Analysis of a unilateral contact problem taking into account adhesion and friction

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Abstract

In this paper, we investigate a contact problem between a viscoelastic body and a rigid foundation, when both the effects of the (irreversible) adhesion and of the friction are taken into account. We describe the adhesion phenomenon in terms of a damage surface parameter according to FRÉMOND's theory, and we model the unilateral contact by Signorini conditions and the friction by a *nonlocal* Coulomb law. All the constraints on the internal variables as well as the contact and the friction conditions are rendered by means of subdifferential operators, whence the highly nonlinear character of the resulting PDE system. Our main result states the existence of a global-in-time solution (to a suitable variational formulation) of the related Cauchy problem. It is proved by an approximation procedure combined with time discretization.

Key words: contact, adhesion, friction, irreversibility, existence.

AMS (MOS) Subject Classification: 35K55, 74A15, 74M15.

1 Introduction

The present analysis is concerned with a highly nonlinear PDE system describing a unilateral contact problem between a viscoelastic body and a support, when the effects of adhesion and of friction are simultaneously taken into account. Referring (see [15]) to FRÉMOND's modeling approach to adhesive contact, we aim to include into the description of the system evolution additional surface effects due to friction. In the papers [4] and [5], frictionless adhesive contact problems have been addressed and global-in-time well-posedness results have been proved. Furthermore, in [5] the long-time behavior of the solutions has been investigated. Then, a model for frictionless adhesive contact, encompassing thermal effects, has been analyzed in the recent contributions [6, 7].

The focus of the present paper is to generalize the model introduced in [4], taking into account both the adhesive and the frictional effects, and to investigate the well-posedness of the related PDE system.

1.1 Derivation of the model and the PDE system

Let us introduce the model and the corresponding initial and boundary-value problem we are dealing with. Let $\Omega \subset \mathbb{R}^3$ a smooth bounded domain, representing the reference configuration of a viscoelastic body which may be in contact with a rigid foundation on a part Γ_c of its boundary. Concerning the surface $\Gamma = \partial\Omega$ of the body, we assume $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_c$, where the sets Γ_i, $i = 1, 2$, are open

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subsets in the relative topology of $\partial\Omega$, with smooth boundary and disjoint one from each other. In particular, the body is fixed on Γ_1, a known traction acts on Γ_2, and Γ_c is the contact surface. We suppose that Γ_c and Γ_1 have strictly positive measures and, without loss of generality, we identify Γ_c with a subset of $\mathbb{R}^2$, i.e., we treat Γ_c as a flat surface.

To describe the mechanical behavior of the system during the time interval $[0,T]$, $T>0$, we introduce the variables of the model, some of which are defined on the domain Ω, while others only on the contact surface Γ_c. We work under the small deformation assumption. As state variables we consider the symmetric linearized strain tensor $\varepsilon(\mathbf{u})$ ($\mathbf{u}$ represents the vector of small displacements), and a surface damage parameter χ related to the state of the bonds responsible for adhesion. For example, if the contact surface is covered by glue, χ describes the state of the fibers of the glue. In particular, χ denotes the fraction of active bonds and satisfies the constraint $\chi \in [0,1]$, the value $\chi = 0$ corresponding to the case of completely broken fibers (no adhesion) and $\chi = 1$ to unbroken bonds (active contact). Taking into account local interactions (in the adhesive and between the adhesive substance and the body), we include the gradient $\nabla\chi$ and the displacement trace $\mathbf{u}_{|\Gamma_c}$ among the state variables on the contact surface.

Now, let us introduce the balance equations. Neglecting any acceleration effect, we consider the quasistatic equation

$$-\operatorname{div}\boldsymbol{\sigma} = \mathbf{f} \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

supplemented by the following boundary conditions

$$\mathbf{u} = \mathbf{0} \text{ in } \Gamma_1 \times (0,T), \quad \boldsymbol{\sigma}\mathbf{n} = \mathbf{g} \text{ in } \Gamma_2 \times (0,T), \tag{1.2}$$

$$\boldsymbol{\sigma}\mathbf{n} + \mathbf{R} = \mathbf{0} \text{ in } \Gamma_c \times (0,T), \tag{1.3}$$

where $\boldsymbol{\sigma}$ denotes the stress tensor, $\mathbf{R}$ the reaction on the contact surface, $\mathbf{f}$ a volume force, $\mathbf{g}$ an assigned traction, and $\mathbf{n}$ the outward unit normal vector to the boundary Γ. Moreover, on the contact surface Γ_c we introduce the following balance equation for microscopic movements

$$B - \operatorname{div}\mathbf{H} = 0 \text{ in } \Gamma_c \times (0,T), \quad \mathbf{H}\cdot\mathbf{n}_s = 0 \text{ in } \partial\Gamma_c \times (0,T), \tag{1.4}$$

B, $\mathbf{H}$ representing interior forces, responsible for the damage of the adhesive bonds between the body and the support, and $\mathbf{n}_s$ the outward unit normal vector to $\partial\Gamma_c$. We refer to [15, 18] for a detailed derivation of (1.1)-(1.4), by a generalized version of the principle of virtual power. Finally, let us point out that, within this paper we do not consider possible damage phenomena occurring in the domain Ω, which was instead done in [8] for a (frictionless) model of adhesive contact.
Constitutive relations for $\boldsymbol{\sigma}$, $\mathbf{R}$, B, and $\mathbf{H}$, split into dissipative and non-dissipative contributions, are recovered from the pseudo-potential and the free energy functionals we are going to introduce.

Notation 1.1 To simplify notation, hereafter, upon considering the trace $\mathbf{u}_{|\Gamma_c}$ of $\mathbf{u}$ on the contact surface Γ_c, we shall avoid the index $|_{\Gamma_c}$, if it is clear from the context that we are working on Γ_c.

Notation 1.2 We shall use the following decompositions of vectors and tensors on Γ_c. We denote by v_{N} and $\mathbf{v}_{\mathrm{T}}$ the normal component and the tangential part of $\mathbf{v}$, defined on Γ_c by

$$v_{\mathrm{N}} := \mathbf{v}\cdot\mathbf{n}, \quad \mathbf{v}_{\mathrm{T}} := \mathbf{v} - v_{\mathrm{N}}\mathbf{n}. \tag{1.5}$$

Analogously, the normal component and the tangential part of the stress tensor $\boldsymbol{\sigma}$ are denoted by σ_{N} and $\boldsymbol{\sigma}_{\mathrm{T}}$, and they are defined by

$$\sigma_{\mathrm{N}} := \boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{n}, \quad \boldsymbol{\sigma}_{\mathrm{T}} := \boldsymbol{\sigma}\mathbf{n} - \sigma_{\mathrm{N}}\mathbf{n}. \tag{1.6}$$

Energy and dissipation functionals and constitutive laws The free energy of the system is given by a volume part Ψ_Ω and a surface one Ψ_{Γ_c}. The volume contribution Ψ_Ω is

$$\Psi_\Omega := \frac{1}{2}\varepsilon(\mathbf{u})K\varepsilon(\mathbf{u}), \tag{1.7}$$

K denoting the elasticity tensor, while the contact surface free energy Ψ_{Γ_c} is chosen as follows

$$\Psi_{\Gamma_c} := \frac{c_{\mathrm{N}}}{2}\chi(u_{\mathrm{N}})^2 + \frac{c_{\mathrm{T}}}{2}\chi|\mathbf{u}_{\mathrm{T}}|^2 + I_{(-\infty,0]}(u_{\mathrm{N}}) + w_s(1-\chi) + \frac{k_s}{2}|\nabla\chi|^2 + I_{[0,1]}(\chi). \tag{1.8}$$

The constants $c_{\mathrm{N}}, c_{\mathrm{T}}, w_s, k_s$ are positive. In particular, c_{N} and c_{T} are the adhesive coefficients for the normal and tangential components, respectively. Note that a priori these coefficients may be different, due to possible anisotropy in the material responses to stresses. The positive constant w_s is related to the internal cohesion of the glue. Indeed, it favors values of χ close to 1 (viz., undamaged glue). Let us now briefly comment on internal constraints. The indicator function $I_{(-\infty,0]}$ enforces the internal constraint $u_{\mathrm{N}} \leq 0$, i.e. it renders the impenetrability condition between the body and the support. Finally, the term $I_{[0,1]}(\chi)$ forces χ to assume only physically admissible values, i.e. $\chi \in [0,1]$.

The non-dissipative contributions for the stress tensor, the internal forces, and the reaction are specified by

$$\boldsymbol{\sigma}^{nd} = \frac{\partial \Psi_\Omega}{\partial \varepsilon(\mathbf{u})}, \tag{1.9}$$

$$B^{nd} = \frac{\partial \Psi_{\Gamma_c}}{\partial \chi}, \quad \mathbf{H}^{nd} = \frac{\partial \Psi_{\Gamma_c}}{\partial \nabla\chi}, \tag{1.10}$$

$$R_{\mathrm{N}}^{nd} = \frac{\partial \Psi_{\Gamma_c}}{\partial u_{\mathrm{N}}}, \quad \mathbf{R}_{\mathrm{T}}^{nd} = \frac{\partial \Psi_{\Gamma_c}}{\partial \mathbf{u}_{\mathrm{T}}}. \tag{1.11}$$

Note that we have split the (non-dissipative) reaction into its normal and tangential components. The derivatives in the constitutive relations (1.10) and (1.11) are taken with respect to (the components of) the traces of $\mathbf{u}$ on Γ_c, even if it is not specified (see Notation 1.1).

Now, we describe the dissipation of the system by a pseudo-potential of dissipation (see (1.12)-(1.13) below), in terms of the dissipative variables $\varepsilon(\partial_t\mathbf{u})$ and $\partial_t\chi$. Concerning the evolution of the adhesion, we prescribe that it is dissipative and irreversible. The resulting pseudo-potential of dissipation is written for a volume part Φ_Ω and a contact one Φ_{Γ_c}, defined on Γ_c. More precisely, we assume

$$\Phi_\Omega := \frac{1}{2}\varepsilon(\partial_t\mathbf{u})K_v\varepsilon(\partial_t\mathbf{u}), \tag{1.12}$$

where K_v denotes the viscosity tensor, and

$$\Phi_{\Gamma_c} := \nu\,|-R_N + u_{\mathrm{N}}\chi|\,j(\partial_t\mathbf{u}) + \frac{c_s}{2}|\partial_t\chi|^2 + I_{(-\infty,0]}(\partial_t\chi), \tag{1.13}$$

where c_s is a positive constant, $\nu > 0$ denotes the friction coefficient, and

$$j(\mathbf{v}) = |\mathbf{v}_{\mathrm{T}}| \quad \text{for all } \mathbf{v} \in \mathbb{R}^3. \tag{1.14}$$

Observe that we are considering an *irreversible* damaging process, as we force $\partial_t\chi \leq 0$ with the indicator function $I_{(-\infty,0]}(\partial_t\chi)$. Hence, the choice of the function j reflects the rate-independent character of frictional dissipation.
The frictional contribution in (1.13) has been chosen according to [26], where a model (close to the present one) describing a unilateral contact problem coupling friction and adhesion has been derived. Let us point out that our choice of the constitutive relations and the fact that the dissipation functional is a pseudo-potential of dissipation (i.e. non-negative, convex w.r.t. the dissipative variables and attaining its minimum when the dissipative variables are 0), ensure the thermodynamical consistency of the model. The constitutive relations for the dissipative contributions are given by

$$\boldsymbol{\sigma}^{d} = \frac{\partial \Phi_\Omega}{\partial \varepsilon(\partial_t\mathbf{u})}, \tag{1.15}$$

$$B^{d} = \frac{\partial \Phi_{\Gamma_c}}{\partial(\partial_t\chi)}, \quad \mathbf{H}^{d} = \mathbf{0}, \tag{1.16}$$

$$R_{\mathrm{N}}^{d} = 0, \quad \mathbf{R}_{\mathrm{T}}^{d} = \frac{\partial \Phi_{\Gamma_c}}{\partial(\partial_t\mathbf{u}_{\mathrm{T}})}. \tag{1.17}$$

As far as the dissipative contributions of the reaction, note that they are involved on its tangential component and not on the normal one.

From now on, to simplify the presentation, but without loss of generality, we take the physical constants $c_{\mathrm{N}} = c_{\mathrm{T}} = c_s = k_s = 1$. Substituting (1.8) into (1.11), and (1.13) into (1.17), we obtain

$$R_{\mathrm{N}} = R_{\mathrm{N}}^{nd} \in \chi u_{\mathrm{N}} + \partial I_{(-\infty,0]}(u_{\mathrm{N}}), \tag{1.18}$$

$$\mathbf{R}_{\mathrm{T}} = \mathbf{R}_{\mathrm{T}}^{nd} + \mathbf{R}_{\mathrm{T}}^{d} \in \chi \mathbf{u}_{\mathrm{T}} + \nu \partial I_{(-\infty,0]}(u_{\mathrm{N}}) \mathbf{d}(\partial_t \mathbf{u}), \tag{1.19}$$

where $\partial I_{(-\infty,0]}$ denotes the subdifferential of $I_{(-\infty,0]}$, while $\mathbf{d} : \mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ the subdifferential of the function j (1.14), namely

$$\mathbf{d}(\mathbf{v}) = \begin{cases} \frac{\mathbf{v}_{\mathrm{T}}}{|\mathbf{v}_{\mathrm{T}}|} & \text{if } \mathbf{v}_{\mathrm{T}} \neq \mathbf{0} \\ \{\mathbf{w}_{\mathrm{T}} \,:\, \mathbf{w} \in \overline{B}_1\} & \text{if } \mathbf{v}_{\mathrm{T}} = \mathbf{0}, \end{cases} \tag{1.20}$$

with $\overline{B}_1$ the closed unit ball in $\mathbb{R}^3$. Similarly, computing (1.9) and (1.15), (1.10) and (1.16), we can write system (1.1)-(1.4) in the following form

$$-\operatorname{div}\,(K\varepsilon(\mathbf{u}) + K_v \varepsilon(\partial_t \mathbf{u})) = \mathbf{f} \quad \text{in } \Omega \times (0,T), \tag{1.21}$$

$$\mathbf{u} = \mathbf{0} \quad \text{in } \Gamma_1 \times (0,T), \quad (K\varepsilon(\mathbf{u}) + K_v \varepsilon(\partial_t \mathbf{u}))\mathbf{n} = \mathbf{g} \quad \text{in } \Gamma_2 \times (0,T), \tag{1.22}$$

$$\sigma_{\mathrm{N}} \in -\chi u_{\mathrm{N}} - \partial I_{(-\infty,0]}(u_{\mathrm{N}}) \quad \text{in } \Gamma_c \times (0,T), \tag{1.23}$$

$$\boldsymbol{\sigma}_{\mathrm{T}} \in -\chi \mathbf{u}_{\mathrm{T}} - \nu \partial I_{(-\infty,0]}(u_{\mathrm{N}}) \mathbf{d}(\partial_t \mathbf{u}) \quad \text{in } \Gamma_c \times (0,T), \tag{1.24}$$

$$\partial_t \chi - \Delta \chi + \partial I_{(-\infty,0]}(\partial_t \chi) + \partial I_{[0,1]}(\chi) \ni w_s - \frac{1}{2}|\mathbf{u}|^2 \quad \text{in } \Gamma_c \times (0,T), \tag{1.25}$$

$$\partial_{\mathbf{n}_s} \chi = 0 \quad \text{in } \partial\Gamma_c \times (0,T). \tag{1.26}$$

Before proceeding, let us comment on the meaning of conditions expressed by (1.23)-(1.24). Indeed, (1.23) can be rephrased as

$$u_{\mathrm{N}} \leq 0, \quad \sigma_{\mathrm{N}} + \chi u_{\mathrm{N}} \leq 0, \quad u_{\mathrm{N}}(\sigma_{\mathrm{N}} + \chi u_{\mathrm{N}}) = 0, \tag{1.27}$$

which, in the case $\chi = 0$, reduce to the classical Signorini conditions. Conversely, when $0 < \chi \leq 1$, σ_{N} can be positive, namely the adhesive action of the glue prevents the separation when a tension is applied.

Moreover, relations (1.23)-(1.24) can be expressed by

$$|\boldsymbol{\sigma}_{\mathrm{T}} + \chi \mathbf{u}_{\mathrm{T}}| \leq \nu |\sigma_{\mathrm{N}} + \chi u_{\mathrm{N}}|, \tag{1.28}$$

$$|\boldsymbol{\sigma}_{\mathrm{T}} + \chi \mathbf{u}_{\mathrm{T}}| < \nu |\sigma_{\mathrm{N}} + \chi u_{\mathrm{N}}| \Longrightarrow \partial_t \mathbf{u}_{\mathrm{T}} = \mathbf{0}, \tag{1.29}$$

$$|\boldsymbol{\sigma}_{\mathrm{T}} + \chi \mathbf{u}_{\mathrm{T}}| = \nu |\sigma_{\mathrm{N}} + \chi u_{\mathrm{N}}| \Longrightarrow \exists \lambda \geq 0 : \ \partial_t \mathbf{u}_{\mathrm{T}} = -\lambda(\boldsymbol{\sigma}_{\mathrm{T}} + \chi \mathbf{u}_{\mathrm{T}}), \tag{1.30}$$

which generalize the *dry friction* Coulomb law, to the case when adhesion effects are taken into account.

1.2 Related literature and our own results

In the present paper we study the Cauchy problem related to a generalized version of system (1.21)-(1.26), where the subdifferentials in (1.23)-(1.25) are replaced by general maximal monotone operators, still enforcing the physical constraints on the variables. Actually, a relevant peculiarity of our system is the fact that all the constraints on the internal variables, as well as the unilateral contact conditions and the friction law, are rendered by means of subdifferential operators. On the one hand, let us point out that the resulting formulation of the problem, in comparison with those based on variational inequalities, enables us to clearly identify forces and reactions. This fact turns out to be of particular interest as it allows to give a physical and consistent meaning to internal forces and reactions (coming by physical constraints). On the other hand, the associated PDE system is highly nonlinear: the simultaneous presence of several multivalued operators yields, in particular, the doubly nonlinear structure of (1.24) and (1.25). Furthermore, the coupling of (1.23)-(1.24) (unilateral contact and dry friction Coulomb law) introduces severe mathematical difficulties which, to our knowledge, remain unresolved, even in the case without adhesion. The main difficulty in the analysis is brought forth by a lack of spatial regularity for $\partial_t \mathbf{u}$. Indeed, note in particular that, due to the non-smooth boundary conditions, we are not in the position of applying elliptic regularity results to (1.21), and deduce a $H^2(\Omega;\mathbb{R}^3)$-regularity for $\mathbf{u}$ and $\partial_t \mathbf{u}$. Consequently, we are not able to control the reaction pointwise. Furthermore, the tangential component of the reaction features the product of two multivalued operators, cf. (1.19) and (1.24). That is why, we are led to regularize it and actually work with a *nonlocal version* of the Coulomb law. The idea is to replace in (1.28)–(1.30) the frictional term

$$\nu|\sigma_{\rm N} + \chi u_{\rm N}| \ \text{ with } \ |\mathcal{R}(\sigma_{\rm N} + \chi u_{\rm N})|, \tag{1.31}$$

or, equivalently, in (1.24) the term $\nu\partial I_{(-\infty,0]}(u_{\rm N})$ with $\mathcal{R}(\partial I_{(-\infty,0]}(u_{\rm N})))$. In (1.31), $\mathcal{R}$ is a regularization operator, taking into account nonlocal interactions on the contact surface. We shall specify later our assumptions on $\mathcal{R}$ (cf. (H2)), and construct explicitly an admissible operator in Example 2.4. It is evident from the analysis we shall develop later on, that the regularization through $\mathcal{R}$ is crucial for the mathematical treatment of the system. We also point out that this choice has been widely adopted in the literature, starting from the paper [13], see also below.

Our main result, Theorem 2.1 below, states the existence of a global-in-time solution to the Cauchy problem related for (a generalized version of) system (1.21)-(1.26), with the regularization (1.31). The proof of Thm. 2.1 relies on a suitable regularization/approximation procedure. Namely, we set up a time-discretization scheme for system (1.21)-(1.26), where we also suitably regularize some of the nonlinear operators therein. We construct discrete solutions by time-incremental minimization, and then define approximate solutions via interpolation of the discrete ones. The passage to the limit in the time-discrete scheme exploits compactness and monotonicity techniques.

As far as uniqueness of solutions is concerned, we point out that it remains an open question, essentially due to the doubly non linear character of (1.24) and (1.25). Actually, uniqueness holds in the more regular framework of the approximated problem (see Proposition 5.1).

We conclude with a short, and with no aim at completeness, review of some results in the literature on contact problems with adhesion and friction. First of all, we recall [10, 11, 26], based on Frémond's model (see [15, 16, 17]). In the already quoted paper [26], a consistent model describing unilateral contact, adhesion and friction is originally derived in the framework of continuum thermodynamics. The related quasistatic problem is written as the coupling between two variational inequalities and a first-order ODE (local interactions in the glue are neglected). Under a smallness hypothesis on the friction coefficient, an existence result for an incremental formulation of the problem is proved and some numerical schemes are given. In [10, 11], based on [26], contact problems with adhesion and friction are considered in the quasistatic elastic case and the dynamic viscoelastic case, respectively. In the latter, reversible adhesion and *nonlocal* friction (a regularization operator on the reaction is considered) are analyzed, an existence result for the related variational formulation is proved and some numerical examples are presented.

As already mentioned, a useful tool to overcome some of the analytical difficulties connected to the coupling between Signorini conditions and Coulomb law, is the regularization of the reaction by an appropriate smoothing (and physically meaningful) operator, actually adopting a nonlocal friction law. This idea, dating back to [13], has been exploited in several papers dealing with (adhesionless) static, quasistatic and dynamic contact problems (see, e.g., [12, 21, 25]). An other approximation tool frequently used to describe contact conditions, especially in the investigation of dynamic problems, is the *normal compliance law* (introduced in [24]). Allowing for the interpenetration of the surface asperities, such a law actually permits to dispense with the unilateral constraint on $u_{\rm N}$. From the analytical point of view, the normal compliance condition may be regarded as a regularization of the Signorini conditions by a penalization argument. In fact, we shall employ it in the formulation of our approximate problem, see (3.2)–(3.7) below. Among the others, we quote [1, 20, 27] and [24, 22] where (adhesionless) frictional contact problems based on normal compliance models have been extensively studied in the quasistatic and in the dynamic case, respectively.

Finally, we mention the approach by C. Eck & J. Jarušek, see the monograph [14] and the references therein. They prove existence results for dynamical contact problems, coupling *dry* friction and Signorini contact, *without* recurring to any regularizing operator. However, they use a different form of Signorini conditions, expressed not in terms of $\mathbf{u}$ but of $\partial_t \mathbf{u}$. They claim that this has still some physical interest. In this setting, with the use of refined techniques they are able to obtain enhanced regularity estimates on $\partial_t \mathbf{u}$, which allow them to deal with the *dry* Coulomb law.

Finally, for a review of the theory of (frictionless) contact problems with damage and adhesion we refer, e.g., to the monograph [29], and the references therein.

Plan of the paper. In Sec. 2 we set up the problem, state all of our assumptions, and the main existence result. We devise the time-discretization scheme (featuring a suitable approximation of some of the nonlinear operators) in Sec. 3, while in Sec. 4 we obtain a priori estimates on the approximate solutions. Finally, in Sec. 5 we conclude the proof of Theorem 2.1, by passing to the limit in the approximation scheme. We also prove in Proposition 5.1 a uniqueness result for a (time-continuous) approximation of system (1.21)-(1.26).

2 Main results

2.1 Set-up

We recall that, throughout the paper we shall assume that Ω is a bounded smooth set of $\mathbb{R}^3$, such that Γ_c is a smooth bounded domain of $\mathbb{R}^2$ (one may think of a flat surface). We shall also refer to the following

Notation 2.1 Given a Banach space X, we denote by $\langle \cdot, \cdot \rangle_X$ the duality pairing between its dual space X' and X itself, by $\| \cdot \|_X$ both the norm in X and in any power of it. In particular, we shall use short-hand notation for some function spaces

$$\mathbf{H} = L^2(\Omega; \mathbb{R}^3), \quad \mathbf{V} = H^1(\Omega; \mathbb{R}^3), \quad \mathbf{W} := \{\mathbf{v} \in \mathbf{V} \, : \, \mathbf{v} = \mathbf{0} \text{ a.e. on } \Gamma_1\}$$

$$\mathbf{H}_{\Gamma_c} := L^2(\Gamma_c; \mathbb{R}^3), \quad \mathbf{Y}_{\Gamma_c} = H^{1/2}_{00,\Gamma_1}(\Gamma_c; \mathbb{R}^3),$$

$$H_{\Gamma_c} = L^2(\Gamma_c), \quad V_{\Gamma_c} = H^1(\Gamma_c), \quad Y_{\Gamma_c} = H^{1/2}_{00,\Gamma_1}(\Gamma_c),$$

where we recall that the space $\mathbf{Y}_{\Gamma_c}$ is defined by

$$H^{1/2}_{00,\Gamma_1}(\Gamma_c; \mathbb{R}^3) = \left\{ \mathbf{w} \in H^{1/2}(\Gamma_c; \mathbb{R}^3) \, : \, \exists \, \tilde{\mathbf{w}} \in H^{1/2}(\Gamma; \mathbb{R}^3) \text{ with } \tilde{\mathbf{w}} = \mathbf{w} \text{ in } \Gamma_c, \, \tilde{\mathbf{w}} = \mathbf{0} \text{ in } \Gamma_1 \right\}.$$

The space $\mathbf{W}$ is endowed with the natural norm induced by $\mathbf{V}$.

Preliminaries of elasticity theory. We recall the definition of the standard bilinear forms of linear viscoelasticity, which are involved in the variational formulation of equation (1.21). Dealing with an anisotropic and inhomogeneous material, we assume that the fourth-order tensors $K = (a_{ijkh})$ and $K_v = (b_{ijkh})$, denoting the elasticity and the viscosity tensor, respectively, satisfy the classical symmetry and ellipticity conditions

$$\begin{aligned}
& a_{ijkh} = a_{jikh} = a_{khij} \, , \qquad b_{ijkh} = b_{jikh} = b_{khij} \, , \quad i,j,k,h = 1,2,3 \\
& \exists \, \alpha_0 > 0 \, : \qquad a_{ijkh}\xi_{ij}\xi_{kh} \geq \alpha_0 \xi_{ij}\xi_{ij} \quad \forall \, \xi_{ij} : \xi_{ij} = \xi_{ji} \, , \quad i,j = 1,2,3 \, , \\
& \exists \, \beta_0 > 0 \, : \qquad b_{ijkh}\xi_{ij}\xi_{kh} \geq \beta_0 \xi_{ij}\xi_{ij} \quad \forall \, \xi_{ij} : \xi_{ij} = \xi_{ji} \, , \quad i,j = 1,2,3 \, ,
\end{aligned}$$

where the usual summation convention is used. Moreover, we require

$$a_{ijkh}, b_{ijkh} \in L^\infty(\Omega) \, , \quad i,j,k,h = 1,2,3.$$

By the previous assumptions on the elasticity and viscosity coefficients, the following bilinear forms $a, b : \mathbf{W} \times \mathbf{W} \to \mathbb{R}$, defined by

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega a_{ijkh} \varepsilon_{kh}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{W},$$
$$b(\mathbf{u}, \mathbf{v}) := \int_\Omega b_{ijkh} \varepsilon_{kh}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{W}$$

turn out to be continuous and symmetric. In particular, we have

$$\exists\, M > 0 : \ |a(\mathbf{u}, \mathbf{v})| + |b(\mathbf{u}, \mathbf{v})| \leq M \|\mathbf{u}\|_{\mathbf{W}} \|\mathbf{v}\|_{\mathbf{W}} \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbf{W}. \tag{2.1}$$

Moreover, since Γ_1 has positive measure, by Korn's inequality we deduce that $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are $\mathbf{W}$-elliptic, i.e., there exist $C_a, C_b > 0$ such that

$$a(\mathbf{u}, \mathbf{u}) \geq C_a \|\mathbf{u}\|^2_{\mathbf{W}} \qquad \text{for all } \mathbf{u} \in \mathbf{W}, \tag{2.2}$$
$$b(\mathbf{u}, \mathbf{u}) \geq C_b \|\mathbf{u}\|^2_{\mathbf{W}} \qquad \text{for all } \mathbf{u} \in \mathbf{W}. \tag{2.3}$$

2.2 Assumptions and variational formulation of the problem

Assumptions on the nonlinearities in the PDE system. We first generalize the constraint on u_{N} in the free energy (1.8). So, let us consider a function

$$\phi : \mathbb{R} \to [0, +\infty] \ \text{ proper, convex and lower semicontinuous, with } \phi(0) = 0, \tag{H1}$$

with effective domain $\mathrm{dom}(\phi)$. Then, we define

$$\varphi : Y_{\Gamma_c} \to [0, +\infty] \ \text{ by } \ \varphi(v) := \begin{cases} \int_{\Gamma_c} \phi(v) & \text{if } \phi(v) \in L^1(\Gamma_c), \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4}$$

Hence, we introduce

$$\boldsymbol{\varphi} : \mathbf{Y}_{\Gamma_c} \to [0, +\infty], \quad \text{defined by } \ \boldsymbol{\varphi}(\mathbf{u}) := \varphi(u_N) \quad \text{for all } \mathbf{u} \in \mathbf{Y}_{\Gamma_c}. \tag{2.5}$$

Since $\boldsymbol{\varphi} : \mathbf{Y}_{\Gamma_c} \to [0, +\infty]$ is a proper, convex and lower semicontinuous functional on $\mathbf{Y}_{\Gamma_c}$, its subdifferential $\partial \boldsymbol{\varphi} : \mathbf{Y}_{\Gamma_c} \rightrightarrows \mathbf{Y}'_{\Gamma_c}$ is a maximal monotone operator. Notice that, when $\mathrm{dom}(\phi) \subseteq (-\infty, 0]$ the operator $\partial \boldsymbol{\varphi}$ then renders the impenetrability condition $u_{\mathrm{N}} \leq 0$ on Γ_c, see also the discussion in [4].

Concerning the regularizing operator $\mathcal{R}$ for frictional reaction, we ask that $\mathcal{R} : L^2(0, T; \mathbf{Y}'_{\Gamma_c}) \to L^2(0, T; \mathbf{H}_{\Gamma_c})$ fulfils

$$\forall\, (\boldsymbol{\eta}_n),\ \boldsymbol{\eta} \in L^2(0, T; \mathbf{Y}'_{\Gamma_c}),\ \boldsymbol{\eta}_n \rightharpoonup \boldsymbol{\eta} \ \text{ in } L^2(0, T; \mathbf{Y}'_{\Gamma_c}) \ \Rightarrow \ \mathcal{R}(\boldsymbol{\eta}_n) \to \mathcal{R}(\boldsymbol{\eta}) \ \text{ in } L^2(0, T; \mathbf{H}_{\Gamma_c}). \tag{H2}$$

In Section 5.2 we are going to show that, if $\mathcal{R}$ satisfies some additional condition (cf. (2.27) below), then a uniqueness result holds for some approximation of system (1.21)-(1.26). In Example 2.4, we shall exhibit an operator $\mathcal{R}$ which complies both with (H2), and with the latter condition (2.27) required for uniqueness.

Furthermore, we consider

$$\begin{gathered} \widehat{\rho} : \mathbb{R} \to [0, +\infty] \ \text{ proper, convex, and lower semicontinuous with } \ \mathrm{dom}(\widehat{\rho}) \subset (-\infty, 0] \text{ and} \\ \text{its subdifferential } \partial \widehat{\rho} : (-\infty, 0] \rightrightarrows \mathbb{R} \text{ fulfilling } 0 \in \partial \widehat{\rho}(0). \end{gathered} \tag{H3}$$

We shall denote by ρ the subdifferential operator $\partial \widehat{\rho}$. In fact, ρ generalizes the operator $\partial I_{(-\infty, 0]}$ in (1.25) yielding irreversible evolution for χ. Analogously, we take

$$\begin{gathered} \widehat{\beta} : \mathbb{R} \to [0, +\infty] \ \text{ proper, convex and lower semicontinuous with } \ \mathrm{dom}(\widehat{\beta}) \subset [0, +\infty) \text{ and} \\ \text{with } \widehat{\beta}(0) = 0 = \min \widehat{\beta}, \end{gathered} \tag{H4}$$

and let $\beta := \partial \widehat{\beta} : [0, +\infty) \rightrightarrows \mathbb{R}$.

Remark 2.2 In [5], [7], it is not supposed that the domain of β is bounded or a subset of $[0,+\infty)$. This corresponds to the fact that the model considered therein could include situations in which repulsive dynamics may occur between the body and the support, leading to $\chi<0$ (and forcing, in this case, separation, due to (1.23)). However, for the present analysis we need to enforce positivity of χ, cf. also the discussion at the beginning of Section 3.

Assumptions on the problem data. We suppose that

$$\mathbf{u}_0 \in \mathbf{W} \text{ and } \mathbf{u}_0 \in \mathrm{dom}(\boldsymbol{\varphi}), \tag{2.6}$$

$$\chi_0 \in H^2(\Gamma_c),\ \partial_{\mathbf{n}_s}\chi_0 = 0 \text{ a.e. in } \partial\Gamma_c, \quad \widehat{\beta}(\chi_0) \in L^1(\Gamma_c) \text{ and } \beta^0(\chi_0) \in L^2(\Gamma_c), \tag{2.7}$$

where $\beta^0(\chi_0)$ denotes the minimal section of $\beta(\chi_0)$. As far as the body force $\mathbf{f}$ and the surface traction $\mathbf{g}$ are concerned, we prescribe that

$$\mathbf{f} \in L^2(0,T;\mathbf{H}), \tag{2.8}$$

$$\mathbf{g} \in L^2(0,T;\mathbf{Y}'_{\Gamma_c}), \tag{2.9}$$

and we define $\mathbf{F}:(0,T)\to\mathbf{W}'$ by

$$\langle \mathbf{F}(t),\mathbf{v}\rangle_{\mathbf{W}} := \int_\Omega \mathbf{f}(t)\cdot\mathbf{v} + \langle \mathbf{g}(t),\mathbf{v}\rangle_{\mathbf{Y}_{\Gamma_c}} \quad \text{for all } \mathbf{v}\in W \quad \text{for a.e. } t\in(0,T). \tag{2.10}$$

Of course, thanks to (2.8)-(2.9), $\mathbf{F}\in L^2(0,T;\mathbf{W}')$.

Variational formulation of the problem. Recall that $\mathbf{d}:\mathbb{R}^3 \rightrightarrows \mathbb{R}^3$ denotes the subdifferential of the function j (1.14).

Problem 2.3 Find $(\mathbf{u},\chi,\boldsymbol{\eta},\boldsymbol{\mu},\xi,\zeta)$ such that

$$\mathbf{u}\in H^1(0,T;\mathbf{W}), \tag{2.11}$$

$$\chi \in L^\infty(0,T;H^2(\Gamma_c))\cap H^1(0,T;V_{\Gamma_c})\cap W^{1,\infty}(0,T;H_{\Gamma_c}), \tag{2.12}$$

$$\boldsymbol{\eta}\in L^2(0,T;\mathbf{Y}'_{\Gamma_c}), \tag{2.13}$$

$$\boldsymbol{\mu}\in L^2(0,T;\mathbf{H}_{\Gamma_c}), \tag{2.14}$$

$$\xi\in L^\infty(0,T;H_{\Gamma_c}), \tag{2.15}$$

$$\zeta\in L^\infty(0,T;H_{\Gamma_c}), \tag{2.16}$$

fulfilling the initial conditions

$$\mathbf{u}(0)=\mathbf{u}_0, \qquad \chi(0)=\chi_0, \tag{2.17}$$

and

$$\begin{aligned} b(\partial_t\mathbf{u},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + \int_{\Gamma_c}\chi\mathbf{u}\cdot\mathbf{v} \\ + \langle\boldsymbol{\eta},\mathbf{v}\rangle_{\mathbf{Y}_{\Gamma_c}} + \int_{\Gamma_c}\boldsymbol{\mu}\cdot\mathbf{v} = \langle\mathbf{F},\mathbf{v}\rangle_{\mathbf{W}} \quad \text{for all } \mathbf{v}\in\mathbf{W} \text{ a.e. in } (0,T), \end{aligned} \tag{2.18}$$

$$\boldsymbol{\eta}\in\partial\boldsymbol{\varphi}(\mathbf{u}) \text{ a.e. in } (0,T), \tag{2.19}$$

$$\boldsymbol{\mu}\in|\mathcal{R}(\boldsymbol{\eta})|\mathbf{d}(\partial_t\mathbf{u}) \text{ a.e. in } \Gamma_c\times(0,T), \tag{2.20}$$

$$\partial_t\chi - \Delta\chi + \xi + \zeta = w_s - \frac{1}{2}|\mathbf{u}|^2 \quad \text{a.e. in } \Gamma_c\times(0,T), \tag{2.21}$$

$$\xi\in\rho(\partial_t\chi) \text{ a.e. in } \Gamma_c\times(0,T), \tag{2.22}$$

$$\zeta\in\beta(\chi) \text{ a.e. in } \Gamma_c\times(0,T), \tag{2.23}$$

$$\partial_{\mathbf{n}_s}\chi = 0 \text{ a.e. in } \partial\Gamma_c\times(0,T). \tag{2.24}$$

Main result.

Theorem 2.1 *Assume* (H1), (H2), (H3), (H4), *and* (2.6)–(2.9). *Then, Problem 2.3 admits a solution* $(\mathbf{u}, \chi, \boldsymbol{\eta}, \boldsymbol{\mu}, \xi, \zeta)$. *More precisely, there exists a function*

$$\eta \in L^2(0,T;Y'_{\Gamma_c}),\ \textit{with}\ \eta(t) \in \partial\varphi(u_N(t)),\ \textit{s.t.}\ \langle \boldsymbol{\eta}(t), \mathbf{v}\rangle_{\mathbf{Y}_{\Gamma_c}} = \langle \eta(t), v_{\mathrm{N}}\rangle_{Y_{\Gamma_c}}\ \textit{for a.a.}\ t \in (0,T), \tag{2.25}$$

and (2.18) *in fact reads for all* $\mathbf{v} \in \mathbf{W}$

$$b(\partial_t \mathbf{u}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \int_{\Gamma_c} \chi \mathbf{u}\cdot\mathbf{v} + \langle \eta, v_{\mathrm{N}}\rangle_{Y_{\Gamma_c}} + \int_{\Gamma_c} \boldsymbol{\mu}\cdot\mathbf{v}_{\mathrm{T}} = \langle \mathbf{F}, \mathbf{v}\rangle_{\mathbf{W}} \quad \textit{a.e. in}\,(0,T).$$

The proof of Theorem 2.1 shall be developed throughout Sections 3–5, by passing to the limit in a suitable time-discretization scheme.

We conclude this section with the construction of an admissible operator $\mathcal{R} : L^2(0,T;\mathbf{Y}'_{\Gamma_c}) \to L^2(0,T;\mathbf{H}_{\Gamma_c})$ fulfilling (H2). The following example highlights the physical meaning of $\mathcal{R}$, as a regularization taking into account nonlocal interactions on the contact surface.

Example 2.4 Fix $f \in \mathrm{C}_0^\infty(\mathbb{R}^2;\mathbb{R}^3)$, with $\mathrm{supp}(f) \subset \Gamma_c$, and for all $\boldsymbol{\eta} \in L^2(0,T;\mathbf{Y}'_{\Gamma_c})$ let

$$\mathcal{R}(\boldsymbol{\eta})(x,t) = \int_0^t \langle \boldsymbol{\eta}(\cdot,s), f(x-\cdot)\rangle_{\mathbf{Y}_{\Gamma_c}}\,\mathrm{d}s \quad \text{for a.a.}\,(x,t) \in \Gamma_c \times (0,T).$$

Then, $\mathcal{R}(\boldsymbol{\eta}) \in L^2(0,T;\mathbf{H}_{\Gamma_c})$: indeed,

$$\int_{\Gamma_c} |\mathcal{R}(\boldsymbol{\eta})(x,t)|^2\,\mathrm{d}x \le \|\boldsymbol{\eta}\|^2_{L^2(0,T;\mathbf{Y}'_{\Gamma_c})} \int_{\Gamma_c} \|f(x-\cdot)\|^2_{\mathbf{Y}_{\Gamma_c}}\,\mathrm{d}x \le C\|\boldsymbol{\eta}\|^2_{L^2(0,T;\mathbf{Y}'_{\Gamma_c})}. \tag{2.26}$$

Hence, $\mathcal{R} : L^2(0,T;\mathbf{Y}'_{\Gamma_c}) \to L^2(0,T;\mathbf{H}_{\Gamma_c})$ is a linear and bounded operator, which further fulfils (H2). Indeed, let $(\boldsymbol{\eta}_n)$, $\boldsymbol{\eta} \in L^2(0,T;\mathbf{Y}'_{\Gamma_c})$ be like in (H2): for almost all $(x,t) \in \Gamma_c \times (0,T)$ we have

$$\begin{aligned}\mathcal{R}(\boldsymbol{\eta}_n)(x,t) &= \int_0^T \langle \boldsymbol{\eta}_n(\cdot,s), \mathbf{1}_{(0,t)} f(x-\cdot)\rangle_{\mathbf{Y}_{\Gamma_c}}\,\mathrm{d}s \\ &\to \int_0^T \langle \boldsymbol{\eta}(\cdot,s), \mathbf{1}_{(0,t)} f(x-\cdot)\rangle_{\mathbf{Y}_{\Gamma_c}}\,\mathrm{d}s = \mathcal{R}(\boldsymbol{\eta})(x,t)\end{aligned}$$

as $n \to \infty$. Moreover, it follows from estimate (2.26) that

$$\|\mathcal{R}(\boldsymbol{\eta}_n)\|^2_{L^2(0,T;\mathbf{H}_{\Gamma_c})} \le C\|\boldsymbol{\eta}_n\|^2_{L^2(0,T;\mathbf{Y}'_{\Gamma_c})} \le C'$$

hence the dominated convergence theorem yields $\mathcal{R}(\boldsymbol{\eta}_n) \to \mathcal{R}(\boldsymbol{\eta})$ in $L^2(0,T;\mathbf{H}_{\Gamma_c})$.

Finally, we observe that for all $\boldsymbol{\eta}_1,\ \boldsymbol{\eta}_2 \in L^2(0,T;\mathbf{H}_{\Gamma_c})$ there holds

$$\begin{aligned}&\int_0^T \int_{\Gamma_c} |\mathcal{R}(\boldsymbol{\eta}_1)(x,t) - \mathcal{R}(\boldsymbol{\eta}_2)(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \int_{\Gamma_c} \left| \int_0^t \int_{\Gamma_c} (\boldsymbol{\eta}_1(y,s) - \boldsymbol{\eta}_2(y,s)) \cdot f(x-y)\,\mathrm{d}y\,\mathrm{d}s \right|^2 \mathrm{d}x\,\mathrm{d}t \\ &\le C \int_0^T \int_{\Gamma_c} \int_0^t \|f(x-\cdot)\|^2_{L^\infty(\Gamma_c;\mathbb{R}^3)} \|\boldsymbol{\eta}_1(\cdot,s) - \boldsymbol{\eta}_2(\cdot,s)\|^2_{\mathbf{H}_{\Gamma_c}}\,\mathrm{d}s\,\mathrm{d}x\,\mathrm{d}t \\ &\le C \left(\int_{\Gamma_c} \|f(x-\cdot)\|^2_{L^\infty(\Gamma_c;\mathbb{R}^3)}\,\mathrm{d}x \right) \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|^2_{L^2(0,T;\mathbf{H}_{\Gamma_c})}.\end{aligned}$$

Therefore, $\mathcal{R}$ has the additional property that

$$\exists C_{\mathcal{R}} > 0\ \forall\, \boldsymbol{\eta}_1,\ \boldsymbol{\eta}_2 \in L^2(0,T;\mathbf{H}_{\Gamma_c}) :\ \|\mathcal{R}(\boldsymbol{\eta}_1) - \mathcal{R}(\boldsymbol{\eta}_2)\|_{L^2(0,T;\mathbf{H}_{\Gamma_c})} \le C_{\mathcal{R}} \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{L^2(0,T;\mathbf{H}_{\Gamma_c})}. \tag{2.27}$$

3 Time discretization scheme and regularization

We fix a time-step $\tau = T/N > 0$, $N \in \mathbb{N}$, inducing a partition $t_0 = 0 < t_1 < \cdots < t_n < \cdots < t_{N-1} < t_N = T$ of the interval $(0, T)$, with $t_n := n\tau$, $n = 0, \ldots, N$.

We shall consider a time-discrete problem featuring a suitable regularization of some of the nonlinear operators appearing in system (2.18)–(2.24). Indeed, we shall replace the operator $\beta : \mathbb{R} \rightrightarrows \mathbb{R}$ by its Yosida regularization β_ε (see e.g. [2, 3, 9]), and exploit that $\beta_\varepsilon : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function. In particular, $\mathrm{dom}(\beta_\varepsilon) = \mathbb{R}$, hence in the approximate system the constraint $\chi \geq 0$ is no longer enforced. Thus, to ensure that adhesion on the boundary is active when the glue is not completely damaged (i.e. there are no repulsive forces), we need to replace the adhesion coefficient χ in (2.18), by its positive part $\chi^+ = \max\{\chi, 0\}$.

Furthermore, we shall replace the function ϕ, which enters in the definition of the functional $\boldsymbol{\varphi}$ through (2.4) and (2.5), by its Yosida approximation $\phi_\varepsilon : \mathbb{R} \to [0, +\infty)$. We recall that ϕ_ε is convex, differentiable, and such that ϕ'_ε is the Yosida regularization of the subdifferential $\partial\phi : \mathbb{R} \rightrightarrows \mathbb{R}$. Therefore, in this way we replace the constraint (2.19) by its regularized version $\boldsymbol{\eta} \in \partial\phi_\varepsilon(u_{\mathrm{N}})\mathbf{n}$ a.e. in $\Gamma_c \times (0, T)$, cf. with (3.7) below.

Finally, we approximate $\mathbf{F}$ (2.10) by local means, namely we set

$$\mathbf{F}_i := \frac{1}{\tau} \int_{t_\tau^{i-1}}^{t_\tau^i} \mathbf{F}(s)\,\mathrm{d}s \quad \text{for all } i = 1, \ldots, N.$$

Time-discrete problem. Given $(\mathbf{u}_0, \chi_0) \in \mathbf{W} \times H^2(\Gamma_c)$, find $\{(\mathbf{u}_i^\varepsilon, \chi_i^\varepsilon)\}_{i=0}^N \subset \mathbf{W} \times H^2(\Gamma_c)$ such that

$$\mathbf{u}_0^\varepsilon = \mathbf{u}_0, \qquad \chi_0^\varepsilon = \chi_0, \tag{3.1}$$

and fulfilling for all $i = 1, \ldots, N$

$$b\left(\frac{\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon}{\tau}, v\right) + a(\mathbf{u}_i^\varepsilon, v) + \int_{\Gamma_c} (\chi_i^\varepsilon)^+ \mathbf{u}_i^\varepsilon \cdot \mathbf{v} + \int_{\Gamma_c} \boldsymbol{\eta}_i^\varepsilon \cdot \mathbf{v} + \int_{\Gamma_c} \boldsymbol{\mu}_i^\varepsilon \cdot \mathbf{v} = \langle \mathbf{F}_i, \mathbf{v} \rangle_{\mathbf{W}} \qquad \text{for all } \mathbf{v} \in \mathbf{W}, \tag{3.2}$$

$$\boldsymbol{\mu}_i^\varepsilon = |\mathcal{R}(\boldsymbol{\eta}_{i-1}^\varepsilon)| \mathbf{z}_i^\varepsilon, \quad \mathbf{z}_i^\varepsilon \in \mathbf{d}\left(\frac{\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon}{\tau}\right) \qquad \text{a.e. in } \Gamma_c, \tag{3.3}$$

$$\frac{\chi_i^\varepsilon - \chi_{i-1}^\varepsilon}{\tau} - \Delta\chi_i^\varepsilon + \xi_i^\varepsilon + \beta_\varepsilon(\chi_i^\varepsilon) = w_s - \frac{1}{2}|\mathbf{u}_{i-1}^\varepsilon|^2 \qquad \text{a.e. in } \Gamma_c, \tag{3.4}$$

$$\xi_i^\varepsilon \in \rho\left(\frac{\chi_i^\varepsilon - \chi_{i-1}^\varepsilon}{\tau}\right) \qquad \text{a.e. in } \Gamma_c, \tag{3.5}$$

$$\partial_{\mathbf{n}_s} \chi_i^\varepsilon = 0 \qquad \text{on } \partial\Gamma_c, \tag{3.6}$$

where

$$\boldsymbol{\eta}_i^\varepsilon := \phi'_\varepsilon((\mathbf{u}_i^\varepsilon)_{\mathrm{N}})\mathbf{n} =: \eta_i^\varepsilon \mathbf{n} \qquad \text{for all } i = 0, \ldots, N. \tag{3.7}$$

Proposition 3.1 *Assume* (2.6)–(2.9). *Then, there exist* $\{(\mathbf{u}_i^\varepsilon, \chi_i^\varepsilon)\}_{i=0}^N \subset \mathbf{W} \times H^2(\Gamma_c)$ *fulfilling* (3.1)–(3.5).

Proof. We proceed by induction on the index i, showing that, given $(\mathbf{u}_{i-1}^\varepsilon, \chi_{i-1}^\varepsilon, \boldsymbol{\eta}_{i-1}^\varepsilon) \in \mathbf{W} \times H^2(\Gamma_c) \times \mathbf{H}_{\Gamma_c}$, there exist a triple $(\mathbf{u}_i, \chi_i, \boldsymbol{\eta}_i^\varepsilon)$ fulfilling (3.2)–(3.5). Notice that (3.4) and (3.2) are decoupled, hence we can tackle each equation separately.

First, we show that, given $(\mathbf{u}_{i-1}^\varepsilon, \chi_{i-1}^\varepsilon)$, there exists $(\chi_i^\varepsilon, \xi_i^\varepsilon) \in H_{\Gamma_c} \times H_{\Gamma_c}$ solving (3.4). Indeed, we choose

$$\chi_i^\varepsilon \in \mathrm{Argmin}_{\,v \in V_{\Gamma_c}} \Big\{ \frac{\tau}{2} \left\| \frac{v - \chi_{i-1}^\varepsilon}{\tau} \right\|_{H_{\Gamma_c}}^2 + \tau \int_{\Gamma_c} \widehat{\rho}\left(\frac{v - \chi_{i-1}^\varepsilon}{\tau}\right) + \frac{1}{2} \int_{\Gamma_c} |\nabla v|^2 + \int_{\Gamma_c} \widehat{\beta}_\varepsilon(v) + \frac{1}{2} \int_{\Gamma_c} (\mathbf{u}_{i-1}^\varepsilon)^2 v \Big\}. \tag{3.8}$$

With the *direct method* of the Calculus of Variations, it can be easily shown that the above minimum problem (3.8) admits at least a solution in V_{Γ_c}. Hence, χ_i^ε fulfils the Euler equation for (3.8), namely (3.4), where $\xi_i^\varepsilon \in H_{\Gamma_c}$ is as in (3.5). Taking into account that the all terms $|\mathbf{u}_{i-1}^\varepsilon|^2$, $\beta_\varepsilon(\chi_{i-1}^\varepsilon)$, and ξ_i^ε are in H_{Γ_c}, a comparison argument in (3.4), joint with standard elliptic regularity results, implies that $\chi_i^\varepsilon \in H^2(\Gamma_c)$.

Analogously, given χ_i^ε and $(\mathbf{u}_{i-1}^\varepsilon, \chi_{i-1}^\varepsilon)$, with $\boldsymbol{\eta}_{i-1}^\varepsilon = \phi_\varepsilon'((\mathbf{u}_{i-1}^\varepsilon)_{\mathrm{N}}\mathbf{n}$, we find $\mathbf{u}_i^\varepsilon \in \mathbf{W}$ and $\boldsymbol{\mu}_i^\varepsilon \in \mathbf{H}_{\Gamma_c}$ fulfilling (3.2), by solving the minimum problem

$$\begin{aligned}\mathbf{u}_i^\varepsilon \in \text{Argmin}_{\mathbf{v}\in\mathbf{W}}\Big\{ &\frac{\tau}{2} b\left(\frac{\mathbf{v}-\mathbf{u}_{i-1}^\varepsilon}{\tau}, \frac{\mathbf{v}-\mathbf{u}_{i-1}^\varepsilon}{\tau}\right) + \tau\int_{\Gamma_c} |\mathcal{R}(\boldsymbol{\eta}_{i-1}^\varepsilon)| j\left(\frac{\mathbf{v}-\mathbf{u}_{i-1}^\varepsilon}{\tau}\right) \\ &+ \frac{1}{2}a(\mathbf{v},\mathbf{v}) + \frac{1}{2}\int_{\Gamma_c}(\chi_i^\varepsilon)^+|\mathbf{v}|^2 + \int_{\Gamma_c}\phi_\varepsilon(v_{\mathrm{N}}) - \langle F_i, \mathbf{v}\rangle_{\mathbf{W}}\Big\},\end{aligned} \tag{3.9}$$

which admits a solution, again via the direct method. □

Notation 3.2 We respectively denote by $\overline{\mathbf{u}}_\tau^\varepsilon$, $\underline{\mathbf{u}}_\tau^\varepsilon$, and $\mathbf{u}_\tau^\varepsilon$, the left-continuous and right-continuous piecewise constant, and the piecewise linear interpolants of the family $\{\mathbf{u}_i^\varepsilon\}_{i=0}^N \subset \mathbf{W}$, viz.

$$\begin{array}{lll}\overline{\mathbf{u}}_\tau^\varepsilon(0) := \mathbf{u}_0, \quad \overline{\mathbf{u}}_\tau^\varepsilon(t) := \mathbf{u}_i^\varepsilon & t \in (t_{i-1}, t_i], & i = 1, ..., N; \\ \underline{\mathbf{u}}_\tau^\varepsilon(0) := \mathbf{u}_0, \quad \underline{\mathbf{u}}_\tau^\varepsilon(t) := \mathbf{u}_{i-1}^\varepsilon & t \in [t_{i-1}, t_i) & i = 1, ..., N; \\ \mathbf{u}_\tau^\varepsilon(0) := \mathbf{u}_0, \quad \mathbf{u}_\tau^\varepsilon(t) := \frac{t-t_{i-1}}{\tau}\mathbf{u}_i^\varepsilon + (1 - \frac{t-t_{i-1}}{\tau})\mathbf{u}_{i-1}^\varepsilon & t \in [t_{i-1}, t_i), & i = 1, ..., N;\end{array}$$

We shall also make use of the interpolants $\overline{\chi}_\tau^\varepsilon$, χ_τ^ε, $\overline{\boldsymbol{\eta}}_\tau^\varepsilon$, $\underline{\boldsymbol{\eta}}_\tau^\varepsilon$, $\overline{\eta}_\tau^\varepsilon$, $\overline{\mu}_\tau^\varepsilon$, $\overline{\mathbf{z}}_\tau^\varepsilon$, $\overline{\xi}_\tau^\varepsilon$, and $\overline{\mathbf{F}}_\tau$, analogously defined. It can be easily checked that

$$\overline{\mathbf{F}}_\tau \to \mathbf{F} \quad \text{in } L^2(0,T;\mathbf{W}') \text{ as } \tau \to 0. \tag{3.10}$$

In terms of the above defined interpolants, system (3.2)–(3.7) reads:

$$b(\partial_t \mathbf{u}_\tau^\varepsilon, \mathbf{v}) + a(\overline{\mathbf{u}}_\tau^\varepsilon, \mathbf{v}) + \int_{\Gamma_c} \overline{\boldsymbol{\eta}}_\tau^\varepsilon \mathbf{v} + \int_{\Gamma_c} \overline{\boldsymbol{\mu}}_\tau^\varepsilon \mathbf{v} + \int_{\Gamma_c} (\overline{\chi}_\tau^\varepsilon)^+ \, \overline{\mathbf{u}}_\tau^\varepsilon \mathbf{v} = \langle \overline{\mathbf{F}}_\tau, \mathbf{v}\rangle_{\mathbf{W}} \qquad \text{for all } \mathbf{v} \in \mathbf{W}, \tag{3.11}$$

$$\overline{\boldsymbol{\eta}}_\tau^\varepsilon = \overline{\eta}_\tau^\varepsilon \mathbf{n}, \quad \overline{\eta}_\tau^\varepsilon = \phi_\varepsilon'(\overline{u_{\mathrm{N}}}_\tau^\varepsilon) \qquad \text{a.e. in } \Gamma_c \times (0,T), \tag{3.12}$$

$$\overline{\boldsymbol{\mu}}_\tau^\varepsilon = |\mathcal{R}(\underline{\boldsymbol{\eta}}_\tau^\varepsilon)| \overline{\mathbf{z}}_\tau^\varepsilon \in |\mathcal{R}(\underline{\boldsymbol{\eta}}_\tau^\varepsilon)| \mathbf{d}(\partial_t \mathbf{u}_\tau^\varepsilon) \qquad \text{a.e. in } \Gamma_c \times (0,T), \tag{3.13}$$

$$\partial_t \chi_\tau^\varepsilon - \Delta \overline{\chi}_\tau^\varepsilon + \overline{\xi}_\tau^\varepsilon + \beta_\varepsilon(\overline{\chi}_\tau^\varepsilon) = w_s - \frac{1}{2}|\underline{\mathbf{u}}_\tau^\varepsilon|^2 \qquad \text{a.e. in } \Gamma_c \times (0,T), \tag{3.14}$$

$$\overline{\xi}_\tau^\varepsilon \in \rho(\partial_t \chi_\tau^\varepsilon) \qquad \text{a.e. in } \Gamma_c \times (0,T), \tag{3.15}$$

$$\partial_{\mathbf{n}_s} \overline{\chi}_\tau^\varepsilon = 0 \qquad \text{a.e. on } \partial\Gamma_c \times (0,T). \tag{3.16}$$

4 A priori estimates

In this section we perform some a priori estimates on the approximate solutions, obtaining bounds which are *independent* of the parameters τ and ε. Hereafter, we shall denote by the same letter c various positive constants, depending only on the problem data, but *neither* on τ *nor* on ε.

Preliminarily, we recall the well-known *discrete by-part integration* formula

$$\sum_{i=1}^m \tau\delta b_i \cdot v_i = b_m v_m - b_0 v_1 - \sum_{i=2}^m \tau b_{i-1}\delta v_i \quad \text{for all } \{b_i\}_{i=1}^m, \{v_i\}_{i=1}^m, \tag{4.1}$$

where we have used the short-hand notation $\delta b_i = (b_i - b_{i-1})/\tau$.

First estimate. We first test (3.2) by $\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon$. Note that, due to (2.3) there holds

$$b\left(\frac{\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon}{\tau}, \mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon\right) \geq C_b \tau \left\| \frac{\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon}{\tau} \right\|_{\mathbf{W}}^2. \tag{4.2}$$

Furthermore, it is straightforward to get

$$a(\mathbf{u}_i^\varepsilon, \mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon) = \frac{1}{2} a(\mathbf{u}_i^\varepsilon, \mathbf{u}_i^\varepsilon) - \frac{1}{2} a(\mathbf{u}_{i-1}^\varepsilon, \mathbf{u}_{i-1}^\varepsilon) + \frac{1}{2} a(\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon, \mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon) \tag{4.3}$$

In addition, the operator ρ in (3.4) forces irreversibility, namely $\mathcal{X}_i^\varepsilon \leq \mathcal{X}_{i-1}^\varepsilon$ a.e. in Γ_c. Therefore, we conclude for all $i = 1, \dots, N$ that $\mathcal{X}_i^\varepsilon \leq \mathcal{X}_0$ a.e. in Γ_c. Taking into account condition (2.7) on $\mathcal{X}_0$, we conclude

$$\begin{aligned} \|(\mathcal{X}_i^\varepsilon)^+\|_{L^\infty(\Gamma_c)} \leq c, \quad \text{whence} \quad \int_{\Gamma_c} (\mathcal{X}_i^\varepsilon)^+ \mathbf{u}_i^\varepsilon \cdot (\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon) &\leq \|(\mathcal{X}_i^\varepsilon)^+\|_{L^\infty(\Gamma_c)} \|\mathbf{u}_i^\varepsilon\|_{L^4(\Gamma_c)} \|\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon\|_{L^4(\Gamma_c)} \\ &\leq c \|\mathbf{u}_i^\varepsilon\|_{\mathbf{W}} \|\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon\|_{\mathbf{W}}, \end{aligned} \tag{4.4}$$

where we have used that $\mathbf{W}$ embeds continuously into $L^4(\Gamma_c; \mathbb{R}^3)$. Exploiting that ϕ_ε is convex, we see that

$$\int_{\Gamma_c} \boldsymbol{\eta}_i^\varepsilon (\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon) \geq \int_{\Gamma_c} \phi_\varepsilon((\mathbf{u}_i^\varepsilon)_{\mathrm{N}}) - \phi_\varepsilon((\mathbf{u}_{i-1}^\varepsilon)_{\mathrm{N}}). \tag{4.5}$$

Finally, the monotonicity of $\mathbf{d}$ yields

$$\int_{\Gamma_c} |\mathcal{R}(\boldsymbol{\eta}_{i-1}^\varepsilon)| \mathbf{d}\left(\frac{\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon}{\tau}\right)(\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon) \geq 0. \tag{4.6}$$

Collecting (4.2)–(4.6) and summing up for $i = 1, ..., m$, $1 \leq m \leq N$, we arrive at

$$\begin{aligned} &\tau C_b \sum_{i=1}^m \left\| \frac{\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon}{\tau} \right\|_{\mathbf{W}}^2 + \frac{C_a}{2} \|\mathbf{u}_m^\varepsilon\|_{\mathbf{W}}^2 + \int_{\Gamma_c} \phi_\varepsilon((\mathbf{u}_m^\varepsilon)_{\mathrm{N}}) \\ &\leq c \left(\|\mathbf{u}_0\|_{\mathbf{W}}^2 + \int_{\Gamma_c} \phi_\varepsilon((\mathbf{u}_0^\varepsilon)_{\mathrm{N}}) + \sum_{i=1}^m \|\mathbf{F}_i\|_{\mathbf{W}'} \|\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon\|_{\mathbf{W}} + \sum_{i=1}^m \|\mathbf{u}_i^\varepsilon\|_{\mathbf{W}} \|\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon\|_{\mathbf{W}} \right). \end{aligned} \tag{4.7}$$

Notice that the first two terms on the right-hand side are bounded in view of (2.6). Concerning the last term, we can exploit Young's inequality to obtain

$$c \sum_{i=1}^m \|\mathbf{u}_i^\varepsilon\|_{\mathbf{W}} \|\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon\|_{\mathbf{W}} \leq \frac{C_b \tau}{4} \sum_{i=1}^m \left\| \frac{\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon}{\tau} \right\|_{\mathbf{W}}^2 + c\tau \sum_{i=1}^m \|\mathbf{u}_i^\varepsilon\|_{\mathbf{W}}^2.$$

Here, in particular, choosing τ sufficiently small, we can write

$$c\tau \sum_{i=1}^m \|\mathbf{u}_i^\varepsilon\|_{\mathbf{W}}^2 \leq \frac{C_a}{4} \|\mathbf{u}_m^\varepsilon\|_{\mathbf{W}}^2 + c \sum_{i=1}^{m-1} \|\mathbf{u}_i^\varepsilon\|_{\mathbf{W}}^2.$$

Analogously, there holds

$$\sum_{i=1}^m \|\mathbf{F}_i\|_{\mathbf{W}'} \|\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon\|_{\mathbf{W}} \leq \frac{C_b \tau}{4} \sum_{i=1}^m \left\| \frac{\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon}{\tau} \right\|_{\mathbf{W}}^2 + c\tau \sum_{i=1}^m \|\mathbf{F}_i\|_{\mathbf{W}'}^2.$$

Eventually, we get

$$\sum_{i=1}^m \tau \left\| \frac{\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon}{\tau} \right\|_{\mathbf{W}}^2 + \|\mathbf{u}_m^\varepsilon\|_{\mathbf{W}}^2 \leq c \left(1 + \sum_{i=1}^{m-1} \|\mathbf{u}_i^\varepsilon\|_{\mathbf{W}}^2 \right).$$

Combining the above calculations with (4.7) and applying a discrete version of the Gronwall lemma (see, e.g., [19, Chap. 2.2]), we are able to deduce

$$\|\partial_t \mathbf{u}_\tau^\varepsilon\|_{L^2(0,T;\mathbf{W})} + \|\overline{\mathbf{u}}_\tau^\varepsilon\|_{L^\infty(0,T;\mathbf{W})} \leq c. \tag{4.8}$$

Second estimate. Now, we test (3.4) by $\chi_i^\varepsilon - \chi_{i-1}^\varepsilon$ and add over $i = 1, ...m$. From the monotonicity of ρ and the fact that $0 \in \rho(0)$, it follows that

$$\int_{\Gamma_c} \xi_i^\varepsilon (\chi_i^\varepsilon - \chi_{i-1}^\varepsilon) \geq 0.$$

Then, we have

$$\int_{\Gamma_c} \beta_\varepsilon(\chi_i^\varepsilon)(\chi_i^\varepsilon - \chi_{i-1}^\varepsilon) \geq \int_{\Gamma_c} \widehat{\beta}_\varepsilon(\chi_i^\varepsilon) - \int_{\Gamma_c} \widehat{\beta}_\varepsilon(\chi_{i-1}^\varepsilon).$$

Moreover,

$$\int_{\Gamma_c} \nabla\chi_i^\varepsilon \nabla(\chi_i^\varepsilon - \chi_{i-1}^\varepsilon) = \frac{1}{2}\int_{\Gamma_c} |\nabla\chi_i^\varepsilon|^2 - \frac{1}{2}\int_{\Gamma_c} |\nabla\chi_{i-1}^\varepsilon|^2 + \frac{1}{2}\int_{\Gamma_c} |\nabla(\chi_i^\varepsilon - \chi_{i-1}^\varepsilon)|^2.$$

Then, adding over $i = 1, ..., m$, we get (cf. (2.7))

$$\begin{aligned}
&\tau\sum_{i=1}^m \left\| \frac{\chi_i^\varepsilon - \chi_{i-1}^\varepsilon}{\tau} \right\|_{H_{\Gamma_c}}^2 + \frac{1}{2}\|\nabla\chi_m^\varepsilon\|_{H_{\Gamma_c}}^2 + \int_{\Gamma_c} \widehat{\beta}_\varepsilon(\chi_m^\varepsilon) \\
&\leq \frac{1}{2}\|\nabla\chi_0\|_{H_{\Gamma_c}}^2 + \int_{\Gamma_c} \widehat{\beta}(\chi_0) + \tau c \sum_{i=1}^m (\|\mathbf{u}_{i-1}^\varepsilon\|_{\mathbf{W}}^2 + 1) \left\| \frac{\chi_i^\varepsilon - \chi_{i-1}^\varepsilon}{\tau} \right\|_{H_{\Gamma_c}} \\
&\leq c\left(1 + \tau\sum_{i=1}^m \|\mathbf{u}_{i-1}^\varepsilon\|_{\mathbf{W}}^4\right) + \frac{\tau}{2}\sum_{i=1}^m \left\| \frac{\chi_i^\varepsilon - \chi_{i-1}^\varepsilon}{\tau} \right\|_{H_{\Gamma_c}}^2.
\end{aligned}$$

Thus, taking into account the previously proved estimates (4.8), we are in the position of concluding

$$\|\partial_t\chi_\tau^\varepsilon\|_{L^2(0,T;H_{\Gamma_c})} + \|\nabla\overline{\chi}_\tau^\varepsilon\|_{L^\infty(0,T;H_{\Gamma_c})} \leq c. \tag{4.9}$$

Third estimate. We test (3.4) by the function $\tau(-\Delta\delta\chi_i^\varepsilon + \delta\beta_\varepsilon(\chi_i^\varepsilon))$, where $\delta\chi_i^\varepsilon = \frac{\chi_i^\varepsilon - \chi_{i-1}^\varepsilon}{\tau}$, and $\delta\beta_\varepsilon(\chi_i^\varepsilon) = \frac{\beta_\varepsilon(\chi_i^\varepsilon) - \beta_\varepsilon(\chi_{i-1}^\varepsilon)}{\tau}$. First, note that, by the monotonicity of β_ε, we have

$$\int_{\Gamma_c} \tau\delta\beta_\varepsilon(\chi_i^\varepsilon)\delta\chi_i^\varepsilon \geq 0.$$

Then, due to the monotonicity of ρ (see also [23, Lemma 4.1]), there holds

$$\int_{\Gamma_c} \xi_i^\varepsilon\ (-\Delta\delta\chi_i^\varepsilon) \geq 0$$

where ξ_i^ε is defined in (3.5). Then, combining the facts that β_ε is Lipschitz monotone, ρ is monotone, and it satisfies $0 \in \rho(0)$, we can infer that

$$\tau\int_{\Gamma_c} \xi_i^\varepsilon\ \delta\beta_\varepsilon(\chi_i^\varepsilon) \geq 0.$$

As previously seen, we have

$$\tau\int_{\Gamma_c} (-\Delta\chi_i^\varepsilon + \beta_\varepsilon(\chi_i^\varepsilon))(-\Delta\delta\chi_i^\varepsilon + \delta\beta_\varepsilon(\chi_i^\varepsilon)) \geq \frac{1}{2}\int_{\Gamma_c} (|-\Delta\chi_i^\varepsilon + \beta_\varepsilon(\chi_i^\varepsilon)|^2 - |-\Delta\chi_{i-1}^\varepsilon + \beta_\varepsilon(\chi_{i-1}^\varepsilon)|^2).$$

Eventually, summing up for $i = 1, ..., m$ and exploiting estimates (4.8) for $\mathbf{u}_i^\varepsilon$, we get

$$\begin{aligned}
&\sum_{i=1}^m \tau\int_{\Gamma_c} |\nabla\delta\chi_i^\varepsilon|^2 + \frac{1}{2}\int_{\Gamma_c} (|-\Delta\chi_m^\varepsilon + \beta_\varepsilon(\chi_m^\varepsilon)|^2 - |-\Delta\chi_0 + \beta_\varepsilon(\chi_0)|^2) \\
&\leq \sum_{i=1}^m \tau\int_{\Gamma_c} \left(w_s - \frac{1}{2}|\mathbf{u}_{i-1}^\varepsilon|^2\right)(-\Delta\delta\chi_i^\varepsilon + \delta\beta_\varepsilon(\chi_i^\varepsilon)) \\
&= w_s\int_{\Gamma_c} (-\Delta\chi_m^\varepsilon + \beta_\varepsilon(\chi_m^\varepsilon)) - w_s\int_{\Gamma_c} (-\Delta\chi_0 + \beta_\varepsilon(\chi_0)) + I_1,
\end{aligned} \tag{4.10}$$

where

$$
\begin{aligned}
I_1 &= -\sum_{i=1}^{m} \tau \int_{\Gamma_c} \frac{1}{2} |\mathbf{u}_{i-1}^\varepsilon|^2 (-\Delta\delta\chi_i^\varepsilon + \delta\beta_\varepsilon(\chi_i^\varepsilon)) \\
&= -\int_{\Gamma_c} \frac{1}{2} |\mathbf{u}_{m-1}^\varepsilon|^2 (-\Delta\chi_m^\varepsilon + \beta_\varepsilon(\chi_m^\varepsilon)) + \int_\Omega \frac{1}{2} |\mathbf{u}_0|^2 (-\Delta\chi_0 + \beta_\varepsilon(\chi_0)) \\
&\quad + \sum_{i=1}^{m-1} \tau \int_{\Gamma_c} (-\Delta\chi_i^\varepsilon + \beta_\varepsilon(\chi_i^\varepsilon)) \left(\frac{1}{2\tau} |\mathbf{u}_i^\varepsilon|^2 - \frac{1}{2\tau} |\mathbf{u}_{i-1}^\varepsilon|^2 \right),
\end{aligned}
\tag{4.11}
$$

where the second equality follows from the discrete integration by parts formula (4.1). Now, we plug (4.11) into (4.10). We absorb the first term on the right-hand side of (4.10) into the left-hand side, and estimate the second term via (2.7). We deal in a similar way with the first two summands on the right-hand side of (4.11), also taking into account estimate (4.8). Finally, we estimate the third term on the right-hand side of (4.11), for short referred to as I_2, in this way:

$$
\begin{aligned}
I_2 &= \sum_{i=1}^{m-1} \tau \int_{\Gamma_c} (-\Delta\chi_i^\varepsilon + \beta_\varepsilon(\chi_i^\varepsilon)) \frac{1}{2\tau} (\mathbf{u}_i^\varepsilon + \mathbf{u}_{i-1}^\varepsilon) \cdot (\mathbf{u}_i^\varepsilon - \mathbf{u}_{i-1}^\varepsilon) \\
&\le \frac{1}{2} \sum_{i=1}^{m-1} \tau \| -\Delta\chi_i^\varepsilon + \beta_\varepsilon(\chi_i^\varepsilon) \|_{H_{\Gamma_c}}^2 + \frac{1}{4} \|\overline{\mathbf{u}}_\tau^\varepsilon\|_{L^\infty(0,T;L^4(\Gamma_c))}^2 \sum_{i=1}^{m-1} \tau \|\delta \mathbf{u}_i^\varepsilon\|_{L^4(\Gamma_c)}^2,
\end{aligned}
\tag{4.12}
$$

where the last estimate follows from the Hölder inequality. Then, we insert (4.12) into (4.10). Now, the second term on the right-hand side of (4.12) is estimated because of (4.8), whereas to deal with the first one, we apply, for τ sufficiently small, the discrete version of the Gronwall lemma in [19, Chap. 2.2]. In this way, we conclude

$$
\|\partial_t \chi_\tau^\varepsilon\|_{L^2(0,T;V_{\Gamma_c})} + \| -\Delta\overline{\chi}_\tau^\varepsilon + \beta(\overline{\chi}_\tau^\varepsilon)\|_{L^\infty(0,T;H_{\Gamma_c})} \le c. \tag{4.13}
$$

Thus, standard elliptic regularity results ensure

$$
\|\overline{\chi}_\tau^\varepsilon\|_{L^\infty(0,T;H^2(\Gamma_c))} + \|\beta_\varepsilon(\overline{\chi}_\tau^\varepsilon)\|_{L^\infty(0,T;H_{\Gamma_c})} \le c. \tag{4.14}
$$

By a comparison in (3.4) we also deduce

$$
\|\overline{\xi}_\tau^\varepsilon\|_{L^2(0,T;H_{\Gamma_c})} \le c. \tag{4.15}
$$

Fourth estimate. Estimate (4.8) implies, by a comparison in (3.2), that the reaction, i.e. the term $|\mathcal{R}(\underline{\boldsymbol{\eta}}_\tau^\varepsilon)|\overline{\mathbf{z}}_\tau^\varepsilon + \overline{\boldsymbol{\eta}}_\tau^\varepsilon$, is bounded. More precisely, we have

$$
\| |\mathcal{R}(\underline{\boldsymbol{\eta}}_\tau^\varepsilon)|\overline{\mathbf{z}}_\tau^\varepsilon + \overline{\boldsymbol{\eta}}_\tau^\varepsilon \|_{L^2(0,T;\mathbf{Y}_{\Gamma_c}')} \le c. \tag{4.16}
$$

In particular, for any test function $\mathbf{v} \in L^2(0,T;\mathbf{W})$, there holds

$$
\int_0^T \langle |\mathcal{R}(\underline{\boldsymbol{\eta}}_\tau^\varepsilon)|\overline{\mathbf{z}}_\tau^\varepsilon + \overline{\boldsymbol{\eta}}_\tau^\varepsilon, \mathbf{v}\rangle_{\mathbf{Y}_{\Gamma_c}} = \int_0^T \int_{\Gamma_c} |\mathcal{R}(\underline{\boldsymbol{\eta}}_\tau^\varepsilon)|\overline{\mathbf{z}}_\tau^\varepsilon \cdot \mathbf{v}_{\mathrm{T}} + \int_0^T \int_{\Gamma_c} \overline{\eta}_\tau^\varepsilon v_{\mathrm{N}} \le c\|\mathbf{v}\|_{L^2(0,T;\mathbf{Y}_{\Gamma_c})}
$$

Thus, by definition of norm, we can infer that

$$
\begin{aligned}
C &\ge \| |\mathcal{R}(\underline{\boldsymbol{\eta}}_\tau^\varepsilon)|\overline{\mathbf{z}}_\tau^\varepsilon + \overline{\boldsymbol{\eta}}_\tau^\varepsilon \|_{L^2(0,T;\mathbf{Y}_{\Gamma_c}')} \\
&= \sup_{\mathbf{v} \in L^2(0,T;\mathbf{Y}_{\Gamma_c})} \frac{\int_0^T \int_{\Gamma_c} |\mathcal{R}(\underline{\boldsymbol{\eta}}_\tau^\varepsilon)|\overline{\mathbf{z}}_\tau^\varepsilon \mathbf{v}_{\mathrm{T}} + \int_{\Gamma_c} \overline{\eta}_\tau^\varepsilon v_{\mathrm{N}}}{\|\mathbf{v}\|_{L^2(0,T;\mathbf{Y}_{\Gamma_c})}} \\
&\ge \sup_{\mathbf{v} \in L^2(0,T;\mathbf{Y}_{\Gamma_c}), \mathbf{v}_{\mathrm{T}} = \mathbf{0} \text{ in } \Gamma_c \times (0,T)} \frac{\int_0^T \langle \overline{\boldsymbol{\eta}}_\tau^\varepsilon, \mathbf{v}\rangle_{Y_{\Gamma_c}}}{\|\mathbf{v}\|_{L^2(0,T;\mathbf{Y}_{\Gamma_c})}}.
\end{aligned}
$$

Thus, we conclude that

$$
\|\overline{\eta}_\tau^\varepsilon\|_{L^2(0,T;Y_{\Gamma_c}')}, \|\overline{\boldsymbol{\eta}}_\tau^\varepsilon\|_{L^2(0,T;\mathbf{Y}_{\Gamma_c}')} \le c. \tag{4.17}
$$

Then, (4.17) and (4.16) lead to

$$\| |\mathcal{R}(\underline{\boldsymbol{\eta}}_\tau^\varepsilon)| \overline{\mathbf{z}}_\tau^\varepsilon \|_{L^2(0,T;\mathbf{Y}'_{\Gamma_c})} \leq c. \tag{4.18}$$

In addition, it is immediately recovered from the definition of $\mathbf{d}$ (cf. (2.20)), that

$$\|\overline{\mathbf{z}}_\tau^\varepsilon\|_{L^\infty(\Gamma_c \times (0,T))} \leq c. \tag{4.19}$$

5 Conclusion of the proof of Theorem 2.1 and uniqueness result

5.1 Passage to the limit

In this section, we detail the passage to the limit in the approximate system (3.11)–(3.16) as *both* τ and ε tend to 0. Since all of estimates proved in Section 4 hold for constants independent of both parameters, we take the limit $\tau, \varepsilon \to 0$ *simultaneously*. The proof is split in some steps.

Compactness. It is straightforward to check that

$$\|\overline{\mathbf{u}}_\tau^\varepsilon - \underline{\mathbf{u}}_\tau^\varepsilon\|_{L^\infty(0,T;\mathbf{W})},\ \|\mathbf{u}_\tau^\varepsilon - \overline{\mathbf{u}}_\tau^\varepsilon|_{L^\infty(0,T;\mathbf{W})} \leq \tau^{1/2}\|\partial_t \mathbf{u}_\tau^\varepsilon\|_{L^2(0,T;\mathbf{W})} \leq c\tau^{1/2} \tag{5.1}$$

$$\|\chi_\tau^\varepsilon - \overline{\chi}_\tau^\varepsilon\|_{L^\infty(0,T;H_{\Gamma_c})} \leq \tau^{1/2}\|\partial_t \chi_\tau^\varepsilon\|_{L^2(0,T;H_{\Gamma_c})} \leq c\tau^{1/2} \tag{5.2}$$

where the last inequality in (5.1) ((5.2), respectively) ensues from estimate (4.8) ((4.9), resp.). Combining estimates (4.8), (4.9) and (4.13)-(4.15), as well as (5.1)-(5.2), with weak compactness results, we conclude that there exist functions $(\mathbf{u}, \chi, \boldsymbol{\eta}, \eta, \boldsymbol{\mu}, \xi, \zeta)$ with the regularity (2.11)–(2.16), and $\eta \in L^2(0,T;Y'_{\Gamma_c})$, such that, along some (not-relabeled) subsequence, the following convergences hold as ε, $\tau \to 0$

$$\mathbf{u}_\tau^\varepsilon \rightharpoonup \mathbf{u} \ \text{ in } H^1(0,T;\mathbf{W}), \qquad \overline{\mathbf{u}}_\tau^\varepsilon \rightharpoonup^* \mathbf{u} \ \text{ in } L^\infty(0,T;\mathbf{W}), \tag{5.3}$$

$$\chi_\tau^\varepsilon \rightharpoonup^* \chi \ \text{ in } H^1(0,T;V_{\Gamma_c}) \cap L^\infty(0,T;H^2(\Gamma_c)), \qquad \overline{\chi}_\tau^\varepsilon \rightharpoonup^* \chi \ \text{ in } L^\infty(0,T;H^2(\Gamma_c)), \tag{5.4}$$

$$\overline{\zeta}_\tau^\varepsilon := \beta_\varepsilon(\overline{\chi}_\tau^\varepsilon) \rightharpoonup^* \zeta \ \text{ in } L^\infty(0,T;H_{\Gamma_c}), \tag{5.5}$$

$$\overline{\xi}_\tau^\varepsilon \rightharpoonup \xi \ \text{ in } L^2(0,T;H_{\Gamma_c}), \tag{5.6}$$

$$\overline{\boldsymbol{\eta}}_\tau^\varepsilon \rightharpoonup \boldsymbol{\eta} \ \text{ in } L^2(0,T;\mathbf{Y}'_{\Gamma_c}), \tag{5.7}$$

$$\overline{\eta}_\tau^\varepsilon \rightharpoonup \eta \ \text{ in } L^2(0,T;Y'_{\Gamma_c}), \tag{5.8}$$

$$\overline{\mathbf{z}}_\tau^\varepsilon \rightharpoonup^* \mathbf{z} \ \text{ in } L^\infty(\Gamma_c \times (0,T)), \tag{5.9}$$

$$|\mathcal{R}(\underline{\boldsymbol{\eta}}_\tau^\varepsilon)|\overline{\mathbf{z}}_\tau^\varepsilon \rightharpoonup \boldsymbol{\mu} \ \text{ in } L^2(0,T;\mathbf{Y}'_{\Gamma_c}). \tag{5.10}$$

Moreover, we also have that

$$\underline{\boldsymbol{\eta}}_\tau^\varepsilon \rightharpoonup \boldsymbol{\eta} \text{ in } L^2(0,T;\mathbf{Y}'_{\Gamma_c}).$$

To check this, it is sufficient to observe that $(\underline{\boldsymbol{\eta}}_\tau^\varepsilon)_\tau$ is bounded in $L^2(0,T;\mathbf{Y}'_{\Gamma_c})$, thus, up to a further subsequence it weakly converges in $L^2(0,T;\mathbf{Y}'_{\Gamma_c})$ to some $\underline{\boldsymbol{\eta}}$. Now, we note that, for every $\mathbf{v} \in \mathrm{C}^0([0,T];\mathbf{Y}_{\Gamma_c})$ there holds

$$\int_0^T \langle \overline{\boldsymbol{\eta}}_\tau^\varepsilon(t) - \underline{\boldsymbol{\eta}}_\tau^\varepsilon(t), \mathbf{v}(t)\rangle_{\mathbf{Y}_{\Gamma_c}} \,\mathrm{d}t = \int_0^T \langle \overline{\boldsymbol{\eta}}_\tau^\varepsilon(t) - \overline{\boldsymbol{\eta}}_\tau^\varepsilon(t-\tau), \mathbf{v}(t)\rangle_{\mathbf{Y}_{\Gamma_c}} \,\mathrm{d}t = \int_0^T \langle \overline{\boldsymbol{\eta}}_\tau^\varepsilon(t), \mathbf{v}(t) - \mathbf{v}(t+\tau)\rangle_{\mathbf{Y}_{\Gamma_c}} \,\mathrm{d}t$$

(where the second equality follows from a change of variables), and that the last integral converges to 0 as $\tau \to 0$. In this way, we conclude that $\int_0^T \langle \boldsymbol{\eta}, \mathbf{v}\rangle_{\mathbf{Y}_{\Gamma_c}} = \int_0^T \langle \underline{\boldsymbol{\eta}}, \mathbf{v}\rangle_{\mathbf{Y}_{\Gamma_c}}$ for all $\mathbf{v} \in \mathrm{C}^0([0,T];\mathbf{Y}_{\Gamma_c})$. This yields, via a density argument, that $\boldsymbol{\eta} = \underline{\boldsymbol{\eta}}$. Hence, in view of condition (H2) on $\mathcal{R}$, we have

$$\mathcal{R}(\underline{\boldsymbol{\eta}}_\tau^\varepsilon) \to \mathcal{R}(\boldsymbol{\eta}) \quad \text{in } L^2(0,T;\mathbf{H}_{\Gamma_c}), \text{ so that } \ \boldsymbol{\mu} = \mathcal{R}(\boldsymbol{\eta})\mathbf{z}. \tag{5.11}$$

Now, strong compactness results (cf., e.g. [28]), joint with (5.1), (5.2) yield as $\varepsilon,\,\tau \to 0$

$$\begin{aligned} &\mathbf{u}_\tau^\varepsilon \to \mathbf{u} \text{ in } C^0([0,T];H^{1-\delta}(\Omega)^3) \text{ for all } 0<\delta<1,\\ &\mathbf{u}_\tau^\varepsilon \to \mathbf{u} \text{ in } C^0([0,T];L^p(\Gamma_c)^3) \text{ for all } 1\le p<4, \end{aligned} \tag{5.12}$$

$$\begin{aligned} &\overline{\mathbf{u}}_\tau^\varepsilon \to \mathbf{u} \text{ in } L^\infty(0,T;H^{1-\delta}(\Omega)^3) \text{ for all } 0<\delta<1,\\ &\overline{\mathbf{u}}_\tau^\varepsilon \to \mathbf{u} \text{ in } L^\infty(0,T;L^p(\Gamma_c)^3) \text{ for all } 1\le p<4, \end{aligned} \tag{5.13}$$

$$\begin{aligned} &\chi_\tau^\varepsilon \to \chi \text{ in } C^0([0,T];H^{2-\delta}(\Gamma_\mathrm{c})) \text{ for all } 0<\delta<2,\\ &\overline{\chi}_\tau^\varepsilon \to \chi \text{ in } L^\infty(0,T;H^{2-\delta}(\Gamma_\mathrm{c})) \text{ for all } 0<\delta<2. \end{aligned} \tag{5.14}$$

Passage to the limit in (3.14)**.** Now, we pass to the limit in (3.14): combining (5.3) with (5.13), one easily verifies that $|\overline{\mathbf{u}}_\tau^\varepsilon|^2 \rightharpoonup^* |\mathbf{u}|^2$ in $L^\infty(0,T;H_{\Gamma_c})$ as $\varepsilon,\,\tau\to 0$. Therefore, also taking into account convergences (5.4), (5.5), and (5.6), we conclude that the quadruple $(\mathbf{u},\chi,\xi,\zeta)$ satisfies (2.21). Combining the weak convergence (5.5) with the strong one (5.14), we have

$$\limsup_{\varepsilon,\,\tau\to 0}\int_0^t\int_{\Gamma_\mathrm{c}}\beta_\varepsilon(\overline{\chi}_\tau^\varepsilon)\overline{\chi}_\tau^\varepsilon \le \int_0^t\int_{\Gamma_\mathrm{c}}\zeta\chi.$$

In view of well-known results from the theory of maximal monotone operators (see, e.g., [3, Prop. II.1.1]), the above inequality is sufficient to conclude that $\zeta\in\beta(\chi)$ a.e. on $\Gamma_\mathrm{c}\times(0,T)$, viz. (2.23) holds. In the same way, to conclude that $\xi\in\rho(\partial_t\chi)$ a.e. on $\Gamma_\mathrm{c}\times(0,T)$, we show that

$$\limsup_{\varepsilon,\,\tau\to 0}\int_0^t\int_{\Gamma_\mathrm{c}}\overline{\xi}_\tau^\varepsilon\partial_t\chi_\tau^\varepsilon \le \int_0^t\int_{\Gamma_\mathrm{c}}\xi\partial_t\chi,$$

again by testing (3.14) by $\partial_t\chi_\tau^\varepsilon$ and taking the limit. Eventually, a comparison argument in (2.21) reveals that $\partial_t\chi\in L^\infty(0,T;H_{\Gamma_c})$ and $\xi\in L^\infty(0,T;H_{\Gamma_c})$.

Passage to the limit in (3.11)**.** Now, we can easily pass to the limit in (3.11). We get

$$b(\partial_t\mathbf{u},\mathbf{v})+a(\mathbf{u},\mathbf{v})+\int_{\Gamma_\mathrm{c}}\chi^+\mathbf{u}\mathbf{v}+\langle\boldsymbol{\eta},\mathbf{v}\rangle_{\mathbf{Y}_{\Gamma_\mathrm{c}}}+\int_{\Gamma_\mathrm{c}}\boldsymbol{\mu}\mathbf{v}=\langle\mathbf{F},\mathbf{v}\rangle_{\mathbf{Y}_{\Gamma_\mathrm{c}}}, \tag{5.15}$$

for all $\mathbf{v}\in\mathbf{W}$, where

$$\langle\boldsymbol{\eta},\mathbf{v}\rangle_{\mathbf{Y}_{\Gamma_\mathrm{c}}}=\langle\eta,v_N\rangle_{Y_{\Gamma_\mathrm{c}}}. \tag{5.16}$$

Notice that, since $\mathrm{dom}(\widehat{\beta})\subset[0,+\infty)$, then $\chi\ge 0$ a.e. on $\Gamma_\mathrm{c}\times(0,T)$, therefore $\chi^+\equiv\chi$ a.e. on $\Gamma_\mathrm{c}\times(0,T)$. It remains to identify $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$, i.e. to show that (2.19), (2.20) hold.

We shall in fact prove that for all $\mathbf{v}\in\mathbf{Y}_{\Gamma_\mathrm{c}}$ there holds

$$\int_0^T\varphi(\mathbf{v})-\varphi(\mathbf{u})\ge\int_0^T\langle\boldsymbol{\eta},\mathbf{v}-\mathbf{u}\rangle_{\mathbf{Y}_{\Gamma_\mathrm{c}}}=\int_0^T\langle\eta,v_N-u_N\rangle_{Y_{\Gamma_\mathrm{c}}}, \tag{5.17}$$

which yields $\boldsymbol{\eta}\in\partial\boldsymbol{\varphi}(\mathbf{u})$ a.e. in $(0,T)$. To do so, we test (3.11) by $\overline{\mathbf{u}}_\tau^\varepsilon$. Exploiting the previously obtained weak and strong convergences (5.3)–(5.4), (5.7)–(5.10), and (5.12)–(5.14), we show that

$$\limsup_{\varepsilon,\tau\to 0}\int_0^T\langle\overline{\eta}_\tau^\varepsilon,\overline{\mathbf{u}}_\tau^\varepsilon\cdot\mathbf{n}\rangle_{Y_{\Gamma_\mathrm{c}}}=\limsup_{\varepsilon,\tau\to 0}\int_0^T\int_{\Gamma_\mathrm{c}}\overline{\eta}_\tau^\varepsilon\overline{\mathbf{u}}_\tau^\varepsilon\cdot\mathbf{n}\le\int_0^T\langle\eta,u_\mathrm{N}\rangle_{Y_{\Gamma_c}}. \tag{5.18}$$

Note that, to prove (5.18) we need not have identified the weak limit $\boldsymbol{\mu}$: in fact, at this level we have only proved (5.15) and concluded (5.11). The proof of (5.18) solely relies on (5.3)–(5.4), (5.7)–(5.10), and (5.12)–(5.14), combined with lower semicontinuity arguments. Then, we have

$$\begin{aligned}\int_0^T\langle\boldsymbol{\eta},\mathbf{v}-\mathbf{u}\rangle_{\mathbf{Y}_{\Gamma_\mathrm{c}}}=\int_0^T\langle\eta,v_N-u_N\rangle_{Y_{\Gamma_\mathrm{c}}}&\le\liminf_{\varepsilon,\tau\to 0}\int_0^T\int_{\Gamma_\mathrm{c}}\overline{\eta}_\tau^\varepsilon(\overline{v}_\tau^\varepsilon-\overline{u}_\tau^\varepsilon)_\mathrm{N}\\ &\le\liminf_{\varepsilon,\tau\to 0}\int_0^T\int_{\Gamma_\mathrm{c}}\phi_\varepsilon((\overline{v}_\tau^\varepsilon)_\mathrm{N})-\phi_\varepsilon((\overline{u}_\tau^\varepsilon)_\mathrm{N})\\ &\le\int_0^T\int_{\Gamma_\mathrm{c}}\phi(v_N)-\phi(u_N)\end{aligned}$$

where the first equality is due to (5.16) and the ensuing inequalities, respectively, to (5.18), the convexity of ϕ_ε, and the fact that $\phi_\varepsilon \leq \phi$, combined with the Mosco convergence of ϕ_ε to ϕ. Therefore, we conclude (5.17) in view of (2.5).

Finally, we have to identify $\boldsymbol{\mu}$ as an element in $\mathcal{R}(\boldsymbol{\eta})\mathbf{d}(\partial_t \mathbf{u})$ almost everywhere in $\Gamma_c \times (0,T)$. For every fixed $\boldsymbol{\eta} \in L^2(0,T;\mathbf{Y}'_{\Gamma_c})$, we introduce the functional $\mathcal{J}_{\boldsymbol{\eta}} : L^2(0,T;\mathbf{H}_{\Gamma_c}) \to [0,+\infty)$ defined for all $\mathbf{v} \in L^2(0,T;\mathbf{H}_{\Gamma_c})$ by

$$\mathcal{J}_{\boldsymbol{\eta}}(\mathbf{v}) := \int_0^T \int_{\Gamma_c} |\mathcal{R}(\boldsymbol{\eta})(x,t)| j(\mathbf{v}(x,t))\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\Gamma_c} |\mathcal{R}(\boldsymbol{\eta})(x,t)||\mathbf{v}_{\mathrm{T}}(x,t)|\,\mathrm{d}x\,\mathrm{d}t.$$

Clearly, $\mathcal{J}_{\boldsymbol{\eta}}$ is a convex and lower semicontinuous functional on $L^2(0,T;\mathbf{H}_{\Gamma_c})$. It can be easily verified that the subdifferential $\partial\mathcal{J}_{\boldsymbol{\eta}} : L^2(0,T;\mathbf{H}_{\Gamma_c}) \rightrightarrows L^2(0,T;\mathbf{H}_{\Gamma_c})$ of $\mathcal{J}_{\boldsymbol{\eta}}$ is given at every $\mathbf{v} \in L^2(0,T;\mathbf{H}_{\Gamma_c})$ by

$$\boldsymbol{\mu} \in \partial\mathcal{J}_{\boldsymbol{\eta}}(\mathbf{v}) \;\Leftrightarrow\; \boldsymbol{\mu} \in L^2(0,T;\mathbf{H}_{\Gamma_c}) \text{ and } \boldsymbol{\mu}(x,t) \in |\mathcal{R}(\boldsymbol{\eta})(x,t)|\mathbf{d}(\mathbf{v}(x,t)) \tag{5.19}$$

for almost all $(x,t) \in \Gamma_c \times (0,T)$, where $\mathbf{d} = \partial j$ is given by (1.20). We shall now prove that (cf. (5.11)), that

$$\mathcal{J}_{\boldsymbol{\eta}}(\mathbf{w}) - \mathcal{J}_{\boldsymbol{\eta}}(\partial_t \mathbf{u}) \geq \int_0^T \int_{\Gamma_c} |\mathcal{R}(\boldsymbol{\eta})|\mathbf{z} \cdot (\mathbf{w} - \partial_t \mathbf{u}) \quad \forall\, \mathbf{w} \in L^2(0,T;\mathbf{H}_{\Gamma_c}). \tag{5.20}$$

To this aim, we first observe that

$$\limsup_{\varepsilon,\tau\to 0} \int_0^T \int_{\Gamma_c} |\mathcal{R}(\underline{\boldsymbol{\eta}}_\tau^\varepsilon)|\overline{\mathbf{z}}_\tau^\varepsilon \cdot \partial_t \mathbf{u}_\tau^\varepsilon \leq \int_0^T \int_{\Gamma_c} |\mathcal{R}(\boldsymbol{\eta})|\mathbf{z} \cdot \partial_t \mathbf{u}, \tag{5.21}$$

which can be shown by testing (3.11) by $\partial_t \mathbf{u}_\tau^\varepsilon$ and passing to the limit via convergences (5.3)–(5.4), (5.7)–(5.10), (5.12)–(5.14), lower semicontinuity arguments, and the Mosco convergence of ϕ_ε. Therefore, we have

$$\begin{aligned}
\int_0^T \int_{\Gamma_c} |\mathcal{R}(\boldsymbol{\eta})|\mathbf{z} \cdot (\mathbf{w} - \partial_t \mathbf{u}) &\leq \liminf_{\varepsilon,\tau\to 0} \int_0^T \int_{\Gamma_c} |\mathcal{R}(\underline{\boldsymbol{\eta}}_\tau^\varepsilon)|\overline{\mathbf{z}}_\tau^\varepsilon \cdot (\mathbf{w} - \partial_t \mathbf{u}_\tau^\varepsilon) \\
&\leq \int_0^T \int_{\Gamma_c} |\mathcal{R}(\underline{\boldsymbol{\eta}}_\tau^\varepsilon)|(|\mathbf{w}_{\mathrm{T}}| - |(\partial_t \mathbf{u}_\tau^\varepsilon)_{\mathrm{T}}|) \\
&\leq \int_0^T \int_{\Gamma_c} |\mathcal{R}(\boldsymbol{\eta})|(|\mathbf{w}_{\mathrm{T}}| - |(\partial_t \mathbf{u})_{\mathrm{T}}|)
\end{aligned}$$

where the first inequality follows from (5.21), the second one from the fact that $|\mathcal{R}(\underline{\boldsymbol{\eta}}_\tau^\varepsilon)|\overline{\mathbf{z}}_\tau^\varepsilon \in |\mathcal{R}(\underline{\boldsymbol{\eta}}_\tau^\varepsilon)|\mathbf{d}(\partial_t \mathbf{u}_\tau^\varepsilon)$ and from (5.19), and the last one from combining the *weak* convergence (5.3) with the *strong* convergence (5.11). Then, (5.20) ensues. Therefore, we conclude that $\boldsymbol{\mu} \in \partial\mathcal{J}_{\boldsymbol{\eta}}(\partial_t \mathbf{u})$ almost everywhere in $\Gamma_c \times (0,T)$, hence (2.20) by (5.19).

5.2 Uniqueness for a regularized problem

It is clear from the arguments developed in Section 5.1 that, passing to the limit as $\tau \to 0$ for $\varepsilon > 0$ *fixed*, yields a triple $(\mathbf{u}, \chi, \boldsymbol{\mu}, \xi)$ complying with (2.11), (2.12), (2.14), (2.15), and fulfilling the *approximate* problem

$$
\begin{aligned}
&b(\partial_t \mathbf{u},\mathbf{v})+a(\mathbf{u},\mathbf{v})+\int_{\Gamma_c}\chi^+\mathbf{u}\cdot\mathbf{v}\\
&\qquad\qquad+\int_{\Gamma_c}\phi_\varepsilon'(u_{\mathrm{N}})\mathbf{n}\cdot\mathbf{v}+\int_{\Gamma_c}\boldsymbol{\mu}\cdot\mathbf{v}=\langle\mathbf{F},\mathbf{v}\rangle_{\mathbf{W}}\quad\text{for all }\mathbf{v}\in\mathbf{W}\ \text{ a.e. in }(0,T)\,,
\end{aligned}\tag{5.22}
$$

$$
\boldsymbol{\mu}\in|\mathcal{R}(\phi_\varepsilon'(u_{\mathrm{N}})\mathbf{n})|\mathbf{d}(\partial_t\mathbf{u})\ \text{a.e. in }\ \Gamma_c\times(0,T),\tag{5.23}
$$

$$
\partial_t\chi-\Delta\chi+\xi+\beta_\varepsilon(\chi)=w_s-\frac{1}{2}|\mathbf{u}|^2\quad\text{a.e. in }\Gamma_c\times(0,T)\,,\tag{5.24}
$$

$$
\xi\in\rho(\partial_t\chi)\ \text{a.e. in }\ \Gamma_c\times(0,T),\tag{5.25}
$$

$$
\partial_{\mathbf{n}_s}\chi=0\ \text{a.e. in }\partial\Gamma_c\times(0,T)\,.\tag{5.26}
$$

For the related Cauchy problem, we have the following well-posedness result; we point out that an elementary adaptation of our arguments also yields a continuous dependence estimate on the problem data $\mathbf{u}_0$, χ_0, $\mathbf{F}$, see the proof of [4, Thm. 2].

Proposition 5.1 *Assume* (H1), (H2), (H3), (H4), *and* (2.6)–(2.9). *In addition, suppose that the operator* $\mathcal{R}$ *also fulfils* (2.27). *Then, the Cauchy problem* (5.22)–(5.26), *supplemented with the initial conditions* (2.17), *has a unique solution* $(\mathbf{u},\chi,\boldsymbol{\mu},\xi)$ *fulfilling* (2.11), (2.12), (2.14), *and* (2.15).

Proof. As we have just observed, the *existence* of a solution to the Cauchy problem for the PDE system (5.22)–(5.26) follows from Section 5.1. For the uniqueness proof, let $(\mathbf{u}_1,\chi_1,\boldsymbol{\mu}_1,\xi_1)$ and $(\mathbf{u}_2,\chi_2,\boldsymbol{\mu}_2,\xi_2)$ be two solutions to the Cauchy problem for (5.22)–(5.26), where for $i=1,2$ $\boldsymbol{\mu}_i=|\mathcal{R}(\phi_\varepsilon'((u_i)_{\mathrm{N}})\mathbf{n})|\mathbf{z}_i$, and $\mathbf{z}_i\in\mathbf{d}(\partial_t\mathbf{u}_i)$ a.e. on $\Gamma_{\mathrm{c}}\times(0,T)$. We set $\widetilde{\mathbf{u}}:=\mathbf{u}_1-\mathbf{u}_2$, $\widetilde{\chi}:=\chi_1-\chi_2$, $\widetilde{\mathbf{z}}:=\mathbf{z}_1-\mathbf{z}_2$, and $\widetilde{\xi}:=\xi_1-\xi_2$.

First, we subtract (5.22) written for $(\mathbf{u}_2,\chi_2,\boldsymbol{\mu}_2)$ from (5.22) written for $(\mathbf{u}_1,\chi_1,\boldsymbol{\mu}_1)$, we test the resulting relation by $\partial_t\widetilde{\mathbf{u}}$ and we integrate from 0 to t, with $0<t<T$. Integrating by parts, we have $\int_0^t a(\widetilde{\mathbf{u}},\partial_t\widetilde{\mathbf{u}})=\frac{1}{2}a(\widetilde{\mathbf{u}}(t),\widetilde{\mathbf{u}}(t))$. Hence, taking into account (2.2) and (2.3), we conclude

$$
\begin{aligned}
&C_b\int_0^t\|\partial_t\widetilde{\mathbf{u}}\|^2_{\mathbf{W}}+\frac{C_a}{2}\|\widetilde{\mathbf{u}}(t)\|^2_{\mathbf{W}}+\int_0^t\int_{\Gamma_c}|\mathcal{R}(\phi_\varepsilon'((u_2)_{\mathrm{N}})\mathbf{n})|\widetilde{\mathbf{z}}\cdot\partial_t\widetilde{\mathbf{u}}\\
&=-\int_0^t\int_{\Gamma_c}(\chi_1^+-\chi_2^+)\mathbf{u}_1\cdot\partial_t\widetilde{\mathbf{u}}-\int_0^t\int_{\Gamma_c}\chi_2^+\widetilde{\mathbf{u}}\cdot\partial_t\widetilde{\mathbf{u}}\\
&\quad-\int_0^t\int_{\Gamma_c}(|\mathcal{R}(\phi_\varepsilon'((u_1)_{\mathrm{N}})\mathbf{n})|-|\mathcal{R}(\phi_\varepsilon'((u_2)_{\mathrm{N}})\mathbf{n})|)\mathbf{z}_1\cdot\partial_t\widetilde{\mathbf{u}}\\
&\quad-\int_0^t\int_{\Gamma_c}(\phi_\varepsilon'((u_1)_{\mathrm{N}})\mathbf{n}-\phi_\varepsilon'((u_2)_{\mathrm{N}})\mathbf{n})\cdot\partial_t\widetilde{\mathbf{u}}\doteq I_1+I_2+I_3+I_4.
\end{aligned}\tag{5.27}
$$

Note that the third term on the left-hand side of (5.27) is non-negative by monotonicity of $\mathbf{d}$. Next, observe that

$$
I_1\le\int_0^t\|\chi_1^+-\chi_2^+\|_{H_{\Gamma_c}}\|\mathbf{u}_1\|_{L^4(\Gamma_c)}\|\partial_t\widetilde{\mathbf{u}}\|_{L^4(\Gamma_c)}\le\frac{C_b}{8}\int_0^t\|\partial_t\widetilde{\mathbf{u}}\|^2_{\mathbf{W}}+c\int_0^t\|\widetilde{\chi}\|^2_{H_{\Gamma_c}}\|\mathbf{u}_1\|^2_{\mathbf{W}},\tag{5.28}
$$

$$
I_2\le\frac{C_b}{8}\int_0^t\|\partial_t\widetilde{\mathbf{u}}\|^2_{\mathbf{W}}+c\int_0^t\|\chi_2\|^2_{H_{\Gamma_c}}\|\widetilde{\mathbf{u}}\|^2_{\mathbf{W}},\tag{5.29}
$$

$$
\begin{aligned}
I_3\ &\le\textstyle\int_0^t\||\mathcal{R}((\phi_\varepsilon'((u_1)_{\mathrm{N}})\mathbf{n}))|-|\mathcal{R}(\phi_\varepsilon'((u_2)_{\mathrm{N}})\mathbf{n})|\|_{H_{\Gamma_c}}\|\mathbf{z}_1\|_{L^\infty(\Gamma_c)}\|\partial_t\widetilde{\mathbf{u}}\|_{H_{\Gamma_c}}\\
&\le\frac{C_b}{8}\int_0^t\|\partial_t\widetilde{\mathbf{u}}\|^2_{\mathbf{W}}+c\int_0^t\||\mathcal{R}((\phi_\varepsilon'((u_1)_{\mathrm{N}})\mathbf{n}))|-|\mathcal{R}(\phi_\varepsilon'((u_2)_{\mathrm{N}})\mathbf{n})|\|^2_{H_{\Gamma_c}}\\
&\le\frac{C_b}{8}\int_0^t\|\partial_t\widetilde{\mathbf{u}}\|^2_{\mathbf{W}}+c\int_0^t\|\phi_\varepsilon'((u_1)_{\mathrm{N}})-\phi_\varepsilon'((u_2)_{\mathrm{N}})\|^2_{H_{\Gamma_c}}\\
&\le\frac{C_b}{8}\int_0^t\|\partial_t\widetilde{\mathbf{u}}\|^2_{\mathbf{W}}+c\int_0^t\|(u_1)_{\mathrm{N}}-(u_2)_{\mathrm{N}}\|^2_{H_{\Gamma_c}}\\
&\le\frac{C_b}{8}\int_0^t\|\partial_t\widetilde{\mathbf{u}}\|^2_{\mathbf{W}}+c\int_0^t\|\widetilde{\mathbf{u}}\|^2_{\mathbf{W}},
\end{aligned}\tag{5.30}
$$

Indeed, for (5.28) and (5.29) we have used that $\mathbf{W}$ embeds continuously into $L^4(\Gamma_c;\mathbb{R}^3)$. Inequality (5.30) follows from the fact that $|\mathbf{z}_1|\leq 1$ a.e. on $\Gamma_c\times(0,T)$, from (2.27), from the Lipschitz continuity of ϕ_ε and the continuous embedding $\mathbf{W}\subset L^2(\Gamma_c;\mathbb{R}^3)$. With analogous calculations, we also have

$$I_4 \leq \frac{C_b}{16}\int_0^t \|\partial_t\widetilde{\mathbf{u}}\|^2_{\mathbf{W}} + c\int_0^t \|\widetilde{\mathbf{u}}\|^2_{\mathbf{W}}. \tag{5.31}$$

Second, we take the difference between (5.24) written for $(\mathbf{u}_1,\chi_1,\xi_1)$ and for $(\mathbf{u}_2,\chi_2,\xi_2)$, we multiply it by $\partial_t\widetilde{\chi}$ and we integrate over $\Gamma_c\times(0,t)$. Thanks to the monotonicity of ρ, the term $\int_0^t\int_{\Gamma_c}\widetilde{\xi}\,\partial_t\widetilde{\chi}$ is non-negative, and, arguing in the very same way as in the proof of [4, Thm. 2], we find

$$\begin{aligned}
&\int_0^t \|\partial_t\widetilde{\chi}\|^2_{H_{\Gamma_c}} + \frac{1}{2}\|\nabla\widetilde{\chi}(t)\|^2_{H_{\Gamma_c}} \\
&\quad\leq -\int_{\Gamma_c}(\beta_\varepsilon(\chi_1)-\beta_\varepsilon(\chi_2))\,\partial_t\widetilde{\chi} - \frac{1}{2}\int_0^t\int_{\Gamma_c}(\mathbf{u}_1+\mathbf{u}_2)\,\widetilde{\mathbf{u}}\,\widetilde{\chi}_t \\
&\quad\leq \int_0^t \|\widetilde{\chi}\|_{H_{\Gamma_c}}\|\partial_t\widetilde{\chi}\|_{H_{\Gamma_c}} + \int_0^t \|\mathbf{u}_1+\mathbf{u}_2\|_{L^4(\Gamma_c)}\|\partial_t\widetilde{\chi}\|_{L^2(\Gamma_c)}\|\widetilde{\mathbf{u}}\|_{L^4(\Gamma_c)} \\
&\quad\leq \frac{1}{2}\int_0^t \|\widetilde{\chi}_t\|^2_{H_{\Gamma_c}} + c\int_0^t \|\widetilde{\chi}\|^2_{H_{\Gamma_c}} + c\|\mathbf{u}_1+\mathbf{u}_2\|^2_{L^\infty(0,t;W)}\int_0^t \|\widetilde{\mathbf{u}}\|^2_{\mathbf{W}}
\end{aligned} \tag{5.32}$$

Finally, we add inequalities (5.27) and (5.32). Taking into account estimates (5.28)–(5.31), and absorbing some terms on the left-hand side, we get

$$\int_0^t \|\partial_t\widetilde{\mathbf{u}}\|^2_{\mathbf{W}} + \|\widetilde{\mathbf{u}}(t)\|^2_{\mathbf{W}} + \int_0^t \|\partial_t\widetilde{\chi}\|^2_{H_{\Gamma_c}} + \|\nabla\widetilde{\chi}(t)\|^2_{H_{\Gamma_c}} \leq cM\left(\int_0^t \|\widetilde{\chi}\|^2_{H_{\Gamma_c}} + \int_0^t \|\widetilde{\mathbf{u}}\|^2_{\mathbf{W}}\right), \tag{5.33}$$

where the constant M is given by $M=\|\mathbf{u}_1\|^2_{L^\infty(0,t;\mathbf{W})}+\|\mathbf{u}_2\|^2_{L^\infty(0,t;\mathbf{W})}+\|\chi_2\|^2_{L^\infty(0,t;H_{\Gamma_c})}$. We use that $\int_0^t\|\widetilde{\chi}\|^2_{H_{\Gamma_c}}\leq\int_0^t\|\partial_t\widetilde{\chi}\|^2_{L^2(0,s;H_{\Gamma_c})}\,\mathrm{d}s$, hence we apply the Gronwall Lemma (see, e.g., [9, Lemme A.3]) in inequality (5.33), and immediately conclude that $\widetilde{\mathbf{u}}=\mathbf{0}$ a.e. in $\Omega\times(0,T)$ and, likewise, that $\widetilde{\chi}=0$ a.e. on $\Gamma_c\times(0,T)$. By comparison, we also have $\widetilde{\boldsymbol{\mu}}=0$ and $\widetilde{\xi}=0$ a.e. on $\Gamma_c\times(0,T)$. □

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