gamma(0) = 0$. Let us suppose that, for every $\xi \in \mathbb{R}^m \setminus \{0\}$, $g^\natural(\xi) = (\gamma \vee l)^\flat(\xi) = \infty$. Then the functional $\mathbf{S}_m$ is l.s.c. on $\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ with respect to the $w^*$-$\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ convergence.*

As we can see, we are not able to find conditions on $g$ and $\gamma$ that are both necessary and sufficient for the lower semicontinuity of $\mathbf{S}_m$: our conjecture is that in Theorem 6.4 the condition $g^\natural = (\gamma \vee l)^\flat = \infty$ could be replaced with the weaker condition $g^\natural = (\gamma \vee l)^\flat$ that we already know, thanks to Theorem 6.3, to be necessary and, as Alicandro, Braides and Cicalese show in [3] Theorem 4.4, to be also sufficient at least in dimension one.

## 6.1 Convex and level convex functionals

This section is devoted to the proof of Theorem 6.1. Let us start considering the implications $(b) \Rightarrow (a)$ and $(ii) \Rightarrow (i)$.

*Proof of Theorem 6.1* $(b) \Rightarrow (a)$. Let us consider in the following $\Omega = Q^n = (0,1)^n$ the unit cube in $\mathbb{R}^n$: clearly $\mathcal{L}^n(Q^n) = 1$. Let us fix $\xi_h, \xi_0 \in \mathbb{R}^m$, $\xi_h \to \xi_0$ and let $\lambda_h = \xi_h \cdot \mathcal{L}^n$, $\lambda_0 = \xi_0 \cdot \mathcal{L}^n$. We have $\lambda_h \to \lambda_0$ in $w^*$-$\mathcal{M}_{\text{loc}}(Q^n, \mathbb{R}^m)$ and then

$$f(\xi_0) = \mathbb{I}_m(\lambda_0, Q^n) \leq \liminf_{h\to\infty} \mathbb{I}_m(\lambda_h, Q^n) = \liminf_{h\to\infty} f(\xi_h),$$

that is $f$ is l.s.c..

Let us consider now $\xi, \eta \in \mathbb{R}^m$, $t \in (0,1)$ and a sequence $\{B_h\}_{h=1}^{\infty} \subseteq \mathcal{B}(Q^n)$ such that, for every $h \in \mathbb{N}$, $\mathcal{L}^n(B_h) = t$ and $1_{B_h}(x) \to t1_{Q^n}(x)$ in $w^*$-$L^\infty(Q^n)$ (see [2] Proposition 4.2 and Remark 4.3). If $\lambda_h = \xi 1_{B_h}(x) \cdot \mathcal{L}^n + \eta 1_{Q^n \setminus B_h}(x) \cdot \mathcal{L}^n$ then $\lambda_h \to \lambda_0 = (t\xi + (1-t)\eta) \cdot \mathcal{L}^n$ in $w^*$-$\mathcal{M}_{\text{loc}}(Q^n, \mathbb{R}^m)$ thus

$$f(t\xi + (1-t)\eta) = \mathbb{I}_m(\lambda_0, Q^n) \leq \liminf_{h\to\infty} \mathbb{I}_m(\lambda_h, Q^n)$$

$$= \liminf_{h\to\infty} \Big[\mathcal{L}^n(B_h) f(\xi) + \mathcal{L}^n(Q^n \setminus B_h) f(\eta)\Big] = t f(\xi) + (1-t) f(\eta),$$

that is $f$ is convex. □

*Proof of Theorem 6.1* $(ii) \Rightarrow (i)$. Let $\xi_h, \xi_0 \in \mathbb{R}^m$, $\xi_h \to \xi_0$ and let $\lambda_h = \xi_h \cdot \mathcal{L}^n$, $\lambda_0 = \xi_0 \cdot \mathcal{L}^n$. We have $\lambda_h \to \lambda_0$ in $w^*$-$\mathcal{M}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ and then

$$g(\xi_0) = \mathbb{S}_m(\lambda_0, \mathbb{R}^n) \leq \liminf_{h\to\infty} \mathbb{S}_m(\lambda_h, \mathbb{R}^n) = \liminf_{h\to\infty} g(\xi_h),$$

that is $g$ is l.s.c..

Let us consider now $\xi, \eta \in \mathbb{R}^m$, $t \in (0,1)$ and a sequence $\{B_h\}_{h=1}^{\infty} \subseteq \mathcal{B}(\mathbb{R}^n)$ such that $1_{B_h}(x) \to t1_{\mathbb{R}^n}(x)$ in $w^*$-$L^\infty(\mathbb{R}^n)$ (see [2] Proposition 4.2 and Remark 4.3). If $\lambda_h = \xi 1_{B_h}(x) \cdot \mathcal{L}^n + \eta 1_{\mathbb{R}^n \setminus B_h}(x) \cdot \mathcal{L}^n$ then $\lambda_h \to \lambda_0 = (t\xi + (1-t)\eta) \cdot \mathcal{L}^n$ in $w^*$-$\mathcal{M}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ thus

$$g(t\xi + (1-t)\eta) = \mathbb{S}_m(\lambda_0, \mathbb{R}^n) \leq \liminf_{h\to\infty} \mathbb{S}_m(\lambda_h, \mathbb{R}^n) = g(\xi) \vee g(\eta),$$

that is $g$ is level convex. □

The proofs of the implications $(a) \Rightarrow (b)$ and $(i) \Rightarrow (ii)$ of Theorem 6.1 are more difficult and for this reason their are developed in three steps. In the first one we prove that, for every $\lambda \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$, $I_m^*(\lambda, \Omega)$ and $S_m^*(\lambda, \Omega)$ can be computed considering only the convolutions of $\lambda$ as in (6.7) and (6.8) (see Theorem 6.6). This fact implies, in a very simple way, the second step that is the proof that $I_m^*$ and $S_m^*$ are l.s.c. on $\mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$ with respect to the $w^*$-$\mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$ convergence (see Proposition 6.7) and the properties of convexity and level convexity on $\mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$ (see Proposition 6.8). At last in the third step we prove the equalities $\mathbb{I}_m = I_m^*$ and $\mathbb{S}_m = S_m^*$ (see Theorems 6.16 and 6.17) that complete the proof of Theorem 6.1.

### 6.1.1 Simple results via convolutions

Let us start studying $I_m^*$ and $S_m^*$. Lemma 6.5 and Theorem 6.6 are completely inspired to Lemma 1 and Theorem 1 of [69] (see also [51]).

**Lemma 6.5.** *Let $f : \mathbb{R}^m \to [0,+\infty]$ be a proper, l.s.c and convex function and $g : \mathbb{R}^m \to [0,+\infty]$ be a proper, l.s.c and level convex function. Let $\lambda \in \mathcal{M}_{\text{loc}}(\Omega, \mathbb{R}^m)$, $\lambda_h \in \mathcal{M}_{\text{loc}}(\Omega_h, \mathbb{R}^m)$, $\lambda_h << \mathcal{L}^n$ such that $\lambda_h \to \lambda$ in $w^*$-$\mathcal{M}_{\text{loc}}(\updownarrow \Omega, \mathbb{R}^m)$. Then, for every $\rho > 0$, we have*

$$I_m\left(\lambda_\rho \cdot \mathcal{L}^n, \Omega_\rho\right) \leq \liminf_{h\to\infty} I_m(\lambda_h, \Omega_h).$$

*and*

$$S_m\left(\lambda_\rho \cdot \mathcal{L}^n, \Omega_\rho\right) \leq \liminf_{h\to\infty} S_m(\lambda_h, \Omega_h).$$

*Proof.* Let us fix $\rho > 0$ and $K \subset\subset \Omega_\rho$. We have that $(\lambda_h)_\rho \to \lambda_\rho$ point-wise on $K$ as $h \to \infty$. Indeed, if $x \in K$ we have that, if $h$ is large enough, $K \subset\subset \Omega_{h,\rho}$ and

$$|(\lambda_h)_\rho(x) - \lambda_\rho(x)| = \left| \int_{B(x,\rho)} k_\rho(x-y)d\lambda_h(y) - \int_{B(x,\rho)} k_\rho(x-y)d\lambda(y) \right|$$

that converges to zero as $h \to \infty$ by definition of $w^*$-$\mathcal{M}_{\rm loc}(\updownarrow \Omega, \mathbb{R}^m)$ convergence.

Let us prove the relation for $I_m$. Let us fix $x \in K$ and apply Theorem 2.18 with $\mu = k_\rho(x-y)\cdot\mathcal{L}^n$: we have

$$f(\lambda_\rho(x)) \le \liminf_{h\to\infty} f\Big((\lambda_h)_\rho(x)\Big) = \liminf_{h\to\infty} f\left(\int_{B(x,\rho)} \lambda_h^a(y)d\mu(y)\right) \le \liminf_{h\to\infty} \int_{B(x,\rho)} f(\lambda_h^a(y))d\mu(y).$$

Then integrating on $K$ and using Fubini's and Fatou's Theorems

$$\int_{x\in K} f(\lambda_\rho(x))dx \le \int_{x\in K} \liminf_{h\to\infty} \left(\int_{B(x,\rho)} k_\rho(x-y)f(\lambda_h^a(y))dy\right)dx$$

$$\le \liminf_{h\to\infty} \int_K \left(\int_{B(x,\rho)} k_\rho(x-y)f(\lambda_h^a(y))dy\right)dx \le \liminf_{h\to\infty} \int_{K+B(0,\rho)} \left(\int_{B(y,\rho)} k_\rho(x-y)dx\right) f(\lambda_h^a(y))dy$$

$$\le \liminf_{h\to\infty} \int_{\Omega_h} f(\lambda_h^a(x))dx = \liminf_{h\to\infty} I_m(\lambda_h, \Omega_h),$$

and we conclude by taking $K \uparrow \Omega_\rho$.

The argument is simpler for $S_m$. Let us fix $x \in K$ and apply Theorem 2.19 and (2.2) with $\mu = k_\rho(x-y)\cdot\mathcal{L}^n << \mathcal{L}^n$, in order to get

$$g(\lambda_\rho(x)) \le \liminf_{h\to\infty} g\Big((\lambda_h)_\rho(x)\Big) \le \liminf_{h\to\infty} \operatorname*{ess\,sup}_{y\in B(x,\rho)} g(\lambda_h^a(y)) \le \liminf_{h\to\infty} S_m(\lambda_h, \Omega_h).$$

Then

$$\sup_{x\in K} g(\lambda_\rho(x)) \le \liminf_{h\to\infty} S_m(\lambda_h, \Omega_h),$$

and we conclude by taking $K \uparrow \Omega_\rho$. □

By means of Lemma 6.5 we find a representation formula for $I_m^*$ and $S_m^*$ via convolutions, proving in this way the first step of the proof of Theorem 6.1.

**Theorem 6.6.** *Let $f : \mathbb{R}^m \to [0, +\infty]$ be a proper, l.s.c. and convex function and $g : \mathbb{R}^m \to [0, +\infty]$ be a proper, l.s.c. and level convex function. Then, for every $\lambda \in \mathcal{M}_{\rm loc}(\Omega, \mathbb{R}^m)$, we have*

$$I_m^*(\lambda, \Omega) = \lim_{\rho\to 0} I_m(\lambda_\rho \cdot \mathcal{L}^n, \Omega_\rho) \quad \text{and} \quad S_m^*(\lambda, \Omega) = \lim_{\rho\to 0} S_m(\lambda_\rho \cdot \mathcal{L}^n, \Omega_\rho). \tag{6.11}$$

*In particular the limits in the right hand sides exist.*

*Proof.* Let us consider $S_m$. For every $\lambda \in \mathcal{M}_{\rm loc}(\Omega, \mathbb{R}^m)$, $\lambda_\rho \cdot \mathcal{L}^n \to \lambda$ in $w^*$-$\mathcal{M}_{\rm loc}(\updownarrow \Omega, \mathbb{R}^m)$ as $\rho \to 0$ and $\lambda_\rho \cdot \mathcal{L}^n << \mathcal{L}^n$ so that, by Lemma 6.5,

$$\limsup_{\rho\to 0} S_m(\lambda_\rho \cdot \mathcal{L}^n, \Omega_\rho) \le S_m^*(\lambda, \Omega) \le \liminf_{\rho\to 0} S_m(\lambda_\rho \cdot \mathcal{L}^n, \Omega_\rho).$$

The same argument prove the equality also for the integral case. □

By means of Theorem 6.6 we are able to prove the two following propositions that complete the second step of the proof of Theorem 6.1.

**Proposition 6.7.** *Let $f : \mathbb{R}^m \to [0, +\infty]$ be a proper, l.s.c. and convex function and $g : \mathbb{R}^m \to [0, +\infty]$ be a proper, l.s.c. and level convex function. Then $I^*_m$ and $S^*_m$ are l.s.c. on $\mathcal{M}_{\rm loc}(\Omega, \mathbb{R}^m)$ with respect to the $w^*$-$\mathcal{M}_{\rm loc}(\Omega, \mathbb{R}^m)$ convergence.*

*Proof.* Since the same proof works in both cases we make it for the functional $S^*_m$. Let $\lambda, \lambda_h \in \mathcal{M}_{\rm loc}(\Omega, \mathbb{R}^m)$ be such that $\lambda_h \to \lambda$ in $w^*$-$\mathcal{M}_{\rm loc}(\Omega, \mathbb{R}^m)$. Then, by Theorem 6.6, there exists a sequence $\rho_h \downarrow 0$ such that, for every $h \in \mathbb{N}$,

$$S_m\Big((\lambda_h)_{\rho_h} \cdot \mathcal{L}^n, \Omega_{\rho_h}\Big) \le S^*_m(\lambda_h, \Omega) + \frac{1}{h}. \tag{6.12}$$

Now we claim that $(\lambda_h)_{\rho_h} \cdot \mathcal{L}^n \to \lambda$ in $w^*$-$\mathcal{M}_{\rm loc}(\updownarrow \Omega, \mathbb{R}^m)$. Indeed, fixed $\varphi \in C_c(\Omega)$, if $h$ is so large that $\mathrm{spt}(\varphi) + B(0, \rho_h) \subset \Omega_{\rho_h}$ and by using the usual properties of the convolutions (see [7] equation (2.3) page 42), we have

$$\left|\int_\Omega \varphi(x)(\lambda_h)_{\rho_h}(x)dx - \int_\Omega \varphi(x)d\lambda(x)\right| \le$$

$$\left|\int_\Omega \varphi(x)(\lambda_h)_{\rho_h}(x)dx - \int_\Omega \varphi(x)d\lambda_h(x)\right| + \left|\int_\Omega \varphi(x)d\lambda_h(x) - \int_\Omega \varphi(x)d\lambda(x)\right| \le$$

$$\int_\Omega |\varphi_{\rho_h}(x) - \varphi(x)|\, d\lambda_h(x) + \left|\int_\Omega \varphi(x)d\lambda_h(x) - \int_\Omega \varphi(x)d\lambda(x)\right|.$$

Now the first term goes to zero as $h \to \infty$ since $\varphi_{\rho_h} \to \varphi$ uniformly on $\Omega$ and there exists a constant $M > 0$ such that, for every $h \in \mathbb{N}$, $|\lambda_h|(\mathrm{spt}(\varphi) + B(0, \rho_h)) \le M$; the second term goes to zero by definition of $\{\lambda_h\}_{h=1}^\infty$.

Then using the definition of $S^*_m$ and (6.12) we obtain its lower semicontinuity since

$$S^*_m(\lambda, \Omega) \le \liminf_{h\to\infty} S_m\Big((\lambda_h)_{\rho_h} \cdot \mathcal{L}^n, \Omega_{\rho_h}\Big) \le \liminf_{h\to\infty} S^*_m(\lambda_h, \Omega),$$

which ends the proof. □

**Proposition 6.8.** *Let $f : \mathbb{R}^m \to [0, +\infty]$ be a proper, l.s.c. and convex function and $g : \mathbb{R}^m \to [0, +\infty]$ be a proper, l.s.c. and level convex function. Then, for every $\lambda_1, \lambda_2 \in \mathcal{M}_{\rm loc}(\Omega, \mathbb{R}^m)$ and for every $t \in (0, 1)$,*

$$I^*_m\Big(t\lambda_1 + (1-t)\lambda_2, \Omega\Big) \le tI^*_m(\lambda_1, \Omega) + (1-t)I^*_m(\lambda_2, \Omega),$$

*that is, $I^*_m$ is convex on the linear space $\mathcal{M}_{\rm loc}(\Omega, \mathbb{R}^m)$, and*

$$S^*_m\Big(t\lambda_1 + (1-t)\lambda_2, \Omega\Big) \le S^*_m(\lambda_1, \Omega) \vee S^*_m(\lambda_2, \Omega),$$

*that is, $S^*_m$ is level convex on the linear space $\mathcal{M}_{\rm loc}(\Omega, \mathbb{R}^m)$.*

*Proof.* Fixed $\rho > 0$, by the convexity of $f$, we have, for every $x \in \Omega_\rho$,

$$f\Big((t\lambda_1 + (1-t)\lambda_2)_\rho(x)\Big) = f\Big(t(\lambda_1)_\rho(x) + (1-t)(\lambda_2)_\rho(x)\Big) \le tf\Big((\lambda_1)_\rho(x)\Big) + (1-t)f\Big((\lambda_2)_\rho(x)\Big),$$

and then

$$I_m\Big((t\lambda_1 + (1-t)\lambda_2)_\rho \cdot \mathcal{L}^n, \Omega_\rho\Big) \le tI_m\Big((\lambda_1)_\rho \cdot \mathcal{L}^n, \Omega_\rho\Big) + (1-t)I_m\Big((\lambda_2)_\rho \cdot \mathcal{L}^n, \Omega_\rho\Big).$$

Analogously, fixed $\rho > 0$, by the level convexity of $g$, we have, for every $x \in \Omega_\rho$,

$$g\Big((t\lambda_1 + (1-t)\lambda_2)_\rho(x)\Big) = g\Big(t(\lambda_1)_\rho(x) + (1-t)(\lambda_2)_\rho(x)\Big) \le g\Big((\lambda_1)_\rho(x)\Big) \vee g\Big((\lambda_2)_\rho(x)\Big),$$

and then

$$S_m\Big((t\lambda_1 + (1-t)\lambda_2)_\rho \cdot \mathcal{L}^n, \Omega_\rho\Big) \le S\Big((\lambda_1)_\rho \cdot \mathcal{L}^n, \Omega_\rho\Big) \vee S\Big((\lambda_2)_\rho \cdot \mathcal{L}^n, \Omega_\rho\Big).$$

Passing to the limit as $\rho \to 0$ we complete the proof applying Theorem 6.6. □

### 6.1.2 The integral functional

In this section we prove the equality $I_m^* = \mathbb{I}_m$. The proof is based on several facts about the relations between sub-linear functions and measures: we propose here a slight revisited form of the results proved by Goffman and Serrin in [49].

Fixed $m \in \mathbb{N}$, let us consider $\mathcal{F} : \mathbb{R}^m \to [0, \infty]$ and the following sets of hypotheses:

- $\mathcal{F}$ is proper, l.s.c. and sub-linear; (6.13)
- $\mathcal{F}$ is positively homogeneous of degree 1; (6.14)
- there exists a constant $C > 0$ such that, for every $\xi \in \mathbb{R}^m, \mathcal{F}(\xi) \leq C|\xi|$. (6.15)

Let us consider now $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, and set, for every $B \in \mathcal{B}(\Omega)$,

$$\mathcal{F}_\lambda(B) = \sup\left\{\sum_{j=1}^r \mathcal{F}\left(\lambda(B_j)\right) : \forall j\ B_j \in \mathcal{B}(\Omega), B_j \subset\subset \Omega, \bigcup_{j=1}^r B_j \subseteq B, B_i \cap B_j = \emptyset \text{ if } i \neq j\right\}. \tag{6.16}$$

Note that if $\mathcal{F}(\xi) = |\xi|$ then $\mathcal{F}_\lambda$ agrees with $|\lambda|$, the total variation of $\lambda$ given by (2.1).

**Proposition 6.9.** *Let $\mathcal{F} : \mathbb{R}^m \to [0, \infty]$ be a function satisfying* (6.13). *Then, for every $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, $\mathcal{F}_\lambda$ is a positive Borel measure. Moreover if $\mathcal{F}$ satisfies* (6.15) *too, $\mathcal{F}_\lambda \in \mathcal{M}^+(\Omega)$.*

*Proof.* Let us note at first that if $B_1, B_2 \in \mathcal{B}(\Omega)$ and $B_1 \subseteq B_2$ then $0 \leq \mathcal{F}_\lambda(B_1) \leq \mathcal{F}_\lambda(B_2)$ and that $\mathcal{F}_\lambda(\emptyset) = 0$. Let us consider now a sequence $\{E_i\}_{i=1}^\infty \subseteq \mathcal{B}(\Omega)$ such that, for every $i \neq j$, $E_i \cap E_j = \emptyset$. Then defined $B = \bigcup_{i=1}^\infty E_i$ let us show that $\mathcal{F}_\lambda(B) = \sum_{i=1}^\infty \mathcal{F}_\lambda(E_i)$. Indeed, since $\mathcal{F} \geq 0$, for every $q \in \mathbb{N}$,

$$\mathcal{F}_\lambda(B) \geq \sum_{i=1}^q \mathcal{F}_\lambda(E_i) \quad \text{and then} \quad \mathcal{F}_\lambda(B) \geq \sum_{i=1}^\infty \mathcal{F}_\lambda(E_i).$$

In order to prove the converse let $\{B_j\}_{j=1}^r \subseteq \mathcal{B}(\Omega)$ as in (6.16). Then, for every $j = \{1, \ldots, r\}$, $\lambda(B_j) = \sum_{i=1}^\infty \lambda(E_i \cap B_j)$ and, by the sub-linearity and the lower semicontinuity of $\mathcal{F}$,

$$\sum_{j=1}^r \mathcal{F}(\lambda(B_j)) = \sum_{j=1}^r \mathcal{F}\left(\lim_{q\to\infty} \sum_{i=1}^q \lambda(E_i \cap B_j)\right) \leq \sum_{j=1}^r \liminf_{q\to\infty} \mathcal{F}\left(\sum_{i=1}^q \lambda(E_i \cap B_j)\right) \leq$$

$$\sum_{j=1}^r \liminf_{q\to\infty} \sum_{i=1}^q \mathcal{F}\left(\lambda(E_i \cap B_j)\right) = \sum_{j=1}^r \sum_{i=1}^\infty \mathcal{F}(\lambda(E_i \cap B_j)) = \sum_{i=1}^\infty \sum_{j=1}^r \mathcal{F}(\lambda(E_i \cap B_j)) \leq \sum_{i=1}^\infty \mathcal{F}_\lambda(E_i),$$

and, taking now the supremum of the left hand side with respect to the families $\{B_j\}_{j=1}^r \subseteq \mathcal{B}(\Omega)$ as in (6.16), we find the opposite inequality. □

**Lemma 6.10.** *Let $\mathcal{F} : \mathbb{R}^m \to [0, \infty]$ be a function satisfying* (6.13). *If $\alpha, \beta \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and $\lambda = \alpha + \beta$ then $\mathcal{F}_\lambda \leq \mathcal{F}_\alpha + \mathcal{F}_\beta$, equality holding when $\alpha \perp \beta$.*

*Proof.* Let us fix $B \in \mathcal{B}(\Omega)$. Let us suppose at first that $\mathcal{F}_\lambda(B) = \infty$. Then, for every $M \in \mathbb{N}$, we can find $\{B_j\}_{j=1}^r$ as in (6.16) such that

$$M \leq \sum_{j=1}^r \mathcal{F}(\lambda(B_j)) \leq \sum_{j=1}^r \mathcal{F}(\alpha(B_j)) + \mathcal{F}(\beta(B_j)) \leq \mathcal{F}_\alpha(B) + \mathcal{F}_\beta(B),$$

then $\mathcal{F}_\alpha(B)+\mathcal{F}_\beta(B)=\infty$. If instead $\mathcal{F}_\lambda(B)<\infty$, for every $\varepsilon>0$, there exists $\{B_j\}_{j=1}^r$ as in (6.16) such that

$$\mathcal{F}_\lambda(B)\le\sum_{j=1}^r\mathcal{F}(\lambda(B_j))+\varepsilon.$$

Then

$$\mathcal{F}_\lambda(B)\le\sum_{j=1}^r\mathcal{F}(\lambda(B_j))+\varepsilon\le\sum_{j=1}^r\mathcal{F}\Big(\alpha(B_j)+\beta(B_j)\Big)+\varepsilon$$

$$\le\sum_{j=1}^r\mathcal{F}(\alpha(B_j))+\sum_{j=1}^r\mathcal{F}(\beta(B_j))+\varepsilon\le\mathcal{F}_\alpha(B)+\mathcal{F}_\beta(B)+\varepsilon,$$

and, letting $\varepsilon\to0$, we find the wanted inequality. If now $\alpha\perp\beta$ we know that we can find two disjoint sets $B_\alpha,B_\beta\in\mathcal{B}(\Omega)$, such that $\alpha$ is concentrated on $B_\alpha$ and $\beta$ is concentrated on $B_\beta$: thus

$$\mathcal{F}_\alpha(B)=\mathcal{F}_\lambda(B\cap B_\alpha)\quad\text{and}\quad\mathcal{F}_\beta(B)=\mathcal{F}_\lambda(B\cap B_\beta),$$

and then, since $\mathcal{F}_\lambda$ is a measure,

$$\mathcal{F}_\alpha(B)+\mathcal{F}_\beta(B)=\mathcal{F}_\lambda(B\cap B_\alpha)+\mathcal{F}_\lambda(B\cap B_\beta)=\mathcal{F}_\lambda(B).$$

This completes the proof. □

**Theorem 6.11.** *Let $\mathcal{F}:\mathbb{R}^m\to[0,\infty]$ be a function satisfying* (6.13) *and* (6.14)*, $\lambda\in\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ and $u\in L^1_{{\rm loc},\lambda}(\Omega)$, $u\ge0$. Then we have, for every $B\in\mathcal{B}(\Omega)$, $B\subset\subset\Omega$,*

$$\mathcal{F}\left(\int_B u(x)d\lambda(x)\right)\le\int_B u(x)d\mathcal{F}_\lambda(x).$$

*Proof.* Let $B\in\mathcal{B}(\Omega)$, $B\subset\subset\Omega$ (so that $\int_B u(x)d|\lambda|(x)<\infty$) and let us suppose at first that $u$ is a simple function on $B$, that is

$$u(x)=\sum_{j=1}^r c_j1_{B_j}(x),$$

where $c_j\ge0$ and $\{B_j\}_{j=1}^r$ is a partition of $B$ (that obviously is admissible in (6.16)). Then by the sub-linearity and the positively homogeneity of degree 1 of $\mathcal{F}$,

$$\mathcal{F}\left(\int_B u(x)d\lambda(x)\right)=\mathcal{F}\left(\sum_{j=1}^r c_j\lambda(B_j)\right)\le\sum_{j=1}^r c_j\mathcal{F}(\lambda(B_j))\le\sum_{j=1}^r c_j\mathcal{F}_\lambda(B_j)=\int_B u(x)d\mathcal{F}_\lambda(x).$$

If now $u$ is not simple then there exists an increasing sequence $\{u_h\}_{h=1}^\infty$ of simple positive functions on $B$ such that, in particular, $\int_B u_h(x)d\lambda(x)\to\int_B u(x)d\lambda(x)$. Then, for every $h\in\mathbb{N}$,

$$\mathcal{F}\left(\int_B u_h(x)d\lambda(x)\right)\le\int_B u_h(x)d\mathcal{F}_\lambda(x)$$

and, passing to the limit for $h\to\infty$, we end by the lower semicontinuity of $\mathcal{F}$ (used to treat the left hand side) and by Beppo Levi's Theorem (used to treat the right hand side). □

**Theorem 6.12.** *Let $\mathcal{F}:\mathbb{R}^m\to[0,\infty]$ be a function satisfying* (6.13) *and* (6.14) *and let $\lambda_h\in\mathcal{M}_{\rm loc}(\Omega_h,\mathbb{R}^m)$, $\lambda\in\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ such that $\lambda_h\to\lambda$ in $w^*$-$\mathcal{M}_{\rm loc}(\updownarrow\Omega,\mathbb{R}^m)$. Then*

$$\mathcal{F}_\lambda(\Omega)\le\liminf_{h\to\infty}\mathcal{F}_{\lambda_h}(\Omega_h).$$

*Proof.* Let us consider $B \in \mathcal{B}(\Omega)$, $B \subset\subset \Omega$ and $\varepsilon > 0$ and find $\{B_j\}_{j=1}^r$ as in (6.16) such that

$$\mathcal{F}_\lambda(B) \leq \sum_{j=1}^r \mathcal{F}(\lambda(B_j)) + \varepsilon.$$

Fixed $k \in \mathbb{N}$ we can find, for every $j \in \{1, \ldots, r\}$, a compact set $F_j^k$ and an open set $G_j^k$ such that

$$F_j^k \subseteq B_j,\ |\lambda|(B_j \setminus F_j^k) < \frac{1}{k}, \quad \text{and} \quad F_j^k \subset\subset G_j^k,\ |\lambda|(G_j^k \setminus F_j^k) < \frac{1}{k}.$$

We can also suppose that $\{G_j^k\}_{j=1}^r$ are disjoint subsets well contained in $\Omega$. Let us consider now, for every $j \in \{1, \ldots, r\}$, a function $\varphi_j^k \in C_c(G_j^k)$ such that $0 \leq \varphi_j^k \leq 1$ and $\varphi_j^k \equiv 1$ on $F_j^k$. Then

$$\left|\lambda(B_j) - \int_\Omega \varphi_j^k(x) d\lambda(x)\right| \leq \left|\lambda(B_j) - \lambda(F_j^k)\right| + \left|\lambda(F_j^k) - \int_\Omega \varphi_j^k(x) d\lambda(x)\right|$$

$$\leq \frac{1}{k} + \left|\int_{G_j^k \setminus F_j^k} \varphi_j^k(x) d\lambda(x)\right| \leq \frac{2}{k}.$$

Then, by considering the lower semicontinuity of $\mathcal{F}$ there exists $k_\varepsilon \in \mathbb{N}$ such that

$$\mathcal{F}(\lambda(B_j)) \leq \mathcal{F}\left(\int_\Omega \varphi_j^{k_\varepsilon}(x) d\lambda(x)\right) + \frac{\varepsilon}{r},$$

and then

$$\mathcal{F}_\lambda(B) \leq \sum_{j=1}^r \mathcal{F}\left(\int_\Omega \varphi_j^{k_\varepsilon}(x) d\lambda(x)\right) + 2\varepsilon.$$

Now there exists $h_\varepsilon \in \mathbb{N}$ such that, for every $h \geq h_\varepsilon$, by the convergence of the measures $\lambda_h$, the lower semicontinuity of $\mathcal{F}$ and Theorem 6.11, we have

$$\mathcal{F}_\lambda(B) \leq \sum_{j=1}^r \mathcal{F}\left(\int_{\Omega_h} \varphi_j^{k_\varepsilon}(x) d\lambda_h(x)\right) + 3\varepsilon \leq \sum_{j=1}^r \int_{\Omega_h} \varphi_j^{k_\varepsilon}(x) d\mathcal{F}_{\lambda_h}(x) + 3\varepsilon$$

$$= \int_{\Omega_h} \sum_{j=1}^r \varphi_j^{k_\varepsilon}(x) d\mathcal{F}_{\lambda_h}(x) + 3\varepsilon \leq \mathcal{F}_{\lambda_h}(\Omega_h) + 3\varepsilon.$$

Then

$$\mathcal{F}_\lambda(B) \leq \liminf_{h\to\infty} \mathcal{F}_{\lambda_h}(\Omega_h) + 3\varepsilon,$$

and we end the proof taking the supremum on $B \subset\subset \Omega$. □

**Lemma 6.13.** *Let $\mathcal{F} : \mathbb{R}^m \to [0, \infty]$ be a function satisfying* (6.13) *and* (6.15). *Then $\mathcal{F}$ is $C$-Lipschitz on $\mathbb{R}^m$ and $\mathcal{F}(0) = 0$.*

*Proof.* Clearly (6.15) implies that $\mathcal{F}(0) = 0$. In order to prove the Lipschitz inequality let us consider $\xi, \eta \in \mathbb{R}^m$: we have

$$\mathcal{F}(\xi) = \mathcal{F}(\eta + \xi - \eta) \leq \mathcal{F}(\eta) + \mathcal{F}(\xi - \eta) \leq \mathcal{F}(\eta) + C|\xi - \eta|,$$

that is $\mathcal{F}(\xi) - \mathcal{F}(\eta) \leq C|\xi - \eta|$. Since the inequality holds for every $\xi, \eta \in \mathbb{R}^m$, it implies the wanted Lipschitz inequality. □

**Lemma 6.14.** *Let $\mathcal{F} : \mathbb{R}^m \to [0, \infty]$ be a function satisfying* (6.13) *and* (6.15) *and let $\lambda_1, \lambda_2 \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$. Then, for every $B \in \mathcal{B}(\Omega)$, $B \subset\subset \Omega$,*

$$|\mathcal{F}_{\lambda_1}(B) - \mathcal{F}_{\lambda_2}(B)| \leq C|\lambda_1 - \lambda_2|(B).$$

*Proof.* Let us consider $B \in \mathcal{B}(\Omega)$, $B \subset\subset \Omega$ (so that $\mathcal{F}_{\lambda_1}(B), \mathcal{F}_{\lambda_1}(B) < \infty$) and $\varepsilon > 0$. We can find (considering suitable refinements) $\{B_j\}_{j=1}^r$ as in (6.16) such that

$$\mathcal{F}_{\lambda_1}(B) \leq \sum_{j=1}^r \mathcal{F}(\lambda_1(B_j)) + \varepsilon \quad \text{and} \quad \mathcal{F}_{\lambda_2}(B) \leq \sum_{j=1}^r \mathcal{F}(\lambda_2(B_j)) + \varepsilon.$$

Then, by Lemma 6.13,

$$\begin{aligned} |\mathcal{F}_{\lambda_2}(B) - \mathcal{F}_{\lambda_1}(B)| &\leq \left| \sum_{j=1}^r \mathcal{F}(\lambda_1(B_j)) - \sum_{j=1}^r \mathcal{F}(\lambda_2(B_j)) + 2\varepsilon \right| \\ &\leq C \sum_{j=1}^r |\lambda_1(B_j) - \lambda_2(B_j)| + 2\varepsilon \leq C|\lambda_1 - \lambda_2|(B) + 2\varepsilon, \end{aligned}$$

and we end by letting $\varepsilon \to 0$. □

**Theorem 6.15.** *Let $\mathcal{F} : \mathbb{R}^m \to [0, \infty]$ be a function satisfying* (6.13)*,* (6.14) *and* (6.15)*, $\mu \in \mathcal{M}^+(\Omega)$ and $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$. Then, for every $B \in \mathcal{B}(\Omega)$,*

$$\mathcal{F}_\lambda(B) = \int_B \mathcal{F}(\lambda^a(x)) d\mu(x) + \mathcal{F}_{\lambda^s}(B),$$

*where $\lambda$ has been decomposed as in (2.3) with respect to $\mu$.*

*Proof.* By Lemma 6.10, $\mathcal{F}_\lambda = \mathcal{F}_{\lambda^a \cdot \mu} + \mathcal{F}_{\lambda^s}$. Thus we have to prove that, for every $B \in \mathcal{B}(\Omega)$, $\mathcal{F}_{\lambda^a \cdot \mu}(B) = \int_B \mathcal{F}(\lambda^a(x)) d\mu(x)$. Let us suppose at first that $B \in \mathcal{B}(\Omega)$, $B \subset\subset \Omega$ (so that $\mathcal{F}_{\lambda^a \cdot \mu}(B) < \infty$) and that $\lambda^a$ is a simple function on $B$, that is

$$\lambda^a(x) = \sum_{i=1}^k c_i 1_{E_i}(x),$$

where $\{E_i\}_{i=1}^k \subseteq \mathcal{B}(\Omega)$ is a partition of $B$ and $\{c_i\}_{i=1}^k \subseteq \mathbb{R}^m$. Let us fix $\varepsilon > 0$ and find $\{B_j\}_{j=1}^r$ as in (6.16) such that

$$\mathcal{F}_{\lambda^a \cdot \mu}(B) \leq \sum_{j=1}^r \mathcal{F}\Big((\lambda^a \cdot \mu)(B_j)\Big) + \varepsilon.$$

Then

$$\sum_{j=1}^r \sum_{i=1}^k \mathcal{F}\Big((\lambda^a \cdot \mu)(E_i \cap B_j)\Big) \leq \mathcal{F}_{\lambda^a \cdot \mu}(B) \leq \sum_{j=1}^r \sum_{i=1}^k \mathcal{F}\Big((\lambda^a \cdot \mu)(E_i \cap B_j)\Big) + \varepsilon. \tag{6.17}$$

But, by (6.14), we have also

$$\begin{aligned} \sum_{j=1}^r \sum_{i=1}^k \mathcal{F}\Big((\lambda^a \cdot \mu)(E_i \cap B_j)\Big) &= \sum_{j=1}^r \sum_{i=1}^k \mathcal{F}\Big(c_i \mu(E_i \cap B_j)\Big) = \sum_{j=1}^r \sum_{i=1}^k \mathcal{F}(c_i)\mu(E_i \cap B_j) \\ &= \sum_{i=1}^k \mathcal{F}(c_i)\mu(E_i) = \int_B \mathcal{F}(\lambda^a(x)) d\mu(x), \end{aligned}$$

and since in (6.17) $\varepsilon$ is arbitrary we end.

If now $\lambda^a$ is not simple let us consider a sequence $\{\lambda_h^a\}_{h=1}^\infty$ of simple functions on $B$ such that $\lambda_h^a \to \lambda^a$ in $L^1_\mu(B,\mathbb{R}^m)$. Then, if we define $\lambda_h = \lambda_h^a \cdot \mu \in \mathcal{M}(B,\mathbb{R}^m)$, we have, by Lemma 6.13,

$$\left|\int_B \mathcal{F}(\lambda^a(x)) - \mathcal{F}(\lambda_h^a(x))d\mu(x)\right| \le C\int_B |\lambda^a(x) - \lambda_h^a(x)|d\mu(x),$$

and, by Lemma 6.14,

$$|\mathcal{F}_{\lambda^a\cdot\mu}(B) - \mathcal{F}_{\lambda_h}(B)| \le C\int_E |\lambda^a(x) - \lambda_h^a(x)|d\mu(x).$$

Since the right hand side of both the inequalities go to zero as $h \to \infty$ and since, for every $h \in \mathbb{N}$, $\mathcal{F}_{\lambda_h}(B) = \int_B \mathcal{F}(\lambda_h^a(x))d\mu(x)$, it follows that also $\mathcal{F}_{\lambda^a\cdot\mu}(B) = \int_B \mathcal{F}(\lambda^a(x))d\mu(x)$.

If now $B$ is not well contained in $\Omega$ we consider an increasing sequence $\{B_i\}_{i=1}^\infty \subseteq \mathcal{B}(\Omega)$ with $B_i \subset\subset \Omega$ and $B_i \uparrow \Omega$ and we end by approximation using the property of the integrals and of the elements of $\mathcal{M}^+(\Omega)$. ☐

Note that applying two times Theorem 6.15 we find in particular

$$\mathcal{F}_\lambda(B) = \int_B \mathcal{F}(\lambda^a(x))d\mu(x) + \int_B \mathcal{F}\left(\frac{d\lambda^s}{d|\lambda^s|}(x)\right) d|\lambda^s|(x).$$

After these preliminary results on the sub-linear functionals we have the right tools to prove the equality between $\mathbb{I}_m$ and $I_m^*$. Theorem 6.16, together with Theorem 6.6 and Propositions 6.7 and 6.8, proves the part of Theorem 6.1 concerning the functionals $\mathbb{I}_m$, $I_m^*$ and $I_m$.

**Theorem 6.16.** *Let $f : \mathbb{R}^m \to [0,\infty]$ be a proper, l.s.c. and convex function. Then, for every $\lambda \in \mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$, $\mathbb{I}_m(\lambda,\Omega) = I_m^*(\lambda,\Omega)$.*

*Proof.* Let us fix $\mu \in \mathcal{M}^+(\Omega)$. Given $\lambda \in \mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ let us consider $(\lambda,\mu) \in \mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^{m+1})$ and define the measure $\hat{f}_{(\lambda,\mu)}$ where $\hat{f} : \mathbb{R}^{m+1} \to [0,\infty]$ is the function given by Proposition 2.25 that satisfies (6.13) and (6.14). We claim at first that, for every $\lambda \in \mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$, $B \in \mathcal{B}(\Omega)$,

$$\hat{f}_{(\lambda,\mu)}(B) = \int_B f(\lambda^a(x))d\mu + f^\infty_{\lambda^s}(B), \tag{6.18}$$

where $f^\infty_{\lambda^s}$ is well defined since $f^\infty$ is l.s.c. and sub-linear (see Proposition 2.20).

Clearly we can write $(\lambda,\mu) = (\lambda^a,1)\cdot\mu + (\lambda^s,0)$ and, by Lemma 6.10,

$$\hat{f}_{(\lambda,\mu)} = \hat{f}_{(\lambda^a,1)\cdot\mu} + \hat{f}_{(\lambda^s,0)} = \hat{f}_{(\lambda^a,1)\cdot\mu} + f^\infty_{\lambda^s}.$$

Thus it remains to show that,

$$\hat{f}_{(\lambda^a,1)\cdot\mu}(B) = \int_B f(\lambda^a(x))d\mu.$$

It is well known that there exists an increasing sequence $\{f_h\}_{h=1}^\infty$ of convex functions such that $f_h \uparrow f$, $|f_h(\xi)| \le C_h\sqrt{1+|\xi|^2}$ and, for every $|\xi| \le h$, $f_h(\xi) = f(\xi)$. Then, for every $\tau \ge 0$, $\hat{f}_h(\xi,\tau) \le C_h|\xi| \le C_h|(\xi,\tau)|$, that is, $\hat{f}_h$ satisfies also (6.15) on $\mathbb{R}^m \times [0,\infty]$: since the $(m+1)$-th component of $(\lambda^a,1)\cdot\mu$ is a positive measure it is simply to verify that also for these functions Theorem 6.15 holds[2], that is,

$$\hat{f}_{h,(\lambda^a,1)\cdot\mu}(B) = \int_B \hat{f}_h(\lambda^a(x),1)d\mu = \int_B f_h(\lambda^a(x))d\mu.$$

[2]Indeed, one could develop all the results till now proved considering $\mathcal{F} : \mathbb{R}^m \times [0,\infty] \to [0,\infty]$ and $\lambda \in \mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^{m+1})$ with $\lambda_{m+1} \in \mathcal{M}^+(\Omega)$.

By Beppo Levi's Theorem we obtain

$$\lim_{h\to\infty}\int_B f_h(\lambda^a(x))d\mu = \int_B f(\lambda^a(x))d\mu$$

We end if we prove that also

$$\lim_{h\to\infty}\hat{f}_{h,(\lambda^a,1)\cdot\mu}(B) = \hat{f}_{(\lambda^a,1)\cdot\mu}(B).$$

Suppose at first that $\hat{f}_{(\lambda^a,1)\cdot\mu}(B) < \infty$. Fixed $\varepsilon > 0$, there exists $\{B_j\}_{j=1}^r \subseteq \mathcal{B}(\Omega)$, $B_j \subset\subset B$, such that

$$\hat{f}_{(\lambda^a,1)\cdot\mu}(B) \le \sum_{j=1}^r \hat{f}\Big((\lambda^a\cdot\mu)(B_j),\mu(B_j)\Big) + \varepsilon.$$

Since $\lambda^a\cdot\mu << \mu$ we have that $\mu(B_j) = 0$ implies $(\lambda^a\cdot\mu)(B_j) = 0$. Thus we claim that there exists $h'$ such that, for every $h \ge h'$, $j \in \{1,\dots,r\}$,

$$\hat{f}\Big((\lambda^a\cdot\mu)(B_j),\mu(B_j)\Big) = \hat{f}_h\Big((\lambda^a\cdot\mu)(B_j),\mu(B_j)\Big).$$

Indeed, for every $|\xi| \le h\tau$, $\hat{f}_h(\xi,\tau) = \hat{f}(\xi,\tau)$ and then, if $h$ is big enough we have, for every $j = \{1,\dots,r\}$, $|\lambda^a\cdot\mu(B_j)| \le h|\mu(B_j)|$ and the claim follows. Therefore

$$\hat{f}_{(\lambda^a,1)\cdot\mu}(\Omega) \le \sum_{j=1}^r \hat{f}\Big((\lambda^a\cdot\mu)(B_j),\mu(B_j)\Big) + \varepsilon$$

$$\le \sum_{j=1}^r \hat{f}_h\Big((\lambda^a\cdot\mu)(B_j),\mu(B_j)\Big) + \varepsilon \le \hat{f}_{h,(\lambda^a,1)\cdot\mu}(\Omega) + \varepsilon,$$

that implies, as $h\to\infty$ and later $\varepsilon\to 0$,

$$\hat{f}_{(\lambda^a,1)\cdot\mu}(\Omega) \le \liminf_{h\to\infty}\hat{f}_{h,(\lambda^a,1)\cdot\mu}(\Omega).$$

The other inequality is obvious since, for every $(\xi,\tau)\in\mathbb{R}^{m+1}$, $\hat{f}_h(\xi,\tau) \le \hat{f}(\xi,\tau)$. The case $\hat{f}_{(\lambda^a,1)\cdot\mu}(E) = \infty$ can be proved using a similar argument so that (6.18) is achieved.

Let us consider now $\lambda \in \mathcal{M}_{\text{loc}}(\Omega,\mathbb{R}^m)$, $\mu = \mathcal{L}^n$ and $\lambda = \lambda^a\cdot\mathcal{L}^n + \lambda^s$: applying the claim just proved both to $f$ and to $f^\infty$ (with $\mu = |\lambda^s|$) we obtain in particular

$$\hat{f}_{(\lambda,\mathcal{L}^n)}(\Omega) = \int_\Omega f(\lambda^a(x))dx + f^\infty_{\lambda^s}(\Omega) = \int_\Omega f(\lambda^a(x))dx + \int_\Omega f^\infty\left(\frac{\lambda^s}{|\lambda^s|}(x)\right)d|\lambda^s|(x) = \mathbb{I}_m(\lambda,\Omega).$$

We end, by means of Theorem 6.6, proving that also

$$\hat{f}_{(\lambda,\mathcal{L}^n)}(\Omega) = \lim_{\rho\to 0}\int_{\Omega_\rho} f(\lambda_\rho(x))dx.$$

Indeed, by Theorem 6.12 (that can be applied to $\hat{f}$ too) and the claim proved we have

$$\hat{f}_{(\lambda,\mathcal{L}^n)}(\Omega) \le \liminf_{\rho\to 0}\hat{f}_{(\lambda_\rho,1)\cdot\mathcal{L}^n}(\Omega_\rho) = \liminf_{\rho\to 0} f_{\lambda_\rho\cdot\mathcal{L}^n}(\Omega_\rho) = \liminf_{\rho\to 0}\int_{\Omega_\rho} f(\lambda_\rho(x))dx.$$

On the other hand, using Theorem 6.11,

$$\int_{\Omega_\rho} f(\lambda_\rho(x))dx = \int_{\Omega_\rho}\hat{f}(\lambda_\rho(x),1)dx = \int_{\Omega_\rho}\hat{f}\left(\int_{B(x,\rho)} k_\rho(x-y)d(\lambda,\mathcal{L}^n)(y)\right)dx$$

$$\leq \int_{\Omega_\rho} \left( \int_{B(x,\rho)} k_\rho(x-y) d\hat{f}_{(\lambda,\mathcal{L}^n)}(y) \right) dx \leq \int_\Omega \left( \int_{B(y,\rho)\cap\Omega_\rho} k_\rho(x-y)dx \right) d\hat{f}_{(\lambda,\mathcal{L}^n)}(y) \leq \hat{f}_{(\lambda,\mathcal{L}^n)}(\Omega),$$

then

$$\limsup_{\rho\to 0} \int_{\Omega_\rho} f(\lambda_\rho(x))dx \leq \hat{f}_{(\lambda,\mathcal{L}^n)}(\Omega),$$

and the proof is finally achieved. □

### 6.1.3 The supremal functional

We prove now the equality $\mathbb{S}_m = S_m^*$. As for the integral case Theorem 6.17, together with Theorem 6.6 and Propositions 6.7 and 6.8, proves the part of Theorem 6.1 involving $S_m$, $S_m^*$ and $\mathbb{S}_m$.

**Theorem 6.17.** *Let $g : \mathbb{R}^m \to [0,\infty]$ be a proper, l.s.c. and level convex function. Then, for every $\lambda \in \mathcal{M}_{\text{loc}}(\Omega,\mathbb{R}^m)$, $\mathbb{S}_m(\lambda,\Omega) = S_m^*(\lambda,\Omega)$.*

*Proof.* Let us suppose at first that $g$ is positively homogeneous of degree 0.

Let us fix $\lambda \in \mathcal{M}_{\text{loc}}(\Omega,\mathbb{R}^m)$ and show at first that $\mathbb{S}_m(\lambda,\Omega) \leq S_m^*(\lambda,\Omega)$. By Theorem 2.2(i) there exists $N \subseteq \Omega$ such that $\mathcal{L}^n(N) = 0$ and for every $x \in \Omega \setminus N$, $\lambda_\rho(x) \to \lambda^a(x)$ as $\rho \to 0$. Then, using Theorem 6.6, for every $x \in \Omega \setminus N$,

$$g(\lambda^a(x)) \leq \liminf_{\rho\to 0} g(\lambda_\rho(x)) \leq \liminf_{\rho\to 0} \sup_{x\in\Omega_\rho} g(\lambda_\rho(x)) = S_m^*(\lambda,\Omega),$$

and then

$$S_m(\lambda,\Omega) = \operatorname*{ess\,sup}_{x\in\Omega} g(\lambda^a(x)) \leq \sup_{x\in\Omega\setminus N} g(\lambda^a(x)) \leq S_m^*(\lambda,\Omega).$$

Thus it remains to show that

$$|\lambda^s|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} g^\natural \left( \frac{d\lambda^s}{d|\lambda^s|}(x) \right) \leq S_m^*(\lambda,\Omega).$$

By Theorem 2.2(ii) there exists $M \subseteq \Omega$ with $|\lambda^s|(M) = 0$ and such that, for every $x \in \Omega \setminus M$, there exists a sequence $\{\rho_h\}_{h=1}^\infty$, depending on $x$ and decreasing to zero, such that,

$$\lim_{h\to\infty} \left| \frac{d\lambda^s}{d|\lambda^s|}(x) - \frac{\lambda_{\rho_h}(x)}{\int_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y)} \right| = 0. \tag{6.19}$$

Then, using Proposition 2.21 and Theorem 6.6, for every $x \in \Omega \setminus M$,

$$g^\natural \left( \frac{d\lambda^s}{d|\lambda^s|}(x) \right) = g \left( \frac{d\lambda^s}{d|\lambda^s|}(x) \right) \leq \liminf_{h\to\infty} g \left( \frac{\lambda_{\rho_h}(x)}{\int_{B(x,\rho_h)} k_{\rho_h}(x-y)d|\lambda^s|(y)} \right)$$

$$= \liminf_{h\to\infty} g\left(\lambda_{\rho_h}(x)\right) \leq \lim_{h\to\infty} S_m\Big(\lambda_{\rho_h} \cdot \mathcal{L}^n, \Omega_{\rho_h}\Big) = S_m^*(\lambda,\Omega).$$

In conclusion,

$$|\lambda^s|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} g^\natural \left( \frac{d\lambda^s}{d|\lambda^s|}(x) \right) \leq \sup_{x\in\Omega\setminus M} g^\natural \left( \frac{d\lambda^s}{d|\lambda^s|}(x) \right) \leq S_m^*(\lambda,\Omega),$$

and we achieve $\mathbb{S}_m(\lambda,\Omega) \leq S_m^*(\lambda,\Omega)$.

In order to prove the converse inequality let us fix $\rho > 0$. By Theorem 2.19 and Propositions 2.17 and 2.21 we have, for every $x \in \Omega_\rho$,

$$g(\lambda_\rho(x)) = g\left( \int_{B(x,\rho)} k_\rho(x-y)d\lambda(y) \right)$$

$$= g\left(\int_{B(x,\rho)} k_\rho(x-y)\lambda^a(y)dy + \int_{B(x,\rho)} k_\rho(x-y)\frac{d\lambda^s}{d|\lambda^s|}(y)d|\lambda^s|(y)\right)$$

$$\leq g\left(\int_{B(x,\rho)} k_\rho(x-y)\lambda^a(y)dy\right) \vee g\left(\int_{B(x,\rho)} k_\rho(x-y)\frac{d\lambda^s}{d|\lambda^s|}(y)d|\lambda^s|(y)\right)$$

$$\leq \left[\operatorname*{ess\,sup}_{y\in B(x,\rho)} g(\lambda^a(y))\right] \vee \left[|\lambda^s|\text{-}\operatorname*{ess\,sup}_{y\in B(x,\rho)} g\left(\frac{d\lambda^s}{d|\lambda^s|}(y)\int_{B(x,\rho)} k_\rho(x-y)d|\lambda^s|(y)\right)\right]$$

$$\leq \left[\operatorname*{ess\,sup}_{x\in\Omega} g(\lambda^a(x))\right] \vee \left[|\lambda^s|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} g^\natural\left(\frac{d\lambda^s}{d|\lambda^s|}(x)\right)\right] = \mathbb{S}_m(\lambda,\Omega),$$

thus,

$$\sup_{x\in\Omega_\rho} g(\lambda_\rho(x)) \leq \mathbb{S}_m(\lambda,\Omega).$$

The wanted inequality $S_m^*(\lambda,\Omega) \leq \mathbb{S}_m(\lambda,\Omega)$ is achieved as $\rho \to 0$, invoking again Theorem 6.6.

In the general case, let us consider the function $\hat{g}$ defined in (2.15): by Proposition 2.26 we know that $\hat{g}$ is l.s.c., positively homogeneous of degree 0 and level convex. Let now fix $\lambda \in \mathcal{M}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$ and consider $(\lambda,\mathcal{L}^n) \in \mathcal{M}(\Omega,\mathbb{R}^{m+1})$: its decomposition with respect to $\mathcal{L}^n$ is clearly given by

$$(\lambda,\mathcal{L}^n) = (\lambda^a(x),1)\cdot\mathcal{L}^n + \left(\frac{d\lambda^s}{d|\lambda^s|}(x),0\right)\cdot|\lambda^s|.$$

We have also that $(\lambda,\mathcal{L}^n)_\rho : \Omega_\rho \to \mathbb{R}^{m+1}$ is given by $(\lambda,\mathcal{L}^n)_\rho(x) = (\lambda_\rho(x),1)$. Then applying the first step to $\hat{g}$ and $(\lambda,\mathcal{L}^n)$ we have

$$\hat{S}_m^*\Big((\lambda,\mathcal{L}^n),\Omega\Big) = \hat{\mathbb{S}}_m\Big((\lambda,\mathcal{L}^n),\Omega\Big),$$

where $\hat{S}_m^*$ and $\hat{\mathbb{S}}_m$ denotes the functionals (6.6) and (6.2) built up considering $\hat{g}$ and defined on the space $\mathcal{M}_{\mathrm{loc}}(\Omega,\mathbb{R}^{m+1})$. But now we have:

$$\hat{S}_m^*\Big((\lambda,\mathcal{L}^n),\Omega\Big) = \lim_{\rho\to0}\sup_{x\in\Omega_\rho}\hat{g}\Big((\lambda,\mathcal{L}^n)_\rho(x)\Big) = \lim_{\rho\to0}\sup_{x\in\Omega_\rho}\hat{g}(\lambda_\rho(x),1) = \lim_{\rho\to0}\sup_{x\in\Omega_\rho} g(\lambda_\rho(x)) = S_m^*(\lambda,\Omega),$$

and, again by Proposition 2.21,

$$\hat{\mathbb{S}}_m\Big((\lambda,\mathcal{L}^n),\Omega\Big) = \left[\operatorname*{ess\,sup}_{x\in\Omega}\hat{g}(\lambda^a(x),1)\right] \vee \left[|\lambda^s|\text{-}\operatorname*{ess\,sup}_{x\in\Omega}\hat{g}\left(\frac{d\lambda^s}{d|\lambda^s|}(x),0\right)\right]$$

$$= \left[\operatorname*{ess\,sup}_{x\in\Omega} g(\lambda^a(x))\right] \vee \left[|\lambda^s|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} g^\natural\left(\frac{d\lambda^s}{d|\lambda^s|}(x)\right)\right] = \mathbb{S}_m(\lambda,\Omega).$$

This achieves the proof. □

## 6.2 Non level convex functionals

In this section we propose the proof of the Theorems 6.3 and 6.4 related to the functional $\mathbf{S}_m$ in (6.10).

### 6.2.1 Necessary conditions

*Proof of Theorem 6.3.* The fact that $g$ is l.s.c. and level convex can be proved using the same argument of the implication $(ii) \Rightarrow (i)$ of Theorem 6.1. In order to prove the other parts of the theorem let us consider $\{\eta_j\}_{j=1}^{\infty} \subseteq \mathbb{R}^m$ such that $g(\eta_j) \downarrow l = \inf\{g(\xi) : \xi \in \mathbb{R}^m\}$.

Let us study the function $\gamma$. Let $x_0 \in \mathbb{R}^n$ and $\xi_h, \xi_0 \in \mathbb{R}^m \setminus \{0\}$ such that $\xi_h \to \xi_0$ and, fixed $j \in \mathbb{N}$, let

$$\lambda_h = \xi_h \cdot \delta_{x_0} + \eta_j \cdot \mathcal{L}^n, \quad \text{and} \quad \lambda_0 = \xi_0 \cdot \delta_{x_0} + \eta_j \cdot \mathcal{L}^n.$$

Then $\lambda_h \to \lambda_0$ in $w^*$-$\mathcal{M}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ and

$$\gamma(\xi_0) \vee g(\eta_j) = \mathbf{S}_m(\lambda_0, \mathbb{R}^n) \leq \liminf_{h\to\infty} \mathbf{S}_m(\lambda_h, \mathbb{R}^n) = \liminf_{h\to\infty}(\gamma(\xi_h) \vee g(\eta_j)).$$

This implies that, for every $j \in \mathbb{N}$, $\gamma \vee g(\eta_j)$ is l.s.c. on $\mathbb{R}^m$ since also the lower semicontinuity in $\xi = 0$ trivially follows from $\gamma(0) = 0$: by using Proposition 2.32, we obtain that the same holds for $\gamma \vee l$.

Let $\xi, \eta \in \mathbb{R}^m \setminus \{0\}$ such that $\xi \neq -\eta$ and $x_h, x_0 \in \mathbb{R}^n$, $x_h \neq x_0$, such that $x_h \to x_0$. Moreover, fixed $j \in \mathbb{N}$, set

$$\lambda_h = \xi \cdot \delta_{x_h} + \eta \cdot \delta_{x_0} + \eta_j \cdot \mathcal{L}^n, \quad \text{and} \quad \lambda_0 = (\xi + \eta) \cdot \delta_{x_0} + \eta_j \cdot \mathcal{L}^n.$$

Since $\lambda_h \to \lambda_0$ in $w^*$-$\mathcal{M}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ we have

$$\gamma(\xi + \eta) \vee g(\eta_j) = \mathbf{S}_m(\lambda_0, \mathbb{R}^n) \leq \liminf_{h\to\infty} \mathbf{S}_m(\lambda_h, \mathbb{R}^n)$$

$$= \liminf_{h\to\infty} \gamma(\xi) \vee \gamma(\eta) \vee g(\eta_j) = \gamma(\xi) \vee \gamma(\eta) \vee g(\eta_j).$$

Then, since $\gamma(0) = 0$, we have that, for every $j \in \mathbb{N}$, $\gamma \vee g(\eta_j)$ is sub-maximal on $\mathbb{R}^m$ that implies, once $j \to \infty$, that $\gamma \vee l$ satisfies the same property too.

Let us prove now the inequality $g^\natural \leq (\gamma \vee l)^\flat$. Let $\xi_0 \in \mathbb{R}^m \setminus \{0\}$ and let $\xi_h \to \xi_0$, $t_h \uparrow \infty$ such that $g(t_h\xi_h) \to g^\natural(\xi_0)$. Thus we have that there exists a sequence of positive numbers $\{r_h\}_{h=1}^{\infty}$ such that, for every $h \in \mathbb{N}$, $t_h\mathcal{L}^n(B(0, r_h)) = 1$. Fixed $j \in \mathbb{N}$ and setting

$$\lambda_h = t_h \left( \xi_h - \frac{\eta_j}{t_h} \right) 1_{B(0,r_h)}(x) \cdot \mathcal{L}^n + \eta_j \cdot \mathcal{L}^n, \quad \text{and} \quad \lambda_0 = \xi_0 \cdot \delta_0 + \eta_j \cdot \mathcal{L}^n,$$

we have $\lambda_h \to \lambda_0$ in $w^*$-$\mathcal{M}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^m)$ and then

$$\gamma(\xi_0) \vee g(\eta_j) = \mathbf{S}_m(\lambda_0, \mathbb{R}^n) \leq \liminf_{h\to\infty} \mathbf{S}_m(\lambda_h, \mathbb{R}^n) = \lim_{h\to\infty} g(t_h\xi_h) \vee g(\eta_j) = g^\natural(\xi_0) \vee g(\eta_j).$$

This fact proves, as $j \to \infty$, that, for every $\xi_0 \in \mathbb{R}^m \setminus \{0\}$, $(\gamma \vee l)(\xi_0) \leq g^\natural(\xi_0)$. If now we consider a sequence $s_h \downarrow 0$ such that $(\gamma \vee l)^\flat(\xi_0) = \lim_{h\to\infty}(\gamma \vee l)(s_h\xi_0)$, since $g^\natural$ is positively homogeneous of degree 0, we have

$$(\gamma \vee l)^\flat(\xi_0) = \lim_{h\to\infty} (\gamma \vee l)(s_h\xi_0) \leq \liminf_{h\to\infty} g^\natural(s_h\xi_0) = g^\natural(\xi_0),$$

and we find the desired inequality.

We end proving $g^\natural \geq (\gamma \vee l)^\flat$. Let us consider $\xi_0 \in \mathbb{R}^m \setminus \{0\}$ and let $t_h \downarrow 0$, $t_h < 1$, such that $\gamma(t_h\xi_0) \to \gamma^\flat(\xi_0)$. Let us fix $\Omega = Q^n = (0, 1)^n$ and define, for every $k \in \mathbb{N}$,

$$G_k = \left\{ x \in Q^n : x_i = \frac{q}{k+1}, q \in \{1, \dots, k\}, i \in \{1, \dots, n\} \right\}:$$

note that[3] $\#(G_k) = k^n$. Fixed now $M \in \mathbb{N}$, we claim that there exists an sequence $\{k_h\}_{h=1}^{\infty} \subseteq \mathbb{N}$, depending on $M$ and such that $k_h \to \infty$ and, for every $h \in \mathbb{N}$, $M \leq t_h k_h^n < 2^n M$. Obviously this happen if, for every $h \in \mathbb{N}$, there exists $k_h \in \mathbb{N}$ such that

$$\left(Mt_h^{-1}\right)^{\frac{1}{n}} \leq k_h < 2\left(Mt_h^{-1}\right)^{\frac{1}{n}},$$

and since $\left(Mt_h^{-1}\right)^{\frac{1}{n}} > 1$ and $\left(Mt_h^{-1}\right)^{\frac{1}{n}} \to \infty$ as $h \to \infty$, $k_h$ can be found. Then, unless to extract a (not relabelled) subsequence, we have $t_h k_h^n \to s_M \in [M, 2^n M]$.

Fixed $j \in \mathbb{N}$, set the following elements of $\mathcal{M}_{\mathrm{loc}}(Q^n, \mathbb{R}^m)$

$$\lambda_h = \sum_{x \in G_{k_h}} \frac{t_h k_h^n}{k_h^n} \xi_0 \cdot \delta_x + \eta_j \cdot \mathcal{L}^n \quad \text{and} \quad \lambda_0 = s_M \xi_0 \cdot \mathcal{L}^n + \eta_j \cdot \mathcal{L}^n.$$

It is a quite standard result that $\lambda_h \to \lambda_0$ in $w^*$-$\mathcal{M}_{\mathrm{loc}}(Q^n, \mathbb{R}^n)$, thus,

$$g(s_M \xi_0 + \eta_j) = \mathbf{S}_m(\lambda_0, Q^n) \leq \liminf_{h\to\infty} \mathbf{S}_m(\lambda_h, Q^n) = \liminf_{h\to\infty} \left[\gamma\left(t_h \xi_0\right) \vee g(\eta_j)\right] = \gamma^\flat(\xi_0) \vee g(\eta_j).$$

Then, letting $j \to \infty$ and using Proposition 2.33, we have

$$g\left(s_M\left(\xi_0 + \frac{\eta_j}{s_M}\right)\right) = g(s_M \xi_0 + \eta_j) \leq \gamma^\flat(\xi_0) \vee l = (\gamma \vee l)^\flat(\xi_0),$$

and since this relation holds for every $M \in \mathbb{N}$, taking the limit for $M \to \infty$ also $s_M \to \infty$ and then we obtain $g^\natural(\xi_0) \leq (\gamma \vee l)^\flat(\xi_0)$. □

### 6.2.2 Sufficient conditions

Here we prove a theorem in which sufficient conditions for the lower semicontinuity of $\mathbf{S}_m$ are found: we treat directly the case in which $g$ and $\gamma$ depends on $x$ too and we will deduce Theorem 6.4 as a corollary of our main result.

The following lemma is based on Lemma 3.6 in [18].

**Lemma 6.18.** *Let $k \in \mathbb{N}$ and*

$$\mathcal{M}^k = \{\lambda \in \mathcal{M}(\Omega, \mathbb{R}^m) : \#(\mathrm{spt}(\lambda)) \leq k\}.$$

*Then $\mathcal{M}^k$ is sequentially closed with respect to the $w^*$-$\mathcal{M}(\Omega, \mathbb{R}^m)$ topology. Moreover if $l \in [0, \infty)$ and $\gamma : \Omega \times \mathbb{R}^m \to [0, \infty]$ is a proper and Borel function such that $\gamma \vee l$ is l.s.c. on $\Omega \times \mathbb{R}^m$ and, for every $x \in \Omega$, $\gamma(x, \cdot) \vee l$ is sub-maximal on $\mathbb{R}^m$ and $\gamma(x, 0) = 0$, then the functional*

$$\Gamma(\lambda, \Omega) = \bigvee_{x \in A_\lambda} \Big(\gamma(x, \lambda(x)) \vee l\Big),$$

*is l.s.c on $\mathcal{M}^k$ with respect to the $w^*$-$\mathcal{M}(\Omega, \mathbb{R}^m)$ topology.*

*Proof.* Let $\{\lambda_h\}_{h=1}^{\infty} \subseteq \mathcal{M}^k$, $\lambda_h \to \lambda$ in $w^*$-$\mathcal{M}(\Omega, \mathbb{R}^m)$. Then, for every $h \in \mathbb{N}$, there exist two sequences $\{x_i^h\}_{i=1}^k \subseteq \Omega$ (with, for every $h \in \mathbb{N}$, $x_i^h \neq x_j^h$ if $i \neq j$) and $\{\xi_i^h\}_{i=1}^k \subseteq \mathbb{R}^m$ such that

$$\lambda_h = \sum_{i=1}^k \xi_i^h \cdot \delta_{x_i^h}.$$

Clearly there exists $M > 0$ such that, for every $i \in \{1, \ldots, k\}$, $h \in \mathbb{N}$, $|\xi_i^h| \leq M$. Following Lemma 3.6 in [18], let $I \subseteq \{1, \ldots, k\}$ the (possibly empty) set of the index $i$ such that $\{x_i^h\}_{h=1}^{\infty}$

[3] If $A$ is a set we denote with $\#(A)$ the cardinality of $A$.

admits a convergent subsequence. Then we can find $\{x_i\}_{i\in I} \subseteq \Omega$, $\{\xi_i\}_{i\in I} \subseteq \mathbb{R}^m$ (note that it could be $\xi_i = 0$) and a (not relabelled) subsequence of $\{\lambda_h\}_{h=1}^{\infty}$ such that, for every $i \in I$, $x_i^h \to x_i$ and $\xi_i^h \to \xi_i$ for $h \to \infty$. On the other hand, if $i \notin I$ it is easy to verify that $\xi_i^h \cdot \delta_{x_i^h} \to 0$ in $w^*$-$\mathcal{M}(\Omega, \mathbb{R}^m)$. Therefore we have

$$\lambda = \sum_{i\in I} \xi_i \cdot \delta_{x_i} \in \mathcal{M}^k,$$

with $\lambda = 0$ if $I = \emptyset$. Note that, in particular, it may happen that there are $i, j \in I$ such that $x_i = x_j$. The lower semicontinuity of $\Gamma$ is trivial if $\lambda = 0$ (since by definition $\Gamma(0, \Omega) = l$), while if $\lambda \neq 0$ it simply follows by using the lower semicontinuity and the sub-maximality of $\gamma \vee l$, indeed

$$\Gamma(\lambda, \Omega) = \bigvee_{x\in A_\lambda} \Big(\gamma(x, \lambda(x)) \vee l\Big) \leq \bigvee_{i\in I} \Big(\gamma(x_i, \xi_i) \vee l\Big) \leq \liminf_{h\to\infty} \bigvee_{i\in I} \Big(\gamma(x_i^h, \xi_i^h) \vee l\Big)$$

$$\leq \liminf_{h\to\infty} \bigvee_{i=1}^{k} \Big(\gamma(x_i^h, \xi_i^h) \vee l\Big) = \liminf_{h\to\infty} \Gamma(\lambda_h, \Omega),$$

and the proof is achieved. □

Later we need a weak version of Theorem 4.1 proved by Acerbi, Buttazzo and Prinari [2].

**Theorem 6.19.** *Let $g : \Omega \times \mathbb{R}^m \to [0, \infty]$ be a Borel function such that, for $\mathcal{L}^n$-a.e. $x \in \Omega$, $g(x, \cdot)$ is l.s.c. and level convex on $\mathbb{R}^m$. Then the functional*

$$S(u, \Omega) = \operatorname*{ess\,sup}_{x\in\Omega} g(x, u(x))$$

*is l.s.c. on $L^\infty(\Omega, \mathbb{R}^m)$ with respect to the $w^*$-$L^\infty(\Omega, \mathbb{R}^m)$ convergence.*

We can now prove the main result of the section. Note that, when we are dealing with $g : \Omega \times \mathbb{R}^m \to [0, \infty]$, $g^\natural(x, \xi)$ means $(g(x, \cdot))^\natural(\xi)$.

**Theorem 6.20.** *Let $g : \Omega \times \mathbb{R}^m \to [0, \infty]$ be a proper and Borel function such that, for $\mathcal{L}^n$-a.e. $x \in \Omega$, $g(x, \cdot)$ is l.s.c. and level convex on $\mathbb{R}^m$ and there exists a function $\theta_\infty : [0, \infty) \to [0, \infty)$ such that, for every $x \in \Omega$ and $\xi \in \mathbb{R}^m$,*

$$g(x, \xi) \geq \theta_\infty(|\xi|) \quad \text{and} \quad \lim_{t\to\infty} \theta_\infty(t) = \infty.$$

*Moreover, setting $l = \inf\{g(x, \xi) : (x, \xi) \in \Omega \times \mathbb{R}^m\}$, let $\gamma : \Omega \times \mathbb{R}^m \to [0, \infty]$ be a proper and Borel function such that $\gamma \vee l$ is l.s.c. on $\Omega \times \mathbb{R}^m$, for every $x \in \Omega$, $\gamma(x, \cdot) \vee l$ is sub-maximal on $\mathbb{R}^m$ and $\gamma(x, 0) = 0$, and there exists a function $\theta_0 : (0, \infty) \to [0, \infty)$ such that, for every $x \in \Omega$, $\xi \in \mathbb{R}^m \setminus \{0\}$,*

$$\gamma(x, \xi) \vee l \geq \theta_0(|\xi|) \quad \text{and} \quad \lim_{t\to 0} \theta_0(t) = \infty.$$

*Then the functional*

$$\mathbf{S}_m(\lambda, \Omega) = \left[\operatorname*{ess\,sup}_{x\in\Omega} g(x, \lambda^a(x))\right] \vee \left[|\lambda^c|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} g^\natural\left(x, \frac{d\lambda^c}{d|\lambda^c|}(x)\right)\right] \vee \left[\bigvee_{x\in A_\lambda} \gamma(x, \lambda^\#(x))\right]$$

*is l.s.c. on $\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ with respect to the $w^*$-$\mathcal{M}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ convergence.*

*Proof.* Let us suppose at first that $\Omega \subseteq \mathbb{R}^n$ is bounded and let us prove the lower semicontinuity of $\mathbf{S}_m$ on $\mathcal{M}(\Omega, \mathbb{R}^m)$ with respect to the $w^*$-$\mathcal{M}(\Omega, \mathbb{R}^m)$ convergence. Consider $\lambda, \lambda_h \in \mathcal{M}(\Omega, \mathbb{R}^m)$

such that $\lambda_h \to \lambda$ in $w^*$-$\mathcal{M}(\Omega, \mathbb{R}^m)$: then there exists $L > 0$ such that $|\lambda_h|(\Omega), |\lambda|(\Omega) \leq L$. Without loss of generality we can suppose

$$\liminf_{h\to\infty} \mathbf{S}_m(\lambda_h, \Omega) = \lim_{h\to\infty} \mathbf{S}_m(\lambda_h, \Omega),$$

and that there exists a constant $M > 0$ such that, for every $h \in \mathbb{N}$, $\mathbf{S}_m(\lambda_h, \Omega) \leq M$. This fact implies that there exists $M' > 0$ such that, for every $h \in \mathbb{N}$,

(i) $\lambda_h^c = 0$,

(ii) for $\mathcal{L}^n$-a.e. $x \in \Omega$, $|\lambda_h^a(x)| \leq M'$,

(iii) for every $x \in A_{\lambda_h}$, $|\lambda_h^{\#}(x)| \geq \frac{1}{M'}$.

By (ii) we obtain $\{\lambda_h^a(x)\}_{h=1}^{\infty} \subseteq L^\infty(\Omega, \mathbb{R}^m)$ and

$$\sup\left\{\operatorname*{ess\,sup}_{x\in\Omega} |\lambda_h^a(x)| : h \in \mathbb{N}\right\} < \infty.$$

Then by Proposition 2.7 there exists a subsequence (not relabelled) and a function $u \in L^\infty(\Omega, \mathbb{R}^m)$, such that

$$\lambda_h^a(x) \to u(x) \quad \text{in} \quad w^*\text{-}L^\infty(\Omega, \mathbb{R}^m). \tag{6.20}$$

Moreover, since, for every $h \in \mathbb{N}$, $|\lambda_h|(\Omega) \leq L$, we have also $|\lambda_h^{\#}|(\Omega) \leq L$ thus, writing $\lambda_h^{\#} = \sum_{x\in A_{\lambda_h}} \lambda_h^{\#}(x) \cdot \delta_x$, it follows, for every $h \in \mathbb{N}$,

$$L \geq \left|\lambda_h^{\#}\right|(\Omega) = \sum_{x\in A_{\lambda_h}} \left|\lambda_h^{\#}(x)\right| \geq \frac{1}{M'} \#\left(A_{\lambda_h}\right).$$

Then there exists $k \in \mathbb{N}$ such that, for every $h \in \mathbb{N}$, $\#(A_{\lambda_h}) \leq k$. Then, for every $h \in \mathbb{N}$, $\lambda_h^{\#} \in \mathcal{M}^k$ and, as said, $|\lambda_h^{\#}|(\Omega) \leq L$. Thus, by Theorem 2.1, there exists a subsequence (not relabelled) and a measure $\nu \in \mathcal{M}(\Omega, \mathbb{R}^m)$ such that

$$\lambda_h^{\#} \to \nu \quad \text{in} \quad w^*\text{-}\mathcal{M}(\Omega, \mathbb{R}^m). \tag{6.21}$$

By Lemma 6.18 we know $\nu \in \mathcal{M}^k$.

We claim now that the condition $\lambda_h \to \lambda$ in $w^*$-$\mathcal{M}(\Omega, \mathbb{R}^m)$ together with (6.20) and (6.21) gives that $\lambda = u \cdot \mathcal{L}^n + \nu$, that is $\lambda^a = u$ and $\lambda^{\#} = \nu$. Indeed, for every $\varphi \in C_0(\Omega)$,

$$\int_\Omega \varphi(x) d\lambda_h(x) = \int_\Omega \varphi(x)\lambda_h^a(x)dx + \int_\Omega \varphi(x) d\lambda_h^{\#}(x),$$

and, by (6.20) (noting that $C_0(\Omega) \subseteq L^1(\Omega)$ since $\mathcal{L}^n(\Omega) < \infty$) and (6.21) we have that $\lambda_h \to u \cdot \mathcal{L}^n + \nu$ in $w^*$-$\mathcal{M}(\Omega, \mathbb{R}^m)$. The uniqueness of the limit allows to achieve the claim.

Now, by Theorem 6.19, we have

$$\operatorname*{ess\,sup}_{x\in\Omega} g(x, \lambda^a(x)) \leq \liminf_{h\to\infty} \operatorname*{ess\,sup}_{x\in\Omega} g(x, \lambda_h^a(x)),$$

while, by Lemma 6.18, we have

$$\bigvee_{x\in A_\lambda} \left(\gamma(x, \lambda^{\#}(x)) \vee l\right) \leq \liminf_{h\to\infty} \bigvee_{x\in A_{\lambda_h}} \left(\gamma(x, \lambda_h^{\#}(x)) \vee l\right).$$

Then

$$\liminf_{h\to\infty} \mathbf{S}_m(\lambda_h, \Omega) = \liminf_{h\to\infty} \left[\operatorname*{ess\,sup}_{x\in\Omega} g(x, \lambda_h^a(x))\right] \vee \left[\bigvee_{x\in A_{\lambda_h}} \left(\gamma(x, \lambda_h^{\#}(x)) \vee l\right)\right]$$

$$\geq \left[\operatorname*{ess\,sup}_{x\in\Omega} g(x,\lambda^a(x))\right] \vee \left[\bigvee_{x\in A_\lambda} \Big(\gamma(x,\lambda^\#(x)) \vee l\Big)\right] = \mathbf{S}_m(\lambda,\Omega),$$

and we get the lower semicontinuity.

In the general case let us consider $\lambda, \lambda_h \in \mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$ such that $\lambda_h \to \lambda$ in $w^*$-$\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^m)$. Then, for every open set $\Omega' \subset\subset \Omega$, $\lambda_h \to \lambda$ in $w^*$-$\mathcal{M}(\Omega',\mathbb{R}^m)$ and then

$$\mathbf{S}_m(\lambda,\Omega') \leq \liminf_{h\to\infty} \mathbf{S}_m(\lambda_h,\Omega') \leq \liminf_{h\to\infty} \mathbf{S}_m(\lambda_h,\Omega).$$

At least, since

$$\sup\{\mathbf{S}_m(\lambda,\Omega') : \Omega' \subset\subset \Omega\} = \mathbf{S}_m(\lambda,\Omega),$$

the proof is finally achieved. □

The proof of Theorem 6.4 stated at the beginning of this chapter easily follows from Theorem 6.20 and Propositions 2.23 and 2.24.

# Chapter 7

# Applications to $BV$

In this chapter we present some consequences of the results proved in Chapter 6 when they are applied to the setting of the functions of bounded variation.

## 7.1 Convex and level convex functionals

### 7.1.1 Lower semicontinuity and relaxation

The following theorem easily follows by Theorem 6.1 and it provides the precise statement of what we announced in the introduction about the extensions of integral and supremal functionals on $BV_{\rm loc}(\Omega)$. The integral part of this theorem was proved in the paper of Goffman and Serrin (see [49] Theorem 5) while the supremal part was proved by Gori in (see [51] Theorem 5).

**Theorem 7.1.** *Let $f:\mathbb{R}^n\to[0,+\infty]$ be a proper, l.s.c and convex function and $g:\mathbb{R}^n\to[0,+\infty]$ be a proper, l.s.c. and level convex function. Let us consider the following functionals defined, for every $u\in BV_{\rm loc}(\Omega)$, as*

$$\mathbb{I}(u,\Omega)=\int_\Omega f(\nabla u(x))dx+\int_\Omega f^\infty\left(\frac{dD^s u}{d|D^s u|}(x)\right)d|D^s u|(x), \tag{7.1}$$

*and*

$$\mathbb{S}(u,\Omega)=\left[\operatorname*{ess\,sup}_{x\in\Omega} g(\nabla u(x))\right]\vee\left[|D^s u|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} g^\natural\left(\frac{dD^s u}{d|D^s u|}(x)\right)\right]. \tag{7.2}$$

*Then $\mathbb{I}$ and $\mathbb{S}$ are lower semicontinuous on $BV_{\rm loc}(\Omega)$ with respect to the $w^*$-$BV_{\rm loc}(\Omega)$ convergence and, for every $u\in BV_{\rm loc}(\Omega)$,*

$$\mathbb{I}(u,\Omega)=I^*(u,\Omega)=\lim_{\rho\to0} I(u_\rho,\Omega_\rho)\quad\text{and}\quad \mathbb{S}(u,\Omega)=S^*(u,\Omega)=\lim_{\rho\to0} S(u_\rho,\Omega_\rho), \tag{7.3}$$

*where the functionals $I^*$ and $S^*$ are given by* (1.7) *and* (1.8)*, while $u_\rho$ denotes the convolution of $u$.*

*Proof.* Since, when $u_h,u\in BV_{\rm loc}(\Omega)$ and $u_h\to u$ in $w^*$-$BV_{\rm loc}(\Omega)$, we have $Du_h\to Du$ in $w^*$-$\mathcal{M}_{\rm loc}(\Omega,\mathbb{R}^n)$ and since, for every $u\in BV_{\rm loc}(\Omega)$, $u_\rho\to u$ in $w^*$-$BV_{\rm loc}(\updownarrow\Omega)$ (and then $\nabla u_\rho\cdot\mathcal{L}^n=(Du)_\rho\to Du$ in $w^*$-$\mathcal{M}_{\rm loc}(\updownarrow\Omega,\mathbb{R}^n)$) we end applying directly Theorem 6.1. □

A simple consequence of the equalities (7.3) is that, in particular, for every $u\in BV(\Omega)$[1],

$$\mathbb{I}(u,\Omega)\le R[w^*\text{-}BV]\,(I)\,(u,\Omega)\quad\text{and}\quad \mathbb{S}(u,\Omega)\le R[w^*\text{-}BV]\,(S)\,(u,\Omega).$$

[1] Here we refer to the notations given in the Introduction (see in particular (1.5)) and we consider $I$ and $S$ defined on $W^{1,1}(\Omega)$ that, as known, is a subset of $BV(\Omega)$ dense with respect to the $w^*$-$BV$ convergence (so as $C^\infty(\Omega)\cap W^{1,1}(\Omega)$; see [7] Theorem 3.9).

The propositions below show two particular cases in which also the opposite inequalities hold, and thus $\mathbb{I}$ and $\mathbb{S}$ are just the relaxed functionals of $I$ and $S$. Compare Proposition 7.2 with Theorem 2 by Serrin [68].

In the following, by $I_{|C^\infty(\overline{\Omega})}$ and $S_{|C^\infty(\overline{\Omega})}$ we mean the functionals $I$ and $S$ restricted to the space $C^\infty(\overline{\Omega})$, that is the set of the infinitely derivable functions with every derivative uniformly continuous on $\Omega$: this space, in the cases considered here, is dense on $BV(\Omega)$ with respect to the $w^*$-$BV(\Omega)$ convergence.

**Proposition 7.2.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and $f : \mathbb{R}^n \to [0, \infty)$ be a convex function such that, for every $\xi \in \mathbb{R}^n$,*

$$f(\xi) \leq c|\xi| + d,$$

*where $c, d > 0$. Then, for every $u \in BV(\Omega)$,*

$$\mathbb{I}(u, \Omega) = R[w^*\text{-}BV] \left( I_{|C^\infty(\overline{\Omega})} \right) (u, \Omega),$$

*thus, in particular, $\mathbb{I}(u, \Omega) = R[w^*\text{-}BV] \, (I) \, (u, \Omega)$.*

*Proof.* Let us fix $u \in BV(\Omega)$ and $\Omega \subset\subset \Omega' \subset\subset \mathbb{R}^n$: then we can consider $\hat{u} \in BV(\Omega')$ an extension of $u$ on $\Omega'$ (see [40] Theorem 1, page 183). Clearly, if $0 < \rho < \rho_0$ small enough, $\hat{u}_\rho \in C^\infty(\overline{\Omega})$, $\hat{u}_\rho \to u$ in $w^*$-$BV(\Omega)$ (see [7] Lemma 3.24) and $\hat{u}_\rho = u_\rho$ on $\Omega_\rho$. We claim now that

$$\lim_{\rho \to 0} |I(\hat{u}_\rho, \Omega) - I(u_\rho, \Omega_\rho)| = 0.$$

Indeed

$$|I(\hat{u}_\rho, \Omega) - I(u_\rho, \Omega_\rho)| \leq \int_{\Omega \setminus \Omega_\rho} f(\nabla \hat{u}_\rho(x)) dx \leq c \int_{\Omega \setminus \Omega_\rho} |\nabla \hat{u}_\rho(x)| dx + \mathcal{L}^n(\Omega \setminus \Omega_\rho).$$

By Theorem 2.2(b) in [7], we know that

$$\int_{\Omega \setminus \Omega_\rho} |\nabla \hat{u}_\rho(x)| dx \leq |D\hat{u}| \Big( (\Omega \setminus \Omega_\rho) + B(0, \rho) \Big), \tag{7.4}$$

where $(\Omega \setminus \Omega_\rho) + B(0, \rho) = \{x \in \Omega' : d(x, \Omega \setminus \Omega_\rho) < \rho\}$. Since $|D\hat{u}|(\Omega') < \infty$ and $(\Omega \setminus \Omega_\rho) + B(0, \rho) \downarrow \emptyset$ as $\rho \to 0$ then the left hand side of (7.4) tends to zero and the claim is achieved. Thus

$$\mathbb{I}(u, \Omega) = \lim_{\rho \to 0} I(u_\rho, \Omega_\rho) = \lim_{\rho \to 0} I(\hat{u}_\rho, \Omega) \geq R[w^*\text{-}BV] \left( I_{|C^\infty(\overline{\Omega})} \right) (u, \Omega)$$

and we end the proof being the opposite inequality satisfied. □

**Proposition 7.3.** *Let $\Omega = B(0, 1) \subseteq \mathbb{R}^n$ and $g : \mathbb{R}^n \to [0, \infty]$ be a proper, l.s.c. and level convex function such that $g(0) = 0$. Then, for every $u \in BV(\Omega)$,*

$$\mathbb{S}(u, \Omega) = R[w^*\text{-}BV] \left( S_{|C^\infty(\overline{\Omega})} \right) (u, \Omega),$$

*thus, in particular, $\mathbb{S}(u, \Omega) = R[w^*\text{-}BV] \, (S) \, (u, \Omega)$.*

*Proof.* Let us fix $u \in BV(\Omega)$, consider the convolution $u_\rho : B(0, 1-\rho) \to \mathbb{R}$ (with $\rho < \frac{1}{4}$) and set, for every $x \in B(0, 1)$, $\hat{u}_\rho(x) = u_\rho((1 - 2\rho)x)$: clearly $\hat{u}_\rho \in C^\infty(\overline{B(0, 1)})$.

We claim at first that $\hat{u}_\rho \to u$ in $w^*$-$BV(B(0, 1))$. Indeed, we simply have that, by means of the properties of the convolutions and the change of variables formula, there exists a constant $C > 0$ such that, for every $0 < \rho < \frac{1}{4}$,

$$\int_{B(0,1)} |\hat{u}_\rho(x)| dx + \int_{B(0,1)} |\nabla \hat{u}_\rho(x)| dx$$

$$\leq \frac{1}{(1-2\rho)^n}\int_{B(0,1)}|u(x)|dx + \frac{1}{(1-2\rho)^{n-1}}|Du|\left(B(0,1)\right)dx \leq C.$$

Thus, by Theorem 2.8 there exists $v \in BV(B(0,1))$ such that $\hat{u}_\rho \to v$ in $w^*$-$BV(B(0,1))$. We prove that, for $\mathcal{L}^n$-a.e. $x \in B(0,1)$, $u(x) = v(x)$ showing that, for $\mathcal{L}^n$-a.e. $x \in B(0,1)$, $\hat{u}_\rho(x) \to u(x)$. Indeed, if $M = \sup\{k(x) : x \in \mathbb{R}^n\}$ we have

$$|\hat{u}_\rho(x) - u(x)| = \left|\int_{B((1-2\rho)x,\rho)} k_\rho((1-2\rho)x - y)u(y)dy - u(x)\right|$$

$$\leq \int_{B((1-2\rho)x,\rho)} k_\rho((1-2\rho)x-y)|u(y)-u(x)|dy \leq \frac{M}{\rho^n}\int_{B(x,3\rho)\cap B(0,1)}|u(y)-u(x)|dy.$$

Now if $\rho$ is small enough (that is $3\rho \leq 1 - |x|$), $B(x,3\rho)\cap B(0,1) = B(x,3\rho)$ and then

$$|\hat{u}_\rho(x) - u(x)| \leq \frac{3^n M}{(3\rho)^n}\int_{B(x,3\rho)}|u(y)-u(x)|dy,$$

that, for $\mathcal{L}^n$-a.e. $x \in B(0,1)$, goes to zero as $\rho \to 0$ by the Lebesgue's points Theorem.

Once the claim is proved, applying Proposition 2.29 to $g$, we have, for every $x \in B(0,1)$,

$$g\left(\nabla\hat{u}_\rho(x)\right) = g\Big((1-2\rho)\nabla u_\rho((1-2\rho)x)\Big) \leq g\Big(\nabla u_\rho((1-2\rho)x)\Big) \leq \sup_{x\in B(0,1-\rho)} g\left(\nabla u_\rho(x)\right),$$

and then

$$\sup_{x\in B(0,1)} g\left(\nabla\hat{u}_\rho(x)\right) \leq \sup_{x\in B(0,1-\rho)} g\left(\nabla u_\rho(x)\right).$$

This allows to achieve the thesis since

$$\mathbb{S}(u,\Omega) = \lim_{\rho\to 0} S(u_\rho,\Omega_\rho) \geq \liminf_{\rho\to 0} S(\hat{u}_\rho,\Omega) \geq R[w^*\text{-}BV]\left(S_{|C^\infty(\overline{\Omega})}\right)(u,\Omega)$$

and being the opposite inequality satisfied. □

**Remark 7.4.** It is quite simple to understand that the proposition above holds also if we set $\Omega$ to be a bounded convex set. We have written down the proof for the unit ball in order to simplify many technical details.

### 7.1.2 Dirichlet problem for supremal functionals

Following the work of Anzellotti, Buttazzo and Dal Maso [8] in which the same problem is considered for the integral functionals, we shall introduce a generalized Dirichlet problem for a particular class of supremal functionals defined on $BV(\Omega)$. The formulation we proposed is similar to the one used in the nonparametric Plateau's problem (see [47]).

We have to underline that the functional (7.5) of Theorem 7.5 does not arise from the relaxation on $BV(\Omega)$, with respect to the $w^*$-$BV(\Omega)$ convergence, of the functional (7.2) defined on $C^1_\varphi(\overline{\Omega})$ (that is the space of the function belonging to $C^1(\overline{\Omega})$ with fixed boundary data $\varphi$) but it is simply suggested by the analogy with the integral setting. However, when the hypotheses of Theorem 7.5 are considered, we suppose it is a good candidate to represent just the functional $R\left[w^*\text{-}BV(\Omega)\right]\left(\mathbb{S}_{|C^1_\varphi(\overline{\Omega})}\right)$.

In the following, given a function $u \in BV(\Omega)$, we will always consider the restriction of $u$ to $\partial\Omega$ in the trace sense (see for instance [40] Theorem 1 page 177).

**Theorem 7.5.** *Let $\Omega$ be an open and bounded subset of $\mathbb{R}^n$ with Lipschitz boundary, $g : \mathbb{R}^n \to [0,\infty]$ be a proper, l.s.c. and level convex function and $\varphi \in L^1_{\mathcal{H}^{n-1}}(\partial\Omega)$. Let us consider, for every $u \in BV(\Omega)$, the functional*

$$\mathbb{S}_\varphi(u,\Omega) = \mathbb{S}(u,\Omega) \ \vee \ \left[\mathcal{H}^{n-1}\text{-}\operatorname*{ess\,sup}_{x\in\partial\Omega} g^\natural\Big((\varphi(x)-u(x))\nu(x)\Big)\right], \tag{7.5}$$

*where $\mathbb{S}$ is given by (7.2) and $\nu : \partial\Omega \to S^{n-1}$ is the vector field of the outer normal vectors to $\partial\Omega$*[2]*.*

*Assume that there exist an open and bounded subset of $\mathbb{R}^n$ with Lipschitz boundary $\Omega' \supset\supset \Omega$ and $w \in W^{1,1}(\Omega' \setminus \Omega)$ such that $\varphi = w_{|\partial\Omega}$ and*

$$\operatorname*{ess\,sup}_{x\in\Omega'\setminus\Omega} g(\nabla w(x)) = \inf\{g(\xi) : \xi \in \mathbb{R}^n\}. \tag{7.6}$$

*Then $\mathbb{S}_\varphi$ is l.s.c. on $BV(\Omega)$ with respect to the $w^*$-$BV(\Omega)$ convergence.*

*Moreover if $s = \sup\{g(\xi) : \xi \in \mathbb{R}^n\} \in [0,\infty]$ and $K_g = \{\xi \in \mathbb{R}^n : g^\natural(\xi) < s\}$ is such that $\operatorname{cl}(K_g)$ does not contain any straight line, then the problem*

$$\min\left\{\mathbb{S}_\varphi(u,\Omega) : u \in BV(\Omega)\right\}, \tag{7.7}$$

*admits at least a solution.*

*Proof.* Let $u, u_h \in BV(\Omega)$ such that $u_h \to u$ in $w^*$-$BV(\Omega)$ and define

$$\hat{u}(x) = \begin{cases} u(x) \text{ if } x \in \Omega, \\ w(x) \text{ if } x \in \Omega' \setminus \Omega, \end{cases} \quad \text{and} \quad \hat{u}_h(x) = \begin{cases} u_h(x) \text{ if } x \in \Omega, \\ w(x) \text{ if } x \in \Omega' \setminus \Omega. \end{cases}$$

It is easy to see that $\hat{u}, \hat{u}_h \in BV(\Omega')$ and $\hat{u}_h \to \hat{u}$ in $w^*$-$BV(\Omega')$. Moreover, since

$$D\hat{u} \llcorner \partial\Omega = (\varphi - u)\nu \cdot \mathcal{H}^{n-1} \llcorner \partial\Omega \quad \text{and} \quad D\hat{u}_h \llcorner \partial\Omega = (\varphi - u_h)\nu \cdot \mathcal{H}^{n-1} \llcorner \partial\Omega$$

(see [47] and [8] Theorem 3.1), we have also

$$\mathbb{S}_\varphi(u,\Omega) = \mathbb{S}(\hat{u},\Omega') \quad \text{and} \quad \mathbb{S}_\varphi(u_h,\Omega) = \mathbb{S}(\hat{u}_h,\Omega').$$

Then, by means of Theorem 7.1, it is

$$\mathbb{S}_\varphi(u,\Omega) = \mathbb{S}(\hat{u},\Omega') \leq \liminf_{h\to\infty} \mathbb{S}(\hat{u}_h,\Omega') = \liminf_{h\to\infty} \mathbb{S}_\varphi(u_h,\Omega)$$

and the first part of the theorem is proved.

Before proving the second part of the theorem we need a remark. Let $\Theta : [0,s] \to [0,\infty]$ be a continuous and strictly increasing function: it is clear that $\bar{u} \in BV(\Omega)$ is a minimum point of $\mathbb{S}_\varphi(u,\Omega)$ if and only if it is a minimum point of $\Theta\left(\mathbb{S}_\varphi(u,\Omega)\right)$. However, a simple computation gives that, for every $u \in BV(\Omega)$,

$$\Theta\left(\mathbb{S}_\varphi(u,\Omega)\right) = \left[\operatorname*{ess\,sup}_{x\in\Omega} (\Theta\circ g)(\nabla u(x))\right] \vee \left[|D^s u|\text{-}\operatorname*{ess\,sup}_{x\in\Omega} (\Theta\circ g^\natural)\left(\frac{dD^s u}{d|D^s u|}(x)\right)\right]$$
$$\vee \ \left[\mathcal{H}^{n-1}\text{-}\operatorname*{ess\,sup}_{x\in\partial\Omega} (\Theta\circ g^\natural)\Big((\varphi(x)-u(x))\nu(x)\Big)\right].$$

Thus, by Proposition 2.30, we have that $\Theta\left(\mathbb{S}_\varphi(u,\Omega)\right)$ is in fact the functional (7.5) related to $\Theta \circ g : \mathbb{R}^n \to [0,\infty]$. Then, invoking now Proposition 2.31, we can suppose, without loss of

[2]This vector field can be defined since $\partial\Omega$ is Lipschitz so that Rademacher's theorem can be applied.

generality, that $g$ is demi-coercive, that is, there exist $a > 0, b \geq 0$ and $\eta \in \mathbb{R}^n$ such that, for every $\xi \in \mathbb{R}^n$,

$$g(\xi) \geq a|\xi| - \langle \eta, \xi \rangle - b.$$

If, for every $u \in BV(\Omega)$, $\mathbb{S}_\varphi(u, \Omega) = \infty$ there is nothing to prove. Thus let us consider any $u \in BV(\Omega)$ such that $\mathbb{S}_\varphi(u, \Omega) < \infty$: we claim that, setting $c = \frac{1}{3\mathcal{L}^n(\Omega)} \wedge \frac{1}{3\mathcal{H}^{n-1}(\partial\Omega)}$,

$$\mathbb{S}_\varphi(u, \Omega) \tag{7.8}$$

$$\geq c \left\{ \int_\Omega g(\nabla u(x))dx + \int_\Omega g^\infty \left( \frac{dD^s u}{d|D^s u|}(x) \right) d|D^s u|(x) + \int_{\partial\Omega} g^\infty \Big( (\varphi(x) - u(x))\nu(x) \Big) \mathcal{H}^{n-1}(x) \right\}.$$

Indeed, clearly it holds

$$\operatorname*{ess\,sup}_{x \in \Omega} g(\nabla u(x)) \geq \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega g(\nabla u(x))dx. \tag{7.9}$$

Moreover since

$$|D^s u|\text{-}\operatorname*{ess\,sup}_{x \in \Omega} g^\natural \left( \frac{dD^s u}{d|D^s u|}(x) \right) < \infty,$$

it holds, for $|D^s u|$-a.e. $x \in \Omega$, $g^\natural \left( \frac{dD^s u}{d|D^s u|}(x) \right) < \infty$ and then, by Proposition 2.28(ii), we have also, for $|D^s u|$-a.e. $x \in \Omega$, $g^\infty \left( \frac{dD^s u}{d|D^s u|}(x) \right) = 0$: then, in particular,

$$\frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega g^\infty \left( \frac{dD^s u}{d|D^s u|}(x) \right) d|D^s u|(x) = 0. \tag{7.10}$$

At last using Proposition 2.28(i) we have

$$\mathcal{H}^{n-1}\text{-}\operatorname*{ess\,sup}_{x \in \partial\Omega} g^\natural \Big( (\varphi(x) - u(x))\nu(x) \Big) \geq \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_{\partial\Omega} g^\infty \Big( (\varphi(x) - u(x))\nu(x) \Big) d\mathcal{H}^{n-1}(x). \tag{7.11}$$

Putting together (7.9), (7.10) and (7.11) we obtain, for every $u \in BV(\Omega)$ such that $\mathbb{S}_\varphi(u, \Omega) < \infty$,

$$\mathbb{S}_\varphi(u, \Omega) \geq \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega g(\nabla u(x))dx \vee \frac{1}{\mathcal{L}^n(\Omega)} \int_\Omega g^\infty \left( \frac{dD^s u}{d|D^s u|}(x) \right) d|D^s u|(x)$$

$$\vee \frac{1}{\mathcal{H}^{n-1}(\partial\Omega)} \int_{\partial\Omega} g^\infty \Big( (\varphi(x) - u(x))\nu(x) \Big) \mathcal{H}^{n-1}(x)$$

$$\geq c \left\{ \int_\Omega g(\nabla u(x))dx + \int_\Omega g^\infty \left( \frac{dD^s u}{d|D^s u|}(x) \right) d|D^s u|(x) + \int_{\partial\Omega} g^\infty \Big( (\varphi(x) - u(x))\nu(x) \Big) d\mathcal{H}^{n-1}(x) \right\},$$

and (7.8) is proved.

By means of (7.8) and Proposition 2.28(iii)[3] we find

$$\mathbb{S}_\varphi(u, \Omega) \geq ac \int_\Omega |\nabla u(x)|dx - c \int_\Omega \langle \eta, \nabla u(x) \rangle dx - bc\mathcal{L}^n(\Omega) + ac|D^s u|(\Omega) +$$

$$-c \int_\Omega \left\langle \eta, \frac{dD^s u}{d|D^s u|}(x) \right\rangle d|D^s u|(x) + ac \int_{\partial\Omega} |\varphi(x) - u(x)| d\mathcal{H}^{n-1}(x) +$$

$$-c \int_{\partial\Omega} \langle \eta, (\varphi(x) - u(x))\nu(x) \rangle d\mathcal{H}^{n-1}(x),$$

[3]Compare with Theorems 2.7 and 3.2 of [8].

and then, using the fact that[4], for every $u \in BV(\Omega)$,

$$\int_\Omega \langle \eta, \nabla u(x)\rangle dx + \int_\Omega \left\langle \eta, \frac{dD^s u}{d|D^s u|}(x)\right\rangle d|D^s u|(x) = \int_{\partial\Omega} \langle \eta, u(x)\nu(x)\rangle d\mathcal{H}^{n-1}(x)$$

and that there exists[5] a constant $A = A(\Omega) > 0$ such that, for every $u \in BV(\Omega)$,

$$\int_\Omega |u(x)|dx \le A\left\{|Du|(\Omega) + \int_{\partial\Omega} |u(x)|d\mathcal{H}^{n-1}(x)\right\},$$

it follows

$$\mathbb{S}_\varphi(u,\Omega) \ge \frac{ac}{2}\left(1 \wedge \frac{1}{A}\right)\left\{|Du|(\Omega) + \int_\Omega |u(x)|dx + \int_{\partial\Omega} |u(x)|d\mathcal{H}^{n-1}(x)\right\} - dc, \tag{7.12}$$

where

$$d = \int_{\partial\Omega} |\varphi(x)|d\mathcal{H}^{n-1}(x) + \int_{\partial\Omega} \langle \eta, \varphi(x)\nu(x)\rangle d\mathcal{H}^{n-1}(x).$$

If now we consider a minimizing sequence $\{u_h\}_{h=1}^\infty$ such that, for every $h \in \mathbb{N}$, $\mathbb{S}_\varphi(u_h,\Omega) < \infty$ we can prove its compactness only noting that (7.12) allows just to apply Theorem 2.8: this fact, together with the lower semicontinuity of the functional $\mathbb{S}_\varphi$, guarantees the existence of a minimum for (7.7). □

**Remark 7.6.** In the hypotheses of Theorem 7.5, if there exists $u \in BV(\Omega)$ such that $\mathbb{S}_\varphi(u,\Omega) < \infty$, then the same holds for every solution $\bar{u}$ of (7.7) too. Then, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial\Omega$,

$$(\varphi(x) - \bar{u}(x))\nu(x) \in K_g = \{\xi \in \mathbb{R}^n : g^\natural(\xi) < \infty\}.$$

Since $g^\natural$ is level convex and positively homogeneous of degree 0 (see Proposition 2.20), $K_g$ is clearly a convex cone and then it follows that, for $\mathcal{H}^{n-1}$-a.e. $x \in \{x \in \partial\Omega : \nu(x) \notin K_g\}$, $\varphi(x) - \bar{u}(x) \le 0$ and, for $\mathcal{H}^{n-1}$-a.e. $x \in \{x \in \partial\Omega : -\nu(x) \notin K_g\}$, $\varphi(x) - \bar{u}(x) \ge 0$.

Therefore we conclude that, for $\mathcal{H}^{n-1}$-a.e. $x \in \{x \in \partial\Omega : \nu(x) \notin K_g \cup (-K_g)\}$, $\varphi(x) = \bar{u}(x)$.

It is also clear that, arguing as in the proof of Theorem 7.5, we cannot avoid the presence of the condition (7.6). Nevertheless we can easily find suitable classes of examples in which (7.6) is satisfied.

Consider for instance the case in which $\Omega = B(0,1) \subseteq \mathbb{R}^2$ and $g : \mathbb{R}^2 \to [0,\infty)$ is defined as $g(\xi_1,\xi_2) = (\xi_1 \vee 0) + (\xi_2 \vee 0)$. If $\varphi$ is the restriction to $S^1$ of a function $w \in C^1(B(0,1+\varepsilon))$ (where $\varepsilon > 0$) such that, for every $x \in B(0,1+\varepsilon) \setminus B(0,1)$, $\nabla w(x) \in \{(\xi_1,\xi_2) : \xi_1 \le 0, \xi_2 \le 0\}$, then (7.6) holds.

## 7.2 Non level convex functionals

### 7.2.1 A minimum problem

In this section we find a simple application of Theorem 6.20 to the $BV$ setting where obviously only the one dimensional case is considered: the result here presented should be compared with the work of Alicandro, Braides and Cicalese [3] in which an analogous problem is deeply analyzed.

For this let $(a,b) \subseteq \mathbb{R}$, $g, w, \gamma : (a,b) \times \mathbb{R} \to [0,\infty]$ be proper Borel functions. We want to minimize the following functional

$$L(u,K) = \left[\operatorname*{ess\,sup}_{x\in(a,b)\setminus K} g(x,u'(x))\right] \vee \left[\operatorname*{ess\,sup}_{x\in(a,b)\setminus K} w(x,u(x))\right] \vee \left[\bigvee_{x\in K} \gamma(x,u(x+) - u(x-))\right], \tag{7.13}$$

[4]See Theorem 1 pg 177 [40] and [47]

[5]See [47] and equation (3.4) of [8].

on the class

$$\mathcal{A} = \Big\{(u, K) : K = \{a = t_0 < \ldots < t_m = b\} \subseteq (a, b), m \in \mathbb{N}, \forall i \in \{1, \ldots, m\}\ u \in W^{1,\infty}(t_{i-1}, t_i)\Big\}.$$

Here $u(x+)$ and $u(x-)$ are the right and the left limit of $u$ in $x$ that always exist for every $x \in K$ when $(u, K) \in \mathcal{A}$. Let us note the analogy between this problem and the classical Mumford-Shah image segmentation one (see [7] Chapter 6).

In order to solve this problem let us note at first that if $(u, K) \in \mathcal{A}$ then $u \in SBV(a, b)$. For this we give a relaxed formulation of the functional $L$ extending its definition on the space $SBV(a, b)$ in the obvious way: for every $u \in SBV(a, b)$, decomposing its distributional derivative as $Du = u' \cdot \mathcal{L}^1 + D^{\#}u \in \mathcal{M}(a, b)$, where $u' \in L^1(a, b)$ and $D^{\#}u$ is purely atomic, we set

$$\mathbb{L}(u, (a, b)) = \left[\operatorname*{ess\,sup}_{x\in(a,b)} g(x, u'(x))\right] \vee \left[\operatorname*{ess\,sup}_{x\in(a,b)} w(x, u(x))\right] \vee \left[\bigvee_{x\in A_{Du}} \gamma(x, D^{\#}u(x))\right]. \tag{7.14}$$

Clearly, when $(u, K) \in \mathcal{A}$, $L(u, K) = \mathbb{L}(u, (a, b))$, thus

$$\inf_{u\in SBV(a,b)} \mathbb{L}(u, (a, b)) \leq \inf_{(u,K)\in\mathcal{A}} L(u, K).$$

The following theorem shows that, with suitable hypotheses on $g, w, \gamma$, the functional $L$ admits a minimum on $\mathcal{A}$. Note that, in order to obtain the compactness we need a particular coercivity hypothesis on $\gamma$, that, as Proposition 2.27 shows, is more regular than it seems.

**Theorem 7.7.** *Let $g, w, \gamma : (a, b) \times \mathbb{R} \to [0, \infty]$ be proper and Borel functions. Let us suppose that, for $\mathcal{L}^1$-a.e. $x \in (a, b)$, $g(x, \cdot), w(x, \cdot)$ are l.s.c. and level convex on $\mathbb{R}$ and there exists a function $\theta_\infty : [0, \infty) \to [0, \infty)$ such that, for every $x \in (a, b)$ and $\xi \in \mathbb{R}$,*

$$g(x, \xi) \wedge w(x, \xi) \geq \theta_\infty(|\xi|) \quad \text{and} \quad \lim_{t\to\infty} \theta_\infty(t) = \infty.$$

*Moreover, setting $l = \inf\{g(x, \xi) : (x, \xi) \in (a, b) \times \mathbb{R}\}$, let us suppose that $\gamma \vee l$ is l.s.c. on $(a, b) \times \mathbb{R}$, for every $x \in (a, b)$, $\gamma(x, \cdot) \vee l$ is sub-maximal on $\mathbb{R}$ and $\gamma(x, 0) = 0$, and there exists a function $\theta_0 : (0, \infty) \to [0, \infty)$ such that,*

$$\begin{cases} \gamma(x, \xi) \geq \theta_0(\xi) \ \text{if}\ x \in (a, b), \xi > 0, \\ \gamma(x, \xi) = \infty \ \text{if}\ x \in (a, b), \xi < 0, \end{cases} \quad \text{and} \quad \lim_{t\to 0} \theta_0(t) = \infty.$$

*Then the problem*

$$\min\{L(u, K) : (u, K) \in \mathcal{A}\}$$

*admits at least a solution.*

*Proof.* We start considering the functional $\mathbb{L}$ defined by (7.14) and proving, by using the Direct Methods, that it has a minimum on $SBV(a, b)$. For this let us consider a minimizing sequence $\{u_h\}_{h=1}^\infty \subseteq SBV(a, b)$: we can suppose that there exists a constant $M > 0$ such that, for every $h \in \mathbb{N}$, $\mathbb{L}(u_h, (a, b)) \leq M$ (we can suppose $\mathbb{L} \not\equiv \infty$). We want to prove at first the compactness of this sequence.

Using the coercivity property of $g, w$ and $\gamma$, we can find a constant $M' > 0$ such that, for every $u \in SBV(a, b)$ such that $\mathbb{L}(u, (a, b)) \leq M$ (in particular, for every $u_h$), we have

(i) for $\mathcal{L}^1$-a.e. $x \in (a, b)$, $|u'(x)| \leq M'$,

(ii) for $\mathcal{L}^1$-a.e. $x \in (a, b)$, $|u(x)| \leq M'$,

(iii) for every $x \in A_{Du}$ (the set of the atoms of $Du$), $D^{\#}u(x) \geq \frac{1}{M'}$: this condition implies, in particular, that on every jump point the function $u$ increases.

Then, considering the sequence $\{u_h\}_{h=1}^{\infty}$, we are in the position to prove the compactness of this sequence using Theorem 2.9 if we are able to prove that there exists $C > 0$ such that, for every $h \in \mathbb{N}$, $\#(A_{Du_h}) \leq C$. In order to prove this we consider the formula, referred to a good representative of $u_h$, given by

$$u_h(b-) - u_h(a+) = \int_a^b u'(x)dx + \sum_{x \in A_{Du_h}} D^{\#}u_h(x)$$

that implies, by means of (iii),

$$\#(A_{Du_h})\frac{1}{M'} \leq \sum_{x \in A_{Du_h}} D^{\#}u_h(x) \leq |u_h(b-) - u_h(a+)| + \int_a^b |u_h'(x)|dx \leq 2M' + (b-a)M' = C.$$

Thus there exists a subsequence (not relabelled) and $\bar{u} \in SBV(a,b) \cap L^{\infty}(a,b)$ such that

$$u_h \to \bar{u} \text{ in } w^*\text{-}BV(a,b), \quad \text{and} \quad u_h \to \bar{u} \text{ in } w^*\text{-}L^{\infty}(a,b).$$

Let us show now the lower semicontinuity of $\mathbb{L}$ on this sequence. Obviously $Du_h \to D\bar{u}$ in $w^*$-$\mathcal{M}(a,b)$ and, by Theorem 6.20, for the functional

$$\left[\operatorname*{ess\,sup}_{x \in (a,b)} g(x, u'(x))\right] \vee \left[\bigvee_{x \in A_{Du}} \gamma(x, D^{\#}u(x))\right],$$

the lower semicontinuity holds. By Theorem 6.19 also the functional

$$\operatorname*{ess\,sup}_{x \in (a,b)} w(x, u(x)),$$

is lower semicontinuous on the considered sequence. Then

$$\mathbb{L}(\bar{u}, (a,b)) \leq \liminf_{h \to \infty} \mathbb{L}(u_h, (a,b)),$$

that is, $\mathbb{L}$ admits a minimum on $SBV(a,b)$ given by $\bar{u}$. Let us note now that since (i), (ii) and (iii) hold for $\bar{u}$ too, clearly $\#(A_{D\bar{u}}) \leq C$ and, for $\mathcal{L}^1$-a.e. $x \in (a,b)$, $|u'(x)| \leq M'$: then $(\bar{u}, A_{D\bar{u}}) \in \mathcal{A}$ and the thesis is achieved. □

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