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A POINTWISE GRADIENT BOUND FOR ELLIPTIC EQUATIONS ON COMPACT MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE

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The goal of this note is to prove the following result:

Theorem 1. *Let M be a smooth, compact, Riemannian manifold with non-negative Ricci curvature and let $f \in C^1(\mathbb{R})$.*

Let $u \in C^3(M)$ be a solution of

$$\Delta_g u + f(u) = 0$$

on M, with $\mathfrak{m} := \inf_M u$, $\mathfrak{M} := \sup_M u$, and let F be a primitive of f.

Then,

$$\frac{1}{2}|\nabla_g u(x)|^2 \le \sup_{r\in[\mathfrak{m},\mathfrak{M}]} F(r) - F(u(x)), \tag{1}$$

for any $x \in M$.

Also, if equality in (1) holds at some point $x_o \in \{\nabla_g u \neq 0\}$, then:

- *equality in (1) holds at all the points of the connected component of $M \cap \{\nabla_g u \neq 0\}$ that contains x_o,*
- $\mathrm{Ric}_g(\nabla_g v, \nabla_g v)$ *vanishes at all the points of the connected component of $M \cap \{\nabla_g u \neq 0\}$ that contains x_o.*

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The notation used here above is the standard one, namely ∇_g is the Riemannian gradient and Δ_g is the Laplace-Beltrami operator, that is, in local coordinates,

$$(\nabla_g \phi)^i = g^{ij}\partial_j \phi$$

and

$$\Delta_g \phi = \mathrm{div}_g(\nabla_g \phi) = \frac{1}{\sqrt{|g|}}\partial_i\Big(\sqrt{|g|}g^{ij}\partial_j\phi\Big),$$

for any smooth function $\phi : M \to \mathbb{R}$.

We remark that when equality in (1) holds on a connected open set U, then u is an isoparametric function in U, see pages 541–548 of [8]. In particular, any level set σ of u has constant mean curvature along $\sigma \cap U$. For a comprehensive description of isoparametric functions, see also [10].

The pointwise estimate of Theorem 1 may be seen as an extension of the one obtained in [6], where a similar result was proven in the case of $\mathbb{R}^n$.

We observe that if F is bounded, (1) implies the following universal estimate:

$$\frac{1}{2}|\nabla_g u(x)|^2 \le \sup_{r\in\mathbb{R}} F(r) - \inf_{r\in\mathbb{R}} F(r).$$

The proof we give here of Theorem 1 uses the technique of [2], where important strengthenings of the work of [6] were performed in the degenerate and singular Euclidean case.

The proof is based on the "P-function technique", i.e. in a convenient use of the maximum principle, applied to a function which solves a degenerate PDE (see [7, 9]).

For related results in the Euclidean setting, see also [4].

Proof of Theorem 1. We recall that, if $\phi \in C^3(M)$,

$$\frac{1}{2}\Delta_g|\nabla_g\phi|^2 = |H_\phi|^2 + \langle \nabla_g\Delta_g\phi, \nabla_g\phi\rangle + \mathrm{Ric}_g(\nabla_g\phi, \nabla_g\phi). \tag{2}$$

Here above, H_ϕ is the Hessian of ϕ: note that (2) is the so-called Bochner-Weitzenböck formula (see, for instance, [1, 11] and references therein).

Moreover, we have that

$$|H_\phi|^2 \ge \big|\nabla_g|\nabla_g\phi|\big|^2 \qquad \text{almost everywhere.} \tag{3}$$

See, for instance, [3] for the simple proof of this fact.

Also, we observe that, since M is compact, if $v \in C^2(M)$ then there exists $x(v) \in M$ which minimizes v, and so

$$\nabla_g v(x(v)) = 0. \tag{4}$$

We now define

$$G(t) := \sup_{r\in[\mathfrak{m},\mathfrak{M}]} F(r) - F(t). \tag{5}$$

We remark that

$$G(t) \ge 0 \tag{6}$$

for any $t \in [\mathfrak{m}, \mathfrak{M}]$.

We also fix $\alpha \in (0,1)$, say $\alpha = 1/2$, and set $[u]_{C^\alpha(M)}$ to be the α-seminorm of u, which is finite by assumption.

Let

$$\begin{aligned} \mathcal{F} := \Big\{ & v \in C^2(M) \text{ solutions of } \Delta_g v = G'(v) \text{ in } M \text{ with } \mathfrak{m} \le v \le \mathfrak{M} \\ & \text{and } [v]_{C^\alpha(M)} \le [u]_{C^\alpha(M)} \Big\}. \end{aligned} \tag{7}$$

Also, given $v \in \mathcal{F}$, following [2], we define

$$P(v,x) := |\nabla_g v(x)|^2 - 2G(v(x)). \tag{8}$$

We now claim that, for any $v \in \mathcal{F}$ and any $x \in M$,

$$|\nabla_g v(x)|^2 \Delta_g P(v,x) - 2G'(v(x)) \langle \nabla_g v(x), \nabla_g P(v,x) \rangle \ge \frac{|\nabla_g P(v,x)|^2}{2}. \tag{9}$$

We remark that (9) may be considered the Riemannian analogue of formula (2.7) in [2], where a similar equality was found in the Euclidean setting.

To prove (9), we use (2) and (7) to obtain that

$$\begin{aligned}
& |\nabla_g v|^2 \Delta_g P - 2G'(v)\langle \nabla_g v, \nabla_g P\rangle - \frac{|\nabla_g P|^2}{2} \\
= \;& |\nabla_g v|^2 \Big(\Delta_g |\nabla_g v|^2 - 2\Delta_g\big(G(v)\big)\Big) + 2f(v)\langle \nabla_g v, \nabla_g P\rangle - \frac{|\nabla_g P|^2}{2} \\
= \;& 2|\nabla_g v|^2 \Big(|H_v|^2 + \langle \nabla_g \Delta_g v, \nabla_g v\rangle + \mathrm{Ric}_g(\nabla_g v, \nabla_g v) - \mathrm{div}_g\big(G'(v)\nabla_g v\big)\Big) \\
& \quad + 2f(v)\langle \nabla_g v, \nabla_g P\rangle - \frac{|\nabla_g P|^2}{2} \\
= \;& 2|\nabla_g v|^2 \Big(|H_v|^2 - \langle \nabla_g (f(v)), \nabla_g v\rangle \\
& \quad + \mathrm{Ric}_g(\nabla_g v, \nabla_g v) - G''(v)|\nabla_g v|^2 - G'(v)\Delta_g v\Big) \\
& \quad + 2f(v)\Big(\langle \nabla_g v, \nabla_g |\nabla_g v|^2\rangle - 2\langle \nabla_g v, \nabla_g\big(G(v)\big)\rangle\Big) \\
& \quad - \frac{\Big|\nabla_g |\nabla_g v|^2 - 2\nabla_g\big(G(v)\big)\Big|^2}{2} \\
= \;& 2|\nabla_g v|^2 \Big(|H_v|^2 - f'(v)|\nabla_g v|^2 \\
& \quad + \mathrm{Ric}_g(\nabla_g v, \nabla_g v) + f'(v)|\nabla_g v|^2 - \big(f(v)\big)^2\Big) \\
& \quad + 2f(v)\Big(\langle \nabla_g v, \nabla_g |\nabla_g v|^2\rangle + 2f(v)|\nabla_g v|^2\Big) \\
& \quad - \frac{\Big|\nabla_g |\nabla_g v|^2 + 2f(v)\nabla_g v\Big|^2}{2} \\
= \;& 2|\nabla_g v|^2 \Big(|H_v|^2 + \mathrm{Ric}_g(\nabla_g v, \nabla_g v)\Big) - \frac{\big|\nabla_g |\nabla_g v|^2\big|^2}{2}.
\end{aligned}$$

Hence, recalling (3) and the fact that the Ricci curvature is nonnegative, we obtain that

$$\begin{aligned}|\nabla_g v|^2\Delta_g P - 2G'(v)\langle\nabla_g v, \nabla_g P\rangle - \frac{|\nabla_g P|^2}{2}\\ &= 2|\nabla_g v|^2\Big(|H_v|^2 - \big|\nabla_g|\nabla_g v|\big|^2 + \mathrm{Ric}_g(\nabla_g v, \nabla_g v)\Big)\\ &\geq 2|\nabla_g v|^2\,\mathrm{Ric}_g(\nabla_g v, \nabla_g v).\end{aligned} \tag{10}$$

We observe that the above quantity is nonnegative, and this proves (9).

Now, we define

$$P_o := \sup_{\substack{v\in\mathcal{F}\\ x\in M}} P(v,x). \tag{11}$$

We observe that, if $v \in \mathcal{F}$,

$$\begin{aligned}|f(v(x)) - f(v(y))| &\leq \|f\|_{C^1([\mathfrak{m},\mathfrak{M}])}|v(x)-v(y)|\\ &\leq \|f\|_{C^1([\mathfrak{m},\mathfrak{M}])}\|v\|_{C^\alpha(M)}|x-y|^\alpha\\ &\leq \|f\|_{C^1([\mathfrak{m},\mathfrak{M}])}\|u\|_{C^\alpha(M)}|x-y|^\alpha\end{aligned}$$

for any x, $y \in M$.

Consequently, by elliptic regularity (see, e.g., [5]), any $v \in \mathcal{F}$ satisfies

$$\|v\|_{C^{2,\alpha}(M)} \leq C_o \tag{12}$$

for a suitable $C_o > 0$ independent on v (more precisely, C_o only depends on f, $\mathfrak{m}$, $\mathfrak{M}$ and $\|u\|_{C^\alpha(M)}$).

In particular, the sup in (11) is finite.

We claim that

$$P_o \leq 0. \tag{13}$$

To check (13), we argue by contradiction. We suppose that

$$P_o > 0 \tag{14}$$

$v_k \in \mathcal{F}$ and $x_k \in M$ in such a way that

$$P_o - \frac{1}{k} \leq P(v_k, x_k) \leq P_o. \tag{15}$$

Since M is compact, we may suppose that x_k converges to some $x_\infty \in M$, up to subsequence.

Also, by (12), v_k converges in $C^2(M)$, up to subsequence, to some v_∞.

Notice that $v_\infty \in \mathcal{F}$ by construction.

Therefore, (15) gives that

$$P(v_\infty, x_\infty) = P_o. \tag{16}$$

From (6), (14) and (16), we obtain that

$$|\nabla_g v_\infty(x_\infty)|^2 \geq |\nabla_g v_\infty(x_\infty)|^2 - 2G(v_\infty(x_\infty)) = P_o > 0$$

and therefore

$$\nabla_g v_\infty(x_\infty) \neq 0. \tag{17}$$

In light of (9), (16) and (17), the Strong Maximum Principle gives that

$$P(v_\infty, x) = P_o \qquad \text{for any } x \in M. \tag{18}$$

In particular, recalling (4) and using (6), (14) and (18), we conclude that

$$\begin{aligned} 0 = |\nabla_g v_\infty(x(v_\infty))|^2 &\geq |\nabla_g v_\infty(x(v_\infty))|^2 - 2G\big(v_\infty(x(v_\infty))\big) \\ &= P\big(v_\infty, x(v_\infty)\big) = P_o > 0. \end{aligned}$$

Since this is a contradiction, the proof of (13) is complete.

Then, by (5) and (13),

$$\begin{aligned} 0 \geq P_o = \sup_{\substack{v\in\mathcal{F}\\ x\in M}} P(v,x) &\geq P(u,x) = |\nabla_g u(x)|^2 - 2G(u(x)) \\ &= |\nabla_g u(x)|^2 - 2\left[\sup_{r\in[\mathfrak{m},\mathfrak{M}]} F(r) - F\big(u(x)\big)\right], \end{aligned}$$

that is (1).

We now suppose that equality in (1) holds at some point

$$x_o \in \{\nabla_g u \neq 0\} \tag{19}$$

and we prove that it holds at all the points of the connected component of $M \cap \{\nabla_g u \neq 0\}$ that contains x_o.

For this, let M' be such connected component. We notice that, by (5) and (13),

$$0 \geq P_o \geq P(u, x_o) = |\nabla_g u(x_o)|^2 - 2\left[\sup_{r\in[\mathfrak{m},\mathfrak{M}]} F(r) - F\big(u(x_o)\big)\right] = 0,$$

and so

$$P(u, x_o) = \max_{x\in M} P(u,x) = 0.$$

Thus, (19) and the Strong Maximum Principle gives that

$$P(u,x) = 0 \qquad \text{for any } x \in M'. \tag{20}$$

This shows that equality in (1) holds at all the points of M'.

Furthermore, from (10) and (20),

$$0 = |\nabla_g v|^2 \Delta_g P - 2G'(v)\langle \nabla_g v, \nabla_g P\rangle - \frac{|\nabla_g P|^2}{2} \geq 2|\nabla_g v|^2 \operatorname{Ric}_g(\nabla_g v, \nabla_g v)$$

and so

$$|\nabla_g v|^2 \operatorname{Ric}_g(\nabla_g v, \nabla_g v) = 0$$

at all the points of M'.

Since $\nabla_g v \neq 0$ in M', this gives that $\operatorname{Ric}_g(\nabla_g v, \nabla_g v)$ vanishes identically in M'.

This completes the proof of Theorem 1. $\square$

Remark 2. We observe that, from (3), (10) and (20), we have also proved that if equality in (1) holds at some point $x_o \in \{\nabla_g u \neq 0\}$, then $|H_v| = \big|\nabla_g|\nabla_g v|\big|$ at all the points of the connected component of $M \cap \{\nabla_g u \neq 0\}$ that contains x_o.

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