A variational model for dislocations in the line tension limit

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Abstract

We study the interaction of a singularly perturbed multiwell energy (with an anisotropic nonlocal regularizing term of $H^{1/2}$ type) and a pinning condition. This functional arises in a phase field model for dislocations which was recently proposed by Koslowski, Cuitiño and Ortiz but is also of broader mathematical interest. In the context of the dislocation model we identify the Γ-limit of the energy in all scaling regimes for the number N_ε of obstacles. The most interesting regime is $N_\varepsilon \approx |\ln \varepsilon|/\varepsilon$, where ε is a nondimensional length scale related to the size of the crystal lattice. In this case the limiting model is of line tension type. One important feature of our model is that the set of energy wells is periodic and hence not compact. A key ingredient in the proof is thus a compactness estimate (up to a single translation) for finite energy sequences, which generalizes earlier results of Alberti, Bouchitté and Seppecher for the two-well problem with an $H^{1/2}$ regularization.

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1 Introduction

We study the functional

$$E_\varepsilon(u) = \frac{1}{\varepsilon}\int_Q W(u)\,dx + \iint_{Q\times Q} K_\nu(x-y)|u(x)-u(y)|^2\,dx\,dy, \tag{1}$$

subject to the pinning condition

$$u = 0 \quad \text{on } B(x_\varepsilon^i, R\varepsilon) = B^i_{R\varepsilon}, \quad \text{for } i = 1,\ldots,N_\varepsilon. \tag{2}$$

Here $Q = (-1/2, 1/2)^2$ is the unit square, W is one-periodic, non-negative and vanishes exactly on the integers $\mathbf{Z}$, a typical choice being

$$W(u) = \operatorname{dist}^2(u, \mathbf{Z}), \tag{3}$$

and the nonlocal part of the energy behaves like the $H^{1/2}$ norm, i.e., $K_\nu(z) \approx |z|^{-3}$. The parameter ν is Poisson's ratio, see (6) below. The points x_ε^i play the role of pinning centres and we will later assume that they are well-separated and sufficiently uniformly distributed, see (16) and (15) below. Finally $\varepsilon > 0$ is a small parameter and we study the limit as $\varepsilon \to 0$ (after suitable rescaling).

The above functional with the choice (3) has recently been proposed by Koslowski, Cuitiño and Ortiz as a phase-field model for dislocations. In this setting ε is the ratio of the spacing of the lattice planes and the size of the physical domain under consideration. In this context our main achievement is that we identify the relevant scaling regimes for the number of obstacles N_ε and the corresponding Γ-limits of the (suitably scaled) energy E_ε. Specifically if $N_\varepsilon \approx \varepsilon^{-1}|\log\varepsilon|$ we show that the limit of $E_\varepsilon/(\varepsilon N_\varepsilon)$ is the so called line-tension limit, i.e., the limit functional is defined on the space $BV(Q;\mathbf{Z})$, and is given by

$$\int_{S_u} \gamma(n)\, d\mathcal{H}^1 + \int_Q D_\nu(u, B_R)\, dx,$$

where S_u is the jump set of u, with normal n, $\gamma(n)$ is an (anisotropic) line energy density, and $D_\nu(u, B_R)$ represents the limiting contribution of the obstacles $B^i_{R\varepsilon}$ (see Theorem 10 below for a precise statement). If $N_\varepsilon \ll \varepsilon^{-1}|\log\varepsilon|$ the line energy contribution dominates, only constant functions $u \equiv a$, with $a \in \mathbf{Z}$, are admissible in the limit and their energy is given by $D_\nu(a, B_R)$. If $N_\varepsilon \gg \varepsilon^{-1}|\log\varepsilon|$ then the line energy becomes negligible, finite energy sequences may only converge weakly and the limit energy is given by the convex envelope $\int_Q D_\nu^{**}(u, B_R)\, dx$, where u may now take values in $\mathbf{R}$, see Corollary 11 below for a precise statement.

From a more general mathematical point of view, the problem we consider combines two features which have been very extensively studied in the last years. The first one is the interaction of singularly perturbed multiwell energies and higher order regularisations. Beginning with the work of Modica and Mortola [15, 16] a large body of work has concentrated on multiwell energies with a compact set of energy minimizing states and a local regularization given by the Dirichlet integral (see, e.g., [14, 9, 7, 5] and the extensive list of references therein).

The more delicate case of a regularizing term which corresponds to the $H^{1/2}$ norm and which requires a logarithmic rescaling was studied by Alberti, Bouchitté and Seppecher [3]. For nonlocal terms with general anisotropic, but regular, kernels see [2, 1].

The second feature is the interaction of a pinning condition like (2) and Dirichlet-type energies. Since the work of Marchenko and Kruslov [13] and of Cioranescu and Murat [4] this interaction has attracted a lot of attention (see for instance [6] for many further references). Roughly speaking, a general theme of this large body of work is that for well separated obstacles the limiting problem has no constraint like (2) but involves an extra energy contribution of the form $\int a(x)|u|^2\, dx$ where $a(x)$ can be viewed as a local 'capacity density' (and where the appropriate notion of 'capacity' is related to the Dirichlet-type energy in the orginal functional). Of course a may be singular or degenerate and instead of $a(x)dx$ one may obtain more general measures which are no longer absolutely continuous with respect to the Lebesgue measure.

Our problem combines both features. We show that the contributions of the singular perturbation (leading to a line energy) and of the pinning constraint are essentially additive. The interaction of pinning and the multiwell structure is reflected in the structure of the limiting pinning energy $\int_Q D_\nu(u, B_R)\, dx$ which is no longer a quadratic expression in u.

Compared with most previous work on singularly perturbed multiwell problems our problem involves an important additional difficulty. The set of wells, i.e., the zero set of W is no longer compact. Hence it is not clear that sequences with finite (rescaled) energy are bounded in L^1. One crucial ingredient in our analysis is a uniform L^2 estimate (up to translation by integers) for sequences for which the rescaled energy $E_\varepsilon/|\log\varepsilon|$ is bounded (see Theorems 12 and 13 below).

Our main result will be stated in terms of Γ-convergence and we have therefore focused on the main part of the energy functional. In order to study nontrivial minimizers one should add a suitable loading term, which can be easily done, see the end of Subsection 1.2 for further discussion.

1.1 A quick review of the phase field theory of Koslowski, Cuitiño and Ortiz

Here we briefly discuss the interpretation of the different terms in (1) and the pinning condition. We refer to [10] for a detailed discussion of the model and to [8] for a discussion of the non-dimensional version of the energy (1). The setting is that of continuum crystal elasticity, with small strains. Consider the displacement field $U : \mathbf{R}^3 \to \mathbf{R}^3$ of an infinite elastic crystal. A specific assumption of the model in [10] is that one allows crystal slip only along the single plane $x_3 = 0$. Moreover one assumes that only a single slip system is active, i.e., the jump $[U]$ of the displacement U across the slip plane is of the form

$$[U](x_1, x_2, 0) = u(x_1, x_2) b\mathbf{e}, \tag{4}$$

where $b = |\mathbf{b}|$ is the length of the Burgers vector $\mathbf{b}$ of the active slip system and $\mathbf{e} = \mathbf{b}/b$ its direction. We choose coordinates such that $\mathbf{e}$ is the first coordinate direction. Given u, the associated elastic energy is obtained by minimizing the usual linear elastic energy away from the slip plane, i.e., by minimizing

$$\int_{\mathbf{R}^3\setminus\{x_3=0\}} \mu\,|e(U)|^2 + \frac{\lambda}{2}\,|\mathrm{tr}\, e(U)|^2 \; dx_1 dx_2 dx_3, \tag{5}$$

where $e(U) = \frac{1}{2}(\nabla U + \nabla U^T)$ is the symmetrized displacement gradient. The constants μ and λ are the elastic moduli which satisfy $\mu > 0$ and $3\lambda + 2\mu > 0$. We now suppose that $u : \mathbf{R}^2 \to \mathbf{R}$ is periodic with periodic cell $Q = (-1/2, 1/2)^2$ so that u can be viewed as a function on the torus T^2. Then the elastic energy per period is obtained by minimizing

$$\int_{T^2\times\mathbf{R}} \mu\,|e(U)|^2 + \frac{\lambda}{2}\,|\mathrm{tr}\, e(U)|^2 \; dx_1 dx_2 dx_3\,.$$

over all $U : T^2 \times \mathbf{R} \to \mathbf{R}^3$ which satisfy (4). This minimization can be carried out by considering the Fourier series of u

$$u(x) = \sum_{k\in(2\pi\mathbf{Z})^2} \widehat{u}(k) e^{ik\cdot x}, \quad \widehat{u}(k) = \int_{T^2} u(x) e^{-ik\cdot x}\, dx$$

and this yields

$$E_{\mathrm{elastic}}(u) = \frac{\mu b^2}{4} \sum_{k\in(2\pi\mathbf{Z})^2} m_\nu(k) |\widehat{u}(k)|^2,$$

where

$$m_\nu(k) = \frac{k_2^2}{|k|} + \frac{1}{1-\nu}\frac{k_1^2}{|k|}, \quad k = (k_1, k_2), \tag{6}$$

and where $\nu \in (-1, 1/2)$ is Poisson's ratio which is given by $\nu := \frac{\lambda}{2(\lambda+\mu)}$.

In real space we have

$$E_{\mathrm{elastic}}(u) = \frac{\mu b^2}{2} \int_{T^2}\int_{T^2} K_\nu(x-y) |u(x) - u(y)|^2\, dx dy, \tag{7}$$

and the kernel K_ν is given by the Fourier series with coefficients $-\frac{1}{4} m_\nu(k)$.

Now we turn to the local contribution in E_ε. If u is a (constant) integer then the jump $[U]$ is a lattice vector and hence the crystal lattice is not perturbed in the immediate neighbourhood of the slip plane. If u is not an integer there is an additional distorsion of the lattice near the slip plane and in the Peierls-Nabarro theory one models this by an extra energy contribution

$$E_{\mathrm{Peierls}}(u) = \int_{T^2} W(u)\, dx,$$

where W is a one-periodic, non-negative function which vanishes exactly on $\mathbf{Z}$. If one fixes a specific form of W, e.g., $W(u) = A\,\mathrm{dist}^2(u, \mathbf{Z})$, or $W = \frac{A}{\pi^2}\sin^2(\pi u)$ one can relate the coefficent A to the shear modulus μ and the properties of the crystal lattice by considering very small shear deformations. This line of reasoning leads to the expression $A = \mu b^2/(2d)$, where d is the distance between two neighbouring slip planes.

Dividing the sum of E_{Peierls} and E_{elastic} by $\mu b^2/2$ we arrive to the energy E_ε in (1) with W given by (3) and $\varepsilon = d$. Thus ε is proportional to the lattice spacing. If one starts more generally from the situation where u is periodic with periodic cell $(-l/2, l/2)^2$ and rescales to the unit cell one finds similarly that $\varepsilon = d/l$. Instead of periodic boundary conditions for u one can also consider other boundary conditions, see also the next subsection.

The regions $B^i_{R\varepsilon}$ represent obstacles (e.g., inclusions of another material) which restrain slip. The condition (2) represents the limiting case of infinitely strong obstacles which permit no slip at all. One can also consider obstacles of finite strength where slip is only possible under sufficiently strong loading. In this case one drops the condition (2) and instead adds an additional term

$$E_{\text{obstacle}} = \sum_i \int \lambda_0 \frac{1}{\varepsilon}\psi\Big(\frac{x - x^i_\varepsilon}{\varepsilon}\Big)|u|\,dx \tag{8}$$

to the energy (see Section 4.3 of [10], and the Appendix of [8] for further details). In [8] we have shown (in the dilute limit) that the consideration of obstacles of finite strength leads to a Γ-limit which has exactly the same form as for infinitely strong obstacles. The difference is that now the limiting pinning energy density $D_\nu(a, B_R)$ no longer has quadratic growth in a, but only grows linear in a, the limiting slope being related to the effective strength.

Finally one can easily include a forcing term

$$-\int_{T^2} S^\varepsilon u\,dx$$

in the energy, where S^ε is the resolved shear stress. In view of the results in Section 3 a natural scaling assumption on the applied force is

$$\frac{1}{\varepsilon N^\varepsilon} S^\varepsilon \to S \quad \text{in } L^2(T^2).$$

In this case the corresponding Γ-limit simply contains the additional term $-\int_{T^2} Su\,dx$. If $S^\varepsilon \ll \varepsilon N^\varepsilon$ then the applied force disappears in the limit. If $S^\varepsilon \gg \varepsilon N^\varepsilon$ then the applied force dominates in the limit. In the case $\varepsilon N^\varepsilon \ll |\log\varepsilon|$ there can still be an interesting interaction between the applied force an the line tension term in the limit, at least if the total applied force $\int_{T^2} S^\varepsilon\,dx$ vanishes or converges to zero sufficiently fast as $\varepsilon \to 0$.

1.2 Possible generalizations

For the rest of this paper we consider the functional (1) with the specific choice $W(u) = \mathrm{dist}^2(u, \mathbf{Z})$, the specific periodic kernel (7) discussed in the previous subsection and the hard pinning condition (2). Many of the results can, however, easily be extended to a more general setting. We briefly discuss some possible generalizations, roughly in the order of increasing difficulty.

- *More general local functions W.* The results can easily be extended to general periodic continuous functions W which are minimized exactly on $\mathbf{Z}$. For the crucial compactness result one can even allow certain nonperiodic W as long as their minimizers remain discrete and W does not degenerate too much near $\pm\infty$. As discussed in Remark 8 below the limiting energy does not depend on W but just on the spacing of its zeroes.

- *Other boundary conditions, other kernels K.* The results can be extended to other boundary conditions (for a sufficiently smooth bounded domain $\Omega \subset \mathbf{R}^2$). The corresponding three-dimensional elastic energy in the cylinder $\Omega \times \mathbf{R}$ has the same singular behaviour as in the periodic case. For Neumann (natural) boundary conditions one obtains the expression $\int_\Omega \int_\Omega K(x,y)|u(x)-u(y)|^2\, dxdy$ where the kernel $K(x,y)$ is smooth and behaves asymptotically for $y \to x$ as K_ν, i.e., $K(x,y) \approx K_\nu(x-y)$. One can also consider abstract kernels $K(x,y)$ as long as they behave like $L(x-y)$ for $y \to x$, where L is a positive function which is homogeneous of degree -3 and sufficiently smooth on the unit sphere.

- *General dimensions.* We focus on the case $n = 2$ because it arises naturally in the dislocation model which motivated this work. The results can, however, easily be extended to general dimensions, using a kernel which behaves like the kernel of the $H^{1/2}$ norm, i.e., $K(x,y) \approx |x-y|^{-n-1}$. In this case the compactness result gives an estimate in $L^{n/(n-1)}$ (for $n \geq 2$). The case $n = 1$ was studied by Kurzke [11]. In this case the optimal estimate is in the Orlicz space e^L.

- *Soft pinning.* As discussed in the previous subsection one can replace the 'hard' pinning condition (2) by a penalty term (8). For the subcritical scaling this is discussed in the appendix of [8] and the argument can be extended to the general setting.

- *Different obstacles and varying obstacle densities.* It is not necessary to assume that the obstacles have all the same size $R\varepsilon$ (or the same strength in the case of soft obstacles). Instead one can consider obstacles of varying size $B_i = B(x_i, R_i\varepsilon)$. Also the density of obstacles need not be constant. In this case we expect that the penalty term due to pinning is of the form $\Lambda(x)D_\nu(u(x), B_1)$ where $\Lambda(x)$ represents the local 'capacity density' (appropriately scaled with N_ε). This is briefly discussed in Remark 13 in [8]. Such extensions are well known in the context of competition between pinning and a local Dirichlet energy. To identify the precise assumptions for such a result to hold in the present context, including a suitable weakening of the condition on equipartition, will require some technical work.

1.3 Outline

In Section 2 we briefly review some known results, in particular the properties of the periodic kernel K_ν and its relation with a similar -3 homogeneous kernel Γ_ν, the convergence result in the dilute case $N_\varepsilon \leq C/\varepsilon$ and the definition of the dislocation capacity $D_\nu(a, B_R)$, and finally the results of Alberti, Bouchitté and Seppecher for the competition between a two-well energy and the $H^{1/2}$ norm under logarithmic rescaling, leading to a line tension limit.

In Section 3 we describe our main results. To emphasize the underlying mathematical structure we state them separately for the functional with and without pinning. The next three sections are devoted to the proof of the result in the critical scaling regime. The central compactness estimate is discussed in Section 4 and the lower and upper bound are discussed in Sections 5 and 6, respectively. In the last section we sketch how the result can be extended to the sub- and supercritical scaling. While the argument in the subcritical scaling is a straightforward extension of the results in the dilute case [8] using the compactness estimate, the lower bound in the supercritical case requires some care. We no longer have compactness but we can still show that the oscillation of u is small on a scale which is large compared to the typical spacing of the pinning sites and this suffices to conclude.

2 Review of some known results

2.1 Properties of the anisotropic kernel

Here we review some properties of the nonlocal term in the energy, see [10, 8] for further details and proofs. We start from the expression

$$\iint_{T^2\times T^2} K_\nu(x-y)|u(x)-u(y)|^2\, dx\, dy\,,$$

for the (suitably normalized) elastic energy introduced above. The Fourier coefficients of the kernel K_ν are given by

$$-\frac{1}{4}\left(\frac{k_2^2}{|k|}+\frac{1}{1-\nu}\frac{k_1^2}{|k|}\right), \quad \forall k \in (2\pi\mathbf{Z})^2,$$

where $-1 < \nu < \frac{1}{2}$ is Poisson's ratio. Since the Fourier coefficents of $K_\nu(t)$ are bounded from above and below by a multiple of $|k|$ the kernel is equivalent to the $H^{\frac{1}{2}}$-kernel, i.e.,

$$\frac{1}{2}[u]_{H^{\frac{1}{2}}(T^2)} \leq \iint_{T^2\times T^2} K_\nu(x-y)|u(x)-u(y)|^2\,dx\,dy \leq [u]_{H^{\frac{1}{2}}(T^2)}\,, \tag{9}$$

where $[u]_{H^{\frac{1}{2}}(T^2)}$ denotes the $H^{\frac{1}{2}}(T^2)$ seminorm defined by

$$[u]^2_{H^{\frac{1}{2}}(T^2)} = \sum_{k\in(2\pi\mathbf{Z})^2} |k||\widehat{u}(k)|^2\,. \tag{10}$$

In real space this seminorm can be written as

$$\frac{1}{2}[u]^2_{H^{\frac{1}{2}}(T^2)} = \iint_{T^2\times T^2} K(x-y)|u(x)-u(y)|^2\,dx\,dy\,, \tag{11}$$

where the kernel $K(t)$ is defined by $\widehat{K}(k) = -\frac{1}{4}|k|$ and satisfies the following properties:

i) $K(t) \sim |t|^{-3}$ as $|t| \to 0$;

ii) $K(t)$ is periodic, i.e., is defined in T^2.

The kernel $K_\nu(t)$ also satisfies properties i) and ii). For small t the term $K_\nu(t)$ is well approximated by the homogeneous and positive kernel

$$\Gamma_\nu(t) = \frac{1}{8\pi(1-\nu)|t|^3}\left(\nu+1-3\nu\frac{t_2^2}{|t|^2}\right), \tag{12}$$

which is the inverse Fourier transform of $-\frac{1}{4}\left(\frac{\lambda_2^2}{|\lambda|}+\frac{1}{1-\nu}\frac{\lambda_1^2}{|\lambda|}\right)$. The following precise relation between K_ν and Γ_ν can be easily established by the Poisson summation formula.

Proposition 1 *There exists a constant $C > 0$ such that*

$$|\Gamma_\nu(t) - K_\nu(t)| \leq C$$

on $\{t \in \mathbf{R}^2 \,:\, |t_i| \leq 3/4\}$. Moreover K_ν is positive.

Remark 2 By Proposition 1 using the homogeneity of Γ_ν we deduce that for every $\delta > 0$

$$\lim_{\varepsilon\to 0} \varepsilon^3 K_\nu(\varepsilon t) = \Gamma_\nu(t)$$

uniformly on $\{t \in \mathbf{R}^2 \,:\, |t| \leq \delta\}$.

Remark 3 The results proved here and in [8] can be extended to more general kernels $\mathcal{K}(z)$ which are bounded from above and below by $|z|^{-3}$ and which satisfy the counterpart of Proposition 1 with a -3 homogeneous kernel Γ which is sufficiently smooth on the unit sphere.

Since $[\cdot]_{H^{\frac{1}{2}}}$ is a trace seminorm we can deduce a Poincaré type inequality for functions in $H^{\frac{1}{2}}(T^2)$ (see [8], Proposition 3).

Proposition 4 *There exists a constant C_0 such that for every $u \in H^{\frac{1}{2}}(T^2)$, with $u = 0$ on $E \subseteq T^2$, we have*

$$\int_{T^2} |u|^2 dx \le C_0 \left(1 + \frac{1}{\mathrm{Cap}(E \times \{0\})}\right) [u]^2_{H^{\frac{1}{2}}(T^2)}\,, \tag{13}$$

where $\mathrm{Cap}(E \times \{0\})$ *denote the harmonic capacity of* $E \times \{0\}$ *as a subset of* $\mathbf{R}^3$.

Remark 5 Let $Q = (-\frac{1}{2}, \frac{1}{2})^2$. Any $H^{\frac{1}{2}}(Q)$ function can be extended by reflection to a periodic function on the square Q_2 of size 2. Applying the above inequality we get that there exists a constant C_1 such that

$$\int_Q |u|^2 dx \le C_1 \left(1 + \frac{1}{\mathrm{Cap}(E \times \{0\})}\right) \int\!\!\int_{Q\times Q} \frac{|u(x) - u(y)|^2}{|x-y|^3} dx\, dy\,. \tag{14}$$

2.2 Subcritical density of obstacles and the dislocation capacity

In [8] we studied the Γ-convergence of the variational model described in the introduction for the dilute case, i.e., in the regime $\varepsilon N_\varepsilon \approx 1$ (we will see in Corollary 11 below that the results actually hold as long as $\varepsilon N_\varepsilon \ll |\log \varepsilon|$). We will always assume that the obstacles $B^i_{R\varepsilon}$ in (2) are uniformly distributed and well separated in the following sense. Let $\mathcal{I}_\varepsilon$ be the index set which parametrizes the pinning centres x^i_ε. For every subset E of $(-\frac{1}{2}, \frac{1}{2})^2$ we denote by $\mathcal{I}_\varepsilon(E) := \{i \in \mathcal{I}_\varepsilon \,:\, x^i_\varepsilon \in E\}$ the indices corresponding to the pinning centres in E and we require the following.

- (*Uniform distribution*) There exists a constant $L > 0$ such that

$$|\#(\mathcal{I}_\varepsilon(Q')) - N_\varepsilon |Q'|| \le L \tag{15}$$

  for every open square $Q' \subset (-\frac{1}{2}, \frac{1}{2})^2$;

- (*Separation*) there exist $\beta \in (0,1)$ and $\varepsilon_0 > 0$ such that

$$\mathrm{dist}(x^i_\varepsilon, x^j_\varepsilon) > 6\varepsilon^\beta \tag{16}$$

  for every $i, j \in \mathcal{I}_\varepsilon$, $i \ne j$, and for every $\varepsilon \in (0, \varepsilon_0)$.

In [8] we proved that there exists a function $D_\nu(a, B_R)$, which is defined for integers a, such that the following result holds.

Theorem 6 *Assume $\varepsilon N_\varepsilon \to \Lambda$ and that the discs $B^i_{R\varepsilon}$ are uniformly distributed and well separated. Then the functional $\mathcal{F}_\varepsilon(u) := E_\varepsilon(u)/N_\varepsilon\varepsilon$, i.e.,*

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \dfrac{1}{N_\varepsilon \varepsilon^2} \displaystyle\int_{T^2} \mathrm{dist}^2(u, \mathbf{Z})\, dx + \frac{1}{N_\varepsilon \varepsilon} \int\!\!\int_{T^2\times T^2} K_\nu(x-y)|u(x) - u(y)|^2\, dx\, dy & \text{if } u \in H^{\frac{1}{2}}(T^2)\,, \\ & u = 0 \text{ on } \bigcup_i B^i_{R\varepsilon} \\ +\infty & \text{otherwise}\,, \end{cases}$$

Γ*-converges, with respect to the strong L^2 topology, to the functional*

$$\mathcal{F}(u) = \begin{cases} D_\nu(u, B_R) & \text{if } u = \mathrm{const.} \in \mathbf{Z}\,, \\ +\infty & \text{otherwise}\,. \end{cases} \tag{17}$$

The limit $D_\nu(u, B_R)$ can be characterized by means of the following cell-problem formula

$$D_\nu(a, B_R) := \inf\left\{ \int_{\mathbf{R}^2} \mathrm{dist}^2(\zeta, \mathbf{Z})\, dx + \int\!\!\int_{\mathbf{R}^2\times\mathbf{R}^2} \Gamma_\nu(x-y)|\zeta(x) - \zeta(y)|^2\, dx\, dy : \right. \tag{18}$$

$$\left. \zeta = a \text{ on } B_R\,,\ \zeta \in L^4(\mathbf{R}^2) \right\}.$$

Here the condition $\zeta \in L^4(\mathbf{R}^2)$ ensures that ζ decays to zero at infinity at least in an average sense. The choice of L^4 is natural in view of the critical embedding $H^{1/2} \hookrightarrow L^4$. More generally for every integer a and for every open bounded set Ω we can define a set function that we call the $H^{\frac{1}{2}}$*-dislocation capacity of an open set* E *with respect to* Ω *at the integer level* $a \in \mathbf{Z}$, as follows

$$D_\nu(a, E, \Omega) := \inf\left\{ \int_{\mathbf{R}^2} \mathrm{dist}^2(\zeta, \mathbf{Z})\, dx + \iint_{\mathbf{R}^2\times\mathbf{R}^2} \Gamma_\nu(x-y)|\zeta(x) - \zeta(y)|^2\, dx\, dy : \right. \qquad (19)$$

$$\left. \zeta = a \text{ on } E\,,\ \zeta = 0 \text{ on } \mathbf{R}^2 \setminus \Omega \right\},$$

where E is an open subset of Ω. In (18) and (19) the infimum is attained and the minimum is called the $H^{\frac{1}{2}}$*-dislocation capacitary potential of* E.

In [8] we proved that $D_\nu(a, \cdot)$ (and $D_\nu(a, \cdot, \Omega)$) is actually a Choquet capacity and moreover it is quadratic in a, as a goes to infinity. More precisely, for every bounded open set E there exist two constants, C_1 and C_2, such that

$$C_1 a^2 \le D_\nu(a, E) \le C_1(a^2 + 2a^{3/2}) + C_2 a, \quad \forall a \in \mathbf{Z} \qquad (20)$$

(see [8], Proposition 8). Another fact which is crucial in using $D_\nu(a, B_R)$ as a cell-problem formula in the study of the asymptotic behaviour of $\mathcal{F}_\varepsilon$ is the following convergence property

$$\lim_{T\to\infty} D_\nu(a, E, B_T) = D_\nu(a, E)\,. \qquad (21)$$

2.3 The $H^{1/2}$ regularisation and logarithmic rescaling

Alberti, Bouchitté and Seppecher studied the functional E_ε (without the pinning condition) for a two-well potential W and the $H^{1/2}$ nonlocal energy.

Theorem 7 ([3]) *The functional*

$$J_\varepsilon(u) = \begin{cases} \dfrac{1}{|\log\varepsilon|\varepsilon} \displaystyle\int_Q \mathrm{dist}^2(u, \{0,1\})\, dx + \dfrac{1}{|\log\varepsilon|} \iint_{Q\times Q} \dfrac{|u(x)-u(y)|^2}{|x-y|^3}\, dx\, dy & \text{if } u \in H^{\frac{1}{2}}(Q)\,, \\ +\infty & \text{otherwise} \end{cases} \qquad (22)$$

Γ*-converges (with respect to the strong topology in* L^1*) to the functional*

$$J(u) = \begin{cases} 4Per_Q(\{u=0\}) & \text{if } u \in BV(Q, \{0,1\})\,, \\ +\infty & \text{otherwise}\,, \end{cases} \qquad (23)$$

where $Per_Q(\{u=0\})$ *denotes the perimeter of the set* $\{u=0\}$ *relative to the set* Q.

Remark 8 We have stated the result for the special two-well energy $\mathrm{dist}^2(u, \{0,1\})$ to emphasize the similarities and differences with the choice (3). The result in [3] is proved for a general local energy W with $W(0) = W(1) = 0$, $W > 0$ otherwise and $W(u) \ge c|u|^2 - C$. Interestingly, and in contrast to the situation with regularizing energy of Dirichlet type [15, 16, 14], the Γ-limit does not depend on the shape of W, but just on the position of its zeros. This is an effect of the logarithmic rescaling. In fact in the proof of the upper bound in the verification of the Γ-limit one has a large amount of freedom. The precise choice of the transition profile by which one approximates a jump does not matter, as long as the length scale of the transition is chosen correctly. We will also exploit this fact in Section 6.

Comparing the above result with Theorem 6 it is natural to conjecture that the critical scaling regime for the number of obstacles N_ε which leads to an interaction of the line energy in the result above and the pinning energy discussed earlier is given by $N_\varepsilon \approx \varepsilon^{-1}|\log\varepsilon|$. This is what we will establish in the next section. As a byproduct we also obtain the limiting energy functionals in the subcritical regime $N_\varepsilon \ll \varepsilon^{-1}|\log\varepsilon|$ and in the supercritical regime $N_\varepsilon \gg \varepsilon^{-1}|\log\varepsilon|$.

3 Main results

We establish compactness and Γ-convergence results for the energy functionals $E_\varepsilon(u)$, both with and without the pinning condition. We first describe the behaviour of the functionals $I_\varepsilon(u) = E_\varepsilon(u)/|\log\varepsilon|$, thus extending the results of [3] to energies with infinitely many wells (and to an anisotropic kernel).

Theorem 9 *The functional*

$$I_\varepsilon(u) = \begin{cases} \dfrac{1}{|\log\varepsilon|\varepsilon}\displaystyle\int_{T^2} \mathrm{dist}^2(u,\mathbf{Z})\,dx + \dfrac{1}{|\log\varepsilon|}\displaystyle\iint_{T^2\times T^2} K_\nu(x-y)|u(x)-u(y)|^2\,dx\,dy \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } u\in H^{\frac12}(T^2)\,, \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise}\,, \end{cases} \tag{24}$$

Γ-converges (with respect to the strong topology of $L^1(T^2)$) to the functional

$$I(u) = \begin{cases} \displaystyle\int_{S_u} \gamma(n)|[u]|d\mathcal{H}^1 & \text{if } u\in BV(T^2,\mathbf{Z})\,, \\ +\infty & \text{otherwise}\,, \end{cases} \tag{25}$$

where n denotes the normal on the jump set S_u of u, where $[u]$ denotes the jump of u across S_u in direction n and where the anisotropic line energy density *$\gamma(n)$ is defined, for any $n\in S^1$, by*

$$\gamma(n) := 2\int_{x\cdot n=1} \Gamma_\nu(x)d\mathcal{H}^1 = 2\lim_{\delta\to0}\delta^2\int_{x\cdot n=\delta} K_\nu(x)d\mathcal{H}^1, \tag{26}$$

with Γ_ν given by (12). More precisely

i) Every sequence $\{u_\varepsilon\}$ such that $\sup_\varepsilon I_\varepsilon(u_\varepsilon) < \infty$ is bounded in $L^2(T^2)$, up to a translation, and is pre-compact in $L^q(T^2)$ for every $q<2$. Every cluster point u of the translated sequence belongs to $BV(T^2,\mathbf{Z})$ and satisfies

$$\int_{T^2} |Du| \le C\liminf_{\varepsilon\to0} I_\varepsilon(u_\varepsilon)\,.$$

ii) Every sequence $\{u_\varepsilon\}$ strongly converging in $L^1(T^2)$ to some function u satisfies

$$\liminf_{\varepsilon\to0} I_\varepsilon(u_\varepsilon) \ge I(u)\,.$$

iii) For every $u\in BV(T^2,\mathbf{Z})$ there exists a sequence $\{u_\varepsilon\}$ which converges strongly to u in $L^2(T^2)$ such that

$$\lim_{\varepsilon\to0} I_\varepsilon(u_\varepsilon) = I(u)\,.$$

The proof of the result above will be obtained as a consequence of a compactness theorem (Theorem 12 in Section 4) a lower bound given in Section 5 (Theorem 17), and an upper bound proved in Section 6 (Theorem 24).

The asymptotic analysis for the functional with the pinning condition in the critical scaling $N_\varepsilon \approx \varepsilon^{-1}|\log\varepsilon|$ is summarized in the following theorem.

Theorem 10 *Assume $N_\varepsilon\varepsilon/|\log\varepsilon| \to \Lambda$, $0<\Lambda<\infty$, and that the discs $B^i_{R\varepsilon}$ are uniformly distributed and well separated. Then the functional*

$$F_\varepsilon(u) = \begin{cases} \dfrac{1}{|\log\varepsilon|\varepsilon}\displaystyle\int_{T^2} \mathrm{dist}^2(u,\mathbf{Z})\,dx + \dfrac{1}{|\log\varepsilon|}\displaystyle\iint_{T^2\times T^2} K_\nu(x-y)|u(x)-u(y)|^2\,dx\,dy \\ \qquad\qquad\qquad\qquad \text{if } u\in H^{\frac12}(T^2)\,, \text{ and } u=0 \text{ on } \bigcup_i B^i_{R\varepsilon}, \\ +\infty \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise}\,, \end{cases} \tag{27}$$

Γ-converges (with respect to the strong topology of $L^1(T^2)$) to the functional

$$F(u) = \begin{cases} \displaystyle\int_{S_u} \gamma(n)|[u]|d\mathcal{H}^1 + \Lambda \int_{T^2} D_\nu(u, B_R)\, dx & \text{if } u \in BV(T^2, \mathbf{Z})\,, \\ +\infty & \text{otherwise}\,. \end{cases} \tag{28}$$

More precisely

i) *Every sequence $\{u_\varepsilon\}$ such that $\sup_\varepsilon F_\varepsilon(u_\varepsilon) < \infty$ is bounded in $L^2(T^2)$ and is pre-compact in $L^q(T^2)$, for every $q < 2$. Every cluster point u belongs to $BV(T^2, \mathbf{Z})$ and satisfies*

$$\int_{T^2} |Du| \le C \liminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon)\,.$$

ii) *Every sequence $\{u_\varepsilon\}$ strongly converging in $L^1(T^2)$ to some function u satisfies*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \ge F(u)\,.$$

iii) *For every $u \in BV(T^2, \mathbf{Z})$ there exists a sequence $\{u_\varepsilon\}$ which converges strongly to u in L^2 such that*

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = F(u)\,.$$

Theorem 10 follows from the compactness result obtained in Section 4 (Proposition 16) and the lower and the upper bounds obtained in Section 5 (Theorem 21) and Section 6 (Theorem 27), respectively.

Finally we extend the result to the subcritical and supercritical scaling, which correspond to $\Lambda = 0$ and $\Lambda = \infty$, respectively. If we consider the rescaling $E_\varepsilon/(N_\varepsilon\varepsilon)$, then formally the limit is given by dividing (28) by Λ. Thus for $\Lambda = 0$ we expect the line energy to dominate, leading to the constraint $u = \text{const}$, while for $\Lambda = \infty$ the line energy becomes negligible so we can no longer expect compactness in L^1 and the limiting energy density functional has to be replaced by its relaxation, given by the convex hull $D^{**}_\nu(u, B_R)$. More precisely

$$\begin{aligned} &u \mapsto D^{**}_\nu(u, B_R) \quad \text{is the convex hull of the function } \widetilde{D}, \text{ where} \\ &\widetilde{D}(u) = D_\nu(u, B_R), \quad \text{if } u \in \mathbf{Z}, \qquad \widetilde{D}(u) = +\infty \quad \text{otherwise.} \end{aligned} \tag{29}$$

Corollary 11 *Assume that the discs $B^i_{R\varepsilon}$ are uniformly distributed and well separated. Consider the functional*

$$\mathcal{F}_\varepsilon(u) = \begin{cases} \displaystyle\frac{1}{\varepsilon^2 N_\varepsilon} \int_{T^2} \text{dist}^2(u, \mathbf{Z})\, dx + \frac{1}{\varepsilon N_\varepsilon} \iint_{T^2\times T^2} K_\nu(x-y)|u(x)-u(y)|^2\, dx\, dy \\ \qquad\qquad \text{if } \ u \in H^{\frac12}(T^2)\,, \text{and } u = 0 \text{ on } \bigcup_i B^i_{R\varepsilon}, \\ +\infty \qquad\qquad \text{otherwise.} \end{cases} \tag{30}$$

(i) *If $N^\varepsilon \to \infty$ and $\varepsilon N_\varepsilon/|\log\varepsilon| \to 0$ then $\mathcal{F}_\varepsilon$ Γ-converges (with respect to the strong topology in $L^2(T^2)$) to the functional*

$$\mathcal{F}(u) = \begin{cases} D_\nu(a, B_R)\, dx & \text{if } u \equiv a,\ a \in \mathbf{Z}, \\ +\infty & \text{otherwise}\,. \end{cases} \tag{31}$$

(ii) *If $\varepsilon N_\varepsilon/|\log\varepsilon| \to \infty$ then $\mathcal{F}_\varepsilon$ Γ-converges (with respect to the weak topology in $L^2(T^2)$) to the functional*

$$\mathcal{F}(u) = \int_{T^2} D^{**}_\nu(u, B_R)\, dx. \tag{32}$$

The proof of Corollary 11 follows closely that of Theorems 9 and 10. In Section 7 we sketch the necessary modifications in the argument. The main difficulty is that in the supercritical case one no longer has L^1 compactness. Nonetheless we will show that on a scale which is large compared to the typical particle separation $1/\sqrt{N_\varepsilon}$ the L^2 oscillation of a finite energy sequence u_ε can still be controlled. This will allow us to locally estimate the energy from below essentially in the same way as in the presence of strong convergence.

4 Compactness

In this section we establish the following compactness result.

Theorem 12 *Let* $\{u_\varepsilon\}$ *be such that*

$$\liminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) < +\infty\,,$$

then there exists a sequence of integers $\{a_\varepsilon\}$ *such that the sequence* $\{u_\varepsilon - a_\varepsilon\}$ *is bounded in* L^2 *and relatively compact in* $L^q(Q)$*, for every* $q < 2$*. Every cluster point* u *of* $\{u_\varepsilon - a_\varepsilon\}$ *belongs to* $BV(T^2, \mathbf{Z})$ *and satisfies*

$$\int_Q |Du| \le C \liminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon)\,. \tag{33}$$

The main difference with the result of Alberti, Bouchitté and Seppecher [3] is that the local part of the energy $\mathrm{dist}^2(u, \mathbf{Z})$ is not coercive. Hence the crucial point of the proof consists in the derivation of a uniform L^2 bound on u_ε (up to a translation). Pointwise convergence can then be deduced from [3] by a truncation argument.

4.1 The L^2-bound

Theorem 13 *Let* $Q = (-\frac{1}{2}, \frac{1}{2})^2$*. Given* $M > 0$ *and* $\varepsilon_0 > 0$ *there exists a constant* $C(M, \varepsilon_0)$ *such that the following holds. Suppose that* $0 < \varepsilon < \varepsilon_0$*,*

$$I_\varepsilon(u_\varepsilon) \le M, \tag{34}$$

and that $|\{u_\varepsilon > 0\}| > \frac{1}{2}|Q|$ *and* $|\{u_\varepsilon < 1\}| > \frac{1}{2}|Q|$*. Then*

$$\int_Q |u_\varepsilon|^2\, dx \le C(M, \varepsilon_0).$$

Remarks

1. The bound is optimal, since the embedding of BV into L^2 is optimal. Indeed if $I_\varepsilon(u_\varepsilon) \le M$ would imply a uniform L^p_{loc} bound for $p > 2$ then, by Theorem 9, every function in $BV(Q, \mathbf{Z})$ would be $L^p_{\mathrm{loc}}(Q)$, which is false.

2. If one considers the analogue of the functional I_ε in n dimensions ($n \ge 2$), with kernel $K(z) \approx |z|^{-(n+1)}$ the argument below gives an $L^{\frac{n}{n-1}}$ bound, which is again optimal. For $n = 1$ Kurzke [11] showed that the optimal bound corresponds to the Orliz space e^L.

3. Kurzke [12] recently found a different proof of Theorem 12 based on slicing and his one-dimensional result [11].

4. The proof does not use the periodicity of K_ν, the lower bound $K_\nu(x-y) \ge c|x-y|^{-3}$ is sufficient.

The idea of the proof of Theorem 12 is simple. From Theorem 9 we expect that $I_\varepsilon(u_\varepsilon)$ behaves asymptotically similar to the BV seminorm. Comparing the super- level sets $\{u_\varepsilon > k\}$ and $\{u_\varepsilon > k-1\}$ we thus expect that the perimeter of $\{u_\varepsilon > k\}$ is controlled by $I_\varepsilon(T_{k-1}u_\varepsilon)$, where $T_{k-1}u_\varepsilon$ is given by $u_\varepsilon \vee (k-1) \wedge k$, i.e., the truncation at levels $k-1$ and k. In combination with the isoperimetric inequality we would get $|\{u_\varepsilon > k\}|^{\frac{1}{2}} \le CI_\varepsilon(T_{k-1}u_\varepsilon)$, and from this the assertion would already follow (see (53) and (54) below).

Unfortunately I_ε does not really control the perimeter of the level sets uniformly in ε. Instead we will establish directly a bound for the level sets $A_k^\varepsilon := \{u_\varepsilon > k+1-\sigma\}$ of the form

$$|A_k^\varepsilon|^{\frac{1}{2}} \le CI_\varepsilon(T_k u_\varepsilon)\,, \tag{35}$$

for a fixed $\sigma \in (0, \frac{1}{4})$. To prove this we first replace the singular kernel by a regular kernel and drop the logarithmic rescaling. Then we bound the singular kernel $\frac{1}{|t|^3}$ from below by a dyadic sum of scaled regular kernels and with a covering argument we get an estimate like (35) at least if $|A_k^\varepsilon|$ is not too small, see (50) below.

If $|A_k^\varepsilon|$ is very small then a scaling argument shows that the term $\mathrm{dist}^2(u_\varepsilon, \mathbf{Z})$ becomes negligible and we directly use the embedding of $H^{\frac{1}{2}}$ in L^4 and conclude after a short calculation.

We begin with the relevant estimates for the regular kernels.

Proposition 14 *Fix $\varphi \in C_c^\infty(B_1(0))$, with $\int \varphi(x)dx = 1$ and $\varphi > 0$ in $B_{\frac{1}{2}}(0)$. For every $\lambda \in (0,1)$ there exists a constant $c(\lambda)$ such that for every subset A of Q with $\lambda \le |A| \le 1-\lambda$ we have*

$$\gamma_\delta(A,Q) := \frac{1}{\delta}\int_A\int_{Q\setminus A} \varphi_\delta(x-x')\,dx\,dx' \ge c(\lambda) \qquad \forall\ \delta \in (0,1)\ , \tag{36}$$

where $\varphi_\delta(x) = \frac{1}{\delta^2}\varphi\left(\frac{x}{\delta}\right)$.

Proof. Fix λ. Assume by contradiction that there exists a sequence of sets $\{A_k\}$ such that

$$\lambda \le |A_k| \le 1-\lambda \tag{37}$$

and a sequence $\{\delta_k\}$, such that

$$\lim_{k\to\infty} \gamma_{\delta_k}(A_k, Q) = 0\,. \tag{38}$$

We may assume that δ_k converges to some δ and the characteristic function $\chi_{A_k}(x)$ of A_k converges to $\theta(x)$ weak* in L^∞ as $k \to \infty$. If $\delta \ne 0$, then from (38) we get

$$\int\!\!\int_{Q\times Q} \theta(x)(1-\theta(x'))\varphi_\delta(x-x')\,dx\,dx' = 0\,.$$

This implies either $\theta = 0$ a.e. in Q or $\theta = 1$ a.e. in Q, which is impossible, since (37) implies $\lambda \le \int_Q \theta(x)\,dx \le 1-\lambda$. If $\delta = 0$, we can rewrite $\gamma_{\delta_k}(A_k)$ as follows

$$\frac{1}{\delta_k}\int_{A_k}\int_{Q\setminus A_k} \varphi_{\delta_k}(x-x')\,dx\,dx' = \frac{1}{2\delta_k}\int_Q\int_Q \varphi_{\delta_k}(x-x')|\chi_{A_k}(x) - \chi_{A_k}(x')|^2\,dx\,dx'$$
$$= \frac{1}{2\delta_k}\int_Q\int_Q \varphi_{\delta_k}(x-x')|\chi_{A_k}(x) - \chi_{A_k}(x')|^2\,dx\,dx' + \frac{1}{2\delta_k}\int_Q \mathrm{dist}^2(\chi_{A_k}, \{0,1\})dx =: \frac{1}{2}\Phi_{\delta_k}(\chi_{A_k})$$

The asymptotics of the functional Φ_δ has been studied by Alberti and Bellettini in [1]. By their compactness result we get that there exists a set A with finite perimeter such that, up to a subsequence, the sequence χ_{A_k} strongly converges to χ_A in L^1. Moreover applying [1], Theorem 1.4, we obtain that there exists a positive constant C such that

$$\liminf_{k\to\infty} \gamma_{\delta_k}(A_k, Q) \ge C\mathrm{Per}_Q(A)\,.$$

By (38) we get $\mathrm{Per}_Q(A) = 0$, but this again contradicts (37). ○

Lemma 15 *Let $Q = (-\frac{1}{2}, \frac{1}{2})^2$. There exist two constants C_1 and C_2 such that for every $A \subseteq Q$ and $B \subseteq Q$ with $A \cap B = \emptyset$ and $|A| \leq \frac{1}{2}$ the following inequality holds (with $E := Q \setminus (A \cup B)$)*

$$\int_A \int_B \frac{1}{|x-y|^3}\, dx\, dy \geq C_1 |A|^{\frac{1}{2}} \log \frac{C_2|A|}{|E|}\,. \tag{39}$$

Proof. Let Q' denote the concentric subsquare $(-\frac{1}{6}, \frac{1}{6})^2$. We first prove the interior estimate

$$\int_A \int_B \frac{1}{|x-y|^3}\, dx\, dy \geq \widetilde{C}_1 |A \cap Q'|^{\frac{1}{2}} \log \frac{\widetilde{C}_2 |A \cap Q'|}{|E|} \tag{40}$$

and then derive the global estimate (39) by a reflection and extension argument.

We first establish the counterpart of (36) for the singular kernel $\frac{1}{|t|^3}$, i.e., we prove that if $A \subseteq Q$ satisfies

$$\lambda \leq |A| \leq 1 - \lambda\,,$$

then there exists a constants $c(\lambda)$ and $C(\lambda)$ such that

$$\int_A \int_B \frac{1}{|x-y|^3}\, dx\, dy \geq C(\lambda) \log \frac{c(\lambda)}{2|E|}\,, \tag{41}$$

$c(\lambda)$ being the constant given by Proposition 14. Clearly we may assume that $|E| = |Q \setminus (A \cup B)| < \frac{c(\lambda)}{2}$. Since $A \cup B \cup E = Q$, by Proposition 14 it is easy to see that

$$\frac{1}{\delta} \int_A \int_B \varphi_\delta(x-y)\, dx\, dy \geq c(\lambda) - \frac{|E|}{\delta} \geq \frac{c(\lambda)}{2} \qquad \text{as long as } \frac{2|E|}{c(\lambda)} < \delta < 1\,. \tag{42}$$

We will get (41) by estimating the singular kernel with a dyadic sum of regular kernels. Take $m \in I\!N$ such that $2^{-m} \leq \frac{2|E|}{c(\lambda)} \leq 2^{-m+1}$, hence $m \geq \log(c(\lambda)/2|E|)/\log 2$. Then take

$$\delta_i = 2^{-i}, \quad \text{for } i = 0, ..., m\,.$$

By the definition of φ_δ and taking into account that $\varphi_\delta(x-y) = 0$ if $|x-y| > \delta$, we have

$$\frac{1}{|x-y|^3} \geq \frac{1}{\delta \sup \varphi} \varphi_\delta(x-y) \qquad \forall\, x, y \in Q\,.$$

More generally we claim that there exists a constant C, depending on φ, such that

$$\frac{1}{|x-y|^3} \geq C \sum_{i=0}^{m} \frac{1}{\delta_i} \varphi_{\delta_i}(x-y) \tag{43}$$

for all x, $y \in Q$. Suppose first that $|x-y| < \delta_m$. Then

$$\sum_{i=0}^{m} \frac{1}{\delta_i} \varphi_{\delta_i}(x-y) \leq \sup \varphi \sum_{i=0}^{m} \frac{1}{\delta_i^3} \leq \frac{8}{7} \sup \varphi \frac{1}{\delta_m^3} \leq C|x-y|^{-3}\,.$$

Now suppose $\delta_{i_0+1} \leq |x-y| < \delta_{i_0}$ for some $i_0 \in 0, \ldots, m-1$. Then

$$\sum_{i=0}^{m} \frac{1}{\delta_i} \varphi_{\delta_i}(x-y) \leq \sup \varphi \sum_{i=0}^{i_0+1} \frac{1}{\delta_i^3} \leq \frac{64}{7} (\sup \varphi) \frac{1}{\delta_{i_0}^3} \leq C|x-y|^{-3}$$

and this finishes the proof of (43). Thus (41) follows from (43) and (42).

If $|A| > \frac{1}{18}$ then (41) immediately gives (39). We may thus assume $|A| \leq \frac{1}{18}$ and we prove (40) by a covering argument. By scaling (41) we get that for every $r > 0$ and for every $A \subseteq Q$ such that

$$\lambda \leq \frac{|A \cap Q_r|}{|Q_r|} \leq 1 - \lambda\,,$$

we have

$$\int_{A\cap Q_r}\int_{B\cap Q_r}\frac{1}{|x-y|^3}\,dx\,dy \geq C(\lambda)r\log\frac{c(\lambda)r^2}{2|E\cap Q_r|}\,, \tag{44}$$

where Q_r denotes a square of size r.

We claim that for almost every $x\in A\cap Q'$ there exists a square $Q_{r_x}(x)$ centered in x and of size r_x such that

$$\frac{1}{4}\leq\frac{|A\cap Q_{r_x}(x)|}{|Q_{r_x}(x)|}\leq\frac{3}{4}\,. \tag{45}$$

Indeed on the one hand a.e. $x\in A\cap Q'$ has density 1 and on the other hand

$$\frac{|A\cap Q_{\frac{1}{3}}(x)|}{|Q_{\frac{1}{3}}(x)|}\leq 9|A|\leq\frac{1}{2}.$$

By the continuity of $\frac{|A\cap Q_r(x)|}{|Q_r(x)|}$ with respect to r we get (45). By the Besicovitch Covering Theorem we obtain a family of disjoint squares $Q_{r_i}(x_i)$, $i\in I$, satisfying (45) such that

$$\sum_{i\in I}r_i^2=\sum_{i\in I}|Q_{r_i}(x_i)|\geq\widetilde{c}|A\cap Q'|\,,$$

where $\widetilde{c}$ is a universal constant. We consider a subfamily J corresponding to squares where $|E|$ has sufficiently low density,

$$J:=\left\{i\in I:\frac{|E\cap Q_{r_i}(x_i)|}{r_i^2}<\frac{2}{\widetilde{c}}\frac{|E|}{|A\cap Q'|}\right\}\,. \tag{46}$$

We have

$$\sum_{i\in J}r_i^2=\sum_{i\in I}r_i^2-\sum_{i\notin J}r_i^2\geq\widetilde{c}|A\cap Q'|-\frac{\widetilde{c}}{2}\sum_{i\notin J}\frac{|E\cap Q_{r_i}(x_i)|}{|E|}|A\cap Q'|\geq\frac{\widetilde{c}}{2}|A\cap Q'|\,. \tag{47}$$

By (44) and (46) we get

$$\begin{aligned}\int_A\int_B\frac{1}{|x-y|^3}\,dx\,dy &\geq \sum_{i\in J}\int_{A\cap Q_{r_i}(x_i)}\int_{B\cap Q_{r_i}(x_i)}\frac{1}{|x-y|^3}\,dx\,dy\\ &\geq \sum_{i\in J}Cr_i\log\frac{cr_i^2}{2|E\cap Q_{r_i}(x_i)|}\geq C\left(\sum_{i\in J}r_i^2\right)^{\frac{1}{2}}\log\frac{c\widetilde{c}|A\cap Q'|}{4|E|}\,.\end{aligned}$$

Together with (47) this yields the interior estimate (40).

To establish the full estimate (39) we extend A to $(-\frac{3}{2},\frac{3}{2})^2$ by reflection along the lines $x_1=\pm\frac{1}{2}$ and $x_2=\pm\frac{1}{2}$. Consider the maps $\pi_{-1}:(-\frac{1}{2},\frac{1}{2})\to(-\frac{3}{2},-\frac{1}{2})$ and $\pi_1:(-\frac{1}{2},\frac{1}{2})\to(\frac{1}{2},\frac{3}{2})$ given by $\pi_{-1}(s):=-1-s$ and $\pi_1(s):=1-s$, respectively, and set $\pi_0(s):=s$, $\pi_{ij}(x_1,x_2):=(\pi_i(x_1),\pi_j(x_2))$ and

$$\widetilde{A}=\bigcup_{i,j=-1}^{1}\pi_{ij}(A),\quad \widetilde{B}=\bigcup_{i,j=-1}^{1}\pi_{ij}(B),\quad \widetilde{E}=\bigcup_{i,j=-1}^{1}\pi_{ij}(E)$$

and $\widetilde{Q}=(-\frac{3}{2},\frac{3}{2})^2$. Note that $|\pi_i(s)-\pi_j(t)|\geq|t-s|$ for all $i,j\in\{-1,0,1\}$ and all $s,t\in(-\frac{1}{2},\frac{1}{2})$. Thus

$$\begin{aligned}&\int_{\widetilde{A}}\int_{\widetilde{B}}\frac{1}{|x-y|^3}\,dx\,dy\\ =\;&\sum_{i,j,k,l=-1}^{1}\int_A\int_B\frac{1}{|\pi_{ij}(x)-\pi_{kl}(y)|^3}\,dx\,dy\\ \leq\;& 81\int_A\int_B\frac{1}{|x-y|^3}\,dx\,dy.\end{aligned} \tag{48}$$

On the other hand the interior estimate (40) yields, after the rescaling $(x,y)\mapsto(3x,3y)$

$$\int_{\widetilde{A}}\int_{\widetilde{B}}\frac{1}{|x-y|^3}\,dx\,dy \geq \widetilde{C}_1|\widetilde{A}\cap Q|^{1/2}\ln\frac{\widetilde{C}_2|\widetilde{A}\cap Q|}{|\widetilde{E}|} = \widetilde{C}_1|A|^{1/2}\ln\frac{\widetilde{C}_2|A|}{9|E|}. \tag{49}$$

Together with (48) this implies (39), with $C_1=\frac{1}{81}\widetilde{C}_1$ and $C_2=\frac{1}{9}\widetilde{C}_2$. ○

Proof of Theorem 13. Fix $\sigma\in(0,\frac14)$ and $k\in\mathbf{Z}$. Denote by A_k^ε and B_k^ε the following two super and sub-level sets

$$A_k^\varepsilon=\{x\in Q\,:\,u_\varepsilon>k+1-\sigma\}\quad\text{and}\quad B_k^\varepsilon=\{x\in Q\,:\,u_\varepsilon<k+\sigma\}\,.$$

We will give the proof in several steps. Assume first that $u_\varepsilon>0$. The idea is to get an estimate for the super-level sets of the following type

$$\sum_{k=0}^{\infty}|A_k^\varepsilon|^{\frac12}\leq C\,,$$

in order to deduce the L^2 estimate.

Step 1. Let $\alpha<\frac13$. Then there exists a constant C depending on α, M and ε_0 such that for all $k\geq1$

$$\int_{A_k^\varepsilon}\int_{B_k^\varepsilon}K_\nu(x-y)\,dx\,dy\geq C|\log\varepsilon||A_k^\varepsilon|^{\frac12}\,, \tag{50}$$

whenever $|A_k^\varepsilon|^{\frac12}>\varepsilon^\alpha$.

To see this denote by E_k^ε the set $\{x\in Q\,:\,\ k+\sigma\leq u_\varepsilon\leq k+1-\sigma\}$. By (34) we deduce that $|E_k^\varepsilon|\leq\frac{M}{\sigma^2}\varepsilon|\log\varepsilon|$. Thus we may assume $|E_k^\varepsilon|\leq\varepsilon^{3\alpha}$. By Lemma 15 and the fact that $K_\nu(t)\geq C/|t|^3$, we get

$$\int_{A_k^\varepsilon}\int_{B_k^\varepsilon}K_\nu(x-y)\,dx\,dy\geq C_1|A_k^\varepsilon|^{\frac12}\log\frac{C_2\varepsilon^{2\alpha}}{\varepsilon^{3\alpha}}$$

which gives (50).

Step 2. Let α be as in Step 1 and assume that $|A_k^\varepsilon|^{\frac12}\leq\varepsilon^\alpha$. Let $\gamma\in(1,2)$. Then there exists a positive constant C (depending on α, γ, M and ε_0) such that

$$|A_k^\varepsilon|^{\frac12}\leq\frac{C}{(k-\sigma)^\gamma}\,. \tag{51}$$

Indeed by the Sobolev inequality and the fact that $|\{u_\varepsilon<1\}|\geq\frac12|Q|$, we have

$$\|(u_\varepsilon-1)_+\|_{L^4(Q)}\leq C[(u_\varepsilon-1)_+]_{H^{\frac12}}\leq C(|\log\varepsilon|I_\varepsilon(u_\varepsilon))^{\frac12}\,.$$

Thus by Hölder's inequality we get

$$\begin{aligned}|A_k^\varepsilon|(k-\sigma) &\leq \int_{A_k^\varepsilon}|(u_\varepsilon-1)_+|dx\\ &\leq |A_k^\varepsilon|^{\frac34}\|(u_\varepsilon-1)_+\|_{L^4(Q)}\leq C|A_k^\varepsilon|^{\frac34}(|\log\varepsilon|I_\varepsilon(u_\varepsilon))^{\frac12}.\end{aligned}$$

Hence

$$|A_k^\varepsilon|^{\frac14}\leq C\frac{1}{(k-\sigma)}(|\log\varepsilon|I_\varepsilon(u_\varepsilon))^{\frac12}\,.$$

Since $|A_k^\varepsilon|^{\frac12}\leq\varepsilon^\alpha$ and since $\varepsilon^{\alpha(\frac{1}{2\gamma}-\frac14)}|\log\varepsilon|^{1/2}$ remains bounded for $\varepsilon\leq\varepsilon_0$ this implies that

$$|A_k^\varepsilon|^{\frac{1}{2\gamma}}\leq C\frac{1}{(k-\sigma)}(I_\varepsilon(u_\varepsilon))^{\frac12}$$

and (51) follows by raising the last inequality to the power γ.

Step 3. There exists a positive constant C (depending on M and ε_0) such that

$$\sum_{k=0}^{\infty} |A_k^\varepsilon|^{\frac{1}{2}} \le C\,. \tag{52}$$

This is a consequence of Step 1 and Step 2. Fix $\alpha \in (0,1/3)$ and $\gamma \in (1,2)$. If $|A_k^\varepsilon|^{\frac{1}{2}} > \varepsilon^\alpha$, then we apply Step 1 and we get

$$|A_k^\varepsilon|^{\frac{1}{2}} \le C I_\varepsilon(T_k u_\varepsilon)\,,$$

where $T_k u_\varepsilon = (u_\varepsilon \vee k) \wedge (k+1)$. If $|A_k^\varepsilon|^{\frac{1}{2}} \le \varepsilon^\alpha$, then we apply Step 2. Thus we get

$$\sum_{k=1}^{\infty} |A_k^\varepsilon|^{\frac{1}{2}} \le C \sum_{k=1}^{\infty} \left(\frac{1}{(k-\sigma)^\gamma} + I_\varepsilon(T_k u_\varepsilon) \right) \le C(1 + I_\varepsilon(u_\varepsilon))$$

and this gives (52).

Step 4. We conclude the proof by noting that for any decreasing sequence a_k of positive numbers we have

$$\sum_{k=1}^{\infty} k a_k \le \sup_{k\ge 1}(k a_k^{\frac{1}{2}}) \sum_{k=1}^{\infty} a_k^{\frac{1}{2}} \le \left(\sum_{k=1}^{\infty} a_k^{\frac{1}{2}} \right)^2 . \tag{53}$$

We apply this inequality with $a_k = |A_{k-2}^\varepsilon|$ and use Step 3 to deduce that

$$\int_0^\infty t|\{u_\varepsilon > t\}|\,dt \le \sum_{k=1}^{\infty} k|\{u_\varepsilon > k-1\}| \le \sum_{k=1}^{\infty} k|A_{k-2}^\varepsilon| \le C\,. \tag{54}$$

Hence

$$\int_{\{u_\varepsilon > 0\}} |u_\varepsilon|^2 dx = \int_0^\infty 2t|\{u_\varepsilon > t\}|\,dt \le C\,.$$

The theorem follows by arguing in a similar way for the negative part of u_ε. ○

4.2 Compactness

Proof of Theorem 12. We may assume that $|\{u_\varepsilon \ge 0\}| \ge \frac{1}{2}|Q|$ and $|\{u_\varepsilon \le 1\}| \ge \frac{1}{2}|Q|$, since otherwise we may replace $\{u_\varepsilon\}$ with $\{u_\varepsilon - a_\varepsilon\}$, where

$$a_\varepsilon = \max\{a \in \mathbf{Z} :\ |\{u_\varepsilon - a > 0\}| \ge \frac{1}{2}|Q|\}\,. \tag{55}$$

Furthermore we may suppose that $I_\varepsilon(u_\varepsilon) \le C_1$ and by passage to a subsequence $\liminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) = \lim_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon)$.

Theorem 13 shows that u_ε is uniformly bounded in $L^2(T^2)$. Hence there exists $u \in L^2(T^2)$ such that, up to extraction of a subsequence,

$$u_\varepsilon \rightharpoonup u \qquad \text{in} \ \ L^2(T^2)\,. \tag{56}$$

To obtain strong convergence in L^1 we consider again the truncation operator $T_a u = (u \vee a) \wedge (a+1)$, for all $a \in \mathbf{Z}$. Clearly by (9)

$$\frac{1}{|\log \varepsilon|} \left([T_a u_\varepsilon]^2_{H^{\frac{1}{2}}(Q)} + \frac{1}{\varepsilon} \int_Q \mathrm{dist}^2(T_a u_\varepsilon, \{a, a+1\})\,dx \right) \le 2 I_\varepsilon(T_a u_\varepsilon) \le 2 I_\varepsilon(u_\varepsilon) \le C\,.$$

By [3], Theorem 4.7, (see also [2]) we have that, up to extraction of a subsequence, for every $a \in \mathbf{Z}$, there exists an L^1 function $u_a \in BV(Q, \{a, a+1\})$ such that

$$T_a u_\varepsilon \to u_a \qquad \text{in} \;\; L^1(T^2) \tag{57}$$

and

$$\int_{T^2} |Du_a| \le C \liminf_{\varepsilon\to 0} I_\varepsilon(T_a u_\varepsilon)\,. \tag{58}$$

Now consider $M \in I\!N$ and the truncation operator $T^M u = (u \vee -M) \wedge M$. By (57) there exists $u^M \in BV(T^2, \mathbf{Z})$ such that

$$T^M u_\varepsilon \to u^M \qquad \text{in} \;\; L^1(T^2)\,.$$

On the set $\{|u_\varepsilon| > M\}$ we have $|u_\varepsilon| \le |u_\varepsilon|^2/M$. Thus weak lower semi-continuity of the L^1 norm and the L^2 bound yield

$$\|u^M - u\|_{L^1(T^2)} \le \liminf_{\varepsilon\to 0} \|T^M u_\varepsilon - u_\varepsilon\|_{L^1(T^2)} \le \liminf_{\varepsilon\to 0} 2 \int_{\{|u_\varepsilon|>M\}} |u_\varepsilon|\, dx \le \frac{C}{M}\,.$$

Now $u_\varepsilon - u = u_\varepsilon - T^M u_\varepsilon + T^M u_\varepsilon - u^M + u^M - u$ and thus

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon - u\|_{L^1(T^2)} \le \frac{2C}{M}\,.$$

Since M was arbitrary this shows that $u_\varepsilon \to u$ in $L^1(T^2)$ (and hence in all $L^q(T^2)$, for $q < 2$) and thus $u_a = T_a u$. It is easy to verify that

$$\sum_{a\in\mathbf{Z}} \iint_{Q\times Q} K_\nu(x-y)|T_a u_\varepsilon(x) - T_a u_\varepsilon(y)|^2\, dx\, dy \le \iint_{Q\times Q} K_\nu(x-y)|u_\varepsilon(x) - u_\varepsilon(y)|^2\, dx\, dy\,.$$

Hence (58) yields (33), after summation over a.

○

4.3 The effect of the pinning

Theorem 12 gives compactness up to translation by integers of any sequence $\{u_\varepsilon\}$ which satisfies $\liminf_{\varepsilon\to 0} I_\varepsilon(u_\varepsilon) < \infty$. If in addition u_ε is also subject to the pinning condition, this eliminates the translation invariance of the problem and yields, via a Poincaré inequality, an L^2 bound and compactness of the sequence u_ε.

Proposition 16 *Let u_ε such that*

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) < +\infty\,,$$

then u_ε is bounded in $L^2(T^2)$ and relatively compact in $L^q(T^2)$, for every $q < 2$. Every cluster point u belongs to $BV(T^2, \mathbf{Z})$ and satisfies

$$\int_{T^2} |Du| \le C \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon)\,. \tag{59}$$

Proof. In view of (9), we may assume without loss of generality that

$$[u_\varepsilon]^2_{H^{\frac{1}{2}}(T^2)} \le C|\log\varepsilon|\,. \tag{60}$$

By Theorem 12 we know that for every ε there exists an integer number a_ε such that $\{u_\varepsilon - a_\varepsilon\}$ is bounded in $L^2(T^2)$, relatively compact in $L^q(T^2)$, for $q < 2$, and every cluster point satisfies (59). In order to conclude it is enough to have an L^2 bound for $\{u_\varepsilon\}$ which yields that the sequence $\{a_\varepsilon\}$ is bounded. This can be obtained through the Poincaré inequality. Fix $\rho_\varepsilon = \sqrt{\frac{(L+1)\varepsilon}{|\log\varepsilon|}}$ (L is the constant given by (15)). With a little abuse of notation we denote by $Q^j_{\rho_\varepsilon}$ the squares of a lattice on Q of size approximately ρ_ε. Applying the Poincaré inequality (14), scaled to the square $Q^j_{\rho_\varepsilon}$, we get

$$\int_{Q^j_{\rho_\varepsilon}} |u_\varepsilon|^2 dx \le C_1\rho_\varepsilon\left(1+\frac{\rho_\varepsilon}{\mathrm{Cap}((\{u_\varepsilon=0\}\cap Q^j_{\rho_\varepsilon})\times\{0\})}\right)[u_\varepsilon]^2_{H^{\frac12}(Q^j_{\rho_\varepsilon})}. \tag{61}$$

By our choice of ρ_ε and assumption (15) we have

$$1 \le \#(\mathcal{I}_\varepsilon(Q^j_{\rho_\varepsilon})) \le 2L+1$$

and thus $\mathrm{Cap}((\{u_\varepsilon=0\}\cap Q^j_{\rho_\varepsilon})\times\{0\}) \ge C'R\varepsilon$. Taking the sum over all j in (61) and using (60) we get

$$\int_Q |u_\varepsilon|^2 dx \le \sum_j C_1\rho_\varepsilon\left(1+\frac{\rho_\varepsilon}{C'R\varepsilon}\right)[u_\varepsilon]^2_{H^{\frac12}(Q^j_{\rho_\varepsilon})} \le C\rho_\varepsilon\left(1+\frac{\rho_\varepsilon}{C'R\varepsilon}\right)|\log\varepsilon| \le C$$

which concludes the proof. ○

5 Lower bound

Theorem 17 *Let $\{u_\varepsilon\}$ be a sequence such that $\liminf_{\varepsilon\to0} I_\varepsilon(u_\varepsilon) < +\infty$. Assume that $\{u_\varepsilon\}$ converges strongly in $L^1(T^2)$ to some function u. Then $u \in BV(T^2,\mathbf{Z})$ and*

$$\int_{T^2}\gamma\left(\frac{Du}{|Du|}\right)|Du| = \int_{S_u}\gamma(n)|[u]|d\mathcal{H}^1 \le \liminf_{\varepsilon\to0} I_\varepsilon(u_\varepsilon), \tag{62}$$

where the anisotropic line energy density $\gamma(n)$ is defined by (26)

The proof of Theorem 17 is based on a blow-up argument. Let $n \in S^1$ and denote by Q^n a square centered at 0, with size 1 and parallel to n. We now estimate from below the energy of a function on Q^n which is close to the characteristic function of the half plane $\{n\cdot x>0\}$. We denote the latter function by $u^n_0 = \chi_{\{n\cdot x>0\}}$.

Lemma 18 *Fix $0<\delta<\frac12$ and $\frac12<\alpha<1$. Then there exist $\delta_0>0$ and $\varepsilon_0>0$ such that for every $n\in S^1$, for every $\varepsilon\in(0,\varepsilon_0)$ and for every $u\in L^1(Q)$ satisfying*

$$\int_{Q^n}|u-u^n_0|\,dx \le \delta_0 \tag{63}$$

and

$$\int_{Q^n}\mathrm{dist}^2(u,\mathbf{Z})\,dx \le \varepsilon^\alpha \tag{64}$$

we have

$$\iint_{Q^n\times Q^n}\Gamma_\nu(x-y)|u(x)-u(y)|^2dx\,dy \ge \gamma(n)\alpha(1-\delta)|\log\varepsilon|, \tag{65}$$

where $\gamma(n)$ is defined by (26).

Proof. After rotation we can easily restrict our analysis to the case $n=e_1$, in which the square Q^n reduces to $Q=(-\frac12,\frac12)^2$. It then suffices to prove the statement with $\gamma(n)=\gamma(e_1)$ and we write from now u_0 instead of $u^{e_1}_0$. By scaling we may also assume that $\gamma(e_1)=2$. Since Γ_ν is homogeneous of degree -3 we have then

$$\Gamma^1_\nu(x_1) := \int_{\mathbf{R}}\Gamma_\nu(x_1,x_2)\,dx_2 \quad = \quad \frac{1}{x_1^2}. \tag{66}$$

We also assume that $0 \le u \le 1$ (otherwise we truncate the function by 0 and 1). Let us consider the following sub and super-level sets

$$A = \left\{ u < \frac{\delta}{8} \right\} \qquad \text{and} \qquad B = \left\{ u > 1 - \frac{\delta}{8} \right\}.$$

Then

$$\iint_{Q\times Q} \Gamma_\nu(x-y)|u(x)-u(y)|^2 dx\, dy \ge 2\left(1-\frac{\delta}{4}\right)^2 \int_A \int_B \Gamma_\nu(x-y)\, dx\, dy\,.$$

Hence it is sufficient to show that

$$\int_A \int_B \Gamma_\nu(x-y)\, dx\, dy \ge \alpha\left(1-\frac{\delta}{2}\right)|\log\varepsilon|\,. \tag{67}$$

Using the change of variables, $y = x+z$, we get

$$\int_A \int_B \Gamma_\nu(x-y)\, dx\, dy = \int_{\mathbf{R}^2} \Gamma_\nu(-z)\left(\int_{A\cap(B-z)} dx\right) dz = \int_{\mathbf{R}^2} \Gamma_\nu(z)|A\cap(B-z)|\, dz\,. \tag{68}$$

Let $\frac{1}{2} < \sigma < \alpha$. By (64) we know that for ε_0 small enough the set $E = Q \setminus (A\cup B)$ satisfies

$$|E| \le \varepsilon^\sigma\,. \tag{69}$$

If the function u is independent of x_2 and E is a simple strip, then $|A\cap(B-z)| \ge z_1 - \varepsilon^\sigma$ and the result can be obtained from (68) by an explicit computation. In general these conditions are not satisfied, but the idea is that they are approximatively satisfied. In order to deal with the general case let us estimate the difference between χ_B and the characteristic function of $\{x_1 > 0\}$, i.e., u_0. If $x_1 > 0$ and $x \in A$, then $|u-u_0| \ge \frac{1}{2}$. Thus

$$|\chi_B - u_0| = |\chi_B - 1| \le \chi_E + \chi_A \le \chi_E + 2|u-u_0| \quad \text{for } x_1 > 0.$$

Similarly, if $x_1 < 0$ and $x \in B$, then $|u-u_0| = |u| \ge \frac{1}{2}$. Hence

$$|\chi_B - u_0| = |\chi_B| \le 2|u-u_0| \quad \text{for } x_1 < 0\,.$$

Combining the previous estimates we get

$$|\chi_B - u_0| \le \chi_E + 2|u-u_0| \qquad \text{in } Q\,. \tag{70}$$

Now consider a concentric subsquare Q' of Q, i.e., $Q' = \rho Q$ with $\rho < 1$. By (69) we have

$$|A\cap Q'| + |B\cap Q'| \ge |Q'| - \varepsilon^\sigma \tag{71}$$

Suppose that z satisfies $|z|_\infty = \max(|z_1|,|z_2|) < (1-\rho)/2$ and $z_1 > 0$. Since $|(B-z)\cap Q'| = |B\cap(Q'+z)|$, we have

$$\begin{aligned} |(B-z)\cap Q'| - |B\cap Q'| &= \int_{Q'+z} \chi_B\, dx - \int_{Q'} \chi_B\, dx \\ &\ge \int_{Q'+z} u_0\, dx - \int_{Q'} u_0\, dx - \int_{(Q'+z)\triangle Q'} |\chi_B - u_0|\, dx \\ &\ge \rho z_1 - \int_U |\chi_B - u_0|\, dx\,, \end{aligned}$$

where U is the annular region $U = (\rho + |z|_\infty)Q \setminus (\rho - |z|_\infty)Q$ and $(Q'+z)\triangle Q' \subset U$. Together with (71) this yields

$$\begin{aligned} m(z) := |A\cap(B-z)| &\ge |A\cap(B-z)\cap Q'| \ge |A\cap Q'| + |(B-z)\cap Q'| - |Q'| \\ &\ge \rho z_1 - \varepsilon^\sigma - \int_U |\chi_B - u_0|\, dx\,. \end{aligned} \tag{72}$$

We now must choose ρ and U properly in order to obtain a sufficiently large bound for the right hand side of (72). We fix $\delta_2 > 0$ and $\rho_0 = 1 - \delta_2$ and we cover $Q \setminus \rho_0 Q$ by annular regions of thickness $2|z|_\infty$, i.e., we take $\rho_i = \rho_0 + 2i|z|_\infty$, $U_i = \rho_i Q \setminus \rho_{i-1} Q$, with $i = 1, ..., k$ and $k < \delta_2/4|z|_\infty \le k+1 \le 2k$. We apply (72) with $\rho = \rho_i$ and $U = U_i$, we sum all the inequalities for i from 1 to k, we divide by k and we get

$$m(z) \ge z_1\rho_0 - \varepsilon^\sigma - \frac{1}{k}\int_Q |\chi_B - u_0|\, dx\,.$$

This, together with (69), (70) and (63) yields

$$m(z) \ge (1-\delta_2)z_1 - \varepsilon^\sigma - \frac{8}{\delta_2}|z|_\infty(2\delta_0 + \varepsilon^\sigma)\,,$$

whenever $|z|_\infty \le \delta_2/2$. We now assume $\varepsilon_0^\sigma \le 2\delta_0$ and choose $\delta_2 = 2\sqrt{\delta_0}$. This gives

$$m(z) \ge (1 - 2\sqrt{\delta_0})z_1 - \varepsilon^\sigma - 16\sqrt{\delta_0}|z|_\infty\,, \qquad \text{if } |z|_\infty \le \sqrt{\delta_0}\,. \tag{73}$$

Then, in order to conclude from here together with (68) and to obtain (67), we must estimate the following integrals:

$$I_1 = \int_{\{\varepsilon^\sigma \le z_1 \le 1\}} \varepsilon^\sigma \Gamma_\nu(z)\, dz_1\, dz_2 \le \varepsilon^\sigma \int_{\varepsilon^\sigma}^1 \Gamma_\nu^1(z_1)\, dz_1 \le 1\,, \tag{74}$$

where we used (66). Since $|z|_\infty \le 2|z|$ and $\int_{\mathbf{R}}(z_1^2 + z_2^2)^{-1} dz_2 = \pi/|z_1|$ we have

$$I_2 = \int_{\{\varepsilon^\sigma \le z_1 \le 1\}} |z|_\infty \Gamma_\nu(z)\, dz_1\, dz_2 \le \int_{\varepsilon^\sigma}^1 \frac{C}{|z_1|}\, dz_1 \le C\sigma|\log\varepsilon|\,. \tag{75}$$

Finally, by the definition of Γ_ν^1 and the fact that $\Gamma_\nu(z) \le C/z_2^3$, we get

$$\begin{aligned} I_3 = \int_{\varepsilon^\sigma}^{\delta_0}\int_{\{|z_2|\le\delta_0\}} z_1\Gamma_\nu(z)\, dz_2\, dz_1 \quad &\ge \quad \int_{\varepsilon^\sigma}^{\delta_0} z_1(\Gamma_\nu^1(z_1) - \frac{C}{\delta_0^2})dz_1 \\ &\ge \quad \log\frac{\delta_0}{\varepsilon^\sigma} - C = \sigma|\log\varepsilon| + \log\delta_0 - C\,. \end{aligned} \tag{76}$$

Thus by (73) - (76) we have

$$\begin{aligned} \frac{1}{|\log\varepsilon|}\int_{\mathbf{R}^2} m(z)\Gamma_\nu(z)\, dz \quad &\ge \quad \frac{1}{|\log\varepsilon|}\Big[(1-2\sqrt{\delta_0})I_3 - I_1 - 16\sqrt{\delta_0}I_2\Big] \\ &\ge \quad (1-2\sqrt{\delta_0})\sigma - C(\sqrt{\delta_0} + \frac{1}{|\log\varepsilon_0|}) \end{aligned}$$

and this proves (67) if $\alpha - \sigma$, δ_0 and ε_0 are chosen sufficiently small.

○

With the following lemma we prove that the measure μ_ε defined on the product $T^2 \times T^2$ by

$$\mu_\varepsilon(A \times B) := \frac{1}{|\log\varepsilon|}\int_A\int_B K_\nu(x-y)|u_\varepsilon(x) - u_\varepsilon(y)|^2 dx\, dy \tag{77}$$

converges weakly to a measure which is supported on the diagonal and which can be estimated from below by a measure concentrated on the jump set of the limit function u.

Lemma 19 *Let u_ε be a sequence which converges strongly in $L^1(T^2)$ to some function $u \in BV(T^2, \mathbf{Z})$ and assume $0 \le u_\varepsilon \le 1$. Let μ_ε be defined by (77) and let μ be its weak*-limit (for a subsequence). Then μ is concentrated on the diagonal $D = \{(x,x) : x \in T^2\}$, i.e., $\mu(E) = 0$ if $E \cap D = \emptyset$. Moreover the measure λ on T^2 which is defined by*

$$\lambda(A) := \mu(\{(x,x) \ : \ x \in A\})\,, \tag{78}$$

satisfies

$$\lambda \geq \gamma(n) d\mathcal{H}^1 \llcorner S_u \,, \tag{79}$$

where S_u is the jump set of u.

Proof. To prove that supp$\mu \subseteq D$ it is enough to show that for any continuous nonnegative function $\varphi : T^2 \times T^2 \to \mathbf{R}$ with supp$\varphi \cap D = \emptyset$ we have $\int \varphi \, d\mu = 0$. Let $\delta = \mathrm{dist}(\mathrm{supp}\varphi, D) > 0$. Since u_ε is bounded by 1 we have

$$\begin{aligned} \int_{T^2} \varphi(x,y)\, d\mu &= \lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \iint_{T^2 \times T^2} \varphi(x,y) K_\nu(x-y) |u_\varepsilon(x) - u_\varepsilon(y)|^2 dx\, dy \\ &\leq \lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|} \frac{C}{\delta^3} \iint_{T^2 \times T^2} \varphi(x,y) dx\, dy = 0\,. \end{aligned}$$

Thus μ is concentrated on D. Hence we can define the measure λ on T^2 by

$$\lambda(A) := \mu(\{(x,x) \ : \ x \in A\}) = \mu(A \times A)\,.$$

In order to conclude it is enough to show that for $\mathcal{H}^1$-a.e. $x_0 \in S_u$ we have

$$\liminf_{r \to 0} \frac{\lambda(Q_r^n(x_0))}{r} = \liminf_{r \to 0} \lim_{\varepsilon \to 0} \frac{\mu_\varepsilon(Q_r^n(x_0) \times Q_r^n(x_0))}{r} \geq \gamma(n)\,, \tag{80}$$

where, with a little abuse of notation, Q_r^n denotes the square centered at x_0, with size r, parallel to the normal n on S_u at x_0.

For $\mathcal{H}^1$-a.e. $x_0 \in S_u$ we have

$$\lim_{r \to 0} \int_{Q_1^n} |u(rx + x_0) - \chi_{\{x \cdot n > 0\}}|\, dx = 0\,.$$

Fix such a $x_0 \in S_u$. We will proceed with a blow-up argument. Consider the sequence v_ε obtained from u_ε by a rescaling, i.e., $v_\varepsilon(x) = u_\varepsilon(rx + x_0)$. By a change of variables we have

$$\frac{1}{r} \mu_\varepsilon(Q_r^n(x_0) \times Q_r^n(x_0)) = \iint_{Q_1^n \times Q_1^n} K_\nu^r(x-y) |v_\varepsilon(x) - v_\varepsilon(y)|^2 dx\, dy\,,$$

where $K_\nu^r(t) = r^3 K_\nu(rt)$. From Proposition 1 and the homogeneity of Γ_ν we get

$$\frac{1}{r} \mu_\varepsilon(Q_r^n(x_0) \times Q_r^n(x_0)) = \iint_{Q_1^n \times Q_1^n} \Gamma_\nu(x-y) |v_\varepsilon(x) - v_\varepsilon(y)|^2 dx\, dy + o(1)\,, \tag{81}$$

as r goes to zero. The idea is to use Lemma 18 with the function v_ε. Fix $0 < \delta < \frac{1}{2}$ and $\frac{1}{2} < \alpha < 1$ and let $\delta_0 > 0$ and $\varepsilon_0 > 0$ be the constants given by Lemma 18. For $r > 0$ small enough we have

$$\int_{Q_1^n} |u(rx + x_0) - \chi_{\{x \cdot n > 0\}}|\, dx \leq \frac{\delta_0}{2}\,.$$

Fix such an r. There exists an ε_0 such that for every $\varepsilon < \varepsilon_0$

$$\frac{1}{r^2} \int_{Q_r^n} |u_\varepsilon(x) - u(x)|\, dx \leq \frac{\delta_0}{2}\,.$$

We then get

$$\int_{Q_1^n} |v_\varepsilon(x) - \chi_{\{x \cdot n > 0\}}|\, dx \leq \int_{Q_1^n} |v_\varepsilon(x) - u(rx + x_0)|\, dx + \int_{Q_1^n} |u(rx + x_0) - \chi_{\{x \cdot n > 0\}}| \leq \delta_0\,.$$

Since v_ε also satisfies (64) (after a possible reduction of ε_0) Lemma 18 yields

$$\frac{1}{|\log\varepsilon|}\iint_{Q_1^n\times Q_1^n}\Gamma_\nu(x-y)|v_\varepsilon(x)-v_\varepsilon(y)|^2dx\,dy\geq\gamma(n)\alpha(1-\delta)\,. \tag{82}$$

Thus by (81) we get

$$\lambda(\overline{Q}_r^n(x_0))\geq\liminf_{\varepsilon\to0}\mu_\varepsilon(Q_r^n(x_0)\times Q_r^n(x_0))\geq r\gamma(n)\alpha(1-\delta)+o(r)\,.$$

Dividing by r and taking the limit $r\to0$ we get (80) since $\alpha\in(\frac{1}{2},1)$ and $\delta>0$ were arbitrary. ○

Proof of Theorem 17. By the compactness result (Theorem 12) we can deduce that $u\in BV(T^2,\mathbf{Z})$. We obtain the result by Lemma 19 using a truncation argument. Let $j\in\mathbf{Z}$ and let us consider the truncations $T_ju_\varepsilon=(u_\varepsilon\vee j)\wedge(j+1)$. Clearly each truncation T_ju_ε converges to T_ju and, up to a translation, it satisfies the assumptions of Lemma 19. Thus we have that for any $j\in\mathbf{Z}$

$$\liminf_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}\iint_{T^2\times T^2}K_\nu(x-y)|T_ju_\varepsilon(x)-T_ju_\varepsilon(y)|^2dx\,dy\geq\int_{S_{T_ju}}\gamma(n)\,d\mathcal{H}^1\,. \tag{83}$$

Note that $|u_\varepsilon(x)-u_\varepsilon(y)|^2\geq\sum_{j\in\mathbf{Z}}|T_ju_\varepsilon(x)-T_ju_\varepsilon(y)|^2$ for every $x,y\in T^2$, that $\cup_{j\in\mathbf{Z}}S_{T_ju}=S_u$ $\mathcal{H}^1$-a.e., and that $|[u](x)|=\sum_{j\in\mathbf{Z}}|[T_ju](x)|$ $\mathcal{H}^1$-a.e. $x\in S_u$. Hence, by (83), we have

$$\begin{aligned}\liminf_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}&\iint_{T^2\times T^2}K_\nu(x-y)|u_\varepsilon(x)-u_\varepsilon(y)|^2dx\,dy\\&\geq\sum_{j\in\mathbf{Z}}\liminf_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}\iint_{T^2\times T^2}K_\nu(x-y)|T_ju_\varepsilon(x)-T_ju_\varepsilon(y)|^2dx\,dy\\&\geq\sum_{j\in\mathbf{Z}}\int_{S_{T_ju}}\gamma(n)\,d\mathcal{H}^1=\int_{S_u}\gamma(n)|[u]|(x)\,d\mathcal{H}^1(x)\,,\end{aligned} \tag{84}$$

which concludes the proof.

○

Remark 20 Reasoning as in the proof above we conclude that for any sequence u_ε converging to u in L^1 and satisfying $I_\varepsilon(u_\varepsilon)\leq C$, and for every continuous non-negative function φ on $T^2\times T^2$ we get

$$\liminf_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}\iint_{T^2\times T^2}\varphi(x,y)K_\nu(x-y)|u_\varepsilon(x)-u_\varepsilon(y)|^2dx\,dy\geq\int_{S_u}\varphi(x,x)\gamma(n)|[u]|(x)\,d\mathcal{H}^1(x)$$

and thus $\lambda\geq\gamma(n)|[u]|\mathcal{H}^1\llcorner S_u$, where $\lambda(A)=\mu(\{(x,x)\,:\,x\in A\})$ and μ is the weak*-limit of the measures μ_ε defined in (77). Moreover it is easy to see that if u_ε is also bounded in L^2, μ is concentrated on the diagonal D, i.e., $\lambda(A)=\mu(A\times A)$.

Combining Theorem 17 and the results proved in [8] we now establish the following lower bound for the Γ-limit of the functional F_ε.

Theorem 21 *Assume that $N_\varepsilon\varepsilon/|\log\varepsilon|\to\Lambda$ and that the discs $B^i_{R\varepsilon}$ are uniformly distributed and well separated. Let u_ε be a sequence such that $\liminf_{\varepsilon\to0}F_\varepsilon(u_\varepsilon)<+\infty$. Assume that u_ε converges strongly in $L^1(T^2)$ to some function u, then $u\in BV(T^2,\mathbf{Z})$ and*

$$\int_{S_u}\gamma(n)|[u]|d\mathcal{H}^1+\Lambda\int_{T^2}D_\nu(u,B_R)\,dx\leq\liminf_{\varepsilon\to0}F_\varepsilon(u_\varepsilon)\,, \tag{85}$$

where $\gamma(n)$ is defined by (26) and $D_\nu(\cdot,B_R)$ is defined by (18).

The proof uses the following two lemmas proved in [8].

Lemma 22 *Given $\mathcal{R} : \mathbf{R}_+ \to \mathbf{R}_+$, with $\mathcal{R}(\varepsilon) \to \infty$ as $\varepsilon \to 0$, there exists a function $\omega : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$, with $\omega(\varepsilon, \delta) \to 0$ as $(\varepsilon, \delta) \to (0, 0)$, such that the following statement holds. Let $a \in \mathbf{Z}$. If $\zeta \in H^{\frac{1}{2}}(B_{\mathcal{R}(\varepsilon)})$ satisfies*

$$\fint_{B_{\mathcal{R}(\varepsilon)}} |\zeta - a| dx \leq \delta \tag{86}$$

and $\zeta = 0$ on B_R, then

$$\int_{B_{\mathcal{R}(\varepsilon)}} \mathrm{dist}^2(\zeta, \mathbf{Z})\, dx + \iint_{B_{\mathcal{R}(\varepsilon)} \times B_{\mathcal{R}(\varepsilon)}} K^\varepsilon(x-y)|\zeta(x) - \zeta(y)|^2\, dx\, dy \geq D_\nu(a, B_R) - \omega(\varepsilon, \delta)\,, \tag{87}$$

where $K^\varepsilon(t) = \varepsilon^3 K_\nu(\varepsilon t)$.

Lemma 23 *There exists a positive constant C such that for every $0 < \rho < \widehat{\rho}$ the following inequality holds*

$$\fint_{B_\rho} |u|\, dx \leq \fint_{B_{\widehat{\rho}}} |u|\, dx + \frac{C}{\sqrt{\rho}} [u]_{H^{\frac{1}{2}}(B_{\widehat{\rho}})} \tag{88}$$

for all $u \in H^{\frac{1}{2}}(B_{\widehat{\rho}})$.

Proof of Theorem 21. The proof is based on a blow-up argument like the proof of Theorem 17. Let μ_ε be the measure defined in (77), let μ be its weak*-limit, let λ be defined by (78) and let η_ε be the following measure

$$\eta_\varepsilon(A) = \frac{1}{\varepsilon|\log \varepsilon|} \int_A \mathrm{dist}^2(u_\varepsilon, \mathbf{Z})\, dx\,. \tag{89}$$

Since η_ε is bounded, it converges weakly* up to a subsequence. Let η be its weak* limit. The main step is now to prove that

$$\lambda + \eta \geq \Lambda D_\nu(u(x), B_R)\, dx\,, \tag{90}$$

i.e., that for a.e. $x \in T^2$ we have

$$\lim_{r \to 0} \liminf_{\varepsilon \to 0} \frac{\mu_\varepsilon(Q_r(x) \times Q_r(x)) + \eta_\varepsilon(Q_r(x))}{|Q_r|} \geq \Lambda D_\nu(u(x), B_R)\,. \tag{91}$$

Let us fix $a \in \mathbf{Z}$ and x_0 be a Lebesgue point of u with $u(x_0) = a$. In order to simplify the notation assume that $x_0 = 0$. The proof of (91) follows using Lemmas 22 and 23 and the same strategy of the proof of the lower bound in [8] taking into account that here the domain is Q_r and the regime for the obstacles is different, i.e., $N_\varepsilon \approx |\log \varepsilon|/\varepsilon$. We repeat the argument here for the convenience of the reader.

Consider on Q_r a lattice of squares, denoted by Q_j^ε, of size approximatively $1/\sqrt{N_\varepsilon}$. Let $\widehat{Q}_j^\varepsilon$ be concentric squares of three times the size. Since each point is contained at most in 9 of the squares $\widehat{Q}_j^\varepsilon$ we have

$$\sum_j \iint_{\widehat{Q}_j^\varepsilon \times \widehat{Q}_j^\varepsilon} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^3}\, dx\, dy \leq C|\log \varepsilon| \approx N_\varepsilon \varepsilon$$

and

$$\sum_j \int_{\widehat{Q}_j^\varepsilon} |u_\varepsilon - a| dx \leq \omega_r |Q_r| + 9||u_\varepsilon - u||_{L^1}\,,$$

where $\omega_r \to 0$ as $r \to 0$. Let $\theta > 0$. Then there exist a set of indices $\mathcal{J}^\varepsilon(Q_r)$ such that $\#(\mathcal{J}^\varepsilon(Q_r)) \geq (1-\theta)N_\varepsilon|Q_r|$ and a constant C_θ such that

$$\iint_{\widehat{Q}_j^\varepsilon \times \widehat{Q}_j^\varepsilon} \frac{|u_\varepsilon(x) - u_\varepsilon(y)|^2}{|x-y|^3}\, dx\, dy \leq C_\theta \varepsilon$$

and

$$\fint_{\widehat{Q}_j^\varepsilon} |u_\varepsilon - a|dx \le C_\theta \left(\omega_r + \frac{||u_\varepsilon - u||_{L^1}}{|Q_r|} \right)$$

for all $j \in \mathcal{J}^\varepsilon(Q_r)$. Let $0 < \delta < 1$. By applying Lemma 23 with $\rho = \varepsilon^\beta$, with $\frac{1}{2} < \beta < 1$, and $\widehat{\rho} = \frac{1}{\sqrt{N_\varepsilon}}$, for each $x_\varepsilon^i \in Q_j^\varepsilon$ we also have

$$\fint_{B^i_{\varepsilon^\beta}} |u_\varepsilon - a|dx \le \delta \qquad \text{if } \varepsilon \le \varepsilon_0(\delta, \theta, r) \text{ and } r \le r_0(\theta)\,. \tag{92}$$

Then by Lemma 22 applied with $\mathcal{R}(\varepsilon) = \varepsilon^{\beta - 1}$ we get (after the scaling $X \to x/\varepsilon$)

$$\mu_\varepsilon(B^i_{\varepsilon^\beta} \times B^i_{\varepsilon^\beta}) + \eta_\varepsilon(B^i_{\varepsilon^\beta}) \ge \frac{\varepsilon}{|\log \varepsilon|} \left(D_\nu(a, B_R) - \omega(\varepsilon, \delta) \right), \quad \text{if } \varepsilon < \varepsilon_0(\delta, \theta, r)$$

for any i such that $x_\varepsilon^i \in Q_j^\varepsilon$. Since the points x_ε^i are assumed to be well separated (see (16)), the balls $B^i_{\varepsilon^\beta}$ are disjoint and summation over all i yields

$$\mu_\varepsilon(Q_r \times Q_r) + \eta_\varepsilon(Q_r) \ge \frac{\varepsilon}{|\log \varepsilon|} \left[\sum_{j \in \mathcal{J}^\varepsilon(Q_r)} \#(\mathcal{I}_\varepsilon(Q_j^\varepsilon)) \right] \left(D_\nu(a, B_R) - \omega(\varepsilon, \delta) \right) .$$

The uniform distribution of the obstacles (see condition (15)) implies that

$$\begin{aligned} \sum_{j \in \mathcal{J}^\varepsilon(Q_r)} \#(\mathcal{I}_\varepsilon(Q_j^\varepsilon)) &= \#(\mathcal{I}_\varepsilon(Q_r)) - \sum_{j \notin \mathcal{J}^\varepsilon(Q_r)} \#(\mathcal{I}_\varepsilon(Q_j^\varepsilon)) \ge \#(\mathcal{I}_\varepsilon(Q_r)) - \sum_{j \notin \mathcal{J}^\varepsilon(Q_r)} (N_\varepsilon |Q_j^\varepsilon| + L) \\ &= \#(\mathcal{I}_\varepsilon(Q_r)) - (L+1)\#(\{j : \ j \notin \mathcal{J}^\varepsilon(Q_r)\})\,. \end{aligned}$$

Since $\#(\{j : \ j \notin \mathcal{J}^\varepsilon(Q_r)\}) \le N_\varepsilon |Q_r| \theta$, by (15) we get

$$\begin{aligned} \mu_\varepsilon(Q_r \times Q_r) + \eta_\varepsilon(Q_r) &\ge \frac{\varepsilon N_\varepsilon |Q_r| \left(1 - \theta(L+1)\right)}{|\log \varepsilon|} \left(D_\nu(a, B_R) - \omega(\varepsilon, \delta) \right) \\ &\ge \Lambda |Q_r| \left(1 - \theta(L+1)\right) \left(D_\nu(a, B_R) - \omega(\varepsilon, \delta) \right) + o(1) \end{aligned}$$

as ε goes to zero. Taking the limits $\varepsilon \to 0$, then $r \to 0$, then $\delta \to 0$ and finally $\theta \to 0$, we obtain (91). Now we conclude easily by (91) and Lemma 19 together with Remark 20 that

$$\lambda + \eta \ge \Lambda D_\nu(u(x), B_R)\, dx \qquad \text{and} \qquad \lambda \ge \gamma(n)|[u]|\mathcal{H}^1 \llcorner S_u\,.$$

Since the two measures $D_\nu(u(x), B_R)\, dx$ and $\gamma(n)|[u]|\mathcal{H}^1 \llcorner S_u$ are mutually singular, we get

$$\lambda + \eta \ge \Lambda D_\nu(u(x), B_R)\, dx + \gamma(n)|[u]|\mathcal{H}^1 \llcorner S_u$$

and this concludes the proof.

○

6 Upper bound

In this section we establish an upper bound for the Γ-limit of I_ε and F_ε. This concludes the proof of the Γ-convergence result in the critical scaling (see Theorems 9 and 10).

Theorem 24 *For every $u \in BV(T^2, \mathbf{Z})$ there exists a sequence $\{u_\varepsilon\}$ converging to u in $L^2(T^2)$ such that*

$$\limsup_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon) \le \int_{S_u} \gamma(n)|[u]| d\mathcal{H}^1\,.$$

Proof. By a standard density argument, we can restrict our analysis to the case of $u \in BV(T^2, \mathbf{Z}) \cap L^\infty(T^2)$ such that S_u is polygonal, with a finite number of sides, and $|[u]| = 1$ $\mathcal{H}^1$-a.e. on S_u. We also choose a fundamental domain Q of the torus such that $|Du|(\partial Q) = 0$. We construct u_ε simply mollifying u at the scale ε.

Fix $\varphi \in C_c^\infty(B_1(0))$, $\varphi \geq 0$ and $\int \varphi \, dx = 1$, and set

$$u_\varepsilon = \varphi_\varepsilon * u \qquad \text{where } \varphi_\varepsilon(x) = \varepsilon^{-2}\varphi\left(\frac{x}{\varepsilon}\right).$$

Since $u_\varepsilon = u$ outside an ε-neighbourhood of S_u we clearly have that

$$\frac{1}{\varepsilon}\int_Q \operatorname{dist}^2(u_\varepsilon, \mathbf{Z})\, dx \leq C\,. \tag{93}$$

The conclusion follows if we prove that

$$\lim_{\varepsilon\to 0}\mu_\varepsilon(Q\times Q) = \lim_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\iint_{Q\times Q} K_\nu(x-y)|u_\varepsilon(x)-u_\varepsilon(y)|^2 dx\, dy = \int_{S_u}\gamma(n)|[u]|d\mathcal{H}^1\,, \tag{94}$$

where the measure μ_ε is defined as in (77). We first prove that the total variation of μ_ε is bounded. Let A be an open subset of Q and denote by $d(A)$ the diameter of A. Using the change of variables $z = y - x$ we get

$$\mu_\varepsilon(A\times A) \leq \frac{C}{|\log\varepsilon|}\int_{|z|\leq d(A)}\frac{1}{|z|^3}\int_A |u_\varepsilon(x+z)-u_\varepsilon(x)|^2 dx\, dz\,. \tag{95}$$

Let us denote now by $\mathcal{N}_\delta$ the δ-neighbourhood of S_u. Since $u \in BV(Q, \mathbf{Z})$ and S_u is polygonal (with a finite number of sides) we have

$$|\mathcal{N}_\delta \cap A| \leq Cd(A)\delta \tag{96}$$

and, by the definition of u_ε, we easily deduce that u_ε is constant on $B_\delta(x)$ if $x \notin \mathcal{N}_{\delta+\varepsilon}$. Thus, using that u is bounded by a constant M, we get

$$\begin{aligned}\int_{2\varepsilon\leq|z|\leq d(A)}\frac{1}{|z|^3}\int_A |u_\varepsilon(x+z)-u_\varepsilon(x)|^2 dx\, dz &\leq 4M^2\int_{2\varepsilon\leq|z|\leq d(A)}\frac{1}{|z|^3}|\mathcal{N}_{3|z|}\cap A|dz\\ &\leq Cd(A)\int_{2\varepsilon\leq|z|\leq d(A)}\frac{1}{|z|^2}\,dz \leq Cd(A)|\log\varepsilon|\,.\end{aligned}$$

On the other hand we have that $|Du_\varepsilon| \leq C/\varepsilon$ and hence

$$\begin{aligned}\int_{|z|\leq 2\varepsilon}\frac{1}{|z|^3}\int_A |u_\varepsilon(x+z)-u_\varepsilon(x)|^2 dx\, dz &\leq C\int_{|z|\leq 2\varepsilon}\frac{1}{|z|^3}|\mathcal{N}_{2\varepsilon}\cap A|\frac{|z|^2}{\varepsilon^2}dz\\ &\leq Cd(A)\int_{|z|\leq 2\varepsilon}\frac{1}{|z|\varepsilon}dz \leq C\,.\end{aligned}$$

By (95) we get

$$\mu_\varepsilon(A\times A) \leq Cd(A) + o(1) \tag{97}$$

as ε tends to zero. In particular μ_ε is bounded and thus, up to a subsequence, converges weakly* to a measure μ. Since u_ε is bounded in L^∞, Remark 20 implies that μ is concentrated on the diagonal. As above, we define the measure $\lambda(A) := \mu(A\times A)$. By the definition of u_ε it is easy to check that supp$\lambda \subseteq S_u$. Moreover, taking the limit $\varepsilon \to 0$ in (97) we get a similar estimate for λ, i.e., $\lambda(A) \leq Cd(A)$. Taking the Radon-Nikodym derivative of λ with respect to $\mathcal{H}^1$, we deduce that

$$\lambda \leq C\mathcal{H}^1 \llcorner S_u\,. \tag{98}$$

We conclude the proof if we prove that for $\mathcal{H}^1$-a.e. $x_0 \in S_u$

$$\lim_{r\to 0} \frac{\lambda(B_r(x_0))}{2r} \le \gamma(n)\,. \tag{99}$$

Indeed, this together with the lower bound implies (94). The latter inequality can be obtained by an explicit calculation. Let us assume, for the sake of simplicity, that $x_0 = 0$ and that the normal n of S_u at x_0 is e_1, i.e., S_u is locally contained in $\{x_1 = 0\}$. Moreover it is not restrictive to assume that $u(x) = \chi_{\{x_1>0\}}$ if $x \in B_r$. Then

$$u_\varepsilon(x) = \psi(\frac{x_1}{\varepsilon})\,, \qquad \text{with} \quad \psi(t) = \int_{\{z_1 \ge t\}} \varphi(z)\,dz \qquad \text{if } x \in B_r\,.$$

Hence ψ is smooth and decreasing and $\psi(t) = 1$ if $t \le -1$, $\psi(t) = 0$ if $t \ge 1$. Set $R = r/\varepsilon$. It follows from Proposition 1 that

$$\frac{1}{2r}\iint_{B_r\times B_r} K_\nu(x-y)|u_\varepsilon(x)-u_\varepsilon(y)|^2 dx\,dy = \frac{1}{2R}\iint_{B_R\times B_R} \Gamma_\nu(x-y)|\psi(x_1)-\psi(y_1)|^2 dx\,dy + O(\varepsilon^3)\,. \tag{100}$$

Now note that since $\Gamma_\nu(t) \ge 0$ we have

$$\begin{aligned}\frac{1}{2R}\int_{-R}^{R}\int_{-R}^{R}\Gamma_\nu(x-y)\,dx_2\,dy_2 &= \frac{1}{2R}\int_{-2R}^{2R}\Gamma_\nu(x_1-y_1,z_2)(2R-|z_2|)\,dz_2\\ &\le \int_{-\infty}^{\infty}\Gamma_\nu(x_1-y_1,z_2)\,dz_2 = \Gamma_\nu^1(x_1-y_1) = \frac{\gamma(e_1)}{2|x_1-y_1|^2}\,.\end{aligned}$$

Thus using the fact that, for $t > 0$, $|\psi(x_1) - \psi(x_1 + t)|$ is zero if $x_1 \le -t - 1$ or if $x_1 \ge 1$ and is bounded by $\mathrm{Lip}\psi\, t$ if t is small and by 1 if t is big, we deduce that

$$\begin{aligned}\frac{1}{2R}\iint_{B_R\times B_R}\Gamma_\nu(x-y)|\psi(x_1)-\psi(y_1)|^2 dx\,dy &\le \frac{\gamma(e_1)}{2}\int_{-R}^{R}\int_{-R}^{R}\frac{1}{|x_1-y_1|^2}|\psi(x_1)-\psi(y_1)|^2\,dx_1\,dy_1\\ &= \gamma(e_1)\int_0^{2R}\frac{1}{t^2}\int_{-R}^{R-t}|\psi(x_1)-\psi(x_1+t)|^2\,dx_1\,dt\\ &\le \gamma(e_1)\int_0^{2R}\frac{1}{t^2}\int_{-t-1}^{1}|\psi(x_1)-\psi(x_1+t)|^2\,dx_1\,dt\\ &\le \gamma(e_1)\left(\int_0^1\frac{(t+2)}{t^2}(\mathrm{Lip}\psi\,t)^2dt + \int_1^{2R}\frac{(t+2)}{t^2}\,dt\right)\\ &\le \gamma(e_1)(C+\log(2R))\,.\end{aligned}$$

Since $\log(2R) = \log(2r) + \log\frac{1}{\varepsilon}$, from (100) and (93) we finally get

$$\frac{\lambda(B_r(x_0))}{2r} \le \liminf_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\frac{1}{2r}\iint_{B_r\times B_r}K_\nu(x-y)|u_\varepsilon(x)-u_\varepsilon(y)|^2dx\,dy \le \gamma(e_1)\,,$$

which concludes the proof.

○

Remark 25 Note that in the construction of the optimal sequence for the Γ-limit (Theorem 24) the precise shape of the profile φ is irrelevant. It does not influence the logarithmic contribution of the $H^{\frac{1}{2}}$ norm of u_ε but only the terms which are of order one.

Remark 26 Together with the lower bound (Theorem 17) and (93), we see that a sequence constructed by convolution as in the above proof is also optimal on any open subset A of Q and, up to a subsequence, satisfies

$$\lim_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\int_A\int_A K_\nu(x-y)|u_\varepsilon(x)-u_\varepsilon(y)|^2dx\,dy = \int_{S_u\cap A}\gamma(n)|[u]|d\mathcal{H}^1\,.$$

In particular we proved that the measure λ defined in (78) satisfies

$$\lambda = \gamma(n) d\mathcal{H}^1 \llcorner S_u \,.$$

With the theorem below we will give the upper bound for the Γ-convergence result stated in Theorem 10.

Theorem 27 *Assume that $N_\varepsilon \varepsilon / |\log \varepsilon| \to \Lambda$ and that the discs $B^i_{R\varepsilon}$ are uniformly distributed and well separated. For every $u \in BV(T^2, \mathbf{Z})$ there exists a sequence $\{v_\varepsilon\}$ converging to u in $L^2(T^2)$ such that $v_\varepsilon \in H^{\frac{1}{2}}(T^2)$, $v_\varepsilon = 0$ a.e. on $\cup_i B^i_{R\varepsilon}$ and*

$$\limsup_{\varepsilon \to 0} F_\varepsilon(v_\varepsilon) \leq \int_{S_u} \gamma(n) |[u]| d\mathcal{H}^1 \; + \int_Q \Lambda D_\nu(u, B_R)\, dx \,. \tag{101}$$

Proof. The general idea of the proof is to consider the optimal sequence obtained by convolution in Theorem 24 and to modify it in order to let it satisfy the pinning condition using the $H^{\frac{1}{2}}$-dislocation capacitary potentials as cut-off functions, as we did for the dilute case in [8]. However the presence of the non local term in the energy makes this argument more involved. In the proof we will always assume that the fundamental domain Q for T^2 is chosen such that $|Du|(\partial Q) = 0$.

Step 1. The statement of the theorem holds for any u constant, $u = a \in \mathbf{Z}$.

In this case the sequence v_ε can be constructed exactly as in [8] using the $H^{\frac{1}{2}}$-dislocation capacitary potentials (see Section 2.2 for the relevant definitions). We give here some details in order to fix some notation for the following steps. Denote by ζ^a_ε the $H^{\frac{1}{2}}$-dislocation capacitary potential of B_R with respect to $B_{\varepsilon^{\alpha-1}}$ at the level a, where $\alpha \in (\frac{\beta+1}{2}, 1)$ and $\beta < 1$ is given by (16). Then the sequence $v_\varepsilon = v^a_\varepsilon$ is defined by

$$v^a_\varepsilon(x) := \begin{cases} a - \zeta^a_\varepsilon\left(\dfrac{x - x^i_\varepsilon}{\varepsilon}\right) & \text{if } x \in \cup_i B^i_{\varepsilon^\alpha}, \\ a & \text{otherwise.} \end{cases}$$

By μ^a_ε and η^a_ε we denote the following measures

$$\mu^a_\varepsilon(A \times B) := \frac{1}{|\log \varepsilon|} \int_A \int_B K_\nu(x-y) |v^a_\varepsilon(x) - v^a_\varepsilon(y)|^2 dx\, dy \tag{102}$$

and

$$\eta^a_\varepsilon(A) := \frac{1}{\varepsilon |\log \varepsilon|} \int_A \mathrm{dist}(v^a_\varepsilon, \mathbf{Z})\, dx \,, \tag{103}$$

by μ^a and η^a we denote their weak*-limits (which exist up to extraction of a subsequence) and by λ^a we denote the measure such that $\lambda^a(A) = \mu^a(A \times A)$. With this notation what we have to prove is $\lambda^a + \eta^a \leq \Lambda D_\nu(a, B_R)\, dx$.

Using the properties of the kernel $K_\nu(t)$ (see Proposition 1) and a rescaling argument we can easily check that

$$\mu^a_\varepsilon(B^i_{\varepsilon^\alpha} \times B^i_{\varepsilon^\alpha}) + \eta^a_\varepsilon(B^i_{\varepsilon^\alpha}) \leq \frac{\varepsilon}{|\log \varepsilon|} (D_\nu(a, B_R, B_{\varepsilon^{\alpha-1}}) + C\varepsilon^{4\alpha-1}) \,. \tag{104}$$

The choice of the exponent α is motivated by the following argument that will permit us to control the non local term in the energy and to show that long range interactions are negligible, i.e.,

$$\mu^a_\varepsilon((Q \times Q) \cap \{|x-y| > \varepsilon^\beta\}) \leq C a^2 \varepsilon^{(2\alpha-\beta-1)} \left(|Q| + \frac{L\varepsilon}{|\log \varepsilon|} \right) \,. \tag{105}$$

As a consequence, in view of (104) and (21), we get

$$\mu^a_\varepsilon(Q \times Q) + \eta^a_\varepsilon(Q) \leq \sum_{I_\varepsilon(Q)} \mu^a_\varepsilon(B^i_{\varepsilon^\alpha} \times B^i_{\varepsilon^\alpha}) + \eta^a_\varepsilon(B^i_{\varepsilon^\alpha}) + o(1) \leq \Lambda D_\nu(a, B_R) + o(1)$$

as ε goes to zero. Once (105) is shown, Step 1 is finished (see [8] for more details).

It only remains to prove (105). Indeed we will prove the following more general statement which will be useful later.

There exists a constant $C>0$ with the following property. If $\{w_\varepsilon\}$ and $\{z_\varepsilon\}$ are two sequences in $H^{\frac12}(Q)$ which are bounded in L^∞ by a constant M and satisfy $w_\varepsilon(x)=z_\varepsilon(x)=const.$ in $Q\setminus\cup_i B^i_{\varepsilon^\alpha}$, then for every $r\in[0,1]$

$$\frac{1}{|\log\varepsilon|}\iint_{\substack{Q_r\times Q_r\\|x-y|>\varepsilon^\beta}} K_\nu(x-y)|w_\varepsilon(x)-z_\varepsilon(y)|^2dx\,dy \le CM^2\varepsilon^{(2\alpha-\beta-1)}\left(|Q_r|+\frac{L\varepsilon}{|\log\varepsilon|}\right). \tag{106}$$

To show (106) it is enough to use the properties of the kernel K_ν and the uniform distribution of the obstacles (cf. (15)). We have

$$\begin{aligned}
\frac{1}{|\log\varepsilon|}\iint_{\substack{Q_r\times Q_r\\|x-y|>\varepsilon^\beta}} K_\nu(x-y)\quad&|w_\varepsilon(x)-z_\varepsilon(y)|^2dx\,dy\\
&\le 2\frac{1}{|\log\varepsilon|}\sum_i\int_{B^i_{\varepsilon^\alpha}}\int_{Q_r}\chi_{|x-y|>\varepsilon^\beta}\,K_\nu(x-y)\,|w_\varepsilon(x)-z_\varepsilon(y)|^2dx\,dy\\
&\le \frac{1}{|\log\varepsilon|}CM^2\#\mathcal{I}_\varepsilon(Q_r)\varepsilon^{2\alpha}\int_{B_{4r}\setminus B_{\varepsilon^\beta}}\frac{1}{|y|^3}\,dy\\
&\le CM^2\frac{(N_\varepsilon|Q_r|+L)}{|\log\varepsilon|}\frac{\varepsilon^{2\alpha}}{\varepsilon^\beta}\le CM^2\varepsilon^{(2\alpha-\beta-1)}\left(|Q_r|+\frac{L\varepsilon}{|\log\varepsilon|}\right).
\end{aligned}$$

Step 2. Let $u\in BV(T^2,\mathbf{Z})\cap L^\infty(T^2)$, with S_u polygonal with a finite number of sides. We will prove that for every $\delta>0$ there exists a sequence v^δ_ε converging to u strongly in $L^2(T^2)$ such that

$$\limsup_{\varepsilon\to0}F_\varepsilon(v^\delta_\varepsilon)\le\int_{S_u}\gamma(n)|[u]|d\mathcal{H}^1\;+\int_{T^2}(1-\chi_{\mathcal{N}_\delta})D_\nu(u(x),B_R)+\chi_{\mathcal{N}_\delta}|u(x)|^2D_\nu(1,B_R)\,dx\,, \tag{107}$$

where $\chi_{\mathcal{N}_\delta}$ denotes the characteristic function of the δ-neighbourhood of S_u.

By our choice of u there exist N integer number a_i, $i=1,...,N$, and N polygons P_i such that

$$u=\sum_{i=1}^N a_i\chi_{P_i}\,.$$

From Theorem 24 we find a sequence u_ε, obtained by convolution, such that $u_\varepsilon=u$ in $Q\setminus\mathcal{N}_\varepsilon$, where $\mathcal{N}_\varepsilon$ denotes an ε-neighbourhood of S_u, and

$$\lim_{\varepsilon\to0}\mu_\varepsilon(Q\times Q)=\lim_{\varepsilon\to0}\frac{1}{|\log\varepsilon|}\iint_{Q\times Q}K_\nu(x-y)|u_\varepsilon(x)-u_\varepsilon(y)|^2\,dx\,dy=\int_{S_u}\gamma(n)|[u]|d\mathcal{H}^1\,.$$

We must modify u_ε in order to let it satisfy the pinning condition. Thus for any point x^j_ε which is not in $\mathcal{N}_\delta$ we modify u_ε exactly as in the previous step using the $H^{\frac12}$-dislocation capacitary potential. If $x^j_\varepsilon\in\mathcal{N}_\delta$ we modify u_ε by multiplying it by an appropriate cut-off function, namely the scaled $H^{\frac12}$-dislocation capacitary potential at level 1. This permits in particular to achieve the pinning condition in $\mathcal{N}_\varepsilon$ where u_ε is not constant. Precisely we define the function w^δ_ε as follows

$$w^\delta_\varepsilon(x)=\begin{cases}a_i-\zeta^{a_i}_\varepsilon\left(\dfrac{x-x^j_\varepsilon}{\varepsilon}\right) & \text{if } x\in B_{\varepsilon^\alpha}(x^j_\varepsilon)\text{ with }x^j_\varepsilon\in P_i\setminus\mathcal{N}_\delta,\\[2ex] u_\varepsilon(x)\left(1-\zeta^1_\varepsilon\left(\dfrac{x-x^j_\varepsilon}{\varepsilon}\right)\right) & \text{if } x\in B_{\varepsilon^\alpha}(x^j_\varepsilon)\text{ with }x^j_\varepsilon\in\mathcal{N}_\delta,\\[2ex] u_\varepsilon(x) & \text{otherwise.}\end{cases} \tag{108}$$

Let us define again, as above, by $\widetilde{\mu}_\varepsilon$ and $\widetilde{\eta}_\varepsilon$ the following measures

$$\widetilde{\mu}_\varepsilon(A\times B):=\frac{1}{|\log\varepsilon|}\int_A\int_B K_\nu(x-y)|w^\delta_\varepsilon(x)-w^\delta_\varepsilon(y)|^2dx\,dy \tag{109}$$

and

$$\widetilde{\eta}_\varepsilon(A):=\frac{1}{\varepsilon|\log\varepsilon|}\int_A \mathrm{dist}(w^\delta_\varepsilon,\mathbf{Z})\,dx\,, \tag{110}$$

and let us denote by $\widetilde{\mu}$ and $\widetilde{\eta}$ their weak*-limits and by $\widetilde{\lambda}$ the measure such that $\widetilde{\lambda}(A)=\widetilde{\mu}(A\times A)$. For the sake of simplicity we drop the dependence on δ in the notation for the measures introduced above. We shall prove that

$$\widetilde{\lambda}+\widetilde{\mu}\le\Lambda[(1-\chi_{\mathcal{N}_\delta}(x))D_\nu(u(x),B_R)+\chi_{\mathcal{N}_\delta}(x)|u(x)|^2D_\nu(1,B_R)]dx+\gamma(n)|[u]|(x)d\mathcal{H}^1\llcorner S_u\,. \tag{111}$$

This clearly implies the assertion of Step 2.

First we prove that $\widetilde{\lambda+\mu}$ is absolutely continuous with respect to the Lebesgue measure outside S_u. Fix $x_0\in Q\setminus S_u$. Since for any $a\in\mathbf{Z}$ and $v\in\mathbf{R}$ we have that $\mathrm{dist}^2(av,\mathbf{Z})\le a^2\mathrm{dist}^2(v,\mathbf{Z})$, by the minimality of the $H^{\frac12}$-dislocation capacitary potentials it is easy to check that

$$D_\nu(a,B_R)\le a^2D_\nu(1,B_R)$$

and for all $x^i_\varepsilon\in Q_r(x_0)$ we have

$$\widetilde{\mu}_\varepsilon(B^i_{\varepsilon^\alpha}\times B^i_{\varepsilon^\alpha})+\widetilde{\eta}_\varepsilon(B^i_{\varepsilon^\alpha})\le|u(x_0)|^2\left(\mu^1_\varepsilon(B^i_{\varepsilon^\alpha}\times B^i_{\varepsilon^\alpha})+\eta^1_\varepsilon(B^i_{\varepsilon^\alpha})\right)$$

where μ^1_ε and η^1_ε have been defined in Step 1 (take $a=1$). Thus in view of (106), applied also for $w_\varepsilon=v^a_\varepsilon$ and $z_\varepsilon=av^1_\varepsilon$, we have

$$\widetilde{\mu}_\varepsilon(Q_r(x_0)\times Q_r(x_0))+\widetilde{\eta}_\varepsilon(Q_r(x_0))\le|u(x_0)|^2\left(\mu^1_\varepsilon(Q_r(x_0)\times Q_r(x_0))+\eta^1_\varepsilon(Q_r(x_0))\right)+o(1)\,.$$

Then taking the limit as ε goes to zero and using Step 1 we see that $\widetilde{\lambda}+\widetilde{\eta}$ is absolutely continuous with respect to the Lebesgue measure outside S_u and satisfies (111) in $\mathcal{N}_\delta\setminus S_u$. In the case $x_0\in Q\setminus\mathcal{N}_\delta$ we shall prove that

$$\widetilde{\lambda}(Q_r(x_0))+\widetilde{\eta}(Q_r(x_0))\le\Lambda D_\nu(u(x_0),B_R))|Q_r|+o(r^2) \tag{112}$$

as r goes to zero. This clearly follows as above by Step 1 and the fact that v_ε coincide with v^a_ε in a neighbourhood of x_0, where $a=u(x_0)$. For r small enough

$$\widetilde{\mu}_\varepsilon(Q_r(x_0)\times Q_r(x_0))+\widetilde{\eta}_\varepsilon(Q_r(x_0))=\mu^a_\varepsilon(Q_r(x_0)\times Q_r(x_0))+\eta^a(Q_r(x_0))\,.$$

Thus outside $\mathcal{N}_\delta$ inequality (111) follows from Step 1.

Let now consider $x_0\in S_u$. We will prove that

$$\limsup_{r\to0}\frac{\widetilde{\lambda}(Q_r(x_0))+\widetilde{\eta}(Q_r(x_0))}{r}\le\gamma(n)|[u](x_0)|\,. \tag{113}$$

It is easy to see that

$$\lim_{r\to0}\frac{\widetilde{\eta}(Q_r(x_0))}{r}=0\,.$$

Indeed, using a change of variable and the definition of w^δ_ε, in particular the fact that u_ε is constant in $Q\setminus\mathcal{N}_\varepsilon$, for any $r<\delta$, we get

$$\begin{aligned}\widetilde{\eta}_\varepsilon(Q_r(x_0)) &\le \widetilde{\eta}_\varepsilon(Q_r(x_0)\cap\mathcal{N}_\varepsilon)+\widetilde{\eta}_\varepsilon(Q_r(x_0)\setminus\mathcal{N}_\varepsilon)\\ &\le \frac{C\varepsilon2r}{\varepsilon|\log\varepsilon|}+\frac{1}{\varepsilon|\log\varepsilon|}\#\mathcal{I}_\varepsilon(Q_r)\varepsilon^2\int_{B_{\varepsilon^{\alpha-1}}}a^2\mathrm{dist}^2(\zeta^1_\varepsilon,\mathbf{Z})\,dx\le\Lambda a^2D_\nu(1,B_R)|Q_r|+o(1)\,,\end{aligned}$$

as ε goes to zero, where we denoted by a the maximum of $|u|$ in Q_r. Thus in order to prove (113) it is enough to show that

$$\limsup_{r\to 0}\frac{\widetilde{\lambda}(Q_r(x_0))}{r}\le \gamma(n)|[u](x_0)|\,. \tag{114}$$

For $r<\delta$ we have $Q_r\subseteq \mathcal{N}_\delta$. Since by definition $w_\varepsilon^\delta(x)=u_\varepsilon(x)v_\varepsilon^1(x)$ in $\mathcal{N}_\delta$, we have, for every $\sigma>0$,

$$\begin{aligned}
\widetilde{\mu}_\varepsilon(Q_r\times Q_r) &= \frac{1}{|\log\varepsilon|}\iint_{Q_r\times Q_r} K_\nu(x-y)|v_\varepsilon^1(x)u_\varepsilon(x)-v_\varepsilon^1(x)u_\varepsilon(y)+v_\varepsilon^1(x)u_\varepsilon(y)-v_\varepsilon^1(y)u_\varepsilon(y)|^2dx\,dy\\
&\le \frac{1}{|\log\varepsilon|}(1+\sigma)\iint_{Q_r\times Q_r} K_\nu(x-y)|v_\varepsilon^1(x)|^2|u_\varepsilon(x)-u_\varepsilon(y)|^2dx\,dy\\
&+ \frac{1}{|\log\varepsilon|}\left(1+\frac{1}{\sigma}\right)\iint_{Q_r\times Q_r} K_\nu(x-y)|u_\varepsilon(y)|^2|v_\varepsilon^1(x)-v_\varepsilon^1(y)|^2dx\,dy\\
&\le (1+\sigma)\mu_\varepsilon(Q_r\times Q_r)+\sup_Q|u|^2\left(1+\frac{1}{\sigma}\right)\mu_\varepsilon^1(Q_r\times Q_r)\,,
\end{aligned}$$

where in the last inequality we used the fact that v_ε^1 is bounded by 1 and $\sup|u_\varepsilon|\le\sup|u|$. Then (114) follows by the above estimate, taking the limit as ε goes to zero. Indeed, in view of Remark 26 and Step 1 we have

$$\frac{1}{r}\lim_{\varepsilon\to 0}\widetilde{\mu}_\varepsilon(Q_r\times Q_r)\le(1+\sigma)\frac{1}{r}\int_{Q_r\cap S_u}\gamma(n)|[u]|\,d\mathcal{H}^1+\left(1+\frac{1}{\sigma}\right)\Lambda\sup_Q|u|^2D_\nu(1,B_R)\frac{|Q_r|}{r}\,.$$

After taking the limit as r goes to zero the estimate (114), and hence (111) and (113), follows by the arbitrariness of σ.

Step 3. In this step we conclude the proof. First we can obtain the upper bound in the theorem for functions $u\in BV(T^2,\mathbf{Z})\cap L^\infty(T^2,\mathbf{Z})$, with S_u polygonal, with a finite number of sides by Step 2 and a diagonalization argument.

By estimate (20) the map $u\mapsto \int_Q D_\nu(u,B_R)\,dx$ is continuous in $L^2(Q,\mathbf{Z})$. This implies that for every $u\in BV(T^2,\mathbf{Z})$ there exist approximations $u^k\in BV(T^2,\mathbf{Z})\cap L^\infty(T^2,\mathbf{Z})$ with S_{u^k} polygonal with a finite number sides such that $u^k\to u$ in L^2 and $F(u^k)\to F(u)$ as $k\to\infty$. ○

7 The sub-critical and super-critical regimes

In this section we will briefly discuss the asymptotic behaviour of $E_\varepsilon(u)/(N_\varepsilon\varepsilon)$ in the sub-critical and in the super-critical regime. In the latter case we also assume $N_\varepsilon\ll\frac{1}{\varepsilon^{2\beta}}$, with $\beta<1$, in oder to keep the obstacles well separated (see condition (16)).

7.1 The sub-critical regime

In the case $N_\varepsilon\ll|\log\varepsilon|/\varepsilon$ the result can be deduced from Theorem 10. Indeed the compactness result given in Theorem 12 implies compactness in $L^q(T^2)$, for $q<2$, of any sequence $\{u_\varepsilon\}$ such that $\frac{E_\varepsilon(u_\varepsilon)}{\varepsilon N_\varepsilon}\le C$. Moreover every cluster point $u\in BV(T^2,\mathbf{Z})$ of such a sequence must satisfy $\int_{T^2}|Du|=0$ and hence is a constant $a\in\mathbf{Z}$. Thus also in this case, as for the dilute case discussed in Theorem 6, the effect of the pinning condition is weaker than the line tension and in a similar way one can deduce that the Γ-limit is given by

$$E(u)=\begin{cases} D_\nu(a,B_R) & \text{if } u=\text{const.}\in\mathbf{Z},\\ +\infty & \text{otherwise.}\end{cases}$$

7.2 The super-critical regime

We now consider the case $\frac{|\log\varepsilon|}{\varepsilon} \ll N_\varepsilon \ll \frac{1}{\varepsilon^{2\beta}}$. It is easy to check that also in this case the Poincaré inequality, as in Proposition 16, yields an $L^2(T^2)$ bound for all sequences with equibounded energy, but in general we cannot expect more than weak convergence up to a subsequence.

The upper bound in the proof of Corollary 11 can be obtained for the class of piecewise constant functions with integer values taking into account that in the super-critical scale the line tension effect becomes negligible. This class is weakly dense in $L^2(T^2,\mathbf{R})$ and then the general case follows by an energy density argument.

The proof of the lower bound is more delicate. By a blow-up argument it is enough to understand the case of a sequence $\{u_\varepsilon\}$ converging to a real constant c weakly in $L^2(T^2)$. The idea in this case is that even if the sequence converges only weakly, it oscillates at a larger scale than the average distance between the obstacles. Then at this scale, we still can find an integer value to which we can apply Lemma 22 and get a local lower bound for the energy.

Proposition 28 *If $u_\varepsilon \rightharpoonup c = \text{const.}$, with $c \in \mathbf{R}$, then*

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon N_\varepsilon} E_\varepsilon(u_\varepsilon) \geq D_\nu^{**}(c, B_R)\,, \tag{115}$$

where $D_\nu^{**}(\cdot, B_R)$ is the convex envelope of $D_\nu(\cdot, B_R)$ as defined in (18).

Proof. *Step 1 (Selection of good pinning sites x_ε^i).* We may assume that $\frac{E_\varepsilon(u_\varepsilon)}{\varepsilon N_\varepsilon} \leq C$. Thus for every $\theta \in (0,1)$ we can find a set of indices $\mathcal{I}_\varepsilon^\theta$ and a constant $C(\theta)$ such that $\#(\mathcal{I}_\varepsilon^\theta) \geq (1-\theta)N_\varepsilon$ and for all $i \in \mathcal{I}_\varepsilon^\theta$ the points x_ε^i satisfy

$$\frac{1}{\varepsilon}\int_{B^i_{\rho_\varepsilon}} \operatorname{dist}^2(u_\varepsilon, \mathbf{Z})\, dx \leq C(\theta)\varepsilon\,, \tag{116}$$

$$\iint_{B^i_{\rho_\varepsilon}\times B^i_{\rho_\varepsilon}} K_\nu(x-y)|u_\varepsilon(x) - u_\varepsilon(y)|^2 dx\, dy \leq C(\theta)\varepsilon \tag{117}$$

and

$$\fint_{B^i_{\rho_\varepsilon}} |u_\varepsilon|^2 dx \leq C(\theta)\,, \tag{118}$$

where $B^i_{\rho_\varepsilon}$ denotes the disc of radius $\rho_\varepsilon = 1/\sqrt{N_\varepsilon}$ and center x^i_ε.
Step 2 (Assignment of a value c^i_ε to each good pinning site). By the (scaling invariant) embedding of $H^{\frac{1}{2}}$ into L^4, there exist constants c^i_ε such that

$$\int_{B^i_{\rho_\varepsilon}} |u_\varepsilon - c^i_\varepsilon|^4 dx \leq C[u_\varepsilon]^4_{H^{\frac{1}{2}}(B^i_{\rho_\varepsilon})} \leq C\varepsilon^2\,.$$

Thus

$$\fint_{B^i_{\rho_\varepsilon}} |u_\varepsilon - c^i_\varepsilon|^4 dx \leq C N_\varepsilon \varepsilon^2 \to 0 \tag{119}$$

and

$$\fint_{B^i_{\rho_\varepsilon}} |u_\varepsilon - c^i_\varepsilon|\, dx \leq C(N_\varepsilon\varepsilon^2)^{\frac{1}{4}} \to 0\,.$$

By the interpolation inequality, given by Lemma 23, applied to $u_\varepsilon - c^i_\varepsilon$ we have

$$\fint_{B^i_{\varepsilon^\beta}} |u_\varepsilon - c^i_\varepsilon|\, dx \leq \fint_{B^i_{\rho_\varepsilon}} |u_\varepsilon - c^i_\varepsilon|\, dx + C\frac{\varepsilon^{\frac{1}{2}}}{\varepsilon^{\frac{\beta}{2}}} \to 0\,, \tag{120}$$

where $\beta < 1$ is chosen as in (16). From (119) we also deduce

$$\fint_{B^i_{\rho_\varepsilon}} |u_\varepsilon - c^i_\varepsilon|^2 dx \le C(N_\varepsilon \varepsilon^2)^{\frac{1}{2}} \to 0\,.$$

Together with (118) this yields

$$|c^i_\varepsilon| \le C\,. \tag{121}$$

Step 3 (Lower bound for each good pinning site). We claim that there exist integers a^i_ε such that

$$\sup_{i \in \mathcal{I}^\theta_\varepsilon} |c^i_\varepsilon - a^i_\varepsilon| \to 0 \tag{122}$$

and

$$\frac{1}{\varepsilon^2} \int_{B^i_{\varepsilon^\beta}} \mathrm{dist}^2(u_\varepsilon, \mathbf{Z})\, dx + \frac{1}{\varepsilon} \iint_{B^i_{\varepsilon^\beta} \times B^i_{\varepsilon^\beta}} K_\nu(x-y)|u_\varepsilon(x) - u_\varepsilon(y)|^2 dx\, dy \ge D_\nu(a^i_\varepsilon, B_R) - o(1)\,. \tag{123}$$

To see this we consider the scaled function $\zeta^i_\varepsilon(x) = u_\varepsilon(x^i_\varepsilon + \varepsilon x)$. Then (120) gives

$$\omega_1(\varepsilon) = \fint_{B_{\varepsilon^{\beta-1}}} |\zeta^i_\varepsilon - c^i_\varepsilon|\, dx \to 0 \tag{124}$$

uniformly in $i \in \mathcal{I}^\theta_\varepsilon$, and the left hand side of (123) can be written as

$$T^i_\varepsilon := \int_{B_{\varepsilon^{\beta-1}}} \mathrm{dist}^2(\zeta^i_\varepsilon, \mathbf{Z})\, dx + \iint_{B_{\varepsilon^{\beta-1}} \times B_{\varepsilon^{\beta-1}}} K^\varepsilon_\nu(x-y)|\zeta^i_\varepsilon(x) - \zeta^i_\varepsilon(y)|^2 dx\, dy\,.$$

By (116) and (117) we know that T^i_ε is bounded. Moreover by (124) the set

$$\{z \in B_{\varepsilon^{\beta-1}} \;:\; |\zeta^i_\varepsilon(z) - c^i_\varepsilon| < 2\omega_1(\varepsilon)\}$$

has measure at least $|B_{\varepsilon^{\beta-1}}|/2$. Hence since T^i_ε is bounded we have

$$\mathrm{dist}(c^i_\varepsilon, \mathbf{Z}) \le \omega_1(\varepsilon) + \frac{C}{|B_{\varepsilon^{\beta-1}}|^{\frac{1}{2}}} \to 0$$

This proves (122). In view of (124) we also get

$$\fint_{B_{\varepsilon^{\beta-1}}} |\zeta^i_\varepsilon - a^i_\varepsilon|\, dx \to 0\,.$$

Hence Lemma 22 and (121) yield

$$T^i_\varepsilon \ge D_\nu(a^i_\varepsilon, B_R) - o(1)\,.$$

Summation of (123) over the good centers yields

$$\frac{1}{\varepsilon N_\varepsilon} E_\varepsilon(u_\varepsilon) \ge (1-\theta) \frac{1}{\#(\mathcal{I}^\theta_\varepsilon)} \sum_{i \in \mathcal{I}^\theta_\varepsilon} D_\nu(a^i_\varepsilon, B_R) + o(1)\,. \tag{125}$$

Step 4 (Relation between the values a^i_ε assigned to the pinning sites and the weak limit c of u_ε). By (119) and (122) we see that u_ε is close to an (integer) constant on balls of radius $\frac{1}{\sqrt{N_\varepsilon}}$ centered on 'good' pinning sites. We now show that u_ε is actually nearly constant on a slightly larger scale r_ε. This will allow us to exploit the uniform distribution of the pinning sites to conclude that $\int_Q u_\varepsilon\, dx \approx \frac{1}{\#(\mathcal{I}^\theta_\varepsilon)} \sum_{i \in \mathcal{I}^\theta_\varepsilon} a^i_\varepsilon$. Since $\int_Q u_\varepsilon\, dx \to c$ the proof is then easily finished.

By assumption $N_\varepsilon \ll \varepsilon^{-2\beta}$. Thus there exists r_ε such that

$$\frac{1}{\sqrt{N_\varepsilon}} \ll r_\varepsilon \ll \frac{1}{\varepsilon^{\frac{1}{2}} N_\varepsilon^{\frac{3}{4}}} . \tag{126}$$

We assume that $\frac{1}{r_\varepsilon}$ is an integer and we cover Q by a lattice of squares $\widetilde{Q}^j_{r_\varepsilon}$ of size r_ε. By $\widehat{Q}^j_{r_\varepsilon}$ we denote the concentric squares of three times the size. Given θ we can find a set of indices $\mathcal{J}^\theta_\varepsilon$ such that $\#(\mathcal{J}^\theta_\varepsilon) \geq (1-\theta) r_\varepsilon^{-2}$ and for all $j \in \mathcal{J}^\theta_\varepsilon$ the squares $\widehat{Q}^j_{r_\varepsilon}$ satisfy

$$\int\!\!\int_{\widehat{Q}^j_{r_\varepsilon} \times \widehat{Q}^j_{r_\varepsilon}} K_\nu(x-y) |u_\varepsilon(x) - u_\varepsilon(y)|^2 dx\, dy \leq C(\theta) \varepsilon N_\varepsilon r_\varepsilon^2 , \tag{127}$$

$$\int_{\widehat{Q}^j_{r_\varepsilon}} |u_\varepsilon|^2 dx \leq C(\theta) r_\varepsilon^2 . \tag{128}$$

By the embedding of $H^{\frac{1}{2}}$ into L^4, there exist constants A^j_ε such that

$$⨍_{\widetilde{Q}^j_{r_\varepsilon}} |u_\varepsilon - A^j_\varepsilon|^4 dx \leq C \varepsilon^2 N_\varepsilon^2 r_\varepsilon^2 \to 0 .$$

Now consider a good pinning site in a good square, i.e., $x^i_\varepsilon \in \widetilde{Q}^j_{r_\varepsilon}$, $i \in \mathcal{I}^\theta_\varepsilon$ and $j \in \mathcal{J}^\theta_\varepsilon$. Then by the interpolation inequality (applied to $u_\varepsilon - A^j_\varepsilon$) and (126)

$$⨍_{B^i_{\rho_\varepsilon}} |u_\varepsilon - A^j_\varepsilon| dx \leq ⨍_{\widetilde{Q}^j_{r_\varepsilon}} |u_\varepsilon - A^j_\varepsilon|\, dx + C \frac{\varepsilon^{\frac{1}{2}} N_\varepsilon^{\frac{1}{2}} r_\varepsilon}{N_\varepsilon^{-\frac{1}{4}}} \leq C o(1) .$$

Thus in view of (119) and (122)

$$|a^i_\varepsilon - A^j_\varepsilon| \leq C o(1) .$$

Since $a^i_\varepsilon \in \mathbf{Z}$ this shows that for good pinning sites in good squares a^i_ε depends only on the square $\widetilde{Q}^j_{r_\varepsilon}$.

Using the uniform distribution of x^i_ε and the fact that $r_\varepsilon \gg N_\varepsilon^{-\frac{1}{2}}$ as well as (118) we deduce that

$$\left| \frac{1}{N_\varepsilon} \sum_{i \in \mathcal{I}^\theta_\varepsilon ,\ x^i_\varepsilon \in \widetilde{Q}^j_{r_\varepsilon}} a^i_\varepsilon - \int_{\widetilde{Q}^j_{r_\varepsilon}} u_\varepsilon\, dx \right| \leq C |\widetilde{Q}^j_{r_\varepsilon}| o(1) + C \frac{\#(\mathcal{I}_\varepsilon(\widetilde{Q}^j_{r_\varepsilon}) \setminus \mathcal{I}^\theta_\varepsilon)}{N_\varepsilon} . \tag{129}$$

Now we sum (129) over all $j \in \mathcal{J}^\theta_\varepsilon$. Let E_ε be the union of the squares $\widetilde{Q}^j_{r_\varepsilon}$, with $j \notin \mathcal{J}^\theta_\varepsilon$. Then $|E_\varepsilon| \leq \theta$ and by the L^2 bound on u_ε we have $\int_{E_\varepsilon} |u_\varepsilon|\, dx \leq C \theta^{\frac{1}{2}}$. Combining this with (121) and (122) we deduce after a short calculation that

$$\lim_{\theta \to 0} \lim_{\varepsilon \to 0} \left| \frac{1}{\#(\mathcal{I}^\theta_\varepsilon)} \sum_{i \in \mathcal{I}^\theta_\varepsilon} a^i_\varepsilon - c \right| = \lim_{\theta \to 0} \lim_{\varepsilon \to 0} \left| \frac{1}{\#(\mathcal{I}^\theta_\varepsilon)} \sum_{i \in \mathcal{I}^\theta_\varepsilon} a^i_\varepsilon - \int_Q u_\varepsilon\, dx \right| = 0$$

Together with (125) and the inequality

$$\lim_{\theta \to 0} \lim_{\varepsilon \to 0} \frac{1}{\#(\mathcal{I}^\theta_\varepsilon)} \sum_{i \in \mathcal{I}^\theta_\varepsilon} D_\nu(a^i_\varepsilon, R) \geq \lim_{\theta \to 0} \lim_{\varepsilon \to 0} \frac{1}{\#(\mathcal{I}^\theta_\varepsilon)} \sum_{i \in \mathcal{I}^\theta_\varepsilon} D^{**}_\nu(a^i_\varepsilon, R) \geq D^{**}_\nu(c, R)$$

this finishes the proof. ○

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