Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators

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Dedicated to Professor L.E. Payne on the occasion of his 80^{th} birthday

Abstract. We investigate the first eigenvalue of a highly nonlinear class of elliptic operators which includes the p–Laplace operator $\Delta_p u = \sum_i \frac{\partial}{\partial x_i}(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i})$, the pseudo–$p$–Laplace operator $\tilde{\Delta}_p u = \sum_i \frac{\partial}{\partial x_i}(|\frac{\partial u}{\partial x_i}|^{p-2}\frac{\partial u}{\partial x_i})$ and others. We derive the positivity of the first eigenfunction, simplicity of the first eigenvalue, Faber-Krahn and Payne-Rayner type inequalities. In another chapter we address the question of symmetry for positive solutions to more general equations. Using a Pohozaev-type inequality and isoperimetric inequalities as well as convex rearrangement methods we generalize a symmetry result of Kesavan and Pacella. Our optimal domains are level sets of a convex function H^o. They have the so-called Wulff shape associated with H and only in special cases they are Euclidean balls.

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1 Introduction

Throughout this paper let $H : I\!R^n \mapsto I\!R$ be a nonnegative convex function of class $C^1(I\!R^n\backslash\{0\})$ which is even and positively homogeneous of degree 1, so that

$$H(t\xi) = |t|H(\xi) \quad \text{for any} \quad t \in I\!R,\ \xi \in I\!R^n. \tag{1.1}$$

A typical example is $H(\xi) = (\sum_i |\xi_i|^q)^{1/q}$ for $q \in [1, \infty)$. Note that there are positive constants α and β such that H satisfies

$$\alpha|\xi| \leq H(\xi) \leq \beta|\xi| \quad \text{for any} \quad \xi \in I\!R^n. \tag{1.2}$$

We can assume without loss of generality that the convex closed set

$$K := \{x \in I\!R^n \ ; \ H(x) \leq 1\}$$

has the same measure $|K| = \omega_n$ as the unit ball in $I\!R^n$ equipped with the Euclidean l_2 norm.

We shall investigate Euler equations which involve functionals containing the expression

$$\int_\Omega (H(\nabla u))^p \, dx \ . \tag{1.3}$$

The differential equations contain operators of the form

$$Qu := \sum_{i=1}^n \frac{\partial}{\partial x_i} \left((H(\nabla u))^{p-1} H_{\xi_i}(\nabla u) \right) \ . \tag{1.4}$$

In particular for $H(\xi) = (\sum_k |\xi_k|^q)^{1/q}$ the operator Q becomes

$$Qu := \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\left(\sum_{k=1}^n \left| \frac{\partial u}{\partial x_k} \right|^q \right)^{(p-q)/q} \left| \frac{\partial u}{\partial x_i} \right|^{q-2} \frac{\partial u}{\partial x_i} \right) \ . \tag{1.5}$$

Note that for $q = 2$ this is the usual p–Laplace operator, while for $p = q$ it is the pseudo–p–Laplace operator that was extensively studied in [5]. For these cases and for $q = \infty$ the equation $-Qu = f(x)$ and its limit as $p \to \infty$ was adressed in [18]. For general H and $p = 2$ the equation $-Qu = f(x)$ in Ω was investigated in [2]. In [2] it became clear that the isoperimetric role that is played by the ball (among all domains Ω of given volume) is now replaced by a set homothetic to the polar set K^o of K. For $H(\xi) = c_q(\sum_i |\xi_i|^q)^{1/q}$ the set K^o is given by $K^o := \{x \in I\!R^n \ ; \ c_q^{-1}(\sum_i |\xi_i|^{q'})^{1/q'} \leq 1\}$, with q' conjugate to q and with

$$c_q^{-1} = \frac{q}{2} \frac{\Gamma(1/2)}{\Gamma(1/q)} \left(\frac{\Gamma(1 + n/q)}{\Gamma(1 + n/2)} \right)^{1/n} \ . \tag{1.6}$$

In this paper we study positive solutions of the general eigenvalue problem $Qu + \lambda_p |u|^{p-2} u = 0$ and of equations such as $Qu = f(u)$ with subcritical growth for f, both under zero boundary conditions.

2 Preliminaries

This section is devoted to recall some results about functions having the properties of the function H introduced in the previous section. Let $H : I\!R^n \to [0,+\infty[$ be a convex function satisfying the homogeneity property:

$$H(tx) = |t|H(x), \qquad \forall x \in I\!R^n,\ \forall t \in I\!R. \tag{2.1}$$

Recall that H satisfies

$$\alpha|\xi| \le H(\xi) \le \beta|\xi|, \qquad \forall \xi \in I\!R^n, \tag{2.2}$$

for some positive constants $\alpha \le \beta$. Because of (2.1) we can assume, without loss of generality, that the convex closed set

$$K = \{x \in I\!R^n : H(x) \le 1\}$$

has measure $|K|$ equal to the measure ω_n of the unit sphere in $I\!R^n$. Sometimes, we will say that H is the gauge of K. If one defines (see [31]) the support function of K as:

$$H^o(x) = \sup_{\xi \in K} < x, \xi >,$$

it is easy to verify that $H^o : I\!R^n \to [0,+\infty[$ is a convex, homogeneous function, and that H, H^o are polar to each other in the sense that:

$$H^o(x) = \sup_{\xi \neq 0} \frac{< x, \xi >}{H(\xi)}, \tag{2.3}$$

and

$$H(x) = \sup_{\xi \neq 0} \frac{< x, \xi >}{H^o(\xi)}. \tag{2.4}$$

For example it follows:

$$| < x, \xi > | \le H(x) H^o(\xi).$$

Clearly $H^o(x)$ itself is the gauge of the set:

$$K^o = \{x \in I\!R^n : H^o(x) \le 1\}.$$

We say that K and K^o are polar to each other. Finally we denote by k_n the measure of K^o. Further details can be found, e.g., in [21], [31].

Let Ω be an open subset of $I\!R^n$. It is possible to give the following definition of the total variation of a function $u \in BV(\Omega)$ with respect to a gauge function H (see [3]):

$$\int_\Omega |Du|_H = \sup\left\{ \int_\Omega u \operatorname{div}\sigma\, dx : \sigma \in C^1_0(\Omega; I\!R^n),\ H^o(\sigma) \le 1 \right\}.$$

This yields the following "generalized" definition of perimeter of a set E with respect to H:

$$P_H(E;\Omega) = \int_\Omega |D\chi_E|_H = \sup\left\{\int_E \operatorname{div}\sigma\, dx : \sigma \in C_0^1(\Omega;I\!R^n),\ H^o(\sigma) \le 1\right\}.$$

The following co-area formula

$$\int_\Omega |Du|_H = \int_0^\infty P_H(\{u > s\};\Omega)\, ds, \qquad \forall u \in BV(\Omega),$$

and the equality

$$P_H(E;\Omega) = \int_{\Omega\cap\partial^* E} H(\nu)\, d\mathcal{H}^{n-1}(x)$$

hold, where $\partial^* E$ is the reduced boundary of E and ν is the outer normal to E (see [3]).

One obtains readily that by definition $P_H(E;\Omega)$ is finite if and only if the usual perimeter:

$$P(E;\Omega) = \sup\left\{\int_E \operatorname{div}\sigma\, dx : \sigma \in C_0^1(\Omega;I\!R^n),\ |\sigma| \le 1\right\}$$

is finite. In fact, (2.1) and (2.2) give

$$\frac{1}{\beta}|\xi| \le H^o(\xi) \le \frac{1}{\alpha}|\xi|, \qquad \forall \xi \in I\!R^n,$$

and then

$$\alpha P(E;\Omega) \le P_H(E;\Omega) \le \beta P(E;\Omega).$$

Finally we quote a result which can be found in [2], [7], [13]

Theorem 2.1 (Wulff theorem)
If E is a set of finite perimeter in $I\!R^n$, then

$$P_H(E;I\!R^n) \ge n k_n^{1/n} |E|^{1-1/n}, \tag{2.5}$$

and equality holds if and only if E has Wulff shape, i.e. E is a sub-level set of H^o, modulo translations.

3 The eigenvalue problem

For $p \in (1,\infty)$ let us consider the following variational problem:

$$\lambda_p(\Omega) := \inf\left\{\int_\Omega (H(\nabla v))^p\ dx\ ;\ v \in W_0^{1,p}(\Omega)\ ,\ \int_\Omega |v|^p\ dx = 1\ \right\} \tag{3.1}$$

The direct method in the calculus of variations provides an existence proof for a minimizer u_p. Without loss of generality the minimizing function is nonnegative (otherwise replace u_p by $|u_p|$). But more can be said.

Theorem 3.1 (The first eigenvalue is simple and the first eigenfunction positive.) *Problem* (3.1) *has a unique positive solution u_p, which solves the Euler-Lagrange equation*

$$\begin{aligned} Qu_p + \lambda_p u_p &= 0 \quad \text{in } \Omega, \\ u_p &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.2}$$

Proof: We observe that u_p minimizes also the nonnegative functional $J_p(v) := \int_\Omega H(\nabla u)^p - \lambda_p(\Omega)|u|^p \, dx$. According to the regularity theory for quasiminima in [16] minimizers of J_p are bounded [16], Thm. 7.5, Hölder continuous [16], Thm. 7.6 and satisfy a strong maximum principle [16], Thm. 7.12. Therefore u_p is positive in Ω. Moreover, if $\partial\Omega \in C^2$, then $u_p \in C^{1,\alpha}(\overline{\Omega})$ according to [33], [32] or [23]. To prove its uniqueness we follow the idea from [4] and assume that there are two positive minimizers u and U of (3.1). For $t \in (0,1)$ we set $\eta = tu^p + (1-t)U^p$ and $u_t = \eta^{1/p}$. Notice that u_t is an admissible function for problem (3.1). To evaluate the functional in (3.1) at u_t we calculate

$$\nabla u_t = \eta^{-1+1/p} \left(tu^{p-1} \nabla u + (1-t) U^{p-1} \nabla U \right) ,$$

so that by the homogeneity of H

$$H(\nabla u_t) = \eta^{1/p} H \left(\frac{tu^p}{\eta} \frac{\nabla u}{u} + \frac{(1-t)U^p}{\eta} \frac{\nabla U}{U} \right) .$$

Now we set $s(x) := tu^p/\eta$, observe that it belongs to the interval $(0,1)$ and exploit first the convexity and then the homogeneity of H to arrive at

$$\begin{aligned} H(\nabla u_t)^p &= \eta H \left(s(x) \frac{\nabla u}{u} + (1 - s(x)) \frac{\nabla U}{U} \right)^p \\ &\leq \eta \left(s(x) H \left(\frac{\nabla u}{u} \right)^p + (1 - s(x)) H \left(\frac{\nabla U}{U} \right)^p \right) \\ &= tu^p H \left(\frac{\nabla u}{u} \right)^p + (1-t) U^p H \left(\frac{\nabla U}{U} \right)^p \\ &= tH(\nabla u)^p + (1-t) H(\nabla U)^p \ . \end{aligned} \tag{3.3}$$

After integration of (3.3) we see that also u_t is a minimizer and that equality holds a.e. in Ω in (3.3). But then $u\nabla U = U \nabla u$ a.e. in Ω, i.e. u/U is constant a.e.. This and the norm constraint for u implies uniqueness and completes the proof of Theorem 3.1

Remark 3.2 The last result was first proved by Lindqvist in [24], [25] for the $p-$Laplacian operator with a different proof.

Theorem 3.3 (Faber-Krahn type inequality)
Among all domains Ω of given volume the shape function $\lambda_p(\Omega)$ is minimized by a set homothetic to K^o, in other words $\lambda_p(\Omega) \geq \lambda_p(\Omega^*)$, where Ω^* has the Wulff shape (of K^o) and the volume of Ω. Moreover, equality holds only if $\Omega = \Omega^*$ modulo translation.

Proof: In Schwarz-symmetrization of functions one replaces their level sets by concentric balls of same measure. In convex symmetrization a la [2] the balls are replaced by equimeasurable centered sets homothetic to K^o. This way a function u with support $\overline{\Omega}$ is transformed into a function u^* with support $\overline{\Omega^*}$. We apply convex symmetrization to the first eigenfunction u_p on Ω. Then u_p^* is an admissible function for the variational problem that characterizes $\lambda_p(\Omega^*)$, because $\int_\Omega |u_p|^p \ dx = \int_{\Omega^*} |u_p^*|^p \ dx$. Now an application of the Pólya-Szegö type inequality [2], Thm. 3.1

$$\int_\Omega H(\nabla v)^p \ dx \geq \int_{\Omega^*} H(\nabla v^*)^p \ dx \qquad \text{for any } v \in W_0^{1,p}(\Omega) \tag{3.4}$$

shows that $\lambda_p(\Omega) \geq \lambda_p(\Omega^*)$. It remains to discuss the sharpness of the inequality. To do this one has to analyze the case of equality in (3.4). Fortunately this was already done in [11], Thm. 5.1 and in [12], and ends the proof of Theorem 3.3.

From Hölder's inequality one can see that for $0 < q < r$ the estimate $||u||_q \leq c(r, q, |\Omega|) \ ||u||_r$ holds for any $u \in L^r(\Omega)$ with $c(r, q, |\Omega|) = |\Omega|^{(r-q)/q}$. If u is an eigenfunction, then one can reverse the inequality sign.

Theorem 3.4 (Payne-Rayner type inequality)
If u_p solves Problem (3.1) then for every $0 < q < r \leq \infty$ there exists a positive constant $\beta(r, q, p, n, \lambda_p(\Omega))$ such that

$$||u_p||_r \leq \beta(r, q, p, n, \lambda_p) \ ||u_p||_q \ . \tag{3.5}$$

Moreover, equality in (3.5) holds if and only if $\Omega = \Omega^*$ modulo translation.

We can adapt the proof from [1], Section 4. We fix q and p and let v_q be a positive eigenfunction on a set B of shape K^o with eigenvalue $\lambda_p(\Omega)$ which is suitably scaled so that

$$||v_q||_{L^q(B)} = ||u_p||_{L^q(\Omega)} \ . \tag{3.6}$$

So v_q solves

$$\begin{cases} Qv_q + \lambda_p v_q & = 0 \qquad \text{in } B, \\ v_q & = 0 \qquad \text{on } \partial B. \end{cases} \tag{3.7}$$

with Q as in (1.4). Because of (3.1) it is clear that $|B| \leq |\Omega|$. Moreover B is given by

$$B = \left\{ x \in I\!R^n \ ; \ H^o(x) \leq (\kappa_p/\lambda_p(\Omega))^{1/p} \right\} \ , \tag{3.8}$$

where κ_p denotes the first eigenvalue of problem (3.7) in the unit ball K^o. The Payne-Rayner inequality is now a consequence of the following comparison result among the onedimensional rearrangements of u_p and v_q.

Theorem 3.5 (Comparison of rearrangements)
If u_p and v_q are as defined above we have

$$\begin{array}{rclll} \displaystyle\int_0^s (u_p^*(t))^q \ dt & \leq & \displaystyle\int_0^s (v_q^*(s))^q \ dt & \quad \text{for } s \in [0, |B|] \text{ and } q \in (0, \infty) & (3.9) \\ u_p^*(s) & \geq & v_\infty^*(s) & \quad \text{for } s \in [0, |B|] \text{ and } q = \infty \ . & \end{array}$$

Moreover, equality in (3.9) *holds if and only if $\Omega = B$ modulo translation.*

The proof of Theorem 3.5 is essentially contained in [1], pp. 446f and can be almost literally copied from there, except that now Lemma 4.2 in [1] has to be replaced by

Lemma 3.6 (Estimate for $-(u^*(s))'$)
If u_p solves Problem (3.1) *then the following inequality holds:*

$$-(u_p^*(s))' \leq \frac{\lambda_p^{1/(p-1)}}{\left(nk_n^{1/n} s^{1-1/n}\right)^{p/(p-1)}} \left(\int_0^s (u_p^*(\sigma))^{p-1} \ d\sigma \right)^{\frac{1}{p-1}} \quad \text{a.e. in } (0, |\Omega|). \tag{3.10}$$

For $p = 2$ the proof of this Lemma can be found in [2], pp. 285f, but we repeat it here for the benefit of the reader and for a more general application later. Suppose that $u \in W_0^{1,p}(\Omega)$ solves

$$\begin{cases} -Qu = f(x) & \quad \text{in } \Omega \\ \quad u = 0 & \quad \text{on } \partial\Omega \end{cases} \tag{3.11}$$

with Qu as in (1.4) and $f \in L^\infty(\Omega)$. A weak solution of (3.11) satisfies

$$\int_\Omega \sum_{i=1}^n \left((H(\nabla u))^{p-1} H_{\xi_i}(\nabla u) \frac{\partial \varphi}{\partial x_i} \right) \ dx = \int_\Omega f \varphi \ dx, \qquad \text{for all } \varphi \in W_0^{1,p}(\Omega).$$

Using the following test function $\varphi_h(x)$ for $h>0$ and $t>0$

$$\varphi_h(x)=\begin{cases} h & \text{if } |u|>t+h \\ \\ (|u|-t)\,\mathrm{sign}\,u & \text{if } t<|u|\le t+h \\ \\ 0 & \text{if } |u|\le t \end{cases}$$

one gets in a standard way,

$$-\frac{d}{dt}\int_{|u|>t}\sum_{i=1}^{n}\left((H(\nabla u))^{p-1}H_{\xi_i}(\nabla u)\frac{\partial u}{\partial x_i}\right)\,dx\le\int_0^{\mu(t)}f^*(s)ds, \tag{3.12}$$

where $\mu(t)=|\{u>t\}|$ is the distribution function of u. Taking the functional underlying (3.11) into account it follows

$$-\frac{d}{dt}\int_{|u|>t}(H(\nabla u))^p\le\int_0^{\mu(t)}f^*(s)ds. \tag{3.13}$$

At this point, one has to use the isoperimetric inequality (2.5)

$$P_H(E;I\!R^n)\ge nk_n^{1/n}|E|^{1-1/n} \tag{3.14}$$

in order to estimate the left-hand side of (3.13) from below. In fact, the estimate

$$P_H(E;I\!R^n)=-\frac{d}{dt}\int_{|u|>t}H(\nabla u)\,dx\le\left(-\frac{d}{dt}\int_{|u|>t}(H(\nabla u))^p\,dx\right)^{1/p}(-\mu'(t))^{1-1/p}$$

and (3.14) give

$$\left(nk_n^{1/n}\mu(t)^{1-1/n}\right)^p\le\left(-\frac{d}{dt}\int_{|u|>t}(H(\nabla u))^p\,dx\right)(-\mu'(t))^{p-1}\;. \tag{3.15}$$

Now a combination of (3.13) and (3.15) yields, for a.e. $t\in[0,\sup u)$,

$$1\le\frac{(-\mu'(t))}{\left(nk_n^{1/n}\mu(t)^{1-1/n}\right)^{p/(p-1)}}\left(\int_0^{\mu(t)}f^*(s)ds\right)^{\frac{1}{p-1}}. \tag{3.16}$$

For $f(x)=\lambda_p|u_p(x)|^{p-2}u_p(x)$ relation (3.16) becomes (3.10). This proves Lemma 3.6.

In contrast to the previous estimates the last estimate in this section is optimal (as $p\to1$) for any shape of Ω and not just for $\Omega=\Omega^*$.

Theorem 3.7 (Cheeger type inequality)
For every $p \in (1, \infty)$ the eigenvalue $\lambda_p(\Omega)$ can be estimated from below as follows:

$$\lambda_p(\Omega) \geq \left(\frac{h(\Omega)}{p} \right)^p . \tag{3.17}$$

Here $h(\Omega) := \inf\{ \ P_H(D, I\!R^n)/|D| \ ; \ D$ simply connected and $D \subset\subset \Omega \ \}$ *is the Cheeger constant associated with Ω. Moreover, $\lambda_p(\Omega)$ converges to $h(\Omega)$ as $p \to 1$.*

For $H(\xi) = |\xi|$ and $p = 2$ this is Cheeger's original estimate [6] and for $H(\xi) = |\xi|$ and general p it can be found in [22]. For a more general H one can easily modify the proof from [22] by using the generalized coarea formula from [11] or [12]. To prove the limiting behaviour of $\lambda_p(\Omega)$ as $p \to 1$ we proceed as in [19] and observe that (3.6) implies $\liminf_{p\to 1} \lambda_p(\Omega) \geq h(\Omega)$. Therefore is suffices to find a suitable upper bound. Let $\{D_k\}_{k=1,2,...}$ be a sequence of admissible domains for which $P_H(D_k, I\!R^n)/|D_k|$ converges to $h(\Omega)$. Approximate the characteristic function of each D_k by a function w_k with the following properties: $w \equiv 1$ on $\overline{D_k}$, $w \equiv 0$ outside an ε–neighborhood of D_k and $|\nabla w_k| = 1/\varepsilon$ in an ε–layer outside D_k. For small ε the function w_k is in $W_0^{1,\infty}(\Omega)$ and provides the upper bound

$$\lambda_p(\Omega) \leq \frac{P_H(D_k, I\!R^n)}{|D_k|} \, (\alpha\varepsilon)^{1-p} . \tag{3.18}$$

Now one sends first $p \to 1$, then $k \to \infty$ to complete the proof of Theorem 3.7.

4 Symmetry of positive solutions

In this section we consider positive solutions to the following problem:

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big((H(\nabla u))^{n-1} H_{\xi_i}(\nabla u)\big) = f(u) & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega = \{x \in I\!R^n : H^o(x) < R\}$, with $R > 0$, $n \geq 2$ has the so-called Wulff-shape. The function $f : I\!R \to I\!R$ is supposed to be continuous and to satisfy the following conditions:

$$f(s) > 0 \qquad \text{for} \quad s > 0, \tag{4.2}$$

$$f(s) \leq c_1|s|^r + c_2 \qquad \text{for} \quad s > 0, \tag{4.3}$$

for some $r > 0$.

Our aim is to show that any positive solution u to (4.1) necessarily satisfies a symmetry property. More precisely we prove that all level sets of u have Wulff shape and are homothetic to Ω. We immediately observe that such a result contains the well known symmetry result proved in [15] in the case that $H(\xi) = H^o(\xi) = |\xi|$ and Ω is a ball of radius R. We have the following:

Theorem 4.1
Suppose that H is strictly convex, that $H(\xi)^n$ is of class $C^2(I\!R^n)$ and that f satisfies the assumptions (4.2) *and* (4.3). *Then the level sets of any positive solution to problem* (4.1) *have Wulff shape and are homothetic to Ω.*

The proof of the above theorem is based on arguments similar to those in [26] (see also [20]), where H had the special structure $H(\xi) = (\sum_k |\xi_k|^n)^{1/n}$. A key point of the proof is a suitable version of the well known identity of Pohožaev [29]. Various extensions of this identity have been given under various assumptions on the structure of the differential operator in (4.1) and on the regularity of the solution u (see, for instance, [30], [17], [9], [14]). We will use the version contained in [9], which can be used because for $n \geq 2$ the function $H(\xi)^n$ is strictly convex.

Theorem 4.2 (Pohožaev identity from [9])
Under the assumptions of Theorem 4.1, suppose $u \in C^1(\bar{\Omega})$ is a weak solution to problem (4.1). *Then the following identity holds true:*

$$\frac{n-1}{n} \int_{\partial\Omega} H(\nabla u)^n < x, \nu > \, d\mathcal{H}^{n-1} = n \int_\Omega F(u)\, dx,$$

where ν is the outer normal to Ω and $F(s) = \displaystyle\int_0^s f(\sigma)\, d\sigma$.

To prove Theorem 4.1 suppose that $u \in W_0^{1,n}(\Omega)$ is a positive solution of (4.1). First we observe that $u \in C^1(\bar{\Omega})$ due to well-known regularity results contained in [10], [16], [23], [32], [33]. Therefore Theorem 4.2 applies and, taking into account the fact that (2.4) implies $< x, \nu >= R\, H(\nu)$ on $\partial\Omega$, we get

$$\frac{n-1}{n} R \int_{\partial\Omega} H(\nabla u)^n H(\nu)\, d\mathcal{H}^{n-1} = n \int_\Omega F(u)\, dx. \tag{4.4}$$

After integration of equation (4.1) on Ω we obtain

$$\begin{aligned} \int_\Omega f(u)\, dx &= -\int_{\partial\Omega} \sum_{i=1}^n \big((H(\nabla u))^{n-1} H_{\xi_i}(\nabla u) \nu_i \, d\mathcal{H}^{n-1} \qquad (4.5) \\ &\leq \left(\int_{\partial\Omega} (H(\nabla u))^n H(\nu)\, d\mathcal{H}^{n-1} \right)^{\frac{n-1}{n}} \left(\int_{\partial\Omega} H(\nu)\, d\mathcal{H}^{n-1} \right)^{\frac{1}{n}} \\ &= (nk_n)^{1/n} R^{\frac{n-1}{n}} \left(\int_{\partial\Omega} (H(\nabla u))^n H(\nu)\, d\mathcal{H}^{n-1} \right)^{\frac{n-1}{n}} , \end{aligned}$$

where we have used the following facts: because of the homogeneity of H we have $\sum_i H_{\xi_i}(\xi)\xi_i = H(\xi)$ and the isoperimetric inequality (2.5) holds as an equality for our special Ω, that is, $P_H(\Omega;I\!R^n) = nk_n^{1/n}|\Omega|^{1-1/n} = nk_nR^{n-1}$.

We can now apply the rearrangement techniques that were already used in the previous section. Arguing as in the proof of Lemma 3.6 in order to get (3.16) we easily obtain, for a.e. $t \in [0, \sup u)$,

$$n^n k_n \mu(t)^{n-1} \le (-\mu'(t))^{n-1} \int_{u>t} f(u)\,dx\ . \tag{4.6}$$

An integration, together with (4.6), yields

$$\begin{aligned}\int_\Omega F(u)\,dx &= \int_0^{\sup u} f(t)\mu(t)\,dt \qquad (4.7)\\ &\le \frac{1}{(nk_n^{1/n})^{\frac{n}{n-1}}}\int_0^{\sup u}\left(\int_{u>t} f(u)\,dx\right)^{\frac{1}{n-1}} f(t)(-\mu'(t))\,dt\\ &= \frac{n-1}{n(nk_n^{1/n})^{\frac{n}{n-1}}}\left(\int_\Omega f(u)\,dx\right)^{\frac{n}{n-1}}\ .\end{aligned}$$

Collecting (4.4), (4.5), (4.6) and (4.7) gives now

$$\int_\Omega f(u)\,dx \le \left(\frac{n^2}{n-1}\right)^{\frac{n-1}{n}} (nk_n)^{\frac{1}{n}}\left(\int_\Omega F(u)\,dx\right)^{\frac{n-1}{n}} \le \int_\Omega f(u)\,dx.$$

This means that all the inequalities we have used are in fact equalities. In particular (4.6) holds as an equality for a.e. $t \in [0, \sup u)$. By standard rearrangement properties we then get (see also the proof of Lemma 3.6):

$$u^*(s) = \frac{1}{\left(nk_n^{1/n}\right)^{n/(n-1)}}\int_s^{|\Omega|}\frac{1}{r}\left(\int_0^r f(u^*(\sigma))\,d\sigma\right)^{\frac{1}{n-1}} dr \quad \text{in } (0,|\Omega|) \tag{4.8}$$

It turns out (see [2]) that the function $u^\#(x) = u^*\big(k_n\big(H^o(x)\big)^n\big)$ solves problem (4.1). Therefore, using u and $u^\#$ as test functions in the differential equations for u and $u^\#$, we have

$$\int_\Omega (H(\nabla u))^n\ dx = \int_\Omega f(u)u\ dx = \int_\Omega f(u^\#)u^\#\ dx = \int_\Omega (H(\nabla u^\#))^n\ dx.$$

At this point we recall a result contained in [11], [12] which states that, provided $|\{x \in \Omega : 0 < u^\#(x) < \sup u,\ |\nabla u|(x) = 0\}| = 0$,

$$\int_\Omega (H(\nabla u))^n\ dx = \int_\Omega (H(\nabla u^\#))^n\ dx$$

if and only if $u = u^{\#}$. Observing that $u^{\#}$ has the required symmetry property the theorem will be completely proved if we verify that $|\{x \in \Omega : 0 < u^{\#}(x) < \sup u,\ |\nabla u|(x) = 0\}| = 0$. Indeed, it is sufficient to observe that from (4.8) and from assumption (4.2) it results that u^* is strictly decreasing in $(0, |\Omega|)$.

Remark 4.3 The approach used in the above proof allows us to obtain a symmetry result under hypotheses which, also in the simple case $H(\xi) = |\xi|$, are weaker than those needed in [15]. On the other hand our proof works only for a problem in the form (4.1) and it is not clear how to extend it to the case the differential operator is in the general form (1.4) with $p \neq n$ (see [26], [20] for similar observations). We also remark that, while in the case $H(\xi) = |\xi|$ the moving plane method turns out to be useful to prove results as Theorem 4.1, in our general context the possibility to apply it does not seem to be immediate.

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