Nonlinear thin-walled beams with rectangular cross-section - Part II

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Abstract

In this paper we report the second part of our results concerning the rigorous derivation of a hierarchy of one-dimensional models for thin-walled beams with rectangular cross-section. Denoting by h and $\delta_h \ll h$ the length of the sides of the cross-section of the beam, we analyse the limit behaviour of a non-linear elastic energy which scales as ε_h^2 when $\varepsilon_h/\delta_h \to 0$.

2001 AMS Mathematics Classification Numbers: 74K20, 74B10, 49J45

Keywords: thin-walled cross-section beams, nonlinear elasticity, Γ-convergence, dimension reduction

1 Introduction

The purpose of this paper is to continue the rigorous derivation started in [8] of a hierarchy of one-dimensional models for thin-walled beams. As explained in Part I, geometrically, a thin-walled beam is a slender structural element whose length is much larger than the diameter of the cross-section which, on its hand, is larger than the thickness of the thin wall. To model it, we consider a beam of length ℓ with a rectangular cross-section of sides h and δ_h with

$$h \to 0 \quad \text{and} \quad \frac{\delta_h}{h} \stackrel{h\to 0}{\longrightarrow} 0.$$

After rescaling the domain the elastic energy rewrites as

$$I^h(y) = \int_\Omega W(\nabla_h y(x))\,dx, \quad \text{with} \quad \nabla_h y = \left(y_{,1}\,, \frac{y_{,2}}{h}, \frac{y_{,3}}{\delta_h}\right),$$

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where W denotes the elastic energy density of the material, while y and $\nabla_h y$ denote, respectively, the deformation and the rescaled deformation gradient.

We let (y^h) be a sequence of deformations for which the energy scales as ε_h^2, where (ε_h) is a sequence of positive numbers; more precisely, we assume that

$$I^h(y^h) \leq C\varepsilon_h^2 \tag{1}$$

and we study the Γ-limit of the sequence of functionals I^h/ε_h^2. The expression of the Γ-limit depends on the behaviour of ε_h with respect to the intrinsic scale δ_h. More precisely, we identify three main regimes:

- subcritical: $\dfrac{\delta_h}{\varepsilon_h} \xrightarrow{h\to 0} 0$;
- critical: $\dfrac{\delta_h}{\varepsilon_h} \xrightarrow{h\to 0} 1$;
- supercritical: $\dfrac{\delta_h}{\varepsilon_h} \xrightarrow{h\to 0} +\infty$.

The subcritical and the critical regimes have been studied in [8]. In this paper we focus on the supercritical case.

Assuming $\varepsilon_h/\delta_h \to 0$, we first show that, if a sequence of deformations (y^h) satisfies (1), then the rescaled gradients $\nabla_h y^h$ must converge, as $h \to 0$, to a constant rotation (Lemma 3.1), which can be assumed to coincide with the identity, up to an orthonormal change of coordinates. Therefore, we expect to have linearization effects in the limiting energy. For this reason we introduce the sequence of displacements (u^h) and of twist functions (ϑ^h) associated with (y^h) and study their compactness properties (see Lemma 3.7). This part of the proof deeply relies on the rigidity estimate obtained by Friesecke, James, and Müller [9].

We then show that the Γ-limit of I^h/ε_h^2, as $h \to 0$, can be expressed in terms of the limit displacement u and of the limit twist function ϑ, and depends on the existence and on the value of the following limit:

$$r := \lim_{h\to 0} \frac{\varepsilon_h}{\delta_h^2}.$$

We distinguish the three regimes $r = 0$, $r = +\infty$, and $r \sim 1$. By a rescaling of the cross-section, the last one can be reduced to the case $r = 1$. If $r \in \{0, 1\}$, we first prove that the limit displacement u must belong to the set $\mathcal{A}^{BN}$ of Bernoulli-Navier displacements (see Definition 3.6). Moreover, in Theorems 3.11 and 3.14 we show that for these values of r the Γ-limit of I^h/ε_h^2 is the functional $I^r : \mathcal{A}^{BN} \times W^{1,2}(0,\ell) \to [0, +\infty)$ defined by

$$I^r(u,\vartheta) := \tfrac{1}{24}\int_0^\ell Q_2(\xi_3'', \vartheta')\,dx_1 + \tfrac{1}{2}\int_0^\ell E\big(\xi_1' + \tfrac{r}{2}(\xi_3')^2\big)^2\,dx_1 + \tfrac{1}{24}\int_0^\ell E\big(\xi_2''\big)^2\,dx_1$$

for every $(u,\vartheta) \in \mathcal{A}^{BN} \times W^{1,2}(0,\ell)$. Here the functions $\xi_1 \in W^{1,2}(0,\ell)$, ξ_2, $\xi_3 \in W^{2,2}(0,\ell)$ are such that

$$u_1(x) = \xi_1(x_1) - x_2\xi_2'(x_1) - x_3\xi_3'(x_1), \quad u_2(x) = \xi_2(x_1), \quad u_3(x) = \xi_3(x_1)$$

for a.e. $x \in \Omega$. The density function Q_2 is a positive definite quadratic form, while E is a positive constant, and they both can be easily computed from the knowledge of W (see (6) and (7)). If the beam is made of an isotropic material, the constant E coincides with the Young modulus of the material.

If, instead, $r = +\infty$, we prove that the limit displacement u must have the following structure: for a.e. $x \in \Omega$

$$u_1(x) = \xi_1(x_1), \quad u_2(x) = 0, \quad u_3(x) = \xi_3(x_1),$$

with $\xi_1,\, \xi_3 \in W^{2,2}(0, \ell)$ satisfying

$$\xi_1' = -\frac{1}{2}(\xi_3')^2.$$

We denote by $\mathcal{A}^\infty$ the set of all displacements in $W^{2,2}(\Omega; \mathbb{R}^3)$ satisfying these conditions. Assuming in addition that

$$\lim_{h\to 0} \frac{h^2 \varepsilon_h}{\delta_h^2} = 0, \tag{2}$$

in Theorems 3.11 and 3.12 we show that, for $r = +\infty$, the Γ-limit of I^h/ε_h^2 is given by the functional

$$I^\infty(u, \vartheta) := \tfrac{1}{24} \int_0^\ell Q_2(u_3'', \vartheta')\, dx_1$$

for every $(u, \vartheta) \in \mathcal{A}^\infty \times W^{1,2}(0, \ell)$.

Assumption (2) is crucial in the construction of the recovery sequence. Heuristically, it allows us to stretch the mid-plane, i.e., the x_1x_2-plane, by deformations of order $\varepsilon_h/(\delta_h/h)^2$. When $\lim_{h\to 0} \varepsilon_h/(\delta_h/h)^2 \neq 0$ the mid-plane must undergo a deformation which is very close to an infinitesimal isometry. For this reason we conjecture that, in this range, the Γ-limit should coincide with the Γ-limit of the geometrically linear Kirchhoff functional for a rectangular plate (see [10]), representing the mid-plane of the beam, when the length of one of the two sides approaches zero.

Γ-convergence results for thin-walled beams were obtained within the theory of linear elasticity in [5, 6, 7], while Γ-convergence results for beams within the nonlinear framework were deduced in [1, 12, 13, 14, 15].

The paper is organized as follows. In Section 2 we recall the setting of the problem and some preliminary results. Section 3 is devoted to the discussion of the supercritical case. Finally, in Section 4 we introduce applied loads and prove convergence of minimizers.

The notation is the same adopted in Part I of the present paper, to which we refer for details.

2 Setting of the problem and preliminaries

Let

$$\Omega_h := (0, \ell) \times \omega_h \subset \mathbb{R}^3,$$

where

$$\omega_h := \{(z_2, z_3) \;:\; |z_2| < h/2, |z_3| < \delta_h/2\} \subset \mathbb{R}^2$$

with $h > 0$, $\delta_1 := 1$ and

$$\lim_{h\to 0} \delta_h/h = 0.$$

Henceforth we shall refer to Ω_h as the reference configuration of the body and denote the elastic energy associated with a deformation $v : \Omega_h \to \mathbb{R}^3$ by

$$E^h(v) := \int_{\Omega_h} W(\nabla v(z))\, dz.$$

We assume that the stored energy density $W : \mathbb{R}^{3\times 3} \to [0, +\infty]$ satisfies the following assumptions:

1. $W \in C^0(\mathbb{R}^{3\times 3})$, W is of class C^2 in a neighborhood of $SO(3)$;
2. W is frame indifferent, i.e., $W(F) = W(RF)$ for every $F \in \mathbb{R}^{3\times 3}$ and $R \in SO(3)$;
3. $W(F) \geq C\,\mathrm{dist}^2(F, SO(3))$, $C > 0$; $W(F) = 0$ if $F \in SO(3)$.

A key role will be played by the following quadratic form:

$$Q_3(F) := \frac{\partial^2 W}{\partial F^2}(I)(F, F) = \sum_{i,j,k,l=1}^{3} \frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}(I) F_{ij} F_{kl}, \qquad F \in \mathbb{R}^{3\times 3}. \tag{3}$$

In view of 3 this form is positive semi-definite and hence convex. Moreover, by 1 and 2 we have that (see, e.g., [11, Section 29])

$$Q_3(F) = Q_3\Big(\frac{F + F^T}{2}\Big). \tag{4}$$

In the special case when the energy density W is isotropic, that is, $W(RFQ) = W(F)$ for all $F \in \mathbb{R}^{3\times 3}$ and $R, Q \in SO(3)$, then it turns out that

$$Q_3(F) = 2\mu|e|^2 + \lambda(\mathrm{tr}\, e)^2, \qquad e = \frac{F + F^T}{2} \tag{5}$$

for some λ, $\mu \in \mathbb{R}$.

The limit problems will be stated in terms of the density function

$$Q_2(\alpha, \beta) := \min\{Q_3(A) \;:\; A \in \mathbb{R}^{3\times 3},\, A^T = A,\, A_{11} = \alpha,\, A_{12} = \beta\}, \tag{6}$$

and of the constant

$$E := \min\{Q_2(1, \beta) \;: \beta \in \mathbb{R}\} = \min\{Q_3(A) \;:\; A \in \mathbb{R}^{3\times 3},\, A^T = A,\, A_{11} = 1\}. \tag{7}$$

Let us remark that Q_2 is a positive definite quadratic form and $E > 0$. Moreover, in the isotropic case where Q_3 takes the form (5), a simple computation shows that

$$Q_2(\alpha, \beta) = 4\mu\beta^2 + E\alpha^2,$$

and $E = \mu\frac{2\mu+3\lambda}{\mu+\lambda}$ is the Young modulus of the material.

To state our results it is convenient to stretch the domain Ω_h along the transverse directions z_2 and z_3 in a way that the transformed domain does not depend on h. Let us therefore set $\omega := \omega_1$, $\Omega := \Omega_1$, and let

$$p_h : \Omega \to \Omega_h$$

be defined by

$$p_h(x) = p_h(x_1, x_2, x_3) = (x_1, hx_2, \delta_h x_3). \tag{8}$$

Let us consider the following 3×3 matrix

$$\nabla_h y := \left(y_{,1}\, , \frac{y_{,2}}{h}, \frac{y_{,3}}{\delta_h} \right), \tag{9}$$

where $y_{,i}$ denotes the column vector of the partial derivatives of y with respect to x_i, $i = 1, 2, 3$. Then we can consider the rescaled energy $I^h : W^{1,2}(\Omega; \mathbb{R}^3) \to [0, +\infty]$ defined by $I^h(y) := \frac{1}{h\delta_h} E^h(y \circ p_h^{-1})$, i.e.,

$$I^h(y) = \int_\Omega W(\nabla_h y(x))\, dx$$

for every $y \in W^{1,2}(\Omega; \mathbb{R}^3)$.

Throughout the rest of the paper (ε_h) will denote a sequence of strictly positive real numbers. We conclude the section by recalling a result proven in [8, Theorem 3.2 and Lemma 3.3] and concerning some general compactness properties for sequences of deformations with equibounded energy.

Lemma 2.1 *Let (y^h) be a sequence in $W^{1,2}(\Omega; \mathbb{R}^3)$ such that*

$$\left(\int_\Omega \mathrm{dist}^2(\nabla_h y^h, SO(3))\, dx \right)^{\frac{1}{2}} \leq C\varepsilon_h \tag{10}$$

for every $h > 0$. Then, there exist two sequences $R^h : (0, \ell) \times (-1/2, 1/2) \to SO(3)$ and

$$\tilde{R}^h \in C^\infty((0, \ell) \times (-\tfrac{1}{2}, \tfrac{1}{2}); \mathbb{R}^{3\times 3})$$

such that

1. $\|\tilde{R}^h - R^h\|_{L^2} \leq C\varepsilon_h, \quad \|\tilde{R}^h - R^h\|_{L^\infty} \leq Ch^{1/2}\varepsilon_h/\delta_h,$
2. $\|\nabla_h y^h - \tilde{R}^h\|_{L^2} \leq C\varepsilon_h,$
3. $\|\tilde{R}^h_{,1}\|_{L^2} \leq C\varepsilon_h/\delta_h, \quad \|\tilde{R}^h_{,2}\|_{L^2} \leq Ch\varepsilon_h/\delta_h,$

where the constant C may change from line to line. Moreover, if in addition $h^{1/2}\varepsilon_h/\delta_h \to 0$, then we can take

4. *$\tilde{R}^h(x_1, x_2) \in SO(3)$ for every $(x_1, x_2) \in (0, \ell) \times (-1/2, 1/2)$ and every $h > 0$.*

3 The supercritical case

This section is devoted to the study of the asymptotic behaviour of a sequence of deformations $(y^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ satisfying

$$I^h(y^h) = \int_\Omega W(\nabla_h y^h)\, dx \le C\varepsilon_h^2 \tag{11}$$

for every $h > 0$, where

$$\lim_{h\to 0} \frac{\varepsilon_h}{\delta_h} = 0. \tag{12}$$

Under these assumptions, properties *3* and *4* of Lemma 2.1 imply that the sequence $\tilde{R}^h$ converges weakly in $W^{1,2}$ to a constant rotation R. In the next lemma we introduce suitable rotations and translations of the coordinate system in such a way to deal with a limit rotation equal to the identity.

Lemma 3.1 *Let (y^h) be a sequence in $W^{1,2}(\Omega;\mathbb{R}^3)$ satisfying* (10) *and let $(\tilde{R}^h)$ be the sequence constructed in Lemma 2.1. Then, there exists a sequence of constant rotations $Q^h \in SO(3)$ such that, setting $\bar{R}^h = {Q^h}^T \tilde{R}^h$ and $\bar{y}^h = {Q^h}^T y^h - c^h$, where c^h is any constant, we have*

1. $\|\nabla_h \bar{y}^h - \bar{R}^h\|_{L^2} \le C\varepsilon_h$,
2. $\|\bar{R}^h{}_{,1}\|_{L^2} \le C\dfrac{\varepsilon_h}{\delta_h}, \quad \|\bar{R}^h{}_{,2}\|_{L^2} \le Ch\dfrac{\varepsilon_h}{\delta_h}$,
3. $\|\bar{R}^h - I\|_{L^2} \le C\dfrac{\varepsilon_h}{\delta_h}$,

where the constant C may change from line to line. Moreover, if in addition (12) *holds, then for every h small enough we can take*

4. *$\bar{R}^h(x_1,x_2) \in SO(3)$ for every $(x_1,x_2) \in (0,\ell)\times(-1/2,1/2)$*

and

$$\int_\Omega (\nabla_h \bar{y}^h - \nabla_h {\bar{y}^h}^T)\, dx = 0. \tag{13}$$

Proof. By Sobolev-Poincaré inequality there exist some constant matrices $\tilde{Q}^h \in \mathbb{R}^{3\times 3}$ such that

$$\|\tilde{R}^h - \tilde{Q}^h\|_{L^2} \le C\|\nabla \tilde{R}^h\|_{L^2} \le C\frac{\varepsilon_h}{\delta_h}, \tag{14}$$

where the last inequality follows from property *3* of Lemma 2.1. Let R^h be the sequence of approximating rotations constructed in Lemma 2.1. The first inequality in *1* of that lemma and (14) yield

$$\|R^h - \tilde{Q}^h\|_{L^2} \le C\frac{\varepsilon_h}{\delta_h},$$

and since $R^h \in SO(3)$, this implies that

$$\text{dist}(\tilde{Q}^h, SO(3)) \le C\frac{\varepsilon_h}{\delta_h}.$$

Thus, there exists $\hat{Q}^h \in SO(3)$ such that

$$|\hat{Q}^h - \tilde{Q}^h| \le C\frac{\varepsilon_h}{\delta_h}.$$

Setting $\hat{R}^h := {\hat{Q}^h}{}^T \tilde{R}^h$ and using (14), we obtain

$$\|\hat{R}^h - I\|_{L^2} = \|\tilde{R}^h - \hat{Q}^h\|_{L^2} \le \|\tilde{R}^h - \tilde{Q}^h\|_{L^2} + \|\tilde{Q}^h - \hat{Q}^h\|_{L^2} \le C\frac{\varepsilon_h}{\delta_h},$$

that is, property *3* of the statement for the sequence $\hat{R}^h$. Moreover, setting $\hat{y}^h := {\hat{Q}^h}{}^T y^h - \hat{c}^h$, where $\hat{c}^h$ is any constant, we deduce properties *1* and *2* of the statement for $(\hat{y}^h)$ and $(\hat{R}^h)$ from properties *2* and *3* of Lemma 2.1.

Assume now (12). Property *4* of the statement for $(\hat{R}^h)$ follows immediately from *4* of Lemma 2.1. In order to satisfy also (13) we need to modify the constructed sequences. Let

$$F^h := \fint_\Omega \nabla_h \hat{y}^h \, dx.$$

Then, from properties *1* and *3* for the sequences with an over-hat, we have

$$\begin{aligned} |F^h - I| &\le \fint_\Omega |\nabla_h \hat{y}^h - I| \, dx \le C\|\nabla_h \hat{y}^h - I\|_{L^2} \\ &\le C(\|\nabla_h \hat{y}^h - \hat{R}^h\|_{L^2} + \|\hat{R}^h - I\|_{L^2}) \le C\frac{\varepsilon_h}{\delta_h}. \end{aligned} \tag{15}$$

By (12) this implies, in particular, that $\det F^h > 0$ for h sufficiently small. Thus, by the polar decomposition theorem, there exist $P^h \in SO(3)$ and a positive symmetric matrix U^h such that $F^h = P^h U^h$. Since $|U^h - I| = \text{dist}(F^h, SO(3)) \le |F^h - I|$, we have

$$|P^h - I| \le |P^h - F^h| + |F^h - I| = |U^h - I| + |F^h - I| \le C\frac{\varepsilon_h}{\delta_h}, \tag{16}$$

where we have used (15). We claim that

$$Q^h := \hat{Q}^h P^h, \quad \bar{R}^h := {P^h}^T \hat{R}^h, \quad \bar{y}^h := {P^h}^T \hat{y}^h = {Q^h}^T y^h - c^h$$

satisfy properties *1*–*4* and (13) of the lemma. Indeed, conditions *1*, *2*, and *4* are immediate, while *3* follows from (16), since we have

$$\|\bar{R}^h - I\|_{L^2} \le \|\bar{R}^h - \hat{R}^h\|_{L^2} + \|\hat{R}^h - I\|_{L^2} = \|P^h - I\|_{L^2} + \|\hat{R}^h - I\|_{L^2}.$$

Finally, since $\nabla_h \bar{y}^h = {P^h}^T \nabla_h \hat{y}^h$, we have

$$\begin{aligned}\int_\Omega (\nabla_h \bar{y}^h - {\nabla_h \bar{y}^h}^T)\, dx &= |\Omega|({P^h}^T F^h - ({P^h}^T F^h)^T)\\ &= |\Omega|(U^h - {U^h}^T) = 0,\end{aligned}$$

hence also (13) is satisfied. □

Remark 3.2 Since the energy density W is frame indifferent, the energy I^h on a deformation y does not change if a rigid motion is superimposed to y; therefore, a sequence of deformations (y^h) satisfying (11) is not, in general, bounded in any reasonable space. In Lemma 3.1 to obtain bounds we have superimposed an appropriate rigid motion $r^h(x) := {Q^h}^T x - c^h$ to each deformation y^h. The motion r^h is not uniquely determined; indeed, if we replace Q^h by $Q^h \exp(\frac{\varepsilon_h}{\delta_h} K)$, then properties *1, 2, 3* are still satisfied and condition *4* may be obtained arguing as in the proof of the lemma. However, one can easily show that if (Q^h) and $(\tilde{Q}^h)$ are two sequences of constant rotations for which the lemma is true, then $|Q^h - \tilde{Q}^h| \le C \frac{\varepsilon_h}{\delta_h}$.

In the next lemma we study the implications of the bounds obtained in Lemma 3.1.

Lemma 3.3 *Assume* (12). *Let* (y^h) *be a sequence in* $W^{1,2}(\Omega;\mathbb{R}^3)$ *satisfying* (10) *and let* $(\bar{R}^h)$ *be the sequence constructed in Lemma 3.1. Then there exist three tensor fields* $A \in W^{1,2}((0,\ell);\mathbb{R}^{3\times3})$, $B \in L^2((0,\ell)\times(-\frac{1}{2},\frac{1}{2});\mathbb{R}^{3\times3})$, *and* $G \in L^2(\Omega;\mathbb{R}^{3\times3})$, *with* A *and* B *skew-symmetric, such that, up to subsequences,*

1. $A^h := \dfrac{\bar{R}^h - I}{\varepsilon_h/\delta_h} \rightharpoonup A$ *in* $W^{1,2}((0,\ell)\times(-\frac{1}{2},\frac{1}{2});\mathbb{R}^{3\times3})$,

2. $\operatorname{sym}\dfrac{\bar{R}^h - I}{(\varepsilon_h/\delta_h)^2} \to \dfrac{A^2}{2}$ *in* $L^2((0,\ell)\times(-\frac{1}{2},\frac{1}{2});\mathbb{R}^{3\times3})$,

3. $B^h := \dfrac{{\bar{R}^h}_{,2}}{h\varepsilon_h/\delta_h} \rightharpoonup B$ *in* $L^2((0,\ell)\times(-\frac{1}{2},\frac{1}{2});\mathbb{R}^{3\times3})$,

4. $G^h := \dfrac{{\bar{R}^h}^T \nabla_h \bar{y}^h - I}{\varepsilon_h} \rightharpoonup G$ *in* $L^2(\Omega;\mathbb{R}^{3\times3})$.

Moreover, we have

5. $A_{,1}e_2 = Be_1$, *hence* $A_{12,1} = B_{12} = 0$ *and* $A_{23,1} = B_{13}$ *a.e. in* $(0,\ell)\times(-\frac{1}{2},\frac{1}{2})$,

6. $G(x)e_1 = x_3 A_{,1}(x_1)e_3 + \tilde{G}(x_1,x_2)e_1$ *for a.e.* $x \in \Omega$,

7. $G(x)e_2 = x_3 B(x_1,x_2)e_3 + \tilde{G}(x_1,x_2)e_2$ *for a.e.* $x \in \Omega$,

for a suitable $\tilde{G} \in L^2((0,\ell) \times (-\frac{1}{2}, \frac{1}{2}); \mathbb{R}^{3\times 3})$.

PROOF. By *3* of Lemma 3.1 the sequence A^h is bounded in L^2, hence it admits a subsequence which converges weakly in L^2. Let A denote this weak limit. By *2* of Lemma 3.1 we have that $A^h_{,2} \to 0$ in L^2, while the derivative with respect to x_1 is bounded in L^2. This implies that, up to subsequences, $A^h \rightharpoonup A$ weakly in $W^{1,2}$ and that the limit A is independent of x_2. Since $\bar{R}^h \in SO(3)$, we have

$$A^h + {A^h}^T = -\frac{\varepsilon_h}{\delta_h} {A^h}^T A^h, \tag{17}$$

and passing to the limit as $h \to 0$, we obtain that $A + A^T = 0$.

We now prove *2*. By (17) we have that

$$\operatorname{sym} \frac{\bar{R}^h - I}{(\varepsilon_h/\delta_h)^2} = \operatorname{sym} \frac{A^h}{\varepsilon_h/\delta_h} = -\frac{{A^h}^T A^h}{2}.$$

The claim now follows from *1*, the compact embedding theorem, and the fact that A is skew-symmetric.

Let us prove *3*. The weak convergence of a subsequence of B^h follows from the second estimate in *2* of Lemma 3.1. Let us call B its weak limit. Since

$$0 = \Big(\frac{{\bar{R}^h}^T \bar{R}^h}{h\varepsilon_h/\delta_h} \Big)_{,2} = \frac{{\bar{R}^h}^T_{,2}}{h\varepsilon_h/\delta_h} \bar{R}^h + {\bar{R}^h}^T \frac{\bar{R}^h{}_{,2}}{h\varepsilon_h/\delta_h},$$

we deduce that $\operatorname{sym} B = 0$ by passing to the limit and using *3* of Lemma 3.1.

Convergence *4* is an immediate consequence of *1* of Lemma 3.1.

Property *5* follows from the equality

$$A^h_{,1} e_2 = \Big(\frac{\bar{R}^h e_2 - \bar{y}^h_{,2}/h}{\varepsilon_h/\delta_h} \Big)_{,1} + \Big(\frac{\bar{y}^h_{,1} - \bar{R}^h e_1}{h\varepsilon_h/\delta_h} \Big)_{,2} + \frac{\bar{R}^h{}_{,2} e_1}{h\varepsilon_h/\delta_h} \quad \text{in } H^{-1}(\Omega; \mathbb{R}^{3\times 3}).$$

Indeed, by using *1* of Lemma 3.1 and *3* to pass to the limit we obtain $A_{,1} e_2 = B e_1$. This rewrites as $A_{i2,1} = B_{i1}$ for $i = 1, 2, 3$, from which the remaining relations in *5* follow by using the fact that A and B are skew-symmetric.

To prove *6* we note that in $H^{-1}(\Omega; \mathbb{R}^{3\times 3})$ there holds

$$(\bar{R}^h G^h e_1)_{,3} = \Big(\frac{\bar{y}^h_{,3}/\delta_h - \bar{R}^h e_3}{\varepsilon_h/\delta_h} \Big)_{,1} + \Big(\frac{\bar{R}^h e_3 - e_3}{\varepsilon_h/\delta_h} \Big)_{,1}$$

and, using *1* and *3* of Lemma 3.1, and *1* and *4* already proven, we find

$$G_{,3} e_1 = A_{,1} e_3,$$

from which we obtain *6*. Equation *7* can be proven similarly. □

Hereafter we assume that the following limit exists:

$$r := \lim_{h \to 0} \frac{\varepsilon_h}{\delta_h^2}. \tag{18}$$

Without loss of generality, we may assume that $r \in \{0, 1, +\infty\}$, by possibly changing the value of the constant C, appearing in (11), and by taking $\omega_h = (-ah/2, ah/2) \times (-b\delta_h/2, b\delta_h/2)$ for appropriate constants a and b.

In the next lemma we take a closer look at the rescaled displacement gradient.

Lemma 3.4 *Under the same assumptions of Lemma 3.3, we have*

i. the sequence $\Big(\dfrac{\nabla_h \bar{y}^h - I}{\varepsilon_h/\delta_h}\Big)$ *admits a subsequence which converges to A in* $L^2(\Omega; \mathbb{R}^{3\times 3})$,

where A is the field introduced in Lemma 3.3. With r defined as in (18)*, the following statements hold:*

ii. if $r = +\infty$, then, up to extracting a subsequence, $\mathrm{sym}\Big(\dfrac{\nabla_h \bar{y}^h - I}{(\varepsilon_h/\delta_h)^2}\Big) \to \dfrac{A^2}{2}$ *in* $L^2(\Omega; \mathbb{R}^{3\times 3})$,

iii. if $r \in \{0, 1\}$, then $\mathrm{sym}\Big(\dfrac{\nabla_h \bar{y}^h - I}{\varepsilon_h}\Big)$ *is bounded in* $L^2(\Omega; \mathbb{R}^{3\times 3})$.

Proof. Statement *i* follows by observing that

$$\frac{\nabla_h \bar{y}^h - I}{\varepsilon_h/\delta_h} = \frac{\nabla_h \bar{y}^h - \bar{R}^h}{\varepsilon_h}\delta_h + \frac{\bar{R}^h - I}{\varepsilon_h/\delta_h}$$

and using *1* of Lemma 3.1 and *1* of Lemma 3.3.

Assume now (18). Statement *ii* follows from

$$\mathrm{sym}\Big(\frac{\nabla_h \bar{y}^h - I}{(\varepsilon_h/\delta_h)^2}\Big) = \mathrm{sym}\Big(\frac{\nabla_h \bar{y}^h - \bar{R}^h}{\varepsilon_h}\Big)\frac{\delta_h^2}{\varepsilon_h} + \mathrm{sym}\Big(\frac{\bar{R}^h - I}{(\varepsilon_h/\delta_h)^2}\Big) \tag{19}$$

and by *1* of Lemma 3.1 and *2* of Lemma 3.3.

Similarly, for *iii* we have

$$\mathrm{sym}\Big(\frac{\nabla_h \bar{y}^h - I}{\varepsilon_h}\Big) = \mathrm{sym}\Big(\frac{\nabla_h \bar{y}^h - \bar{R}^h}{\varepsilon_h}\Big) + \mathrm{sym}\Big(\frac{\bar{R}^h - I}{(\varepsilon_h/\delta_h)^2}\Big)\frac{\varepsilon_h}{\delta_h^2}, \tag{20}$$

and again the claim follows from *1* of Lemma 3.1 and *2* of Lemma 3.3. □

We now define two sets of displacements which will play a crucial role in what follows.

Definition 3.5 *Let $\mathcal{A}^\infty$ be the class of all displacements $u \in W^{2,2}((0,\ell); \mathbb{R}^3)$ satisfying the following property: there exist ξ_1, $\xi_3 \in W^{2,2}(0,\ell)$ such that*

$$u_1 = \xi_1, \quad u_2 = 0, \quad u_3 = \xi_3,$$

with

$$\xi_1' = -\frac{1}{2}(\xi_3')^2. \tag{21}$$

Definition 3.6 *Let $\mathcal{A}^{BN}$ be the set of (Bernoulli-Navier) displacements $u \in W^{1,2}(\Omega;\mathbb{R}^3)$ satisfying the following condition: there exist $\xi_1 \in W^{1,2}(0,\ell)$, ξ_2, $\xi_3 \in W^{2,2}(0,\ell)$ such that*

$$u_1 = \xi_1 - x_2\xi_2' - x_3\xi_3', \quad u_2 = \xi_2, \quad u_3 = \xi_3.$$

Let

$$\mathcal{A}^r := \begin{cases} \mathcal{A}^\infty & \text{if } r = +\infty, \\ \mathcal{A}^{BN} & \text{if } r \in \{0,1\}. \end{cases}$$

In the next lemma we introduce the twist of the cross-section and we study its convergence together with the convergence of the displacements.

Lemma 3.7 *Under the same assumptions of Lemma 3.3, let $\vartheta^h : (0,\ell) \to \mathbb{R}$ be defined by*

$$\vartheta^h := \frac{1}{I_0}\frac{1}{\varepsilon_h}\int_\omega \big(\frac{\delta_h}{h}x_2\bar{y}_3^h - x_3\bar{y}_2^h\big)\, dx_2dx_3,$$

where

$$I_0 := \int_\omega (x_2^2 + x_3^2)\, dx_2dx_3 = \frac{1}{6}.$$

Let A, G, and $\tilde{G}$ be the fields introduced in Lemma 3.3. Then

$$\vartheta^h \rightharpoonup \vartheta := A_{32} \text{ in } W^{1,2}(0,\ell),$$

and for a.e. $x \in \Omega$

$$\begin{aligned} G_{12}(x) &= -x_3\vartheta'(x_1) + \tilde{G}_{12}(x_1,x_2), \\ G_{21}(x) &= -x_3\vartheta'(x_1) + \tilde{G}_{21}(x_1,x_2). \end{aligned} \tag{22}$$

Let r be as in (18)*. Then the following statements hold:*

i. if $r = +\infty$, then, for a suitable choice of the constants c^h in Lemma 3.1, the sequence of displacements $u^h : \Omega \to \mathbb{R}^3$ defined by

$$\begin{aligned} u_1^h &:= \frac{\bar{y}_1^h - x_1}{(\varepsilon_h/\delta_h)^2}, \\ u_2^h &:= \frac{\bar{y}_2^h - hx_2}{\varepsilon_h/\delta_h}, \\ u_3^h &:= \frac{\bar{y}_3^h - \delta_h x_3}{\varepsilon_h/\delta_h}, \end{aligned} \tag{23}$$

admits a subsequence which converges in $W^{1,2}(\Omega;\mathbb{R}^3)$ to a function $u \in \mathcal{A}^\infty$. Moreover,

$$G_{11}(x) = -x_3\xi_3''(x_1) + \tilde{G}_{11}(x_1,x_2) \tag{24}$$

for a.e. $x \in \Omega$.

ii. If $r \in \{0,1\}$, then, for a suitable choice of the constants c^h in Lemma 3.1, the sequence of displacements $u^h : \Omega \to \mathbb{R}^3$ defined by

$$\begin{aligned} u_1^h &:= \frac{\bar{y}_1^h - x_1}{\varepsilon_h}, \\ u_2^h &:= \frac{\bar{y}_2^h - hx_2}{\varepsilon_h/h}, \\ u_3^h &:= \frac{\bar{y}_3^h - \delta_h x_3}{\varepsilon_h/\delta_h}, \end{aligned} \tag{25}$$

admits a subsequence which converges weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$ to a function $u \in \mathcal{A}^{BN}$. Moreover,

$$G_{11}(x) = \xi_1'(x_1) - x_2\xi_2''(x_1) - x_3\xi_3''(x_1) + \frac{r}{2}(\xi_3'(x_1))^2 \tag{26}$$

for a.e. $x \in \Omega$.

Proof. Since $A = A(x_1)$, the convergence in *i* of Lemma 3.4 implies that

$$\begin{aligned} \frac{1}{h\varepsilon_h/\delta_h}\Big(\bar{y}_3^h - \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{y}_3^h \, dx_2\Big) \quad &\to \quad A_{32}x_2 \quad \text{ in } L^2(\Omega), \\ \frac{1}{\varepsilon_h}\Big(\bar{y}_2^h - \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{y}_2^h \, dx_3\Big) \quad &\to \quad -A_{32}x_3 \quad \text{ in } L^2(\Omega). \end{aligned}$$

Since ϑ^h can be written as

$$\begin{aligned} \vartheta^h \quad = \quad & \frac{1}{I_0}\frac{1}{h\varepsilon_h/\delta_h}\int_\omega x_2\Big(\bar{y}_3^h - \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{y}_3^h \, dx_2\Big) \, dx_2 dx_3 \\ & - \frac{1}{I_0}\frac{1}{\varepsilon_h}\int_\omega x_3\Big(\bar{y}_2^h - \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{y}_2^h \, dx_3\Big) \, dx_2 dx_3, \end{aligned}$$

it is clear that ϑ^h converges to $\vartheta := A_{32}$ strongly in L^2. The convergence is actually weak in $W^{1,2}$, as $(\vartheta^h)'$ is bounded in L^2. Indeed, using the fact that $\bar{R}^h$ is independent of x_3, we obtain

$$\begin{aligned} (\vartheta^h)' \quad = \quad & \frac{1}{I_0}\frac{1}{h\varepsilon_h/\delta_h}\int_\omega x_2(\bar{y}_{3,1}^h - \bar{R}^h{}_{31}) \, dx_2 dx_3 \\ & + \frac{1}{I_0}\frac{1}{h\varepsilon_h/\delta_h}\int_\omega x_2\Big(\bar{R}^h{}_{31} - \int_{-\frac{1}{2}}^{\frac{1}{2}} \bar{R}^h{}_{31} \, dx_2\Big) \, dx_2 dx_3 \\ & - \frac{1}{I_0}\frac{1}{\varepsilon_h}\int_\omega x_3(\bar{y}_{2,1}^h - \bar{R}^h{}_{21}) \, dx_2 dx_3, \end{aligned}$$

where the first and the last term on the right-hand side are bounded in L^2 by *1* of Lemma 3.1, while the second term is bounded in L^2 by Poincaré-Wirtinger inequality and the second estimate in *2* of Lemma 3.1.

Finally, by *5–7* of Lemma 3.3 we deduce (22).

Proof of i. For $r = +\infty$, let us choose the constants c^h in Lemma 3.1 in such a way that $\bar{y}^h - (x_1, hx_2, \delta_h x_3)$ has zero average, and let us define $\hat{u}^h : \Omega \to \mathbb{R}^3$ by

$$\hat{u}^h := \frac{\bar{y}^h - (x_1, hx_2, \delta_h x_3)}{\varepsilon_h/\delta_h}. \tag{27}$$

From *i* of Lemma 3.4 we have that $\nabla_h \hat{u}^h = \frac{\nabla_h \bar{y}^h - I}{\varepsilon_h/\delta_h}$ admits a subsequence, not relabeled, converging to A in $L^2(\Omega; \mathbb{R}^{3\times 3})$. Hence $\nabla \hat{u}^h$ is a Cauchy sequence in $L^2(\Omega; \mathbb{R}^{3\times 3})$ and, since $\bar{y}^h - (x_1, hx_2, \delta_h x_3)$ has zero average, we have that

$$\hat{u}^h \to \hat{u} \text{ in } W^{1,2}(\Omega; \mathbb{R}^3).$$

Moreover, since $\nabla_h \hat{u}^h$ is bounded in $L^2(\Omega; \mathbb{R}^{3\times 3})$ and $\nabla_h \hat{u}^h e_1 = \hat{u}^h_{,1}$, we deduce that $\hat{u} = \hat{u}(x_1)$ and $Ae_1 = \hat{u}_{,1}$. In particular, since the matrix A is skew-symmetric, we deduce that $\hat{u}_{1,1} = A_{11} = 0$, $\hat{u}_{2,1} = A_{21} = -A_{12}$ and $\hat{u}_{3,1} = A_{31} = -A_{13}$. By *5* of Lemma 3.3 we have that $A_{12,1} = 0$ and hence, $\hat{u}_{2,11} = 0$. Putting these information together, and using also the fact that $\hat{u}$ has zero average, we obtain that there exist a constant α and a function $\xi_3 \in W^{2,2}(0, \ell)$, with $\int_0^\ell \xi_3(x_1)\, dx_1 = 0$, such that

$$\hat{u}_1 = 0, \quad \hat{u}_2 = \alpha\big(x_1 - \frac{\ell}{2}\big), \quad \hat{u}_3 = \xi_3. \tag{28}$$

On the other hand, using (27) and (13), we have

$$\int_\Omega \big(\hat{u}^h_{2,1} - \frac{1}{h}\hat{u}^h_{1,2}\big)\, dx = \frac{1}{\varepsilon_h/\delta_h} \int_\Omega \big(\bar{y}^h_{2,1} - \frac{1}{h}\bar{y}^h_{1,2}\big)\, dx = 0,$$

hence, taking the limit as $h \to 0$, we obtain

$$0 = \int_\Omega \big(\hat{u}_{2,1} - A_{12}\big)\, dx = 2 \int_\Omega \hat{u}_{2,1}\, dx,$$

which, in turn, implies that $\alpha = 0$ in (28). Hence

$$\hat{u}_1 = 0, \quad \hat{u}_2 = 0, \quad \hat{u}_3 = \xi_3. \tag{29}$$

Let now $\bar{u}^h : \Omega \to \mathbb{R}^3$ be defined by

$$\bar{y}^h_1 =: x_1 + \big(\frac{\varepsilon_h}{\delta_h}\big)^2 \bar{u}^h_1, \quad \bar{y}^h_2 =: hx_2 + \big(\frac{\varepsilon_h}{\delta_h}\big)^2 \frac{\bar{u}^h_2}{h}, \quad \bar{y}^h_3 =: \delta_h x_3 + \big(\frac{\varepsilon_h}{\delta_h}\big)^2 \frac{\bar{u}^h_3}{\delta_h},$$

so that

$$\frac{\nabla_h \bar{y}^h - I}{(\varepsilon_h/\delta_h)^2} = \begin{pmatrix} \bar{u}^h_{1,1} & \bar{u}^h_{1,2}/h & \bar{u}^h_{1,3}/\delta_h \\ \bar{u}^h_{2,1}/h & \bar{u}^h_{2,2}/h^2 & \bar{u}^h_{2,3}/(h\delta_h) \\ \bar{u}^h_{3,1}/\delta_h & \bar{u}^h_{3,2}/(h\delta_h) & \bar{u}^h_{3,3}/\delta_h^2 \end{pmatrix}. \tag{30}$$

By (13) we have that

$$\int_\Omega (\nabla \bar{u}^h - \nabla \bar{u}^{h^T})\, dx = 0. \tag{31}$$

Since we have also that $\int_\Omega \bar{u}^h\,dx = 0$, by Korn inequality there exists a constant C_K such that

$$\|\operatorname{sym}\nabla\bar{u}^h\|_{L^2} \ge C_K\|\bar{u}^h\|_{W^{1,2}}. \tag{32}$$

By (30) and *ii* of Lemma 3.4 we have that $\operatorname{sym}\nabla\bar{u}^h$ admits a Cauchy subsequence in $L^2(\Omega;\mathbb{R}^3)$; hence inequality (32) implies that there exists $\bar{u} \in W^{1,2}(\Omega;\mathbb{R}^3)$ such that, up to a subsequence,

$$\bar{u}^h \to \bar{u} \text{ in } W^{1,2}(\Omega;\mathbb{R}^3).$$

Moreover, from (30) and *ii* of Lemma 3.4 it follows that $\bar{u}_{i,\alpha} + \bar{u}_{\alpha,i} = 0$ for every $i = 1,2,3$ and $\alpha = 2,3$, hence $\bar{u}$ is a Bernoulli-Navier displacement. In other words, there exist $\bar{\xi}_1 \in W^{1,2}(0,\ell)$ and $\bar{\xi}_2, \bar{\xi}_3 \in W^{2,2}(0,\ell)$ such that

$$\bar{u}_1 = \bar{\xi}_1 - x_2\bar{\xi}_2' - x_3\bar{\xi}_3', \quad \bar{u}_2 = \bar{\xi}_2, \quad \bar{u}_3 = \bar{\xi}_3. \tag{33}$$

Noticing that $\bar{u}_3^h = \hat{u}_3^h\,\delta_h^2/\varepsilon_h$ and recalling that $r = \lim_{h\to 0}\frac{\varepsilon_h}{\delta_h^2} = +\infty$ yield that $\bar{u}_3^h \to 0$ in $W^{1,2}(\Omega)$; hence, $\bar{u}_3 = \bar{\xi}_3 = 0$. Thus, (33) reduces to

$$\bar{u}_1 = \bar{\xi}_1 - x_2\bar{\xi}_2', \quad \bar{u}_2 = \bar{\xi}_2, \quad \bar{u}_3 = 0. \tag{34}$$

By (30) and *ii* of Lemma 3.4 we deduce that $\bar{u}_{1,1} = \frac{1}{2}(A^2)_{11}$. Thus, recalling that A is skew-symmetric, we find

$$\bar{u}_{1,1} = -\frac{1}{2}((A_{21})^2 + (A_{31})^2) = -\frac{1}{2}((\hat{u}_{2,1})^2 + (\hat{u}_{3,1})^2), \tag{35}$$

and using (29) and (34), we deduce

$$\bar{\xi}_1' - x_2\bar{\xi}_2'' = -\frac{1}{2}(\xi_3')^2.$$

Since the right-hand side depends only on x_1, this implies

$$\bar{\xi}_2'' = 0 \quad \text{and} \quad \bar{\xi}_1' = -\frac{1}{2}(\xi_3')^2. \tag{36}$$

From $\int_\Omega \bar{u}_2\,dx = 0$ and $\bar{\xi}_2'' = 0$, we deduce that $\bar{\xi}_2 = \bar{k}(x_1 - \frac{\ell}{2})$ for some constant $\bar{k}$. But, as a consequence of (31), we have that

$$\int_\Omega (\bar{u}_{1,2} - \bar{u}_{2,1})\,dx = 0,$$

which implies $\bar{k} = 0$. Hence, we conclude that

$$\bar{u}_1 = \bar{\xi}_1, \quad \bar{u}_2 = 0, \quad \bar{u}_3 = 0.$$

Moreover, since $\xi_3 \in W^{2,2}(0,\ell)$, we deduce by (36) that $\bar{\xi}_1 \in W^{2,2}(0,\ell)$.

The proof of the statement concerning the convergence of u^h follows now by the analysis above after setting $u_1^h := \bar{u}_1^h$, $u_2^h := \hat{u}_2^h$, $u_3^h := \hat{u}_3^h$, and $\xi_1 := \bar{\xi}_1$. Finally, since $A_{31} = \hat{u}_{3,1} = \xi_3'$, equality *6* of Lemma 3.3 implies (24).

Proof of ii. Let now $r \in \{0, 1\}$. The proof of this case is very similar to a part of the proof of *i*, thus we only sketch it. Noticing that

$$\bar{y}^h_1 = x_1 + \varepsilon_h u^h_1, \quad \bar{y}^h_2 = hx_2 + \varepsilon_h \frac{u^h_2}{h}, \quad \bar{y}^h_3 = \delta_h x_3 + \varepsilon_h \frac{u^h_3}{\delta_h},$$

we have

$$\frac{\nabla_h \bar{y}^h - I}{\varepsilon_h} = \begin{pmatrix} u^h_{1,1} & u^h_{1,2}/h & u^h_{1,3}/\delta_h \\ u^h_{2,1}/h & u^h_{2,2}/h^2 & u^h_{2,3}/(h\delta_h) \\ u^h_{3,1}/\delta_h & u^h_{3,2}/(h\delta_h) & u^h_{3,3}/\delta_h^2 \end{pmatrix}. \tag{37}$$

By part *iii* of Lemma 3.4 and by Korn inequality we deduce that $u^h \rightharpoonup u$ in $W^{1,2}(\Omega; \mathbb{R}^3)$ with $u \in \mathcal{A}^{BN}$. Moreover, from (37) it also follows that

$$\frac{(\nabla_h \bar{y}^h)_{11} - 1}{\varepsilon_h} \rightharpoonup u_{1,1} \text{ in } L^2(\Omega).$$

Multiplying both sides of (37) by δ_h and using *i* of Lemma 3.4, we obtain that $A_{12} = 0$ and $A_{13} = u_{1,3}$. Passing to the limit in the identity

$$(\bar{R}^h G^h)_{11} = \frac{(\nabla_h \bar{y}^h)_{11} - 1}{\varepsilon_h} - \frac{\bar{R}^h{}_{11} - 1}{(\varepsilon_h/\delta_h)^2} \frac{\varepsilon_h}{\delta_h^2},$$

after recalling *3* of Lemma 3.1, *2* and *4* of Lemma 3.3, and the definition of r, we find

$$G_{11} = u_{1,1} - r\frac{(A^2)_{11}}{2} = u_{1,1} + r\frac{(A_{12})^2 + (A_{13})^2}{2},$$

and this completes the proof. □

Remark 3.8 The definitions of u^h and ϑ^h in Lemma 3.7 are given in terms of the deformations $\bar{y}^h$, which in turn depend on the sequence of constant rotations (Q^h), introduced in Lemma 3.1. By Remark 3.2 any two sequences of constant rotations satisfying Lemma 3.1 have difference going to zero, as $h \to 0$. Using this fact, one can show that the limits of ∇u^h and ϑ^h are in fact independent of the choice of (Q^h).

Remark 3.9 We give here a geometrical interpretation of G_{11} and of the constraint (21). By explicitly writing $\varepsilon_h G_h^T G_h \to 0$ in $L^1(\Omega; \mathbb{R}^{3\times3})$ we deduce that

$$\frac{1}{2\varepsilon_h}(\nabla_h y^{h^T} \nabla_h y^h - I) \rightharpoonup \operatorname{sym} G \;\text{ in } L^1(\Omega; \mathbb{R}^{3\times3}). \tag{38}$$

Hence $\operatorname{sym} G$ is the limit of a rescaled sequence of Green-St. Venant strain tensors; thus, G_{11} measures the length's variation of fibers parallel to the axis of the beam (see [2]). The component of (38) on the first row and first column can be rewritten as

$$\frac{(\nabla_h y^h - I)_{11}}{\varepsilon_h} + \frac{1}{2}\frac{\varepsilon_h}{\delta_h^2}\Big[\frac{(\nabla_h y^h - I)^T}{\varepsilon_h/\delta_h}\frac{(\nabla_h y^h - I)}{\varepsilon_h/\delta_h}\Big]_{11} \rightharpoonup G_{11} \;\text{ in } L^1(\Omega). \tag{39}$$

This equation highlights the fact that G_{11} is "generated" by a linear and a quadratic term in $\nabla_h y^h - I$.

In the case $r \in \{0, 1\}$, by using (25) and i of Lemma 3.4, we find

$$u_{1,1} + \frac{1}{2} r (A^T A)_{11} = G_{11}$$

that is,

$$G_{11}(x) = \xi_1'(x_1) - x_2 \xi_2''(x_1) - x_3 \xi_3''(x_1) + \frac{r}{2}(\xi_3'(x_1))^2,$$

which is exactly (26). We note that when the energy is "small", i.e., $r = 0$, the quadratic term in $\nabla_h y^h - I$ does not give any contribution in G_{11}. According to our geometrical interpretation of G_{11}, we deduce that the length's variation along the axis of the beam is given by ξ_1' for $r = 0$ and $\xi_1' + \frac{1}{2}(\xi_3')^2$ for $r = 1$.

In the case $r = +\infty$, that is, when $\varepsilon_h/\delta_h^2 \to +\infty$, we deduce from (39), after multiplication by δ_h^2/ε_h, that

$$\frac{(\nabla_h y^h - I)_{11}}{(\varepsilon_h/\delta_h)^2} + \frac{1}{2}\Big[\frac{(\nabla_h y^h - I)^T}{\varepsilon_h/\delta_h}\frac{(\nabla_h y^h - I)}{\varepsilon_h/\delta_h}\Big]_{11} \to 0 \ \text{ in } L^1(\Omega).$$

As before, by using (23) and i of Lemma 3.4, we find

$$u_{1,1} + \frac{1}{2}(A^T A)_{11} = 0$$

that is, $\xi_1' + \frac{1}{2}(\xi_3')^2 = 0$, which is exactly the constraint (21). This implies that in this regime the axis of the beam is inextensible.

3.1 A liminf inequality

In this subsection we prove a lower bound of the limit energy. We start by recalling a result proven in [8, Lemma 3.4].

Lemma 3.10 *Assume that* $\lim_{h\to 0} \varepsilon_h = 0$. *Let* $(y^h) \subset W^{1,2}(\Omega; \mathbb{R}^3)$, $(R^h) \subset SO(3)$ *and*

$$G^h := \frac{R^h \nabla_h y^h - I}{\varepsilon_h} \rightharpoonup G \ \text{in } L^2(\Omega; \mathbb{R}^{3\times 3}).$$

Then

$$\liminf_{h\to 0} \frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h y^h)\, dx \geq \frac{1}{2}\int_\Omega Q_3(G)\, dx,$$

where Q_3 *is the quadratic form introduced in* (3).

We now state and prove the following liminf inequality.

Theorem 3.11 *Assume* (12). *Let* $y^h \in W^{1,2}(\Omega; \mathbb{R}^3)$ *be a sequence of deformations satisfying* (11). *Then, there exist rotations* $Q^h \in SO(3)$ *and constants* $c^h \in \mathbb{R}$ *such that, setting* $\bar{y}^h := {Q^h}^T y^h - c^h$, *1.* $\bar{y}^h \to x_1 e_1$ *in* $W^{1,2}(\Omega;\mathbb{R}^3)$ *and* $\nabla_h \bar{y}^h \to I$ *in* $L^2(\Omega;\mathbb{R}^{3\times 3})$.

Under assumption (18)*, setting*

$$\begin{aligned} u_1^h &:= \begin{cases} \dfrac{1}{(\varepsilon_h/\delta_h)^2}(\bar{y}_1^h - x_1) & \text{if } r = +\infty, \\ \dfrac{1}{\varepsilon_h}(\bar{y}_1^h - x_1) & \text{if } r \in \{0,1\}, \end{cases} \\ u_2^h &:= \begin{cases} \dfrac{1}{\varepsilon_h/\delta_h}(\bar{y}_2^h - hx_2) & \text{if } r = +\infty, \\ \dfrac{1}{\varepsilon_h/h}(\bar{y}_2^h - hx_2) & \text{if } r \in \{0,1\}, \end{cases} \\ u_3^h &:= \frac{1}{\varepsilon_h/\delta_h}(\bar{y}_3^h - \delta_h x_3), \\ \vartheta^h &:= \frac{1}{I_0}\frac{1}{h\varepsilon_h}\int_\omega (\delta_h x_2 \bar{y}_3^h - hx_3 \bar{y}_2^h)\, dx_2 dx_3, \end{aligned}$$

we have that

2. *up to subsequences, there exists* $u \in \mathcal{A}^r$ *such that* $u^h \rightharpoonup u$ *in* $W^{1,2}(\Omega;\mathbb{R}^3)$*; if* $r = +\infty$*, the convergence is actually strong in* $W^{1,2}(\Omega;\mathbb{R}^3)$*;*
3. *up to subsequences, there exists* $\vartheta \in W^{1,2}(0,\ell)$ *such that* $\vartheta^h \rightharpoonup \vartheta$ *in* $W^{1,2}(0,\ell)$.

Moreover,

$$\liminf_{h\to 0} \frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h y^h)\, dx \geq I^r(u,\vartheta), \tag{40}$$

where $I^r : \mathcal{A}^r \times W^{1,2}(0,\ell) \to [0,+\infty)$ *is defined by*

$$I^r(u,\vartheta) := \tfrac{1}{24}\int_0^\ell Q_2(u_3'', \vartheta')\, dx_1$$

if $r = +\infty$*, and by*

$$I^r(u,\vartheta) := \tfrac{1}{24}\int_0^\ell Q_2(\xi_3'', \vartheta')\, dx_1 + \tfrac{1}{2}\int_0^\ell E\big(\xi_1' + \tfrac{r}{2}(\xi_3')^2\big)^2\, dx_1 + \tfrac{1}{24}\int_0^\ell E\big(\xi_2''\big)^2\, dx_1$$

if $r \in \{0,1\}$*. Here* ξ_1*,* ξ_2*, and* ξ_3 *are as in the definition of* $\mathcal{A}^r$ *(see Definition 3.6).*

Proof. Take as Q^h the sequence of rotations constructed in Lemma 3.1 and as c^h a sequence of constants chosen as in Lemma 3.7. Then, statement *1* follows from *1* and *3* of Lemma 3.1 and from the fact that $\bar{y}^h - (x_1, hx_2, \delta_h x_3)$ has zero average. Statements *2* and *3* follow from Lemma 3.7.

Let us prove (40). Using the frame indifference of W and the definition of $\bar{y}^h$ we have that

$$W(\nabla_h y^h) = W(\nabla_h \bar{y}^h). \tag{41}$$

Let $\bar{R}^h$ be the sequence of approximating rotations of Lemma 3.1 and let

$$G^h := \frac{\bar{R}^{h^T} \nabla_h \bar{y}^h - I}{\varepsilon_h}.$$

By *4* of Lemma 3.3 we have that, up to subsequences, $G^h \rightharpoonup G$ in $L^2(\Omega; \mathbb{R}^{3\times 3})$.

Working with the corresponding subsequence (not relabeled) of $\nabla_h y^h$, and taking into account (41), Lemma 3.10, and (4), we get

$$\begin{aligned} \liminf_{h\to 0} \frac{1}{\varepsilon_h^2} \int_\Omega W(\nabla_h y^h)\, dx &\geq \frac{1}{2} \int_\Omega Q_3(G)\, dx = \frac{1}{2} \int_\Omega Q_3(\operatorname{sym} G)\, dx \\ &\geq \frac{1}{2} \int_\Omega Q_2(G_{11}, \tfrac{1}{2}(G_{12} + G_{21}))\, dx, \qquad (42) \end{aligned}$$

where the last inequality follows from the definition (6) of Q_2.

By Lemma 3.7 we have that for every $r \in \{0, 1, +\infty\}$ there exist $g, \tilde{g} \in L^2((0,\ell) \times (-\frac{1}{2}, \frac{1}{2}))$ such that

$$G_{11}(x) = -x_3 \xi_3''(x_1) + g(x_1, x_2),$$
$$\tfrac{1}{2}(G_{12}(x) + G_{21}(x)) = -x_3 \vartheta'(x_1) + \tilde{g}(x_1, x_2)$$

for a.e. $x \in \Omega$. Since Q_2 is a quadratic form, we obtain

$$\begin{aligned} \int_\Omega Q_2(G_{11}, \tfrac{1}{2}(G_{12} + G_{21}))\, dx &= \int_\Omega Q_2(-x_3\xi_3'' + g, -x_3\vartheta' + \tilde{g})\, dx \\ &= \int_\Omega x_3^2 Q_2(\xi_3'', \vartheta')\, dx + \int_\Omega Q_2(g, \tilde{g})\, dx. (43) \end{aligned}$$

If $r = +\infty$, we simply deduce

$$\int_\Omega Q_2(G_{11}, \tfrac{1}{2}(G_{12} + G_{21}))\, dx \geq \int_\Omega x_3^2 Q_2(\xi_3'', \vartheta')\, dx,$$

hence, by (42)

$$\liminf_{h\to 0} \frac{1}{\varepsilon_h^2} \int_\Omega W(\nabla_h y^h)\, dx \geq \tfrac{1}{24} \int_0^\ell Q_2(\xi_3'', \vartheta')\, dx_1.$$

If $r \in \{0, 1\}$, by (26) we have that

$$g(x_1, x_2) = \xi_1'(x_1) - x_2 \xi_2''(x_1) + \frac{r}{2}(\xi_3'(x_1))^2,$$

hence, using the definition of E (see (7)) we have

$$\begin{aligned} \int_\Omega Q_2(g, \tilde{g})\, dx &\geq \int_\Omega E\big(\xi_1' - x_2\xi_2'' + \tfrac{r}{2}(\xi_3')^2\big)^2\, dx \\ &= \int_\Omega E\big(\xi_1' + \tfrac{r}{2}(\xi_3')^2\big)^2\, dx + \int_\Omega x_2^2 E\big(\xi_2''\big)^2\, dx \\ &= \int_0^\ell E\big(\xi_1' + \tfrac{r}{2}(\xi_3')^2\big)^2\, dx_1 + \tfrac{1}{12} \int_0^\ell E\big(\xi_2''\big)^2\, dx_1. \qquad (44) \end{aligned}$$

Thus, combining (42) – (44), we conclude that

$$\liminf_{h\to 0} \frac{1}{\varepsilon_h^2} \int_\Omega W(\nabla_h y^h)\, dx \geq$$
$$\geq \tfrac{1}{24} \int_0^\ell Q_2(\xi_3'', \vartheta')\, dx_1 + \tfrac{1}{2} \int_0^\ell E\big(\xi_1' + \tfrac{r}{2}(\xi_3')^2\big)^2\, dx_1 + \tfrac{1}{24} \int_0^\ell E\big(\xi_2''\big)^2\, dx_1.$$

□

3.2 Recovery sequences

Here we shall prove that the lower bound obtained in the previous subsection is achieved. For clarity we shall discuss the cases $r = \infty$ and $r \in \{0, 1\}$ in two different subsections.

3.2.1 The recovery sequence in the case $r = \infty$

In this subsection we consider the case in which

$$\lim_{h\to 0} \frac{\varepsilon_h}{\delta_h} = 0 \quad \text{and} \quad \lim_{h\to 0} \frac{\varepsilon_h}{\delta_h^2} = r = +\infty, \tag{45}$$

and we further assume that

$$\lim_{h\to 0} \frac{h^2 \varepsilon_h}{\delta_h^2} = 0. \tag{46}$$

Theorem 3.12 *Assume* (45) *and* (46)*. Then for every* $(u, \vartheta) \in \mathcal{A}^\infty \times W^{1,2}(0, \ell)$ *there exists a sequence of deformations* $y^h \in W^{1,2}(\Omega; \mathbb{R}^3)$ *such that, setting*

$$\begin{aligned}
u_1^h &:= \frac{1}{(\varepsilon_h/\delta_h)^2}(y_1^h - x_1), \\
u_2^h &:= \frac{1}{\varepsilon_h/\delta_h}(y_2^h - h x_2), \\
u_3^h &:= \frac{1}{\varepsilon_h/\delta_h}(y_3^h - \delta_h x_3), \\
\vartheta^h &:= \frac{1}{I_0} \frac{1}{h\varepsilon_h} \int_\omega (\delta_h x_2 y_3^h - h x_3 y_2^h)\, dx_2 dx_3,
\end{aligned}$$

we have that $\nabla_h y^h \to I$ *in* $L^2(\Omega; \mathbb{R}^{3\times 3})$*,* $u^h \to u$ *in* $W^{1,2}(\Omega; \mathbb{R}^3)$*,* $\vartheta^h \to \vartheta$ *in* $W^{1,2}(0, \ell)$*, and*

$$\limsup_{h\to 0} \frac{1}{\varepsilon_h^2} \int_\Omega W(\nabla_h y^h)\, dx \leq I^\infty(u, \vartheta), \tag{47}$$

where

$$I^\infty(u, \vartheta) := \tfrac{1}{24} \int_0^\ell Q_2(u_3'', \vartheta')\, dx_1.$$

Proof. Let us fix $(u,\vartheta)\in\mathcal{A}^\infty\times W^{1,2}(0,\ell)$ smooth enough and let ξ_1 and ξ_3 be as in the definition of $\mathcal{A}^\infty$, see Definition 3.5. For every $t\in[0,\ell]$ we define

$$A(t):=\begin{pmatrix} 0 & 0 & -\xi_3'(t)\\ 0 & 0 & -\vartheta(t)\\ \xi_3'(t) & \vartheta(t) & 0\end{pmatrix}.$$

To simplify notation we set $\eta_h:=\varepsilon_h/\delta_h$, which tends to 0 by (45). Let $R^h\colon[0,\ell]\to\mathbb{R}^{3\times3}$ be the solution of the Cauchy problem

$$\begin{cases} X'=\eta_h XA' & \text{in } [0,\ell],\\ X(0)=\exp(\eta_h A(0)).\end{cases}$$

It is easy to see that $R^h(t)\in SO(3)$ for every t; indeed, $R^h(0)\in SO(3)$ and $(R^h(R^h)^T)'=0$ on $[0,\ell]$ from the equation. Moreover, the function

$$Q^h(t):=I+\eta_h A(t)+\eta_h^2\int_0^t A(s)A'(s)\,ds+\tfrac12\eta_h^2A^2(0),$$

solves the problem

$$\begin{cases} (Q^h)'=\eta_h Q^hA'+\eta_h^3L & \text{in } [0,\ell],\\ Q^h(0)=\exp(\eta_h A(0))+O(\eta_h^3),\end{cases}$$

where

$$L(t)=-\Big(\int_0^t A(s)A'(s)\,ds+\frac12A^2(0)\Big)A'(t).$$

Therefore, by Gronwall Lemma we have that $|R^h-Q^h|=O(\eta_h^3)$ uniformly on $[0,\ell]$; in other words,

$$R^h(t)=I+\eta_h A(t)+\eta_h^2\int_0^t A(s)A'(s)\,ds+\tfrac12\eta_h^2A^2(0)+O(\eta_h^3). \tag{48}$$

Finally, let us fix $\varphi\in C^\infty([0,\ell])$ and $\gamma\in C^\infty([0,\ell];\mathbb{R}^3)$, and define

$$\begin{aligned}\beta^h(x) \;:=\; & \varepsilon_hR^h(x_1)\begin{pmatrix} -\frac12h^2x_2^2x_3\varphi'(x_1)\\ -hx_2x_3\varphi(x_1)\\ \frac12\frac{h^2}{\delta_h}x_2^2\varphi(x_1)\end{pmatrix}\\ & -\varepsilon_hhx_2x_3\vartheta'(x_1)R^h(x_1)e_1+\tfrac12\varepsilon_h\delta_hx_3^2R^h(x_1)\gamma(x_1).\end{aligned}$$

We consider the sequence of three-dimensional deformations $y^h:\Omega\to\mathbb{R}^3$ given by

$$y^h(x):=\int_0^{x_1}R^h(t)e_1\,dt+R^h(x_1)\begin{pmatrix}0\\ hx_2\\ \delta_hx_3\end{pmatrix}+\beta^h(x)+c^h,$$

where $c_1^h := \eta_h^2\xi_1(0)$, $c_2^h := 0$ and $c_3^h := \eta_h\xi_3(0)$.

We first check the convergence of the tangential and normal displacements u^h. The expansion (48) and the definition of $\mathcal{A}^\infty$ imply that

$$\begin{aligned} R_{11}^h(x_1) &= 1-\eta_h^2\int_0^{x_1}\xi_3'(s)\xi_3''(s)\,ds-\tfrac{1}{2}\eta_h^2\xi_3'(0)^2+O(\eta_h^3) \\ &= 1-\tfrac{1}{2}\eta_h^2\xi_3'(x_1)^2+O(\eta_h^3) \\ &= 1+\eta_h^2\xi_1'(x_1)+O(\eta_h^3), \end{aligned} \tag{49}$$

and

$$R_{12}^h(x_1)=O(\eta_h^2), \qquad R_{13}^h(x_1)=O(\eta_h), \tag{50}$$

as $h\to 0$, uniformly in $[0,\ell]$. Since $R^h = I + O(\eta_h)$, we have that $\beta_1^h = o(\eta_h^2)$ and $\nabla\beta_1^h = o(\eta_h^2)$. Combining these two facts with (48)–(50), we obtain that

$$\begin{aligned} u_1^h(x) &= \xi_1(x_1)+o(1), \\ \nabla u_1^h(x) &= \xi_1'(x_1)e_1+o(1). \end{aligned}$$

Therefore, we can conclude that u_1^h converges to u_1 strongly in $W^{1,2}(\Omega)$.

Similar computations show that u_k^h converges to u_k strongly in $W^{1,2}(\Omega)$ for $k=2,3$.

Finally, we note that

$$\vartheta^h=\vartheta+\frac{1}{I_0}\frac{1}{h\varepsilon_h}\int_\omega\Big(\delta_h x_2\beta_3^h-hx_3\beta_2^h\Big)\,dx_2dx_3+O(\eta_h)=\vartheta+o(1),$$

which gives the desired convergence.

Let us prove now the convergence of energies (47). By differentiation we obtain

$$\nabla_h y^h = R^h(x_1)+(R^h)'(x_1)\begin{pmatrix}0\\ hx_2\\ \delta_h x_3\end{pmatrix}\otimes e_1+\nabla_h\beta^h.$$

Since by definition

$$R^{h^T}(R^h)'=\eta_h A' \tag{51}$$

and $\eta_h\delta_h=\varepsilon_h$, we deduce that

$$R^{h^T}\nabla_h y^h = I+\varepsilon_h\begin{pmatrix}-x_3\xi_3''\\ -x_3\vartheta'\\ \frac{h}{\delta_h}x_2\vartheta'\end{pmatrix}\otimes e_1+R^{h^T}\nabla_h\beta^h.$$

Using property (51) and the orthogonality of R^h, a direct computation shows that

$$\begin{aligned} R^{h^T}\nabla_h\beta^h &= \varepsilon_h\begin{pmatrix}0 & -x_3\vartheta' & -\frac{1}{2}\frac{h^2}{\delta_h}x_2^2\varphi'-\frac{h}{\delta_h}x_2\vartheta' \\ 0 & -x_3\varphi & -\frac{h}{\delta_h}x_2\varphi \\ \frac{1}{2}\frac{h^2}{\delta_h}x_2^2\varphi' & \frac{h}{\delta_h}x_2\varphi & 0\end{pmatrix} \\ &\quad+\varepsilon_h x_3(\gamma\otimes e_3)+\tfrac{1}{2}\tfrac{h^2\varepsilon_h^2}{\delta_h^2}x_2^2\varphi A'e_3\otimes e_1+o(\varepsilon_h). \end{aligned} \tag{52}$$

Under the assumption (46) the second term in the second line of the previous formula is of order $o(\varepsilon_h)$. Hence

$$R^{h^T}\nabla_h y^h = I + \varepsilon_h B^h(x) + o(\varepsilon_h),$$

where

$$B^h := \begin{pmatrix} -x_3\xi_3'' & -x_3\vartheta' & -\frac{1}{2}\frac{h^2}{\delta_h}x_2^2\varphi' - \frac{h}{\delta_h}x_2\vartheta' + x_3\gamma_1 \\ -x_3\vartheta' & -x_3\varphi & -\frac{h}{\delta_h}x_2\varphi + x_3\gamma_2 \\ \frac{h}{\delta_h}x_2\vartheta' + \frac{1}{2}\frac{h^2}{\delta_h}x_2^2\varphi' & \frac{h}{\delta_h}x_2\varphi & x_3\gamma_3 \end{pmatrix}.$$

Applying the identity $(I+B)^T(I+B) = I + 2\mathrm{sym}\,B + B^TB$ and observing that $\varepsilon_h^2(B^h)^TB^h = O(\varepsilon_h^2h^2/\delta_h^2) = o(\varepsilon_h)$ by (46), we obtain

$$(\nabla_h y^h)^T\nabla_h y^h = I + 2\varepsilon_h x_3 Z + o(\varepsilon_h)$$

where

$$Z = \begin{pmatrix} -\xi_3'' & -\vartheta' & \frac{1}{2}\gamma_1 \\ -\vartheta' & -\varphi & \frac{1}{2}\gamma_2 \\ \frac{1}{2}\gamma_1 & \frac{1}{2}\gamma_2 & \gamma_3 \end{pmatrix}.$$

By frame-indifference we have

$$W(\nabla_h y^h) = W\big(\sqrt{(\nabla_h y^h)^T\nabla_h y^h}\big) = W(I + \varepsilon_h x_3 Z + o(\varepsilon_h)).$$

As Z is bounded in L^∞, for h small enough the matrix $I+\varepsilon_h x_3 Z + o(\varepsilon_h)$ belongs to the neighborhood of $SO(3)$ where W is of class C^2, so that, by expanding W around the identity, we have

$$\varepsilon_h^{-2}W(\nabla_h y^h) \to \tfrac{1}{2}x_3^2 Q_3(Z) \text{ a.e. in } \Omega,$$
$$\varepsilon_h^{-2}|W(\nabla_h y^h)| \le C(|Z|^2+1).$$

By the dominated convergence theorem this implies

$$\lim_h \frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h y^h)\,dx = \tfrac{1}{24}\int_0^\ell Q_3(Z)\,dx_1. \tag{53}$$

Consider now the general case. Let $(u,\vartheta) \in \mathcal{A}^\infty \times W^{1,2}(0,\ell)$, and let ξ_1 and ξ_3 be as in the definition of $\mathcal{A}^\infty$. Let also $\varphi \in L^2(0,\ell)$ and $\gamma \in L^2((0,\ell);\mathbb{R}^3)$ be such that

$$Q_2(u_3'',\vartheta') = Q_3\begin{pmatrix} -\xi_3'' & -\vartheta' & \frac{1}{2}\gamma_1 \\ -\vartheta' & -\varphi & \frac{1}{2}\gamma_2 \\ \frac{1}{2}\gamma_1 & \frac{1}{2}\gamma_2 & \gamma_3 \end{pmatrix} \tag{54}$$

a.e. in $(0,\ell)$. Finally, let us consider sequences ξ_3^k, ϑ^k, $\varphi^k \in C^\infty([0,\ell])$ and $\gamma^k \in C^\infty([0,\ell];\mathbb{R}^3)$ such that $\xi_3^k \to \xi_3$ strongly in $W^{2,2}(0,\ell)$, $\vartheta^k \to \vartheta$ strongly in

$W^{1,2}(0,\ell)$, $\varphi^k \to \varphi$ strongly in $L^2(0,\ell)$, and $\gamma^k \to \gamma$ strongly in $L^2((0,\ell);\mathbb{R}^3)$. Setting $u^k := (\xi_1^k, 0, \xi_3^k)$, where

$$\xi_1^k(x_1) := -\frac{1}{2}\int_0^{x_1} \big((\xi_3^k)'(t)\big)^2\, dt + \xi_1(0)$$

for every $x_1 \in (0,\ell)$, it is immediate to see that $u^k \in \mathcal{A}^\infty \cap C^\infty([0,\ell];\mathbb{R}^3)$ and $u^k \to u$ in $W^{2,2}((0,\ell);\mathbb{R}^3)$. By the previous argument for every $k \in \mathbb{N}$ we can construct a sequence of three-dimensional deformations, whose associated displacement and twist function converge to (u^k, ϑ^k), as $h \to 0$, and satisfying (53) with Z replaced by

$$Z^k := \begin{pmatrix} -(\xi_3^k)'' & -(\vartheta^k)' & \frac{1}{2}\gamma_1^k \\ -(\vartheta^k)' & -\varphi^k & \frac{1}{2}\gamma_2^k \\ \frac{1}{2}\gamma_1^k & \frac{1}{2}\gamma_2^k & \gamma_3^k \end{pmatrix}.$$

Using a diagonal argument, the continuity of the left-handside of (53) with respect to the L^2 convergence, and equality (54), we deduce the Γ-limsup inequality (47). □

Remark 3.13 The assumption (46) used in Theorem 3.12 is crucial in the construction of the recovery sequence since it allows us to control the stretch of the mid-plane, i.e., the x_1x_2-plane. For instance, in (52) it permits to drop the term

$$\tfrac{1}{2}\tfrac{h^2\varepsilon_h^2}{\delta_h^2}x_2^2\varphi A' e_3 \otimes e_1 = \tfrac{1}{2}\tfrac{h^2\varepsilon_h^2}{\delta_h^2}x_2^2\varphi(-\xi_3'' e_1 \otimes e_1 - \vartheta' e_2 \otimes e_1),$$

that clearly represents a mid-plane deformation.

We note also that (46) coincides with the assumption that was required in the analysis of the critical regime, developed in [7]. Indeed, in this case we have $\lim_{h\to 0}\frac{\varepsilon_h}{\delta_h} = 1$, so that (46) is equivalent to

$$\lim_{h\to 0}\frac{h^2}{\delta_h} = 0,$$

which coincides with condition (5.6) in [7].

3.2.2 The recovery sequence in the case $r \in \{0,1\}$

In this subsection we consider the case in which

$$\lim_{h\to 0}\frac{\varepsilon_h}{\delta_h} = 0 \quad \text{and} \quad \lim_{h\to 0}\frac{\varepsilon_h}{\delta_h^2} = r \in \{0,1\}. \tag{55}$$

Theorem 3.14 *Assume* (55). *Then for every* $(u,\vartheta)\in\mathcal{A}^{BN}\times W^{1,2}(0,\ell)$ *there exists a sequence of deformations* $y^h\in W^{1,2}(\Omega;\mathbb{R}^3)$ *such that, setting*

$$\begin{aligned}
u_1^h &:= \frac{1}{\varepsilon_h}(y_1^h - x_1),\\
u_2^h &:= \frac{1}{\varepsilon_h/h}(y_2^h - hx_2),\\
u_3^h &:= \frac{1}{\varepsilon_h/\delta_h}(y_3^h - \delta_h x_3),\\
\vartheta^h &:= \frac{1}{I_0}\frac{1}{h\varepsilon_h}\int_\omega (\delta_h x_2 y_3^h - hx_3 y_2^h)\,dx_2 dx_3,
\end{aligned}$$

we have that $\nabla_h y^h\to I$ *in* $L^2(\Omega;\mathbb{R}^{3\times3})$, $u^h\to u$ *in* $W^{1,2}(\Omega;\mathbb{R}^3)$, $\vartheta^h\to\vartheta$ *in* $W^{1,2}(0,\ell)$, *and*

$$\limsup_{h\to0}\frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h y^h)\,dx \le I^r(u,\vartheta), \tag{56}$$

where

$$I^r(u,\vartheta) := \tfrac{1}{24}\int_0^\ell Q_2(\xi_3'',\vartheta')\,dx_1 + \tfrac{1}{2}\int_0^\ell E\big(\xi_1' + \tfrac{r}{2}(\xi_3')^2\big)^2\,dx_1 + \tfrac{1}{24}\int_0^\ell E\big(\xi_2''\big)^2\,dx_1$$

with ξ_1,ξ_2 *and* ξ_3 *as in the definition of* $\mathcal{A}^{BN}$.

Proof. Let us fix $(u,\vartheta)\in\mathcal{A}^{BN}\times W^{1,2}(0,\ell)$ smooth enough and let ξ_1, ξ_2, and ξ_3 be as in the definition of $\mathcal{A}^{BN}$. Let us fix $\alpha,\gamma,\sigma\in C^\infty([0,\ell];\mathbb{R}^{3\times3}_{sym})$.

We consider the sequence of three-dimensional deformations $y^h:\Omega\to\mathbb{R}^3$ given by

$$y^h(x) := \begin{pmatrix} x_1\\ hx_2\\ \delta_h x_3\end{pmatrix} + \varepsilon_h\begin{pmatrix} \xi_1 - x_2\xi_2' - x_3\xi_3' - hx_2x_3\vartheta'\\ \frac{1}{h}\xi_2 - x_3\vartheta\\ \frac{1}{\delta_h}\xi_3 + \frac{h}{\delta_h}x_2\vartheta\end{pmatrix} + \varepsilon_h\beta^h(x),$$

where

$$\begin{cases}
\beta_1^h := \delta_h(x_3^2\alpha_{13} + 2x_2x_3\gamma_{13} + 2x_3\sigma_{13}) + \frac{1}{2}h^2x_2^2x_3\alpha_{22}' + h(x_2^2\gamma_{12} + 2x_2\sigma_{12}),\\
\beta_2^h := h(x_2x_3\alpha_{22} + \frac{1}{2}x_2^2\gamma_{22} + x_2\sigma_{22}) + \delta_h(x_3^2\alpha_{23} + 2x_3\sigma_{23}),\\
\beta_3^h := \delta_h(\frac{1}{2}x_3^2\alpha_{33} + x_2x_3\gamma_{33} + x_3\sigma_{33}) + hx_2^2\gamma_{23} - \frac{1}{2}\frac{h^2}{\delta_h}x_2^2\alpha_{22}.
\end{cases}$$

It is easy to see that the displacement and the twist function associated with y^h converge to u and ϑ in $W^{1,2}$. Moreover, we have that

$$\nabla_h y^h = I + \varepsilon_h\big(M^h + \frac{1}{\delta_h}A + \nabla_h\beta^h\big), \tag{57}$$

where

$$M^h := \begin{pmatrix} u_{1,1} - hx_2x_3\vartheta'' & -\xi_2'/h - x_3\vartheta' & -hx_2\vartheta'/\delta_h\\ \xi_2'/h - x_3\vartheta' & 0 & 0\\ hx_2\vartheta'/\delta_h & 0 & 0\end{pmatrix}$$

and

$$A := \begin{pmatrix} 0 & 0 & -\xi_3' \\ 0 & 0 & -\vartheta \\ \xi_3' & \vartheta & 0 \end{pmatrix}.$$

Also,

$$\begin{aligned}\nabla_h \beta^h \quad &= \begin{pmatrix} 0 & x_2\gamma_{12} + \sigma_{12} & x_3\alpha_{13} + x_2\gamma_{13} + \sigma_{13} \\ & x_3\alpha_{22} + x_2\gamma_{22} + \sigma_{22} & x_3\alpha_{23} + x_2\gamma_{23} + \sigma_{23} \\ \mathrm{sym} & & x_3\alpha_{33} + x_2\gamma_{33} + \sigma_{33} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & x_2\gamma_{12} + \sigma_{12} & x_3\alpha_{13} + x_2\gamma_{13} + \sigma_{13} + h^2x_2^2\alpha_{22}'/(2\delta_h) \\ & 0 & hx_2\alpha_{22}/\delta_h + x_3\alpha_{23} - x_2\gamma_{23} + \sigma_{23} \\ \mathrm{skw} & & 0 \end{pmatrix} \\ &+ O(h) + O(\delta_h/h).\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\nabla_h y^{h^T} \nabla_h y^h \;&=\; I + 2\varepsilon_h \,\mathrm{sym}\big(M^h + \frac{1}{\delta_h} A + \nabla_h \beta^h\big) + \frac{\varepsilon_h^2}{\delta_h^2} A^T A \\ &\quad + O\big(\frac{\varepsilon_h^2}{h\delta_h}\big) + O\big(\frac{h\varepsilon_h^2}{\delta_h^2}\big) \\ &=\; I + 2\varepsilon_h \,\mathrm{sym}\,(M^h + \nabla_h\beta^h + \tfrac{r}{2} A^T A) + o(\varepsilon_h),\end{aligned}$$

where the last equality follows from (55). From the above relations we deduce that

$$\mathrm{sym}\,(M^h + \nabla_h\beta^h + \tfrac{r}{2}A^TA) = Z_1 + x_2Z_2 + x_3Z_3 + O(h) + O(\delta_h/h),$$

where

$$Z_1 := \begin{pmatrix} \xi_1' + \frac{r}{2}(\xi_3')^2 & \frac{r}{2}\xi_3'\vartheta + \sigma_{12} & \sigma_{13} \\ & \frac{r}{2}\vartheta^2 + \sigma_{22} & \sigma_{23} \\ \mathrm{sym} & & \frac{r}{2}((\xi_3')^2 + \vartheta^2) + \sigma_{33} \end{pmatrix}, \tag{58}$$

$$Z_2 := \begin{pmatrix} -\xi_2'' & \gamma_{12} & \gamma_{13} \\ & \gamma_{22} & \gamma_{23} \\ \mathrm{sym} & & \gamma_{33} \end{pmatrix}, \qquad Z_3 := \begin{pmatrix} -\xi_3'' & -\vartheta' & \alpha_{13} \\ & \alpha_{22} & \alpha_{23} \\ \mathrm{sym} & & \alpha_{33} \end{pmatrix}. \tag{59}$$

By frame-indifference we have

$$W(\nabla_h y^h) = W\big(\sqrt{(\nabla_h y^h)^T \nabla_h y^h}\big) = W(I + \varepsilon_h(Z_1 + x_2Z_2 + x_3Z_3) + o(\varepsilon_h)).$$

As $(Z_1 + x_2Z_2 + x_3Z_3) \in L^\infty(\Omega;\mathbb{R}^{3\times3})$, for h small enough the matrix $I + \varepsilon_h(Z_1 + x_2Z_2 + x_3Z_3) + o(\varepsilon_h)$ belongs to the neighborhood of $SO(3)$ where W is of class C^2, so that by Taylor expansion we have

$$\varepsilon_h^{-2} W(\nabla_h y^h) \to \tfrac{1}{2}Q_3(Z_1 + x_2Z_2 + x_3Z_3) \text{ a.e. in } \Omega,$$
$$\varepsilon_h^{-2}|W(\nabla_h y^h)| \le C(|Z_1 + x_2Z_2 + x_3Z_3|^2 + 1).$$

By the dominated convergence theorem this implies

$$\begin{aligned}
\lim_h \frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h y^h)\,dx &= \\
&= \tfrac{1}{2}\int_\Omega Q_3(Z_1 + x_2 Z_2 + x_3 Z_3)\,dx \\
&= \tfrac{1}{2}\int_\Omega Q_3(Z_1) + x_2^2 Q_3(Z_2) + x_3^2 Q_3(Z_3)\,dx \\
&= \tfrac{1}{24}\int_0^\ell Q_3(Z_3)\,dx_1 + \tfrac{1}{2}\int_0^\ell Q_3(Z_1)\,dx_1 + \tfrac{1}{24}\int_0^\ell Q_3(Z_2)\,dx_1.
\end{aligned}$$

Consider now the general case. Let $(u,\vartheta) \in \mathcal{A}^{BN} \times W^{1,2}(0,\ell)$, and let ξ_1, ξ_2, and ξ_3 be as in the definition of $\mathcal{A}^{BN}$. Let also $\alpha, \gamma, \sigma \in L^2((0,\ell);\mathbb{R}^{3\times3}_{sym})$ be such that

$$E\big(\xi_1' + \tfrac{r}{2}(\xi_3')^2\big)^2 = Q_3(Z_1), \quad E\big(\xi_2''\big)^2 = Q_3(Z_2), \quad Q_2(\xi_3'', \vartheta') = Q_3(Z_3),$$

a.e. in $(0,\ell)$, where the Z_i are defined as in (58)–(59). Arguing as in the proof of Theorem 3.12, we deduce the Γ-limsup inequality (56) by density. □

4 Convergence of minimizers

In this section we introduce a sequence of forces and characterize the asymptotic behaviour, as $h \to 0$, of minimizers (or almost minimizers) of the total energy. This is made precise in the following theorem.

Theorem 4.1 *Let (α_h) be a sequence of strictly positive real numbers such that*

$$\lim_{h\to 0} \frac{\alpha_h}{\delta_h^2} = 0 \tag{60}$$

and let $\varepsilon_h := \alpha_h/\delta_h$. Let $f^h, f \in L^2((0,\ell)\times(-\frac{1}{2},\frac{1}{2}))$ be such that

$$\int_{(0,\ell)\times(-\frac{1}{2},\frac{1}{2})} f^h\,dx_1dx_2 = \int_{(0,\ell)\times(-\frac{1}{2},\frac{1}{2})} x_i f^h\,dx_1dx_2 = 0, \qquad i = 1,2 \tag{61}$$

and

$$\frac{1}{\alpha_h} f^h \rightharpoonup f \text{ weakly in } L^2((0,\ell)\times(-\tfrac{1}{2},\tfrac{1}{2})). \tag{62}$$

Let

$$J^h(y) := I^h(y) - \int_\Omega f^h e_3 \cdot y\,dx$$

for $y \in W^{1,2}(\Omega;\mathbb{R}^3)$.

Then the following statements hold:

i. $|\inf J^h| \le C\varepsilon_h^2$ *for any $h > 0$ small enough.*

ii. *If* $(y^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *is a minimizing sequence of* $\frac{1}{\varepsilon_h^2}J^h$ *in the following sense*

$$\lim_{h\to 0}\Big(\frac{1}{\varepsilon_h^2}J^h(y^h)-\inf\frac{1}{\varepsilon_h^2}J^h\Big)=0, \tag{63}$$

then there exist some constants $Q^h\in SO(3)$ *and* $c^h\in\mathbb{R}^3$ *such that, setting* $\bar{y}^h := {Q^h}^T y^h - c^h$, *we have that*

$$\bar{y}^h \to x_1 e_1 \quad \textit{strongly in } W^{1,2}(\Omega;\mathbb{R}^3). \tag{64}$$

iii. *Assume in addition that*

$$\lim_{h\to 0}\frac{\alpha_h}{\delta_h^3}=\lim_{h\to 0}\frac{\varepsilon_h}{\delta_h^2}=r\in\{0,1,+\infty\}. \tag{65}$$

If $r=+\infty$, *assume also that*

$$\lim_{h\to 0}\frac{h^2\varepsilon_h}{\delta_h^2}=0. \tag{66}$$

Then, for

$$\begin{aligned}
u_1^h &:= \begin{cases}\dfrac{1}{(\varepsilon_h/\delta_h)^2}(\bar{y}_1^h-x_1) & \textit{if } r=+\infty,\\[2mm] \dfrac{1}{\varepsilon_h}(\bar{y}_1^h-x_1) & \textit{if } r\in\{0,1\},\end{cases}\\
u_2^h &:= \begin{cases}\dfrac{1}{\varepsilon_h/\delta_h}(\bar{y}_2^h-hx_2) & \textit{if } r=+\infty,\\[2mm] \dfrac{1}{\varepsilon_h/h}(\bar{y}_2^h-hx_2) & \textit{if } r\in\{0,1\},\end{cases}\\
u_3^h &:= \frac{1}{\varepsilon_h/\delta_h}(\bar{y}_3^h-\delta_h x_3),\\
\vartheta^h &:= \frac{1}{I_0}\frac{1}{h\varepsilon_h}\int_\omega(\delta_h x_2\bar{y}_3^h-hx_3\bar{y}_2^h)\,dx_2dx_3,
\end{aligned}$$

we have that, up to subsequences, $u^h\to\bar{u}$ *in* $W^{1,2}(\Omega;\mathbb{R}^3)$, $\vartheta^h\rightharpoonup\bar{\vartheta}$ *weakly in* $W^{1,2}(0,\ell)$ *and* $Q^h\to\bar{Q}$, *where* $(\bar{u},\bar{\vartheta},\bar{Q})\in\mathcal{A}^r\times W^{1,2}(0,\ell)\times SO(3)$ *minimizes the functional*

$$J^r(u,\vartheta,R):=\begin{cases} I^r(u,\vartheta)-R_{32}\displaystyle\int_0^\ell \bar{f}u_2\,dx_1-R_{33}\int_0^\ell\bar{f}u_3\,dx_1 & \textit{if } r=+\infty,\\[2mm] I^r(u,\vartheta)-R_{33}\displaystyle\int_0^\ell\bar{f}u_3\,dx_1 & \textit{if } r\in\{0,1\},\end{cases}$$

among all $(u,\vartheta,R)\in\mathcal{A}^r\times W^{1,2}(0,\ell)\times SO(3)$. *Here we have set* $\bar{f}(x_1):=\int_{-\frac12}^{\frac12}f(x_1,x_2)\,dx_2$. *Furthermore, we have*

$$\lim_{h\to 0}\frac{1}{\varepsilon_h^2}\inf J^h=\lim_{h\to 0}\frac{1}{\varepsilon_h^2}J^h(y^h)=J^r(\bar{u},\bar{\vartheta},\bar{Q})=\min J^r. \tag{67}$$

Remark 4.2 The two conditions in (61) guarantee that

$$\int_\Omega f^h e_3 \cdot Qx\, dx = 0$$

for every $Q \in SO(3)$ and every $h > 0$.

PROOF OF THEOREM 4.1. Let (y^h) be a minimizing sequence of $\frac{1}{\varepsilon_h^2} J^h$ in the sense of (63) and let $p_h(x) = (x_1, hx_2, \delta_h x_3)$ be defined as in (8). Applying Lemma 3.1 with $\eta_h^2 := I^h(y^h)$ and η_h in place of ε_h, and using Poincaré-Wirtinger inequality we find some constants $P^h \in SO(3)$ and $d^h \in \mathbb{R}^3$ such that for $\tilde{y}^h := {P^h}^T y^h - d^h - p_h$ there holds

$$\|\tilde{y}^h\|_{L^2}^2 + \|\nabla_h \tilde{y}^h\|_{L^2}^2 \le C\delta_h^{-2} I^h(y^h). \tag{68}$$

Moreover, by (61) we have

$$\inf J^h \le J^h(p_h) = -\int_\Omega \delta_h x_3 f^h(x_1, x_2)\, dx = 0. \tag{69}$$

Using (61) – (63), (68), and (69), we obtain

$$\begin{aligned} I^h(y^h) &= J^h(y^h) + \int_\Omega f^h {P^h}^T e_3 \cdot \tilde{y}^h\, dx \\ &\le C\varepsilon_h^2 + \alpha_h \|\tilde{y}^h\|_{L^2} \le C\varepsilon_h^2 + C\frac{\alpha_h}{\delta_h}\big(I^h(y^h)\big)^{1/2}. \end{aligned}$$

Recalling that $\varepsilon_h = \alpha_h/\delta_h$, the previous inequality yields

$$I^h(y^h) \le C\varepsilon_h^2. \tag{70}$$

Moreover, using again (61), (62), and (68), we deduce

$$J^h(y^h) \ge -\int_\Omega f^h {P^h}^T e_3 \cdot \tilde{y}^h\, dx \ge -\alpha_h \|\tilde{y}^h\|_{L^2} \ge -C\varepsilon_h^2.$$

Since (y^h) is a minimizing sequence, the last inequality together with (69) implies that $|\inf J^h| \le C\varepsilon_h^2$.

By (70) and by Theorem 3.11 we deduce statement *ii* of the theorem.

Assume now (65) and, if $r = +\infty$, (66). By Theorem 3.11 we deduce that there exists $(\bar{u}, \bar{\vartheta}) \in \mathcal{A}^r \times W^{1,2}(0,\ell)$ such that, up to subsequences, $u^h \rightharpoonup \bar{u}$ weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$ (strongly if $r = +\infty$) and $\vartheta^h \rightharpoonup \bar{\vartheta}$ weakly in $W^{1,2}(0,\ell)$. Moreover, up to subsequences, we also have that Q^h converges to some $\bar{Q} \in SO(3)$. By (40) and (62) we obtain that

$$\begin{aligned} \liminf_{h\to 0} \frac{1}{\varepsilon_h^2} J^h(y^h) &\ge I^r(\bar{u}, \bar{\vartheta}) + \liminf_{h\to 0} \Big(-\int_\Omega \frac{1}{\alpha_h} f^h {Q^h}^T e_3 \cdot \frac{\delta_h}{\varepsilon_h}(\bar{y}^h - p_h)\, dx \Big) \\ &= J^r(\bar{u}, \bar{\vartheta}, \bar{Q}). \end{aligned} \tag{71}$$

Let now $(u, \vartheta, R) \in \mathcal{A}^r \times W^{1,2}(0,\ell) \times SO(3)$. By Theorems 3.12 and 3.14 there exists a sequence $(\hat{y}^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ such that the corresponding displacement $\hat{u}^h$ and twist function $\hat{\vartheta}^h$ satisfy $\hat{u}^h \to u$ in $W^{1,2}(\Omega;\mathbb{R}^3)$, $\hat{\vartheta}^h \to \vartheta$ in $W^{1,2}(0,\ell)$, and

$$\limsup_{h\to 0} \frac{1}{\varepsilon_h^2} \int_\Omega W(\nabla_h y^h)\,dx \le I^r(u,\vartheta).$$

This implies that

$$\begin{aligned}
\limsup_{h\to 0} \frac{1}{\varepsilon_h^2} J^h(y^h) &= \limsup_{h\to 0} \Big(\frac{1}{\varepsilon_h^2} \inf J^h\Big) \le \limsup_{h\to 0} \frac{1}{\varepsilon_h^2} J^h(R\,\hat{y}^h) \\
&= \limsup_{h\to 0} \Big(\frac{1}{\varepsilon_h^2} I^h(\hat{y}^h) - \frac{1}{\varepsilon_h^2}\int_\Omega f^h R^T e_3 \cdot \hat{y}^h\,dx\Big) \\
&\le I^r(u,\vartheta) + \limsup_{h\to 0} \Big(- \int_\Omega \frac{1}{\alpha_h} f^h R^T e_3 \cdot \frac{\delta_h}{\varepsilon_h}(\hat{y}^h - p_h)\,dx\Big) \\
&= J^r(u,\vartheta,R). \qquad (72)
\end{aligned}$$

Combining (71) and (72) we deduce the minimality of $(\bar{u}, \bar{\vartheta}, \bar{Q})$ and the convergence of the energies (67).

To conclude it remains to show that $u^h \to \bar{u}$ strongly in $W^{1,2}(\Omega;\mathbb{R}^3)$ for $r \in \{0,1\}$. Arguing as in [10, Subsection 7.2], one can infer from (67) that $\frac{1}{\varepsilon_h}\,\mathrm{sym}(\nabla_h \bar{y}^h - I)$ converges strongly in $L^2(\Omega;\mathbb{R}^{3\times 3})$. Repeating the proof of part *ii* of Lemma 3.7, one can show that this implies strong convergence in $W^{1,2}(\Omega;\mathbb{R}^3)$ of the sequence of displacements u^h. □

Acknowledgements. LF and RP acknowledge the support of MIUR under PRIN 2008 "Complex materials and structural models in advanced problems of engineering". MGM was partially supported by GNAMPA under the project "Metodi e modelli variazionali in scienza dei materiali" 2010 and by MIUR under the project PRIN 2008 "Problemi variazionali con scale multiple". RP is also grateful to Regione Autonoma della Sardegna for the support given under the project "Modellazione Multiscala della Meccanica dei Materiali Compositi (M4C)".

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