CONVERGENCE OF HAMILTON-JACOBI EQUATIONS FOR SEQUENCES OF OPTIMAL CONTROL PROBLEMS

Ariela Briani
Dipartimento di Matematica, Università di Pisa,
Via F. Buonarroti 2, 56127 PISA (Italy),
e-mail: briani@dm.unipi.it.

ABSTRACT: We consider a sequence of optimal control problems $(\mathcal{P}_h)$ and we study the asymptotic behavior (as $h \to \infty$) of the optimal pairs (u_h, y_h), the value functions v_h and the Hamilton-Jacobi equations associated to these problems. The Γ-convergence and the "viscosity solutions" theories are used.
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1 INTRODUCTION

In this paper we deal with sequences of optimal control problems of the form

$$(\mathcal{P}_h) \qquad \min\left\{\int_\Omega f_h(t,y,u)dt \ : \ (y,u) \in \Lambda_h\right\}$$

where

$$\Lambda_h = \{(y,u) \in Y \times \mathcal{U} : y'(t) = a_h(t,y) + B_h(t,u), \ \ y(0) = y_h^0\}.$$

In some previous papers (see Buttazzo and Cavazzuti [3] and Buttazzo and Dal Maso [4]) the asymptotic behaviour (as $h \to \infty$) of the optimal pairs (y_h, u_h) of $(\mathcal{P}_h)$ has been studied; more precisely, a new limit problem $(\mathcal{P}_\infty)$ is constructed such that if (y_h, u_h) is an optimal pair of $(\mathcal{P}_h)$ and (y_h, u_h) tends to (y,u) then (y,u) is an optimal pair of $(\mathcal{P}_\infty)$. In particular, in [3] it is shown that if the functions B_h converge strongly, the limit problem $(\mathcal{P}_\infty)$ has the same form of problem $(\mathcal{P}_h)$ while if the functions B_h are only weakly convergent, in the limit problem the state equation may disappear and may be replaced by an integral contribution in the cost functional.
Since this behaviour is somewhat unexpected, we look for a justification of the limit problem via Hamilton-Jacobi equation's theory. In fact we show that the Hamiltonians of the problem $(\mathcal{P}_h)$ converge in a natural way to the Hamiltonian of the problem $(\mathcal{P}_\infty)$ without any Γ-convergence tool.
To obtain this result we start by associating to every problem $(\mathcal{P}_h)$ its value function v_h and its Hamilton-Jacobi equation characterized by a Hamiltonian H_h. We define in the same way the value function v_∞ of the limit problem $(\mathcal{P}_\infty)$ and its Hamiltonian H_∞. We observe then that by the Γ-convergence theory we can conclude that the value functions v_h converge pointwise to v_∞ and we prove that the Hamiltonians H_h converge to H_∞. In order to conclude we need to show that for each fixed h the value function v_h is the solution of the corresponding Hamilton-Jacobi equation and that v_∞ is the solution of the Hamilton-Jacobi equation having H_∞ as Hamiltonian. We do this using the definition of viscosity solution given by Ishii in [9] for the case of

continuous value function and measurable time dependent Hamiltonian.
We can resume this construction in the following scheme

$$\begin{array}{ccccc} (\mathcal{P}_h) & \Longrightarrow & (v_h) & \Longrightarrow & (H_h) \\ \downarrow & & \downarrow & & \Downarrow \\ (\mathcal{P}_\infty) & \Longrightarrow & (v_\infty) & \Longrightarrow & (H_\infty) \end{array}$$

where with a simple line we denote the results following by Γ-convergence and by a double line we represent our work.
We remark that, since we are considering an unbounded control set, the proof of the continuity of the value function it is not a standard result. Moreover, since the Hamiltonian is only measurable in time, we can not apply the results available in the literature to show that v_h and v_∞ are viscosity solutions of the respective Hamilton-Jacobi equations, so we prove it directly (see Theorems 3.3 and 3.4). For all details about the theory of viscosity solution of Hamilton-Jacobi equations we refer to [1], [2] and [8].
Our global result can be seen also as a stability result for Hamilton-Jacobi equations arising in optimal control theory (see Corollary 3.10).

The paper is organized as follows. In section 2 we describe the control problems in all details and we prove the convergence results for the Hamiltonians and for the value functions. In Section 3 we recall the definition of viscosity solution and we prove that for every fixed h the value function v_h is a viscosity solution of the Hamilton-Jacobi equation associated to our problem $(\mathcal{P}_h)$. We write then the Hamilton-Jacobi equation for the limit problem $(\mathcal{P}_\infty)$ and we observe that we can apply the same result to show that the value function v_∞ is a viscosity solution of this equation. The last section is devoted to the details of an interesting example and to some open problems.

2 THE FRAMEWORK AND THE CONVERGENCE RESULT

We consider $\Omega = (0, T)$ with $T < \infty$. Fixed two positive integers n, m our state space Y is the Sobolev space $W^{1,1}(\Omega; \mathbb{R}^n)$ endowed with the $L^\infty(\Omega; \mathbb{R}^n)$ topology and our control space $\mathcal{U}$ is $L^p(\Omega; \mathbb{R}^m)$ endowed with its weak topology. We suppose $1 < p < \infty$.
We want to minimize the cost functional

$$J_h(y, u) = \int_\Omega f_h(t, y, u)\, dt \tag{2.1}$$

where $f_h : \Omega \times \mathbb{R}^n \times \mathbb{R}^m \to [0, \infty)$ are Borel functions and the state equations are

$$\begin{cases} y'(t) = a_h(t, y) + B_h(t, u) \\ y(0) = y_h^0 \end{cases} \tag{2.2}$$

where $a_h : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ and $B_h : \Omega \times \mathbb{R}^m \to \mathbb{R}^n$ are Borel functions. Therefore the optimal control problems we are concerned are

$$(\mathcal{P}_h) \qquad \min\left\{\int_\Omega f_h(t, y, u)\, dt \;:\; (y, u) \in \Lambda_h\right\}$$

where

$$\Lambda_h = \{(y, u) \in Y \times \mathcal{U} : y'(t) = a_h(t, y) + B_h(t, u),\;\; y(0) = y_h^0\}. \tag{2.3}$$

We denote by $\|\cdot\|_r$ the $L^r(\Omega;\mathbb{R}^n)$ norm for $r\in[1,\infty)$ and by χ_E the indicator function of the set E defined by

$$\chi_E(x)=\begin{cases}0 & \text{if } x\in E\\ +\infty & \text{if } x\notin E.\end{cases}$$

We make the following assumptions on a_h , B_h and f_h.

a) There exists a sequence of functions M_h from Ω into $\mathbb{R}$ with $\| M_h \|_1\le M<\infty$ such that, for every $t\in\Omega$

$$|a_h(t,0)|\le M_h(t)\quad \forall h\in\mathbb{N}; \tag{2.4}$$

b) there exists a sequence of functions α_h from Ω into $\mathbb{R}$ with $\| \alpha_h \|_1\le \alpha<\infty$ such that, for every $t\in\Omega$ and for every $y_1,y_2\in\mathbb{R}^n$

$$|a_h(t,y_1)-a_h(t,y_2)|\le \alpha_h(t)|y_1-y_2|\quad \forall h\in\mathbb{N}; \tag{2.5}$$

c) there exist $q>1$, a sequence of functions $G_h\in L^q(\Omega;\mathbb{R}^n)$ with $\| G_h \|_q\le G$, and a constant $D\ge 0$ such that, for every $(t,u)\in\Omega\times\mathbb{R}^m$

$$|B_h(t,u)|\le G_h(t)+D|u|^{p/q}\quad \forall h\in\mathbb{N}; \tag{2.6}$$

d) for a.e. $t\in\Omega$ and for every $y\in\mathbb{R}^n$

$$f_h(t,y,\cdot) \text{ is l.s.c. on } \mathbb{R}^m\quad \forall h\in\mathbb{N}; \tag{2.7}$$

e) there exists a function $C(t,v,w)$, from $\Omega\times\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}$, integrable with respect to t, increasing and continuous in v and w such that, for every $u\in\mathbb{R}^m$ and for every $y,z\in\mathbb{R}^n$

$$|f_h(t,y,u)-f_h(t,z,u)|\le C(t,|y|,|z|)|y-z|\quad \forall h\in\mathbb{N}; \tag{2.8}$$

f) there exist two constants $a>0$ and $b>0$ such that, for every $(t,y,u)\in\Omega\times\mathbb{R}^n\times\mathbb{R}^m$

$$f_h(t,y,u)\ge a|u|^p-b\quad \forall h\in\mathbb{N}; \tag{2.9}$$

g) there exists a control function $u_0\in\mathcal{U}$ such that, for every $y\in Y$ the sequence of functions

$$f_h(\cdot,y(\cdot),u_0(\cdot)) \tag{2.10}$$

is bounded in $L^1(\Omega;\mathbb{R})$;

h) there exists a sequence of control functions $u_h\in L^\infty(\Omega;\mathbb{R}^m)$ such that, for every $y\in Y$

$$B_h(\cdot,u_h(\cdot))\in L^\infty(\Omega;\mathbb{R}^n) \text{ and } \int_\Omega f_h(t,y(t),u_h(t))dt<\infty. \tag{2.11}$$

To characterize the limit control problem we set

$$g_h(t,y,u,w) = f_h(t,y,u) + \chi_{\{B_h(t,u)=w\}}(t,u,w)$$

and we define

$$g_h^*(t,y,u^*,w^*) = \sup_{u\in\mathbb{R}^m\ w\in\mathbb{R}^n} [u\cdot u^* + w\cdot w^* - g_h(t,y,u,w)]. \tag{2.12}$$

If

$$a_h(\cdot,y) \longrightarrow a(\cdot,y) \text{ weakly } \ L^1(\Omega;\mathbb{R}^n) \ \ \forall y\in\mathbb{R}^n, \tag{2.13}$$

$$y_h^0 \to y_0 \ \text{ in } \ \mathbb{R}^n, \tag{2.14}$$

$$g_h^*(\cdot,y,u^*,w^*) \to \varphi(\cdot,y,u^*,w^*) \text{ weakly } L^q(\Omega;\mathbb{R}^n) \tag{2.15}$$

for every $(y,u^*,w^*)\in\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^n$, then the limit problem is given by (for the proof see [3], Proposition 3.2 and 3.3 page 158 or [5], Section 5)

$$(\mathcal{P}_\infty) \qquad \min\left\{\int_\Omega \varphi^*(t,y,u,y'-a(t,y))\,dt \ :\ (y,u)\in Y\times\mathcal{U},\ y(0)=y_0\right\}.$$

Note that, when the coefficients are highly oscillating, the state equation in the limit problem may disappear. This is the case, for instance, when $B_h(t,u) = sin(ht)u$ and $f_h(t,y,u) = |u|^2 + |y-\phi(t)|^2$ which yields

$$(\mathcal{P}_\infty) \quad \min_{Y\times\mathcal{U}}\left\{\int_\Omega |u|^2 + |y-\phi(t)|^2 + 2|y'-a(t,y)|^2\,dt,\ (y,u)\in Y\times\mathcal{U},\ y(0)=y_0\right\}.$$

Our purpose is now to associate to the sequence of problems $(\mathcal{P}_h)$ a sequence of Hamiltonians H_h and to show that it converges to the Hamiltonian H_∞ associated to the limit problem $(\mathcal{P}_\infty)$. By the standard theory of Hamilton-Jacobi equations (see [1] or [8]) it is known that the Hamiltonians associated to problems $(\mathcal{P}_h)$ are

$$H_h(t,x,p) = \sup_{u\in\mathbb{R}^m}\{-p\cdot B_h(t,u) - f_h(t,x,u)\} - p\cdot a_h(t,x) \tag{2.16}$$

while the value functions are

$$v_h(t_0,x_0) = \inf\left\{\int_{t_0}^T f_h(t,y^0(t;u),u(t))\,dt \ :\ u\in\mathcal{U}\right\} \tag{2.17}$$

where $y^0(t;u)$ is the solution of

$$\begin{cases} y'(t) = a_h(t,y) + B_h(t,u) & t\in\Omega \\ y(t_0) = x_0. \end{cases} \tag{2.18}$$

If we look at the limit problem $(\mathcal{P}_\infty)$ we see that the state equation disappears and the derivative $y'(t)$ appears now in the cost. In order to write the value function and the Hamiltonian of this problem we introduce a new variable as follows. We take $w\in L^p(\Omega;\mathbb{R}^n)$ and we consider as control the vector $U=(u,w)$; the space of controls is then $\mathcal{U} = L^p(\Omega;\mathbb{R}^m\times\mathbb{R}^n)$ and the state equation is given by

$$\begin{cases} y'(t) = a(t,y) + CU & t\in\Omega \\ y(t_0) = x_0 \end{cases} \tag{2.19}$$

where $C : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ is the linear mapping $C(z_1, z_2) = z_2$. It follows that the value function is

$$v_\infty(t_0, x_0) = \inf\left\{\int_{t_0}^{T} \alpha(t, y^0(t; U), U(t))\, dt \;:\; U \in \mathcal{U}\right\} \tag{2.20}$$

where $y^0(t; U)$ is the solution of (2.19) and $\alpha(t, y, U) = \varphi^*(t, y, u, w)$ while the Hamiltonian is

$$H_\infty(t, x, p) = \sup_{u\in\mathbb{R}^n\ w\in\mathbb{R}^n} \{-p \cdot w - \varphi^*(t, y, u, w)\} - p \cdot a(t, x). \tag{2.21}$$

Our convergence result for the Hamiltonians is the following.

Theorem 2.1 *Under the assumptions (2.13) and (2.15)*

$$H_h(\cdot, x, p) \to H_\infty(\cdot, x, p) \quad \textit{weakly} \quad L^1(\Omega; \mathbb{R}) \tag{2.22}$$

for every $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proof. By definition of g_h and g_h^* we can write

$$H_h(t, x, p) = \sup_{u\in\mathbb{R}^m\ w\in\mathbb{R}^n} \{-p \cdot w - g_h(t, x, u, w)\} - p \cdot a_h(t, x) =$$

$$= g_h^*(t, x, 0, -p) - p \cdot a_h(t, x)$$

so that, by hypotheses (2.13) and (2.15),

$$H_h(\cdot, x, p) \to \varphi(\cdot, x, 0, -p) - p \cdot a(\cdot, x) \quad \text{weakly} \quad L^1(\Omega; \mathbb{R})$$

for every $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$. Since by (2.21),

$$H_\infty(\cdot, x, p) = \varphi^{**}(\cdot, x, 0, -p) - p \cdot a(\cdot, x) = \varphi(\cdot, x, 0, -p) - p \cdot a(\cdot, x) \tag{2.23}$$

the proof is achieved. □

Remark 2.2 We conclude this paragraph by stating the convergence result for the value functions:

$$\lim_{h\to\infty} v_h(t, x) = v_\infty(t, x) \quad \text{for all } (t, x) \in \Omega \times \mathbb{R}^n \tag{2.24}$$

this is nothing but a different way of writing the Γ-convergence result of (P_h) to (P_∞) (see [4], Proposition 2.1, page 388).

3 THE HAMILTON-JACOBI EQUATION

In order to link the convergence results of the previous section (Theorem 2.1 and Remark 2.2) we have to show that v_h is a viscosity solution of the Hamilton-Jacobi equation having H_h as Hamiltonian and that v_∞ is a viscosity solution of an Hamilton-Jacobi equation having as Hamiltonian H_∞.
For every fixed $h \in \mathbb{N}$ we consider the Hamilton-Jacobi equation

$$\begin{cases} -\frac{\partial}{\partial t} v_h(t_0, x_0) + H_h(t_0, x_0, Dv_h(t_0, x_0)) = 0 & \text{in } \Omega \times \mathbb{R}^n \\ v_h(T, x_0) = 0 & \text{in } \mathbb{R}^n \end{cases} \tag{3.1}$$

where $Dv_h(t,x)$ denotes the gradient with respect to the space variables of $v_h(t,x)$, and the Hamiltonian is given in (2.16). We use the following definition of viscosity solution.

Definition 3.1 *Let $v_h(t,x)$ be a continuous function on $\Omega \times \mathbb{R}^n$. We say that:*
$v_h(t,x)$ is a viscosity subsolution *of (3.1) in $\Omega \times \mathbb{R}^n$ if the following condition is satisfied. If $b \in L^1(0,T)$, $F(t,x,p) \in C([0,T] \times \mathbb{R}^n \times \mathbb{R}^n)$, and F satisfies for some $\delta > 0$, $(t_0,x_0) \in (0,T) \times \mathbb{R}^n$, $\phi \in C^1([0,T] \times \mathbb{R}^n)$,*

$$-b(t) + F(t,x,p) \leq H(t,x,p) \tag{3.2}$$

for $|x - x_0| < \delta$, $|p - D\phi(t_0,x_0)| < \delta$, a.e. $|t - t_0| < \delta$ and if

$$v_h(t,x) + \int_0^t b(s)ds - \phi(t,x)$$

has a local maximum at (t_0,x_0) then

$$-\frac{\partial}{\partial t}\phi(t_0,x_0) + F(t_0,x_0,D\phi(t_0,x_0)) \leq 0.$$

$v_h(t,x)$ is a viscosity supersolution *of (3.1) in $\Omega \times \mathbb{R}^n$ if the following condition is satisfied. If $b \in L^1(0,T)$, $F(t,x,p) \in C([0,T] \times \mathbb{R}^n \times \mathbb{R}^n)$, and F satisfies for some $\delta > 0$, $(t_0,x_0) \in (0,T) \times \mathbb{R}^n$, $\phi \in C^1([0,T] \times \mathbb{R}^n)$,*

$$-b(t) + F(t,x,p) \geq H(t,x,p) \tag{3.3}$$

for $|x - x_0| < \delta$, $|p - D\phi(t_0,x_0)| < \delta$, a.e. $|t - t_0| < \delta$ and if

$$v_h(t,x) + \int_0^t b(s)ds - \phi(t,x)$$

has a local minimum at (t_0,x_0) then

$$-\frac{\partial}{\partial t}\phi(t_0,x_0) + F(t_0,x_0,D\phi(t_0,x_0)) \geq 0.$$

$v_h(t,x)$ is a viscosity solution *of (3.1) in $\Omega \times \mathbb{R}^n$ if it is both viscosity subsolution and viscosity supersolution of (3.1) in $\Omega \times \mathbb{R}^n$ and if*

$$v_h(T,x) = 0 \ \forall \ x \in \mathbb{R}^n.$$

Remark 3.2 This is the first definition of viscosity solution for discontinuous Hamiltonians and has been introduced by Ishii in [9]. Starting from this work a lot of definitions have been given, depending on the regularity of the Hamiltonian and of the value function. For our case (continuous value function and measurable time-dependent Hamiltonian) we recall the work of Lions and Perthame [10]. In this work they show that their definition and Ishii's one are in fact equivalent and they obtain various equivalent forms of these notions. More recently Cardaliguet and Plaskacz in [7] introduced a different kind of definition coming from differential games but it is difficult to compare it directly with the ones given in [10].

We prove now that the value function (2.17) is a viscosity solution of the Hamilton-Jacobi equation (3.1).

Theorem 3.3 *If we suppose (2.4), (2.5), (2.6), (2.8), (2.9) and if there exists a control function $u_0 \in \mathbb{R}^m$ such that, for every $y \in Y$ and for every $h \in \mathbb{N}$*

$$f_h(\cdot, y(\cdot), u_0) \in L^1(\Omega; \mathbb{R}^n), \tag{3.4}$$

then, for every fixed $h \in \mathbb{N}$, $v_h(t_0, x_0)$ is a continuous viscosity solution of (3.1) in $\Omega \times \mathbb{R}^n$.

We need to prove this theorem because in the case of measurable time dependent Hamiltonians it does not fit into classical results (see for example [8], Section II.7). Before proving Theorem 3.3 we observe that it applies also to the limit problem.

Theorem 3.4 *Assume that*

$$M_h \to \overline{M} \quad \textit{weakly } L^1(\Omega; \mathbb{R})$$

$$\alpha_h(t) \to \overline{\alpha} \quad \textit{weakly } L^1(\Omega; \mathbb{R})$$

$$G_h(t) \to \overline{G} \quad \textit{weakly } L^1(\Omega; \mathbb{R})$$

and that for the u_0 given by (3.4) the sequence

$$f_h(\cdot, y(\cdot), u_0)$$

is bounded in $L^1(\Omega; \mathbb{R})$ for every $y \in Y$. Then v_∞ is a viscosity solution of

$$\begin{cases} -\frac{\partial}{\partial t} v_\infty(t_0, x_0) + H_\infty(t_0, x_0, Dv_\infty(t_0, x_0)) = 0 & \textit{in } \Omega \times \mathbb{R}^n \\ v_\infty(T, x_0) = 0 & \textit{in } \mathbb{R}^n \end{cases} \tag{3.5}$$

where the Hamiltonian is now given by (2.21).

Proof. It can be easily verified that $a(t, y)$ and $\alpha(t, y, U)$ satisfy the hypotheses (2.4), (2.5), (2.6), (2.8), (2.9) and (3.4), so Theorem 3.3 applies to this case too. □

In the proof of Theorem 3.3 we will need some preliminary results that we state below.

Lemma 3.5 *Under the hypotheses (2.4), (2.5) and (2.6), the solution $y_h^0(t; u)$ of the state equation (2.18) satisfies the inequality*

$$|y_h^0(t; u)| \le C_h(|x_0|, u) \tag{3.6}$$

where $C_h(|x_0|, u) = A_h\{|x_0| + M + \| B_h(\cdot, u(\cdot)) \|_1\}$ and $A_h = 1 + \| \alpha_h \|_1 \, e^{\|\alpha_h\|_1}$.

Moreover, if we denote by $y_h^1(t; u)$ the solution of (2.18) with initial condition $y(t_1) = x_1$ we have

$$|y_h^0(t; u) - y_h^1(t; u)| \le A_h \left\{ |x_0 - x_1| + \int_{t_0}^{t_1} \gamma_h(s, |x_0|, u)\, ds \right\} \quad \textit{if } \ t_0 < t_1$$

$$|y_h^0(t; u) - y_h^1(t; u)| \le A_h \left\{ |x_0 - x_1| + \int_{t_1}^{t_0} \gamma_h(s, |x_1|, u)\, ds \right\} \quad \textit{if } \ t_1 < t_0$$

where $\gamma_h(s, |x|, u) = \alpha_h(s) C_h(|x|, u) + M_h(s) + |B_h(s, u(s))|$.

Proof. The proof is a standard application of Gronwall's Lemma and it is left to the reader. □

Remark 3.6 It is easy to see that $C_h(|x|, u)$ and $\gamma_h(s, |x|, u)$ are both increasing functions on the variable $|x|$, and $\gamma_h(\cdot, x, u) \in L^1(\Omega; \mathbb{R})$ for every fixed $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$.

A basic tool in the proof of Theorem 3.3 is the Dynamic Programming Principle. Our assumption does not include the continuity of $a_h(\cdot, x)$, $B_h(\cdot, u)$, and $f_h(\cdot, y, u)$, but this result can be proved as in the standard case (see [8], page 9).
We recall the definition of value function:

$$v_h(t, x) = \inf_{u \in \mathcal{U}} \{J_h(t, x, u)\} = \inf_{u \in \mathcal{U}} \left\{ \int_t^T f_h(s, y_h^{x,t}(s; u), u(s))\, dt \right\} \tag{3.7}$$

where we denote by $y_h^{x,t}(s; u)$ (or $y_h^{x,t}(s)$) the solution of (2.18) with initial condition $y(t) = x$ and control u at any time $s \in \Omega$.

Theorem 3.7 *(Dynamic Programming Principle). If (2.4) and (2.5) are fulfilled and the space of controls is $\mathcal{U} = L^p(\Omega; \mathbb{R}^m)$ $(p \in (1, \infty))$ then for every $\tau \in [t, T]$*

$$v_h(t, x) = \inf_{u \in \mathcal{U}} \left\{ \int_t^\tau f_h(s, y_h^{x,t}(s; u), u(s))\, ds + v_h(\tau, y_h^{x,t}(\tau)) \right\}. \tag{3.8}$$

From now on we will denote by $\omega(\delta, x)$ a function such that $\lim_{\delta \to 0} \omega(\delta, x) = 0$ and that is monotone increasing on $|x|$ and by $\omega(\delta)$ a function such that $\lim_{\delta \to 0} \omega(\delta) = 0$. In the same way we will denote by $K(|x|)$ a function monotone increasing on $|x|$ and by K an arbitrary constant. We remark that ω and K may possibly vary from line to line. We prove now a useful estimate.

Lemma 3.8 *Under the hypotheses (2.4), (2.5), (2.6), (2.8), (2.9), (2.10), for every $t \in [0, T]$, for every $x \in \mathbb{R}$, for every $\varepsilon > 0$ if the control u_ε is such that*

$$v_h(t, x) + \varepsilon > \int_t^T f_h(s, y_h^{x,t}(s, u_\varepsilon), u_\varepsilon(s)) ds \tag{3.9}$$

we have the following estimate

$$\| B_h(\cdot, u_\varepsilon(\cdot)) \|_{L^q(t,T)} \leq K(|x|) + K\varepsilon \tag{3.10}$$

where the constant K does not depend on u_ε and t. Moreover

$$\| B_h(\cdot, u_\varepsilon(\cdot)) \|_{L^1(t,T)} \leq K(|x|) + K\varepsilon. \tag{3.11}$$

Proof. We first verify that $v_h(t, x)$ is locally bounded in $\Omega \times \mathbb{R}^n$. Let u_0 given by hypothesis (2.10), by definition (3.7) and by (2.8) we can write

$$v_h(t, x) \leq \int_t^T C(s, |y_h^{x,t}(s; u_0)|, 0)|y_h^{x,t}(s; u_0)|\, ds + \int_t^T f_h(s, 0, u_0(s))\, ds.$$

Applying (3.6) in Lemma 3.5 we have that

$$v_h(t, x) \leq \int_t^T C(s, C_h(|x|, u_0), 0) C_h(|x|, u_0)\, ds + \int_t^T f_h(s, 0, u_0(s))\, ds$$

which, together with Remark 3.6, tells us that $v_h(t,x)$ is locally bounded in $\Omega \times \mathbb{R}^n$, i.e.

$$v_h(t,x) \leq K(|x|). \tag{3.12}$$

Let u_ε satisfy (3.9); by (2.6) and (2.9)

$$\int_t^T |B_h(s,u_\varepsilon(s))|^q ds \leq 2^q \int_t^T (G_h(s)^q + D^q |u_\varepsilon(s)|^p)\, ds \leq$$

$$\leq 2^q \left\{ G^q + \frac{D^q}{a} \left(\int_t^T f_h(s, y_h^{x,t}(s,u_\varepsilon), u_\varepsilon(s)) ds + bT \right) \right\}$$

then, by hypothesis (3.9) and by (3.12) we can write

$$\int_t^T |B_h(s,u_\varepsilon(s))|^q ds < 2^q \left\{ G^q + \frac{D^q}{a} (v_h(t,x) + \varepsilon + bT) \right\} \leq$$

$$\leq 2^q \left(G^q + \frac{D^q}{a} (K(|x|) + \varepsilon + b) \right) = K(|x|) + K\varepsilon.$$

which gives us the estimate (3.10).
Finally the (3.11) is a consequence of the Hölder's inequality. □

We are now in position to prove that the value function is continuous in both variables.

Theorem 3.9 *If we suppose (2.4), (2.5), (2.6), (2.8), (2.9), (2.10), then the value function $v_h(t,x)$ is a continuous function on $[0,T] \times \mathbb{R}^n$.*

Proof. We split the proof in 3 steps.
Step 1. (Continuity in x).
Fix $\varepsilon > 0$ and let u_ε be a control such that

$$v_h(t,w) + \varepsilon > \int_t^T f_h(s, y_{h,\varepsilon}^{w,t}(s), u_\varepsilon(s)) ds$$

where $y_{h,\varepsilon}^{w,t}(s)$ is the solution of (2.18) with control u_ε and initial condition w at time t. If, similarly, $y_{h,\varepsilon}^{x,t}(s)$ is the solution of (2.18) with control u_ε but initial condition x at time t, by (3.7), (2.8) and using Lemma 3.5,

$$v_h(t,x) - v_h(t,w) \leq A_h |w-x| \int_t^T C(s, C_h(|w|,u_\varepsilon), C_h(|x|,u_\varepsilon)) ds + \varepsilon.$$

By definition of $C_h(|w|,u_\varepsilon)$ (see Lemma 3.5), and by Lemma 3.8

$$C_h(|w|,u_\varepsilon) \leq A_h\{|w| + M + K(|x|) + K\varepsilon\}.$$

Since w will tend to x we can suppose that there exists a constant K such that $|x| + |x-w| \leq K$, then, using this in the previous estimate we get

$$C_h(|w|,u_\varepsilon) \leq K(|x|, |x-w|) + K\varepsilon.$$

With a similar argument we have also

$$C_h(|x|,u_\varepsilon) \leq A_h\{|x| + M + K(|x|) + K\varepsilon\} = K(|x|) + K\varepsilon.$$

Our estimate becomes

$$v_h(t,x) - v_h(t,w) \le A_h|w-x| \int_t^T C(s, K(|x|,|x-w|) + K\varepsilon, K(|x|) + K\varepsilon))ds + \varepsilon \le$$

$$\le A_h|w-x| \int_0^T C(s, K(|x|,|x-w|) + K\varepsilon, K(|x|) + K\varepsilon))ds + \varepsilon.$$

Let ε going to zero we obtain, using also hypothesis (2.8),

$$v_h(t,x) - v_h(t,w) \le A_h|w-x| \int_0^T C(s, K(|x|,|x-w|), K(|x|))ds =$$

$$= K(|x|,|x-w|)|w-x|.$$

With a similar argument we can easily get the reverse inequality, to obtain the following estimate

$$|v_h(t,x) - v_h(t,w)| \le K(|x|,|w-x|)|w-x| \tag{3.13}$$

which gives us the continuity in x of the value function.
Step 2. (Continuity in t).
We begin proving the following inequality:

$$v_h(t,x) - v_h(t+\delta,x) \le K(|x|)\omega(\delta,x). \tag{3.14}$$

We consider first the case $\delta > 0$. By Dynamic Programming Principle we have the following inequality:

$$v_h(t,x) \le \int_t^{t+\delta} f_h(s, y_h^{x,t}(s;u_0), u_0)ds + v_h(t+\delta, y_h^{x,t}(t+\delta;u_0))$$

where the control u_0 satisfies hypothesis (2.10). Then

$$v_h(t,x) - v_h(t+\delta,x) \le \int_t^{t+\delta} f_h(s, y_h^{x,t}(s;u_0), u_0)ds + v_h(t+\delta, y_h^{x,t}(t+\delta;u_0)) - v_h(t+\delta,x). \tag{3.15}$$

Using hypothesis (2.8) we can estimate

$$\int_t^{t+\delta} f_h(s, y_h^{x,t}(s;u_0), u_0)ds \le$$

$$\le \int_t^{t+\delta} C(s, |y_h^{x,t}(x;u_0)|, 0)|y_h^{x,t}(x;u_0)|ds + \int_t^{t+\delta} f_h(s,0,u_0)ds \le$$

$$\le C_h(|x|,u_0) \int_t^{t+\delta} C(s, C_h(|x|,u_0), 0)ds + \int_t^{t+\delta} f_h(s,0,u_0)ds$$

where we used (3.6). Thus, by (2.8) and (2.10) we obtain:

$$\int_t^{t+\delta} f_h(s, y_h^{x,t}(s;u_0), u_0)ds \le K(|x|)\omega(\delta,x). \tag{3.16}$$

In order to estimate the second part of (3.15) we need the following inequalities which can be easily deduced from Gronwall's Lemma, Hölder's inequality and hypotheses (2.4), (2.5):

$$|y_h^{x,t}(t+\delta;u_0) - x| \le \omega(\delta,x) \quad , \quad |y_h^{x,t}(t+\delta;0) - x| \le K(|x|). \tag{3.17}$$

Thus, using also the continuity in x of the value function, in particular (3.13), we get

$$v_h(t+\delta, y_h^{x,t}(t+\delta;u_0)) - v_h(t+\delta,x) \le \omega(\delta,x)K(|x|). \tag{3.18}$$

Finally we can write the estimate for $\delta > 0$ inserting (3.16) and (3.18) in (3.15), i.e.

$$v_h(t,x) - v_h(t+\delta,x) \le K(|x|)\omega(\delta,x). \tag{3.19}$$

We consider now the case $\delta < 0$. Fix $\varepsilon > 0$, let u_ε^δ be the control such that

$$v_h(t+\delta,x) + \varepsilon > \int_{t+\delta}^T f_h(s, y_h^{x,t+\delta}(s;u_\varepsilon^\delta), u_\varepsilon^\delta)ds.$$

Then

$$v_h(t,x) - v_h(t+\delta,x) \le$$

$$\le \int_t^T \left(C(s, |y_h^{x,t}(s;u_\varepsilon^\delta)|, |y_h^{x,t+\delta}(s;u_\varepsilon^\delta)|)|y_h^{x,t}(s;u_\varepsilon^\delta) - y_h^{x,t+\delta}(s;u_\varepsilon^\delta)| \right) ds + \varepsilon \tag{3.20}$$

where we used (2.8) and the fact that f_h is a positive function.
By Lemma 3.5 and (3.11) we estimate

$$|y_h^{x,t}(s;u_\varepsilon^\delta)| \le C_h(|x|, u_\varepsilon^\delta) \le A_h \left(|x| + M + \| B_h(\cdot, u_\varepsilon^\delta(\cdot)) \|_{L^1(t+\delta,T)} \right) \le$$

$$\le A_h\left(|x| + M + (K(|x|) + K\varepsilon)\right) = K(|x|) + K\varepsilon \tag{3.21}$$

and

$$|y_h^{x,t+\delta}(s;u_\varepsilon^\delta)| \le A_h\left(|x| + M + (K(|x|) + K\varepsilon)\right) = K(|x|) + K\varepsilon. \tag{3.22}$$

By Lemma 3.5 we can estimate

$$|y_h^{x,t}(s;u_\varepsilon^\delta) - y_h^{x,t+\delta}(s;u_\varepsilon^\delta)| \le$$

$$\le A_h \left\{ \int_{t+\delta}^t (\alpha_h(s)(K(|x|) + K\varepsilon) + M_h(s))ds + (-\delta)^{1/q'}(K(|x|) + K\varepsilon) \right\} \tag{3.23}$$

were we used also (3.22) and Lemma 3.8. Plugging (3.21), (3.22), and (3.23) in (3.20) we have

$$v_h(t,x) - v_h(t+\delta,x) \le$$

$$\le \int_t^T (C(s, K(|x|) + K\varepsilon, K(|x|) + K\varepsilon))\, ds \,\cdot$$

$$\cdot \left(A_h \left\{ \int_{t+\delta}^t (\alpha_h(s)(K(|x|) + K\varepsilon) + M_h(s))ds + (-\delta)^{1/q'}(K(|x|) + K\varepsilon) \right\} \right) + \varepsilon$$

and letting ε go to zero we obtain

$$v_h(t,x) - v_h(t+\delta,x) \le$$

$$\le \int_t^T (C(s, K(|x|), K(|x|)))\, ds \cdot$$

$$\cdot \left(A_h \left\{ \int_{t+\delta}^t (\alpha_h(s)K(|x|) + M_h(s))ds + (-\delta)^{1/q'} K(|x|) \right\} \right).$$

By hypotheses (2.4), (2.5) and (2.8) we can conclude

$$v_h(t,x) - v_h(t+\delta,x) \le K(|x|)(\omega(\delta,x) + \omega(\delta)) = K(|x|)\omega(\delta,x)$$

which, with (3.19) gives us the required inequality. With the same arguments we obtain the reverse inequality, i.e.

$$v_h(t+\delta,x) - v_h(t,x) \leq K(|x|)\omega(\delta,x). \tag{3.24}$$

Step 3. (Final estimate).
Let $t \in [0,T]$, $\delta > 0$ and $x, w \in \mathbb{R}^n$, by (3.13), (3.14) and (3.24) we obtain

$$|v_h(t,x) - v_h(t+\delta,w)| \leq K(|x|)\omega(\delta,x) + K(|x|,|x-w|)|x-w|.$$

This implies

$$\lim_{\delta\to 0, w\to x} |v_h(t,x) - v_h(t+\delta,w)| = 0$$

which completes the proof. □

We are finally able to give the proof of Theorem 3.3.

Proof of theorem 3.3. We observe that by definition of $v_h(t,x)$ we have

$$v_h(T,x_0) = 0 \quad \forall x_0 \in \mathbb{R}^n.$$

In order to show that $v_h(t,x)$ is a viscosity solution of (3.1) in $\Omega\times\mathbb{R}^n$ we have to prove that it is a viscosity subsolution and a viscosity supersolution of (3.1) in $\Omega\times\mathbb{R}^n$.
To prove that $v_h(t,x)$ is a viscosity supersolution of (3.1) in $\Omega\times\mathbb{R}^n$ we take $\phi \in C^1([0,T]\times\mathbb{R}^n)$ and we choose $b \equiv 0 \in L^1(0,T)$ and a function $F(t,x,p) \in C([0,T]\times\mathbb{R}^n\times\mathbb{R})$ such that (3.3) is fulfilled. Let (t_0,x_0) be a local minimum for $v_h(t,x)-\phi(t,x)$, so there exists a $\delta > 0$ such that

$$|t-t_0| \leq \delta \;\; , \;\; |x-x_0| \leq \delta \Rightarrow \phi(t_0,x_0) - \phi(t,x) \geq v_h(t_0,x_0) - v_h(t,x). \tag{3.25}$$

Applying the Dynamic Programming Principle for every $\varepsilon > 0$ there exists a u_ε such that, denoting by $y^0_{h,\varepsilon}(t)$ the solution of (2.18) with initial condition $y_h(t_0) = x_0$ and control u_ε,

$$v_h(t_0,x_0) + \varepsilon > \int_{t_0}^{\tau} f_h(t, y^0_{h,\varepsilon}(t), u_\varepsilon(t))\, dt + v_h(\tau, y^0_{h,\varepsilon}(\tau)) \tag{3.26}$$

for every $\tau \in [t_0,T]$. Then in (3.25) we have for $|x-x_0| \leq \delta$ and $|t-t_0| \leq \delta$

$$\phi(t_0,x_0) - \phi(t,x) \geq$$

$$\geq \int_{t_0}^{\tau} f_h(t, y^0_{h,\varepsilon}(t), u_\varepsilon(t))\, ds - \varepsilon + v_h(\tau, y^0_{h,\varepsilon}(\tau)) - v_h(t,x). \tag{3.27}$$

For any $\tau \in [t_0,T]$, by the Hölder's inequality we have

$$\int_{t_0}^{\tau} |B_h(t,u_\varepsilon(t))|\, dt \leq \left\{ \int_{t_0}^{\tau} |B_h(t,u_\varepsilon(t))|^q\, dt \right\}^{1/q} (\tau - t_0)^{1/q'}$$

(where $1/q + 1/q' = 1$). From hypotheses (2.6) and (2.9) we deduce then

$$\int_{t_0}^{\tau} |B_h(t,u_\varepsilon(t))|^q\, dt \leq 2^q G^q + \frac{b}{a} 2^q(\tau - t_0) + \frac{2^q\, D^q}{a} \left(\int_{t_0}^{\tau} f_h(t, y^0_{h,\varepsilon}(t), u_\varepsilon(t))\, dt \right) \leq$$

$$\leq 2^q G^q + \frac{b}{a} 2^q(\tau - t_0) + \frac{2^q\ D^q}{a}(v_h(t_0,x_0)+\varepsilon)$$

where we used the (3.26). Moreover, since $v_h(t,x)$ is locally bounded, (see Lemma 3.8),

$$\int_{t_0}^{\tau} |B_h(t,u_\varepsilon(t))|\ dt \leq (\overline{B} + \frac{2^q\ D^q}{a}\varepsilon)^{1/q}\ (\tau - t_0)^{1/q'} \tag{3.28}$$

where we posed $\overline{B} = 2^q(G^q + \frac{D^q}{a}\ K(|x_0|) + \frac{b}{a}T)$. Applying Gronwall's Lemma, from (2.4), (2.5), (2.6) and (3.28) it follows that

$$|y^0_{h,\varepsilon}(\tau) - x_0| \leq A_h \int_{t_0}^{\tau} (\alpha_h(s)|x_0| + M_h(s))\ ds + (\overline{B} + \frac{2^q\ D^q}{a}\varepsilon)^{1/q}\ (\tau - t_0)^{1/q'}$$

since it is not restrictive to suppose $\varepsilon \leq 1$, we can fix a $\tau \in]t_0, t_0+\delta]$ such that $|y^0_{h,\varepsilon}(t) - x_0| \leq \delta$ for all $t \in]t_0,\tau]$. So, in the inequality (3.27) we can take $x = y^0_{h,\varepsilon}(\tau)$ and $t = \tau$ to obtain

$$\phi(t_0,x_0) - \phi(\tau, y^0_{h,\varepsilon}(\tau)) \geq \int_{t_0}^{\tau} f_h(t, y^0_{h,\varepsilon}(t), u_\varepsilon(t))\ dt - \varepsilon. \tag{3.29}$$

Since

$$\phi(t_0,x_0) - \phi(\tau, y^0_{h,\varepsilon}(\tau)) =$$

$$= \phi(t_0,x_0) - \phi(\tau,x_0) + \int_{t_0}^{\tau} -D\phi(\tau, y^0_{h,\varepsilon}(t)) \cdot (a_h(t, y^0_{h,\varepsilon}(t)) + B_h(t,u_\varepsilon(t)))\ dt$$

(3.29) becomes

$$\int_{t_0}^{\tau} \left[-D\phi(\tau, y^0_{h,\varepsilon}(t)) \cdot (a_h(t, y^0_{h,\varepsilon}(t)) + B_h(t,u_\varepsilon(t))) - f_h(t, y^0_{h,\varepsilon}(t), u_\varepsilon(t))\right]\ dt +$$

$$+\phi(t_0,x_0) - \phi(\tau,x_0) \geq -\varepsilon. \tag{3.30}$$

Before passing to the limit as $\varepsilon \to 0$ we observe that $y^0_{h,\varepsilon}$ depends on ε, so we first estimate its distance from x_0 in all the terms. By hypothesis (2.8), inequality (3.30) becomes

$$\phi(t_0,x_0) - \phi(\tau,x_0) + \int_{t_0}^{\tau} (-f_h(t,x_0,u_\varepsilon(t)) + C(t,|x_0|+\delta,|x_0|)\ \delta)\ dt +$$

$$+ \int_{t_0}^{\tau} \Phi|a_h(t,x_0) - a_h(t,y^0_{h,\varepsilon}(t))|\ dt + \int_{t_0}^{\tau} \omega(\delta)|a_h(t,x_0) + B_h(t,u_\varepsilon(t))|\ dt +$$

$$+ \int_{t_0}^{\tau} -D\phi(\tau,x_0) \cdot (a_h(t,x_0) + B_h(t,u_\varepsilon(t)))\ dt \geq -\varepsilon$$

where Φ is such that $|D\phi(y^0_{h,\varepsilon}(t),\tau)| \leq \Phi$. Using the estimate (3.28) and recalling the definition of the Hamiltonian (2.16) we obtain

$$\phi(t_0,x_0) - \phi(\tau,x_0) + \int_{t_0}^{\tau} H_h(t,x_0,D\phi(\tau,x_0))\ dt + \varepsilon + \delta \int_{t_0}^{\tau} \eta(t)\ dt +$$

$$\int_{t_0}^{\tau} \omega(\delta)|a_h(t,x_0)|\ dt + \omega(\delta)(\overline{B} + \frac{2^q\ D^q}{a}\varepsilon)^{1/q}(\tau - t_0)^{1/q'} \geq 0$$

where $\eta(t) = C(t, |x_0| + \delta, |x_0|) + \Phi\alpha_h(t) \in L^1(\Omega; \mathbb{R})$ and $|a_h(t, x_0)| \in L^1(\Omega; \mathbb{R})$.
It easy to check that $H_h(t, x_0, D\phi(\tau, x_0)) \in L^1(\Omega; \mathbb{R})$, then passing to the limit as $\varepsilon \to 0$ we obtain

$$\phi(t_0, x_0) - \phi(\tau, x_0) + \int_{t_0}^{\tau} H_h(t, x_0, D\phi(\tau, x_0))\, dt +$$

$$+\delta \int_{t_0}^{\tau} \eta(t)\, dt + \int_{t_0}^{\tau} \omega(\delta)|a_h(t, x_0)|\, dt + \omega(\delta) C_\phi \overline{B}^{1/q} (\tau - t_0)^{1/q'} \geq 0. \tag{3.31}$$

Finally, dividing by $\tau - t_0$ and passing to the limit as $\delta \to 0$ (this implies $\tau \to t_0$), the inequality (3.31) becomes

$$-\frac{\partial}{\partial t}\phi(t_0, x_0) + H_h(t_0, x_0, D\phi(t_0, x_0)) \geq 0 \;\text{ a.e. with respect to }\; t_0.$$

This implies

$$-\frac{\partial}{\partial t}\phi(t_0, x_0) + F(t_0, x_0, D\phi(t_0, x_0)) \geq 0$$

which proves that $v_h(t, x)$ is a viscosity supersolution of (3.1). With the same arguments we obtain that $v_h(t, x)$ is a viscosity subsolution of (3.1) and that complete the proof. □

We observe that using Theorems 3.3 and 3.4 and Remark 2.2, we have proved the following result.

Corollary 3.10 *We assume (2.13) and (2.15) (in the linear case (2.15) can be replaced by (4.4) and (4.5)); if v_h is the unique viscosity solution of (3.1) and v_∞ is the unique viscosity solution of (3.5), then*
$\lim_{h\to\infty} v_h(t, x) = v_\infty(t, x)$ *for all* $(t, x) \in [0, T] \times \mathbb{R}$.

This result is in fact a stability result for Hamilton-Jacobi equations related to optimal control problems. We observe that it does not fit into standard results (see for example Theorem 4.1, page 85 of [2] or Theorem 7.1, page 293 of [8]) since the assumptions we use here only guarantee weak L^1 convergence of the Hamiltonians, while in the literature pointwise convergence is required.

4 FURTHER REMARKS

We consider here an example with a state equation linear in the control variable u and a cost functional quadratic on the control u. The Hamiltonians can be explicitely written in this case, and the convergence result becomes more clear. The control problems we consider are

$$\min \left\{ \int_\Omega \left(\frac{|u(t)|^2}{2} + \psi(t, y) \right)\, dt \;:\; (y, u) \in \Lambda_h \right\} \tag{4.1}$$

where

$$\Lambda_h = \{(y, u) \in Y \times \mathcal{U} : y'(t) = a_h(t, y) + b_h(t)u(t), \;\; y(0) = y_h^0\}.$$

Our control space is $\mathcal{U} = L^2(\Omega, \mathbb{R}^m)$, and we take the $n \times m$ matrices b_h with coefficients in $L^\infty(\Omega; \mathbb{R})$ (we denote this space $L^\infty(\Omega; \mathbb{R}^{n\times m})$), so that (2.6) is fulfilled with $p = q = 2$. We suppose that $\{a_h\}$ verifies (2.4), (2.5), and that $\psi : \Omega \times \mathbb{R}^n \to [0, \infty]$ is a Borel function such that

i) $$\psi(\cdot, 0) \in L^1(\Omega; [0, \infty]);$$

ii) there exists a function $C_\psi(t, v, w)$, from $\Omega \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$, integrable with respect to t, increasing in v and w, such that for every $y, z \in \mathbb{R}^n$

$$|\psi(t, y) - \psi(t, z)| \le C_\psi(t, |y|, |z|)|y - z|.$$

It is easy to see that (2.6), (2.7), (2.8), (2.9), (2.10), and (2.11) are fulfilled. In order to characterize the limit problem we assume (2.13) and (2.14) for the convergence respectively of $\{a_h\}$ and $\{y_h^0\}$ and we consider two different hypotheses on the convergence of $\{b_h\}$. If we suppose that $\{b_h\}$ is bounded in $L^\infty(\Omega; \mathbb{R}^{n\times m})$ and

$$b_h(t) \to b(t) \quad \text{for a.e.} \quad t \in \Omega, \tag{4.2}$$

then the limit problem takes the form

$$\min \left\{ \int_\Omega \left(\frac{|u(t)|^2}{2} + \psi(t, y) \right) dt \ : \ (y, u) \in \Lambda \right\} \tag{4.3}$$

where

$$\Lambda = \{(y, u) \in Y \times \mathcal{U} : y'(t) = a(t, y) + b(t)u(t), \ \ y(0) = y_0\}.$$

On the contrary, if we suppose only that

$$b_h(t) \to b(t) \quad \text{weakly}^* \quad L^\infty(\Omega; \mathbb{R}^{n\times m}) \tag{4.4}$$

with

$$b_h(t)b_h(t)^T \to \beta(t) \quad \text{weakly}^* \quad L^\infty(\Omega; \mathbb{R}^{n\times m}) \tag{4.5}$$

(where $b_h(t)^T$ denotes the transpose matrix of $b_h(t)$), the limit problem becomes

$$\min \left\{ \int_\Omega \left(\frac{|u(t)|^2}{2} + \frac{1}{2} e(t) \cdot k(y)^{-1} e(t) + \psi(t, y) \right) dt \ : \ (y, u) \in Y \times \mathcal{U} \right\} \tag{4.6}$$

where $k(t)$ denotes the matrix $\beta(t) - b(t)b(t)^T$ and $e(t)$ is the vector $y'(t) - a(t, y) - b(t)u(t)$. This example is interesting because we can explicitly calculate the Hamiltonians in all cases; indeed, for problem (4.1) the (2.16) becomes

$$H_h(t, x, p) = \frac{1}{2}\, b_h(t)b_h(t)^T p \cdot p - \psi(t, x) - p \cdot a_h(t, x);$$

the (2.21) in the first case (problem (4.3)) is

$$H_\infty(t, x, p) = \frac{1}{2}\, b(t)b(t)^T p \cdot p - \psi(t, x) - p \cdot a(t, x)$$

while in the second (problem (4.6)) is

$$H_\infty(t, x, p) = \frac{1}{2}\, \beta(t) p \cdot p - \psi(t, x) - p \cdot a(t, x).$$

Remark 4.1 The theory of Γ-limits for sequences of optimal control problems in more general cases has been studied by Buttazzo and Freddi in [6]. They consider a linear state equation on the control u but they suppose only that the $|b_h|^{p'}$ are bounded in L^1, then measures naturally appear in the limit problem. If we associate a Hamilton-Jacobi equation to this limit problem, to understand whether or not we still have a convergence result for Hamiltonians and in which sense the limit Hamilton-Jacobi equation should be understood are open problems.

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