Approximation of anisotropic perimeter functionals by homogenization

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Abstract. We show that all anisotropic perimeter functionals of the form $\int_{\partial^* E\cap\Omega} \varphi(\nu_E)\, d\mathcal{H}^{n-1}$ (φ convex and positively homogeneous of degree one) can be approximated in the sense of Γ-convergence by (limits of) isotropic but inhomogeneous perimeter functionals of the form $\int_{\partial^* E\cap\Omega} a(x/\varepsilon)\, d\mathcal{H}^{n-1}$ (a periodic).

Riassunto. Si dimostra che funzionali perimetro anisotropi della forma $\int_{\partial^* E\cap\Omega} \varphi(\nu_E)\, d\mathcal{H}^{n-1}$ (φ convessa e positivamente omogenea di grado 1) possono essere approssimati via Γ-convergenza attraverso (limiti) di funzionali perimetro isotropi ma inomogenei della forma $\int_{\partial^* E\cap\Omega} a(x/\varepsilon)\, d\mathcal{H}^{n-1}$ (a periodica).

1 Introduction

Object of this paper is the approximation for anisotropic and crystalline energies of the form

$$\mathcal{F}(E) = \int_{\partial^* E\cap\Omega} \varphi(\nu_E)\, d\mathcal{H}^{n-1} \tag{1.1}$$

defined on sets E with finite perimeter on an open set $\Omega \subset \mathbb{R}^n$. Here and henceforth $\partial^* E$ and ν_E are the boundary and the inner normal of E in the usual measure theoretic sense and φ is convex, even, and positively homogeneous of degree one. In other words, φ is a norm on $\mathbb{R}^n$. We do not assume that φ is smooth or isotropic. More precisely, we address the problem of approximating anisotropic functionals of the form (1.1) by locally isotropic but inhomogeneous perimeter functionals of the form

$$\mathcal{G}_\varepsilon(E) = \int_{\partial^* E\cap\Omega} a\Big(\frac{x}{\varepsilon}\Big)\, d\mathcal{H}^{n-1}\,, \tag{1.2}$$

with a a 1-periodic function.

Functionals of the form (1.1) are object of active research, especially in connection with crystalline motion by curvature (see Almgren and Taylor [2], Taylor [22]-[25] and the works by Bellettini, Goglione and Novaga [7], Bellettini and Novaga[8]).

Our approximation suggests an indirect way to deal with crystalline problems where anisotropy is replaced by inhomogeneity and a passage to the limit.

In A. Braides, M. Maslennikov, L. Sigalotti [14] it has been shown that energies of the form (1.2) converge to energies of the form (1.1) (see also Ambrosio-Braides [4]). Here we show that, conversely, all anisotropic energies can be approximated by (limits of) energies of the form (1.2) in the sense of Γ-convergence.

In this paper we suggest two way to approximate φ. In Section 3 given a target φ, $0 < \alpha \leq \varphi \leq \beta < +\infty$, we define a as

$$a(x) = \begin{cases} \varphi(\nu_j) & \text{if } x \in A_j \setminus \Big(\bigcup_{\substack{h\in\mathbb{N}\\ h\neq j}} A_h\Big),\ j \in \mathbb{N} \\ \beta & \text{otherwise in } \mathbb{R}^n\,, \end{cases}$$

for $\mathcal{H}^{n-1}$ a.e. x, where $\{\nu_j\}$ is a dense sequence in S^{n-1} such that $\nu_h \neq \pm\nu_j$ for $h \neq j$, $A_j = \mathbb{Z}^n + \Sigma_j$ and Σ_j is the hyperplane through the origin and orthogonal to ν_j. The idea behind the construction of the function a is that the optimal sequences of sets $E_\varepsilon \to E$ will have boundaries that avoid the sets where the coefficient of a is β; on the contrary these boundaries will lie on hyperplanes A_j, on which $a(x/\varepsilon) = \varphi(\nu_j) = \varphi(\nu_{E_\varepsilon})$, so that indeed $\mathcal{G}_\varepsilon(E_\varepsilon) = \mathcal{F}(E_\varepsilon) \to \mathcal{F}(E)$.

In Section 4 in order to improve the regularity of a a number of technical difficulties must be overcome. First we need to split our construction by considering a finite set $\{\nu_1, \cdots, \nu_k\}$ of rational directions before letting $k \to +\infty$, and at the same time regularize our function a to obtain a continuous integrand. In this way we obtain a Γ-limit depending on k that is a candidate for an approximation of $\mathcal{F}$. The identification of the energy density of this Γ-limit requires the introduction of some carefully constructed piecewise-constant comparison energy densities on which to use the representation formulas for the homogenization of perimeters in [14].

Our result has some connections with a paper by Braides, Buttazzo and Fragalà [11] where (smooth) isotropic Riemannian metrics are shown to be dense in (lower semi-continuous) Finsler metrics in the sense similar to that stated above. Previously Acerbi and Buttazzo [1] proved that the class of Riemannian metrics is not closed in the class of all Finsler metrics with respect to the Γ-convergence of energy integrals. The result in [11] has been generalized to Borel Finsler metrics by Davini [16] (see also [17]).

A possible application of our result is the approximation of perimeter functionals by elliptic energies as in Modica-Mortola [20] (see also [10]) using a double-scale procedure as in Ansini, Braides and Chiadò Piat [6]. In fact, upon identifying a set E with its characteristic functions $u = \chi_E$, the results in [6] show that energies (1.2) can be substituted by energies

$$\mathcal{J}_{\varepsilon,\delta}(u) = \int_\Omega \frac{W(u)}{\delta} + \delta a^2\Big(\frac{x}{\varepsilon}\Big)|Du|^2\,dx$$

defined on $H^1(\Omega)$ where W is a 'double-well energy' and a is periodic.

2 Notation and preliminaries

Let Ω be an open subset of $\mathbb{R}^n$. We denote the Lebesgue n-dimensional measure and the Hausdorff $(n-1)$-dimensional measure of a set $E \subset \mathbb{R}^n$ by $|E|$ and $\mathcal{H}^{n-1}(E)$, respectively, and we set

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

We say that a sequence $\{E_j\}$ of measurable sets of Ω converges to a measurable set $E \subset \Omega$, and we write $E_j \to E$, if $|E_j \triangle E| \to 0$. Let E be a Lesbegue measurable subset of $\mathbb{R}^n$. We denote the *essential boundary* of E by $\partial^* E$; *i.e.,*

$$\partial^* E = \Big\{x \in \mathbb{R}^n : \limsup_{\rho \to 0^+} \frac{|B_\rho(x) \setminus E|}{\rho^n} > 0 \text{ and } \limsup_{\rho \to 0^+} \frac{|B_\rho(x) \cap E|}{\rho^n} > 0\Big\}.$$

We say that E is a set of finite perimeter in Ω, or a *Caccioppoli set*, if it is measurable and

$$P(E, \Omega) := \sup\Big\{\int_E \operatorname{div} g \, dx : \ g \in C_0^1(\Omega; \mathbb{R}^n), |g| \le 1\Big\} < +\infty;$$

the number $P(E, \Omega)$ is called perimeter of E in Ω. We denote the class of sets with finite perimeter in Ω by $\mathcal{P}(\Omega)$ and the class of sets of locally finite perimeter in $\mathbb{R}^n$ by

$$\mathcal{P}_{\text{loc}}(\mathbb{R}^n) = \{F \subset \mathbb{R}^n \ : \ F \in \mathcal{P}(\Omega), \text{ for any open set } \Omega \subset\subset \mathbb{R}^n\}.$$

Let χ_E be the *characteristic function* of E. For any set $E \in \mathcal{P}(\Omega)$ the essential boundary of E, $\partial^* E$, is $\mathcal{H}^{n-1}$-rectifiable; *i.e.,* there exists a countable family (Γ_i) of graphs of Lipschitz functions of $(n-1)$ variables such that $\mathcal{H}^{n-1}(\partial^* E \setminus \bigcup_{i=1}^\infty \Gamma_i) = 0$ and $\mathcal{H}^{n-1}(\partial^* E \cap \Omega) < +\infty$. Moreover, the distributional derivative $D\chi_E$ is an $\mathbb{R}^n$-valued finite Radon measure in Ω, $P(E, \Omega) = |D\chi_E|(\Omega)$ and a generalized Gauss-Green formula holds

$$\int_E \operatorname{div} g \, dx = -\int_\Omega \langle \nu_E, g \rangle \, d|D\chi_E|, \qquad g \in C_0^1(\Omega; \mathbb{R}^n),$$

where $D\chi_E = \nu_E \, |D\chi_E|$ is the polar decomposition of $D\chi_E$ (see Theorem 3.36 in [5]). If E has smooth boundary, the Gauss-Green theorem implies that $D\chi_E = \nu_E \, \mathcal{H}^{n-1} \llcorner \partial^* E$, where ν_E is the inner normal to E. This representation of the distributional derivative was generalized by De Giorgi and Federer as follows:

$$\exists \ \nu_E(x) := \lim_{\rho \to 0^+} \frac{D\chi_E(B_\rho(x))}{|D\chi_E|(B_\rho(x))} \in S^{n-1} \quad \mathcal{H}^{n-1}\text{- a.e. } x \in \partial^* E$$

and

$$D\chi_E = \nu_E \, \mathcal{H}^{n-1} \llcorner \partial^* E.$$

In particular, for every set $E \in \mathcal{P}(\Omega)$, we have that $P(E, \Omega) = \mathcal{H}^{n-1}(\partial^* E \cap \Omega)$.

We refer to the books by Ambrosio, Fusco and Pallara [5] and Federer [19] for the complete exposition of the theory of sets with finite perimeter.

Let $\nu \in S^{n-1}$, let Q^ν be an open cube of $\mathbb{R}^n$ centered at the origin having side length 1 and one face orthogonal to ν, and let $\Pi^\nu_\pm = \{x \in \mathbb{R}^n : \langle x, \pm\nu\rangle > 0\}$. $\partial_\pm Q^\nu$ denote the side of ∂Q^ν orthogonal to ν and included in $\Pi^\nu_\pm$, respectively, while $\partial_L Q^\nu = \partial Q^\nu \setminus (\partial_+ Q^\nu \cup \partial_- Q^\nu)$ is the lateral part of the boundary; *i.e.,* the union of the sides of Q^ν that are parallel to ν.

2.1 Preliminary results

In this section we recall some results that we will use in the sequel.

Theorem 2.1 *Let $\varphi : S^{n-1} \to [0, +\infty)$ be a bounded Borel function and*

$$\mathcal{F}(E) = \int_{\partial^* E\cap\Omega} \varphi(\nu_E)\, d\mathcal{H}^{n-1}$$

for every $E \in \mathcal{P}(\Omega)$. Then the functional $\mathcal{F}$ is lower semicontinuous, in the sense that for every sequence $\{E_h\} \in \mathcal{P}(\Omega)$ and $E \in \mathcal{P}(\Omega)$

$$\lim_{h\to+\infty} |(E_h \triangle E) \cap \Omega| = 0 \quad \Longrightarrow \quad \mathcal{F}(E) \le \liminf_{h\to+\infty} \mathcal{F}(E_h)\,,$$

if and only if the positively one-homogeneous extension of φ from S^{n-1} to $\mathbb{R}^n$ is convex.

The proof of the necessity of Theorem 2.1 is due to Ambrosio-Braides [3] while for the sufficiency we recall the Reshetnyak's theorem (see e.g. [10]).

For simplicity in the following we will say that a real valued function defined on S^{n-1} is convex if its positively one-homogeneous extension from S^{n-1} to $\mathbb{R}^n$

$$p \mapsto \varphi\Big(\frac{p}{|p|}\Big)\, |p|$$

is convex.

Definition 2.2 *Let A be an open set with bounded Lipschitz boundary and let F and G be sets with finite perimeter in A. Let $\omega \subset \partial A$, we say that*

$$G = F \quad \text{on} \quad \omega$$

if and only if the trace (in the usual sense of BV functions) of χ_F and χ_G coincide for $\mathcal{H}^{n-1}$-almost every $x \in \omega$.

Remark 2.3 By Theorem 2.1 and by a simple rescaling argument, for every convex function $\varphi : S^{n-1} \to [0, \infty)$ we have that

$$T^{n-1}\varphi(\nu) \le \int_{\partial^* E\cap TQ^\nu} \varphi(\nu_E)\, d\mathcal{H}^{n-1}\,,$$

for every $T>0$ and every $E\in\mathcal{P}_{\rm loc}(\mathbb{R}^n)$ such that $E=\Pi^\nu_+$ on $T\partial Q^\nu$.

Similarly, for every convex function $\varphi: S^{n-1}\to[0,\infty)$ we have also that

$$T^{n-1}\varphi(\nu)\le\int_{\partial^* E\cap TQ^\nu}\varphi(\nu_E)\,d\mathcal{H}^{n-1}\,,$$

for every $E\in\mathcal{P}_{\rm loc}(\mathbb{R}^n)$ such that $E+T\eta_i=E$, $i=1,\cdots,n-1$, and $E=\Pi^\nu_+$ on $T\partial_\pm Q^\nu$.

Theorem 2.4 (Homogenization of perimeters [14]) *Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary and let $f:\mathbb{R}^n\to[\alpha,\beta]$, with $0<\alpha<\beta<+\infty$, be a 1-periodic Borel function. Then, there exists the limit*

$$\Gamma\text{-}\lim_{\varepsilon\to 0}\int_{\partial^* E\cap\Omega} f\Big(\frac{x}{\varepsilon}\Big)\,d\mathcal{H}^{n-1}=\int_{\partial^* E\cap\Omega} f_{\rm hom}(\nu_E)\,d\mathcal{H}^{n-1}\tag{2.1}$$

for every $E\in\mathcal{P}(\Omega)$. Moreover, there exists the limit

$$\lim_{\varepsilon\to 0}\quad\inf\Big\{\int_{\partial^* F\cap Q^\nu} f\Big(\frac{x}{\varepsilon}\Big)\,d\mathcal{H}^{n-1}:\ F\in\mathcal{P}(Q^\nu),\quad F=\Pi^\nu_+\ \text{on}\ \partial Q^\nu\Big\}$$

for every $\nu\in S^{n-1}$, the function $f_{\rm hom}$ is convex and satisfies the asymptotic formula

$$f_{\rm hom}(\nu)=\lim_{\varepsilon\to 0}\quad\inf\Big\{\int_{\partial^* F\cap Q^\nu} f\Big(\frac{x}{\varepsilon}\Big)\,d\mathcal{H}^{n-1}:\ F\in\mathcal{P}(Q^\nu),\quad F=\Pi^\nu_+\ \text{on}\ \partial Q^\nu\Big\},\tag{2.2}$$

for every $\nu\in S^{n-1}$.

(See also [4]).

Proposition 2.5 (Periodic boundary conditions) *Let f be as in Theorem 2.4. Then*

$$\begin{aligned}f_{\rm hom}(\nu)\ &=\ \lim_{T\to+\infty}\frac{1}{T^{n-1}}\inf\Big\{\int_{\partial^* F\cap TQ^\nu} f(x)\,d\mathcal{H}^{n-1}:\ F\in\mathcal{P}_{\rm loc}(\mathbb{R}^n),\ F=\Pi^\nu_+\ \text{on}\ T\partial_\pm Q^\nu,\\ &\qquad\qquad F+T\eta_i=F\quad i=1,\cdots,n-1\Big\}\end{aligned}\tag{2.3}$$

for every $\nu\in S^{n-1}$ where $(\eta_1,\cdots,\eta_{n-1})$ are linearly independent vectors orthogonal to the faces of Q^ν other than ν.

Proof. Let us define

$$\begin{aligned}g^p_T(\nu)\ &=\ \frac{1}{T^{n-1}}\inf\Big\{\int_{\partial^* F\cap TQ^\nu} f(x)\,d\mathcal{H}^{n-1}:\ F\in\mathcal{P}_{\rm loc}(\mathbb{R}^n),\ F=\Pi^\nu_+\ \text{on}\ T\partial_\pm Q^\nu,\\ &\qquad\qquad F+T\eta_i=F\quad i=1,\cdots,n-1\Big\}\\ &=\ \inf\Big\{\int_{\partial^*\frac{1}{T}F\cap Q^\nu} f(Tx)\,d\mathcal{H}^{n-1}:\ F\in\mathcal{P}_{\rm loc}(\mathbb{R}^n),\ \frac{1}{T}F=\Pi^\nu_+\ \text{on}\ \partial_\pm Q^\nu,\\ &\qquad\qquad \frac{1}{T}F+\eta_i=\frac{1}{T}F\quad i=1,\cdots,n-1\Big\}\end{aligned}\tag{2.4}$$

and

$$g_T(\nu) \;=\; \inf\Big\{\int_{\partial^* E\cap Q^\nu} f(Tx)\, d\mathcal{H}^{n-1} :\; E\in\mathcal{P}(Q^\nu), \quad E=\Pi^\nu_+ \text{ on } \partial Q^\nu\Big\}\,,$$

for every $T\in\mathbb{N}$. Note that the limit of (2.4), as $T\to+\infty$, exists since it is an infimum on $\mathbb{N}$.

Let $F_T\in\mathcal{P}_{\rm loc}(\mathbb{R}^n)$ be such that $F_T+T\eta_i=F_T$, $i=1,\cdots,n-1$, $F_T=\Pi^\nu_+$ on $T\partial_\pm Q^\nu$ and

$$\int_{\partial^*\frac{1}{T}F_T\cap Q^\nu} f(Tx)\, d\mathcal{H}^{n-1}\le g^p_T(\nu)+o(1) \tag{2.5}$$

as $T\to+\infty$. Hence, if we denote by $E_T:=(1/T)F_T$ then the sequence $\{E_T\}$ converges to Π^ν_+ as T tends to ∞. Reasoning as in [14] Lemma 3.2 we may construct a new sequence $\{\widetilde{E}_T\}$ still converging to Π^ν_+ such that $\widetilde{E}_T=\Pi^\nu_+$ in a neighborhood of ∂Q^ν and

$$\lim_{T\to+\infty}\mathcal{H}^{n-1}((\partial^*\widetilde{E}_T\triangle\partial^*E_T)\cap Q^\nu)=0\,. \tag{2.6}$$

In fact, let us define

$$\widetilde{E}_T=\begin{cases} E_T & \text{on } Q^\nu\setminus Q^\nu_\delta\\[2ex] \Pi^\nu_+ & \text{on } Q^\nu_\delta\end{cases} \tag{2.7}$$

where $Q^\nu_\delta=\{x\in Q^\nu:\; d(x):=\operatorname{dist}(x,\mathbb{R}^n\setminus Q^\nu)<\delta\}$, for all $\delta>0$. Note that $\widetilde{E}_T$ is a test set for g_T. Moreover,

$$\begin{aligned}(\partial^*\widetilde{E}_T\triangle\,\partial^*E_T)\cap Q^\nu &= (\partial^*\widetilde{E}_T\triangle\,\partial^*E_T)\cap Q^\nu\setminus Q^\nu_\delta))\cup(\partial^*\widetilde{E}_T\triangle\,\partial^*E_T)\cap Q^\nu_\delta)\\ &= (E_T\triangle\,\Pi^\nu_+)\cap\{d(x)=\delta\}\cup(\partial^*E_T\triangle\,\Pi^\nu)\cap Q^\nu_\delta)\,. \end{aligned}\tag{2.8}$$

Since $E_T\to\Pi^\nu_+$, by Coarea formula we have that

$$\begin{aligned}0 &= \lim_{j\to+\infty}|Q^\nu_\delta\cap(E_T\triangle\,\Pi^\nu_+)|\\ &= \lim_{j\to+\infty}\int_{Q^\nu_\delta\cap(E_T\triangle\,\Pi^\nu_+)}|\nabla d|\,dx\\ &= \lim_{j\to+\infty}\int_0^\delta\mathcal{H}^{n-1}(\{d(x)=t\}\cap(E_T\triangle\,\Pi^\nu_+))\,dt\,.\end{aligned}$$

By a suitable choice of $\delta=\delta_T\to0$ there exists $t_T\in(0,\delta_T)$ such that

$$\lim_{T\to+\infty}\mathcal{H}^{n-1}(\{d(x)=t_T\}\cap(E_T\triangle\,\Pi^\nu_+))=0\,.$$

Hence, by replacing δ with t_T in (2.7) and (2.8) we get (2.6).

Now we can compare g_T^p with g_T. In fact,

$$\int_{\partial^* \widetilde{E}_T \cap Q^\nu} f(Tx)\, d\mathcal{H}^{n-1} = \int_{\partial^* E_T \cap Q^\nu} f(Tx)\, d\mathcal{H}^{n-1} - \int_{(\partial^* E_T \setminus \partial^* \widetilde{E}_T) \cap Q^\nu} f(Tx)\, d\mathcal{H}^{n-1} + \int_{(\partial^* \widetilde{E}_T \setminus \partial^* E_T) \cap Q^\nu} f(Tx)\, d\mathcal{H}^{n-1}.$$

By (2.2), (2.5) and (2.6) we have that $f_{\text{hom}}(\nu) \le \lim_{T\to+\infty} g_T^p$. The other inequality easily follows by definition of g_T and g_T^p. $\square$

3 Approximation of φ

In this section we prove that given a target φ we can construct a suitable function a such that

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \int_{\partial^* E \cap \Omega} a\Big(\frac{x}{\varepsilon}\Big)\, d\mathcal{H}^{n-1} = \int_{\partial^* E \cap \Omega} \varphi(\nu_E)\, d\mathcal{H}^{n-1}$$

for every $E \in \mathcal{P}(\Omega)$.

Theorem 3.1 *Let $\varphi : S^{n-1} \to [\alpha, \beta]$, with $0 < \alpha < \beta$, be a Borel function such that the positively one-homogeneous extension of φ to $\mathbb{R}^n$*

$$p \mapsto \varphi\Big(\frac{p}{|p|}\Big)\, |p|$$

is convex and even. Let $\{\nu_j\}$ be a dense sequence in S^{n-1} such that $\nu_h \ne \pm\nu_j$ for $h \ne j$ and let $A_j = \mathbb{Z}^n + \Sigma_j$ where Σ_j is the hyperplane through the origin and orthogonal to ν_j, for every $j \in \mathbb{N}$. Let $a : \mathbb{R}^n \to [\alpha, \beta]$ be a Borel function defined by

$$a(x) = \begin{cases} \varphi(\nu_j) & \text{if } x \in A_j \setminus \Big(\bigcup_{\substack{h\in\mathbb{N} \\ h\ne j}} A_h\Big),\ j \in \mathbb{N} \\ \alpha & \text{if } x \in \bigcup_{\substack{h\in\mathbb{N} \\ h\ne j}} (A_j \cap A_h),\ j \in \mathbb{N} \\ \beta & \text{otherwise in } \mathbb{R}^n\,. \end{cases} \tag{3.1}$$

Then,

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \int_{\partial^* E \cap \Omega} a\Big(\frac{x}{\varepsilon}\Big)\, d\mathcal{H}^{n-1} = \int_{\partial^* E \cap \Omega} \varphi(\nu_E)\, d\mathcal{H}^{n-1}$$

for every $E \in \mathcal{P}(\Omega)$

$$\varphi(\nu) = \lim_{T\to+\infty} \frac{1}{T^{n-1}} \inf\Big\{ \int_{\partial^* F \cap TQ^\nu} a(x)\, d\mathcal{H}^{n-1} :\ F \in \mathcal{P}(TQ^\nu),\ F = \Pi_+^\nu \text{ on } T\partial Q^\nu \Big\}.$$

Proof. We recall that by Theorem 2.4 we have that

$$\Gamma\text{-}\lim_{\varepsilon\to 0}\int_{\partial^* E\cap\Omega} a\Big(\frac{x}{\varepsilon}\Big)\,d\mathcal{H}^{n-1} = \int_{\partial^* E\cap\Omega} a_{\rm hom}(\nu_E)\,d\mathcal{H}^{n-1}$$

for every $E\in\mathcal{P}(\Omega)$, where $a_{\rm hom}$ is convex and

$$a_{\rm hom}(\nu) \quad = \quad \lim_{T\to+\infty}\frac{1}{T^{n-1}}\inf\Big\{\int_{\partial^* F\cap TQ^\nu} a(x)\,d\mathcal{H}^{n-1}:\ F\in\mathcal{P}(TQ^\nu),\ F=\Pi^\nu_+ \text{ on } T\partial Q^\nu\Big\}.$$

Hence, it remain to prove then that $\varphi = a_{\rm hom}$.

We first deal with the inequality: $a_{\rm hom}(\nu)\geq\varphi(\nu)$. Let $F\in\mathcal{P}(TQ^\nu)$ such that $F=\Pi^\nu_+$ on $T\partial Q^\nu$. The essential boundary $\partial^* F$ may intersect A_j in a set of positive $\mathcal{H}^{n-1}$ measure, which means that a relevant part of $\partial^* F$ coincides with a part of A_j; hence,

$$\nu_j = \pm\nu_F \qquad \mathcal{H}^{n-1}\text{- a.e. in } \partial^* F\cap A_j\,.$$

Since $\varphi(\nu)=\varphi(-\nu)\in[\alpha,\beta]$, by (3.1) we get then

$$\begin{aligned}
&\int_{\partial^* F\cap TQ^\nu} a(x)\,d\mathcal{H}^{n-1}\\
=\;& \sum_{j\in\mathbb{N}}\int_{\partial^* F\cap A_j\cap TQ^\nu}\varphi(\nu_j)\,d\mathcal{H}^{n-1} + \int_{(\partial^* F\setminus\cup_j A_j)\cap TQ^\nu}\beta\,d\mathcal{H}^{n-1}\\
\geq\;& \sum_{j\in\mathbb{N}}\int_{\partial^* F\cap A_j\cap TQ^\nu}\varphi(\nu_F)\,d\mathcal{H}^{n-1} + \int_{(\partial^* F\setminus\cup_j A_j)\cap TQ^\nu}\varphi(\nu_F)\,d\mathcal{H}^{n-1}\\
=\;& \int_{\partial^* F\cap TQ^\nu}\varphi(\nu_F)\,d\mathcal{H}^{n-1}\,.
\end{aligned}\tag{3.2}$$

By (3.2) and Remark 2.3, we can conclude that

$$\int_{\partial^* F\cap TQ^\nu} a(x)\,d\mathcal{H}^{n-1}\geq T^{n-1}\varphi(\nu)$$

and by definition of $a_{\rm hom}$

$$a_{\rm hom}(\nu)\geq\varphi(\nu)\,. \tag{3.3}$$

By (3.1), we have that

$$\int_{\Sigma_j\cap TQ^{\nu_j}} a(x)\,d\mathcal{H}^{n-1} = T^{n-1}\varphi(\nu_j)\,;$$

hence, $a_{\rm hom}(\nu_j)=\varphi(\nu_j)$ for every $j\in\mathbb{N}$. To conclude the proof of the theorem it remains to show that $a_{\rm hom}(\nu)=\lim_{j\to\infty}a_{\rm hom}(\nu_j)$, and this is an easy consequence of the convexity of $a_{\rm hom}$.

□

The following proposition allows us to describe φ also by a homogenization formula with periodic boundary conditions.

Proposition 3.2

$$\varphi(\nu) = \lim_{T\to+\infty} \frac{1}{T^{n-1}} \inf\Big\{\int_{\partial^* F\cap TQ^\nu} a(x)\, d\mathcal{H}^{n-1} : F \in \mathcal{P}_{\rm loc}(\mathbb{R}^n),\ F = \Pi^\nu_+ \text{ on } T\partial_\pm Q^\nu,$$
$$F + T\eta_i = F \quad i = 1, \cdots, n-1\Big\} \quad (3.4)$$

for every $\nu \in S^{n-1}$ where $(\eta_1, \cdots, \eta_{n-1})$ are linearly independent vectors orthogonal to the faces of Q^ν other than ν.

Proof. By Theorem 3.1 we have that $a_{\rm hom}(\nu) = \varphi(\nu)$; hence, by Proposition 2.5 we get (3.4). □

4 Approximation scheme for φ by regularization

In this section we suggest another way to approximate φ by regularizing a.

Theorem 4.1 *Let $\varphi : S^{n-1} \to [\alpha, \beta]$, with $0 < \alpha < \beta$, be a Borel function such that the positively homogeneous of degree one extension of φ to $\mathbb{R}^n$*

$$p \mapsto \varphi\Big(\frac{p}{|p|}\Big)\, |p|$$

is convex and even. Then there exists a family of functions $a_{k,\lambda} : \mathbb{R}^n \mapsto [\alpha, \beta]$, 1-periodic and λ-Lipschitz, such that

$$\Gamma\text{-}\lim_{\varepsilon\to0} \int_{\partial^* E\cap\Omega} a_{k,\lambda}\Big(\frac{x}{\varepsilon}\Big)\, d\mathcal{H}^{n-1} = \int_{\partial^* E\cap\Omega} \varphi_{k,\lambda}(\nu_E)\, d\mathcal{H}^{n-1}$$

for every $\lambda \in \mathbb{R}^+$, $k \in \mathbb{N}$ and $E \in \mathcal{P}(\Omega)$. Moreover,

$$\lim_{k\to+\infty} \lim_{\lambda\to+\infty} \varphi_{k,\lambda}(\nu) = \varphi(\nu)$$

for every $\nu \in S^{n-1}$ and

$$\Gamma\text{-}\lim_{k\to+\infty}\Big(\Gamma\text{-}\lim_{\lambda\to+\infty} \int_{\partial^* E\cap\Omega} \varphi_{k,\lambda}(\nu_E)\, d\mathcal{H}^{n-1}\Big) = \int_{\partial^* E\cap\Omega} \varphi(\nu_E)\, d\mathcal{H}^{n-1}$$

for every $E \in \mathcal{P}(\Omega)$.

Proof. Let Ξ be the set of unit rational directions; *i.e.*, $\Xi = \{\nu = \frac{x}{|x|} \in S^{n-1} : x \in \mathbb{Z}^n \setminus \{0\}\}$. Since Ξ is dense and countable in S^{n-1} we consider a dense sequence $\{\nu_j\}_{j\in\mathbb{N}} \in \Xi$ such that $\nu_h \neq \pm\nu_j$ for $h \neq j$. We define $A_j = \mathbb{Z}^n + \Sigma_j$ where Σ_j is the hyperplane through the origin and orthogonal to ν_j, for every $j \in \mathbb{N}$. The set A_j is closed and 1-periodic with respect to the canonical basis $(e_1, \cdots, e_n)$ of $\mathbb{R}^n$. We fix $k \in \mathbb{N}$ and we consider the first k directions $(\nu_1, \cdots, \nu_k) \subset \{\nu_j\}_{j\in\mathbb{N}}$. We define

$$a_k(x) = \begin{cases} \varphi(\nu_j) & \text{if } x \in A_j \setminus \left(\bigcup_{\substack{h=1\\h\neq j}}^k A_h\right), \quad j = 1, \cdots, k \\ \\ \alpha & \text{if } x \in \bigcup_{\substack{h=1\\h\neq j}}^k (A_j \cap A_h), \quad j = 1, \cdots, k \\ \\ \beta & \text{otherwise in } \mathbb{R}^n \end{cases} \tag{4.1}$$

and we denote by $a_{k,\lambda}$ the Yosida transform of a_k; *i.e.,*

$$a_{k,\lambda}(x) = \inf_{y\in\mathbb{R}^n} \{a_k(y) + \lambda|x-y|\}, \qquad \lambda \in \mathbb{R}^+.$$

Hence, $a_{k,\lambda}$ is λ-Lipschitz. Moreover, since a_k is lower semicontinuous and 1-periodic, we have that $a_{k,\lambda}$ is also 1-periodic and the sequence $\{a_{k,\lambda}\}_\lambda$ converges increasingly to a_k as $\lambda \to +\infty$; *i.e.,*

$$a_k(x) = \sup_{\lambda\geq 0} a_{k,\lambda}(x) \tag{4.2}$$

(see e.g. [13] Remark 1.6 and Proposition 1.7). For any fixed $k \in \mathbb{N}$ and $\lambda \in \mathbb{R}^+$ the function $a_{k,\lambda}$ is continuous, bounded and 1-periodic. By Theorem 2.4 we have that

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \int_{\partial^* E\cap\Omega} a_{k,\lambda}\left(\frac{x}{\varepsilon}\right) d\mathcal{H}^{n-1} = \int_{\partial^* E\cap\Omega} \varphi_{k,\lambda}(\nu_E)\, d\mathcal{H}^{n-1}$$

and

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \int_{\partial^* E\cap\Omega} a_k\left(\frac{x}{\varepsilon}\right) d\mathcal{H}^{n-1} = \int_{\partial^* E\cap\Omega} \varphi_k(\nu_E)\, d\mathcal{H}^{n-1}$$

for every $E \in \mathcal{P}(\Omega)$ where $\varphi_{k,\lambda}$ and φ_k are convex functions. By Proposition 2.5, $\varphi_{k,\lambda}$ and φ_k can be also described by the following formulas

$$\begin{aligned} \varphi_{k,\lambda}(\nu) \quad = \quad & \lim_{T\to+\infty} \frac{1}{T^{n-1}} \inf\Big\{ \int_{\partial^* F\cap TQ^\nu} a_{k,\lambda}(x)\, d\mathcal{H}^{n-1} : \ F \in \mathcal{P}_{\rm loc}(\mathbb{R}^n), \\ & F = \Pi^\nu_+ \text{ on } T\partial_\pm Q^\nu, \ F + T\eta_i = F \quad i = 1, \cdots, n-1 \Big\} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \varphi_k(\nu) \quad = \quad & \lim_{T\to+\infty} \frac{1}{T^{n-1}} \inf\Big\{ \int_{\partial^* F\cap TQ^\nu} a_k(x)\, d\mathcal{H}^{n-1} : \ F \in \mathcal{P}_{\rm loc}(\mathbb{R}^n), \\ & F = \Pi^\nu_+ \text{ on } T\partial_\pm Q^\nu, \ F + T\eta_i = F \quad i = 1, \cdots, n-1 \Big\}, \end{aligned} \tag{4.4}$$

for every $\nu \in S^{n-1}$. Our aim is to study the pointwise convergence of $\{\varphi_{k,\lambda}\}_{k,\lambda}$ letting first λ and then k go to $+\infty$. In the following we first prove that $\{\varphi_{k,\lambda}\}_\lambda$ pointwise converges to

φ_k as λ tends to $+\infty$; then, we show that $\{\varphi_k\}$ pointwise converges to φ as k tends to $+\infty$. Therefore to conclude the proof of the theorem it remains to observe that the pointwise convergence of the convex integrands $\{\varphi_{k,\lambda}\}_\lambda$ and $\{\varphi_k\}_k$ implies the Γ-convergence of the corresponding families of functionals. In fact, the pointwise convergence of convex functions implies the uniform convergence on S^{n-1}. Hence, we have that

$$\Gamma\text{-}\lim_{\lambda\to+\infty}\int_{\partial^* E\cap\Omega}\varphi_{k,\lambda}(\nu_E)\,d\mathcal{H}^{n-1}=\int_{\partial^* E\cap\Omega}\varphi_k(\nu_E)\,d\mathcal{H}^{n-1}$$

and

$$\Gamma\text{-}\lim_{k\to+\infty}\int_{\partial^* E\cap\Omega}\varphi_k(\nu_E)\,d\mathcal{H}^{n-1}=\int_{\partial^* E\cap\Omega}\varphi(\nu_E)\,d\mathcal{H}^{n-1}$$

which concludes the proof of the theorem.

Let us deal with the pointwise convergence of $\{\varphi_{k,\lambda}\}_\lambda$ to φ_k. By (4.2) we have that $\varphi_k(\nu)\geq\varphi_{k,\lambda}(\nu)$. Hence, if we prove that

$$\liminf_{\lambda\to+\infty}\varphi_{k,\lambda}(\nu)\geq\varphi_k(\nu) \tag{4.5}$$

then we can conclude that there exists the limit, as λ tends to $+\infty$, and

$$\lim_{\lambda\to+\infty}\varphi_{k,\lambda}(\nu)=\varphi_k(\nu) \tag{4.6}$$

for every $\nu\in S^{n-1}$. To obtain (4.5) we need to introduce some auxiliary functions $\tilde a_{k,\lambda}$.

Definition of the auxiliary functions. Let $k\geq n$. We define

$$\begin{aligned} S(k) \;=\; &\{s=(s_1,\cdots,s_k)\,:\; s_j=0 \text{ or } j\,,\quad j=1,\cdots,k\quad\text{and}\\ &\quad\text{at least two of}\quad(s_1,\cdots,s_k)\quad\text{are different from }0\}\,. \end{aligned}$$

For every fixed $s\in S(k)$ we define then

$$H_s^d=\bigcap_{\substack{s_j\neq 0\\ j=1,\cdots,k}} A_{s_j}$$

where $d=0,\cdots,n-2$ denotes the dimension of H_s^d. Note that for any fixed d the sets H_s^d, $s\in S(k)$, may be not disjoints. Moreover, the intersection between $\{H_s^d\}_{d,s}$ and TQ^ν gives rise to a finite number of sets.

Around any H_s^d and A_j we construct suitable neighborhoods and we define the following sets

$$U=\bigcup_{\substack{s\in S(k)\\ d=0,\cdots,n-2}}\left\{x\in\mathbb{R}^n\,:\,\operatorname{dist}(x,H_s^d)<\lambda^{-1/n-d}\right\}$$

and

$$U_j = \Big\{x \in \mathbb{R}^n : \mathrm{dist}\,(x, A_j) < \lambda^{-\gamma}\Big\} \setminus U$$

with $1/2 < \gamma < 1$ and $j = 1, \cdots, k$. Since the sets A_j, $j = 1, \cdots, k$ are closed and pairwise disjoint, the sets U_j are pairwise disjoint for λ large enough. Note that $\lambda^{-\gamma} < \lambda^{-1/2} < \cdots < \lambda^{-1/n-d} < \cdots < \lambda^{-1/n}$. Finally, we define for λ large enough

$$\tilde{a}_{k,\lambda}(x) = \begin{cases} \varphi(\nu_j) & \text{if } x \in U_j, \quad j = 1, \cdots, k \\ \alpha & \text{if } x \in U \\ \beta & \text{otherwise in } \mathbb{R}^n\,; \end{cases} \tag{4.7}$$

where U_j and U are defined as above.

The choice of the radii in the definition of U and U_j allows to compare easily $a_{k,\lambda}$ and $\tilde{a}_{k,\lambda}$ and prove that

$$a_{k,\lambda}(x) \geq \tilde{a}_{k,\lambda}(x) \tag{4.8}$$

for every $x \in \mathbb{R}^n$ and for λ big enough. In fact, if $x \in U$ the inequality is trivial since $\tilde{a}_{k,\lambda}(x) = \alpha$ while, by definition, $a_k(y) \geq \alpha$ for every $y \in \mathbb{R}^n$. If $x \in U_j$ then $\tilde{a}_{k,\lambda}(x) = \varphi(\nu_j)$ and there exists H^d_s such that

$$a_{k,\lambda}(x) = \min\{\alpha + \lambda|x - \Pi^d_s x|,\, \varphi(\nu_j) + \lambda|x - \Pi_j x|,\, \beta\}$$

where Π^d_s denotes the orthogonal projection on H^d_s and Π_j is the orthogonal projection on A_j. The sets A_j are not convex. The orthogonal projection makes no sense globally. By the way, in most formulas only $|x - \Pi_j x|$ is used. This is the distance from A_j, which is always well defined. In order to prove the inequality (4.8) it is sufficient to exclude that $\alpha + \lambda|x - \Pi^d_s x|$ can be the minimum, for example, showing that

$$\alpha + \lambda|x - \Pi^d_s x| > \beta\,. \tag{4.9}$$

In fact,

$$\alpha + \lambda|x - \Pi^d_s x| \geq \alpha + \lambda^{(1-1/n-d)} > \beta$$

for λ big enough. Similarly, if $x \in \mathbb{R}^n \setminus (U \cup \bigcup_{j=1}^k U_j)$, then $\tilde{a}_{k,\lambda}(x) = \beta$ and $a_{k,\lambda}(x)$ is essentially reduced to be

$$a_{k,\lambda}(x) = \min\{\alpha + \lambda|x - \Pi^d_s x|, \varphi(\nu_h) + \lambda|x - \Pi_h x|, \beta\}$$

for a suitable choice of A_h and H^d_s. But also in this case (4.9) is satisfied and

$$\varphi(\nu_h) + \lambda|x - \Pi_h x| \geq \varphi(\nu_h) + \lambda^{1-\gamma} > \beta$$

for λ big enough, which implies (4.8).

In order to prove (4.5) we now have to compare $\tilde{a}_{k,\lambda}$ with a_k. To this end in the following Steps we introduce the functions $\phi^d_\lambda : \mathbb{R}^n \mapsto \mathbb{R}^n$, for $d = 0, \cdots, n-1$, such that if we compose them then we get a function $\phi_\lambda = \phi_\lambda^{n-1} \circ \cdots \circ \phi_\lambda^0$ that "projects" the sets U, U_j into H_s^d, A_j, respectively, for every $j = 1, \cdots, k$, $d = 0, \cdots, n-1$ and $s \in S(k)$.

Step 0 ($d = 0$). For every $s \in S(k)$ the set H_s^0 is a sequence of points $\{x_p^s\}_p$. We denote by $B_p^s(r) = B(x_p^s, r)$ and we define ϕ_λ^0 as follows:
(a) ϕ_λ^0 projects $B_p^s(\lambda^{-1/n})$ into the center x_p^s; *i.e.*,

$$\phi_\lambda^0(x) = x_p^s\,, \qquad x \in B_p^s(\lambda^{-1/n})\,, \quad \forall\, p \ \text{ and } \ s \in S(k)\,;$$

(b) ϕ_λ^0 dilates $B_p^s(\lambda^{-1/n+1}) \setminus B_p^s(\lambda^{-1/n})$ into $B_p^s(\lambda^{-1/n+1})$ keeping fixed $\partial B_p^s(\lambda^{-1/n+1})$; *i.e.*,

$$\phi_\lambda^0(x) = \frac{1}{1-\lambda^{-1/n(n+1)}}\left(|x - x_p^s| - \lambda^{-1/n}\right)^+ \frac{x - x_p^s}{|x - x_p^s|} + x_p^s$$

for every $x \in B_p^s(\lambda^{-1/n+1}) \setminus B_p^s(\lambda^{-1/n})$, p and $s \in S(k)$;
(c) ϕ_λ^0 is the identity on $\mathbb{R}^n \setminus \bigcup_{s\in S(k),p} B_p^s(\lambda^{-1/n+1})$; *i.e.*,

$$\phi_\lambda^0(x) = x\,, \qquad x \notin \bigcup_{s\in S(k),p} B_p^s(\lambda^{-1/n+1})\,.$$

Note that ϕ_λ^0 is a Lipschitz function with constant

$$\mathrm{Lip}\,(\phi_\lambda^0) = \frac{1}{1-\lambda^{-1/n(n+1)}}\,.$$

In dimension $d \geq 1$, ϕ_λ^d is the natural generalization of ϕ_λ^0 but we have also to take into account that ϕ_λ^d is applied to sets already modified by $\phi_\lambda^{d-1} \circ \cdots \circ \phi_\lambda^0$.

Step d ($d = 1, \cdots, n-2$). We denote by

$$I_s^d = \phi_\lambda^{d-1} \circ \cdots \circ \phi_\lambda^0(\{x \in \mathbb{R}^n \,:\, \mathrm{dist}\,(x, H_s^d) < \lambda^{-1/n-d}\})$$

and

$$N_s^d = \left\{x \in \mathbb{R}^n \,:\, \mathrm{dist}\,(x, H_s^d) < \lambda^{-1/(n-d+1)}\right\}.$$

Let $\widetilde{\Pi}_s^d$ be the orthogonal projection on ∂I_s^d; $\widetilde{\Pi}_s^d$ is defined only locally. We recall that Π_s^d is the orthogonal projection on H_s^d. Then we define ϕ_λ^d as follows:

(a) ϕ_λ^d projects I_s^d into H_s^d; *i.e.*,

$$\phi_\lambda^d(x) = \Pi_s^d(x)\,, \qquad x \in I_s^d\,, \quad \forall s\,;$$

(b) ϕ_λ^d dilates $N_s^d \setminus I_s^d$ into N_s^d keeping fixed ∂N_s^d; *i.e.*,

$$\phi_\lambda^d(x) = \frac{\lambda^{-1/(n-d+1)}}{\lambda^{-1/(n-d+1)} - |\widetilde{\Pi}_s^d(x) - \Pi_s^d(x)|}\left(|x - \Pi_s^d(x)| - |\widetilde{\Pi}_s^d(x) - \Pi_s^d(x)|\right)^+ \frac{x - \Pi_s^d(x)}{|x - \Pi_s^d(x)|} + \Pi_s^d(x)$$

for every $x \in N_s^d \setminus I_s^d$ and $s \in S(k)$;

(c) ϕ_λ^d is the identity on $\mathbb{R}^n \setminus \bigcup_{s\in S(k)} N_s^d$; *i.e.*,

$$\phi_\lambda^d(x) = x\,, \qquad x \notin \bigcup_{s\in S(k)} N_s^d\,.$$

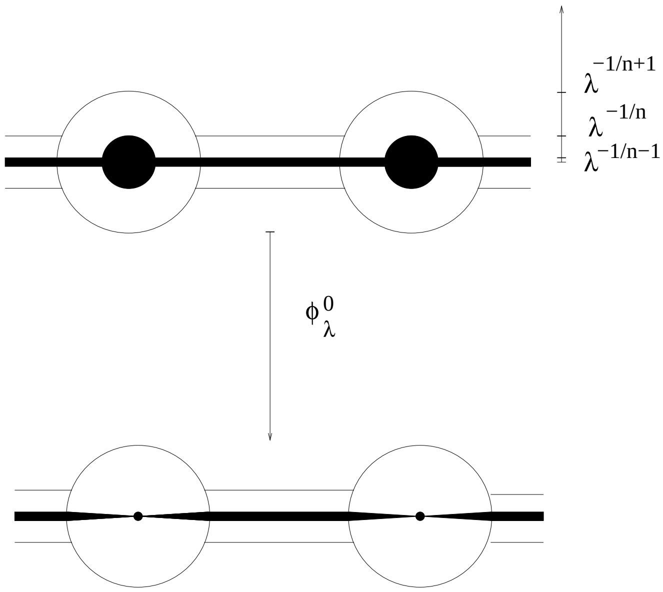

Figure 1: ϕ_λ^0 applied to the neighborhoods of H_s^0 and H_s^1

Note that

$$I_s^d = \quad \phi_\lambda^{d-1}\circ\cdots\circ\phi_\lambda^0(\{x\in\mathbb{R}^n \,:\, \mathrm{dist}\,(x,H_s^d) < \lambda^{-1/n-d}\}) \\ \bigcap \\ \{x\in\mathbb{R}^n \,:\, \mathrm{dist}\,(x,H_s^d) < \lambda^{-1/n-d}\}$$

then

$$\frac{\lambda^{-1/(n-d+1)}}{\lambda^{-1/(n-d+1)} - |\widetilde{\Pi}_s^d(x) - \Pi_s^d(x)|} \le \frac{\lambda^{-1/(n-d+1)}}{\lambda^{-1/(n-d+1)} - \lambda^{-1/(n-d)}} = \frac{1}{1-\lambda^{-1/(n-d)(n-d+1)}}\,.$$

Therefore, ϕ_λ^d is a Lipschitz function with constant

$$\mathrm{Lip}\,(\phi_\lambda^d) = \frac{1}{1-\lambda^{-1/(n-d)(n-d+1)}}\,.$$

Moreover, by definition, any function ϕ_λ^d maps H_s^d and A_j in themselves for every $d = 0, \cdots, n-2$, $s \in S(k)$ and $j = 1, \cdots, k$.

Step $(n-1)$. Finally, we define ϕ_λ^{n-1} as the function that dilates

$$\mathbb{R}^n \setminus \Big(\phi_\lambda^{n-2} \circ \cdots \circ \phi_\lambda^0 \Big(\cup_{j=1}^k U_j \cup U\Big)\Big)$$

such that

$$\begin{cases} \phi_\lambda^{n-1} \circ \phi_\lambda^{n-2} \circ \cdots \circ \phi_\lambda^0 \Big(\cup_{j=1}^k U_j \cup U\Big) = \cup_{j=1}^k A_j \,, \\ \phi_\lambda^{n-1}(A_j) = A_j & j = 1, \cdots, k \\ \operatorname{Lip} \phi_\lambda^{n-1} = 1 + (c_k/\lambda^\gamma) \,. \end{cases}$$

We define then

$$\phi_\lambda = \phi_\lambda^{n-1} \circ \cdots \circ \phi_\lambda^0$$

that is a Lipschitz function with constant

$$\operatorname{Lip}(\phi_\lambda) = \Big(1 + \frac{c_k}{\lambda^\gamma}\Big) \prod_{d=0}^{n-2} \Big(\frac{1}{1 - \lambda^{-1/(n-d)(n-d+1)}}\Big) \,, \tag{4.10}$$

and

$$\lim_{\lambda \to +\infty} \operatorname{Lip}(\phi_\lambda) = 1 \,. \tag{4.11}$$

Now we use the construction of ϕ_λ and of the auxiliaries functions $\tilde{a}_{k,\lambda}$ to prove the inequality (4.5). Let $G \in \mathcal{P}_{\mathrm{loc}}(\mathbb{R}^n)$ such that $G = \Pi_+^\nu$, on $T\partial_\pm Q^\nu$, $G + T\eta_i = G$ for $i = 1, \cdots, (n-1)$; by (4.7) we have that

$$\begin{aligned} & \int_{\partial^* G \cap TQ^\nu} \tilde{a}_{k,\lambda}(x)\, d\mathcal{H}^{n-1} \\ = & \sum_{j=1}^k \varphi(\nu_j)\, \mathcal{H}^{n-1}(\partial^* G \cap TQ^\nu \cap U_j) + \alpha\, \mathcal{H}^{n-1}(\partial^* G \cap TQ^\nu \cap U) \\ & + \beta\, \mathcal{H}^{n-1}(\partial^* G \cap V_j) \,, \end{aligned} \tag{4.12}$$

where $V_j = TQ^\nu \setminus (U_j \cup U)$. Note that by definition of ϕ_λ we have that $\mathcal{H}^{n-1}(\phi_\lambda(\partial^* G \cap TQ^\nu \cap U)) = 0$. Hence, by (4.12), (4.10) and the property of the Hausdorff measure with respect to a Lipschitz function (see [5] Proposition 2.49 (iv)) we get that

$$\begin{aligned} & \operatorname{Lip}(\phi_\lambda)^{n-1} \int_{\partial^* G \cap TQ^\nu} \tilde{a}_{k,\lambda}(x)\, d\mathcal{H}^{n-1} \\ \geq & \sum_{j=1}^k \varphi(\nu_j)\, \mathcal{H}^{n-1}(\phi_\lambda(\partial^* G \cap TQ^\nu \cap U_j)) + \beta\, \mathcal{H}^{n-1}(\phi_\lambda(\partial^* G \cap V_j)) \\ \geq & \int_{\partial^* \tilde{G} \cap TQ^\nu} a_k(x)\, d\mathcal{H}^{n-1} \,, \end{aligned} \tag{4.13}$$

where $\tilde{G} \in \mathcal{P}_{\rm loc}(\mathbb{R}^n)$ such that

$$\partial^* \tilde{G} \cap TQ^\nu \subseteq \phi_\lambda(\partial^* G \cap TQ^\nu)\,. \tag{4.14}$$

Note that, since A_j is 1-periodic with respect to the canonical basis $(e_1, \cdots, e_n)$ of $\mathbb{R}^n$, we have that, in general, $A_j + T\eta_i \neq A_j$ for every $i = 1, \cdots, n-1$ and $j = 1, \cdots, k$. Hence, when we apply ϕ_λ we may have that $\tilde{G} + T\eta_i \neq \tilde{G}$ for some $i = 1, \cdots, n-1$. By definition, any change due to ϕ_λ remains in a neighborhood of $\partial^* G$ with radius smaller than $\lambda^{-1/n+1}$ (see Steps $d = 0, \cdots, n-1$). Hence, we may slightly modify $\tilde{G}$ close to $T\partial_L Q^\nu$ to match the periodic boundary conditions. We denote by $S \in \mathcal{P}_{\rm loc}(\mathbb{R}^n)$ this new set such that $S = \Pi_+^\nu$ on $T\partial_\pm Q^\nu$, $S + T\eta_i = S$, for every $i = 1, \cdots, n-1$ and such that

$$\Big|\mathcal{H}^{n-1}(\partial^* S \cap TQ^\nu) - \mathcal{H}^{n-1}(\partial^* \tilde{G} \cap TQ^\nu)\Big| = O(\lambda^{-1/n+1}\, T^{n-2})\,. \tag{4.15}$$

By (4.13) and (4.15), since a_k is bounded, we get then

$$\begin{aligned}
&\frac{\mathrm{Lip}(\phi_\lambda)^{n-1}}{T^{n-1}} \int_{\partial^* G \cap TQ^\nu} \tilde{a}_{k,\lambda}(x)\, d\mathcal{H}^{n-1} \\
\geq\ & \frac{1}{T^{n-1}} \int_{\partial^* S \cap TQ^\nu} a_k(x)\, d\mathcal{H}^{n-1} + O(\lambda^{-1/n+1}\, T^{-1}) \\
\geq\ & \frac{1}{T^{n-1}} \inf\Big\{ \int_{\partial^* F \cap TQ^\nu} a_k(x)\, d\mathcal{H}^{n-1} :\ F \in \mathcal{P}_{\rm loc}(\mathbb{R}^n),\ F = \Pi_+^\nu \text{ on } T\partial_\pm Q^\nu, \\
&\qquad\qquad\qquad\qquad\qquad\qquad F + T\eta_i = F \quad i = 1, \cdots, n-1 \Big\} \\
&+ O(\lambda^{-1/n+1}\, T^{-1})\,.
\end{aligned} \tag{4.16}$$

Hence, by (4.8), (4.16) and the definition of $\varphi_{k,\lambda}$ and φ_k (see(4.3) and (4.4)) passing to the limit as T tends to $+\infty$ we have that

$$\varphi_{k,\lambda}(\nu) \geq \frac{1}{\mathrm{Lip}(\phi_\lambda)^{n-1}}\ \varphi_k(\nu)$$

for every $\nu \in S^{n-1}$. Finally, by (4.11) passing to the limit as λ tends to $+\infty$ we get (4.5) which implies, as already observed, the pointwise convergence of $\{\varphi_{k,\lambda}\}_\lambda$ to φ_k.

It remains to study the pointwise convergence of $\{\varphi_k\}$ to φ as k tends to $+\infty$. By Remark 2.3 and (4.1) we have that

$$\varphi(\nu) \leq \frac{1}{T^{n-1}} \int_{\partial^* F \cap TQ^\nu} \varphi(\nu_F)\, d\mathcal{H}^{n-1} \leq \frac{1}{T^{n-1}} \int_{\partial^* F \cap TQ^\nu} a_k(x)\, d\mathcal{H}^{n-1}\,,$$

for every $F \in \mathcal{P}_{\rm loc}(\mathbb{R}^n)$ such that $F + T\eta_i = F$, $i = 1, \cdots, n-1$, and $F = \Pi_+^\nu$ on $T\partial_\pm Q^\nu$. By definition of φ_k we have then

$$\varphi(\nu) \leq \varphi_k(\nu) \tag{4.17}$$

while

$$\varphi(\nu_j) = \varphi_k(\nu_j)\,, \qquad\qquad j = 1, \cdots, k\,. \tag{4.18}$$

We define

$$\psi_k(z) = \begin{cases} \varphi(z) & \text{if } z \in \cup_{j=1}^{k} \mathbb{R}\,\nu_j \\ \\ \beta|z| & \text{otherwise}\,, \end{cases}$$

then, $\varphi(\nu) \leq \psi_k(\nu)$. Let $co\,\psi_k$ be the convex envelope of ψ_k. Since φ is convex we have $\varphi(\nu) \leq co\,\psi_k(\nu)$. Moreover, $\varphi(\nu_j) \leq co\,\psi_k(\nu_j) \leq \psi_k(\nu_j) = \varphi(\nu_j)$; hence,

$$\varphi(\nu_j) = co\,\psi_k(\nu_j) \tag{4.19}$$

for $j = 1, \cdots, k$. The functions $(co\,\psi_k)$ are equi-lipschitz on compact sets of $\mathbb{R}^n$ (see Section 5.1, Chapter 5 in [13]); hence, by (4.19) and the density of $\{\nu_j\}$ we get that

$$\lim_{k\to+\infty} co\,\psi_k(\nu) = \varphi(\nu) \tag{4.20}$$

for every $\nu \in S^{n-1}$. By (4.18) and the definition of ψ_k we have then

$$\varphi_k(\nu) \leq \psi_k(\nu)\,. \tag{4.21}$$

We recall that by Theorem 2.4 the functions $\varphi_{k,\lambda}$ are convex then by (4.6) φ_k is also convex for every $k \in \mathbb{N}$. By (4.21) it follows then

$$\varphi_k(\nu) \leq co\,\psi_k(\nu) \tag{4.22}$$

for every $\nu \in S^{n-1}$.

By (4.17), (4.22) and (4.20) we can prove that there exists the limit as k tends to $+\infty$ and

$$\lim_{k\to+\infty} \varphi_k(\nu) = \varphi(\nu)$$

for every $\nu \in S^{n-1}$ which concludes the proof. □

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