Regularity and compactness for the DiPerna–Lions flow

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1 Flow of non smooth vector fields: the regular Lagrangian flow

When $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded smooth vector field, the *flow of* b is the smooth map $X : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\begin{cases} \dfrac{dX}{dt}(t,x) \;=\; b(t,X(t,x))\,, \quad t \in [0,T] \\ \\ X(0,x) \;=\; x\,. \end{cases} \tag{1}$$

Existence and uniqueness of the flow are guaranteed by the classical Cauchy–Lipschitz theorem. The study of (1) out of the smooth context is of great importance (for instance, in view of the possible applications to conservation laws or to the theory of the motion of fluids) and has been studied by several authors. What can be said about the well–posedness of (1) when b is only in some class of weak differentiability? We remark from the beginning that no generic uniqueness results (i.e. for a.e. initial datum x) are presently available.

This question can be, in some sense, "relaxed" (and this relaxed problem can be solved, for example, in the Sobolev or BV framework): we look for a *canonical selection principle*, i.e. a strategy that "selects", for a.e. initial datum x, a solution $X(\cdot,x)$ in such a way that this selection is stable with respect to smooth approximations of b. This in some sense amounts to redefine our notion of solution: we add some conditions which select a "relevant" solution of our equation. This is encoded in the following definition: we consider only the flows such that there are no concentrations of the trajectories. We will denote by $\mathcal{L}^d$ the d-dimensional Lebesgue measure in $\mathbb{R}^d$.

Definition 1 (Regular Lagrangian flow). *Let $b \in L^1_{\rm loc}([0,T] \times \mathbb{R}^d;\mathbb{R}^d)$. We say that a map $X : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a* regular Lagrangian flow *for the vector field b if* *(i) for a.e. $x \in \mathbb{R}^d$ the map $t \mapsto X(t,x)$ is an absolutely continuous integral solution of $\dot\gamma(t) = b(t,\gamma(t))$ for t in $[0,T]$, with $\gamma(0)=x$;*

(ii) there exists a constant L independent of t such that

$$\mathcal{L}^d\big(X(t,\cdot)^{-1}(A)\big) \le L\mathcal{L}^d(A) \qquad \text{for every Borel set } A \subset \mathbb{R}^d. \tag{2}$$

The constant L in (ii) will be called the compressibility constant *of X.*

2 The link with the transport equation

Existence, uniqueness and stability of regular Lagrangian flows have been proved in [DPL89] by DiPerna and Lions for Sobolev vector fields with bounded divergence. In a recent groundbreaking paper (see [Amb04]) this result has been extended by Ambrosio to BV coefficients with bounded from below divergence.

The arguments of the DiPerna–Lions theory are quite indirect and they exploit (via the theory of characteristcs) the connection between (1) and the Cauchy problem for the *transport equation*

$$\begin{cases} \partial_t u(t,x) + b(t,x)\cdot\nabla_x u(t,x) \;=\; 0 \\ \\ u(0,\cdot) \;=\; \bar u\,. \end{cases} \tag{3}$$

Assuming that the divergence of b is in L^1 we can define bounded distributional solutions of (3) using the identity $b\cdot\nabla_x u = \nabla_x\cdot(bu) - u\nabla_x\cdot b$. Following DiPerna and Lions we say that a distributional solution $u \in L^\infty([0,T]\times\mathbb{R}^d)$ of (3) is a *renormalized solution* if

$$\begin{cases} \partial_t[\beta(u(t,x))] + b(t,x)\cdot\nabla_x[\beta(u(t,x))] \;=\; 0 \\ \\ [\beta(u)](0,\cdot) \;=\; \beta(\bar u) \end{cases} \tag{4}$$

holds in the sense of distributions for every function $\beta \in C^1(\mathbb{R};\mathbb{R})$. In their seminal paper DiPerna and Lions showed that, if the vector field b has Sobolev regularity with respect to the space variable, then every bounded solution is renormalized. Ambrosio [Amb04] extended this result to BV vector fields with divergence in L^1. Under suitable compressibility assumptions (for instance $\nabla_x \cdot b \in L^\infty$), the renormalization property gives *uniqueness* and *stability* for (3) (the existence follows in a quite straightforward way from standard approximation procedures).

In turn, this uniqueness and stability property for (3) can be used to show existence, uniqueness and stability of regular Lagrangian flows (we refer to [DPL89] for the original proofs and to [Amb04] for a different derivation of the same conclusions).

3 Properties of the regular Lagrangian flow

Having defined our notion of solution of (1) and having shown well–posedness for this problem under suitable regularity assumptions on the vector field, it is interesting to investigate some further properties of this solution. In particular we are interested to

- the *regularity* of the regular Lagrangian flow with respect to the initial datum: this amounts to the study of the map $x \mapsto X(t,x)$;
- the *compactness* of regular Lagrangian flows corresponding to vector fields satisfying natural uniform bounds.

The differentiability of the regular Lagrangian flow with respect to x has been first studied by Le Bris and Lions in [LBL04]. For vector fields with Sobolev regularity, they are able to show (using an extension of the theory of renormalized solutions) the existence of measurable maps $W_t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\frac{X(t,x+\varepsilon y) - X(t,x) - \varepsilon W_t(x,y)}{\varepsilon} \to 0 \quad \text{locally in measure in } \mathbb{R}^d_x \times \mathbb{R}^d_y. \tag{5}$$

We recall that a sequence of Borel maps $\{f_n\}$ is said to be *locally convergent in measure* to f in $\mathbb{R}^k$ if

$$\lim_{n\to\infty} \mathcal{L}^k\big(\{x \in B_R(0) \ : \ |f_n(x) - f(x)| > \delta\}\big) = 0$$

for every $R > 0$ and every $\delta > 0$. If the sequence $\{f_n\}$ is locally equibounded in L^∞, then the local convergence in measure is equivalent to the strong convergence in $L^1_{\rm loc}$.

However, it turns out (see [AM05]) that the differentiability property expressed in (5) does not imply the classical approximate differentiability. We recall that a map $f : \mathbb{R}^k \to \mathbb{R}^m$ is said to be *approximately differentiable at* $x \in \mathbb{R}^k$ if there exists a linear map $L(x) : \mathbb{R}^k \to \mathbb{R}^m$ such that

$$\frac{f(x+\varepsilon y) - f(x) - \varepsilon L(x)y}{\varepsilon} \to 0 \quad \text{locally in measure in } \mathbb{R}^d_y.$$

Notice also that this concept has a pointwise meaning, while the one in (5) is global. Moreover, it is possible to show that the map f is approximately differentiable a.e. in $\Omega \subset \mathbb{R}^k$ if and only if the following Lusin–type approximation with Lipschitz maps holds: for every $\varepsilon > 0$ it is possible to find a set $\Omega' \subset \Omega$ with $\mathcal{L}^k(\Omega \setminus \Omega') \leq \varepsilon$ such that $f|_{\Omega'}$ is Lipschitz.

Approximate differentiability for regular Lagrangian flows relative to $W^{1,p}$ vector fields, with $p > 1$, has been first proved by Ambrosio, Lecumberry and Maniglia in [ALM05]. The need for considering only the case $p > 1$ comes from the fact that some tools from the theory of maximal functions are used, as will be explained in the next section. In [ALM05] the strategy is no more an extension of the theory of renormalized solutions: the authors introduce some new estimates along the flow, inspired by the remark that, at a formal level, we can control the time derivative of $\log(|\nabla X(t,x)|)$ with $|\nabla b|(t, X(t,x))$. The strategy of [ALM05] allows to make this remark rigorous: it is possible to consider some integral quantities which contain a discretization of the space gradient of the flow and prove some estimates along the flow, in fact using the PDE formulation of the problem presented in the previous section. Then, the application of Egorov theorem allows the passage from integral estimates to pointwise estimates on big sets, and from this it is possible to recover Lipschitz regularity on big sets, and eventually one gets the approximate differentiability. However, the application of Egorov theorem implies a loss of quantitative informations: this strategy does not allow a control of the Lipschitz constant in terms of the size of the "neglected" set.

4 Quantitative estimates for $W^{1,p}$ vector fields

Starting from the result of Ambrosio, Lecumberry and Maniglia [ALM05], the main point of [CDL06] is a modification of the estimates in such a way that quantitative informations are not lost. Let X be a regular Lagrangian flow relative to a bounded vector field $b \in L^1(W^{1,p})$ for some $p > 1$. For every $R > 0$ we define the quantity

$$A_p(R,X) := \left\| \sup_{0\le t\le T} \sup_{0<r<R} \fint_{B_r(x)} \log\left(\frac{|X(t,x)-X(t,y)|}{r}+1\right) dy \right\|_{L^p_x(B_R(0))}.$$

The strategy of the proof is based on two main steps.

4.1 A–priori estimate

It is possible to give an a–priori estimate for the functional $A_p(R,X)$ in terms of the $L^1_t(L^p_x)$ norm of Db and the compressibility constant of the flow. Trying to estimate the quantity

$$\frac{d}{dt} \fint_{B_r(x)} \log\left(\frac{|X(t,x)-X(t,y)|}{r}+1\right) dy \tag{6}$$

we get some difference quotients of the vector field computed along the flow. Here comes into play the theory of maximal functions.

We recall that, for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^k;\mathbb{R}^m)$, we can define the *maximal function* of f as

$$Mf(x) := \sup_{r>0} \fint_{B_r(x)} |f(y)|\, dy\,.$$

It is well–known (see for example [Ste70]) that for every $p > 1$ the strong estimate

$$\|Mf\|_{L^p(\mathbb{R}^k)} \leq C_{k,p}\|f\|_{L^p(\mathbb{R}^k)} \tag{7}$$

holds, while this is not true in the limit case $p = 1$. Moreover, if f has Sobolev regularity, we can estimate the increments using the maximal function of the derivative: there exists a negligible set $N \subset \mathbb{R}^k$ such that for every x, $y \in \mathbb{R}^k \setminus N$ we have

$$|f(x) - f(y)| \leq C_k|x - y|\big(MDf(x) + MDf(y)\big)\,.$$

Going back to (6), we see that it is possible to estimate the difference quotients which appear in the time differentiation using the maximal function of Db, computed along the flow. After, we integrate with respect to the time, we pass to the supremums and eventually we take the L^p norm in order to reconstruct the quantity $A_p(R, X)$. Then, changing variable (and for this we just pay a factor given by the compressibility constant L) and using the strong estimate (7) in order to express the bound in term of Db, we finally get the a–priori quantitative estimate

$$A_p(R, X) \leq C\big(R, L, \|Db\|_{L^1(L^p)}\big)\,. \tag{8}$$

4.2 Quantitative Lipschitz property

From the bound (8) we can obtain a *quantitative Lusin–type Lipschitz approximation* of the regular Lagrangian flow. This means that we are able to estimate the growth of the Lipschitz constant in terms of the size of the neglected set. For every fixed $\varepsilon > 0$ and every $R > 0$ we apply Chebyshev inequality to get a constant

$$M = M(\varepsilon) = \frac{A_p(R, X)}{\varepsilon^{1/p}}$$

and a set $K \subset B_R(0)$ with $\mathcal{L}^d(B_R(0) \setminus K) \leq \varepsilon$ such that for every $x \in K$

$$\sup_{0\leq t\leq T}\ \sup_{0<r<2R}\ \fint_{B_r(x)} \log\left(\frac{|X(t,x) - X(t,y)|}{r} + 1\right) dy \leq M\,.$$

From this it easily follows that

$$|X(t,x) - X(t,y)| \leq \exp\left(\frac{c_d A_p(R, X)}{\varepsilon^{1/p}}\right)|x - y|\,,$$

i.e. we obtain the following *explicit* control of the Lipschitz constant:

$$\mathrm{Lip}\,(X(t,\cdot)|_K) \sim \exp\left(C\varepsilon^{-1/p}\right)\,. \tag{9}$$

5 Corollaries and remarks

5.1 Immediate consequences

Approximate differentiability

Recalling the equivalence stated immediately after the definition of approximate differentiability, it follows that the regular Lagrangian flow is approximately differentiable with respect to the space variable a.e. in $\mathbb{R}^d$. We notice that the quantitative result expressed in (9) is not strictly necessary for this first consequence.

Compactness

The quantitative version of the Lusin–type Lipschitz approximation can be used to show the precompactness in L^1_{loc} for the regular Lagrangian flows $\{X_n\}$ generated by a sequence $\{b_n\}$ of vector fields equibounded in L^∞ and in $L^1(W^{1,p})$ (for $p > 1$), under the assumption that the compressibility constants of the regular Lagrangian flows are equibounded. We illustrate here the main idea to get this result.

On every ball $B_R(0)$, the regular Lagrangian flows are equibounded. We fix $\varepsilon > 0$. For every n we apply (9) to find a set K_n with $\mathcal{L}^d(B_R(0) \setminus K_n) \leq \varepsilon$ such that the Lipschitz constants of the maps $X_n|_{K_n}$ are equibounded. Then we can extend every map $X_n|_{K_n}$ to a map $\widetilde{X}_n$ defined on all $B_R(0)$ in such a way that the sequence $\{\widetilde{X}_n\}$ is equibounded and equicontinuous over $B_R(0)$. Hence we can apply Ascoli–Arzelà theorem to this sequence, getting strong compactness in L^∞. But since every map $\widetilde{X}_n$ coincides with the regular Lagrangian flow X_n out of a small set, it is simple to check that this implies strong compactness in L^1 for the regular Lagrangian flows $\{X_n\}$.

A merit of this approach is the fact that the compactness result holds under an assumption of equiboundedness of the compressibility constants: we do not need a uniform bound on the divergence of the vector fields (this would be in general a stronger condition). Some compactness results under uniform bounds on the divergence are already present in the literature: see for example [DPL89].

We finally remark that an extension of our strategy to the case $p = 1$ would give a positive answer to a conjecture proposed by Bressan in [Bre03B]. See also [Bre03A] for a related conjecture on mixing flows.

5.2 Quantitative stability

With similar techniques it is possible to show a result of quantitative stability for regular Lagrangian flows relative to $W^{1,p}$ vector fields (here we need again the assumption $p > 1$). The stability results in [DPL89] and [Amb04] were obtained using some abstract compactness arguments, hence they do not give a rate of convergence. It is indeed possibile to show that, for the regular Lagrangian flows X_1 and X_2 relative to bounded vector fields b_1 and b_2 belonging to $L^1(W^{1,p})$, the following estimate holds:

$$\|X_1(T,\cdot) - X_2(T,\cdot)\|_{L^1(B_r)} \leq C\Big|\log\big(\|b_1 - b_2\|_{L^1([0,T]\times B_R)}\big)\Big|^{-1}.$$

The constant C and R depend on the usual uniform bounds on the vector fields. We remark that this estimate also gives a new direct proof of the uniqueness of the regular Lagrangian flow.

5.3 A regularity result for the PDE

Using again (9) it is possible to show a result relative to solutions to the transport equation (3). We can prove that for bounded vector fields belonging to $L^1(W^{1,p})$ (as usual with $p > 1$) and with bounded divergence, solutions of (3) propagate a mild regularity, the same one of the corresponding regular Lagrangian flow expressed by (9).

6 An a–priori estimates approach

The estimates we have presented give a new possible approach to the theory of regular Lagrangian flows. In particular, we can develop (in the $W^{1,p}$ context with $p > 1$) a theory of ODEs completely independent from the associated PDEs theory. The general scheme is the following:

- the compactness we have illustrated can be used to show *existence* of the regular Lagrangian flow, via regularization, for vector fields with bounded divergence;
- the *uniqueness* comes together with the *stability*, which is recovered in a new quantitative fashion;
- in addition to these results, we can show the quantitative *regularity* expressed in (9), which implies the *approximate differentiability*;
- finally, a new *compactness* result is obtained.

All this results are obtained at the Lagrangian level, with no mention to the transport equation theory.

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