A VARIATIONAL APPROACH TO THE STOCK-RECRUITMENT RELATIONSHIP IN FISH POPULATION DYNAMIC

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Abstract. We deal with a variational model of the Stock-Recruitment (S-R) relationship in fish dynamic. In this model the S-R relationship is characterized as a minimizer of a suitable integral energy functional. By basic tools of Calculus of Variations a necessary condition is derived. As application, the derived condition is used to test the equilibrium of the recruitment level. An exploratory numerical procedure is also discussed.

Keywords. Calculus of Variations, Eulero-Lagrange equation, Fish Popoulation Dynamic, Stock-Recruitment relationship.
MSC 2000. 49K05. 49K99. 65K10. 65K99.

1. Introduction

The cycle of regeneration of a population is crucial to maintain the population at a stable level. For fish populations, a crucial step in this cycle is the Recruitment phase, i.e. the age at which the fishes born become vulnerable to the fishing activities. The Stock-Recruitment (S-R) problem concern the relationship between the spawning fish stock biomass (SSB) and the subsequent recruitment. Despite the fundamental importance of the S-R process in fish dynamic, the mathematical modeling of this subject is yet not really satisfactory. The matter is that a typical S-R plot is quite spread and it is not immediately evident if a S-R relationship actually holds or what its mathematical expression has to be. Of course, various models has been proposed to describe the S-R relationship, such as Beverton-Holt model, Ricker model, hockey stick model etc... We refer the reader to [2, 3, 9, 14, 16, 20] and the reference therein for an account on the S-R problem. In this paper we propose a variational approach based on prime principles. A basic observation is that many fish species appear to maintain a constant mean recruitment level, at least in some SSB range. This is the reason for the assumption of constant recruitment in many fish population models. We assume this fact as a prime principle which could be mathematically stated by asking that any S-R relationship, say $r = h(s)$, has a fixed integral mean in a reference SSB range $[s_1, s_2]$. At this step, there are infinitely many possible relations of the type $r = h(s)$ satisfying this principle, and we need some criterion in order to select the function h among all the possible ones. To this aim, we suppose that the biological phenomenon of recruitment, being submitted to natural selection, produces a S-R relationship more advantages as possible. Therefore, in such a way the function h is selected as the best as possible to guarantee the well survivor of the fish stock. Introducing a cost function $W(s, r)$, which denotes the positive cost for the amount s of SSB which yields the recruitment r, we assume as another prime principle that the S-R relationship h is characterized as a minimum, or at least as a critical point, of the following functional

$$J(h) = \int_{s_1}^{s_2} \left(\int_0^{h(s)} W(s, r) \; dr \right) \; ds.$$

In Section 2 we discuss existence and uniqueness of an S-R relationship by using basic tools of the Calculus of Variations. In Section 3 we show as the most common S-R relationships considered in the literature could be characterized by these variational principles. Moreover, we derive the following necessary condition which has to be satisfied by any S-R relationship h:

$$W(s, h(s)) = constant.$$

The above condition is quite simple and could be used to test the equilibrium of the recruitment level. More precisely, under external pressure such as fishing, pollution etc... the fish stock can leave its optimal previous configuration. Although this occurrence could be not detectable, at least not below a certain SSB level, from a direct inspection of the S-R process, it could be evident by inspection of the above necessary condition. Indeed, also if the observed S-R values do not show any detectable trend in the recruitment, it is sufficient to test the validity of the equation $W(s, h(s)) = constant$ to control the state of the recruitment. If this equation is not satisfied, we infer that the optimality condition is violated, and then the previous equilibrium of the S-R process is lost. In Section 4 we describe an exploratory numerical procedure based on this variational framework. Although rudimentary, the analysis performed seems in our opinion promising. Therefore we believe that the variational model proposed deserves to be deeper investigate both from the theoretical and the applicative point of view.

2. The Variational framework

In this section we will introduce the key ideas to model the main problem of relating the SSB to the subsequent recruitment based on the Calculus of Variations. In particular, instead of searching an explicit S-R relationship, we look for some basic theoretic principles in order to develop a variational model. Since we are primarily interested in cases in which there are no detectable trends in the recruitment, we will confine our treatment on the range of the SSB which corresponds to an oscillating recruitment around a medium value r_m. If we denote by $r = h(s)$ the hypothetical functional S-R relationship and $[s_1, s_2]$ is the equilibrium range of SSB, our first principle can be stated as the following

Principle 1. *The function $h(s)$ satisfy the following integral mean condition*

$$\int_{s_1}^{s_2} h(s)\ ds = r_m(s_1 - s_2).$$

At this point we need some criterion in order to select the function h among all the possible ones. To this aim, we suppose that the biological phenomenon of recruitment, being submitted to natural selection, produces a S-R relationship more advantages as possible. Therefore, in such a way the function h is selected as the best as possible to guarantee the well survivor of the stock. Of course, we need some more precise statement in order to translates these hypothesis into a useful mathematical principle. We assume that, whatever the recruitment process actually works, the stock, in his equilibrium configuration, operates by minimizing an *energy-like functional*, or at least by selecting critical points. More precisely, we ask that the S-R relationship realizes the minimum, or at least a critical point, among all other possible configurations. To define a suitable functional we denote by $W(s, r)$ the positive cost for the amount s of SSB which yields the recruitment r. It is reasonable to suppose that the total cost in $[s_1, s_2]$ depends on the area spanned by the function h. In other words, very expensive recruitment levels are possible but only at a very small range of SSB. Therefore, the total cost associated to the function h could be defined as

$$J(h) = \int_{s_1}^{s_2} \left(\int_0^{h(s)} W(s, r)\ dr \right)\ ds.$$

We can now state a second variational principle

Principle 2. *The function $r = h(s)$ for a S-R relationship realizes the minimum, or at least a critical point, of the functional*

$$J(h) = \int_{s_1}^{s_2} \left(\int_0^{h(s)} W(s,r)\ dr \right)\ ds. \tag{2.1}$$

Our main aim is then to derive some consequences from these two prime principles which could be useful to handle with the S-R relationship. Observe that the function h and the cost W in (2.1) are in general unknown. Moreover, the total cost $J(h)$ depends only on the amount of the recruitment. Of course the S-R phenomenon could be also depend on the geometric shape of h, for instance from the rapidity at which variations on the recruitment happens. It is possible to take into account of these requirements by introducing terms in (2.1) which depends on the first derivative h'. For a similar approach applied to the control of industrial producing of flat surfaces see the forthcoming paper [4].

2.1. **The basic equation.** In this section we propose to derive a control equation by using the basic techniques of the Calculus of Variations. For an introduction to this topic we refer for instance to [5, 11, 13, 15, 21]. The problem is to study the behavior of the energy-type functional (2.1) under area preserving perturbations of the function h. In this section we suppose that the functions W and h satisfy all the regularity assumption needed. Consider a smooth function $\eta : [s_1, s_2] \to \mathbb{R}$ such that $\int_{s_1}^{s_2} \eta(s)\ ds = 0$. To fix ideas, it is possible to consider functions of the type $\eta(s) = \sin\left(2k\pi \frac{s-s_1}{s_2-s_1}\right)$. For $\varepsilon > 0$, we consider the perturbations

$$h_\varepsilon(s) = h(s) + \varepsilon\eta(s).$$

With this choice the functions h_ε are admissible according to our Principle 1. We address to evaluate the first variation of the functional J. By definition, the first variation of J is given by

$$\delta J(h,\eta) := \lim_{\varepsilon \to 0^+} \frac{J(h_\varepsilon) - J(h)}{\varepsilon}.$$

Observe that

$$\begin{aligned} \frac{J(h_\varepsilon) - J(h)}{\varepsilon} &= \frac{1}{\varepsilon} \int_{s_1}^{s_2} \left(\int_0^{h_\varepsilon(s)} W(s,r)\ \ dr - \int_0^{h(s)} W(s,r)\ \ dr \right)\ ds \\ &= \int_{s_1}^{s_2} \left(\frac{1}{\varepsilon} \int_{h(s)}^{h_\varepsilon(s)} W(s,r)\ \ dr \right)\ ds. \end{aligned}$$

Letting $\varepsilon \to 0$, by the fundamental theorem of calculus we get

$$\delta J(h,\eta) = \int_{s_1}^{s_2} W(s,h(s))\ \eta(s)\ \ ds.$$

According to our Principle 2, the recruitment processing selects a critical point of J if this first variation vanishes. This is the case if we require the condition

$$W(s,h(s)) = \text{constant} \tag{2.2}$$

since in such case

$$\delta J(h,\eta) = C \int_{s_1}^{s_2} \eta(s)\ ds = 0, \tag{2.3}$$

recalling that the perturbations η are area preserving. Therefore in this case the configuration $h(s)$ is a critical point for J. Vice versa, as standard in Calculus of Variations (see for instance [5, 11, 21]) $\delta J = 0$ for every η implies condition (2.2). Of course, condition

(2.2) can be investigated by the Implicit Function Theorem. However, in such case the function h is defined just locally. Moreover, if $h \in \mathcal{C}^1$, differentiation yields the following equation

$$\partial_s W(s, h(s)) + \partial_r W(s, h(s))h'(s) = 0, \tag{2.4}$$

which could be solved just for special forms of the cost W. Therefore, critical points of J could be in general difficult to find. Anyway, condition (2.2) could be taken as a characterization of the S-R relationship once the cost density function $W(s, r)$ is given, by biological derivation or by experimental inspection. In particular, condition (2.2) could be useful to test if an observed recruitment is in equilibrium with respect to the correspondent SSB. In the next section we will see how the basic S-R relationship considered in literature can be regarded as minimizers of suitable cost functional J.

2.2. **Existence of a S-R relationship.** The above variational paradigm make sense only if optimal configuration h in fact exists. Since we are dealing with an infinite dimensional problem this is not a trivial fact. Applying the so called direct methods of the Calculus of Variations, it is not difficult to find conditions under with such a minimum exists. Basically, the matter is to have a compactness condition on the space of admissible functions h and lower semicontinuity (l.s.) of the functional J. In such case, given a minimizing sequence, i.e. a sequence h_n such that

$$\lim_{n\to+\infty} J(h_n) = \inf J(h) = m,$$

by compactness we find (by passing to a subsequence) an admissible function h such that $h_n \to h$ as $n \to +\infty$. Then, by l.s. we have

$$J(h) \le \liminf_{n\to+\infty} J(h_n) = m.$$

Hence h is a minimizer for the functional J. To apply direct methods, denoting by $L(s, r) = \int_0^r W(s, t)\, dt$, the functional J can be written in the standard form

$$J(h) = \int_{s_1}^{s_2} L(s, h(s))\, ds. \tag{2.5}$$

Typically, to get existence we have to impose conditions on the function $L(s, r)$ and/or on the space of admissible functions h. Note that the functional J is l.s. with respect to the strong convergence of the Lebesgue space $L^p([s_1, s_2])$. Indeed, every convergent sequence $h_n \to h$ in $L^p([s_1, s_2])$ admits a subsequence, denoted again by h_n, such that $h_n(s) \to h(s)$ for almost every (a.e.) $s \in [s_1, s_2]$. Applying the Fatou's Lemma we get

$$J(h) = \int_{s_1}^{s_2} L(s, h(s))\, ds \le \liminf_{n\to+\infty} \int_{s_1}^{s_2} L(s, h_n(s))\, ds = \liminf_{n\to+\infty} J(h_n).$$

Since the above argument can be applied to every subsequence, the l.s. of J follows. Observe that the above argument actually establishes the l.s of J with respect to the pointwise convergence. However, compactness with respect to the strong topology in L^p spaces in general is not easy to recover. Observe that in this framework the condition (2.2) is understood for a.e. $s \in [s_1, s_2]$. Under some conditions on the integrand function, a standard existence result can be stated.

Theorem 2.1 (Existence Theorem I). *Let $W(s, r)$ be a positive continuous cost satisfying the following conditions:*

(1) *(Growth condition) There exists continuous functions $K(s), H(s) > 0$ and $1 < p < +\infty$ such that $L(s, r) \ge H(s)|r|^p - K(s)$;*
(2) *(Convexity condition) Fixed s, the function $L(s, r)$ is convex with respect to the r-variable;*

Then, the functional $J(h)$ admits minimizers on $L^p([s_1, s_2])$. Moreover, every solution of (2.2) is in fact a minimizer of J. If $W(s,\cdot)$ is one-to-one then minimizers are a.e. continuous and there is uniqueness of minimizers of J among continuous functions. Finally, if $L(s,\cdot)$ is strictly convex then we have a unique minimizer of J.

Proof. By the growth condition we infer the compactness with respect to the weak topology of $L^p([s_1, s_2])$. Observe also that the weak convergence preserves the constraint stated in Principle 1. On the other hand, the convexity assumption ensures that the functional J is convex. Since convex and l.s. functional are l.s. with respect to the weak topology, the direct method yields the existence of a minimizer $h \in L^p([s_1, s_2])$ for $J(h)$. By the convexity condition, for every $s, r, \hat{r}$ the following inequality holds

$$L(s,\hat{r}) \geq L(s,r) + \partial_r L(s,r)(\hat{r} - r). \tag{2.6}$$

Fixed two admissible functions $r = h(s), \hat{r} = \hat{h}(s)$, recalling that $L(s,r) = \int_0^r W(s,t)\ dt$, integrating the above inequality we obtain

$$J(\hat{h}) \geq J(h) + \int_{s_1}^{s_2} W(s,h(s)) \left(\hat{h}(s) - h(s)\right)\ ds.$$

Therefore, because of Principle 1, if h satisfy the necessary condition (2.2) then $J(\hat{h}) \geq J(h)$ for every $\hat{h}$, and then h is a minimizer of J. First we show uniqueness among continuous functions. For if, consider h_1, h_2 two continuous minimizers of J. Since h_1, h_2 are both minimizer of J we have

$$J(h_1) = J(h_2) \Rightarrow \int_{s_1}^{s_2} \left(\int_{h_2(s)}^{h_1(s)} W(s,r)\ dr \right)\ ds = 0.$$

By the Integral Mean Value Theorem there exist $s^* \in [S_1, S_2], r_{s^*} \in (h_1(s^*), h_2(s^*))$ such that

$$W(s^*, r_{s^*})(s_2 - s_1)(h_1(s^*) - h_2(s^*)) = 0.$$

Therefore, there exists at least one point s^* such that $h_1(s^*) = h_2(s^*)$. By the necessary condition (2.2), because of continuity we get

$$W(s, h_1(s)) = W(s^*, h_1(s^*)) = W(s^*, h_2(s^*)) = W(s, h_2(s)).$$

Then, by the injectivity of $W(s,\cdot)$ the equality $h_1 = h_2$ follows. Let us now check that minimizers are in fact a.e. continuous. Indeed, consider a sequence such that $s_n \to s$ as $n \to +\infty$. By passing to a subsequence we may assume that $h(s_n) \to l$. By continuity of W we infer $W(s_n, h(s_n)) \to W(s,l)$. Therefore, by using condition (2.2) and the injectivity of W, for a.e. $s \in [s_1, s_2]$ we have $h(s) = l$. Finally, given two minimizers u, v, by convexity it follows that also $\frac{u+v}{2}$ is a minimizer. Therefore we have

$$\int_{s_1}^{s_2} \left(\frac{1}{2}L(s,u) + \frac{1}{2}L(s,v) - L\left(s, \frac{u+v}{2}\right) \right)\ ds = 0.$$

However, by convexity the integrand function is positive. Hence, for almost every $s \in [s_1, s_2]$ it results

$$\frac{1}{2}L(s,u) + \frac{1}{2}L(s,v) - L\left(s, \frac{u+v}{2}\right) = 0.$$

If $L(s,\cdot)$ is strictly convex the above formula leads to a contradiction and then J must admit a unique minimizer. □

The assumptions of Theorem 2.1 can be weakened in several ways, see [12, 15]. Observe that if $W(s,\cdot) \in \mathcal{C}^1$, the strictly convexity assumption on L is equivalent to the condition $\partial_r W > 0$, namely, as one can aspect, the cost W is strictly increasing with respect to variations on the recruitment at a fixed SSB level. Moreover, if $W \in \mathcal{C}^k$ and $\partial_r W \neq 0$, by using the Implicit Function Theorem it results that every solution of (2.2), and then every minimizer of J, is in fact of class $\mathcal{C}^k$. Observe that the assumptions of Theorem 2.1 are conditions on the integrand function $L(s,r)$ and then they impose some restrictions on the expression of the cost function $W(s,r)$. For instance, a suitable growth condition on W could be as the following

$$W(s,r) \geq \alpha(s)r^q + \beta(s)$$

for continuous functions $\alpha(s), \beta(s) > 0$ and $q > 0$. Therefore, since the cost function $W(s,r)$ is an unknown of the model, it could be preferable to make some more assumption on the space of admissible function. To this aim, under minor regularity assumption on $W(s,r)$ (continuity is in fact enough), is not hard to check that the functional J is l.s., in fact continuous, with respect to the pointwise convergence. To get compactness we need some more conditions on the admissible functions. A first condition could be derived by observing that by biological reasons, the recruitment is bounded by a large constant K. Therefore, we may assume to deal with equibounded functions. Moreover, it could be reasonable to aspect that a biological system has a limited capacity to perform variations of his state, for instance in responding to environmental changes. Moreover, since the S-R phenomenon works at a discrete scale, we may assume that the rapidity at which the recruitment changes along the process is bounded by a constant M. Therefore, we could also assume a uniform bound on the first derivative of the admissible functions. Hence, an appropriate space of admissible functions is the space $Lip_M\left([s_1,s_2], \|\cdot\|_\infty\right)$ of Lipschitz functions with Lipschitz constant not greater than M. Then, the compactness is guaranteed by the Ascoli-Arzelà compactness Theorem. Actually, a weaker condition on a p-norm of the first derivative is enough by considering the space $AC([s_1,s_2])$ of absolutely continuous functions. We summarize this discussion in the following

Theorem 2.2 (Tonelli's Compactness Theorem). *Every sequence h_n in $AC([S_1,S_2])$ such that*

$$\|h_n\|_\infty \leq K, \quad \|h_n'\|_p \leq M$$

for $1 < p \leq +\infty$, admits a uniformly convergent subsequence to a function $h \in AC([s_1,s_2])$ satisfying the same bound $\|h\|_\infty \leq K$, $\|h'\|_p \leq M$.

Proof. If $p = +\infty$, h_n is an equibounded and equi-Lipischitz sequence. Then the result follows by the Ascoli-Arzelà Theorem. Consider now the case $1 < p < +\infty$. Let q be the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. For every $s,t \in [s_1,s_2]$, by the Holder inequality we get

$$|h_n(s) - h_n(t)| \leq \int_s^t |h_n'(u)|\, du \leq \left(\int_s^t |h_n'(u)|^p\, du\right)^{\frac{1}{p}} (t-s)^{\frac{1}{q}} \leq M(t-s)^{\frac{1}{q}}.$$

Hence, h_n is an equi-Holder sequence and therefore h_n is equicontinuous. Since h_n is equibounded by hypothesis, the Ascoli-Arzelà Theorem ensures the existence of a subsequence, denoted again by h_n, which converge uniformly to a continuous function h such that $\|h\|_\infty \leq K$. By the condition $\|h_n'\|_p \leq M$, we find a function $g \in L^p([s_1,s_2])$ and a subsequence w_n' of h_n' such that $w_n' \to g$ weakly in $L^p([s_1,s_2])$, namely

$$\forall \varphi \in L^q([s_1,s_2]) : \int_{s_1}^{s_2} w_n'(s)\varphi(s)\, ds \to \int_{s_1}^{s_2} g(s)\varphi(s)\, ds$$

as $n \to +\infty$. For $\bar{s} \in [s_1,s_2]$, consider the characteristic function $\varphi := \chi_{[s_1,\bar{s}]} \in L^q([s_1,s_2])$. Since $w_n \in AC([s_1,s_2])$ we have

$$w_n(\bar{s}) - w_n(s_1) = \int_{s_1}^{\bar{s}} w_n'(s)\, ds = \int_{s_1}^{s_2} w_n'(s)\varphi(s)\, ds.$$

Letting $n \to +\infty$ we get

$$h(\bar{s}) - h(s_1) = \int_{s_1}^{s_2} g(s)\varphi(s)\, ds = \int_{s_1}^{\bar{s}} g(s)\, ds.$$

Therefore $h \in AC([s_1, s_2])$ and $h' = g$ almost everywhere. Since the norm is l.s. with respect to the weak convergence we finally get

$$\|h'\|_p = \|g\|_p \le \liminf_{n\to+\infty} \|w_n'\|_p \le M.$$

□

We then state the following

Theorem 2.3 (Existence Theorem II). *Let $W(s,r) > 0$ be a continuous cost function. Then the functional $J(h)$ defined by (2.1) admits minimizers h on the set*

$$\Lambda = \left\{ h \in AC([s_1, s_2]) \ : \ \|h\|_\infty \le K,\ \|h'\|_p \le M,\ 1 < p \le +\infty,\ \int_{s_1}^{s_2} h(s)\, dS = r_m(s_1 - s_2) \right\}.$$

Proof. The existence follows combining Theorem 2.2 with the fact that J is continuous with respect to the pointwise convergence. □

Although in Theorem 2.3 we have almost no restrictions on the cost function W and we find regular minimizers of the functional J, in general in this case we cannot expect the validity of the condition (2.2). Indeed, in such case the variations $h_\varepsilon = h + \varepsilon\eta$ in general do not belong to Λ.

3. Remarkable examples

In this section we show that the basic S-R relationships considered in the literature can be reformulated in a variational framework.

3.1. **Constant recruitment.** The constant recruitment $h(s) = r_m$ satisfy the necessary condition (2.2) if and only if the cost function $W = W(r)$ does not depend explicitly on the SSB. In such case the constant recruitment is the only possible equilibrium configuration. Therefore, if one aspect that the cost to produce the recruitment has to depend on the underlying SSB, we have to aspect a non-constant, maybe fluctuating around a medium level, S-R relationship.

3.2. **Beverton-Holt.** The classical Beverton and Holt model correspond to the configuration $h(s) = \frac{\alpha s}{1+\beta s}$ where α, β are model parameters. In order to satisfy condition (2.2) a simple choice is for

$$W(s,r) = r\frac{1+\beta s}{s}.$$

Therefore the cost is directly proportional to the recruitment produced and inversely proportional to the underlying SSB. Then, for a given recruitment, the cost is very high at small SSB levels, while it decreases for higher SSB level. This last behavior could be not realistic since one may aspect an increasing cost for high SSB levels to produce the same recruitment level. The parameters α, β could be chosen in order to satisfy our Principle 1. With this choice, observing that $L(s,r) = \frac{1}{2}r^2\left(\frac{1+\beta s}{s}\right)$, by applying Theorem 2.1 it follows that h is the unique minimizer of the functional J.

3.3. **Ricker.** In the Ricker model the S-R relationship has the form $h(s) = \alpha s e^{-\beta s}$ where as above we consider parameters in order to satisfy Principle 1. The obvious choice is for

$$W(s,r) = r\frac{e^{\beta s}}{s}.$$

This cost has the advantage to penalize both low and high level of SSB. As above, the Ricker configuration is the unique minimizer of the functional J.

4. An application attempt

In this section we sketch a quantitative use of the variational model. Our aim is just to illustrate a possible way to use the model to investigate the S-R relationship in fish dynamic. In such a way we propose further developments to realize useful tools responding to the needed of the scientists involved in this field. Therefore, we maintain our discussion to a quite rough level without getting inside in deep statistical or numerical details.

Suppose to have a fish population in its natural equilibrium configuration. This means that if we consider a S-R plot, the correspondent points will be disposed according to our variational principles. Therefore, no detectable trend will be apparent for the recruitment (Principle 1). If the stock is subject to a stress factor, such as an higher fishing pressure, pollution, and so on, we aspect a modification of the stock equilibrium. However, in such case the S-R plot could be not evidence this stress factors, showing no-trend or paradoxically a positive trend in the recruitment as it is actually happened in the collapse of some fish stock in the past. We claim that the necessary condition

$$W(s, h(s)) = constant \tag{4.1}$$

could be useful to investigate these occurrences. In fact, if by biological inspection we recover the expression of the cost function W, and by the S-R plot we infer the expression of h, then we can test the validity of (4.1). Actually, if (4.1) is not satisfied, this means that some of the variational principles of the model are violated. In particular the recruitment configuration leaves its optimal configuration (Principle 2) despite to a no detectable trend in the S-R plot (Principle 1).

There are several difficulties into realize the above program. In fact, by starting from a stock equilibrium configuration, the matter is to infer a suitable expression for the cost function which has to be constant on the configuration $r = h(s)$. This step of course requires a large number of S-R plots to test the cost W. However, in general for a given fish stock just few data are available. This obstacle could be surrounded by a numerical simulator. Here we use the ALADYM (actually ALADYM-r) model developed in [1]. ALADYM is a single species simulator which, by starting from the main parameters of the population (growth parameters, natural and fishing mortality, etc...), produces the evolution of a simulated population for the time prescribed by the user. The relative software is licensed as open source under GPL2 and can be freely downloaded from the Fisboat web-site (www.ifremer.fr/drvecohal/fisboat). We refer to [1] and to the software documentation for the details concerning the ALADYM model. Here we have chosen parameter which could be representative of female of M. Merluccius, generating an evolution population over a period of twenty years with two different fishing efforts (corresponding to the parameter *Fishing coefficient* F of the ALADYM model equal to 1 or to 1.5). Anyway, for reader convenience we provide an example of input sheet for the ALADYM model in the Appendix. Afterwords, we use the generated population to estimate the SSB and the subsequent recruitment. To this aim we have used another software tool called SURBA based on [7]. For the description of the SURBA model we refer the reader also to [10, 17, 18]. SURBA is currently under development and test versions are available from the Marine Laboratory, Aberdeen. (Contact Coby Needle: needlec@marlab.ac.uk).We have limited our simulation to 15 age classes. We provide an example of input sheet for the SURBA model in the Appendix. In particular, we have chosen the SSB and recruitment estimate evaluated by the 50-th percentile of bootstrapped runs performed by SURBA. In this way we have generated various S-R plot corresponding to the two different fishing coefficient F. Here, see Figure 4.1 and Figure 4.2, we furnish some of these plot (the values are rescaled by the factor 10^6). To measure the trend on the Recruitment

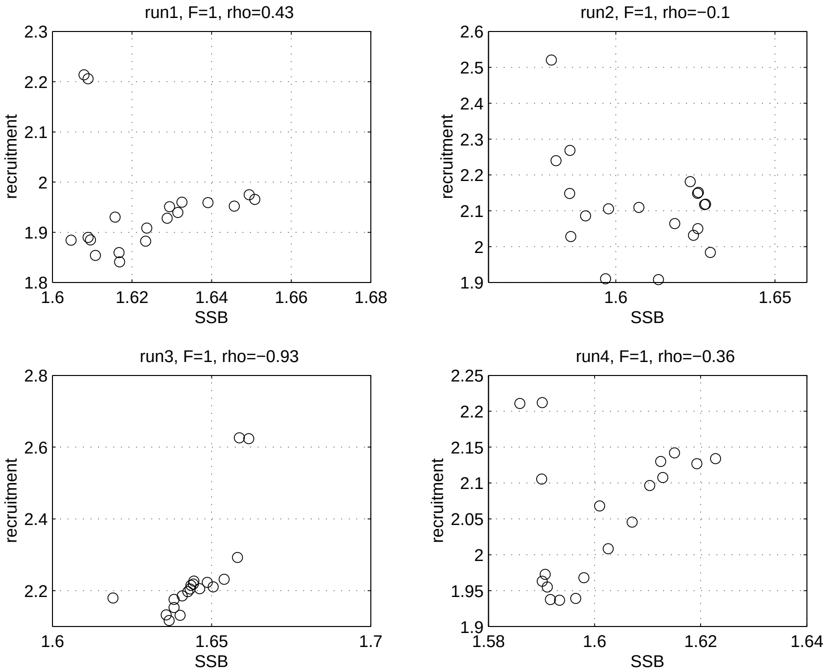

FIGURE 4.1. Four different S-R plot with $F = 1$

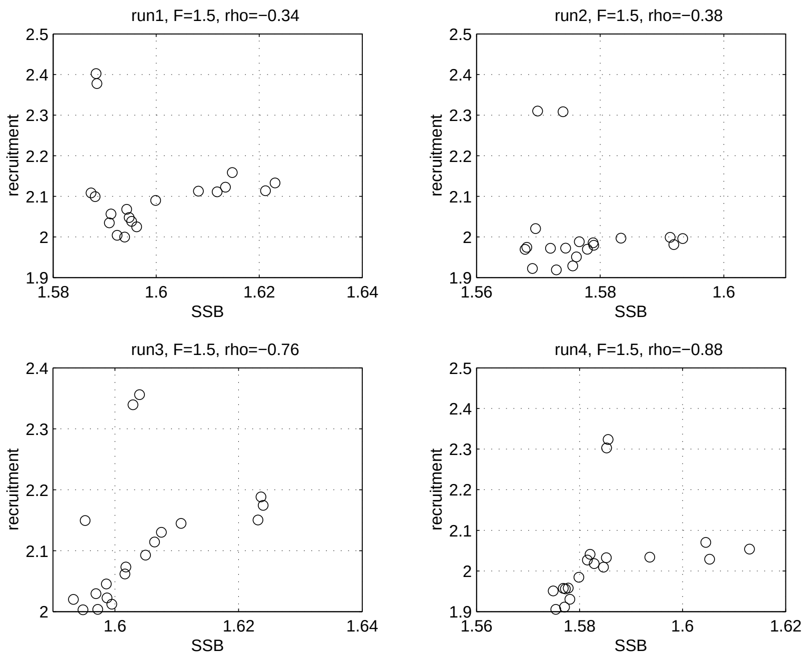

FIGURE 4.2. Four different S-R plot with $F = 1.5$

we have chosen the non-parametric Spearman's rank correlation coefficient, denoted by ρ.

We refer to [8] for a preliminary discussion on the subject. The ρ coefficient satisfy the inequality $-1 \leq \rho \leq 1$ and furnishes a measure of how well an arbitrary monotonic function describes the relationship between two variables. Here we confine ourself to observe that values of ρ close to 1 or -1 denotes an increasing or decreasing trend, otherwise we may assume that no trend is detectable. Here we calculate the ρ coefficient by the simplified formula

$$\rho = 1 - \frac{6 \sum_{i=1}^{n} d_i^2}{n(n^2-1)} \tag{4.2}$$

where $d_i = x_i - y_i$ is the difference between the ranks of the corresponding values s, r of the S-R plot, while n is the total number of points of the plot. Therefore, despite the changing in the fishing pressure, in general there is no detectable trend in the recruitment (see Figure 4.1 and 4.2). At this point we want to use condition (4.1) to get a more sensitive analysis. Indeed, if the stock leaves for some reason its equilibrium configuration, then plotting the values given by (4.1) we aspect to observe a remarkable trend. Suppose for instance that the first plot in Figure 4.1 corresponds to an equilibrium configuration. We have to choose a cost function satisfying (4.1). Here we make very simple choices in order to use just basic routine of calculus programs such as MATLAB or R software. Usually, just few points on a S-R plot are available. However, condition (4.1) is a global condition. Therefore it would be more accurate to infer an expression of the configuration $r = h(s)$ and then to evaluate h on a given grid of points which will remain fixed for the evaluation of the configurations coming from all the other plot. Therefore we have fixed a grid of ten points by the MATLAB command $z = linspace(x(1), x(end), 10)$, where x denotes the abscissa vector of the points belonging to the first plot of Figure 4.1, to generate the vector z. Then we recover the configuration h by a simple linear interpolation. The evaluation of h on the grid z could be done by the MATLAB command $h_1 = interp1(x1, y1, z, 'linear')$ where x_1, y_1 denotes the coordinates of the points of the first plot. We have now to choose a function W such that evaluating W on h_1 returns a same constant value. A simple choice is a polynomial having h_1 as vector of roots. This choice is problematic from a numerical point of view but is immediate by the MATLAB command $w = poly(h_1)$ which furnishes the coefficients w of such polynomial. Of course here we have to suppose that such a polynomial is an approximation in some sense of a suitable cost function W. Anyway, if also the other plot in Figure 4.1 correspond to equilibrium configurations, then our choice is meaningful if and only if the plot of points $(z_i, W(h(z_i)))$ does not present any detectable trend. By evaluating $h(z_i)$ as above by linear interpolation, and the values $W(h(z_i))$ by the MATLAB command $polyval(w, h(z_i))$, we can evaluate the Spearman's coefficient for the obtained new plot. Of course, we aspect that this procedure will produce a detectable trend starting from the plot of Figure 4.2, showing that the previous equilibrium configuration of the stock is lost. The results obtained are summarized in Figure 4.3 and Figure 4.4. The plot presented are representative of what happens in general run, i.e. more spread plot with a low ρ coefficient in the $F = 1$ case, a monotonic plot with high ρ factor in the $F = 1.5$ case. Therefore, Figure 4.3 and Figure 4.4 could suggest a compromised S-R process despite no trend is detectable from Figure 4.1 and Figure 4.2. Here we have restricted to run which involve range of SSB overlapping with the interval corresponding to z. As one can aspect, otherwise the plot obtained are not accurate.

Therefore, although rudimentary, in our opinion these exploratory evaluations seem promising. Hence we believe that the variational model proposed deserves to be deeper investigated both from the theoretical and applicative point of view.

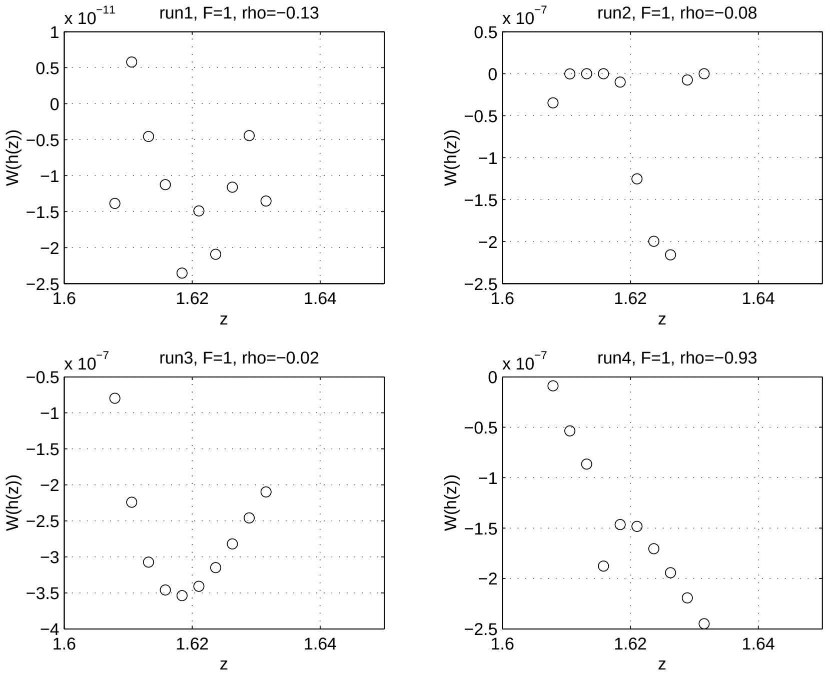

Figure 4.3

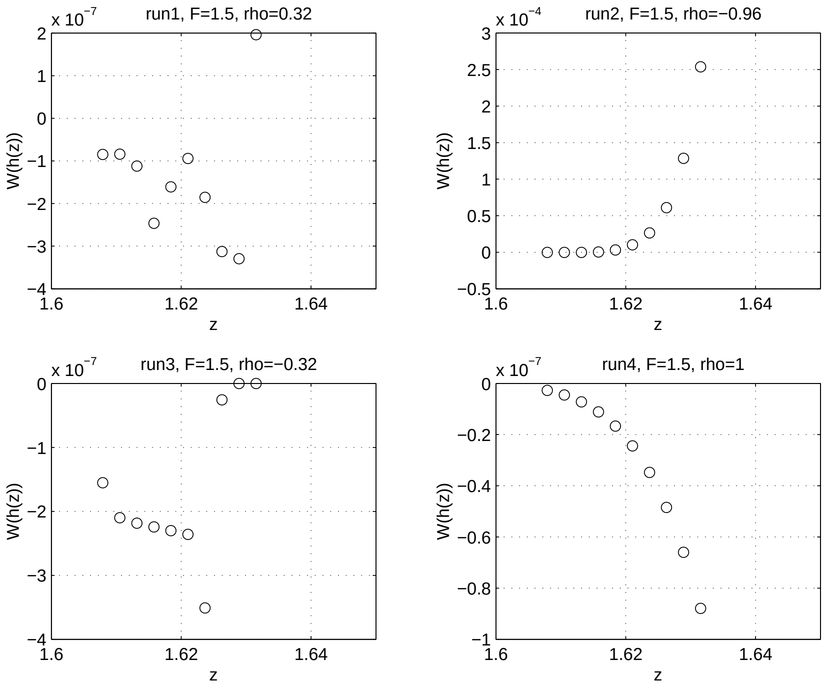

Figure 4.4

5. Discussion

We have proposed a variational model of the Stock-Recruitment (S-R) relationship in fish dynamic. In this model the S-R relationship is characterized as a minimizer of the integral energy functional (2.1). By basic tools of Calculus of Variations we derived the necessary condition (2.2) which a Stock-Recruitment relationship has to satisfy. As application, this necessary condition is used to test the equilibrium of the recruitment level. Precisely, an exploratory numerical procedure is performed by using some software tools to produce simulated data relative to a fish population corresponding to different fishing impact. The calculation proposed seems to show that by increasing the fishing impact the necessary condition (2.2) is violated despite no trend is detectable by the Stock-Recruitment configuration. Therefore, the method outlined in the paper could be useful to test the equilibrium of the Recruitment level in apparent stationary configurations. Therefore, we believe that the variational model proposed deserves to be deeper investigate both from the theoretical and the applicative point of view.

6. Appendix

We provide an example of input sheets for the ALADYM and SURBA runs.

6.1. **Input for ALADYM.** The Input data for the ALADYM model are collected in an EXCEL sheet. Here we present the parameters utilized. In particular we have chosen the natural mortality according to the Chen-Watanabe model (see [6]) and a constant recruitment in order to built the simulated population.

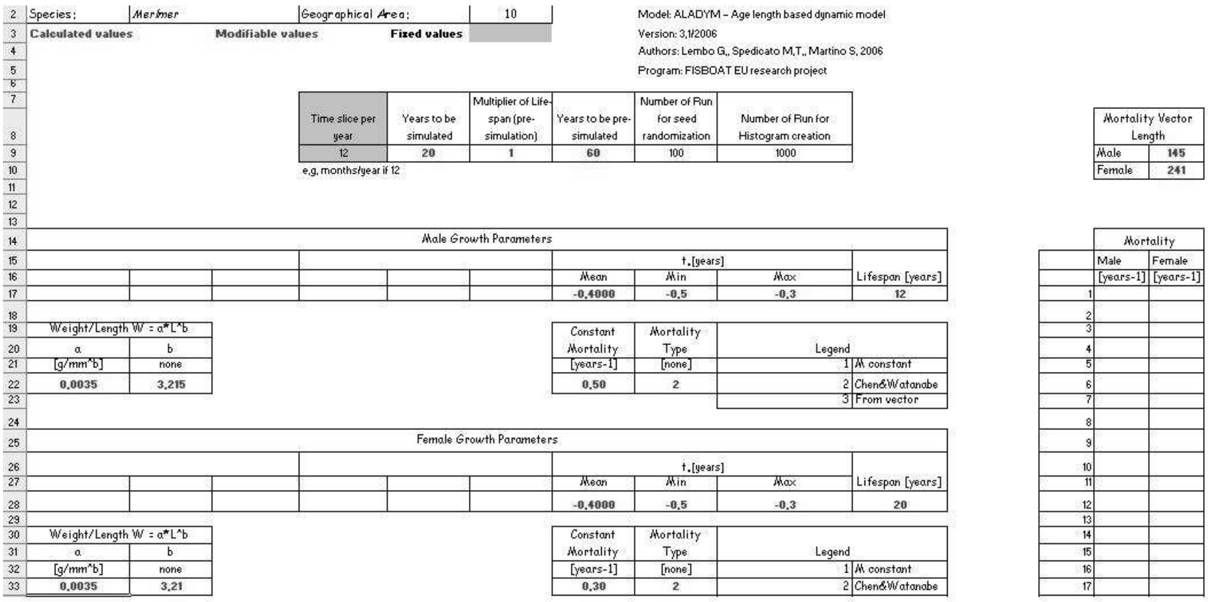

Figure 6.1. ALADYM inputs

6.2. **Input for SURBA.** The Input data for the SURBA (we have used the version SURBA 2.20) model could be collected in a text file .dat. Here we present the parameters utilized. We have utilized the female exploited population coming from the ALADYM model. In particular, since ALADYM works at a month scale, just to simplify the data interpretation, we have converted the population data to a year scale to produce the index for the SURBA model. We run Surba with constant vectors at age for natural mortality, maturity and stock weights, using the estimates generated by the ALADYM model. Moreover we have chosen a smoothing factor equal to 1 and a λ factor equal to zero in every SURBA run.

Acknowledgments. The author wishes to thank G. Lembo and M.T. Spedicato for useful discussions on the subject and for attracting its attention to the ALADYM and SURBA model during a stay of the author at COISPA *Tecnologia e Ricerca*, Bari (Italy).

	A	B	C	D	E	F	G	H	I	J
55										
56	Proportion of offspring/month									
57		January	February	March	April	May	June		Delay for SS calculation	
58	Total	0,10	0,30	0,10	0,05	0,15	0,10			
59		July	August	September	October	November	December		[month]	
60	1,00	0,10	0,05	0,05	0,00	0,00	0,00		1	
61										
62										
63	Time slice Parameters		(total rows) =	240	+ seed run					
64										
65		QZ				Offspring Variability		Offspring	Sex Ratio	Fishing
66	Parameter	Male	Female			-%	+%		Female/Total	Coefficient
67		[years-1]	[years-1]			none	none	none	none	none
68	seed run	1,0	1,0			20,00	20,00	10000000	0,50	1,50
69	1 month	1,0	1,0			20,00	20,00	10000000	0,50	1,50
70	2 month	1,0	1,0			20,00	20,00	10000000	0,50	1,50
71	3 month	1,0	1,0			20,00	20,00	10000000	0,50	1,50
72	4 month	1,0	1,0			20,00	20,00	10000000	0,50	1,50
73	5 month	1,0	1,0			20,00	20,00	10000000	0,50	1,50
74	6 month	1,0	1,0			20,00	20,00	10000000	0,50	1,50
75	7 month	1,0	1,0			20,00	20,00	10000000	0,50	1,50
76	8 month	1,0	1,0			20,00	20,00	10000000	0,50	1,50
77	9 month	1,0	1,0			20,00	20,00	10000000	0,50	1,50
78	10 month	1,0	1,0			20,00	20,00	10000000	0,50	1,50

Figure 6.2. ALADYM inputs

Parameters for the random runs					
Parameter	Distribution	Min	Max	A	B
Offspring R [none]	1	5000000	15000000	18,30	0,55
Male Growth K [years^{-1}]	3	0,200	0,300	0,250	0,0700
Female Growth K [years^{-1}]	3	0,130	0,140	0,147	0,0700
Male Growth $L_{infinity}$ [mm]	3	480,00	520,00	586,00	28,30
Female Growth $L_{infinity}$ [mm]	3	950,00	1.000,00	884,00	35,30
Male Maturity Ogive L50% [mm]	4	180,00	200,00	0,00	0,00
Male Maturity Ogive L75%-L25% [mm]	4	11,00	12,00	0,00	0,00
Female Maturity Ogive L50% [mm]	4	319,00	339,00	0,00	0,00
Female Maturity Ogive L75%-L25% [mm]	4	11,00	12,00	0,00	0,00

Legend			
	Distribution	A	B
1	Lognormal	Mean ln(x)	Ds ln(x)
2	Gamma	Shape	Scale
3	Normal	Mean (x)	Ds (x)
4	Uniform	None	None

Figure 6.3. ALADYM inputs

References

[1] A. Abella, F. Fiorentino, G. Lembo, S. Martino, M.T. Spedicato, Aladym: An age and length-based single species simulator for exploring alternative management strategies, Aquatic Living Resources **22** (2009), 233-241.

[2] N.J. Barrowman, R.A. Myers, Still more spawner-recruitment curves: the hockey stick and its generalizations. Can. J. Fish Aquat. Sci., **57** (2000), 665-676.

[3] N.J. Barrowman, R.A. Myers, Is fish recruitment related to spawner abundance? Fish. Bull. **94** (1996), 707-724.

[4] A. Brancolini, M. Chimienti, M. Dassisti, L. Granieri, F. Maddalena, Variational techniques for assessing technological memory on flat surfaces, work in progress.

Select Gear	2	
1	trawler-ogive	S(L)= 1 / (1 + exp((log(9) / (L75p - L25p) * (L50p - L)))
2	trawler-ogive-des	S(L)=1/(1+exp((2*ln(3)/SR)*(L50%-L))*1/(1+exp((-2*ln(3)/DSR)*(D50%-L)

Parameter	L50%	Selection range (L75%-L25%)	D50%	L75%	D75%	DSR	Male Z Observed	Female Z Observe	Fmax
	[mm]	[mm]	[mm]	[mm]	[mm]	[mm]	[years-1]	[years-1]	[years-1]
seed run	100,00	36	500	118	482	36			
1 month	100,00	36	500	118	482	36			
2 month	100,00	36	500	118	482	36			
3 month	100,00	36	500	118	482	36			
4 month	100,00	36	500	118	482	36			
5 month	100,00	36	500	118	482	36			
6 month	100,00	36	500	118	482	36			
7 month	100,00	36	500	118	482	36			
8 month	100,00	36	500	118	482	36			
9 month	100,00	36	500	118	482	36			
10 month	100,00	36	500	118	482	36			
11 month	100,00	36	500	118	482	36			
12 month	100,00	36	500	118	482	36			
13 month	100,00	36	500	118	482	36			
14 month	100,00	36	500	118	482	36			
15 month	100,00	36	500	118	482	36			
16 month	100,00	36	500	118	482	36			

FIGURE 6.4. ALADYM inputs

```
---Title--------------------------------------
Surba test merlmer female
---Number of ages-----------------------------
15
---Number of years----------------------------
20
---First age----------------------------------
0
---First year---------------------------------
2000
---plus group flag 1 = plus group-------------
0
---Start and end period of survey-------------
0.5 0.75
--- Index--------------------------------------
2297036.51  848042.68  485646.33  321408.28  229515.18  171373.52  135596.69  109491.89  86147.52
2343059.18  852910.94  487644.92  322221.73  229809.63  171675.80  136063.08  110329.13  86442.87
2337807.86  850473.01  487922.32  322383.98  229828.19  172200.86  136042.43  110745.79  87480.75
2324308.52  845875.45  486981.07  321852.63  229446.88  171843.50  135897.52  110681.17  87784.95
2342729.25  845916.78  487231.67  321876.07  229396.30  171836.63  136551.51  110793.10  89296.70
2332519.10  844491.20  487278.35  321882.29  229361.86  171623.52  137000.77  111238.52  88594.42
2331061.54  840957.85  486996.19  321665.69  229196.75  170774.06  136921.90  111260.53  88613.74
2325943.10  835769.17  485789.27  320949.85  228724.00  170358.37  137266.80  111192.04  88566.78
2319486.98  831469.89  484437.47  320116.74  228172.57  169933.18  137053.36  111526.49  87438.53
2296557.28  824911.18  482104.37  318741.69  227282.06  169327.91  136722.88  111369.26  86646.41
2277529.63  817966.41  479362.09  317065.45  226206.37  169459.02  136422.14  111234.70  86534.33
2261226.68  811116.58  476494.12  315275.39  225075.33  168856.21  136690.02  110633.59  86424.56
2270372.42  807671.00  475210.85  314298.03  224413.94  168560.26  136589.67  110656.59  87153.67
2341277.19  810172.03  476609.17  315273.74  224817.78  168823.70  136399.99  110611.10  87776.25
2348967.19  812009.90  476094.32  315922.46  225153.51  168691.70  135168.01  110162.83  87794.12
2330278.84  812303.43  474784.22  315969.80  225152.65  168413.22  134305.69  110050.73  86624.40
2343927.13  815381.24  474908.04  316357.51  225352.02  168369.86  134211.33  110098.23  86710.16
2345556.91  814694.45  474450.49  316604.99  225487.02  168874.62  133496.96  110044.85  86385.25
2349068.07  816058.31  473059.52  316768.41  225576.01  168759.87  133280.50  109979.00  86427.85
2329097.60  817559.56  471102.58  316474.60  225381.51  167682.97  132954.37  109817.74  87086.60
```

FIGURE 6.5. SURBA inputs

[5] G. Buttazzo, M. Giaquinta, S. Hildebrandt, One-Dimensional Variational Problems. An Introduction. Oxford Lecture Series in Mathematics and its Applications, 15. The Clarendon Press, Oxford University Press, New York, 1998.

[6] S. Chen, S. Watanabe, Age dependence of natural mortality coefficient in fish population dynamics. Nippon Suisan Gakkaishi, **55**(2) (1989), 205-208.

[7] R.M. Cook, Stock trends in six North Sea stocks as revealed by an analysis of research vessel surveys, ICES Journal of Marine Science, **54** (1997), 924-933.

[8] G. Corder, D.I. Foreman, Nonparametric Statistic for Non-Staticians: A Step-by-Step Approach. Wiley, 2009.

[9] R.B. Deriso, T.J. Quinn, Quantitative Fish Dynamics. Oxford University Press, Oxford, 1999.

[10] A.J.R. Cotter, R.J.B. Fryer, B. Mesnil, C.L. Needle, V.M. Trenkel, A review of fishery independent assessment models, and initial evaluation based on simulated data, Aquatic Living Resources **22** (2009), 207-216.

[11] B. Dacorogna, Introduction to the Calculus of Variations. Imperial College Press, 2004.

[12] I. Fonseca, G. Leoni, Modern Methods in the Calculus of Variations: L^p spaces. Springer, 2007.

```
--age weights----------------------------------------
1   1.00   1.00   1.00   1.00   1.00   1   1 1 1 1 1 1 1 1
---Default catchabilities----------------------
1   1.00   1.00   1.00   1.00   1   1 1 1 1 1 1 1 1 1
---Mean F range--------------------------------
1    6
--- num yrs for mean F. M. W. Mat. Rec. Forecasts-
3        3        3      3       3        2
---Nat. morta at age---------------------------
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
1.242  0.610  0.422  0.333  0.281  0.247  0.224  0.207  0.194  0.184  0.176  0.169  0.165  0.161  0.157
---Maturity at age-----------------------------
0.00000 0.00000 0.09970 0.99998 1.00000 1.00000 1.00000 1 1 1 1 1 1 1 1
0.00000 0.00000 0.09970 0.99998 1.00000 1.00000 1.00000 1 1 1 1 1 1 1 1
0.00000 0.00000 0.09970 0.99998 1.00000 1.00000 1.00000 1 1 1 1 1 1 1 1
0.00000 0.00000 0.09970 0.99998 1.00000 1.00000 1.00000 1 1 1 1 1 1 1 1
0.00000 0.00000 0.09970 0.99998 1.00000 1.00000 1.00000 1 1 1 1 1 1 1 1
```

Figure 6.6. SURBA inputs

```
---Stock weights at age -----------------------
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
0.005  0.045  0.143  0.305  0.523  0.788  1.084  1.399  1.722  2.044  2.357  2.657  2.940  3.204  3.448
```

Figure 6.7. SURBA inputs

[13] M. Giaquinta, S. Hildebrandt, Calculus of Variations. I. The Lagrangian Formalism. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 310. Springer-Verlag, Berlin, 1996.

[14] D.J. Gilbert, Towards a new recruitment paradigm for fish stocks. Can. J. Fish. Aquat. Sci. **54** (1997): 969-977.

[15] J. Jost, X. Li-Jost, Calculus of Variations. Cambridge University Press, 1999.

[16] R.A. Myers, Comment and reanalysis: paradigms for recruitment studies, Can. J. Fish. Aquat. Sci. **54** (1997), 978-981.

[17] C.L. Needle, Survey-based assessments with SURBA, Working Document to the ICES Working Group on Methods of Fish Stock Assessment, Copenhagen, 2003.

[18] C.L. Needle, SURBA 3.0: Technical Manual (first draft), 2005.

[19] A. Quarteroni, F. Saleri, Introduzione al Calcolo Scientifico. Esercizi e Problemi Risolti con MATLAB. Springer, 2006.

[20] P. Sparre, S.C. Venema, Introduction to Tropical Fish Stock Assessment. Part 1, manual. FAO Fish. Techn. Pap., **306**(1) Rev. 2, 1998.

[21] J. Troutman, Variational Calculus with Elementary Convexity. Springer, 1983.

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