DOMAINS WITH THE CONE PROPERTY IN CARNOT–CARATHÉODORY SPACES

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Abstract. We study John domains in the metric space associated with a system of diagonal vector fields. We also discuss some applications.

1. Introduction

In this paper we study domains with the twisted cone property in $\mathbb{R}^n$ equipped with the control distance generated by a family of vector fields $X_1, ..., X_m$. Bounded domains satisfying similar geometric conditions are relevant, for instance, in the global theory of second order PDEs related to subelliptic operators of the form $\mathcal{L} = \sum_{j=1}^m X_j^2$.

The twisted cone property was introduced in the Euclidean setting by John in his seminal paper [Joh] on the rigidity of quasiisometric maps in $\mathbb{R}^n$. Besides its importance in geometric function theory, this property plays a central role in the theory of first order Sobolev spaces, see e.g. the papers by Reshetnyak [R], Besov [Be], Martio [M], Bojarski [Bo] and the monograph by Maz'ya and Poborchi [MP]. More recent related references are [SS], [BK], [HK1], [KOT]. It is worth recalling that also other classes of domains appearing in more refined questions in harmonic analysis and PDEs can be defined through cone conditions, such as uniform domains (introduced by Martio and Sarvas [MS] and by Jones [Jon]) and non–tangentially accessible (NTA) domains (introduced by Jerison and Kenig [JK]).

Domains with the cone property can be defined in any metric space and in particular in a Carnot–Carathéodory space. Given a metric space (M, d), a rectifiable path $\gamma : [0, 1] \to M$ and a positive number $\varepsilon > 0$, the twisted cone with core at γ and aperture ε is the set $\mathcal{C}_{\gamma,\varepsilon} = \bigcup_{0<t<1} B\big(\gamma(t), \varepsilon \,\mathrm{length}(\gamma|_{(0,t)})\big)$. A bounded domain $\Omega \subset M$ is a John domain with John constant ε and center $x_0 \in \Omega$ if for any $x \in \Omega$ there is a cone $\mathcal{C}_{\gamma,\varepsilon} \subset \Omega$ such that $\gamma(0) = x$ and $\gamma(1) = x_0$. It is an easy exercise to show that in metric spaces with geodesics balls are John domains. Note that in the Euclidean setting twisted cone domains are more general than domains with the "rigid" cone property, the von Koch snowflake in the plane being an example.

Several theorems involving the cone property have been generalized from the Euclidean to the Carnot–Caratéodory setting. We briefly recall some of them. If a bounded open set Ω satisfies an exterior cone property with respect to the distance generated by a family of Hörmander vector fields, then the Wiener criterion is satisfied at any boundary point and Ω is regular in the sense of Perron, Wiener, Brelot and Bony for the Dirichlet problem for the subelliptic Laplacian $\mathcal{L}$: see the papers

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by Negrini and Scornazzani [NS], Hueber [Hu], Danielli [D], and for an approach in metric spaces see also the paper by Björn, MacManus and Shanmugalingam [BMS].

Moreover, the interior cone property is related to chaining conditions that are useful in the proof of subelliptic Sobolev–Poincaré inequalities. This fact was recognized by Jerison in [Je], and later used by several other authors; see the contributions by Lu [L], Franchi, Gutierrez and Wheeden [FGW], Garofalo and Nhieu [GN1], Franchi, Lu and Wheeden [FLW], Buckley, Koskela and Lu [BKL], Hajłasz and Koskela [HK2].

Despite all these results, no easy condition is known which ensures the John property for a domain Ω in a metric space associate with a family of vector fields. This problem, which is sometimes eluded in the literature, requires a precise knowledge of the structure of control balls. To emphasize the nontrivial nature of the situation, we mention an example of [Je], in which it is shown that in the apparently simple situation of the vector fields $X_1 = \partial_1$, $X_2 = x_1^3\partial_2$ in the plane, the Poincaré inequality may fail for smooth domains, which, therefore, need not be John domains for the related control metric. Thus, in Carnot–Carathéodory spaces the Euclidean regularity is not sufficient to ensure the cone property and an even more striking example of this fact is that in homogeneous groups "gauge balls" may fail to be John domains if the group has step greater than 2 (see [MM2]).

Aim of this paper is to give a contribution to the description of the structure of domains with the cone property in Carnot–Carathéodory spaces. Before describing our results we briefly recall what is already known. Capogna and Garofalo proved in [CG] that $C^{1,1}-$domains in step 2 homogeneous groups are John domains (see also [MM2], where the NTA property is proved under the same assumptions). In the same setting, examples of John and NTA domains are provided by Capogna and Tang [CT], Greshnov [G2] and Capogna, Garofalo and Pauls [CGP]. For the pair of vector fields in the plane $X_1 = \partial_1$, $X_2 = |x_1|^\alpha\partial_2$, $\alpha > 0$, it is not difficult to check that $\alpha-$admissible domains in the sense of [MM1] (see also [FF1] and the discussion in [DGN]) are John domains (compare Jerison's example mentioned above). Finally, the authors give in [MM2] a sufficient condition ensuring the John property for open sets in step 3 homogeneous groups. The analysis becomes considerably harder in groups of higher step and, to our knowledge, no result is known in general homogeneous groups and in Carnot–Carathéodory spaces.

In this paper we study domains with the cone property in a class of Carnot–Carathéodory spaces with no underlying group structure. We consider a system of diagonal vector fields in $\mathbb{R}^n$ of the form $X_1 = \lambda_1(x)\partial_1, ..., X_n = \lambda_n(x)\partial_n$, whose control metric, under suitable assumptions on the functions λ_j, has been studied in detail by Franchi and Lanconelli [FL]. The basic model case we shall study can be exemplified in $\mathbb{R}^3$ by the following vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = x_1^{\alpha_1}\frac{\partial}{\partial x_2}, \quad X_3 = x_1^{\alpha_1}x_2^{\alpha_2}\frac{\partial}{\partial x_3}, \quad \alpha_1, \alpha_2 \in \mathbb{N}. \tag{1.1}$$

Consider an open set in $\mathbb{R}^3$ of the form $\Omega = \{x_3 > \varphi(x)\}$, $x = (x_1, x_2) \in \mathbb{R}^2$, where $\varphi \in C^1(\mathbb{R}^2)$. By the results of [FL], control balls can be written as $Q(x,r) \times$

$]x_3 - F_3(x,r), x_3 + F_3(x,r)[$, where $Q(x,r)$ are suitable rectangles in the plane and $F_3(x,r) > 0$. We say that the boundary $\partial\Omega$ is *admissible* if

$$\sum_{i=1,2} \operatorname{osc}(X_i\varphi, Q(x,r)) \le C\Big(r \sum_{i=1,2} |X_i\varphi(x)|^m + \operatorname{osc}(\lambda_3; Q(x,r))\Big). \tag{1.2}$$

This inequality is a requirement on the oscillation of the derivatives of the function φ along the vector fields X_1 and X_2. The first term in the right hand side vanishes in the characteristic set (i.e. at points where X_1, X_2 and X_3 are tangent to $\partial\Omega$), while the second one gives an amount of oscillation admitted also at characteristic points. This latter is determined by the oscillation of the function $\lambda_3(x) = x_1^{\alpha_1} x_2^{\alpha_2}$, which is strictly related to the size of control balls in the vertical direction. The correct balance between the two terms is given by the power $m = \frac{\alpha_1+\alpha_2+\alpha_1\alpha_2-1}{\alpha_1+\alpha_2+\alpha_1\alpha_2}$. This choice is a key point. In Definition 2.8, generalizing (1.2), we introduce a class of *domains with admissible boundary* in the $n-$dimensional situation. The main result of the paper is the following

Theorem 1.1. *Domains with admissible boundary are John domains.*

The class of domains with admissible boundary is nonempty. In Theorem 2.10 we show that the open set $\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : \big(|x_1|^{2(\alpha_1+1)} + x_2^2\big)^{1+\alpha_2} + x_3^2 < 1\}$ has admissible boundary for the vector fields in (1.1). The basic tool in the proof of this result is the following criterion for checking the admissibility condition (1.2).

Theorem 1.2. *Let $N(x) = |x_1|^{2(\alpha_1+1)} + x_2^2$ and assume that $\varphi(x) = g(N(x))$, where $g \in C^2(0,+\infty)$ is a function such that for some constant $C > 0$*

$$0 \le g'(t) \le C t^{\frac{\alpha_2-1}{2}}, \quad |g''(t)| \le C\frac{g'(t)}{t}, \quad g'(2t) \le C g'(t), \quad t > 0.$$

Then the surface $\{x_3 = \varphi(x_1,x_2)\}$ is admissible.

Our results, together with those proved in [FF1] (which are in a "two dimensional" setting) are the only known examples of regular domains for diagonal vector fields.

The last section of the paper is devoted to some applications of the cone property in the setting of Hörmander vector fields. We prove a Hardy inequality for domains satisfying an exterior "fatness" condition (see Theorem 4.1), a growth estimate near the boundary for $\mathcal{L}$–harmonic functions (see Proposition 4.2) and the integrability of the distance function from the boundary in John domains (see Theorem 4.4). The last two results are used in [MM3].

Before closing the introduction we would like to mention some more papers where notions of regularity for domains are analyzed. Examples of regular and irregular boundary points for the Dirichlet problem have been studied by Hansen and Hueber in [HH]. The notion of uniform domain (or (ε,δ)–domain) for a family of vector fields has been introduced and studied by Garofalo and Nhieu in [GN2], and by Vodop'yanov and Greshnov in [VG] and in [G1]. They generalized Jones extension theorem to the setting of subelliptic Sobolev spaces. Capogna and Garofalo [CG] introduced the notion of NTA domain in Carnot–Carathéodory spaces and studied the boundary behavior of positive $\mathcal{L}$–harmonic functions. The papers [CGN] and [G2] deal with examples of uniform and NTA domains in homogeneous groups of step 2. In the interesting paper [DGN] the problem of the trace of Sobolev functions in Carnot–Carathéodory spaces is studied (see also [MM1]). In the papers [FF1] and [FF2] a new class of domains, called $\varphi-$Harnack domains, is introduced and studied. In the Heisenberg group, the problem of boundary accessibility through rectifiable curves is examined in [BM]. Finally, we mention [MM3], which is the continuation of the research initiated in the present work. In this paper we shall prove that "admissible domains" in the sense of Definition 2.8 are NTA domains.

The plan of the paper is the following. In Section 2 we introduce "admissible domains" and we show that the class is nonempty. In Section 3 we prove that these domains are John domains. Section 4 is devoted to the applications.

Notation. If $u, v \geq 0$, we write $u \lesssim v$ for $u \leq Cv$, where $C \geq 1$ is an absolute constant. Analogously, $u \simeq v$ stands for $u \lesssim v$ and $v \lesssim u$. By d we denote the control metric induced on $\mathbb{R}^n$ by a system of vector fields. For $K \subset \mathbb{R}^n$ we write $\operatorname{diam}(K) = \sup_{x,y \in K} d(x, y)$ and $\operatorname{dist}(x, K) = \inf_{y \in K} d(x, y)$. The Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ will be denoted by $|E|$. If $\gamma : [0, 1] \to \mathbb{R}^n$ is a curve and $0 \leq a \leq b \leq 1$, we denote by $\gamma|_{[a,b]}$ the restriction of γ to the interval $[a, b]$. The coordinate versors of $\mathbb{R}^n$ are denoted by $e_1, \dots, e_n$. Finally, the symbol σ always denotes the "aperture" of a cone. We adopt a slight abuse of notation writing σ to denote an absolute constant which tends to zero as $\sigma \to 0$.

2. Geometry of diagonal vector fields and flat surfaces

In this section we describe the geometry of diagonal vector fields and introduce the basic definitions of regularity for surfaces in the metric space associated with them. Then we provide examples of regular surfaces.

2.1. **Preliminaries and definition of admissible surface.** Consider the vector fields

$$X_j = \lambda_j(x) \frac{\partial}{\partial x_j}, \qquad j = 1, \dots, n \tag{2.1}$$

$$\text{where} \quad \lambda_1(x) = 1 \quad \text{and} \quad \lambda_j(x) = \prod_{i=1}^{j-1} |x_i|^{\alpha_i}, \quad j = 2, ..., n. \tag{2.2}$$

The real numbers α_i satisfy the following condition

$$\alpha_i = 0 \quad \text{or} \quad \alpha_i \in [1, \infty[. \tag{2.3}$$

This condition ensures that the functions λ_j, and thus the vector fields X_j, are locally Lipschitz continuous. If the numbers α_i are integers then we can change the functions λ_j in (2.2) writing $x_i^{\alpha_i}$ instead of $|x_i|^{\alpha_i}$. The vector fields become a Hörmander system.

We introduce some notation, following [FL]. For all $j = 1, ..., n$ define inductively the functions $F_j : \mathbb{R}^n \times [0, +\infty) \to [0, +\infty)$ by

$$\begin{aligned} &F_1(x, r) = r, \qquad F_2(x, r) = r\lambda_2(|x_1| + F_1(x, r)), \quad \dots \\ &F_j(x, r) = r\lambda_j\big(|x_1| + r, |x_2| + F_2(x, r), \dots, |x_{j-1}| + F_{j-1}(x, r)\big). \end{aligned} \tag{2.4}$$

An inspection of the explicit form (2.2) of the functions λ_j shows immediately that

$$F_{j+1}(x, r) = F_j(x, r)\big(|x_j| + F_j(x, r)\big)^{\alpha_j}, \quad j = 1, \dots, n-1. \tag{2.5}$$

Note that $F_j(x, r)$ actually depends only on $x_1, \dots, x_{j-1}$. Moreover, $r \mapsto F_j(x, r)$ is increasing and it satisfies the following doubling property

$$F_j(x, 2r) \le CF_j(x, r), \quad x \in \mathbb{R}^n,\ 0 < r < \infty \tag{2.6}$$

for all $j = 1, ..., n$. This also implies

$$F_j(x, r + s) \le C(F_j(x, r) + F_j(x, s)), \quad 0 < r, s < \infty. \tag{2.7}$$

Finally, for any fixed $x \in \mathbb{R}^n$ the function $F_j(x, \cdot)$ is strictly increasing and maps $]0, +\infty[$ onto itself. We denote its inverse by $G_j(x, \cdot) = F_j(x, \cdot)^{-1}$.

For each $j = 1, ..., n$ define inductively the real number d_j by

$$d_1 = 1, \quad d_2 = 1 + \alpha_1, \quad \dots \quad , d_j = 1 + \sum_{i=1}^{j-1} d_i\alpha_i = (1 + \alpha_1) \cdot ... \cdot (1 + \alpha_{j-1}). \tag{2.8}$$

We say that d_j is the *degree of the variable* x_j. Note that $F_j(0, r) = r^{d_j}$.

It is well known that the vector fields (2.1) induce on $\mathbb{R}^n$ a metric d in the following way (see [FL], [FP] and [NSW]). A Lipschitz continuous curve $\gamma : [0, T] \to \Omega$, $T \ge 0$, is *subunit* if there exists a vector of measurable functions $h = (h_1, ..., h_n) : [0, T] \to \mathbb{R}^n$ such that $\dot\gamma(t) = \sum_{j=1}^n h_j(t)X_j(\gamma(t))$, $|h(t)| \le 1$ for a.e. $t \in [0, T]$. Define $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty)$ by setting

$$\begin{aligned} d(x, y) = \inf\{T \ge 0 :\ &\text{there exists a subunit curve } \gamma : [0, T] \to \mathbb{R}^n \\ &\text{such that } \gamma(0) = x \text{ and } \gamma(T) = y\}. \end{aligned} \tag{2.9}$$

Notice that the definition of the metric d still makes sense for any system of smooth vector fields $X_1, ..., X_m$, even with $m < n$, provided that $d(x, y)$ is finite for all x, y (this happens for instance if the vector fields satisfies the Hörmander condition, see [NSW]). We denote by $B(x, r)$ the balls in $\mathbb{R}^n$ defined by the metric d.

The structure of the control balls can be described by means of the following boxes:

$$\mathrm{Box}(x, r) := \{x + h : |h_j| < F_j(x, r),\ j = 1, ..., n\}. \tag{2.10}$$

The following theorem is proved in [FL].

Theorem 2.1. *There exists a constant $C > 0$ such that:*

$$\begin{aligned} &\mathrm{Box}(x, C^{-1}r) \subset B(x, r) \subset \mathrm{Box}(x, Cr), \qquad x \in \mathbb{R}^n, r \in]0, +\infty[\\ &C^{-1}d(x, y) \le \sum_{j=1}^n G_j(x, |y_j - x_j|) \le Cd(x, y), \qquad x, y \in \mathbb{R}^n. \end{aligned} \tag{2.11}$$

Theorem 2.1, the triangle inequality and the doubling property (2.6) (or a direct computation) show the estimate

$$F_j(x+F_k(x,r)e_k,s) \le CF_j(x,s) \quad x\in\mathbb{R}^n,\ 0<r\le s<+\infty. \tag{2.12}$$

Before proceeding we introduce the following convention. If $j=1,...,n$ and $x=(x_1,...,x_n)\in\mathbb{R}^n$ we write $\hat{x}_j=(x_1,...,x_{j-1},x_{j+1},...,x_n)\equiv(x_1,...,x_{j-1},0,x_{j+1},...,x_n)$.

In the following section we shall study regularity properties connected with the boundary of an open set $\Omega\subset\mathbb{R}^n$ of class C^1. Here we introduce the basic assumptions on $\partial\Omega$. Given a point $x\in\partial\Omega$, we write locally $\partial\Omega$ as a graph of the form $x_j=\varphi(\hat{x}_j)$ for some $j=1,\dots,n$. We first discuss the case $j=n$, which is the most significant.

Introduce the $(n-1)$–dimensional box

$$\mathrm{Box}_n(\hat{x}_n,r)=\{\hat{x}_n+\hat{h}_n : |h_i|<F_i(\hat{x}_n,r),\ i=1,\dots,n-1\}, \tag{2.13}$$

and let $\Lambda_n(\hat{x}_n,r)=\sup\limits_{\hat{y}_n\in\mathrm{Box}_n(\hat{x}_n,r)}|\lambda_n(\hat{y}_n)-\lambda_n(\hat{x}_n)|$.

The following proposition shows some properties of Λ_n that will be needed in the following and in the paper [MM3]. The proof is postponed to the Appendix. It relies upon the following well known equivalence, which will be used several times in the sequel. Let $\alpha\ge1$. Then

$$(t+r)^\alpha-t^\alpha\simeq\alpha r(t+r)^{\alpha-1},\quad t\ge0,\quad r\ge0. \tag{2.14}$$

Proposition 2.2. *Assume that at least one of the numbers α_j, $j=1,\dots,n$, is non zero. There exists a constant $\eta>0$ such that for all $\hat{x}_n\in\mathbb{R}^{n-1}$, $R>0$ and $a\in\,]0,1]$*

$$\Lambda_n(\hat{x}_n,aR)\le h(a)\Lambda_n(\hat{x}_n,R),\quad \textit{where}\quad h(a)=\frac{a}{a+\eta(1-a)}. \tag{2.15}$$

Moreover, $\Lambda_n(\hat{x}_n,r))\ge r^{d_n-1}$, $\Lambda_n(\hat{x}_n,r)\le(C/r)F_n(x,r)$ and $\Lambda_n(x,2r)\le C\Lambda_n(x,r)$ for some constant $C>0$, and for all $r>0$ and $\hat{x}_n\in\mathbb{R}^{n-1}$.

To introduce our definition of "admissible surface" we proceed as follows. First of all we give the definition for a graph of the form $x_n=\varphi(\hat{x}_n)$. This is the most degenerate case and contains all the difficulties of the problem. Then we will show that a graph of the form $x_j=\varphi(\hat{x}_j)$ with $j\ne n$ can be studied reducing to the previous case. Finally, in Definition 2.8 we introduce the notion of open set with admissible boundary. Recall the standard notation $\mathrm{osc}(f,A):=\sup\limits_{x,y\in A}|f(x)-f(y)|$.

Definition 2.3. Let $\varphi\in C^1(\mathbb{R}^{n-1})$. The surface $\{x_n=\varphi(\hat{x}_n)\}$ is said to be *admissible* if there exist $C>0$ and $r_0>0$ such that for all $\hat{x}_n\in\mathbb{R}^{n-1}$, $r\in\,]0,r_0]$

$$\sum_{i\ne n}\mathrm{osc}(X_i\varphi,\mathrm{Box}_n(\hat{x}_n,r))\le C\Big(r\sum_{i\ne n}|X_i\varphi(\hat{x}_n)|^{\frac{d_n-2}{d_n-1}}+\Lambda_n(\hat{x}_n,r)\Big). \tag{2.16}$$

Note that, if we are not in the Euclidean case, i.e. at least one of the numbers α_i is non zero, then $d_n\ge2$ (compare (2.3)) and the exponent $\frac{d_n-2}{d_n-1}$ is non negative.

We are actually interested in surfaces which are the boundary of bounded sets. Definition 2.3 can be stated also for a bounded graph $x_n=\varphi(\hat{x}_n)$, letting $\hat{x}_n$ belong to a bounded open set of $\mathbb{R}^{n-1}$.

Proposition 2.4. *Let $\varphi \in C^1(\mathbb{R}^{n-1})$ satisfy* (2.16). *Then there exists $C > 0$ such that for all $\hat{x}_n \in \mathbb{R}^{n-1}$, $r \in]0, r_0]$*

$$\operatorname{osc}(\varphi, \operatorname{Box}_n(\hat{x}_n, r)) \le C\Big(r \sum_{i \neq n} |X_i\varphi(\hat{x}_n)| + r\Lambda_n(\hat{x}_n, r)\Big). \tag{2.17}$$

Proof. Fix $\hat{x}_n, \hat{y}_n \in \mathbb{R}^{n-1}$ and let $\delta = d(\hat{x}_n, \hat{y}_n)$. Then there is a subunit curve $\gamma : [0,\delta] \to \mathbb{R}^{n-1} = \mathbb{R}^{n-1} \times \{0\}$ such that $\gamma(0) = \hat{x}_n$ and $\gamma(\delta) = \hat{y}_n$. Then we have

$$|\varphi(\hat{x}_n) - \varphi(\hat{y}_n)| \le \int_0^\delta \sum_{i\neq n} |X_i\varphi(\gamma(t))| dt \le \delta \sup_{\operatorname{Box}_n(\hat{x}_n,\delta)} \sum_{i \neq n} |X_i\varphi|. \tag{2.18}$$

By (2.16)

$$\begin{aligned}
\sup_{\operatorname{Box}_n(\hat{x}_n,\delta)} \sum_{i\neq n} |X_i\varphi| &\le \sum_{i\neq n} |X_i\varphi(\hat{x}_n)| + \sum_{i\neq n} \operatorname{osc}(X_i\varphi, \operatorname{Box}_n(\hat{x}_n,\delta)) \\
&\lesssim \sum_{i\neq n} |X_i\varphi(\hat{x}_n)| + \delta \sum_{i\neq n} |X_i\varphi(\hat{x}_n)|^{\frac{d_n-2}{d_n-1}} + \Lambda_n(\hat{x}_n,\delta) \\
&\lesssim \sum_{i\neq n} |X_i\varphi(\hat{x}_n)| + \Lambda_n(\hat{x}_n,\delta).
\end{aligned}$$

Here we used Hölder inequality $\delta|X_i\varphi(\hat{x}_n)|^{\frac{d_n-2}{d_n-1}} \lesssim \delta^{d_n-1} + |X_i\varphi(\hat{x}_n)|$ and the inequality $\delta^{d_n-1} \le \Lambda_n(\hat{x}_n,\delta)$ proved in Proposition 2.2. Now, (2.17) follows from Proposition 2.1 and from the doubling property $\Lambda_n(\hat{x}_n, 2r) \le C\Lambda_n(\hat{x}_n, r)$ proved in Proposition 2.2. □

Remark 2.5. It can be shown that given $\varphi \in C^2(\mathbb{R}^{n-1})$ satisfying

$$\sum_{i,j=1}^{n-1} |X_iX_j\varphi(\hat{x}_n)| \le C \sum_{i=1}^{n-1} |X_i\varphi(\hat{x}_n)|^{\frac{d_n-2}{d_n-1}} \tag{2.19}$$

for all $\hat{x}_n \in \mathbb{R}^{n-1}$ with $x_1x_2 \cdot \ldots \cdot x_{n-1} \neq 0$, then the surface $\{x_n = \varphi(\hat{x}_n)\}$ is admissible. It is not difficult to see that condition (2.19) implies (2.16). This can be checked with a suitable adaptation of Gronwall inequality. Condition (2.19) is also easier to check than (2.16). Surfaces of the form $x_n = |x_1|^m$, where m is large enough satisfy (2.19). The drawback is that this condition is not refined enough in order to find examples of admissible surfaces which are boundaries of bounded sets (see Theorems 2.9 and 2.10).

Now we introduce admissible surfaces of the type $\{x_j = \varphi(\hat{x}_j)\}$ when $j \neq n$. We would like to give a definition similar to Definition 2.3. The set $\operatorname{Box}_n(\hat{x}_n, r)$ is the intersection of $\operatorname{Box}(x, r)$ with the plane $\{y \in \mathbb{R}^n : y_n = x_n\}$. When $j \neq n$ the intersection of $\operatorname{Box}(x, r)$ with the plane $\{y \in \mathbb{R}^n : y_j = x_j\}$ varies with x_j. Thus (2.13) can not be trivially generalized. But, roughly speaking, the vector fields $X_{j+1}, \ldots, X_n$ are "more degenerate" than X_j, and this suggests that the dependence of the function $\varphi(\hat{x}_j)$ on $x_{j+1}, \ldots, x_n$ needs a less careful control than the dependence on $x_1, \ldots, x_{j-1}$.

In order to make this remark rigorous we can proceed in the following way. Define new functions and vector fields

$$\widetilde{\lambda}_i(x) = \begin{cases} \lambda_i(x) & \text{if } i \le j, \\ \lambda_j(x) & \text{if } i \ge j. \end{cases} \qquad \text{and } \widetilde{X}_i = \widetilde{\lambda}_i \partial_i, \quad i = 1, \dots, n. \tag{2.20}$$

In this position we can view the variable x_j as the $n-$th variable with respect to the new vector fields. All previous results hold for these vector fields. The functions $\widetilde{F}_i(x,r)$ are defined exactly as in (2.4). Set $\widetilde{\mathrm{Box}}(x,r) = \{x+h : |h_i| < \widetilde{F}_i(x,r),\, i = 1,...,n\}$ and denote by $\widetilde{d}$ the metric constructed as in (2.9) using subunit curves with respect to the vector fields $\widetilde{X}_j$. Let $\widetilde{B}(x,r)$ be the corresponding balls. In the following proposition we list some easy relations between the distances d and $\widetilde{d}$.

Proposition 2.6. *For any $C_1 > 0$ there is $C_2 > 0$ such that:*

(i) *if $|x_j|, |y_j|, r < C_1$ then $B(x,r) \subset \widetilde{B}(x, C_2 r)$ and $\widetilde{d}(x,y) \le C_2 d(x,y)$;*
(ii) *writing $x' = (x_1, ..., x_j)$ and $x'' = (x_{j+1}, ..., x_n)$, we have $d((x',x''),(y',x'')) \simeq \widetilde{d}((x',x''),(y',x''))$.*

Proof. We have $\widetilde{F}_i(\hat{x}_j, r) = \widetilde{F}_i(\hat{x}_j, r)$, if $i \le j$, while for $i > j$ it is $\widetilde{F}_i(x,r) = F_j(x,r)$. Then, if $i > j$,

$$\begin{aligned} F_i(x,r) &= F_j(x,r)\big(|x_j| + F_j(x,r)\big)^{\alpha_j} \cdots \big(|x_{i-1}| + F_{i-1}(x,r)\big)^{\alpha_{i-1}} \\ &\le C F_j(x,r) \le F_j(x, Cr) = \widetilde{F}_i(x, Cr), \end{aligned}$$

as soon as $|x|, r \le C$. Then $\mathrm{Box}(x,r) \subset \widetilde{\mathrm{Box}}(x, Cr)$. Thus (i) follows by Theorem 2.1.

In order to see (ii) recall that the function $\widetilde{G}_i(x,\cdot)$ is the inverse of $\widetilde{F}_i(x,\cdot)$. Moreover, if $i \le j$ then $\widetilde{F}_i(x,r) = F_i(x,r)$. Thus Theorem 2.1 gives

$$d\big((x',x''),(y',x'')\big) \simeq \sum_{i=1}^{j} G_i(x', |x_i - y_i|) = \sum_{i=1}^{j} \widetilde{G}_i(x', |x_i - y_i|) \simeq d\big((x',x''),(y',x'')\big)$$

This concludes the proof of (ii). ☐

The sections of the boxes $\widetilde{\mathrm{Box}}(x,r)$ with the planes $\{y \in \mathbb{R}^n : y_j = x_j\}$ do not depend on x_j. Thus we can set with unambiguous meaning

$$\widetilde{\mathrm{Box}}_j(\hat{x}_j, r) = \{\hat{x}_j + \hat{h}_j : |h_i| < \widetilde{F}_i(\hat{x}_j, r),\, i \ne j\} \tag{2.21}$$

$$\text{and} \quad \widetilde{\Lambda}_j(\hat{x}_j) = \sup_{\hat{y}_j \in \widetilde{\mathrm{Box}}_j(\hat{x}_j, r)} |\widetilde{\lambda}_j(\hat{y}_j) - \widetilde{\lambda}_j(\hat{x}_j)|. \tag{2.22}$$

The function $\widetilde{\Lambda}_j$ enjoys the properties of Proposition 2.2, the subscript j replacing the subscript n.

We are ready to give the general definition of admissible surface and of set with admissible boundary.

Definition 2.7. Let $\varphi \in C^1(\mathbb{R}^{n-1})$. The surface $\{x_j = \varphi(\hat{x}_j)\}$ is said to be *admissible* if there exist $C > 0$ and $r_0 > 0$ such that for all $\hat{x}_j \in \mathbb{R}^{n-1}$, $r \in]0, r_0]$

$$\sum_{i \neq j} \operatorname{osc}(\widetilde{X}_i\varphi, \widetilde{\operatorname{Box}}_j(\hat{x}_j, r)) \leq C\Big(r \sum_{i \neq j} |\widetilde{X}_i\varphi(\hat{x}_j)|^{\frac{d_j-2}{d_j-1}} + \widetilde{\Lambda}_j(\hat{x}_j, r)\Big). \tag{2.23}$$

Definition 2.8 (Domain with admissible boundary)**.** A connected bounded open set $\Omega \subset \mathbb{R}^n$ is said to be *with admissible boundary* if it is of class C^1 and for all $x \in \partial\Omega$ there exists a neighborhood $\mathcal{U}$ of x such that $\partial\Omega \cap \mathcal{U}$ is an admissible surface according to Definitions 2.3 and 2.7.

2.2. **Examples of admissible domains in $\mathbb{R}^3$.** We shall study examples of admissible surfaces and of sets with admissible boundary in $\mathbb{R}^3$. Consider the functions $\lambda_1 \equiv 1$, $\lambda_2 = |x_1|^{\alpha_1}$, $\lambda_3 = |x_1|^{\alpha_1}|x_2|^{\alpha_2}$ and the corresponding diagonal vector fields

$$X_1 = \partial_1, \quad X_2 = |x_1|^{\alpha_1}\partial_2, \quad X_3 = |x_1|^{\alpha_1}|x_2|^{\alpha_2}\partial_3. \tag{2.24}$$

We consider the case $\alpha_i \geq 1$, $i = 1, 2$. The degrees of the variables x_1, x_2 and x_3 are respectively $d_1 = 1$, $d_2 = 1 + \alpha_1$, $d_3 = (1 + \alpha_1)(1 + \alpha_2)$.

We begin with the study of admissible surfaces of the form $\{x_3 = \varphi(x_1, x_2)\}$. We write $x = (x_1, x_2)$ and $|X\varphi| = |X_1\varphi| + |X_2\varphi|$. If $\varphi \in C^1(\mathbb{R}^2)$ condition (2.16) reads

$$\sum_{i=1,2} \operatorname{osc}(X_i\varphi, \operatorname{Box}_3(x, r)) \lesssim r|X\varphi(x)|^{\frac{d_3-2}{d_3-1}} + \Lambda_3(x, r), \tag{2.25}$$

where $\operatorname{Box}_3(x, r) = \{(x_1 + u_1F_1(x, r), x_2 + u_2F_2(x_1, r)) : |u_1|, |u_2| \leq 1\}$ and $\Lambda_3(x, r) = \sup_{\operatorname{Box}_3(x,r)} |\lambda_3 - \lambda_3(x)|$. Here $F_1(x, r) = r$ and $F_2(x, r) = r(|x_1| + r)^{\alpha_1}$.

We can write explicitly, by (2.14) (see also (A.2) in the Appendix)

$$\begin{aligned} \Lambda_3(x) &\gtrsim r\big(|x_1| + r\big)^{\alpha_1-1}\big(|x_2| + F_2(x_1, r)\big)^{\alpha_2} + |x_1|^{\alpha_1}F_2(x_1, r)\big(|x_2| + F_2(x_1, r)\big)^{\alpha_2-1} \\ &\gtrsim r\big(|x_1| + r\big)^{\alpha_1-1}\big(|x_2| + F_2(x_1, r)\big)^{\alpha_2}. \end{aligned} \tag{2.26}$$

Theorem 2.9. *Let $N(x) = |x_1|^{2d_2} + x_2^2$ and assume that $\varphi(x) = g(N(x))$, where $g \in C^2(0, +\infty)$ is a function such that for some constant $C > 0$*

$$0 \leq g'(t) \leq Ct^{\frac{d_3}{2d_2}-1} \equiv Ct^{\frac{\alpha_2-1}{2}}, \quad |g''(t)| \leq C\frac{g'(t)}{t}, \quad g'(2t) \leq Cg'(t), \quad t > 0. \tag{2.27}$$

Then the surface $\{x_3 = \varphi(x_1, x_2)\}$ is admissible.

Proof. We check (2.25). Note that in this example inequality (2.19) fails. Without loss of generality we assume $x_1, x_2 > 0$. A short computation gives

$$\begin{aligned} |X_1\varphi(x)| &\simeq x_1^{2\alpha_1+1}g'(N(x)) = x_1^{\alpha_1}\{x_1^{d_2}g'(N(x))\} := x_1^{\alpha_1}h_1(x), \\ |X_2\varphi(x)| &\simeq x_1^{\alpha_1}\{x_2g'(N(x))\} := x_1^{\alpha_1}h_2(x). \end{aligned} \tag{2.28}$$

Note that $|h(x)| = |(h_1(x), h_2(x))| = N(x)^{1/2} g'(x)$, and then we get $|X\varphi(x)| \simeq |x_1|^{\alpha_1} N(x)^{1/2} g'(N(x))$. Moreover

$$\begin{aligned}
\mathrm{osc}(X_i\varphi, \mathrm{Box}_3(x,r)) &\lesssim |(x_1+r)^{\alpha_1} h_i(x+F(x,r)) - x_1^{\alpha_1} h_i(x)| \\
&\le ((x_1+r)^{\alpha_1} - x_1^{\alpha_1}) h_i(x) + (x_1+r)^{\alpha_1}(h_i(x+F(x,r)) - h_i(x)) \\
&\lesssim \alpha_1 r (x_1+r)^{\alpha_1-1} h_i(x) + (x_1+r)^{\alpha_1}(h_i(x+F(x,r)) - h_i(x)),
\end{aligned}$$

where we used (2.14). Writing $h = (h_1, h_2)$ we find the estimate from above for the oscillation

$$\sum_{i=1,2} \mathrm{osc}(X_i\varphi, \mathrm{Box}_3(x,r)) \lesssim r(x_1+r)^{\alpha_1-1}|h(x)| + (x_1+r)^{\alpha_1}|h(x+F(x,r)) - h(x)|. \tag{2.29}$$

We already know that $|h(x)| \simeq N(x)^{1/2} g'(N(x))$. In order to estimate the last term in the right hand side we use the following inequality (as in (2.18))

$$|h_i(x+F(x,r)) - h_i(x)| \lesssim r \sup_{y\in \mathrm{Box}_3(x,r),\, k=1,2} |X_k h_i(y)|.$$

The computation of the second derivatives and condition $g''(t) \le C g'(t)/t$ give

$$\begin{aligned}
X_1 h_1(x) &\simeq x_1^{\alpha_1}\{g'(N(x)) + x_1^{2(\alpha_1+1)} g''(N(x))\} \simeq x_1^{\alpha_1} g'(N(x)) \\
X_2 h_1(x) &\simeq x_2 x_1^{2\alpha_1+1} g''(N(x)) \lesssim \frac{x_1^{\alpha_1+1} x_2}{N(x)} x_1^{\alpha_1} g'(N(x)) \lesssim x_1^{\alpha_1} g'(N(x)) \\
X_1 h_2(x) &= x_2 x_1^{2\alpha_1+1} g''(N(x)) \lesssim x_1^{\alpha_1} g'(N(x)) \\
X_2 h_2(x) &= x_1^{\alpha_1}\{g'(N(x)) + x_2^2 g''(N(x))\} \simeq x_1^{\alpha_1} g'(N(x)).
\end{aligned}$$

Hence we find

$$|h(x+F(x,r)) - h(x)| \lesssim r(x_1+r)^{\alpha_1} g'\big(N(x+F(x,r))\big).$$

Coming back to (2.29), we see that condition (2.25) is guaranteed by

$$\begin{aligned}
&r(x_1+r)^{\alpha_1-1} N(x)^{1/2} g'(N(x)) + r(x_1+r)^{2\alpha_1} g'(N(x+F(x,r))) \\
&\quad\lesssim r\big\{x_1^{\alpha_1} N(x)^{1/2} g'(N(x))\big\}^{\frac{d_3-2}{d_3-1}} + r\big(x_1+r\big)^{\alpha_1-1}\big(x_2 + F_2(x_1,r)\big)^{\alpha_2},
\end{aligned} \tag{2.30}$$

where the first term in the right hand side is provided by (2.28) and the second one comes from (2.26).

Now two cases need to be distinguished: (A) $x_2 \ge x_1^{\alpha_1+1}$; (B) $x_2 < x_1^{\alpha_1+1}$.

Study of Case (A). We ignore the contribution of the first term in the right hand side of (2.30) and we consider the second one only. Thus (2.30) is implied by

$$N(x)^{1/2} g'(N(x)) + (x_1+r)^{\alpha_1+1} g'(N(x+F(x,r))) \lesssim (x_2 + F_2(x_1,r))^{\alpha_2}. \tag{2.31}$$

Notice that in Case (A) $N(x) \simeq x_2^2$.

We distinguish the following two subcases: (A1) $x_2 \le r^{\alpha_1+1}$; (A2) $x_2 > r^{\alpha_1+1}$.

Case (A1). We majorize the left hand side of (2.31) using $x_1 \le r$ and $x_2 \le r^{\alpha_1+1}$, and we set $x = 0$ in the right hand side obtaining the stronger condition

$r^{\alpha_1+1}g'(r^{2(\alpha_1+1)}) \lesssim r^{\alpha_2(\alpha_1+1)}$, which can be rewritten as $g'(r^{2d_2}) \lesssim r^{d_3-2d_2}$. This last inequality is satisfied by assumption (2.27).

Case (A2). Condition (2.31) is implied by

$$N(x)^{1/2}g'(N(x)) + (x_1+r)^{\alpha_1+1}g'(N(x+F(x,r))) \lesssim x_2^{\alpha_2}.$$

We can use $x_1^{\alpha_1+1}, r^{\alpha_1+1} \le x_2$ and $N(x) \simeq N(x+F(x,r)) \simeq x_2^2$. This gives $x_2 g'(x_2^2) \lesssim x_2^{\alpha_2}$, i.e. $g'(x_2^2) \lesssim x_2^{\frac{d_3}{d_2}-2}$. The latter inequality holds by assumption.

Study of Case (B). Here we have $N(x) \simeq x_1^{2(\alpha_1+1)}$. Two subcases must be distinguished: (B1) $x_1 \le r$; (B2) $x_1 > r$.

Case (B1). In this case we ignore the contribution of the first term in the right hand side of (2.30) and consider the second term only. Condition (2.30) is guaranteed by

$$N(x)^{1/2}g'(N(x)) + (x_1+r)^{\alpha_1+1}g'(N(x+F(x,r))) \lesssim (x_2+F_2(x_1,r))^{\alpha_2}. \tag{2.32}$$

Set $x=0$ in the right hand side of (2.32) and use $x_1 \le r$ and $x_2 < r^{\alpha_1+1}$ to find the stronger inequality $r^{\alpha_1+1}g'(r^{2(\alpha_1+1)}) \le r^{\alpha_2(\alpha_1+1)}$, i.e. $g'(r^{2d_2}) \le r^{d_3-2d_2}$.

Case (B2). We use here the contribution of the first term in the right hand side of (2.30) and we get the stronger inequality

$$\begin{aligned}(x_1+r)^{\alpha_1-1}N(x)^{1/2}g'(N(x)) + (x_1+r)^{2\alpha_1}g'(N(x+F(x,r)))& \\ \lesssim \left\{x_1^{\alpha_1}N(x)^{1/2}g'(N(x))\right\}^{\frac{d_3-2}{d_3-1}}&.\end{aligned}$$

Since $r \le x_1$ and $x_2 \le x_1^{\alpha_1+1}$, we finally find the stronger condition $x_1^{2\alpha_1}g'(x_1^{2d_2}) \lesssim \{x_1^{\alpha_1+d_2}g'(x_1^{2d_2})\}^{\frac{d_3-2}{d_3-1}}$, i.e. $g'(x_1^{2d_2}) \lesssim x_1^{d_3-2d_2}$. The proof is ended. □

Finally, we give an example in $\mathbb{R}^3$ of bounded open set with admissible boundary.

Theorem 2.10. *The open set* $\Omega = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : \big(|x_1|^{2(\alpha_1+1)} + x_2^2\big)^{1+\alpha_2} + x_3^2 < 1\}$ *has admissible boundary.*

Proof. Let $\varepsilon \in (0,1)$ be fixed. The surface $\partial\Omega \cap \{|x_3| > \varepsilon\}$ can be studied by means of Theorem 2.9. Indeed, the lower cap can be written in the form

$$x_3 = -\big(1 - N(x_1,x_2)^{1+\alpha_2}\big)^{1/2} = g(N(x_1,x_2))$$

where $N(x_1,x_2) = |x_1|^{2(\alpha_1+1)} + x_2^2$, and for any fixed $t_0 < 1$ it is easy to see that the function $g(t) = -\big(1-t^{1+\alpha_2}\big)^{1/2}$ satisfies conditions (2.27) for $t \in (0,t_0)$.

The surface $\partial\Omega \cap \{|x_3| < \varepsilon\}$ is non characteristic, and hence admissible, away from a neighborhood of its intersection with the plane $x_1 = 0$. To complete the proof of the theorem it is enough to show that $\partial\Omega$ is admissible in a neighborhood of $(0,1,0)$. Here, $\partial\Omega$ can be parameterized as follows

$$x_2 = \Big(\big(1-x_3^2\big)^{\frac{1}{1+\alpha_2}} - x_1^{2(\alpha_1+1)}\Big)^{1/2} := \varphi(x_1,x_3).$$

We check that the function φ satisfies condition (2.23). To this aim, as suggested by (2.20), we consider the vector fields $\widetilde{X}_1 = \partial_1$, $\widetilde{X}_2 = |x_1|^{\alpha_1}\partial_2$, $\widetilde{X}_3 = |x_1|^{\alpha_1}\partial_3$. We have to check

$$\sum_{i=1,3} \operatorname{osc}(\widetilde{X}_i\varphi, \widetilde{\operatorname{Box}}_2(\hat{x}_2, r)) \lesssim r(|\widetilde{X}_1\varphi(\hat{x}_2)| + |\widetilde{X}_3\varphi(\hat{x}_2)|)^{\frac{d_2-2}{d_2-1}} + \widetilde{\Lambda}_2(\hat{x}_2, r), \tag{2.33}$$

where $d_2 = 1 + \alpha_1$ and

$$\widetilde{\operatorname{Box}}_2(\hat{x}_2, r) = \{\hat{x}_2 + \hat{h}_2 : |h_1| < \widetilde{F}_1(\hat{x}_2, r) = r,\ |h_3| < \widetilde{F}_3(\hat{x}_2, r) = r(|x_1| + r)^{\alpha_1}\}.$$

Write $x = (x_1, x_3)$, $\widetilde{F} = (\widetilde{F}_1, \widetilde{F}_3)$. An easy computation shows that

$$|\widetilde{X}_1\varphi(x)| = h_1(x)|x_1|^{2\alpha_1+1} \quad \text{and} \quad |\widetilde{X}_2\varphi(x)| = h_2(x)|x_3|\,|x_1|^{\alpha_1},$$

where h_1 and h_2 are positive Lipschitz continuous functions in a neighborhood of the origin (we do not need their explicit form here). Assume without loss of generality $x_1, x_3 \geq 0$.

We estimate the left hand side of (2.33). First, we have

$$\begin{aligned} \operatorname{osc}(\widetilde{X}_1\varphi, \widetilde{\operatorname{Box}}_2(\hat{x}_2, r)) &\lesssim \big|h_1(x + \widetilde{F}(x,r))(x_1 + r)^{2\alpha_1+1} - h_1(x)x_1^{2\alpha_1+1}\big| \\ &\lesssim \big|h_1(x + \widetilde{F}(x,r)) - h_1(x)\big|\, x_1^{2\alpha_1+1} \\ &\qquad + h_1(x + \widetilde{F}(x,r))\,\big|(x_1 + r)^{2\alpha_1+1} - x_1^{2\alpha_1+1}\big| \\ &\lesssim r x_1^{2\alpha_1+1} + r(x_1 + r)^{2\alpha_1} \lesssim r(x_1 + r)^{2\alpha_1}. \end{aligned} \tag{2.34}$$

We used the Lipschitz continuity of h_1 and the estimate $|\widetilde{F}(x,r)| \lesssim r$. Moreover

$$\begin{aligned} \operatorname{osc}(\widetilde{X}_3\varphi, \widetilde{\operatorname{Box}}(\hat{x}_2, r)) &\lesssim \big|h_2(x + \widetilde{F}(x,r))(x_1 + r)^{\alpha_1}(x_3 + \widetilde{F}_3(x_1, r)) - h_2(x)x_1^{\alpha_1}x_3\big| \\ &\lesssim \big|h_2(x + \widetilde{F}(x,r)) - h_2(x)\big|\, x_1^{\alpha_1}x_3 \\ &\qquad + h_2(x + \widetilde{F}(x,r))\,\big|(x_1 + r)^{\alpha_1}(x_3 + \widetilde{F}_3(x_1, r)) - x_1^{\alpha_1}x_3\big| \\ &\lesssim r x_1^{\alpha_1}x_3 + \big|(x_1 + r)^{\alpha_1}(x_3 + \widetilde{F}_3(x_1, r)) - x_1^{\alpha_1}x_3\big|. \end{aligned} \tag{2.35}$$

The last term can be evaluated as follows

$$\begin{aligned} &(x_1 + r)^{\alpha_1}(x_3 + \widetilde{F}_3(x_1, r)) - x_1^{\alpha_1}x_3 \\ &\qquad \lesssim (x_1 + r)^{\alpha_1} - x_1^{\alpha_1}(x_3 + \widetilde{F}_3(x_1, r)) + x_1^{\alpha_1}\big((x_3 + \widetilde{F}_3(x_1, r)) - x_3\big) \\ &\qquad \lesssim r(x_1 + r)^{\alpha_1-1} + \widetilde{F}_3(x_1, r)x_1^{\alpha_1} \lesssim r(x_1 + r)^{\alpha_1-1}. \end{aligned} \tag{2.36}$$

Taking into account (2.34), (2.35), (2.36) and the equivalence $\widetilde{\Lambda}_2(\hat{x}_2, r) \simeq r(x_1 + r)^{\alpha_1-1}$, condition (2.33) is implied by $(x_1+r)^{2\alpha_1} + x_1^{\alpha_1}x_3 + (x_1+r)^{\alpha_1-1} \lesssim (x_1+r)^{\alpha_1-1}$, which is trivially satisfied. □

3. Domains with admissible boundary are John domains

Before proving that admissible domains are John domains, we briefly recall the definition of John domain.

Definition 3.1. A bounded open set $\Omega \subset (\mathbb{R}^n, d)$ is a *John domain* if there exist $x_0 \in \Omega$ and $\sigma > 0$ such that for all $x \in \Omega$ there exists a continuous curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = x_0$ and

$$\operatorname{dist}\big(\gamma(t), \partial\Omega\big) \geq \sigma \operatorname{diam}\big(\gamma|_{[0,t]}\big). \tag{3.1}$$

A curve satisfying (3.1) will be called a *John curve*, x_0 will be called the *center* and σ the *constant* of Ω.

In order to avoid a possible confusion concerning definitions, we stress that in general metric spaces the definition of John domain is given with $\operatorname{length}(\gamma|_{[0,t]})$ replacing $\operatorname{diam}(\gamma|_{[0,t]})$ in (3.1). Anyway, by a general result due to Martio and Sarvas [MS, Theorem 2.7], such definitions are in fact equivalent in doubling metric spaces with geodesics. Moreover, in our proofs we shall always work with John curves γ satisfying $\operatorname{diam}(\gamma|_{[0,t]}) \simeq d(\gamma(t), \gamma(0))$.

For the proof of the following easy proposition we refer to [MM2].

Proposition 3.2. *Let $\Omega \subset (\mathbb{R}^n, d)$ be a bounded open set and for any $r > 0$ define $\Omega_r = \{y \in \Omega : \operatorname{dist}(y, \partial\Omega) > r\}$. Assume that there exist $r > 0$ and $\sigma > 0$ such that Ω_r is arcwise connected and such that for any $x \in \Omega$ there exists a continuous curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) \in \Omega_r$ and*

$$\operatorname{dist}(\gamma(t), \partial\Omega) \geq \sigma \operatorname{diam}(\gamma|_{[0,t]}) \tag{3.2}$$

for all $t \in [0,1]$. Then Ω is a John domain.

Now we are able to state our main theorem.

Theorem 3.3. *If Ω is an open set with admissible boundary according to Definition 2.8, then it is a John domain in the metric space $(\mathbb{R}^n, d)$.*

Proof. We use Proposition 3.2. Given $\bar{x} \in \partial\Omega$ we show that there exists a neighborhood $\mathcal{U}$ of $\bar{x}$ and $\sigma > 0$ such that for all $x \in \Omega \cap \mathcal{U}$ there exists a curve γ starting from x and satisfying (3.2). The claim follows choosing by compactness a finite covering of $\partial\Omega$.

Fix $\bar{x} \in \Omega$ and write locally $\partial\Omega$ as a graph of the form $x_j = \varphi(\hat{x}_j)$ for some $j = 1, \dots, n$, where φ is a C^1 function. We begin with the basic case $j = n$. Let $\varphi \in C^1(\mathbb{R}^{n-1})$ be a function satisfying the admissibility condition (2.16) and assume for the sake of simplicity that $\Omega = \{x_n > \varphi(\hat{x}_n)\}$.

We have to construct a John curve starting from a point $x = \hat{x}_n + x_n e_n \in \Omega$. To this aim two different situations need to be distinguished:

$$\begin{aligned} &\max_{i<n} |X_i \varphi(\hat{x}_n)| \leq \lambda_n(\hat{x}_n) \qquad \text{(Case 1)},\\ &\max_{i<n} |X_i \varphi(\hat{x}_n)| > \lambda_n(\hat{x}_n) \qquad \text{(Case 2)}. \end{aligned} \tag{3.3}$$

In Case 1, the characteristic case, we construct a John curve starting from x of the form $x + te_n$, $t \geq 0$. In Case 2 the path must be split into two pieces. The first one starts from x in the coordinate direction e_k, where $k < n$ is such that the derivative $|X_k\varphi(\hat{x}_n)|$ is "maximal" among all the $|X_i\varphi(\hat{x}_n)|$, $i = 1, \dots, n-1$, and moves in this direction for a time $\bar{t} = \bar{t}(x)$ which must be established in a careful way (compare (3.7)). The second part of the path will be of the form $\gamma(\bar{t}) + (t - \bar{t})e_n$.

First of all we introduce the following notation

$$\nu_i = \nu_i(\hat{x}_n) = -\partial_i\varphi(\hat{x}_n), \quad N_i = \frac{\nu_i}{|\nu_i|}, \quad \text{if } \nu_i \neq 0, \quad i \neq n,$$
$$\text{and} \quad w(\hat{x}_n) = \sum_{i \neq n} |X_i\varphi(\hat{x}_n)|.$$

Case 1. Define

$$\gamma(t) = x + te_n = \hat{x}_n + (x_n + t)e_n \quad \text{and} \quad \delta \equiv \delta(t) = G_n(\hat{x}_n, t) \simeq d(\gamma(t), \gamma(0)). \tag{3.4}$$

Consider for small $\sigma > 0$

$$\mathrm{Box}(\gamma(t), \sigma\delta) = \Big\{ \big(x_n + t + u_n F_n(x, \sigma\delta)\big)e_n + \hat{x}_n + \hat{u}_n\hat{F}_n(x, \sigma\delta) : |u_i| \leq 1, \, i = 1, ..., n \Big\}.$$

We used $F_i(x + te_n, \delta) = F_i(x, \delta)$.

We claim that there exists $\sigma > 0$ independent from x such that $\mathrm{Box}(\gamma(t), \sigma\delta) \subset \Omega$, i.e. such that the following condition holds:

$$x_n + t + u_n F_n(x, \sigma\delta) > \varphi\big(\hat{x}_n + \hat{u}_n\hat{F}_n(x, \sigma\delta)\big), \quad \delta > 0, \, |u_i| \leq 1. \tag{3.5}$$

This is the John condition (3.2). Since $x \in \Omega$ then $x_n - \varphi(\hat{x}_n) \geq 0$. Take the worse case $u_n = -1$ in (3.5). Moreover, $F_n(x, \sigma\delta) \leq F_n(x, \delta) = F_n(x, G_n(x, t)) = t$. Thus, condition (3.5) is easily seen to be implied by

$$\big|\varphi\big(\hat{x}_n + \hat{u}_n\hat{F}_n(x, \sigma\delta)\big) - \varphi(\hat{x}_n)\big| \leq (1 - \sigma)t, \quad \delta > 0, \, |\hat{u}_n| \leq 1.$$

Using the control (2.17) for the oscillation of φ, Case 1 and Proposition 2.2 we may estimate the left hand side as follows

$$\begin{aligned} \big|\varphi\big(\hat{x}_n + \hat{u}_n\hat{F}_n(x, \sigma\delta)\big) - \varphi(\hat{x}_n)\big| &\leq \mathrm{osc}(\varphi, \mathrm{Box}_n(\hat{x}_n, \sigma\delta)) \\ &\lesssim \sigma\delta w(x) + \sigma\delta\Lambda_n(\hat{x}_n, \sigma\delta)) \\ &\lesssim \sigma\delta\lambda_n(\hat{x}_n) + \sigma F_n(\hat{x}_n, \delta) \simeq \sigma F_n(\hat{x}_n, \delta). \end{aligned}$$

In the last equivalence we used the trivial estimate $\delta\lambda_n(\hat{x}_n) \leq F_n(x, \delta)$. Thus, (3.5) is implied by $\sigma F_n(x, \delta) \leq \sigma_0 t$, where σ_0 is a small but absolute constant. Since $t = F_n(x, \delta)$, this inequality holds as soon as $\sigma \leq \sigma_0$.

Case 2. Assume that x satisfies Case 2 in (3.3). Take any $k = 1, \dots, n-1$ such that

$$|X_k\varphi(\hat{x}_n)| \geq \frac{1}{2} \max_{i<n} |X_i\varphi(\hat{x}_n)| > \frac{1}{4}\lambda_n(\hat{x}_n). \tag{3.6}$$

(The factors $\frac{1}{2}$ and $\frac{1}{4}$ will become relevant in [MM3], where we prove that admissible domains are non–tangentially accessible).

Fix $\varepsilon_0 > 0$ and let $\bar\delta = \bar\delta(x)$ be the solution of the following equation

$$\Lambda_n(\hat x_n, \bar\delta) = \varepsilon_0 |X_k \varphi(\hat x_n)|. \tag{3.7}$$

(The function $\Lambda_n(x, \cdot)$ is a homeomorphism of $[0, \infty[$ onto itself). The number ε_0 will be fixed and will become an absolute constant in (3.16). Finally, set $\bar t = F_k(x, \bar\delta)$.

We define the first piece of the John curve letting for $t \in [0, \bar t]$

$$\gamma(t) = x + tN_k e_k, \qquad \text{and} \quad \delta = \delta(t) = G_k(x,t) \simeq d(\gamma(0), \gamma(t)).$$

For $\sigma > 0$ consider the $\mathrm{Box}(\gamma(t), \sigma\delta) = \Big\{ \big(tN_k e_k + \sum_{i=1}^n \big(x_i + u_i F_i(x + tN_k e_k, \sigma\delta)\big) e_i : |u_i| \le 1,\ i = 1, ..., n \Big\}$. We claim that there exist $\varepsilon_0, \sigma > 0$ independent of x such that the following John condition holds

$$\mathrm{Box}(\gamma(t), \sigma\delta(t)) \subset \Omega, \quad t \in [0, \bar t]. \tag{3.8}$$

Points of the box belong to Ω provided that for all u, $|u_i| \le 1$, $i = 1, \dots, n$, we have

$$x_n + u_n F_n(x + tN_k e_k, \sigma\delta) > \varphi\Big(tN_k e_k + \sum_{i \ne n} \big(x_i + u_i F_i(x + tN_k e_k, \sigma\delta)\big) e_i \Big).$$

Take the worse case $u_n = -1$ and use $x_n > \varphi(\hat x_n)$. The inequality above is implied by

$$\varphi\Big(tN_k e_k + \sum_{i \ne n} \big(x_i + u_i F_i(x + tN_k e_k, \sigma\delta)\big) e_i \Big) - \varphi(\hat x_n) + F_n(x + tN_k e_k, \sigma\delta) \le 0,$$

which can be rewritten as

$$\mathrm{I} + \mathrm{II} + F_n(x + tN_k e_k, \sigma\delta) \le 0, \tag{3.9}$$

where we set

$$\begin{aligned} \mathrm{I} &= \varphi\Big(tN_k e_k + \sum_{i \ne n} \big(x_i + u_i F_i(x + tN_k e_k, \sigma\delta)\big) e_i \Big) - \varphi(tN_k e_k + \hat x_n), \\ \mathrm{II} &= \varphi(tN_k e_k + \hat x_n) - \varphi(\hat x_n). \end{aligned}$$

We claim that ε_0 in (3.7) can be fixed independently from x in such a way that

$$\mathrm{II} \le -\frac{1}{2} t |\nu_k| \quad \text{ for all } t \in [0, \bar t]. \tag{3.10}$$

Indeed, by the mean value theorem there exists $\vartheta \in [0,1]$ such that

$$\begin{aligned} \varphi(\hat x_n + tN_k e_k) - \varphi(\hat x_n) &= \partial_k \varphi(\hat x_n + \vartheta t N_k e_k) t N_k \\ &= \partial_k \varphi(\hat x_n) t N_k + \{\partial_k \varphi(\hat x_n + \vartheta t N_k e_k) - \partial_k \varphi(\hat x_n)\} t N_k \\ &= -|\nu_k| t + \{\partial_k \varphi(\hat x_n + \vartheta t N_k e_k) - \partial_k \varphi(\hat x_n)\} t N_k. \end{aligned}$$

Notice that Case 2 in (3.3) ensures $\nu_k \neq 0$ and $\lambda_k(\hat{x}_n) \neq 0$. The curly brackets can be estimated by (2.16) as follows (notice that λ_k does not depend on x_k and $t = F_k(x, \delta)$)

$$\begin{aligned}\big|\{\partial_k\varphi(\hat{x}_n + \vartheta t N_k e_k) - \partial_k\varphi(\hat{x}_n)\}\big| &= \frac{1}{\lambda_k(\hat{x}_n)}\big|X_k\varphi(\hat{x}_n + \vartheta t N_k e_k) - X_k\varphi(\hat{x}_n)\big| \\ &\leq \frac{1}{\lambda_k(\hat{x}_n)} \operatorname{osc}(X_k\varphi, \operatorname{Box}_n(\hat{x}_n, \delta)) \\ &\lesssim \frac{1}{\lambda_k(\hat{x}_n)}\big(\delta w(\hat{x}_n)^{(d_n-2)/(d_n-1)} + \Lambda_n(\hat{x}_n, \delta)\big) \\ &\lesssim \frac{1}{\lambda_k(\hat{x}_n)}\big(\delta |X_k\varphi(\hat{x}_n)|^{(d_n-2)/(d_n-1)} + \Lambda_n(\hat{x}_n, \delta)\big).\end{aligned}$$

In the last inequality we used (3.6). Now, (3.10) is guaranteed by

$$C_0\big(\delta|X_k\varphi(\hat{x}_n)|^{(d_n-2)/(d_n-1)} + \Lambda_n(\hat{x}_n, \delta)\big) \leq \frac{1}{2}|\nu_k|\lambda_k(\hat{x}_n) = \frac{1}{2}|X_k\varphi(\hat{x}_n)|, \tag{3.11}$$

where C_0 is a big but absolute constant. By Hölder inequality and Proposition 2.2

$$C_0\delta|X_k\varphi(\hat{x}_n)|^{(d_n-2)/(d_n-1)} \leq \frac{1}{4}|X_k\varphi(\hat{x}_n)| + C_1\delta^{d_n-1} \leq \frac{1}{4}|X_k\varphi(\hat{x}_n)| + C_2\Lambda_n(\hat{x}_n, \delta),$$

where C_2 is a new big absolute constant. Using $\Lambda_n(\hat{x}_n, \delta) \leq \Lambda_n(\hat{x}_n, \bar{\delta}) = \varepsilon_0|X_k\varphi(\hat{x}_n)|$ (this is (3.7)) we see that (3.11) is guaranteed by a choice of $\varepsilon_0 > 0$ such that $4(C_0 + C_2)\varepsilon_0 \leq 1$.

Now, by the estimate on II, inequality (3.9) is implied by

$$\mathrm{I} + F_n(x + tN_k e_k, \sigma\delta) \leq \frac{1}{2}t|\nu_k|, \quad t \in [0, \bar{t}]. \tag{3.12}$$

We claim that this inequality holds as soon as $\sigma > 0$ is small enough independently of x.

First of all by (2.17) we find

$$\begin{aligned}\mathrm{I} &\leq \Big|\varphi\Big(tN_k e_k + \sum_{i\neq n}\big(x_i + u_i F_i(x + tN_k e_k, \sigma\delta)\big)e_i\Big) - \varphi(tN_k e_k + \hat{x}_n)\Big| \\ &\leq \operatorname{osc}\big(\varphi, \operatorname{Box}_n(\hat{x}_n + tN_k e_k, \sigma\delta)\big) \\ &\lesssim \sigma\delta w(\hat{x}_n + tN_k e_k) + \sigma\delta\Lambda_n(\hat{x}_n + tN_k e_k, \sigma\delta),\end{aligned}$$

and by (2.16)

$$\begin{aligned}w(\hat{x}_n + tN_k e_k) &= \sum_{i\neq n}|X_i\varphi(\hat{x}_n + tN_k e_k)| \\ &\leq \sum_{i\neq n}|X_i\varphi(\hat{x}_n)| + \sum_{i\neq n}|X_i\varphi(\hat{x}_n + tN_k e_k) - X_i\varphi(\hat{x}_n)| \\ &\leq w(\hat{x}_n) + \sum_{i\neq n}\operatorname{osc}(X_i\varphi, \operatorname{Box}_n(\hat{x}_n, \delta)) \\ &\lesssim w(\hat{x}_n) + \delta w(\hat{x}_n)^{(d_n-2)/(d_n-1)} + \Lambda_n(\hat{x}_n, \delta) \\ &\lesssim w(\hat{x}_n) + \Lambda_n(\hat{x}_n, \delta).\end{aligned} \tag{3.13}$$

We used once again Hölder inequality and Proposition 2.2. Since $\Lambda_n(\hat{x}_n+tN_k e_k,\sigma\delta)\lesssim\Lambda_n(\hat{x}_n,\delta)$, using $w(\hat{x}_n)\le|X_k\varphi(\hat{x}_n)|$ we finally get

$$\mathrm{I}\lesssim\sigma\delta(|X_k\varphi(\hat{x}_n)|+\Lambda_n(\hat{x}_n,\delta)). \tag{3.14}$$

Now we show that the second term in the left hand side of (3.12) satisfies the same estimate (we will need (2.12)):

$$\begin{aligned}F_n(x+tN_ke_k,\sigma\delta)&\le\sigma F_n(x+tN_ke_k,\delta)\lesssim\sigma F_n(x,\delta)\\&=\sigma\delta\lambda_n(\hat{x}_n+\hat{F}_n(x,\delta))\le\sigma\delta(\Lambda_n(\hat{x}_n,\delta)+\lambda_n(\hat{x}_n))\\&\lesssim\sigma\delta(\Lambda_n(\hat{x}_n,\delta)+|X_k\varphi(\hat{x}_n)|).\end{aligned} \tag{3.15}$$

We used (3.6).

Taking into account (3.14) and (3.15), and recalling that $\Lambda_n(\hat{x}_n,\delta)\le\Lambda_n(\hat{x}_n,\bar{\delta})=\varepsilon_0|X_k\varphi(\hat{x}_n)|$, we see that condition (3.12) holds as soon as $\sigma\delta|X_k\varphi(\hat{x}_n)|\lesssim t|\nu_k|$, i.e. $C_0\sigma\delta|X_k\varphi(\hat{x}_n)|\le t|\nu_k|$, i.e. $C_0\sigma\delta\lambda_k(\hat{x}_n)\le t$ for all $t>0$. Here $C_0>0$ is a big but absolute constant. This inequality holds if $\sigma>0$ is chosen in such a way that $\sigma C_0\le 1$, because $\delta\lambda_k(\hat{x}_n)\le F_k(\hat{x}_n,\delta)=t$. This proves (3.12), and hence claim (3.8), as well.

So far we have defined a John curve γ staring from a point $x\in\Omega$ satisfying Case 2 in (3.3) for a time $t\in[0,\bar{t}]$, where

$$\bar{t}=F_k(x,\bar{\delta})\quad\text{and }\bar{\delta}\text{ solves}\quad\Lambda_n(\hat{x}_n,\bar{\delta})=\varepsilon_0|X\varphi(\hat{x}_n)|. \tag{3.16}$$

The constant $\varepsilon_0>0$ is from now on fixed. Now we define γ for times $t\ge\bar{t}$. Let

$$\gamma(t)=x+\bar{t}N_ke_k+(t-\bar{t})e_n,\qquad t\ge\bar{t}. \tag{3.17}$$

We shall write $s=t-\bar{t}$. Set

$$\delta=\delta(t)=\bar{\delta}+G_n(x,t-\bar{t})=\bar{\delta}+G_n(x,s)\simeq d(\gamma(0),\gamma(t)). \tag{3.18}$$

For $\sigma>0$ consider the box

$$\mathrm{Box}(\gamma(t),\sigma\delta)=\Big\{se_n+\bar{t}N_ke_k+\sum_{i=1}^{n}(x_i+u_iF_i(x+\bar{t}N_ke_k,\sigma\delta))e_i:|u_i|\le1\Big\}.$$

Since $\varphi(\hat{x}_n)\le x_n$, taking the worse case $u_n=-1$, the John condition $\mathrm{Box}(\gamma(t),\sigma\delta)\subset\Omega$ is implied by

$$\mathrm{J}+\mathrm{JJ}+F_n(x+\bar{t}N_ke_k,\sigma\delta)\le s, \tag{3.19}$$

where we set

$$\begin{aligned}\mathrm{J}&=\varphi\Big(\bar{t}N_ke_k+\sum_{i\neq n}(x_i+u_iF_i(x+\bar{t}N_ke_k,\sigma\delta))e_i\Big)-\varphi(\bar{t}N_ke_k+\hat{x}_n)\\\mathrm{JJ}&=\varphi(\bar{t}N_ke_k+\hat{x}_n)-\varphi(\hat{x}_n).\end{aligned}$$

By (3.10) with $t=\bar{t}$ we have $\mathrm{JJ}\le-\frac{1}{2}|\nu_k|\bar{t}$. Hence, (3.19) is guaranteed by

$$\mathrm{J}+F_n(x+\bar{t}N_ke_k,\sigma\delta)\le s+\frac{1}{2}|\nu_k|\bar{t}. \tag{3.20}$$

We begin with the estimate of J. By (2.17)

$$\begin{aligned}\mathrm{J} &\leq \operatorname{osc}\big(\varphi, \mathrm{Box}_n(\hat{x}_n + \bar{t}N_k e_k, \sigma\delta)\big)\\ &\lesssim \sigma\delta\big(w(\hat{x}_n + \bar{t}e_kN_k) + \Lambda_n(\hat{x}_n + \bar{t}e_kN_k, \sigma\delta)\big),\end{aligned}$$

and by (3.13), (3.6) and Proposition 2.2 (use $\bar{\delta} \leq \delta$)

$$\delta w(\hat{x}_n + \bar{t}e_kN_k) \lesssim \delta w(\hat{x}_n) + \delta\Lambda_n(\hat{x}_n, \bar{\delta}) \lesssim \delta|\nu_k|\lambda_k(\hat{x}_n) + F_n(\hat{x}_n, \delta).$$

On the other hand, by Proposition 2.2, (2.12), (3.18) and (2.7)

$$\begin{aligned}\delta\Lambda_n(\hat{x}_n + \bar{t}e_kN_k, \delta) &\lesssim F_n(\hat{x}_n + \bar{t}e_kN_k, \delta)\\ &\lesssim F_n(\hat{x}_n, \delta) = F_n(\hat{x}_n, \bar{\delta} + G_n(\hat{x}_n, s))\\ &\lesssim F_n(\hat{x}_n, \bar{\delta}) + F_n(\hat{x}_n, G_n(\hat{x}_n, s)) = F_n(\hat{x}_n, \bar{\delta}) + s.\end{aligned}$$

Thus (3.20) is ensured by the inequality

$$\sigma(\delta|\nu_k|\lambda_k(\hat{x}_n) + F_n(\hat{x}_n, \bar{\delta}) + s) \lesssim s + \frac{1}{2}|\nu_k|\bar{t},$$

which reduces to (recall that $\delta - \bar{\delta} = G_n(x, \delta)$, by (3.18))

$$\sigma C_0\big(\bar{\delta}|\nu_k|\lambda_k(\hat{x}_n) + G_n(\hat{x}_n, s)\lambda_k(\hat{x}_n) + F_n(\hat{x}_n, \bar{\delta})\big) \leq s + |\nu_k|\bar{t} \tag{3.21}$$

for some absolute big constant $C_0 > 0$. If in (3.21) we put $s = 0$ we get

$$\sigma C_0(\bar{\delta}|\nu_k|\lambda_k(\hat{x}_n) + F_n(\hat{x}_n, \bar{\delta}))) \leq |\nu_k|\bar{t},$$

which holds for σ small enough (we have already proved it when we proved (3.12), see also (3.14), (3.15) and (3.7)).

To complete the estimate (3.21) it will be enough to show that

$$\sigma G_n(\hat{x}_n, s)\lambda_k(\hat{x}_n)|\nu_k| \leq s + |\nu_k|\bar{t}, \quad \text{for all } s > 0, \tag{3.22}$$

as soon as $\sigma > 0$ is small enough independently of x. Now, (3.22) is equivalent to

$$G_n(\hat{x}_n, s) \leq \frac{s + |\nu_k|\bar{t}}{\sigma\lambda_k|\nu_k|} \quad \Leftrightarrow \quad s \leq F_n\Big(\hat{x}_n, \frac{s + |\nu_k|\bar{t}}{\sigma\lambda_k|\nu_k|}\Big).$$

Notice that the function $f_n(\hat{x}_n, r) = \frac{1}{r}F_n(\hat{x}_n, r)$ is increasing in the variable r. From

$$s + |\nu_k|\bar{t} \geq |\nu_k|\bar{t} = |\nu_k|F_k(\hat{x}_n, \bar{\delta}) \geq |\nu_k|\bar{\delta}\lambda_k(\hat{x}_n) \geq \sigma|\nu_k|\bar{\delta}\lambda_k(\hat{x}_n)$$

it follows

$$f_n\Big(\hat{x}_n, \frac{s + |\nu_k|\bar{t}}{\sigma\lambda_k|\nu_k|}\Big) \geq f_n(x, \bar{\delta}) \gtrsim \Lambda_n(\hat{x}_n, \bar{\delta}),$$

by Proposition 2.2. Finally, recalling (3.7) we find out that (3.22) is implied by

$$s \leq \frac{s + |\nu_k|\bar{t}}{\sigma\lambda_k(\hat{x}_n)|\nu_k|}\Lambda_n(\hat{x}_n, \bar{\delta}) = \frac{\varepsilon_0}{\sigma}(s + |\nu_k|\bar{t}),$$

which holds for all $s > 0$ as soon as $\sigma \leq \varepsilon_0$. Ultimately, this proves (3.19) and ends the discussion of Case 2 and of the parameterization $x_n = \varphi(\hat{x}_n)$.

Now assume that $\bar{x} \in \partial\Omega$ is a point such that for a neighborhood $\mathcal{U}$ of $\bar{x}$ the piece of boundary $\partial\Omega \cap \mathcal{U}$ is a surface of type $\{x_j = \varphi(\hat{x}_j)\}$ for some $j \neq n$ and for some function φ of class C^1 which satisfies the admissibility condition (2.23). We explain how to construct a John curve starting from points $x \in \mathcal{U} \cap \{x_j > \varphi(\hat{x}_j)\}$.

The functions $\widetilde{\lambda}_i$ and the vector fields $\widetilde{X}_i$ are defined in (2.20). By $\widetilde{d}$ we denote the metric induced on $\mathbb{R}^n$ by the vector fields $\widetilde{X}_i$. The boxes $\widetilde{\mathrm{Box}}_j(\hat{x}_j, r)$ and the function $\widetilde{\Lambda}_j$ have been defined in (2.21) and (2.22), respectively. Without loss of generality, we can assume $\mathcal{U} \subset \{|x_i| < 1 : i = 1, ..., n\}$ and $|\partial_i \varphi(\hat{x}_j)| \leq 1$, $i > j$. Then $|\widetilde{X}_i \varphi(\hat{x}_j)| \leq \widetilde{\lambda}_j(\hat{x}_j)$, for all $i > j$. Thus the distinction of cases (3.3) simply is

$$\max_{i<j} |\widetilde{X}_i \varphi(\hat{x}_j)| \leq \widetilde{\lambda}_j(\hat{x}_j) \qquad \text{(Case 1)},$$

$$\max_{i<j} |\widetilde{X}_i \varphi(\hat{x}_j)| > \widetilde{\lambda}_j(\hat{x}_j) \qquad \text{(Case 2)}.$$

In Case 1 we define a curve γ moving directly in the direction e_j, analogously to (3.4). In Case 2 we first define $\gamma(t) = x + tN_k$, where $k = 1, ..., j-1$ is any index such that

$$|\widetilde{X}_k \varphi(\hat{x}_j)| \geq \frac{1}{2} \max_{i<j} |\widetilde{X}_i \varphi(\hat{x}_j)| > \frac{1}{4} \widetilde{\lambda}_j(\hat{x}_j),$$

and $t \in [0, \bar{t}]$, where now $\bar{t} = F_k(\hat{x}_j, \bar{\delta})$ and $\bar{\delta}$ solves $\widetilde{\Lambda}_j(\hat{x}_j, \bar{\delta}) = \varepsilon_0 |\widetilde{X}_k \varphi(\hat{x}_j)|$ instead of (3.7). Then we let γ move in the direction e_j, analogously to (3.17).

The curve γ so defined satisfies, for some $\sigma > 0$ independent of x, the John condition with respect to the metric $\widetilde{d}$, i.e. $\widetilde{B}(\gamma(t), \sigma \widetilde{\mathrm{diam}}(\gamma|_{[0,t]})) \subset \Omega$, $\quad t \in [0,1]$, where $\widetilde{B}$ denote balls in the metric $\widetilde{d}$. The proof of this is exactly the same as for the case $j = n$. By Proposition 2.6 (ii) it follows that $\widetilde{\mathrm{diam}}(\gamma|_{[0,t]}) \simeq \mathrm{diam}(\gamma|_{[0,t]})$ and by (i) it also follows that $B(\gamma(t), \sigma \mathrm{diam}(\gamma|_{[0,t]})) \subset \Omega$. This remark ends the proof of the Theorem. □

4. Applications

In this section we study some applications of the cone property in metric spaces associated with a system $X_1, ..., X_m$ of smooth vector fields in $\mathbb{R}^n$ satisfying the Hörmander condition (see [Ho]). The vector fields in (2.1) with monomial functions λ_j are of this type. A family of Hörmander vector fields induces a metric d on $\mathbb{R}^n$ as in (2.9), which has been studied in [NSW]. This metric is of doubling type: for any compact set $K \subset \mathbb{R}^n$ there exists a constant $C > 0$ such that $|B(x, 2r)| \leq C|B(x, r)|$ for all $x \in K$ and $r \in]0, r_0]$. Here $|\cdot|$ stands for Lebesgue measure and $B(x, r)$ are balls in the metric d. We shall write $|Xu|^2 = \sum_{j=1}^m (X_j u)^2$. The following Poincaré inequality has been proved in [Je] (see also [LM]). Given a bounded set there exist constants $C, r_0 > 0$ such that

$$\int_B |u - u_B| dx \leq Cr \int_B |Xu| dx, \tag{4.1}$$

for all balls B with center in the given bounded set and for all $u \in C^1(B)$. Here u_B is the average of the function u over B and $r \leq r_0$ is the radius of B.

4.1. **Hardy Inequality.** It is not difficult to see that a bounded domain $\Omega \subset (\mathbb{R}^n, d)$ satisfying an exterior cone condition also enjoys the following property. There exist constants $r_0 > 0$ and $C > 0$ such that

$$|B(x,r)| \le C|B(x,r) \cap \Omega^c|, \quad \text{for all } x \in \partial\Omega \text{ and } r \in \,]0, r_0]\,. \tag{4.2}$$

In the following Theorem we prove a Hardy inequality in the setting of Hörmander vector fields. Our proof essentially follows the one given by Hajłasz in [Ha] (see also [BMS] for the case of metric measure spaces).

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set satisfying (4.2). For any $p \in (1,\infty)$ there exists $C > 0$ such that*

$$\int_\Omega \Big(\frac{|u(x)|}{\varrho(x)}\Big)^p dx \le C \int_\Omega |Xu(x)|^p dx, \quad \textit{for all } u \in C_0^1(\Omega). \tag{4.3}$$

Here, $\varrho(x) = \mathrm{dist}(x, \partial\Omega)$ is the distance from the boundary in the metric d.

Proof. We recall the definition of the maximal function. If f is locally integrable define $Mf(x) = \sup_{r>0} \fint_{B(x,t)} |f(y)|dy = \sup_{r>0} \dfrac{1}{|B(x,r)|} \int_{B(x,t)} |f(y)|dy$. The Maximal Theorem in doubling metric spaces states that for any $p > 1$ there exists a constant $C > 0$ such that $\|Mf\|_p \le C\|f\|_p$ (see, for instance, [HK2]). Then it will be enough to establish the following "pointwise" Hardy inequality

$$\frac{|u(x)|}{\varrho(x)} \le CM|Xu|(x), \quad x \in \Omega.$$

Take $x \in \Omega$ and choose $\bar{x} \in \partial\Omega$ such that $d(x,\bar{x}) = \varrho(x)$. Write $B = B(\bar{x}, \varrho(x))$, $B_0 = B(x, 2\varrho(x))$ and $B_j = 2^{-j}B_0 = B(x, 2^{1-j}\varrho(x))$. Then $|u(x)| \le |u(x) - u_{B_0}| + |u_{B_0} - u_B| + |u_B|$. We estimate the first term using the doubling condition and the Poincaré inequality (4.1)

$$\begin{aligned}
|u(x) - u_{B_0}| &= \lim_{j\to\infty} |u_{B_j} - u_{B_0}| \le \sum_{j=1}^{+\infty} |u_{B_j} - u_{B_{j-1}}| \le \sum_{j=1}^{+\infty} \fint_{B_j} |u - u_{B_{j-1}}|dy \\
&\le C \sum_{j=1}^{+\infty} \fint_{B_{j-1}} |u - u_{B_{j-1}}|dy \le C\varrho(x) \sum_{j=1}^{+\infty} 2^{-j} \fint_{B_{j-1}} |Xu|dy \\
&\le C\varrho(x) M|Xu|(x).
\end{aligned}$$

Since $B_0 \subset 4B$, from the doubling property it follows that $|B_0| \le C|B|$, and the estimate of the second term easily follows from the Poincaré inequality

$$\begin{aligned}
|u_{B_0} - u_B| &\le \fint_B |u - u_{B_0}|dy \le C \fint_{B_0} |u - u_{B_0}|dy \\
&\le C\varrho(x) \fint_{B_0} |Xu|dy \le C\varrho(x) M|Xu|(x).
\end{aligned}$$

Finally, since $u_{B\cap\Omega^c} = 0$ using (4.2) we have

$$|u_B| = |u_B - u_{B\cap\Omega^c}| \le \fint_{B\cap\Omega^c} |u - u_B|dy \le C \fint_B |u - u_B|dy \le C\varrho(x) M|Xu|(x).$$

This ends the proof. □

4.2. **Growth of positive harmonic functions.** A system $X_1, ..., X_m$ of Hörmander vector fields define the second order differential operator $\mathcal{L} = -\sum_{j=1}^m X_j^* X_j$ where X_j^* is the adjoint operator of X_j. A smooth function u is $\mathcal{L}$–harmonic in an open set Ω if $\mathcal{L}u = 0$. In the following theorem we prove growth estimates near the boundary for positive $\mathcal{L}$–harmonic function in a John domain. The proof in the Euclidean case is due to Suzuki [Su]. This result and the one in the following subsection are used in [MM3].

Proposition 4.2. *Let Ω be a John domain in $(\mathbb{R}^n, d)$. There are constants $\alpha, \beta > 0$ such that, for any positive $\mathcal{L}$–harmonic function u in Ω there are $C_1, C_2 > 0$ such that*

$$C_1 \varrho(x)^\alpha \le u(x) \le C_2 \varrho(x)^{-\beta}, \quad x \in \Omega, \ \varrho(x) = \mathrm{dist}(x, \partial\Omega). \tag{4.4}$$

Proof. The proof is based on the iteration of the following Harnack inequality: there exists a constant $C_H > 1$ such that $C_H^{-1} u(y) \le u(x) \le C_H u(y)$ for all positive $\mathcal{L}$–harmonic functions u in Ω and for all $x, y \in B$, where B is any ball such that $4B \subset \Omega$ (see for example [CDG]).

We prove the inequality in the left hand side of (4.4) (the other one is analogous). Let x_0 be the John center and σ be the John constant of Ω. Then for any $x \in \Omega$ there is a rectifiable curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = x$, $\gamma(1) = x_0$ and $\varrho(\gamma(t)) \ge \sigma \mathrm{length}(\gamma|_{(0,t)})$ (recall Definition 3.1 and the remark following it). Write $L = \mathrm{length}(\gamma)$ and without loss of generality assume that $L \ge \varrho(x)$ and $\varrho(x_0) \le 4d(x, x_0)$. All points x near the boundary satisfy these conditions. Note that, by the John property, $L \le \sigma^{-1}\varrho(x_0) \le 4\sigma^{-1} d(x, x_0) \le 4\sigma^{-1} L$. Thus L is equivalent to a constant.

Fix a number $\eta \in (0,1)$ such that $8(1-\eta) = \sigma$, take an integer m such that $4\eta^{m+1} L \le \varrho(x) < 4\eta^m L$, and define times $1 = t_0 > t_1 > t_2 > \cdots > t_m > t_{m+1}$ such that

$$\mathrm{length}\big(\gamma|_{(0,t_j)}\big) = \eta^j L, \quad j = 1, \dots, m+1.$$

Then

$$\begin{aligned} d(\gamma(t_j), \gamma(t_{j+1})) &\le \mathrm{length}\big(\gamma|_{(0,t_j)}\big) - \mathrm{length}\big(\gamma|_{(0,t_{j+1})}\big) \\ &= (\eta^j - \eta^{j+1}) \mathrm{length}(\gamma) = (1-\eta)\mathrm{length}\big(\gamma|_{(0,t_j)}\big) \\ &\le (1-\eta)\sigma^{-1} \varrho(\gamma(t_j)) = \frac{1}{8} \varrho(\gamma(t_j)), \end{aligned}$$

and therefore $\gamma(t_{j+1}) \in B\big(\gamma(t_j), \frac{1}{4}\varrho(\gamma(t_j))\big)$. Moreover we have $d(x, \gamma(t_{m+1})) \le \mathrm{length}\big(\gamma|_{(0,t_{m+1})}\big) = \eta^{m+1} L \le \frac{1}{4}\varrho(x)$. Thus $\gamma(t_{m+1}) \in B(x, \frac{1}{4}\varrho(x))$. Iterating $m+2$ times the Harnack inequality we find

$$u(x_0) \ge C_H^{-1} u(\gamma(t_1)) \ge ... \ge C_H^{-(m+1)} u(\gamma(t_{m+1})) \ge C_H^{-(m+2)} u(x).$$

Now the claim follows from the fact that, since $\varrho(x) < 4\eta^m L$, then $m \le \log_\eta \frac{\varrho(x)}{L}$. Therefore, $C_H^{-m} \ge C_H^{-\log_\eta(\varrho(x)/(4L))} = \Big(\dfrac{\varrho(x)}{4L}\Big)^{-\log_\eta C_H}$. The result follows with $\alpha =$

$-\log_\eta C_H$. As noted above, the length L is equivalent to a constant not depending on x, whenever x is near the boundary. $\square$

If the John domain Ω satisfies also an *exterior* twisted cone condition, then for any $x_0 \in \Omega$ there is a Green function $G(x_0, \cdot)$ with singularity at x_0 for the operator $\mathcal{L}$. An easy adaptation of the argument in Proposition 4.2 gives the following result:

Corollary 4.3. *If a John domain Ω satisfies an exterior twisted cone condition, then for some $\alpha > 0$ its Green function satisfies $G(x_0, x) \geq C\varrho(x)^\alpha$ for all x near the boundary.*

4.3. **Integrability of the distance from the boundary.** Here $X_1, \dots, X_m$ is a system of Hörmander vector fields. In the following theorem of Trotsenko–type (see [T]) we show that the control distance from the boundary of a John domain has a negative power of integrability. Our proof essentially follows the Euclidean proof in [A2, Lemma 5].

Theorem 4.4. *Let Ω be a bounded domain. Assume that Ω is a John domain for the control distance and let $\varrho(x) = \mathrm{dist}(x, \partial\Omega)$. There exists $\tau \in (0,1)$ such that*

$$\int_\Omega \frac{dx}{\varrho(x)^\tau} < +\infty.$$

Proof. We denote by x_0 the John center and by $\sigma > 0$ the John constant of Ω. For $j \in \mathbb{N}$ set $t_j = 2^{-j}$ and define $\Omega_0 = \{x \in \Omega : \varrho(x) \geq 1/2\}$ and

$$\Omega_j = \{x \in \Omega : t_{j+1} \leq \varrho(x) < t_j\}, \quad j \in \mathbb{N}.$$

Clearly $\Omega = \bigcup_{j=0}^{+\infty} \Omega_j$, and without loss of generality we can assume $x_0 \in \Omega_0$ (if $\Omega_0 = \varnothing$, then it suffices to let $t_j = b2^{-j}$, where $b > 0$ is a sufficiently small constant). Since

$$\int_{\{\varrho<1/2\}} \frac{dx}{\varrho(x)^\tau} = \sum_{j=1}^{+\infty} \int_{\Omega_j} \frac{dx}{\varrho(x)^\tau} \leq \sum_{j=1}^{+\infty} |\Omega_j| 2^{(j+1)\tau},$$

it suffices to show that the last sum converges for some $\tau \in (0,1)$.

Take a point $x \in \Omega$ with $\varrho(x) < t_{j+1}$ and let $\gamma : [0,1] \to \Omega$ be a John curve such that $\gamma(0) = x$, $\gamma(1) = x_0$ and $\varrho(\gamma(t)) \geq \sigma d(\gamma(t), x)$. There exists a point $\xi \in \gamma([0,1])$ such that $\varrho(\xi) = t_j$ and since $\varrho(\xi) \geq \sigma d(\xi, x)$ it follows that $x \in B(\xi, 2t_j/\sigma)$, and therefore $\bigcup_{i=j+1}^{+\infty} \Omega_i \subset \bigcup_{\varrho(\xi)=t_j} B(\xi, 2t_j/\sigma)$. Since $(\mathbb{R}^n, d)$ is a doubling metric space, by the $5r$-covering theorem (see, for instance, [He]) we can extract from the family of balls $\{B(\xi, 2t_j/\sigma) : \varrho(\xi) = t_j\}$ a countable, disjoint subfamily $\{B(\xi_k, 2t_j/\sigma) : k \in \mathbb{N}\}$ such that

$$\bigcup_{i=j+1}^{+\infty} \Omega_i \subset \bigcup_{k\in\mathbb{N}} B(\xi_k, 10t_j/\sigma). \tag{4.5}$$

Take a point $\xi \in \Omega$ with $\varrho(\xi) = t_j$. There exists $\bar\xi \in \partial\Omega$ such that $d(\bar\xi, \xi) = t_j$. Take a geodesic between ξ and $\bar\xi$ and fix a point ξ^* on this geodesic in such a way that $d(\xi^*, \xi) = 3t_j/8$ and $d(\xi^*, \bar\xi) = 5t_j/8$. With such choices we have $B(\xi^*, t_j/8) \subset \Omega_j \cap$

$B(\xi, 2t_j/\sigma)$. Indeed, take a point $x \in B(\xi^*, t_j/8)$. Then $d(x,\xi) \le d(x,\xi^*) + d(\xi^*,\xi) < t_j/8 + 3t_j/8 = t_j/2 \le 2t_j/\sigma$ as soon as $\sigma \le 4$. This shows that $B(\xi^*, t_j/8) \subset B(\xi, 2t_j/\sigma)$. Moreover,

$$\varrho(x) < d(x,\xi^*) + \varrho(\xi^*) \le d(x,\xi^*) + d(\xi^*,\bar{\xi}) \le \frac{1}{8}t_j + \frac{5}{8}t_j < t_j,$$
$$\varrho(x) \ge \varrho(\xi) - d(x,\xi) \ge \varrho(\xi) - d(x,\xi^*) - d(\xi^*,\xi) \ge t_j - \frac{1}{8}t_j - \frac{3}{8}t_j = \frac{1}{2}t_j = t_{j+1}.$$

This shows that $B(\xi^*, t_j/8) \subset \Omega_j$.

From the inclusion $B(\xi, 10t_j/\sigma) \subset B(\xi^*, 3t_j/8 + 10t_j/\sigma)$ and from the doubling property there exists a constant $C > 0$ not depending on j such that

$$|B(\xi, 10t_j/\sigma)| \le |B(\xi^*, 3t_j/8 + 10t_j/\sigma)| \le C|B(\xi^*, t_j/8)| \le C|\Omega_j \cap B(\xi, 2t_j/\sigma)|.$$

This inequality holds with $\xi = \xi_k$ for all $k \in \mathbb{N}$. Then by (4.5) we get

$$\sum_{i=j+1}^{+\infty} |\Omega_i| \le \sum_{k\in\mathbb{N}} |B(\xi_k, 10t_j/\sigma)| \le C \sum_{k\in\mathbb{N}} |\Omega_j \cap B(\xi_k, 2t_j/\sigma)| \le C|\Omega_j|, \tag{4.6}$$

because the balls $B\big(\xi_k, 2t_j/\sigma\big)$ are disjoint.

Let $r > 1$ be a parameter to be determined and take $m \in \mathbb{N}$. The above inequality gives $\sum_{j=1}^m \sum_{i=j+1}^{+\infty} r^j|\Omega_i| \le C\sum_{j=1}^m r^j|\Omega_j|$, and moreover

$$\sum_{j=1}^{m}\sum_{i=j+1}^{+\infty} r^j|\Omega_i| = \sum_{i=2}^{+\infty}\sum_{j=1}^{\min\{i-1,m\}} r^j|\Omega_i| \ge \sum_{i=2}^{m}\sum_{j=1}^{i-1} r^j|\Omega_i| = \sum_{i=1}^{m}\frac{r^i - r}{r-1}|\Omega_i|.$$

Ultimately, we find $\left(\frac{1}{r-1} - C\right)\sum_{i=1}^m r^i|\Omega_i| \le \frac{r}{r-1}\sum_{i=1}^m |\Omega_i|$, and now we choose $r < 1 + 1/C$. Letting m go to infinity in this last inequality and recalling (4.6) we obtain

$$\sum_{i=1}^{\infty} r^i|\Omega_i| \le C\sum_{i=1}^{\infty}|\Omega_i| \le C|\Omega| < +\infty,$$

because Ω is bounded. Choosing $\tau = \log_2 r$ our claim is proved. ☐

Appendix

Proof of Proposition 2.2. Let $\hat{x}_n \in \mathbb{R}^{n-1}$ and assume without loss of generality $x_i \ge 0$, $i = 1, ..., n-1$. Take $a < 1$. The “reverse doubling” property (2.15) is equivalent to

$$\frac{\Lambda_n(x,R) - \Lambda_n(x,aR)}{\Lambda_n(x,aR)} \ge \frac{\eta(1-a)}{a}.$$

It is easy to realize that $\Lambda_n(x,t) = \prod_{j=1}^{n-1}\big(x_j + F_j(x,t)\big)^{\alpha_j} - \prod_{j=1}^{n-1} x_j^{\alpha_j}$. Then

$$\frac{\Lambda_n(x,R) - \Lambda_n(x,aR)}{\Lambda_n(x,aR)} = \frac{\prod_{j=1}^{n-1}\big(x_j + F_j(x,R)\big)^{\alpha_j} - \prod_{j=1}^{n-1}\big(x_j + F_j(x,aR)\big)^{\alpha_j}}{\prod_{j=1}^{n-1}\big(x_j + F_j(x,aR)\big)^{\alpha_j} - \prod_{j=1}^{n-1}\big(x_j\big)^{\alpha_j}} =: \frac{N}{D}.$$

To write N recall that given non negative numbers $m_j \le M_j$, $j = 1, \dots p$, the difference of their products can be written as follows

$$M_1 M_2 \cdots M_p - m_1 m_2 \cdots m_p = \sum_{k=1}^{p} (M_k - m_k) \prod_{i=1}^{k-1} M_i \prod_{i=k+1}^{p} m_i \tag{A.1}$$

Then

$$\begin{aligned} N &= \sum_{k=1}^{n-1} \Big\{ \big(x_k + F_k(x,R)\big)^{\alpha_k} - \big(x_k + F_k(x,aR)\big)^{\alpha_k} \Big\} \\ &\qquad \cdot \prod_{i=1}^{k-1} \big(x_i + F_i(x,R)\big)^{\alpha_i} \prod_{i=k+1}^{n-1} \big(x_i + F_i(x,aR)\big)^{\alpha_i} \\ &\gtrsim \sum_{k=1}^{n-1} \alpha_k \big(F_k(x,R) - F_k(x,aR)\big)\big(x_k + F_k(x,R)\big)^{\alpha_k - 1} \prod_{i=1}^{k-1} \big(x_i + F_i(x,aR)\big)^{\alpha_i} \prod_{i=k+1}^{n-1} x_i^{\alpha_i}. \end{aligned}$$

Now note that, letting $F_k(x,t) = t f_k(x,t)$ (see (2.4)) we get

$$\begin{aligned} F_k(x,R) - F_k(x,aR) &= R f_k(x,R) - aR f_k(x,aR) \\ &\ge R(1-a) f_k(x,aR) = \frac{1-a}{a} F_k(x,aR). \end{aligned}$$

Then

$$N \gtrsim \frac{1-a}{a} \sum_{k=1}^{n-1} \alpha_k F_k(x,aR)\big(x_k + F_k(x,aR)\big)^{\alpha_k - 1} \prod_{i=1}^{k-1} \big(x_i + F_i(x,aR)\big)^{\alpha_i} \prod_{i=k+1}^{n-1} x_i^{\alpha_i} \simeq \frac{1-a}{a} D,$$

again by (A.1) and (2.14). Thus $N/D \ge \sigma \frac{1-a}{a}$. Therefore the proof of (2.15) is concluded.

To prove the remaining statements use again (A.1) and (2.14) to write

$$\Lambda_n(x,R) \simeq \sum_{k=1}^{n-1} \alpha_k F_k(x,R)\big(x_k + F_k(x,R)\big)^{\alpha_k - 1} \prod_{i=1}^{k-1} \big(x_i + F_i(x,R)\big)^{\alpha_i} \prod_{i=k+1}^{n-1} x_i^{\alpha_i}. \tag{A.2}$$

Now, for any $k = 1, ..., n-1$

$$\begin{aligned} \alpha_k F_k(x,R)(x_k + F_k(x,R))^{\alpha_k - 1} \prod_{i=1}^{k-1} (x_i + F_i(x,R))^{\alpha_i} \prod_{i=k+1}^{n-1} x_i^{\alpha_i} &\le \\ \le \alpha_k \prod_{i=1}^{n-1} (x_i + F_i(x,R))^{\alpha_i} &= \frac{1}{R} F_n(x,R). \end{aligned}$$

Then the second statement follows. Incidentally, note that the explicit estimate of $\Lambda_n(x,r)$ given in (A.2), together with (2.6), shows the doubling property $\Lambda_n(x,2r) \le C\Lambda_n(x,r)$.

Finally, in order to prove that $\Lambda_n(\hat{x}_n, R)) \ge R^{d_n - 1}$ it is enough to estimate from below the right hand side of (A.2) using $x_j + F_j(x,R) \ge F_j(x,R) \ge F_j(0,R)$, $j =$

$1, \ldots, n$

$$\Lambda_n(x,R) \gtrsim \sum_{k=1}^{n-1} \big(F_k(0,R)\big)^{\alpha_k} \prod_{i=1}^{k-1} \big(F_i(0,R)\big)^{\alpha_i} \prod_{i=k+1}^{n-1} \big(F_i(0,R)\big)^{\alpha_i} = CR^{d_n-1}.$$

This ends the proof. $\square$

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