Variational analysis of the asymptotics of the XY model

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Abstract

In this paper we consider the XY (N-dimensional possibly anisotropic) spin type model and, by comparison with a Ginzburg-Landau type functional, we perform a variational analysis in the limit when the number of particles diverges. In particular we show how the appearance of vortex-like singularities can be described by properly scaling the energy of the system through a Γ-convergence procedure. We also address the problem in the case of long range interactions and solve it in 2-dimensions.

1 Introduction

Phase transitions are a striking feature of many natural phenomena. They are characterized by a strong dependence of macroscopic properties of physical systems on external parameters such as temperature or pressure. At a microscopic scale they can be seen as a result of non-linear cooperative phenomena leading to long-range order. Working inside this framework, we aim at giving a variational description of some features which are peculiar of those phase-transition phenomena which occur without breaking the symmetry of the system (e.g. those taking place in films of superfluid helium, superconducting materials as well as in certain magnetic or liquid-crystal systems). This type of phase transition has been first studied in the seminal papers by Berezinskii [8], Kosterlitz [19] and Kosterlitz and Thouless [20] concerning the so-called two-dimensional XY model. This model turns out to be the easiest model that contains all the interesting features of this class of phase transitions. It is constructed on the two dimensional square lattice $\mathbf{Z}^2$ whose points i are occupied by a spin confined to a plane $u(i) \in S^1$. For a given configuration, the energy of the system is

$$F(u) = -\sum_{n.n.} u(i)u(j), \tag{1.1}$$

where $n.n.$ means that the summation is taken over all nearest neighbors; i.e. those sites $i,\ j$ such that $|i-j|$ equals the lattice spacing. Loosely speaking the scenario that the Berezinskii-Kosterlitz-Thouless theory proposes is that the phase transition phenomenon is mediated by the formation of topological defects or vortices. Above some critical temperature T_c, this vortices behave like 'topological charges' that remain unbounded and make the system disordered, while below T_c they bound together in pairs which become the relevant degrees of freedom of the system. We are interested in this low-temperature regime where the cost of small fluctuations of the spin field around the uniform ground state is usually conveniently calculated by coarse-graining on a scale much larger than the lattice spacing. The energy of the resulting model is known as the Ginzburg-Landau energy.

In the present paper we prove that the coarse-graining procedure of the XY model in the thermodynamic limit can be made rigorous and that it leads to a Ginzburg-Landau (GL) energy in the regime in which the so called GL coherence length (here denoted by ε) is extremely small ($\varepsilon \to 0$). In this case the GL energy can be conveniently written as

$$G_\varepsilon(u) = \frac{1}{2} \int |\nabla u|^2 + \frac{1}{\varepsilon^2}(1 - |u|^2)^2. \tag{1.2}$$

The analysis of the energy in (1.2), and in particular the appearance of vortex-like singularities associated to energy concentration phenomena, has been successfully addressed by many authors both from the PDE and the calculus of variations point of view (see e.g. [1], [9], [15], [16], [17]). To this end, since, to leading term, the cost of a vortex singularity is of order $|\log \varepsilon|$, the right energy scaling to be taken into account is $\frac{G_\varepsilon(u)}{|\log \varepsilon|}$. In this regime the key idea to address the problem of the concentration of the energy has been provided by Jerrard in [15] where it has been shown that the relevant tool to track energy concentration is the asymptotic analysis of the Jacobians of sequences u_ε equibounded in energy. The variational analysis of the asymptotics of $\frac{G_\varepsilon(u)}{|\log \varepsilon|}$ has been performed by Jerrard and Soner in [16] and by Alberti, Baldo e Orlandi in [1] in the general N-dimensional case.

By relating the XY model to the GL model we may explain some geometric and topological properties of the spin-field singularities appearing in the XY system in the thermodynamic limit by means of the same variational techniques successfully exploited to study the GL functional. In this way we aim at interpreting some of the results contained in the huge physical literature on the XY model (and variations) (see e.g. [21], [24] and references therein) from the point of view of calculus of variations.

To set up the general N-dimensional problem in the framework of discrete-to-continuum variational limits (see for example [3], [4], [5], [10]) we scale the energy in (1.1) to a fixed domain $\Omega \subset \mathbf{R}^N$. Taking into account the interactions between nearest-neighbors on the lattice $\varepsilon\mathbf{Z}^N \cap \Omega$ (the lattice spacing ε will go to zero in the continuum limit), the scaled energy reads:

$$F_\varepsilon(u) = -\sum_{n.n.} \varepsilon^N u(\varepsilon i) \cdot u(\varepsilon j).$$

Upon identifying the functions $u : \varepsilon\mathbf{Z}^N \cap \Omega \to S^1$ with proper piecewise-constant interpolations, the energies can be considered as being defined on $L^\infty(\Omega)$ and can be described by a Γ-limit (see [10] and [12] for basic definitions and properties) in that framework as ε goes to 0. The bulk scaling we have chosen for F_ε renders its Γ-limit trivially the constant value $-|\Omega|$, the only constraint being $|u| \leq 1$. Such a limit is the same even in the case of Ising-type models ($u \in \{-1, +1\}$) and summarizes the fact that any configuration of spins u_ε can slightly differ from the uniform one on a mesoscopic scale without changing the asymptotic energy. Such a poor description of the ground states can be improved by considering other scalings. In particular we can select sequences that realize the minimum value with a sharper precision; i.e.

$$F_\varepsilon(u_\varepsilon) = \min F_\varepsilon + O(k_\varepsilon)$$

where $k_\varepsilon \to 0$ as $\varepsilon \to 0$. In the Ising case it has been proven ([3]) that the relevant scaling is $k_\varepsilon = \varepsilon$. This scaling yields to a selection criterion of minimal-interface type, in the sense that, on such sequences u_ε, the limit is an interfacial-type energy reflecting the symmetries of the underlying lattice structure. In the present case we show that no interface-type selection can be obtained by a surface scaling (see Example 4.1) and focus on a different scaling, namely $k_\varepsilon = \varepsilon^2|\log\varepsilon|$, which implies a selection criterion of topological nature. The sequence of scaled functionals we consider is the following:

$$\begin{aligned} E_\varepsilon(u) = \frac{F_\varepsilon(u) - \min F_\varepsilon}{\varepsilon^2|\log\varepsilon|} \quad &= \quad \frac{1}{|\log\varepsilon|}\sum_{n.n.}(1 - u(\varepsilon i)\cdot u(\varepsilon j)) \\ &= \quad \frac{1}{2|\log\varepsilon|}\sum_{n.n.}\varepsilon^2\left|\frac{u(\varepsilon i) - u(\varepsilon j)}{\varepsilon}\right|^2. \end{aligned}$$

By associating to any given spin field u the function $v = A(u)$ defined as a continuous piecewise-affine interpolation of u on the cells of the lattice (see (4.19)), we have

$$E_\varepsilon(u) \sim \frac{1}{2|\log\varepsilon|}\int_\Omega |\nabla v|^2. \tag{1.3}$$

Once we prove a key lemma (see Lemma 4.4) which asserts that the singular term in the Ginzburg-Landau energy is controlled by $E_\varepsilon(u)$, that is

$$\frac{1}{\varepsilon^2|\log\varepsilon|}\int_\Omega (|v|^2 - 1)^2 \leq CE_\varepsilon(u),$$

we recognize in the right hand side of (1.3) the leading term of $\frac{G_\varepsilon(v)}{|\log\varepsilon|}$. This argument suggests an analogy between the two models and leads us to describe the formation of vortex-like singularities, associated to a sequence u_ε equibounded in energy, through the convergence in a 'suitable sense' of the Jacobians $J(v_\varepsilon)$ of $v_\varepsilon = A(u_\varepsilon)$. Here $J(v_\varepsilon)$ is meant as the 2-form $dv_\varepsilon^1 \wedge dv_\varepsilon^2$, dv_ε^i being the differential

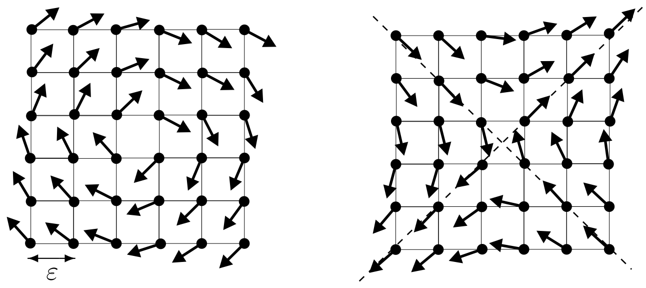

Figure 1: Discrete vortices: optimizing sequences for $M = \delta_{x_0}$ (+1 charged vortex) (left) and $M = -\delta_{x_0}$ (−1 charged vortex) (right).

of the i-th component of v_ε. A key result to describe the structure of the vortices is the following compactness result:

Compactness. Let (u_ε) be a sequence of functions such that $E_\varepsilon(u_\varepsilon) \le C$ and let $v_\varepsilon = A(u_\varepsilon)$. Then we can extract a subsequence (not relabeled) such that $\mathbf{F}_\Omega(\star J(v_\varepsilon) - \pi M) \to 0$, where M is an $(N-2)$-dimensional integral boundary in Ω.

In the previous statement we have denoted by $\star J(v_\varepsilon)$ the $(N-2)$-current, that is a L^1 map on Ω valued in $(N-2)$-vectors, obtained from $J(v_\varepsilon)$ by the standard identification $\star$ of k-covectors with $(N-k)$-vectors. For any current T $\mathbf{F}_\Omega(T)$ denotes its flat norm (see definition (2.6)) and the limit current M is a $(N-2)$-dimensional boundary in the sense that, loosely speaking, it is supported on a $(N-2)$-dimensional rectifiable set which is also a boundary. This set represents the set of the vortex-type singularities of the spin field u_ε as ε goes to zero.

The energy-concentration phenomenon on the set of singularities is described by the following Γ-convergence result which we state explicitly since it does not rigorously fit the usual formulation of that convergence:

Lower bound inequality - Let (u_ε) be a sequence of functions such that $\mathbf{F}_\Omega(\star J(v_\varepsilon) - \pi M) \to 0$, where M is an $(N-2)$-dimensional integral boundary in Ω. Then

$$\liminf_{\varepsilon} E_\varepsilon(u_\varepsilon) \ge \pi \|M\|;$$

Upper bound inequality - Let M be an $(N-2)$-dimensional integral boundary in Ω. Then there exists a sequence (u_ε) such that $\mathbf{F}_\Omega(\star J(v_\varepsilon) - \pi M) \to 0$ and

$$\lim_{\varepsilon} E_\varepsilon(u_\varepsilon) = \pi \|M\|. \tag{1.4}$$

To better explain the previous results consider the case $N = 2$. In this case the limit current M is a finite sum of Dirac masses, that is $M = \sum_{k=1}^{n} d_k \delta_{x_k}$, where $n \in \mathbf{N}$, $x_k \in \Omega$ represent the centers of the vortices and $d_k \in \mathbf{Z}$ are the winding numbers of the spin field around each x_k and are also called charges of the topological singularity. In Figure 1 we have displayed two types of discrete vortices, that is the microscopic configurations of the spin fields leading, in the continuum limit, to $M = +\delta_{x_0}$ and $M = -\delta_{x_0}$.

All the previous results are contained in Theorem 4.2 (see also Remark 4.3) which can be considered as the discrete analogue of Theorem 1.1 in [2]. In this sense, the Ginzburg-Landau energy can be seen as the continuous counterpart of the XY energy within the logarithmic scaling regime. We point out that the limit energy in (1.4) does not reflect the underlying geometry of the lattice. This is explained by the fact that the parallel between the XY and the GL model carries on to the characteristic length scale where energy concentrates that is much larger than the lattice spacing ε.

We remark that the result described above in the two-dimensional case has a number of elements in common with the result obtained by Ponsiglione [22] about the discrete-to-continuum passage in modeling the elastic properties, in the framework of anti-planar linear elasticity, of vertical screw dislocations in a cylindrical crystal. The analogy between the two problems was already known, at least formally, since the pioneering paper by Kosterlitz and Thouless [20]. Indeed it will be proved in a forthcoming paper [6] that the Γ-convergence theorems, independently proved for the two models, can be obtained one from the other rigorously.

Starting from the previous analysis, we have also addressed the problem of the variational description of the continuum limit of the XY model in the anisotropic and in the long-range case. In the first case nearest-neighbors interactions between spins in points i and j of the lattice are differently weighed according to the direction of the vector $i - j$. In Theorem 4.7 we prove the analogue of Theorem 4.2 pointing out how the geometry of the vortex type configurations is affected by the anisotropy of the model.

The long range case, which is the case when the interactions between all the spins are taken into account, is much harder. In Theorem 4.8 we prove a Γ-convergence result asserting that, in 2-dimensions, the limit energy is still of the form (1.4). The set of hypotheses we make is quite general and includes the case in which the interactions satisfy a decay assumption and are isotropic (see Remark 4.9). The general N-dimensional problem is open. In Theorem 4.11 we provide an N-dimensional version of the result stated in Theorem 4.8 by assuming a more technical hypothesis on the interactions. In the proof of these results, besides the argument exploited in proving Theorem 4.2, a key ingredient is the idea, well known to people working in statistical mechanics, and here used in a variational form, of decoupling the order parameter on weakly interacting systems.

Finally we underline that many challenging problems remain open in this framework. Among them we mention the variational description of the XY model in presence of an external magnetic field, the study of the vortex interaction energy or even the dynamic of vortices (see [23] and references therein for the theory developed in the Ginzburg-Landau case). Many of the ideas contained in this paper will be useful to address some of the previous problems.

The paper is organized as follows:

Contents

2 Notation and preliminary results

In what follows $\Omega \subset \mathbf{R}^N$ will be a bounded open set with $\mathcal{L}^N(\partial\Omega) = 0$. We define as $\mathcal{A}(\Omega)$ the class of all open bounded subsets of Ω. For all $B \in \mathbf{R}^N$ we define $\mathbf{Z}_\varepsilon(B) =: \{i \in \mathbf{Z}^N : \ \varepsilon i \in B\}$. We will also make use of the notation $|\cdot|$ to denote the euclidean norm. Given $A,\ B \in \mathbf{R}^N$ ($A \in \mathcal{M}^{d\times N}$ and $B \in \mathcal{M}^{N\times k}$, respectively) we will denote as $A\cdot B$ the scalar product in $\mathbf{R}^N$ (the row times column product, respectively). For any $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2$ we denote by $\xi^\perp = (-\xi_2, \xi_1)$ the vector perpendicular to ξ. We denote by S^1 and B^2 the unit sphere and the unit ball in $\mathbf{R}^2$, respectively.

2.1 Currents

We recall here some basic definitions about currents and forms (for more details we refer to [13], [14]). For every $h = 0, 1, \ldots, N$, an h-form on Ω is a map from Ω into the space of h-covectors $\bigwedge^h(\mathbf{R}^N)$ while an h-current a map from Ω into the space of h-vectors $\bigwedge_h(\mathbf{R}^N)$. By the standard duality between vectors and covectors, the space $\mathcal{D}_h(\Omega)$ of h-currents is identified with the dual of the space $\mathcal{D}^h(\Omega)$ of all smooth h-forms with compact support in Ω. The boundary of an h-current is the $h-1$-current defined by the identity $\partial T[\omega] = T[d\omega]$ for every $\omega \in \mathcal{D}^{h-1}(\Omega)$ where $d\omega$ is the differential of ω. We call boundary any current which is also a boundary. Moreover we say that a current T is a boundary locally in Ω if the restriction $T\lfloor_{\Omega'}$ is a boundary for any $\Omega' \subset\subset \Omega$. A current T is said to have (locally) finite mass when it can be represented as a (locally) bounded Borel measure valued in $\bigwedge_h(\mathbf{R}^N)$. In this case we denote by $|T|$ the variation of the measure T and by $\|T\| = |T|(\Omega)$ the mass of T. The restriction of T to a set E will be denoted by $T\lfloor E$ as for measures.

A set M in Ω is h-rectifiable if it can be covered, up to a $\mathcal{H}^h$-negligible set, by countably many h-surfaces of class C^1. An orientation of M is a map which associates to $\mathcal{H}^h$-*a.e.* $x \in M$ a simple unitary h-vector which spans $Tan(M, x)$ which is the tangent space to M at x defined in a measure theoretic sense. For every given M h-rectifiable set, τ_M orientation and $\sigma_M \in L^1_{loc}(\mathcal{H}^h\lfloor M)$, we define the current

$$T[\omega] := \int_M \sigma_M(\omega \cdot \tau_M)d\mathcal{H}^h. \tag{2.5}$$

We call σ_M the multiplicity of T and observe that, in this case, $\|T\| = \int_M |\sigma_M| d\mathcal{H}^h$. A current T of the type (2.5) is said to be *rectifiable* if σ_M is integer-valued and it is said to be *integral* if both T and ∂T are rectifiable. A sum of finitely many h-currents associated, as in of (2.5), to h-dimensional simplices in $\mathbf{R}^N$, endowed with constant orientations and constant real (integral) multiplicities is a real (integral) *polyhedral* current in $\mathbf{R}^N$. Polyhedral currents in Ω are defined by restriction.

For any given linear map $L : \mathbf{R}^N \to \mathbf{R}^m$ and $\beta \in \bigwedge^h(\mathbf{R}^m)$, then $L^\# \beta \in \bigwedge^h(\mathbf{R}^N)$ is defined by $(L^\#\beta) \cdot (v_1 \wedge \ldots \wedge v_h) = \beta \cdot (Lv_1 \wedge \ldots \wedge Lv_h)$ for every simple vector $v_1 \wedge \ldots \wedge v_h$. Given an open set $\Omega' \subset \mathbf{R}^N$ and a map $f : \Omega \to \Omega'$ we define the *pull-back* of an h-form ω on Ω' according to f to be the $h-$form $f^\#\omega$ on Ω defined by $f^\#\omega(x) = (Df(x))^\#\omega(f(x))$ for every x. The *push-forward* of an h-current T on Ω is the h-current $f_\# T$ on Ω' defined by $f_\# T[\omega] = T[f^\#\omega]$. We observe that if T is associated to (M, τ_M, σ_M) as in (2.5), then $f_\# T$ is the current associated to $(M', \tau_{M'}, \sigma_{M'})$ where $M' = f(M)$, $\sigma_{M'}(y)$ is the sum of $\sigma_M(x)$ for all $x \in f^{-1}(y)$ computed taking the orientation into account. In the rest of the paper, given an $N \times N$ matrix A, we will make use of the shorter notation $A_\# T$ to denote the push forward of T according to $f(x) = Ax$.

We define the *flat norm* of a current $T \in \mathcal{D}_h(\Omega)$ as

$$\mathbf{F}_\Omega(T) := \inf\{\|S\|,\ S \in \mathcal{D}_{h+1}(\Omega)\ T = \partial S\} \tag{2.6}$$

where the infimum is taken to be $+\infty$ if T is not a boundary. Here we recall an approximation result for integral boundaries by polyhedral boundaries with respect to the flat norm (see Proposition 8.6 [2]).

Proposition 2.1 *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^N$, and let T be an integral boundary in Ω with dimension $h < N$ and finite mass. Then there exists a sequence of polyhedral boundaries T_n in $\mathbf{R}^N$ with multiplicity 1 and $|T_n|(\partial\Omega) = 0$, such that $\mathbf{F}_\Omega(T_n\lfloor\Omega - T) \to 0$ and $\|T_n\lfloor\Omega\| \to \|T\|$.*

As a simple consequence of the previous result we also state a local version of it which holds without requiring any regularity on Ω.

Proposition 2.2 *Let Ω be a bounded domain in $\mathbf{R}^N$, and let T be an integral boundary locally in Ω with dimension $h < N$ and finite mass in Ω. Then there exists a sequence of polyhedral boundaries T_n in $\mathbf{R}^N$ with multiplicity 1 and $\operatorname{spt}(T_n) \subset\subset \Omega$, such that $\mathbf{F}_{\Omega'}(T_n\lfloor\Omega' - T) \to 0$ for any $\Omega' \subset\subset \Omega$ and $\|T_n\lfloor\Omega\| \to \|T\|$.*

2.2 Jacobian of Sobolev maps as currents

For every $u \in W^{1,2}(\Omega;\mathbf{R}^2)$ we define the *Jacobian* of $u = (u_1, u_2)$ as the 2-form defined as

$$Ju := du_1 \wedge du_2,$$

where $du_i = \sum_j D_j u_i dx_j$. Since

$$Ju = \frac{1}{2} d(u_1 du_2 - u_2 du_1), \tag{2.7}$$

and the last term makes sense as a distribution even if $u \in L^\infty \cap W^{1,1}(\Omega;\mathbf{R}^2)$, (2.7) can be taken as a definition of Jacobian in this case. We can regard the Jacobian as a $N-2$-current using the identification between forms and currents defined below. Given an h-form $\omega \in L^1_{loc}(\Omega)$, $\star\omega$ is the $N-h$-current on Ω defined by

$$\star\omega[\omega'] = \int_\Omega (\omega' \wedge \omega) \cdot \tau_\Omega \quad \omega' \in \mathcal{D}^{N-h}(\Omega),$$

where $\tau_\Omega = e_1 \wedge \ldots \wedge e_N$ is the standard orientation of Ω. Since

$$\partial(\star\omega) = (-1)^{N-h} \star (d\omega), \tag{2.8}$$

by (2.7) we have that $\star Ju$ is a boundary. Moreover $\star J$ is a continuous operator from $W^{1,1}(\Omega;\mathbf{R}^2)$ into the space of $(N-2)$-dimensional boundaries endowed with the flat norm.

In section 4.3 we will make use of the following lemma.

Lemma 2.3 *Let $U \subset \mathbf{R}^N$ be a bounded open set and let u_ε and v_ε be two sequences belonging to $W^{1,2}(U,\mathbf{R}^2)$. If there exists a constant $C > 0$ such that*

$$(i) \quad \int_U |u_\varepsilon - v_\varepsilon|^2 \, dx \le C\varepsilon^2 |\log\varepsilon|, \qquad (ii) \quad \int_U |\nabla u_\varepsilon - \nabla v_\varepsilon|^2 \, dx \le C|\log\varepsilon|,$$

then $\mathbf{F}_U(\star J(u_\varepsilon) - \star J(v_\varepsilon)) \to 0$.

Proof. Observe that for $u_\varepsilon = (u^1_\varepsilon, u^2_\varepsilon)$ and $v_\varepsilon = (v^1_\varepsilon, v^2_\varepsilon)$, it holds

$$\star J(u_\varepsilon) - \star J(v_\varepsilon) = \star J(u_\varepsilon - v_\varepsilon) + \star J(u^1_\varepsilon - v^1_\varepsilon, v^2_\varepsilon) + \star J(v^1_\varepsilon, u^2_\varepsilon - v^2_\varepsilon) \qquad (2.9)$$

where, for $f = (f^1, f^2)$, we have used the notation $\star J(f)$ or $\star J(f^1, f^2)$ to denote its Jacobian. By triangular inequality, from (2.9) if follows

$$\begin{aligned} \mathbf{F}_U(\star J(u_\varepsilon) - \star J(v_\varepsilon)) \quad &\leq \quad \mathbf{F}_U(\star J(u_\varepsilon - v_\varepsilon)) + \mathbf{F}_U(\star J(u^1_\varepsilon - v^1_\varepsilon, v^2_\varepsilon)) \\ &\quad + \mathbf{F}_U(\star J(v^1_\varepsilon, u^2_\varepsilon - v^2_\varepsilon)). \end{aligned} \qquad (2.10)$$

Let now $g_\varepsilon = (g^1_\varepsilon, g^2_\varepsilon)$ be one of the functions $u_\varepsilon - v_\varepsilon$, $(u^1_\varepsilon - v^1_\varepsilon, v^2_\varepsilon)$ or $(v^1_\varepsilon, u^2_\varepsilon - v^2_\varepsilon)$. Since $Jg_\varepsilon = d(g^1_\varepsilon dg^2_\varepsilon)$, by (2.8) and the definition of the flat norm we get that

$$\mathbf{F}_U(\star Jg_\varepsilon) \leq \| \star (g^1_\varepsilon dg^2_\varepsilon)\|(U) \leq \sum_{i=1}^N \int_U \left| g^1_\varepsilon \frac{\partial g^2_\varepsilon}{\partial x_i} \right| \, dx.$$

Then, thanks to Hölder inequality, by hypotheses (i) and (ii), we get that

$$\mathbf{F}_U(\star Jg_\varepsilon) \leq C \left(\int_\Omega |g_\varepsilon|^2) \right)^{\frac{1}{2}} \left(\int_\Omega |\nabla g_\varepsilon|^2) \right)^{\frac{1}{2}} \leq C\varepsilon |\log \varepsilon|. \qquad (2.11)$$

Hence, the conclusion follows from (2.10) and (2.11). □

2.3 Ginzburg-Landau type energies

Here we recall the main result of [2] about the variational convergence of Ginzburg-Landau type energies (see also [16]). For $\varepsilon > 0$ set

$$G_\varepsilon(u) = \frac{1}{|\log \varepsilon|} \int_\Omega \left(\frac{1}{2} |\nabla u|^2 + \frac{1}{\varepsilon^2} W(u) \right) dx \quad u \in W^{1,2}(\Omega; \mathbf{R}^2), \qquad (2.12)$$

where W is a continuous function which vanishes on S^1, it is strictly positive elsewhere and it satisfies

$$\liminf_{|y| \to 1} \frac{W(y)}{(1 - |y|)^2} > 0 \quad \liminf_{|y| \to \infty} \frac{W(y)}{|y|^2} > 0.$$

Theorem 2.4 *Let Ω be a bounded Lipschitz domain in $\mathbf{R}^N$. The following statements hold.*

(i) Compactness and lower-bound inequality. *Let (u_ε) be a sequence of functions such that $G_\varepsilon(u_\varepsilon) \leq C$. Then we can extract a subsequence (not relabeled) such that $\mathbf{F}_\Omega(\star J(u_\varepsilon) - \pi M) \to 0$ where M is an $(N-2)$-dimensional integral boundary locally in Ω. Moreover*

$$\liminf_\varepsilon G_\varepsilon(u_\varepsilon) \geq \pi \|M\|. \qquad (2.13)$$

(ii) Upper bound inequality. *Let M be an $(N-2)$-dimensional integral boundary locally in Ω, there exists a sequence (u_ε) such that $\mathbf{F}_\Omega(\star J(u_\varepsilon) - \pi M) \to 0$ and*

$$\lim_{\varepsilon} G_\varepsilon(u_\varepsilon) = \pi \|M\|. \tag{2.14}$$

Remark 2.5 *Let A be an invertible $N \times N$ matrix. If in (2.12) we replace $|\nabla u|^2$ by $|\nabla u \cdot A|^2$ then, by a change of variables, it can be easily shown that the same results stated in Theorem 2.4 hold with $|\mathrm{det} A| \|(A^{-1})_\# M\|$ in place of $\|M\|$ in (2.13) and (2.14).*

3 XY model: bulk scaling

In this section we introduce and study the asymptotics of a class of bulk scaled energies governing the XY model in the general N-dimensional, anisotropic, possibly long range case.

Given $\varepsilon > 0$ and $\{c^\xi\}_\xi$ a family of positive constants labeled with $\xi \in \mathbf{Z}^N$ and such that $\sum_\xi c^\xi < +\infty$, we introduce the energy

$$F_\varepsilon(u) = - \sum_{i,j \in \mathbf{Z}_\varepsilon(\Omega)} \varepsilon^N c^{(i-j)} u(\varepsilon i) \cdot u(\varepsilon j)$$

where $u : \varepsilon \mathbf{Z}^N \cap \Omega \to S^1$. It is convenient to rewrite this energy regrouping the interactions in the same direction as follows

$$F_\varepsilon(u) = - \sum_{\xi \in \mathbf{Z}^N} \sum_{i,\ i+\xi \in \mathbf{Z}_\varepsilon(\Omega)} \varepsilon^N c^\xi \ u(\varepsilon i) \cdot u(\varepsilon i + \varepsilon \xi).$$

Let

$$C_\varepsilon(\Omega) := \{u : \mathbf{R}^N \to S^1 : u(x) = u(\varepsilon i) \ \forall x \in \{\varepsilon(i + Q)\} \cap \Omega\}, \tag{3.15}$$

where

$$Q = \Big[0, 1\Big)^N.$$

Then any function $u : \varepsilon \mathbf{Z}^N \cap \Omega \to \mathbf{R}^2$ can be identified with its piecewise interpolation belonging to $C_\varepsilon(\Omega)$.

Then we may regard F_ε as defined on $C_\varepsilon(\Omega)$ and extend it on all $L^\infty(\Omega; \mathbf{R}^2)$ by setting $F_\varepsilon : L^\infty(\Omega; \mathbf{R}^2) \to \mathbf{R} \cup \{+\infty\}$

$$F_\varepsilon(u) = \begin{cases} - \sum_{\xi \in \mathbf{Z}^N} c^\xi \sum_{i,\ i+\xi \in \mathbf{Z}_\varepsilon(\Omega)} \varepsilon^N u(\varepsilon i) \cdot u(\varepsilon i + \varepsilon \xi) & \text{if } u \in C_\varepsilon(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{3.16}$$

Theorem 3.1 *Let $F_\varepsilon : L^\infty(\Omega; \mathbf{R}^2) \to \mathbf{R} \cup \{+\infty\}$ be defined as in (3.16), then F_ε Γ-converges with respect to the w^*-topology of $L^\infty(\Omega; \mathbf{R}^2)$ to the functional $F : L^\infty(\Omega; \mathbf{R}^2) \to \mathbf{R} \cup \{+\infty\}$ defined as*

$$F(u) = \begin{cases} -|\Omega| \sum_\xi c^\xi & \text{if } u \in L^\infty(\Omega; B^2) \\ \\ +\infty & \text{otherwise.} \end{cases}$$

PROOF. If $u_\varepsilon \in C_\varepsilon(\Omega)$ and $u_\varepsilon \to u$ weakly in $L^\infty(\Omega; \mathbf{R}^2)$, then $u \in L^\infty(\Omega; B^2)$. Thus the Γ-limit is finite only on $L^\infty(\Omega; B^2)$. The lower bound inequality is straightforward since

$$F_\varepsilon(u) \geq -\sum_{\xi \in \mathbf{Z}^N} c^\xi\ \varepsilon^N \#\{i,\ i + \xi \in \mathbf{Z}_\varepsilon(\Omega)\} \to -|\Omega| \sum_{\xi \in \mathbf{Z}^N} c^\xi.$$

In order to provide the upper bound inequality, it suffices, by density, to exhibit an optimizing sequences u_ε when the target function u is piecewise-constant. We will show how to construct u_ε when u is constant; the construction can be easily repeated for piecewise-constant u on each set where it is constant.

Let then $u \equiv u_0$, with $u_0 \in B^2$. Thus there exist $u_1,\ u_2 \in S^1$ and $t \in [0, 1]$ such that $u_0 = tu_1 + (1 - t)u_2$. Choose a mesoscopic scale $\delta = \delta(\varepsilon)$ with $\varepsilon << \delta << 1$ and let

$$u_\varepsilon(\varepsilon i) = \begin{cases} u_1 & \text{if } 0 < \varepsilon i_1 \leq t \text{ modulo } \delta \\ u_2 & \text{if } t < \varepsilon i_1 \leq 1 \text{ modulo } \delta \end{cases}$$

(i_1 is the first component of $i \in \mathbf{Z}^N$). Then $u_\varepsilon \to u$ weakly in $L^\infty(\Omega; \mathbf{R}^2)$. Moreover a simple computation show that for any $\xi \in \mathbf{Z}^N$

$$\lim_{\varepsilon \to 0} \sum_{i,\ i+\xi \in \mathbf{Z}_\varepsilon(\Omega)} \varepsilon^N u_\varepsilon(\varepsilon i) \cdot u_\varepsilon(\varepsilon i + \varepsilon \xi) = |\Omega| \left(-1 + C|\xi| \lim_{\varepsilon \to 0} \frac{\varepsilon}{\delta} \right) = -|\Omega|.$$

We then get the conclusion by summing over $\xi \in \mathbf{Z}^N$ and taking into account that $\sum_\xi c_\xi < \infty$. □

4 An higher order description: vortex like singularities

In what follows, in order to study non trivial asymptotic properties of energies of the form (3.16), we will address the problem of finding a relevant scaling of the energies as in the framework of development by $\Gamma-$convergence (see [11]), and perform an asymptotic analysis in that regime.

4.1 Isotropic nearest-neighbors XY model

Let us specialize the energies defined in (3.16) to the case of isotropic nearest-neighbors XY model that is when

$$c^\xi = \begin{cases} 1 & \text{if } \xi = e_k,\ k = 1, \dots, N \\ 0 & \text{otherwise.} \end{cases}$$

Then the energy becomes

$$F_\varepsilon(u) = \begin{cases} - \sum\limits_{\substack{i,j \in \mathbf{Z}_\varepsilon(\Omega) \\ |i-j|=1}} \varepsilon^N u(\varepsilon i) \cdot u(\varepsilon j) & u \in C_\varepsilon(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Let

$$m_\varepsilon := \min F_\varepsilon(u) = -|\Omega| + o(1),$$

and, given a family of positive numbers δ_ε converging to 0 as $\varepsilon \to 0$, we consider the scaled energies

$$E_\varepsilon^{\delta_\varepsilon}(u) := \frac{F_\varepsilon(u) - m_\varepsilon}{\delta_\varepsilon}.$$

Observe that, since $m_\varepsilon = - \sum\limits_{\substack{i,j \in \mathbf{Z}_\varepsilon(\Omega) \\ |i-j|=1}} \varepsilon^N$, we have that, for $u \in C_\varepsilon(\Omega)$,

$$E_\varepsilon^{\delta_\varepsilon}(u) = \frac{\varepsilon^N}{\delta_\varepsilon} \sum_{\substack{i,j \in \mathbf{Z}_\varepsilon(\Omega) \\ |i-j|=1}} (1 - u(\varepsilon i) \cdot u(\varepsilon j)).$$

Note that, despite of Ising type models considered in [3] where the scaling $\delta_\varepsilon = \varepsilon$ gives rise to interfacial surface energies in the limit, here this phenomenon does not occur for any δ_ε such that $\lim_{\varepsilon \to 0} \frac{\delta_\varepsilon}{\varepsilon^2} = +\infty$ as the following example shows.

Example 4.1 Suppose for simplicity that $0 \in \Omega$. Given $a, b \in S^1$, let

$$u(x) = \begin{cases} a & \text{if } x_1 \le 0 \\ b & \text{if } x_1 > 0 \end{cases},$$

where x_1 is the first component of $x \in \mathbf{R}^N$. Then we can construct $u_\varepsilon \to u$ in L^1 such that $E_\varepsilon^{\delta_\varepsilon}(u_\varepsilon) \to 0$. In fact, let $\theta_a, \theta_b \in [0, 2\pi)$ be such that $a = (\cos\theta_a, \sin\theta_a)$, $b = (\cos\theta_b, \sin\theta_b)$, and set

$$\theta_\varepsilon(t) := (\theta_a - \theta_b)(1 - \frac{t}{\eta_\varepsilon}) + \theta_b,$$

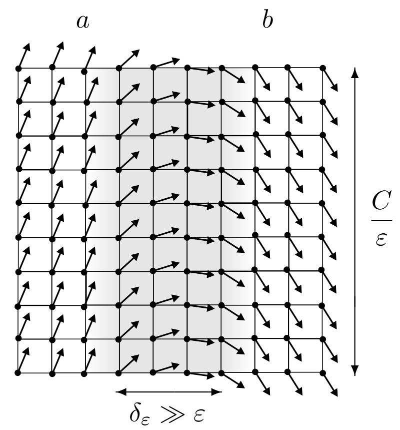

Figure 2: The spin filed u_ε in Example 4.1: the mesoscopic relaxation prevents the formation of domains.

where $\eta_\varepsilon = o(\varepsilon)$ is a scale to be determined a posteriori. Let, then, $u_\varepsilon \in C_\varepsilon(\Omega)$ be defined as

$$u_\varepsilon(\varepsilon i) = \begin{cases} a & \text{if } i_1 \leq 0, \\ (\cos\theta(\varepsilon i_1), \sin\theta(\varepsilon i_1)) & \text{if } i_1 \in (0, \eta_\varepsilon], \\ b & \text{if } i_1 > \eta_\varepsilon. \end{cases}$$

Then, an easy computation gives that

$$E_\varepsilon^{\delta_\varepsilon}(u_\varepsilon) \sim \left(1 - \cos\frac{\varepsilon}{\eta_\varepsilon}(\theta_a - \theta_b)\right)\frac{\eta_\varepsilon}{\delta_\varepsilon} \sim \frac{\varepsilon^2}{\eta_\varepsilon \delta_\varepsilon}.$$

Hence, it suffices to choose η_ε such that $\frac{\varepsilon^2}{\eta_\varepsilon\delta_\varepsilon} \to 0$.

In the sequel of the paper we will focus on the scaling $\delta_\varepsilon = \varepsilon^2|\log\varepsilon|$. With such a scaling, non topologically trivial ground states will appear in the continuum limit.

Let $E_\varepsilon : L^\infty(\Omega;\mathbf{R}^2) \to \mathbf{R} \cup \{+\infty\}$ be defined by

$$E_\varepsilon(u) = \begin{cases} \dfrac{1}{|\log\varepsilon|} \displaystyle\sum_{\substack{i,j \in \mathbf{Z}_\varepsilon(\Omega) \\ |i-j|=1}} \varepsilon^{N-2}(1 - u(\varepsilon i)\cdot u(\varepsilon j)) & \text{if } u \in C_\varepsilon(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.17}$$

Note that, for any $u \in C_\varepsilon(\Omega)$, we have

$$E_\varepsilon(u) = \frac{1}{2|\log\varepsilon|} \sum_{\substack{i,j \in \mathbf{Z}_\varepsilon(\Omega) \\ |i-j|=1}} \varepsilon^N \left|\frac{u(\varepsilon i) - u(\varepsilon j)}{\varepsilon}\right|^2$$

$$\geq \frac{1}{2|\log\varepsilon|}\sum_{k=1}^{N}\sum_{\substack{i\in\mathbf{Z}_\varepsilon(\Omega):\\ \varepsilon(i+Q)\in\ \Omega}} \varepsilon^N|D_\varepsilon^{e_k}u(\varepsilon i)|^2 \tag{4.18}$$

For any $u \in C_\varepsilon(\Omega)$, let us define $v = A(u)$ a piecewise affine interpolation of u on the cells of the lattice as follows: let $\{p_1, p_2, \ldots, p_{N!}\}$ be the set of the $N!$ permutations of $\{1, 2, \ldots, N\}$ and let $\{T^1, T^2, \ldots, T^{N!}\}$ be a partition of the unit cube into N-dimensional simplices defined by

$$T^k := \{x \in [0,1]^N : x_{p_k(1)} \geq x_{p_k(2)} \geq \ldots \geq x_{p_k(N)}\}, \quad k \in \{1, 2, \ldots, N!\}.$$

Then $v = A(u)$ is defined as

$$v(x) = u(\varepsilon i) + \sum_{l=1}^{N} D_\varepsilon^{e_{p_j(l)}} u\left(\varepsilon\left(i + \sum_{m=1}^{l-1} e_{p(l)}\right)\right)(x_{p(l)} - \varepsilon i_{p(l)}),\ x \in \varepsilon(i + T^k)\tag{4.19}$$

One can easily show that

$$\frac{\partial v}{\partial x_l}(x) = D_\varepsilon^{e_l}u(\varepsilon i), \quad x \in \varepsilon(i + P_l),$$

where

$$P_l = \{x :\ 0 \leq x_l \leq 1,\ x_l - 1 \leq x_m \leq x_l,\ l \neq m\}. \tag{4.20}$$

Thus, from (4.18), we easily deduce that, for ε small enough,

$$E_\varepsilon(u) \geq \frac{1}{2|\log\varepsilon|}\int_{\Omega_\varepsilon} |\nabla v|^2\, dx, \tag{4.21}$$

where

$$\Omega_\varepsilon := \{x \in \Omega :\ \mathrm{dist}(x, \partial\Omega) > \sqrt{N}\varepsilon\}. \tag{4.22}$$

By the previous estimate and Lemma 4.4 below, we infer that

$$E_\varepsilon(u) \geq F_\varepsilon(v),$$

where F_ε is a Ginzburg-Landau energy of the form (2.12), whose limiting behavior is described in Theorem 2.4. Actually, the following theorem, which is the main result of the paper, yields that E_ε and F_ε are asymptotically equivalent.

Theorem 4.2 *The following statements hold:*

(i) Compactness and lower-bound inequality. *Let (u_ε) be a sequence of functions such that $E_\varepsilon(u_\varepsilon) \leq C$ and let $v_\varepsilon = A(u_\varepsilon)$ be defined by (4.19). Then we can extract a subsequence (not relabeled) such that, for any $\Omega' \subset\subset \Omega$, $\mathbf{F}_{\Omega'}(\star J(v_\varepsilon) - \pi M\lfloor_{\Omega'}) \to 0$, where M is an $(N-2)$-dimensional integral boundary locally in Ω. Moreover*

$$\liminf_\varepsilon E_\varepsilon(u_\varepsilon) \geq \pi\|M\|.$$

(ii) Upper bound inequality. *Let M be an $(N-2)$-dimensional integral boundary locally in Ω. There exists a sequence (u_ε) such that, for any $\Omega' \subset\subset \Omega$, $\mathbf{F}_{\Omega'}(\star J(v_\varepsilon) - \pi M\lfloor_{\Omega'}) \to 0$ and*

$$\lim_{\varepsilon} E_\varepsilon(u_\varepsilon) = \pi \|M\|. \tag{4.23}$$

Remark 4.3 In the case Ω is a bounded Lipschitz domain, we can prove that the compactness and approximation result stated in Theorem 4.2 holds with respect to the convergence in the flat norm $\mathbf{F}_\Omega$ (and not only $\mathbf{F}_{\Omega'}$ for any $\Omega' \subset\subset \Omega$). In fact let us set $\overline{\mathbf{Z}}_\varepsilon(\Omega) := \{i \in \mathbf{Z}^N : \ \varepsilon(i + [-1,1]^N) \cap \Omega \neq \emptyset\}$ and, for any $u : \varepsilon\overline{\mathbf{Z}}_\varepsilon \to S^1$, set

$$\overline{E}_\varepsilon(u) = \frac{1}{|\log\varepsilon|} \sum_{\substack{i,j \in \overline{\mathbf{Z}}_\varepsilon(\Omega) \\ |i-j|=1}} (1 - u(\varepsilon i)\cdot u(\varepsilon j)).$$

Then, thanks to the Lipschitz regularity of Ω, for any $u \in C_\varepsilon(\Omega)$ it is possible to define a suitable extension $\overline{u} : \varepsilon\overline{\mathbf{Z}}_\varepsilon \to S^1$ such that

$$\overline{E}_\varepsilon(\overline{u}) \leq C(\Omega, N) E_\varepsilon(u). \tag{4.24}$$

Note that the piecewise affine function $\overline{v} = A(\overline{u})$, constructed as in (4.19), is well defined in all the set Ω.

Then, by (4.24), it can be easily proved that if u_ε and M are as in Theorem 4.2 (i), then the family of piecewise affine functions $\overline{v}_\varepsilon = A(\overline{u}_\varepsilon)$, $\overline{u}_\varepsilon$ being the extension of u_ε defined as above, satisfy (up to subsequences): $\mathbf{F}_\Omega(\star J(\overline{v}_\varepsilon) - \pi M) \to 0$. In particular this implies that M is a $N-2$ integral boundary in Ω. Moreover given any $N-2$ integral boundary M in Ω, the optimizing sequence u_ε satisfying (4.23) can be chosen in such a way that $\mathbf{F}_\Omega(\star J(\overline{v}_\varepsilon) - \pi M) \to 0$.

Eventually, we underline that the limit boundary M does not depend on the type of extension of u_ε chosen, in the sense that if $\tilde{u}_\varepsilon$ is any other extension of u_ε satisfying (4.24) and $\tilde{v}_\varepsilon = A(\tilde{u}_\varepsilon)$, then $\mathbf{F}_\Omega(\star J(\overline{v}_\varepsilon) - \star J(\tilde{v}_\varepsilon)) \to 0$.

In the next Lemma we show that, for any $u \in C_\varepsilon(\Omega)$ the penalization term of the Ginzburg-Landau energy of its affine interpolation can be controlled by $E_\varepsilon(u)$.

Lemma 4.4 *Let $u \in C_\varepsilon(\Omega)$ and let $v = A(u)$ be given by (4.19). Then*

$$\frac{1}{\varepsilon^2|\log\varepsilon|}\int_{\Omega_\varepsilon} (|v|^2 - 1)^2\, dx \leq C E_\varepsilon(u),$$

where Ω_ε is defined in (4.22).

Proof. The Lemma is proved if we show that $\forall i \in \mathbf{Z}_\varepsilon(\Omega)$ such that $\varepsilon(i + [0,1]^N) \subset \Omega$ and for all $k \in \{1, 2, \ldots, N!\}$, we get

$$\sup_{x \in \varepsilon(i+T^k)} (|v|^2 - 1)^2 \leq C \sum_{l=1}^{N} \left| D_\varepsilon^{e_{p_k(l)}} u \left(\varepsilon\left(i + \sum_{m=1}^{l-1} e_{p_k(l)}\right)\right)\right|^2 \varepsilon^2. \tag{4.25}$$

In fact, if (4.25) hold true, we get

$$
\begin{aligned}
\frac{1}{\varepsilon^2|\log\varepsilon|} \quad & \int_{\Omega_\varepsilon} (|v|^2-1)^2\,dx \\
\leq \quad & \frac{1}{\varepsilon^2|\log\varepsilon|} \sum_i \sum_{k=1}^{N!} \int_{\varepsilon(i+T^K)} (|v|^2-1)^2\,dx \\
\leq \quad & \frac{1}{\varepsilon^2|\log\varepsilon|} \sum_i \sum_{k=1}^{N!} \varepsilon^N \sup_{x\in\varepsilon(i+T^k)} (|v|^2-1)^2 \\
\leq \quad & \frac{C}{2|\log\varepsilon|} \sum_i \sum_{k=1}^{N!} \varepsilon^N \sum_{l=1}^{N} \left| D_\varepsilon^{e_{p_k(l)}} u\left(\varepsilon\left(i+\sum_{m=1}^{l-1} e_{p_k(l)}\right)\right)\right|^2 \\
\leq \quad & \frac{C}{2|\log\varepsilon|} \sum_{\substack{i,j\in\mathbf{Z}_\varepsilon(\Omega)\\|i-j|=1}} \varepsilon^N \left|\frac{u(\varepsilon i)-u(\varepsilon j)}{\varepsilon}\right|^2 = CE_\varepsilon(u). \qquad (4.26)
\end{aligned}
$$

We now prove (4.25). Fix $i \in \mathbf{Z}_\varepsilon(\Omega)$ such that $\varepsilon(i+[0,1]^N) \subset \Omega$ and $k \in \{1,2,\dots,N!\}$, and set

$$
\alpha_l = u(\varepsilon(i+\sum_{m=1}^{l} e_{p_k(l)})).
$$

Then, for $x \in \varepsilon(i+T^k)$ we can write

$$
v(x) = \alpha_1 + \sum_{l=1}^{N} (\alpha_{l+1}-\alpha_l)(\frac{x_l}{\varepsilon} - i_l).
$$

Then, since $|\alpha_l| = 1$ and $|\frac{x_l}{\varepsilon} - i_l| \leq 1$, we easily get that

$$
(|v(x)|^2-1)^2 \leq C\sum_{l=1}^{N}(|\alpha_{l+1}-\alpha_l|^2).
$$

□

Proof of Theorem 4.2 (i) (Compactness and lower-bound inequality)
Let u_ε be such that $E_\varepsilon(u_\varepsilon) \leq C$, and let $\Omega_n \subset\subset \Omega$ be a sequence of Lipschitz open sets such that $\Omega_n \nearrow \Omega$ as $n \to +\infty$. Then, by Lemma 4.4 and (4.21), we have that, for any $t > 0$ and $n \in \mathbf{N}$

$$
(1+t)E_\varepsilon(u_\varepsilon) \geq \frac{1}{2|\log\varepsilon|}\int_{\Omega_n} |\nabla v_\varepsilon|^2\,dx + \frac{t}{C}\int_{\Omega_n}(|v_\varepsilon|^2-1)^2\,dx. \qquad (4.27)
$$

Then, by Theorem 2.4 we deduce that, for all n we can extract a subsequence (not relabeled) and find a integral boundary M_n, such that $\mathbf{F}_{\Omega_n}(\star J(v_\varepsilon) - \pi M_n) \to 0$.

Note that if $n' > n$ then $M_{n'}\lfloor_{\Omega_n} = M_n$. Letting $n \to +\infty$, by a diagonalization argument we can extract a further subsequence such that we have that $\mathbf{F}_{\Omega_n}(\star J(v_\varepsilon) - \pi M) \to 0$ for all $n \in \mathbf{N}$, where M is such that $M\lfloor_{\Omega_n} = M_n$. Moreover, by (4.27) and Theorem 2.4 (i), we get

$$\liminf_{\varepsilon} E_\varepsilon(u_\varepsilon) \geq \frac{\pi}{1+t}|M|(\Omega_n).$$

Eventually, by letting $n \to \infty$ and $t \to 0$, we obtain the conclusion.

□

In the proof of the upper bound inequality we will make use of the following technical lemma (see [2] Lemma 8.3).

Lemma 4.5 *Let $S \subset \mathbf{R}^N$ be a bounded set contained in a finite union of Lipschitz surfaces of codimension h, and for every $t > 0$, denote by S_t the t-neighborhood of S. There exists a finite constant C (depending on S) such that $\mathcal{L}^N(S_t) \leq Ct^h$ for every $t \geq 0$ and*

$$\int_{S_t} \frac{dx}{(\mathrm{dist(x, S)})^{\mathrm{p}}} \leq \frac{C}{h-p} t^{h-p} \qquad \textit{for every } p < h \textit{ and } t \geq 0.$$

Proof of Theorem 4.2 (ii) (Upper bound inequality)
For reader's convenience we divide the proof in three steps.

step 1. We first consider the case when $M = \tau_{\mathcal{M}} \mathcal{H}^{N-2}\lfloor_{\mathcal{M}}$ where $\mathcal{M} = \{x \in \Omega : x_1 = x_2 = 0\}$ and $\tau_{\mathcal{M}}$ is any constant orientation of $\mathcal{M}$. The function $u : \mathbf{R}^N \to S^1$, given by

$$u(x) = \frac{(x_1, x_2)}{\sqrt{x_1^2 + x_2^2}}, \tag{4.28}$$

is such that $\star Ju = \pi M$, up to a change of the orientation. Moreover $u \in W^{1,1}_{loc}(\mathbf{R}^N; S^1)$ and one can easily check that

$$\begin{aligned} \lim_{\varepsilon} \frac{1}{|\log \varepsilon|} \int_{\Omega \setminus D_\varepsilon} |\nabla u|^2\, dx &= \lim_{\varepsilon} \frac{1}{|\log \varepsilon|} \int_{\Omega \setminus D_\varepsilon} \frac{1}{x_1^2 + x_2^2}\, dx \\ &= \pi \mathcal{H}^{n-2}(\mathcal{M}) = \pi \|M\|, \end{aligned} \tag{4.29}$$

where $D_\varepsilon = \{x \in \Omega : \sqrt{x_1^2 + x_2^2} < \varepsilon\}$. By Lemma 6.1 there exists $y_\varepsilon \in (0,1)^N$ such that, set $u_\varepsilon(x) = T_\varepsilon^{y_\varepsilon} u(x)$, the function $v_\varepsilon = A(u_\varepsilon)$ converges to u strongly in $W^{1,1}_{loc}(\mathbf{R}^N, \mathbf{R}^m)$ which implies that $\mathbf{F}_\Omega(\star J(v_\varepsilon) - \star J(u)) \to 0$. Moreover, as in the proof of Lemma 6.1, we may suppose that, for all $k \in \{1, 2, \ldots, N\}$ and $i \in \mathbf{Z}^N$, $D_\varepsilon^{e_k} u_\varepsilon(\varepsilon i) = \int_0^1 \frac{\partial u}{\partial x_k}(\varepsilon(i + y_\varepsilon + te_k))\, dt$. Indeed in this case one could have chosen as optimizing sequence

$$w_\varepsilon(\varepsilon i) = \begin{cases} u(\varepsilon i) & \text{if } (i_1, i_2) \neq (0,0) \\ u_0 & \text{otherwise,} \end{cases}$$

for some $u_0 \in S^1$. We have preferred the previous construction as it works also in the most general case (see step 3). Let us consider $i \in \mathbf{Z}^N$ such that $i_1 \vee i_2 > 3$ in order that, for any $k \in \{1,2\}$ and for all $t \in (0,1)$ it holds that $\mathrm{dist}(\varepsilon \mathrm{i} + \varepsilon \mathrm{y}_\varepsilon + \mathrm{te}_\mathrm{k}, \mathcal{M}) \geq \varepsilon$. We have that, by Jensen's inequality,

$$
\begin{aligned}
|D_\varepsilon^{e_k} u_\varepsilon(\varepsilon i)|^2 &= \left| \int_0^1 \frac{\partial u}{\partial x_k}(\varepsilon(i + y_\varepsilon + te_k))\, dt \right|^2 \\
&\leq \int_0^1 \left| \frac{\partial u}{\partial x_k}(\varepsilon(i + y_\varepsilon + te_k)) \right|^2 dt \\
&= \left| \frac{\partial u}{\partial x_k}(x) \right|^2 + \int_0^1 \left(\left| \frac{\partial u}{\partial x_k}(\varepsilon(i + y_\varepsilon + te_k)) \right|^2 - \left| \frac{\partial u}{\partial x_k}(x) \right|^2 \right) dt,
\end{aligned}
\tag{4.30}
$$

for any $x \in \varepsilon(i + [0,1)^N)$. By an explicit calculation we get

$$
\left| \nabla \left(\left| \frac{\partial u}{\partial x_k} \right|^2 \right)(x) \right| \leq \frac{C}{(x_1^2 + x_2^2)^{\frac{3}{2}}}.
$$

Hence, by (4.30), we have

$$
|D_\varepsilon^{e_k} u(\varepsilon i)|^2 \leq \left| \frac{\partial u}{\partial x_k}(x) \right|^2 + \frac{C\varepsilon}{(x_1^2 + x_2^2)^{\frac{3}{2}}}.
$$

Thus

$$
\begin{aligned}
\sum_{i_1 \vee i_2 > 3} \varepsilon^N |D_\varepsilon^{e_k} u(\varepsilon i)|^2 &\leq \sum_{i_1 \vee i_2 > 1} \int_{\varepsilon(i+Q)} \left| \frac{\partial u}{\partial x_k}(x) \right|^2 + \frac{C}{(x_1^2 + x_2^2)^{\frac{3}{2}}}\, dx \\
&\leq \int_{\Omega \setminus D_\varepsilon} \left| \frac{\partial u}{\partial x_k}(x) \right|^2 dx + C\varepsilon \int_{\Omega \setminus D_\varepsilon} \frac{1}{(x_1^2 + x_2^2)^{\frac{3}{2}}}\, dx
\end{aligned}
$$

By the previous estimate, observing that the energy accounting for the interaction at a distance of order ε from the singularity is negligible in the limit, we get

$$
E_\varepsilon(u_\varepsilon) \leq \frac{1}{|\log \varepsilon|} \int_{\Omega \setminus D_\varepsilon} |\nabla u|^2\, dx + o(1)
$$

and then the conclusion follows by (4.29).

step 2. Let now $M = \tau_\mathcal{M} \mathcal{H}^{N-2} \lfloor_\mathcal{M}$ where $\mathcal{M} = A \cap \Omega$ with A any $(N-2)$-dimensional affine space in $\mathbf{R}^N$. Then there exists an isometry $T : \mathbf{R}^N \to \mathbf{R}^N$ such that $A = T\{x_1 = x_2 = 0\}$. Then the function $u^T = u \circ T^{-1}$ where u is given by (4.28), is such that $\star J u^T = M$. Moreover $u^T \in W^{1,1}_{loc}(\mathbf{R}^N; S^1)$ and (4.29) holds with u^T in place of u and

$$
D_\varepsilon = \{x \in \Omega : \mathrm{dist}(\mathrm{x}, \mathrm{A}) < \varepsilon\}.
$$

Let $u_\varepsilon = T_\varepsilon^{y_\varepsilon} u^T$ as in step 1. Observing that now, for all $k \in \{1, 2, \dots, N\}$,

$$\left| \nabla \left(\left| \frac{\partial u}{\partial x_k} \right|^2 \right) (x) \right| \leq \frac{C}{(\mathrm{dist(x,A)})^{\frac{3}{2}}},$$

estimate (4.30) for u^T yields to

$$\sum_{i:\mathrm{dist}(\varepsilon \mathrm{i,A}) > (\sqrt{\mathrm{N}}+2)\varepsilon} \varepsilon^N \left| D_\varepsilon^{e_k} u^T(\varepsilon i) \right|^2 \quad \leq \int_{\Omega \setminus D_\varepsilon} \left| \frac{\partial u^T}{\partial x_k}(x) \right|^2 dx + C\varepsilon \int_{\Omega \setminus D_\varepsilon} \frac{1}{(\mathrm{dist(x,A)})^{\frac{3}{2}}} \, dx.$$

The conclusion follows as before.

step 3 In the general case, by Proposition 2.2, we may reduce to prove the upper bound inequality for M a polyhedral boundary such that $\mathrm{spt(M)} \subset\subset \Omega$. Let L be a $(N-1)$-dimensional polyhedral current such that $\partial L = M$. Then there exists $\tilde{u} : \overline{\Omega} \to S^1$, a finite union S of $(N-3)$-dimensional simplices in $\mathbf{R}^N$ which contains all $N-3$-dimensional faces of L and $\delta, \lambda > 0$ such that

(i) $\tilde{u} \in W^{1,1}_{loc}(\mathbf{R}^n, S^1)$ and $\star J\tilde{u} = \pi M$;

(ii) $\tilde{u}$ is locally Lipschitz in $\mathbf{R}^N \setminus (S \cup M)$ and there exists $p < \frac{3}{2}$ such that

$$|D\tilde{u}| = O(1/\mathrm{dist(x,M)}) + \mathrm{O}(1/(\mathrm{dist(x,S)})^{\mathrm{p}}); \tag{4.31}$$

(iii) for every $(N-2)$-dimensional face F of ∂L,

$$\tilde{u}(x) = \frac{x''}{|x''|} \quad \text{for } x \in U(F, \delta, \gamma) \cap \Omega, \tag{4.32}$$

where $U(F,\delta,\gamma) := \{x \in \mathbf{R}^N : \ \mathrm{dist(x,F)} \leq \left(\delta \wedge \frac{\gamma}{\sqrt{1+\gamma^2}} \right) \mathrm{dist(x,\partial F)}\}$.

Let $u_\varepsilon(x) = T_\varepsilon^{y_\varepsilon} \tilde{u}(x)$ as in step 1 and 2. The claim follows if we prove that

$$\limsup_\varepsilon E_\varepsilon(u_\varepsilon) \leq \pi \|M\|.$$

To this aim we localize our energies as follows. For any $V \in \mathbf{R}^N$ we set, for all $u \in C_\varepsilon(\Omega)$,

$$E_\varepsilon(u, V) \quad = \quad \frac{1}{2|\log \varepsilon|} \sum_{\substack{i,j \in \mathbf{Z}_\varepsilon(V) \\ |i-j|=1}} \varepsilon^N \left| \frac{u(\varepsilon i) - u(\varepsilon j)}{\varepsilon} \right|^2. \tag{4.33}$$

For any $\eta > 0$ and $D \subset \mathbf{R}^N$, we define $D^\eta = \{x \in \Omega : \ \mathrm{dist(x,D)} < \eta\}$. Let U be an open set of $\mathbf{R}^N$ such that

$$M \cup S \subset U \subset \bigcup_{F \in \partial L} U(F,\delta,\gamma) \cup S^\eta,$$

then

$$E_\varepsilon(u_\varepsilon) \le E_\varepsilon(u_\varepsilon, S^\eta) + \sum_{F\in\partial L} E_\varepsilon(u_\varepsilon, U(F,\delta,\eta)) + E_\varepsilon(u_\varepsilon, \Omega\setminus U).$$

In what follows, by the arbitrariness of η, we get the conclusion if we prove that

(1) $\limsup_\varepsilon E_\varepsilon(u_\varepsilon, S^\eta) \le O(\eta)$,

(2) $\limsup_\varepsilon E_\varepsilon(u_\varepsilon, \Omega\setminus U) = 0$,

(3) $\limsup_\varepsilon \sum_{F\in\partial L} E_\varepsilon(u_\varepsilon, U(F,\delta,\eta)) \le \pi\|M\|$.

(1) For $i \in \mathbf{Z}_\varepsilon(S^\eta)$ such that $\mathrm{dist}(\varepsilon i, S\cup M) > (\sqrt{N}+2)\varepsilon$, we have that, for any $k\in\{1,2,\dots,N\}$ and $s\in(0,1)$, $\mathrm{dist}(\varepsilon(i+y_\varepsilon+se_k), S\cup M)\ge\varepsilon$. Then, by (4.1), we get

$$\begin{aligned}\varepsilon^N|D_\varepsilon^{e_k}u_\varepsilon(\varepsilon i)|^2 &\le \varepsilon^N\int_0^1 |\frac{\partial\tilde u}{\partial x_k}(\varepsilon(i+y_\varepsilon+se_k))|^2\,ds\\ \le C\varepsilon^N\int_0^1 &\frac{1}{(\mathrm{dist}(\varepsilon(i+y_\varepsilon+se_k),S))^{2p}} + \frac{1}{(\mathrm{dist}(\varepsilon(i+y_\varepsilon+se_k),M))^2}\,ds\\ &\le C\int_{\varepsilon i+[0,\varepsilon)^N}\frac{1}{(\mathrm{dist}(x,S))^{2p}}+\frac{1}{(\mathrm{dist}(x,M))^2}\,dx.\end{aligned}\tag{4.34}$$

Then for $n\in\mathbf{N}$ large enough, by Lemma 4.5,

$$\begin{aligned}E_\varepsilon(u_\varepsilon,S^\eta) &\le E_\varepsilon(u_\varepsilon, S^\eta\cap(S\cup M)^{(n+2)\varepsilon}) + E_\varepsilon(u_\varepsilon, S^\eta\setminus(S\cup M)^{n\varepsilon})\\ &\le o(1)+\frac{1}{|\log\varepsilon|}\int_{S^\eta\setminus(S\cup M)^\varepsilon}\frac{1}{(\mathrm{dist}(x,S))^{2p}}+\frac{1}{(\mathrm{dist}(x,M))^2}\,dx\\ &\le o(1)+C\|M\|(S^\eta),\end{aligned}$$

from which we deduce (1).

(2) is straightforward since $\tilde u$ is locally Lipschitz in $\mathbf{R}^N\setminus U$, from which we get

$$E_\varepsilon(u_\varepsilon,\Omega\setminus U)\le\frac{C}{|\log\varepsilon|}.$$

(3) follows easily since, for any $F\in\partial L$, by step 2

$$\limsup_\varepsilon E_\varepsilon(u_\varepsilon, U(F,\delta,\eta))\le\pi\|M\|(F).$$

□

Remark 4.6 (Antiferromagnetic XY model) Let us consider the antiferromagnetic XY model. In this case the energy is $\tilde F_\varepsilon(u) = -F_\varepsilon(u)$ with F_ε given by (3.16). The opposite sign in the interaction energy now favors oppositely directed

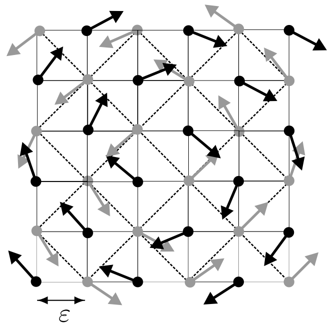

Figure 3: Antiferromagnetic case. Vortex type 2-dimensional singularities: $+1$ charged vortex. In black and grey the two interpenetrating ferromagnetic vortices.

nearest neighboring spins in the volume scaling. This case can be reduced to the previous ferromagnetic one by using the variables

$$w(\varepsilon i) = (-1)^{i_1+i_2+\ldots+i_N} u(\varepsilon i).$$

In this way $\tilde{F}_\varepsilon(u) = F_\varepsilon(w)$ and it is possible to perform the whole analysis we have done so far by using this new variable. In particular we can get some information on the geometry of the vortex type singularities of the antiferromagnetic scaled model. As an example we observe that the configuration of the spin field for a ferromagnetic vortex in the variable w is associated to a spin field in the original variable u which can be described as the superposition of two oppositely directed vortices on two interpenetrating double spaced sublattices (see Fig 4.1).

4.2 Anisotropic nearest-neighbors XY model

In this section we consider the anisotropic nearest-neighbors XY model in which the interactions in different directions are differently weighed, that is in formula (3.16) $c^\xi = 0$ if $\xi \neq e_k$ for all $k = 1, \ldots, N$. Then the energy becomes

$$F_\varepsilon^{an}(u) = \begin{cases} -\displaystyle\sum_{k=1}^{N} c^{e_k} \sum_{i,\ i+e_k \in \mathbf{Z}_\varepsilon(\Omega)} \varepsilon^N u(\varepsilon i) \cdot u(\varepsilon i + \varepsilon e_k) & \text{if } u \in C_\varepsilon(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

Arguing as in the previous section the significant scaled energy becomes

$$E_\varepsilon^{an}(u) = \begin{cases} \dfrac{1}{|\log \varepsilon|} \displaystyle\sum_{k=1}^{N} c^{e_k} \sum_{i,\ i+e_k \in \mathbf{Z}_\varepsilon(\Omega)} \varepsilon^{N-2}(1 - u(\varepsilon i) \cdot u(\varepsilon i + \varepsilon e_k)) & \text{if } u \in C_\varepsilon(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \tag{4.35}$$

Theorem 4.7 *Let $c^{e_k} > 0$ for all $k = 1, \ldots, N$ and set $B = \mathrm{diag}(\sqrt{c^{e_1}}, \ldots, \sqrt{c^{e_N}})$. Then the following statements hold:*

(i) Compactness and lower-bound inequality. *Let (u_ε) be a sequence of functions such that $E_\varepsilon^{an}(u_\varepsilon) \le C$ and let $v_\varepsilon = A(u_\varepsilon)$ be defined by (4.19). Then we can extract a subsequence (not relabeled) such that, for any $\Omega' \subset\subset \Omega$, $\mathbf{F}_{\Omega'}(\star J(v_\varepsilon) - \pi M\lfloor_{\Omega'}) \to 0$, where M is an $(N-2)$-dimensional integral boundary locally in Ω. Moreover*

$$\liminf_{\varepsilon} E_\varepsilon(u_\varepsilon) \ge \det(B)\pi \|B_{\#}^{-1} M\|.$$

(ii) Upper bound inequality. *Let M be an $(N-2)$-dimensional integral boundary locally in Ω, there exists a sequence (u_ε) such that, for any $\Omega' \subset\subset \Omega$, $\mathbf{F}_{\Omega'}(\star J(v_\varepsilon) - \pi M\lfloor_{\Omega'}) \to 0$ and*

$$\lim_{\varepsilon} E_\varepsilon(u_\varepsilon) = \det(B)\pi \|B_{\#}^{-1} M\|.$$

Proof. (Compactness and lower-bound inequality) We first note that, for any $u \in C_\varepsilon(\Omega)$, we have

$$E_\varepsilon(u) \ge \frac{1}{2|\log\varepsilon|} \sum_{k=1}^{N} c^{e_k} \sum_{\substack{i \in \mathbf{Z}_\varepsilon(\Omega): \\ \varepsilon(i+Q) \subset \Omega}} \varepsilon^N |D_\varepsilon^{e_k} u(\varepsilon i)|^2.$$

Thus by the definition of $v_\varepsilon = A(u_\varepsilon)$ we easily deduce that, for ε small enough,

$$E_\varepsilon(u) \ge \frac{1}{2|\log\varepsilon|} \int_{\Omega_\varepsilon} |\nabla v \cdot B|^2 \, dx, \tag{4.36}$$

where

$$\Omega_\varepsilon := \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \sqrt{N}\varepsilon\}.$$

Hence, by (4.36) and Remark 2.5, we prove (i) arguing as in the proof of Theorem 4.2.

(Upper-bound inequality) Since the proof relies on the same argument used in that of Theorem 4.2 (ii), here we only highlight the main differences by exhibiting the optimizing sequence in the case $M = \tau_{\mathcal{M}} \mathcal{H}^{N-2} \lfloor_{\mathcal{M}}$ where $\mathcal{M} = V \cap \Omega$ with V any $(N-2)$-dimensional affine space in $\mathbf{R}^N$. Let us set $\tilde{u} = u^T$ where u^T is the function introduced in step 2 of the proof of Theorem 4.2 and $T : \mathbf{R}^N \to \mathbf{R}^N$ is the isometry such that $B^{-1}(V) = T\{x_1 = x_2 = 0\}$. We remind that $\star J(\tilde{u}) = \pi B_{\#}^{-1} M$ and easily infer that

$$\frac{1}{|\log\varepsilon|} \int_{B^{-1}(\Omega \setminus D_\varepsilon)} |\nabla \tilde{u}|^2 \ dx \to \pi \|B_{\#}^{-1} M\|,$$

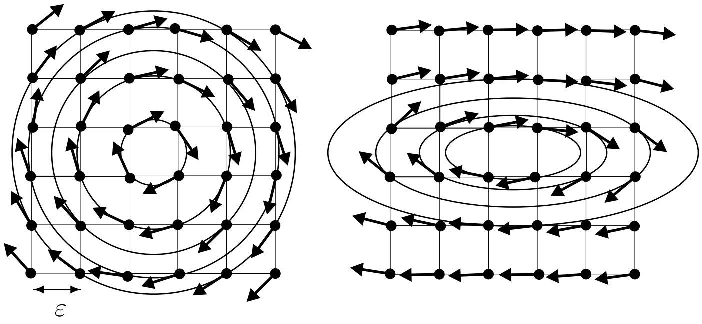

Figure 4: Deformation of the geometry of the singularity by anisotropy for a 2-dimensional $+1$ charged vortex: circular symmetry in the isotropic $c^{e_1} = c^{e_2}$ case (left), elliptical symmetry in the anisotropic $c^{e_1} > c^{e_2}$ case (right).

where $D_\varepsilon = \{x \in \Omega : \mathrm{dist}(\mathrm{x}, \mathrm{V}) < \varepsilon\}$. Setting $u = \tilde{u} \circ B^{-1}$ one can easily verify that $\star J(u) = \pi M$ and, by a change of variable, that

$$\frac{1}{|\log \varepsilon|} \int_{\Omega \setminus D_\varepsilon} |\nabla u \cdot B|^2 \; dx \to \det(B)\pi \|B_{\#}^{-1} M\|.$$

Let $u_\varepsilon = T_\varepsilon^{y_\varepsilon} u$ as in Lemma 6.1. Arguing as in the step 2 of the proof of Theorem 4.2 we have

$$E_\varepsilon^{an}(u_\varepsilon) \leq \frac{1}{|\log \varepsilon|} \int_{\Omega \setminus D_\varepsilon} |\nabla u \cdot B|^2 \; dx + o(\varepsilon)$$

from which the conclusion follows. □

4.3 Long range XY model

In this section we will focus on the long range XY model that is the case in which all interactions are taken into account. We will mainly deal with the $2-$dimensional case and then discuss the extension of the results to higher dimensions. By scaling the energy in (3.16) as in the previous cases, we obtain

$$E_\varepsilon^{lr}(u) = \begin{cases} \dfrac{1}{|\log \varepsilon|} \displaystyle\sum_{\xi \in \mathbf{Z}^N} c^\xi \sum_{i, i+\xi \in \mathbf{Z}_\varepsilon(\Omega)} \varepsilon^{N-2}(1 - u(\varepsilon i) \cdot u(\varepsilon i + \varepsilon \xi)) & \text{if } u \in C_\varepsilon(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

It is convenient to regard the energies above in terms of different quotients. For $u \in C_\varepsilon(\Omega)$ we set

$$D_\varepsilon^\xi u(\varepsilon i) = \frac{u(\varepsilon i + \varepsilon \xi) - u(\varepsilon i)}{\varepsilon |\xi|}$$

and rewrite the energy as

$$E_\varepsilon^{lr}(u) = \frac{1}{2|\log\varepsilon|}\sum_{\xi\in\mathbf{Z}^N}|\xi|^2 c^\xi \mathcal{E}_\varepsilon^\xi(u) \tag{4.37}$$

where

$$\mathcal{E}_\varepsilon^\xi(u) \quad = \sum_{i,i+\xi\in\mathbf{Z}_\varepsilon(\Omega)} \varepsilon^N \left|D_\varepsilon^\xi u(\varepsilon i)\right|^2 .$$

4.3.1 The 2-dimensional case

In the following theorem we prove the analogue of Theorem 4.2 in the 2-dimensional case under the main assumption that $c^\xi = c^{\xi^\perp}$ which implies the isotropic behavior of the energies in the limit.

Theorem 4.8 *Let $N = 2$ and let $\{c^\xi\}_\xi$ be a family of non negative constants such that $c^\xi = c^{\xi^\perp}$, $c^{e_1} > 0$ and $\sum_{\xi\in\mathbf{Z}^2}|\xi|^2 c^\xi < +\infty$. The following statements hold:*

(i) Compactness and lower-bound inequality. *Let (u_ε) be a sequence of functions such that $E_\varepsilon^{lr}(u_\varepsilon) \le C$ and let $v_\varepsilon = A(u_\varepsilon)$ be defined by (4.19). Then we can extract a subsequence (not relabeled) such that, for any $\Omega' \subset\subset \Omega$, $\mathbf{F}_{\Omega'}(\star J(v_\varepsilon) - \pi M\lfloor_{\Omega'}) \to 0$, where M is of the form*

$$M = \sum_{k=1}^{N} d_k \delta_{x_k}, \tag{4.38}$$

for some $N \in \mathbf{N}$, $d_k \in \mathbf{Z}$ and $x_k \in \Omega$. Moreover

$$\liminf_\varepsilon E_\varepsilon^{lr}(u_\varepsilon) \ge \frac{\pi}{2}\sum_{\xi\in\mathbf{Z}^2}|\xi|^2 c^\xi \|M\|. \tag{4.39}$$

(ii) Upper bound inequality. *Let M be of the form (4.38). Then there exists a sequence (u_ε) such that, for any $\Omega' \subset\subset \Omega$, $\mathbf{F}_{\Omega'}(\star J(v_\varepsilon) - \pi M\lfloor_{\Omega'}) \to 0$ and*

$$\lim_\varepsilon E_\varepsilon^{lr}(u_\varepsilon) = \frac{\pi}{2}\sum_{\xi\in\mathbf{Z}^2}|\xi|^2 c^\xi \|M\|.$$

Remark 4.9 The condition $c^\xi = c^{\xi^\perp}$ is in particular implied by a more natural condition widely exploited in the physical literature of the the long range XY-model, namely that $c^\xi = c(|\xi|)$.

Proof of Theorem 4.8 (i) (Compactness and lower-bound inequality)
Since $c^{e_1} > 0$, the compactness is straightforward consequence of Theorem 4.2. Given $\xi \in \mathbf{Z}^2$, we may partition $\mathbf{Z}^2$ as follows:

$$\mathbf{Z}^2 = \bigcup_{h=1}^{|\xi|^2} (z_h + \mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp) \tag{4.40}$$

with $z_h \in \{z \in \mathbf{Z}^2 : \ 0 \le z \cdot \xi < |\xi|, \ 0 \le z \cdot \xi^\perp < |\xi|\}$. Then for $u \in C_\varepsilon$ we may write

$$\mathcal{E}^\xi_\varepsilon(u) = \sum_{h=1}^{|\xi|^2} \mathcal{E}^{\xi,h}_\varepsilon(u)$$

with

$$\mathcal{E}^{\xi,h}_\varepsilon(u) = \sum_{i, i+\xi \in \mathbf{Z}^{\xi,h}_\varepsilon(\Omega)} \varepsilon^2 \left| D^\xi_\varepsilon u(\varepsilon i) \right|^2,$$

where

$$\mathbf{Z}^{\xi,h}_\varepsilon(\Omega) =: \{i \in z_h + \mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp : \ \varepsilon i \in \Omega\}.$$

For all $h \in \{1, \dots, |\xi|^2\}$ and $u \in C_\varepsilon(\Omega)$, let us introduce $v^{\xi,h} = A^{\xi,h}(u)$ a piecewise affine interpolation of u on the cells of the lattice $z_h + \mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp$. To this end, for all $\xi \in \mathbf{R}^2$, we set

$$\begin{aligned} T^-_\xi &= \{x \in \mathbf{R}^2 : \ 0 \le x \cdot \xi^\perp \le x \cdot \xi \le |\xi|\}, \\ T^+_\xi &= \{x \in \mathbf{R}^2 : \ 0 \le x \cdot \xi \le x \cdot \xi^\perp \le |\xi|\}, \\ Q_\xi &= T^-_\xi \cup T^+_\xi. \end{aligned}$$

Then, for $i \in z_h + \mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp$, $v^{\xi,h} = A^{\xi,h}(u)$ is defined as

$$v^{\xi,h}(x) = u(\varepsilon i) + D^\xi_\varepsilon u(\varepsilon i) \left((x - \varepsilon i) \cdot \frac{\xi}{|\xi|} \right) + D^{\xi^\perp}_\varepsilon u(\varepsilon i + \varepsilon\xi) \left((x - \varepsilon i) \cdot \frac{\xi^\perp}{|\xi|} \right)$$

for $x \in \varepsilon(i + T^-_\xi)$, and

$$v^{\xi,h}(x) = u(\varepsilon i) + D^{\xi^\perp}_\varepsilon u(\varepsilon i) \left((x - \varepsilon i) \cdot \frac{\xi^\perp}{|\xi|} \right) + D^\xi_\varepsilon u(\varepsilon i + \varepsilon\xi^\perp) \left((x - \varepsilon i) \cdot \frac{\xi}{|\xi|} \right)$$

for $x \in \varepsilon(i + T^+_\xi)$. Note that

$$\begin{aligned} \frac{\partial v^{\xi,h}}{\partial \xi}(x) &= D^\xi_\varepsilon u(\varepsilon i), \quad x \in \varepsilon(i + P_\xi), \\ \frac{\partial v^{\xi,h}}{\partial \xi^\perp}(x) &= D^{\xi^\perp}_\varepsilon u(\varepsilon i), \quad x \in \varepsilon(i + P_{\xi^\perp}) \end{aligned}$$

where

$$\begin{aligned} P_\xi &= \{x \in \mathbf{R}^2 : -|\xi| < (x\cdot\xi) - |\xi| < (x\cdot\xi^\perp) < (x\cdot\xi) < |\xi|\}, \\ P_{\xi^\perp} &= \{x \in \mathbf{R}^2 : -|\xi| < (x\cdot\xi^\perp) - |\xi| < (x\cdot\xi) < (x\cdot\xi^\perp) < |\xi|\}. \end{aligned}$$

Note that $v_\varepsilon^{e_1,1} = v_\varepsilon$. Then, as in the proof of Theorem 4.2, we have that, for ε small enough,

$$\mathcal{E}_\varepsilon^{\xi,h}(u) \geq \frac{1}{2|\log\varepsilon|}\int_{\Omega_\varepsilon^\xi}\left|\frac{\partial v_\varepsilon^{\xi,h}}{\partial\xi}\right|^2 + \left|\frac{\partial v_\varepsilon^{\xi,h}}{\partial\xi^\perp}\right|^2 \, dx = \frac{1}{2|\log\varepsilon||\xi|^2}\int_{\Omega_\varepsilon^\xi}|\nabla v_\varepsilon^{\xi,h}|^2 \, dx, \tag{4.41}$$

where

$$\Omega_\varepsilon^\xi := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \sqrt{2}|\xi|\varepsilon\}.$$

Given u_ε be such that $E_\varepsilon^{lr}(u_\varepsilon) \leq C$, we now verify that, for every $U \subset\subset \Omega$, $\xi \in \mathbf{Z}^2$ and $h \in \{1, 2, \ldots, |\xi|^2\}$, $\mathbf{F}_U(\star J(v_\varepsilon^{\xi,h}) - \star J(v_\varepsilon)) \to 0$. To this end we show that the sequences $v_\varepsilon^{\xi,h}$ and v_ε satisfy the hypotheses of Lemma 2.3. Let U' be such that $U \subset\subset U' \subset\subset \Omega$. By (4.41) we have that

$$\frac{1}{|\log\varepsilon|}\int_{U'}|\nabla v_\varepsilon^{\xi,h}|^2 + |\nabla v_\varepsilon|^2 \, dx \leq C. \tag{4.42}$$

Set $w_\varepsilon = v_\varepsilon^{\xi,h} - v_\varepsilon$ it holds

$$\int_{U'}|\nabla w_\varepsilon|^2 \, dx \leq C|\log\varepsilon|. \tag{4.43}$$

Moreover, since $w_\varepsilon(\varepsilon i) = 0$ for all $i \in z_h + \mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp$, we have that, for every $x \in \varepsilon i + \varepsilon Q_\xi$,

$$w_\varepsilon(x) = \int_0^1 \nabla w_\varepsilon(\varepsilon i + t(x - \varepsilon i)) \cdot (x - \varepsilon i) \, dt$$

by which we get that

$$|w_\varepsilon(x)|^2 \leq \int_0^1 |\nabla w_\varepsilon(\varepsilon i + t(x - \varepsilon i)) \cdot (x - \varepsilon i)|^2 \, dt. \tag{4.44}$$

Set $t_0 = \frac{1}{\sqrt{2}|\xi|}$. With given $x \in \varepsilon i + \varepsilon Q_\xi$, if $t \leq t_0$, we have that $t|x - \varepsilon i| \leq t\varepsilon\sqrt{2}|\xi| \leq \varepsilon$ which implies, by the construction of the piecewise affine interpolations, that $\nabla w_\varepsilon(\varepsilon i + t(x - \varepsilon i)) \cdot (x - \varepsilon i)$ is constant on $(0, t_0)$. Then the following estimate holds true

$$\int_0^1 |\nabla w_\varepsilon(\varepsilon i + t(x - \varepsilon i)) \cdot (x - \varepsilon i)|^2 \, dt \leq 2\int_{\frac{t_0}{2}}^1 |\nabla w_\varepsilon(\varepsilon i + t(x - \varepsilon i)) \cdot (x - \varepsilon i)|^2 \, dt.$$

Integrating (4.44) over $\varepsilon i + \varepsilon Q_\xi$ and using the previous estimate, we get

$$\int_{\varepsilon i+\varepsilon Q_\xi} |w_\varepsilon(x)|^2 \, dx \leq C\varepsilon^2|\xi|^2 \int_{\frac{t_0}{2}}^1 \int_{\varepsilon i+\varepsilon Q_\xi} |\nabla w_\varepsilon(\varepsilon i + t(x-\varepsilon i))|^2 \, dx \, dt.$$

By the change of variables $y = \varepsilon i + t(x - \varepsilon i)$ we have the following estimate

$$\int_{\varepsilon i+\varepsilon Q_\xi} |w_\varepsilon(x)|^2 \, dx \leq \frac{C\varepsilon^2|\xi|^2}{t_0^2} \int_{\varepsilon i+\varepsilon Q_{t\xi}} |\nabla w_\varepsilon(y)|^2 \, dy \leq C\varepsilon^2|\xi|^2 \int_{\varepsilon i+\varepsilon Q_\xi} |\nabla w_\varepsilon|^2 \, dx.$$

Finally, summing over $i \in \left(z_h + \mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp\right) \cap U$, by (4.43) we get

$$\int_U |w_\varepsilon|^2 \, dx \leq \sum_{i \in (z_h + \mathbf{Z}\xi \oplus \mathbf{Z}\xi^\perp)\cap U} \int_{\varepsilon i+\varepsilon Q_\xi} |w_\varepsilon|^2 \, dx \leq C\varepsilon^2 \int_{U'} |\nabla w_\varepsilon|^2 \, dx \leq C\varepsilon^2 |\log \varepsilon|.$$

Thus we may apply Lemma 2.3 and get, by the compactness result, that there exists a subsequence (not relabeled) such that, for every $U \subset\subset \Omega$ $\mathbf{F}_U(\star J(v_\varepsilon^{\xi,h}) - \pi M) \to 0$ for all $\xi \in \mathbf{Z}^2$ and $h \in \{1, 2, \dots, |\xi|^2\}$ where M is of the form (4.38). Then by (4.41) and by arguing as in the proof of the lower bound inequality in Theorem 4.2, we obtain the lower bound inequality

$$\begin{aligned} \liminf_\varepsilon E_\varepsilon^{lr}(u_\varepsilon) &\geq \sum_\xi |\xi|^2 c^\xi \sum_{h=1}^{|\xi|^2} \liminf_\varepsilon \mathcal{E}_\varepsilon^{\xi,h}(u) \\ &\geq \sum_\xi |\xi|^2 c^\xi \|M\|. \end{aligned}$$

$\square$

In the following proof the construction of the optimizing sequence is the same as in the isotropic nearest-neighbors case. Here we only provide the main estimates of the energy in order to obtain the upper bound.

Proof of Theorem 4.8 (ii) (Upper bound inequality)
For any given $A \in \mathcal{A}(\Omega)$ we localize the energies defining

$$E_\varepsilon^{lr}(u, A) = \sum_{\xi \in \mathbf{Z}^2} |\xi|^2 c^\xi \mathcal{E}_\varepsilon^\xi(u, A)$$

where

$$\mathcal{E}_\varepsilon^\xi(u, A) = \sum_{i, i+\xi \in \mathbf{Z}_\varepsilon(A)} \varepsilon^2 \left| D_\varepsilon^\xi u(\varepsilon i) \right|^2.$$

Assume for simplicity of notation that, by translation, $0 \in \Omega$. Let $\mathcal{M} = \{0\}$ and $M = \delta_0$, the general case being easily deduced by arguing as in the proof of

Theorem 4.2. Let $u(x) = \frac{x}{|x|}$ and let $u_\varepsilon(x) = T_\varepsilon^{y_\varepsilon} u(x)$ be as in step 1 of the proof of Theorem 4.2 (ii). Let $\varepsilon i,\ \varepsilon(i+\xi) \in B(0, 4\varepsilon|\xi|)$. Since $|D_\varepsilon^\xi u(\varepsilon i)| \le \frac{2}{\varepsilon|\xi|}$ we have that

$$\mathcal{E}_\varepsilon^\xi(u, B(0, 5\varepsilon|\xi|)) \le \frac{C}{|\log \varepsilon|}.$$

Let $\varepsilon i,\ \varepsilon(i+\xi) \in \Omega \setminus B(0, 4\varepsilon|\xi|)$. As in (4.30), for every $\xi \in \mathbf{Z}^2$, we have:

$$|D_\varepsilon^\xi u(\varepsilon i)|^2 \le \frac{1}{|\xi|^2}|\nabla u(x)\cdot\xi|^2 + \frac{1}{|\xi|^2}\int_0^1 \left(|\nabla u(x)\cdot\xi|^2 - |\nabla u(\varepsilon i + \varepsilon t\xi)\cdot\xi|^2\right)\, dt$$

and

$$\left|\nabla\left(|\nabla u\cdot\xi|^2\right)(x)\right| \le \frac{C|\xi|^2}{|x|^3}.$$

Thus, by the two previous estimates, we get

$$\begin{aligned}
\sum_{i, i+\xi \in \mathbf{Z}_\varepsilon(\Omega\setminus B(0,4\varepsilon|\xi|))} |D_\varepsilon^\xi u(\varepsilon i)|^2 &\le \frac{1}{|\xi|^2}\int_{\Omega\setminus B(0,\varepsilon)} |\nabla u(x)\cdot\xi|^2\, dx + C\varepsilon|\xi| \int_{\Omega\setminus B(0,\varepsilon|\xi|)} \frac{1}{|x|^3}\, dx \\
&\le \frac{1}{|\xi|^2}\int_{\Omega\setminus B(0,\varepsilon)} |\nabla u(x)\cdot\xi|^2\, dx + C.
\end{aligned}$$

Since

$$\begin{aligned}
E_\varepsilon^{lr}(u) &\le \frac{1}{2|\log\varepsilon|}\sum_\xi |\xi|^2 c^\xi \mathcal{E}_\varepsilon^\xi(u, B(0, 5\varepsilon|\xi|)) \\
&\quad + \frac{1}{2|\log\varepsilon|}\sum_\xi |\xi|^2 c^\xi \mathcal{E}_\varepsilon^\xi(u, \Omega\setminus B(0, 4\varepsilon|\xi|)),
\end{aligned} \tag{4.45}$$

by the previous estimates we get that

$$\begin{aligned}
E_\varepsilon^{lr}(u_\varepsilon) &\le \frac{C}{|\log\varepsilon|}\sum_{\xi\in\mathbf{Z}^2} |\xi|^2 c^\xi \\
&\quad + \frac{1}{2}\sum_{\xi\in\mathbf{Z}^2} |\xi|^2 c^\xi \frac{1}{2|\log\varepsilon|}\left(\mathcal{E}_\varepsilon^\xi(u, \Omega\setminus B(0,\varepsilon)) + \mathcal{E}_\varepsilon^{\xi^\perp}(u, \Omega\setminus B(0,\varepsilon))\right) \\
&\le \frac{C}{|\log\varepsilon|}\sum_{\xi\in\mathbf{Z}^2} |\xi|^2 c^\xi + \frac{1}{2}\sum_{\xi\in\mathbf{Z}^2} |\xi|^2 c^\xi \frac{1}{2|\log\varepsilon|}\int_{\Omega\setminus B(0,\varepsilon)} |\nabla u(x)|^2\, dx.
\end{aligned}$$

By the finiteness of $\sum_\xi |\xi|^2 c^\xi$, passing to the limit as $\varepsilon \to 0$ and recalling that

$$\lim_\varepsilon \frac{1}{2|\log\varepsilon|}\int_{\Omega\setminus B(0,\varepsilon)} |\nabla u(x)|^2\, dx = \pi,$$

we get the conclusion .

□

4.3.2 Generalization to higher dimensions

In this section we discuss the long range problem in any dimension. We will state the analogue of Theorem 4.8 under an N-dimensional condition which extends the assumption $c^\xi = c^{\xi^\perp}$. The main drawback of this hypothesis is that it turns out to be more abstract and in particular not implied by the physical relevant condition $c^\xi = c(|\xi|)$ in dimensions higher than 2 (an exception is provided by the case $N = 2^k$ as we point out in Remark 4.10 below).

To detail this point we need some more definitions. Let $\{B_l : \ l \in \mathbf{N}\}$ be a countable family of orthogonal bases in $\mathbf{R}^N$ of vectors belonging to $\mathbf{Z}^N$ such that

$$\mathbf{Z}^N = \bigcup_l B_l, \qquad \#\{l : \ \xi \in B_l\} < +\infty \quad \forall \xi \in \mathbf{Z}^N. \tag{4.46}$$

In particular such a family is provided by $\{B_l : \ l \in \mathbf{N}\} = \{B_\xi : \ \xi \in \mathbf{Z}^N\}$ where, for any $\xi \in \mathbf{Z}^N$ B_ξ denotes an orthogonal base of $\mathbf{R}^N$ such that $\xi \in B_\xi$ and $\eta \in \mathbf{Z}^N$ and $|\eta| \geq |\xi|$ for all $\eta \in B_\xi$. It can be easily proved that such a family exists and satisfies (4.46).

We can then rewrite the energies in (4.37) as follows

$$E_\varepsilon^{lr}(u) = \frac{1}{2|\log\varepsilon|}\sum_l \sum_{\xi \in B_l} \tilde{c}_\xi \mathcal{E}_\varepsilon^\xi(u),$$

where

$$\tilde{c}_\xi = \frac{c^\xi|\xi|^2}{\#\{l \in \mathbf{N} : \ \xi \in B_l\}}.$$

The main hypothesis we make here is the following N-dimensional discrete isotropy condition:

$$\forall \eta \in \mathbf{Z}^N, \ l \in \mathbf{N} \ \exists c_l : \ c_l = \tilde{c}_\eta \ \forall \eta \in B_l. \tag{4.47}$$

Note that, if $N = 2$, (4.47) turns out to be the 2-dimensional condition $c^\xi = c^{\xi^\perp}$ under the choice

$$B_\xi = \{\xi, \xi^\perp\}.$$

Remark 4.10 In dimension $N = 2^k$ to any $\xi \in \mathbf{Z}^N$ it is possible to associate $N-1$ vectors $\xi_i^\perp \in \mathbf{Z}^N$, $i \in \{1, 2, \ldots, N-1\}$, such that $|\xi_i^\perp| = |\xi|$ and that $B_\xi = \{\xi, \xi_1^\perp, \ldots, \xi_{N-1}^\perp\}$ is an orthogonal base in $\mathbf{R}^N$. By choosing such a B_ξ, condition (4.47) turns out to be $c^\xi = c^{\xi_1^\perp} = \ldots = c^{\xi_{N-1}^\perp}$ and again includes the case $c^\xi = c(|\xi|)$.

The following result holds true:

Theorem 4.11 *Let $\{c^\xi\}_\xi$ be a family of non negative constants satisfying hypothesis (4.47) and such that $c^{e_i} > 0$ for all $i \in \{1, 2, \ldots, N\}$ and $\sum_{\xi \in \mathbf{Z}^N} |\xi|^2 c^\xi < +\infty$. The following statements hold:*

(i) Compactness and lower-bound inequality. *Let* (u_ε) *be a sequence of functions such that* $E^{lr}_\varepsilon(u_\varepsilon) \le C$ *and let* $v_\varepsilon = A(u_\varepsilon)$ *be defined by (4.19). Then we can extract a subsequence (not relabeled) such that, for any* $\Omega' \subset\subset \Omega$, $\mathbf{F}_{\Omega'}(\star J(v_\varepsilon) - \pi M\lfloor_{\Omega'}) \to 0$, *where* M *is an* $(N-2)$*-dimensional integral boundary locally in* Ω. *Moreover*

$$\liminf_{\varepsilon} E_\varepsilon(u_\varepsilon) \ge \frac{\pi}{N} \sum_{\xi \in \mathbf{Z}^N} |\xi|^2 c^\xi \|M\|. \tag{4.48}$$

(ii) Upper bound inequality. *Let* M *be an* $(N-2)$*-dimensional integral boundary locally in* Ω*, there exists a sequence* (u_ε) *such that, for any* $\Omega' \subset\subset \Omega$, $\mathbf{F}_{\Omega'}(\star J(v_\varepsilon) - \pi M\lfloor_{\Omega'}) \to 0$ *and*

$$\lim_{\varepsilon} E_\varepsilon(u_\varepsilon) = \frac{\pi}{N} \sum_{\xi \in \mathbf{Z}^N} |\xi|^2 c^\xi \|M\|.$$

The proof of this result is more technical than that of Theorem 4.8 but relies on the same arguments. Here we prefer to omit the details and focus on the key ideas of the proof.

Sketch of the proof of Theorem 4.11. The compactness result is straightforward as in the 2-dimensional case. To get the lower bound inequality we will single out the contribution to the energy due to the interactions along the directions of each base of the family $\{B_l : l \in \mathbf{N}\}$. Let us fix $l \in \mathbf{N}$ and let $B_l = \{\xi^1, \xi^2, \dots, \xi^N\}$. Then we may partition $\mathbf{Z}^N$ as

$$\mathbf{Z}^N = \bigcup_{h=1}^{\prod_{j=1}^N |\xi^j|} (z_h + \bigoplus_{j=1}^N \mathbf{Z}\xi^j) \tag{4.49}$$

with $z_h \in \{z \in \mathbf{Z}^N : 0 \le z \cdot \xi^j < |\xi^j|, \ j = 1, 2, \dots, N\}$. By the hypothesis (4.47), it is possible to rewrite the energy as

$$E^{lr}_\varepsilon(u) = \frac{1}{2|\log \varepsilon|} \sum_l c_l \mathcal{E}^l_\varepsilon(u)$$

where

$$\mathcal{E}^l_\varepsilon(u) = \sum_{h=1}^{\prod_{j=1}^N |\xi^j|} \sum_{j=1}^N \mathcal{E}^{j,h}_\varepsilon(u)$$

accounts for the energy of the interactions in the directions of B_l and where

$$\mathcal{E}^{j,h}_\varepsilon(u) = \sum_{i, i+\xi^j \in \mathbf{Z}^{j,h}_\varepsilon(\Omega)} \varepsilon^N \left| D^{\xi^j}_\varepsilon u(\varepsilon i) \right|^2,$$

is the energy of the interactions in the direction ξ^j of the points of the lattice

$$\mathbf{Z}_\varepsilon^{j,h}(\Omega) =: \{i \in z_h + \bigoplus_{i=1}^N \mathbf{Z}\xi^j : \ \varepsilon i \in \Omega\}.$$

Let u_ε be such that $E_\varepsilon^{lr}(u_\varepsilon) \leq C$. Proceeding as in the 2-dimensional case, we may define the piecewise affine interpolations $v_\varepsilon^{l,h}$ of u_ε on the cell of the lattice $z_h + \bigoplus_{j=1}^N \mathbf{Z}\xi^j$ and get

$$\sum_{j=1}^N \mathcal{E}_\varepsilon^{j,h}(u) \geq \frac{1}{2|\log\varepsilon| \prod_{j=1}^N |\xi^j|} \int_{\Omega'} |\nabla v_\varepsilon^{l,h}|^2 \, dx, \tag{4.50}$$

for all $\Omega' \subset\subset \Omega$. Moreover, by applying Lemma 2.3 we have, by the compactness result, that there exists a subsequence (not relabeled) such that, for every $\Omega' \subset\subset \Omega$ $\mathbf{F}_{\Omega'}(\star J(v_\varepsilon^{l,h}) - \pi M) \to 0$ for all $l \in \mathbf{N}$ and $h \in \{1, 2, \ldots, \prod_{j=1}^N |\xi^j|\}$ where M is an $N-2$-dimensional integral boundary locally in Ω. The lower bound inequality follows as in the proof of Theorem 4.8.

The upper bound inequality is obtained by the same recovery sequence of the isotropic nearest-neighbors case. More precisely let M be a polyhedral boundary such that spt(M) $\subset\subset \Omega$ and let $\tilde{u}$ and $u_\varepsilon = T_\varepsilon^{y_\varepsilon}(\tilde{u})$ as in step 3 of the proof of Theorem 4.2 (ii). Then by the orthogonality of each base B_l one can show that

$$\mathcal{E}_\varepsilon^l(u_\varepsilon) \leq \frac{1}{2|\log\varepsilon|} \int_{\Omega \setminus D_\varepsilon} |\nabla u|^2 \ dx + o(\varepsilon)$$

where D_ε is a suitable ε-neighborhood of M. Then passing to the limit one obtain

$$\limsup_\varepsilon \mathcal{E}_\varepsilon^l(u_\varepsilon) \leq \pi \|M\|.$$

The conclusion follows by summing over l and observing that

$$\sum_l c_l = \frac{1}{N} \sum_\xi c^\xi |\xi|^2.$$

□

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5 Appendix A: an alternative proof in the 2-d case

Here we restate the convergence result of Theorem 4.2 in the 2-dimensional case and provide a different proof of the upper bound inequality using only discrete estimates.

Theorem 5.1 *The following statements hold:*

(i) Compactness and lower-bound inequality. *Let (u_ε) be a sequence of functions such that $E_\varepsilon(u_\varepsilon) \le C$ and let $v_\varepsilon = A(u_\varepsilon)$ be defined by (4.19). Then we can extract a subsequence (not relabeled) such that, for any $\Omega' \subset\subset \Omega$, $\mathbf{F}_{\Omega'}(\star J(v_\varepsilon) - \pi M\lfloor_{\Omega'}) \to 0$, where $M = \sum_{k=1}^n d_k \delta_{x_k}$ for some $n \in \mathbf{N}$, $d_k \in \mathbf{Z}$ and $x_k \in \Omega$. Moreover*

$$\liminf_{\varepsilon} E_\varepsilon(u_\varepsilon) \ge \pi \|M\| = \pi \sum_{k=1}^n |d_k|. \tag{5.1}$$

(ii) Upper bound inequality. *Let $M = \sum_{k=1}^n d_k \delta_{x_k}$. Then there exists a sequence (u_ε) such that, for any $\Omega' \subset\subset \Omega$, $\mathbf{F}_{\Omega'}(\star J(v_\varepsilon) - \pi M\lfloor_{\Omega'}) \to 0$ and*

$$\lim_{\varepsilon} E_\varepsilon(u_\varepsilon) = \pi \|M\| = \pi \sum_{k=1}^n |d_k|.$$

Proof of Theorem 5.1 (ii) (Upper bound inequality)
step 1. Let $x_0 \in \Omega$ and $\mu = \delta_{x_0}$. For simplicity we may assume that $x_0 = (0,0)$ Then $u(x) = \frac{x}{|x|}$ is such that $\star J(u) = M$. Let $u_\varepsilon \in C_\varepsilon(\Omega)$ be defined by

$$u_\varepsilon^i = \begin{cases} u(\varepsilon i) & \text{if } \varepsilon i \neq (0,0) \\ u_0 & \text{otherwise,} \end{cases}$$

for some $u_0 \in S^1$. Observe that $v_\varepsilon = A(u_\varepsilon) \to u$ strongly in $W^{1,1}(\Omega; \mathbf{R}^2)$ and then $\star J(v_\varepsilon) \to M$ in the flat norm. In fact, by a simple calculation, one can prove that, for any $\delta > 0$, $\nabla v_\varepsilon \to \nabla u$ uniformly in $\Omega \setminus B_\delta$ and

$$\int_{B_\delta} |\nabla v_\varepsilon| \, dx \le C \int_{B_\delta} |\nabla u| \, dx.$$

Let us consider $i \in \mathbf{Z}^2$ such that $i_1, i_2 > 0$. We have that, by Jensen's inequality,

$$\begin{aligned} \left| \frac{u(\varepsilon(i+e_1)) - u(\varepsilon i)}{\varepsilon} \right|^2 &= \left| \int_0^1 \nabla u(\varepsilon(i + t e_1)) \cdot e_1 \, dt \right|^2 \\ &\le \int_0^1 |\nabla u(\varepsilon(i+te_1)) \cdot e_1|^2 \, dt = \int_0^1 \frac{i_2^2}{\varepsilon^2 |i + t e_1|^4} \, dt \le \frac{i_2^2}{\varepsilon^2 |i|^4}. \end{aligned} \tag{5.2}$$

Analogously

$$\left|\frac{u(\varepsilon(i+e_2))-u(\varepsilon i)}{\varepsilon}\right|^2 \le \frac{i_1^2}{\varepsilon^2|i|^4}. \tag{5.3}$$

Then, by the previous inequality and the radial symmetry of u, we have that

$$E_\varepsilon(u_\varepsilon) \le \frac{2}{|\log\varepsilon|}\sum_{i_1=1}^{d_\varepsilon}\sum_{i_2=1}^{d_\varepsilon}\frac{1}{i_1^2+i_2^2},$$

where $d_\varepsilon = \left[\frac{\mathrm{diam}(\Omega)}{\varepsilon}\right]+1$. Since $d_\varepsilon = O(\frac{1}{\varepsilon})$, by Lemma 6.2 it holds

$$\lim_\varepsilon \frac{1}{|\log\varepsilon|}\sum_{i_1=1}^{d_\varepsilon}\sum_{i_2=1}^{d_\varepsilon}\frac{1}{i_1^2+i_2^2} = \frac{\pi}{2}$$

we get

$$\limsup_\varepsilon E_\varepsilon(u_\varepsilon) \le \pi.$$

step 2. Let $M = \sum_{k=1}^n \delta_{x_k}$. Then $u(x) = \prod_{k=1}^n \frac{x-x_k}{|x-x_k|}$, where the product is given by the identification of $\mathbf{R}^2$ with $\mathbf{C}$, is such that $\star J(u) = M$. Then the optimizing sequence is given by

$$u_\varepsilon^i = \begin{cases} u(\varepsilon i) & \text{if } \varepsilon i \ne x_k \quad k\in\{1,2,\dots,n\} \\ u_0 & \text{otherwise,}\end{cases}$$

for some $u_0 \in S^1$. In fact, set $u_k(x) = \frac{x-x_k}{|x-x_k|}$, one can easily prove the following estimate

$$|\nabla u|^2 \le |\nabla u_h|^2 + |\nabla \prod_{k\ne h} u_k|^2 + 2|\nabla u_h||\nabla \prod_{k\ne h} u_k|$$

for all $h\in\{1,2,\dots,n\}$. Note that $|\nabla u_k|$ is bounded in $\Omega\setminus B(x_k,r)$ for any $r>0$. Estimating the contribution of u_ε to the energy around each x_k and using (5.2), (5.3), we get

$$E_\varepsilon(u_\varepsilon) \le \frac{2n}{|\log\varepsilon|}\sum_{i_1=1}^{d_\varepsilon}\sum_{i_2=1}^{d_\varepsilon}\frac{1}{i_1^2+i_2^2} + \frac{C}{|\log\varepsilon|}\Big(1+\sum_{i_1=1}^{d_\varepsilon}\sum_{i_2=1}^{d_\varepsilon}\frac{1}{\sqrt{i_1^2+i_2^2}}\Big).$$

Hence, we conclude that

$$\lim_\varepsilon E_\varepsilon(u_\varepsilon) = \pi n = \pi\|M\|.$$

step 3. In the general case $M = \sum_{k=1}^n d_k\delta_{x_k}$ one can reduce to the previous step by a diagonalization argument. In fact, the function $u(x) = \prod_{k=1}^n \left(\frac{x-x_k}{|x-x_k|}\right)^{d_k}$ is such that $\star J(u) = M$. Then, given $x_l^m \in B(x_k, \frac{1}{m})$, the sequence

$$u_m(x) = \prod_{k=1}^{n} \Big(\prod_{l=1}^{d_k} \Big(\frac{x - x_l^m}{|x - x_l^m|} \Big) \Big)$$

converges to u strongly in $W^{1,1}(\Omega; \mathbf{R}^2)$ and thus $\star J(u_m) \to M$ in the flat norm. The conclusion follows observing that $\star J(u_m)$ is as in step 2. $\square$

6 Appendix B: technical lemmas

For any $y \in [0,1]^N$ and $\varepsilon > 0$, we denote by T_y^ε the operator which maps $u : \mathbf{R}^n \to \mathbf{R}^m$ in $T_y^\varepsilon u : \mathbf{R}^n \to \mathbf{R}^m$ defined as

$$T_y^\varepsilon u(x) = u(\varepsilon y + \varepsilon \left[\frac{x}{\varepsilon} \right]),$$

where, for all $z = (z_1, z_2, \ldots, z_N) \in \mathbf{R}^N$, $[z] = ([z_1], [z_2], \ldots, [z_N])$, $[z_i]$ being the integer part of z_i. Note that $T_y^\varepsilon u$ is constant on each cell of the lattice $\varepsilon \mathbf{Z}^N$ and thus can be identified with a discrete function mapping $\varepsilon \mathbf{Z}^N$ into $\mathbf{R}^m$.

The following approximation result has been used in the proof of the upper bound inequality in Theorems We state it here in a more general form than needed because we think it could be useful for other applications in the discrete to continuum framework.

Lemma 6.1 *Let $u_\varepsilon \to u$ in $W^{1,1}_{loc}(\mathbf{R}^N, \mathbf{R}^m)$. Then, for any compact set K of $\mathbf{R}^N$, there holds*

$$\lim_\varepsilon \int_{[0,1]^N} \|A(T_y^\varepsilon u_\varepsilon) - u\|_{W^{1,1}(K)} \, dy = 0,$$

where $A(T_y^\varepsilon u_\varepsilon)$ is the piecewise affine interpolation of $T_y^\varepsilon u_\varepsilon$ defined in (4.19).

Proof. By Lemma 2.11 in [7], we get

$$\lim_\varepsilon \int_{[0,1]^N} \|T_y^\varepsilon u_\varepsilon - u\|_{L^1(K)} \, dy = 0.$$

Noting that in each cell of the lattice $\varepsilon \mathbf{Z}^N$ $A(T_y^\varepsilon u_\varepsilon)$ is a convex combination of the values of $T_y^\varepsilon u_\varepsilon$ in the nodes of the cell itself, we easily infer that

$$\lim_\varepsilon \int_{[0,1]^N} \|A(T_y^\varepsilon u_\varepsilon) - u\|_{L^1(K)} \, dy = 0.$$

Set $v_y^\varepsilon(x) = A(T_y^\varepsilon u_\varepsilon(x))$, the Lemma is proved if we show that

$$\lim_\varepsilon \int_{[0,1]^N} \|\nabla v_y^\varepsilon - \nabla u\|_{L^1(K)} \, dy = 0.$$

Let us observe that

$$T_y^\varepsilon u_\varepsilon(x) = u_\varepsilon(\varepsilon y + \varepsilon i), \qquad \forall x \in \varepsilon(i + [0,1)^N)$$

and that

$$\frac{\partial v_y^\varepsilon}{\partial x_l}(x) = \frac{u_\varepsilon(\varepsilon y + \varepsilon i + \varepsilon l) - u_\varepsilon(\varepsilon y + \varepsilon i)}{\varepsilon}, \qquad \forall x \in \varepsilon(i + P_l),$$

where P_l is defined as in (4.20). Then, for almost every $y \in [0,1]^N$

$$\frac{\partial v_y^\varepsilon}{\partial x_l}(x) = \int_0^1 \frac{\partial u_\varepsilon}{\partial x_l}(\varepsilon y + \varepsilon i + \varepsilon t e_l)\ dt.$$

Let $K \subset\subset K' \subset\subset K'' \subset\subset \mathbf{R}^N$ be fixed. By using Fubini's Theorem and the change of variables $z = y + i + te_l - \frac{x}{\varepsilon}$ we get, for ε small enough,

$$\begin{aligned}
&\int_{(0,1)^N} dy \int_K \left| \frac{\partial v_y^\varepsilon}{\partial x_l} - \frac{\partial u}{\partial x_l} \right|\ dx \\
&\quad \leq \int_{(0,1)^N} dy \sum_{i \in \mathbf{Z}^N \cap K'} \int_{\varepsilon i + P_l} \left| \frac{\partial v_y^\varepsilon}{\partial x_l} - \frac{\partial u}{\partial x_l} \right|\ dx \\
&\quad \leq \int_{(0,1)^N} dy \sum_{i \in \mathbf{Z}^N \cap K'} \int_{\varepsilon i + P_l} dx \int_0^1 \left| \frac{\partial u_\varepsilon}{\partial x_l}(\varepsilon y + \varepsilon i + \varepsilon t e_l) - \frac{\partial u}{\partial x_l}(x) \right|\ dt \\
&\quad = \sum_{i \in \mathbf{Z}^N \cap K'} \int_{\varepsilon i + P_l} dx \int_0^1 dt \int_{(0,1)^N} \left| \frac{\partial u_\varepsilon}{\partial x_l}(\varepsilon y + \varepsilon i + \varepsilon t e_l) - \frac{\partial u}{\partial x_l}(x) \right|\ dy \\
&\quad \leq \sum_{i \in \mathbf{Z}^N \cap K'} \int_{\varepsilon i + P_l} dx \int_{(-1,2)^N} \left| \frac{\partial u_\varepsilon}{\partial x_l}(\varepsilon z + x) - \frac{\partial u}{\partial x_l}(x) \right|\ dz \\
&\quad \leq \int_{(-1,2)^N} dz \int_{K''} \left| \frac{\partial u_\varepsilon}{\partial x_l}(\varepsilon z + x) - \frac{\partial u}{\partial x_l}(x) \right|\ dx \\
&\quad \leq \int_{(-1,2)^N} dz \int_{K''} \left| \frac{\partial u_\varepsilon}{\partial x_l}(\varepsilon z + x) - \frac{\partial u_\varepsilon}{\partial x_l}(x) \right|\ dx + C \int_{K''} \left| \frac{\partial u_\varepsilon}{\partial x_l}(x) - \frac{\partial u}{\partial x_l}(x) \right|\ dx
\end{aligned}$$

The conclusion follows letting ε go to 0 thanks to the uniform continuity of the translation operator for strongly converging families in L^1. □

The equality stated in the following lemma is obtained by a simple computation.

Lemma 6.2 *Let $(d_\varepsilon)_\varepsilon$ be a family of positive integer numbers such that $d_\varepsilon = O(\frac{1}{\varepsilon})$. Then the following equality holds:*

$$\lim_\varepsilon \frac{1}{|\log \varepsilon|} \sum_{i_1=1}^{d_\varepsilon} \sum_{i_2=1}^{d_\varepsilon} \frac{1}{i_1^2 + i_2^2} = \frac{\pi}{2}.$$

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