A NEW APPROXIMATION RESULT FOR BV-FUNCTIONS

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Abstract -This note is devoted to obtain an approximation result for BV-functions by means of a *quasi-polyhedral* sequence of BV-functions. This approximation could have interesting applications in some problems of the Calculus of Variations.

1. Introduction

In this note, we prove an approximation result for BV-functions. In this direction, the first classical result is due to Anzellotti-Giaquinta, who proved that BV-functions can be approximated by means of smooth functions which are essentially obtained by mollification, so that their main interest lies on the $\mathcal{C}^\infty$-regularity.

In the framework of the Calculus of Variations, this type of approximation has been usefully applied in various problems concerning relaxation and Γ-convergence, where $\{u_\varepsilon\}$ plays the role of the "recovery sequence".

However, in some recent problems it seems more useful to have an approximation of BV-functions which takes into account not so much the $\mathcal{C}^\infty$-regularity of the approximating sequence, as the geometric properties of its discontinuity set.

This idea has been firstly developped by Dibos and Séré, in the context of the approximation of minimizers for the Mumford-Shah functional (see [5]). They proved an approximation result for SBV-functions by means of functions, still belonging to SBV, with their jump set contained in a finite union of smooth hypersurfaces, included in hyperplanes.

Following the outlines of Dibos and Séré, Cortesani and Toader in [4] (see also [3]) proved that those functions, which have a polyhedral jump set and are of class $\mathcal{C}^\infty$ outside, are dense in $SBV^p \cap L^\infty$, in an appropriate sense connected with the Mumford-Shah functional.

In view of similar possible applications, we propose a new approximation result for general BV-functions, which therefore could also be of Cantor type. This implies that the approximating functions are slightly more general than those proposed by Cortesani and Toader. More precisely, given $u \in BV \cap L^\infty$ we construct a sequence of BV-functions, strictly converging to u, such that their set of approximate discontinuity points is "almost" a polyhedron, in the sense that the $\mathcal{H}^{N-1}$-measure of the non-polyhedral part is small. This result is obtained by refining a classical theorem, due to Federer, of approximation of a countable $\mathcal{H}^{N-1}$-rectifiable set by means of smooth compact manifolds which are arbitrarily close to a polyhedron, apart from a set of small $\mathcal{H}^{N-1}$-measure.

2. Preliminaries

In the following, we assume that $N \geq 2$ is a fixed integer. Given $x_0 \in \mathbb{R}^N$ and $r > 0$, $B_r(x_0)$ denotes the ball in $\mathbb{R}^N$ centered in x_0 with radius r. Let Ω be an open bounded set in $\mathbb{R}^N$. We denote by $\mathcal{M}(\Omega;\mathbb{R}^N)$ the space of the $\mathbb{R}^N$-valued Radon measures on Ω. Let $\mathcal{L}^N$ be the Lebesgue measure on $\mathbb{R}^N$, and $\mathcal{H}^{N-1}$ be the Hausdorff measure of dimension $N-1$ on $\mathbb{R}^N$. We denote by ω_N the Lebesgue measure of the unit ball in $\mathbb{R}^N$, so that $\mathcal{L}^N(B_r(x_0)) = \omega_N r^N$, and by S^{N-1} the spherical surface of the unit ball.

Let $u \in L^1_{\rm loc}(\Omega)$; we say that u has an *approximate limit* at $x \in \Omega$, if there exists a unique value $\widetilde{u}(x) \in \mathbb{R}$ such that

$$\lim_{r\to 0^+} \fint_{B_r(x)} |u(y) - \widetilde{u}(x)|\, dx = 0\ , \tag{2.1}$$

where $\fint_{B_r(x)}$ stands for $\frac{1}{\mathcal{L}^N(B_r(x))}\int_{B_r(x)}$. Let S_u be the set of points where the previous property does not hold, the so-called *approximate discontinuity set.* Note that S_u is a Borel set and $\widetilde{u} : \Omega \setminus S_u \to \mathbb{R}$ is a Borel function. Clearly, if x is a Lebesgue point of u, then (2.1) holds with $\widetilde{u}(x)$ replaced by $u(x)$. Moreover, we recall that $\mathcal{L}^N$-almost every $x \in \Omega$ is a Lebesgue point of u.

The space $\mathrm{BV}(\Omega)$ is defined as the space of those functions $u : \Omega \to \mathbb{R}$ belonging to $L^1(\Omega)$ whose distributional gradient Du is an $\mathbb{R}^N$-valued Radon measure with total variation $|Du|$ bounded in Ω. We indicate by $D_a u$ and $D_s u$ the absolutely continuous and the singular part of the measure Du with respect to the Lebesgue measure. We recall that $D_a u$ and $D_s u$ are mutually singular, moreover we can write

$$Du = D_a u + D_s u \qquad \text{and} \qquad D_a u = \nabla u\ \mathcal{L}^N,$$

where ∇u is the Radon-Nikodým derivative of $D_a u$ with respect to the Lebesgue measure. In particular,

$$D_s u = D_c u + (u^+ - u^-)\nu_u\ \mathcal{H}^{N-1}\lfloor_{S_u}$$

where S_u is defined above and it can be decomposed into two subsets J_u and $S_u \setminus J_u$, with J_u countable $\mathcal{H}^{N-1}$-rectifiable and $\mathcal{H}^{N-1}(S_u \setminus J_u) = 0$. The set J_u is the so-called jump set of u. Finally, $D_c u$ is the Cantor part of Du.

Let us recall a useful property for the composition of BV-functions with Lipschitz functions.

THEOREM 2.1 (see, [2, Theorem 3.16]). *Let Ω, Ω' be open subsets of $\mathbb{R}^N$ and $\phi : \Omega' \to \Omega$ be a bijective Lipschitz function, whose inverse is Lipschitz, too. Let $u \in \mathrm{BV}(\Omega)$ and $v = u \circ \phi$. Then $v \in \mathrm{BV}(\Omega')$ and*

$$\frac{1}{[\mathrm{Lip}(\phi)]^{N-1}} |Du|\big(\phi(B)\big) \leq |Dv|(B) \leq [\mathrm{Lip}(\phi^{-1})]^{N-1} |Du|\big(\phi(B)\big)$$

for every Borel subset B of Ω'.

For a general survey on measures and BV-functions we refer to [2], [6], [7], [8], [9].

3. Main results

In the first theorem, we improve a fine property of countable $\mathcal{H}^{N-1}$-rectifiable sets stated in [7, Theorem 4.2.19] (see also [1, Theorem 3.2]). This result is the crucial tool in order to obtain the quasi-polyhedral approximation of BV-functions, which will be stated in Theorem 3.3.

THEOREM 3.1. *Let B_R be a ball in $\mathbb{R}^N$ and $S \subset B_R$ be a countable $\mathcal{H}^{N-1}$-rectifiable set. Then for every $\varepsilon > 0$ there exists a diffeomorphism $\phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ satisfying the following properties:*

(i) $\phi_\varepsilon : B_R \to B_R$ *and, outside* B_R, $\phi_\varepsilon = Id$, *(where* Id *denotes the identity function* $Id(x) = x$*);*

(ii) $\mathrm{Lip}(\phi_\varepsilon),\ \mathrm{Lip}(\phi_\varepsilon^{-1}) \leq 1 + \varepsilon$*;*

(iii) there exists a polyhedron $K_\varepsilon \subset B_R$*, composed by a finite number of* $(N-1)$*-cubes, such that*

$$\mathcal{H}^{N-1}(S \bigtriangleup \phi_\varepsilon(K_\varepsilon)) < \varepsilon \ ;$$

(iv) as $\varepsilon \to 0^+$*,* $\phi_\varepsilon \to Id$*, uniformly on* $\mathbb{R}^N$*;*

(v) as $\varepsilon \to 0^+$*,* $|J(\phi_\varepsilon^{-1})|$ *and* $|J(\phi_\varepsilon)|$ *tend to* 1 *uniformly on* Ω*, where* $J(\phi_\varepsilon^{-1})$ *and* $J(\phi_\varepsilon)$ *denote the determinants of the Jacobian matrices of* ϕ_ε^{-1} *and* ϕ_ε*, respectively.*

Proof. The existence of the diffeomorphism ϕ_ε with the properties (i)–(iii) is proved in [7, Theorem 4.2.19] (see also, [1, Theorem 3.2]). We will show (iv) and (v). By (ii), it follows

$$\frac{1}{1+\varepsilon}|x-y| \leq |\phi_\varepsilon(x) - \phi_\varepsilon(y)| \leq (1+\varepsilon)|x-y|$$

and

$$|\phi_\varepsilon(x)| \leq |\phi_\varepsilon(x) - \phi_\varepsilon(y_0)| + |\phi_\varepsilon(y_0)| \leq (1+\varepsilon)|x-y_0| + |y_0| \leq 2R + 3|y_0|$$

where $x \in B_R$ and $y_0 \in \mathbb{R}^N \setminus B_R$; the same holds for ϕ_ε^{-1}. Hence, the sequences $\{\phi_\varepsilon\}$, $\{\phi_\varepsilon^{-1}\}$ are equibounded and equicontinuous, then there exist $\phi_0,\ \psi_0 : \mathbb{R}^N \to \mathbb{R}^N$ such that, up to a subsequence, $\phi_\varepsilon \to \phi_0$ and $\phi_\varepsilon^{-1} \to \psi_0$ uniformly on B_R (and so on the whole of $\mathbb{R}^N$, since $\phi_\varepsilon = \phi_\varepsilon^{-1} = Id$ outside B_R).
It is easy to see that $\phi_0(\psi_0(x)) = \psi_0(\phi_0(x)) = x$, so that $\psi_0 = \phi_0^{-1}$. Moreover,

$$|\phi_0(x) - \phi_0(y)| = |x-y| \ , \tag{3.1}$$

which implies that ϕ_0 (and hence ϕ_0^{-1}, too) is a linear isometric map. Indeed, assume firstly that $\phi_0(0) = 0$; then, by (3.1), we have

$$|\phi_0(x)| = |x| \ . \tag{3.2}$$

Equalities (3.1) and (3.2) implies also that

$$\langle x, y \rangle = \langle \phi_0(x), \phi_0(y) \rangle \ , \tag{3.3}$$

where $\langle \cdot, \cdot, \rangle$ denotes the usual scalar product in $\mathbb{R}^N$; hence, if $\{e_i\}_{i=1,\ldots,N}$ is an orthonormal basis in $\mathbb{R}^N$, then $\{\phi_0(e_i)\}_{i=1,\ldots,N}$ is an orthonormal basis in $\mathbb{R}^N$, too. Given $x = \sum \alpha_i e_i$, we can write $\phi_0(x) = \sum \beta_i \phi_0(e_i)$, where, by (3.3),

$$\alpha_i = \langle x, e_i \rangle = \langle \phi_0(x), \phi_0(e_i) \rangle = \beta_i \ .$$

This proves that ϕ_0 is a linear isometric map, which coincides with the identity outside B_R, hence $\phi_0 \equiv Id$. If $\phi_0(0) = x_0 \neq 0$, it is enough to replace ϕ_0 with $\widetilde{\phi}_0(\cdot) = \phi_0(\cdot) - x_0$. As before, $\widetilde{\phi}_0$ results to be a linear isometric map, which coincides with a pure translation of x_0 outside B_R. So x_0 must be zero and $\widetilde{\phi}_0 = \phi_0 = Id$; this implies also that the whole sequence $\{\phi_\varepsilon\}$, and not only a subsequence, tends to Id.
In order to prove (v), we note that, by (ii) and recalling that for every invertible matrix A, the $det(A^{-1}) = 1/det(A)$, it follows

$$\frac{1}{(1+\varepsilon)^N} \leq |J(\phi_\varepsilon)| \leq (1+\varepsilon)^N \qquad \text{and} \qquad \frac{1}{(1+\varepsilon)^N} \leq |J(\phi_\varepsilon^{-1})| \leq (1+\varepsilon)^N \ ,$$

which concludes the proof. ∎

In order to state our main result, we need the following definition.

DEFINITION 3.2. Given $u \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$, we say that a sequence $\{u_\varepsilon\} \subseteq \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$ is a *quasi-polyhedral* approximation of u, if the following properties hold:

- the sequence $\{u_\varepsilon\}$ is equibounded in $L^\infty(\Omega)$;
- $u_\varepsilon \to u$ strongly in $L^1(\Omega)$, as $\varepsilon \to 0$;
- $|Du_\varepsilon|(\Omega) \to |Du|(\Omega)$, as $\varepsilon \to 0$;
- there exists a sequence of polyhedron $\{K_\varepsilon\} \subseteq \Omega$ such that $\mathcal{H}^{N-1}(S_{u_\varepsilon} \triangle K_\varepsilon) \to 0$, as $\varepsilon \to 0$;
- $\mathcal{H}^{N-1}(S_{u_\varepsilon}) \to \mathcal{H}^{N-1}(S_u)$, as $\varepsilon \to 0$.

The following theorem contains our main result; i.e., the *quasi-polyhedral* approximation of bounded BV-functions.

THEOREM 3.3. *Let $\Omega = B_R$. For every $u \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$, there exists a quasi-polyhedral approximation $\{u_\varepsilon\}$ of u. More precisely, for every $\varepsilon > 0$, setting $u_\varepsilon := u \circ \phi_\varepsilon$ (where ϕ_ε is given in Theorem 3.1), it follows that $u_\varepsilon \in \mathrm{BV}(\Omega) \cap L^\infty(\Omega)$ and*

(i) $\|u_\varepsilon\|_\infty = \|u\|_\infty$ *and* $\|u_\varepsilon\|_{\mathrm{BV}} \le (1+\varepsilon)^N \|u\|_{\mathrm{BV}}$;
(ii) $u_\varepsilon \to u$ *strongly in* $L^1(\Omega)$, *as* $\varepsilon \to 0$;
(iii) $|Du_\varepsilon|(\Omega) \to |Du|(\Omega)$, *as* $\varepsilon \to 0$;
(iv) there exists a polyhedron $K_\varepsilon \subset \Omega$ *and a constant* $c > 0$ *such that* $\mathcal{H}^{N-1}(S_{u_\varepsilon} \triangle K_\varepsilon) \le c\varepsilon$;
(v) $(1+\varepsilon)^{1-N}\mathcal{H}^{N-1}\big(S_u\big) \le \mathcal{H}^{N-1}(S_{u_\varepsilon}) \le (1+\varepsilon)^{N-1}\mathcal{H}^{N-1}\big(S_u\big)$.

Proof. For every $\varepsilon > 0$, define $u_\varepsilon := u \circ \phi_\varepsilon$, where $\phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ is the diffeomorphism given in Theorem 3.1, with $S := S_u$, which is a countable $\mathcal{H}^{N-1}$-rectifiable subset of Ω. Clearly, by definition, $\|u_\varepsilon\|_\infty = \|u\|_\infty$. By Theorem 2.1 the function u_ε belongs to $\mathrm{BV}(\Omega) \cap L^\infty(\Omega)$ and, for every $0 < \varepsilon \le 1$,

$$\|u_\varepsilon\|_{\mathrm{BV}} = \int_\Omega |u_\varepsilon|\ dx + |Du_\varepsilon|(\Omega) \le \int_\Omega |u||J\phi_\varepsilon^{-1}|\ dy + (1+\varepsilon)^{N-1}|Du|(\Omega) \le (1+\varepsilon)^N \|u\|_{\mathrm{BV}} \le c\ ,$$

where $|J\phi_\varepsilon^{-1}|$ denotes the determinant of the Jacobian matrix of ϕ_ε^{-1} and c is a positive constant independent of ε. Hence, there exists $u_0 \in \mathrm{BV}(\Omega)$ such that, up to a subsequence,

$$u_\varepsilon \to u_0 \qquad \text{strongly in } L^1(\Omega) \qquad \text{and} \qquad Du_\varepsilon \rightharpoonup Du_0 \qquad \text{weakly}^* \text{ in } \mathcal{M}(\Omega;\mathbb{R}^N)\ .$$

Firstly, we will show that $u_0 = u$, which implies also that the whole sequence, not only a subsequence, converges to u.

Let $N \subset \Omega$, with $\mathcal{L}^N(N) = 0$, be such that for every $x \in \Omega \setminus N$, up to another subsequence, $u_\varepsilon(x) \to u_0(x)$ and

$$u_0(x) = \lim_{r\to 0} \frac{1}{\omega_N r^N} \int_{B_r(x)} u_0(y)\ dy \qquad \text{and} \qquad u(x) = \lim_{r\to 0} \frac{1}{\omega_N r^N} \int_{B_r(x)} u(y)\ dy\ . \tag{3.4}$$

Recalling Theorem 3.1, we have that $|J\phi_\varepsilon^{-1}|$ and $|J\phi_\varepsilon|$ tend to 1, as $\varepsilon \to 0^+$, uniformly on Ω. Moreover, for every $r > 0$ and every $x \in \Omega \setminus N$, we have

$$\begin{aligned}|u_\varepsilon(x) - u(x)| &\le \left| u_\varepsilon(x) - \frac{1}{\omega_N r^N}\int_{B_r(x)} u_\varepsilon(y)\, dy\right| + \left|\frac{1}{\omega_N r^N}\int_{B_r(x)} u_\varepsilon(y)\, dy - u(x)\right| \\ &\le \left| u_\varepsilon(x) - \frac{1}{\omega_N r^N}\int_{B_r(x)} u_\varepsilon(y)\, dy\right| + \left|\frac{1}{\omega_N r^N}\int_{\phi_\varepsilon(B_r(x))} u(y)[|J\phi_\varepsilon^{-1}(y)| - 1]\, dy\right| \\ &+ \left|\frac{1}{\omega_N r^N}\left[\int_{\phi_\varepsilon(B_r(x))} u(y)\, dy - \int_{B_r(x)} u(y)\, dy\right]\right| + \left|\frac{1}{\omega_N r^N}\int_{B_r(x)} u(y)\, dy - u(x)\right| .\end{aligned}$$

Passing to the limsup as $\varepsilon \to 0^+$ and taking into account that $\{u_\varepsilon\}$ tends to u_0 strongly in $L^1(\Omega)$ and pointwise in $\Omega \setminus N$, we obtain

$$\begin{aligned}&\limsup_{\varepsilon\to 0^+} |u_\varepsilon(x) - u(x)| \\ &\le \left| u_0(x) - \frac{1}{\omega_N r^N}\int_{B_r(x)} u_0(y)\, dy\right| + \frac{\|u\|_\infty}{\omega_N r^N}\limsup_{\varepsilon\to 0^+}\mathcal{L}^N(\phi_\varepsilon(B_r(x)))\left[\sup_\Omega \left||J\phi_\varepsilon^{-1}(y)| - 1\right|\right] \\ &+ \frac{\|u\|_\infty}{\omega_N r^N}\limsup_{\varepsilon\to 0^+}\mathcal{L}^N(\phi_\varepsilon(B_r(x)) \triangle B_r(x)) + \left|\frac{1}{\omega_N r^N}\int_{B_r(x)} u(y)\, dy - u(x)\right| \\ &= \left| u_0(x) - \frac{1}{\omega_N r^N}\int_{B_r(x)} u_0(y)\, dy\right| + \left|\frac{1}{\omega_N r^N}\int_{B_r(x)} u(y)\, dy - u(x)\right| ,\end{aligned}$$

for every $r > 0$. Now, letting $r \to 0^+$, it follows

$$\limsup_{\varepsilon\to 0^+} |u_\varepsilon(x) - u(x)| \le \limsup_{r\to 0^+}\left[\left| u_0(x) - \frac{1}{\omega_N r^N}\int_{B_r(x)} u_0(y)\, dy\right| + \left|\frac{1}{\omega_N r^N}\int_{B_r(x)} u(y)\, dy - u(x)\right|\right] = 0 ,$$

where we take into account (3.4). This implies that $u_\varepsilon(x) \to u(x)$ for a.e. $x \in \Omega$ and since we have also that $u_\varepsilon(x) \to u_0(x)$ for every $x \in \Omega \setminus N$, it follows that $u = u_0$ almost everywhere and that the whole sequence $\{u_\varepsilon\}$ tends to u in $L^1(\Omega)$. Now, taking into account the lower semicontinuity of the total variation, (i) and (ii) of Theorem 3.1, and Theorem 2.1, we have

$$\begin{aligned}|Du|(\Omega) &\le \liminf_{\varepsilon\to 0^+} |Du_\varepsilon|(\Omega) \le \limsup_{\varepsilon\to 0^+} |Du_\varepsilon|(\Omega) \\ &\le \limsup_{\varepsilon\to 0^+}(1+\varepsilon)^{N-1}|Du|(\phi_\varepsilon(\Omega)) = \limsup_{\varepsilon\to 0^+}(1+\varepsilon)^{N-1}|Du|(\Omega) = |Du|(\Omega) .\end{aligned}$$

Hence, the previous inequality is actually an equality and (iii) is proven. Finally, by (iii) of Theorem 3.1, there exists a polyhedron $K_\varepsilon \subset \Omega$ such that

$$\mathcal{H}^{N-1}(S_u \triangle \phi_\varepsilon(K_\varepsilon)) < \varepsilon . \tag{3.5}$$

Hence, taking into account that $S_{u_\varepsilon} = \phi_\varepsilon^{-1}(S_u)$, the properties of Hausdorff measures (see Proposition 2.49 in [2]), and (ii) of Theorem 3.1, it follows

$$\begin{aligned}\mathcal{H}^{N-1}(S_{u_\varepsilon} \triangle K_\varepsilon) &= \mathcal{H}^{N-1}(\phi_\varepsilon^{-1}(S_u) \triangle K_\varepsilon) = \mathcal{H}^{N-1}\big(\phi_\varepsilon^{-1}(S_u \triangle \phi_\varepsilon(K_\varepsilon))\big) \\ &\le (1+\varepsilon)^{N-1}\mathcal{H}^{N-1}\big(S_u \triangle \phi_\varepsilon(K_\varepsilon)\big) \le (1+\varepsilon)^{N-1}\varepsilon ,\end{aligned}$$

where the last inequality is due to (3.5). In a similar way we obtain

$$\frac{1}{(1+\varepsilon)^{N-1}}\mathcal{H}^{N-1}\big(S_u\big) \le \mathcal{H}^{N-1}(S_{u_\varepsilon}) \le (1+\varepsilon)^{N-1}\mathcal{H}^{N-1}\big(S_u\big),$$

which concludes the proof. □

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