On the variational derivation of the kinematics for thin-walled closed section beams

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Abstract. The kinematics of thin-walled closed cross section beams is studied by comparing the behavior of a closed section with an open section which differs from the former by a "cut" on one side.

1 Introduction

The definition of Γ-convergence given by De Giorgi [6], see also [4] and [5], has found significant applications in dimension reduction problems in mechanics. Strings, beams, membranes, plates and shells have found rigorous justifications, [1, 10, 9, 3]. Anzellotti et. al. [2], see also Percivale [14], starting from the three-dimensional linear theory of elasticity deduced, by Γ-convergence, the De Saint-Venant beam theory. The three dimensional body considered in these works is a cylinder with diameter much smaller than the length. In engineering applications, to minimize the weight of the structure, quite often are used beams with cross section having "walls" of thickness much smaller than the diameter of the cross section: the so called thin-walled beams. This kind of beams have been studied by Rodriguez and Viano [15, 16] and, starting from De Saint-Venant problem, by Morassi [11, 12, 13]. Only recently the fully three-dimensional problem has been studied by means of Γ-convergence. In [7] the authors of this note have considered a cantilever beam of finite length with a rectangular cross section of sides proportional to ε and ε^2, with $0 < \varepsilon < 1$, so to model the "thin-wall". Like in the previous papers, also in [7] the Γ-limit, as the parameter ε goes to zero, was found to be the De Saint-Venant beam model. More complex open sections have been studied in [8], where it is shown that the Γ–limit is either the De Saint-Venant theory or Vlassov theory according to the shape of the cross section. Closed thin-walled cross sections, which have not been considered in this last paper, are the main concern of the present note. We do not address the Γ-convergence problem here, but we study what can be considered as a preliminary step: the compactness of the displacements or, in mechanical terms, the kinematics of the model. To outline the differences between closed and open section we consider two sections: the first closed, and the second, which simply differs from the first by "a cut", open. We then outline the main steps needed to derive the kinematical description of the beam. In doing so we omit proofs by heavily relying on the similarity of the problem considered in this note with the one considered in [8]. Briefly, the section is decomposed in four rectangles and it is shown that each rectangle undergoes to a Bernoulli-Navier type of displacement. Thus the motion of each rectangle is described by four kinematical fields, hence, since the section comprises four rectangles, the motion of the beam, at this stage, is fully determined by means of sixteen fields. Relations between these fields are obtained by studying the kinematics on the regions where the rectangles overlap. Here is the main difference between the two sections considered: in the open section the rectangles overlap in three regions, while in the closed section they overlap in four regions (the fourth region gives rise to a compatibility condition that we call the supplementary junction condition). From the study of these junction conditions we deduce that in the open cross section thin-walled beam the displacement in the plane of the section is a rigid motion, while in the longitudinal direction it is the sum of a Bernoulli-Navier displacement and a quantity proportional to the derivative of the angle of rotation of the section. In other words, the open closed section undergoes to displacements of the type considered in the Vlassov theory [17]. The results found for the open cross section still hold for the closed section once that also the supplementary junction condition is satisfied. We show that this further condition imposes that the angle of rotation of the cross section to be equal to sero. We interpret this result not as a statement that the section does not rotate about the axis of the beam, but instead as a sign that, at the scale that we are looking at, the rotations are too small to be captured. In other words we believe that the sequence that generates the rotation field in the limit problem should be rescaled differently from the one considered in the open section case. This is mechanically evident: it is much harder to twist closed sections than open sections.

2 The 3-dimensional problem

The aim of this paper is to discuss the differences between open and closed section thin walled beams. Accordingly we consider two cylindrical three-dimensional bodies with cross sections as in Figure 2. The open section that we consider differs from the closed section just by "a cut". Given the similarity of the two sections we shall describe in some detail only one of them.

Let us denote by $\Omega_\varepsilon \subset \mathbb{R}^3$ the reference configuration of the thin walled beam with closed section. We can write $\Omega_\varepsilon := \omega_\varepsilon \times (0,\ell)$, and $\omega_\varepsilon := \cup_{i=1}^4 \omega_\varepsilon^{(i)}$, where

$$\omega_\varepsilon^{(1)} := (\varepsilon q_1 - \varepsilon b/2, \varepsilon q_1 + \varepsilon b/2) \times (\varepsilon q_2 - \varepsilon h/2, \varepsilon q_2 - \varepsilon h/2 + \varepsilon^2 s),$$

$$\omega_\varepsilon^{(2)} := (\varepsilon q_1 + \varepsilon b/2 - \varepsilon^2 s, \varepsilon q_1 + \varepsilon b/2) \times (\varepsilon q_2 - \varepsilon h/2, \varepsilon q_2 + \varepsilon h/2),$$

$$\omega_\varepsilon^{(3)} := (\varepsilon q_1 - \varepsilon b/2, \varepsilon q_1 + \varepsilon b/2) \times (\varepsilon q_2 + \varepsilon h/2 - \varepsilon^2 s, \varepsilon q_2 + \varepsilon h/2),$$

$$\omega_\varepsilon^{(4)} := (\varepsilon q_1 - \varepsilon b/2, \varepsilon q_1 - \varepsilon b/2 + \varepsilon^2 s) \times (\varepsilon q_2 - \varepsilon h/2, \varepsilon q_2 + \varepsilon h/2),$$

are four non-empty rectangles.

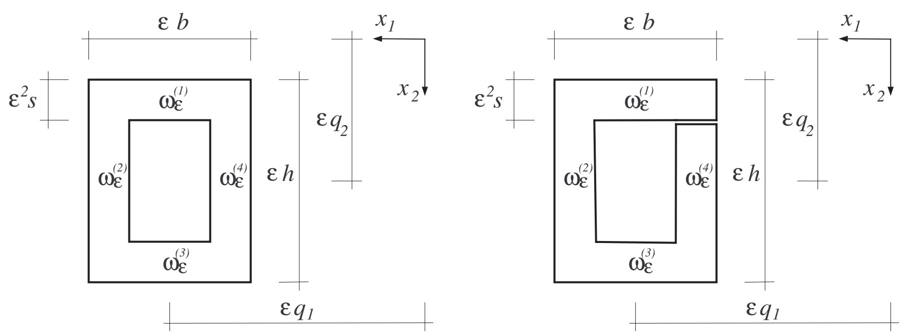

Fig. 1. The closed and the open sections

For later convenience we also set

$$\Omega_\varepsilon^{(i)} := \omega_\varepsilon^{(i)} \times (0, \ell), \quad i = 1, 2, 3, 4,$$

and we note that $\Omega_\varepsilon = \cup_{i=1}^4 \Omega_\varepsilon^{(i)}$ and that they are not pairwise disjoint.

We assume the beam clamped on one of its ends: we thus consider the spaces

$$H^1_\#(\Omega_\varepsilon; \mathbb{R}^3) := \left\{ \mathbf{w} \in H^1(\Omega_\varepsilon; \mathbb{R}^3) : \mathbf{w} = \mathbf{0} \text{ on } \omega_\varepsilon \times \{0\} \right\},$$

and $H^1_\#(\Omega_\varepsilon^{(i)}; \mathbb{R}^3)$ defined in a similar way. We further denote by

$$\begin{aligned} \mathbf{Ew}(\mathbf{x}) &:= \mathrm{sym}(D\mathbf{w}(\mathbf{x})) := \frac{D\mathbf{w}(\mathbf{x}) + D\mathbf{w}^T(\mathbf{x})}{2}, \\ \mathbf{Ww}(\mathbf{x}) &:= \mathrm{skw}(D\mathbf{w}(\mathbf{x})) := \frac{D\mathbf{w}(\mathbf{x}) - D\mathbf{w}^T(\mathbf{x})}{2}, \end{aligned} \tag{1}$$

the strain of $\mathbf{w} : \Omega_\varepsilon \to \mathbb{R}^3$ and the skew symmetric part of the jacobian $D\mathbf{w}$.

Hereafter we denote by $\mathbf{u}^\varepsilon \in H^1_\#(\Omega_\varepsilon; \mathbb{R}^3)$ the solution of an equilibrium problem (not stated for brevity) posed on Ω_ε.

To discuss the convergence of the displacements $\mathbf{u}^\varepsilon$ it is convenient to work on domains which do not depend on ε. We denote by $\omega^{(i)} := \omega_1^{(i)}$ and $\Omega^{(i)} := \Omega_1^{(i)}$, and we let

$$p_\varepsilon^{(i)} : \Omega^{(i)} \to \Omega_\varepsilon^{(i)}, \quad i = 1, 2, 3, 4,$$

be defined by

$$\begin{aligned}
p_\varepsilon^{(1)}(y_1, y_2, y_3) &= \big(\varepsilon y_1, \varepsilon^2(y_2 - q_2 + \frac{h}{2}) + \varepsilon q_2 - \varepsilon\frac{h}{2}, y_3\big),\\
p_\varepsilon^{(2)}(y_1, y_2, y_3) &= \big(\varepsilon^2(y_1 - q_1 - \frac{b}{2}) + \varepsilon q_1 + \varepsilon\frac{b}{2}, \varepsilon y_2, y_3\big),\\
p_\varepsilon^{(3)}(y_1, y_2, y_3) &= \big(\varepsilon y_1, \varepsilon^2(y_2 - q_2 - \frac{h}{2}) + \varepsilon q_2 + \varepsilon\frac{h}{2}, y_3\big),\\
p_\varepsilon^{(4)}(y_1, y_2, y_3) &= \big(\varepsilon^2(y_1 - q_1 + \frac{b}{2}) + \varepsilon q_1 - \varepsilon\frac{b}{2}, \varepsilon y_2, y_3\big).
\end{aligned}$$

For each $\varepsilon > 0$, from the solutions $\mathbf{u}^\varepsilon \in H^1_\#(\Omega_\varepsilon; \mathbb{R}^3)$ we define four functions $\mathbf{u}_\varepsilon^{(i)} \in H^1_\#(\Omega^{(i)}; \mathbb{R}^3)$ by

$$\mathbf{u}_\varepsilon^{(i)} := \mathbf{u}^\varepsilon \circ p_\varepsilon^{(i)}, \quad i = 1, 2, 3, 4.$$

Of course in the regions where the domains overlap we have

$$\begin{aligned}
\mathbf{u}_\varepsilon^{(1)} \circ p_\varepsilon^{(1)^{-1}} &= \mathbf{u}_\varepsilon^{(2)} \circ p_\varepsilon^{(2)^{-1}} \quad \text{in } \Omega_\varepsilon^{(1)} \cap \Omega_\varepsilon^{(2)},\\
\mathbf{u}_\varepsilon^{(3)} \circ p_\varepsilon^{(3)^{-1}} &= \mathbf{u}_\varepsilon^{(2)} \circ p_\varepsilon^{(2)^{-1}} \quad \text{in } \Omega_\varepsilon^{(3)} \cap \Omega_\varepsilon^{(2)},\\
\mathbf{u}_\varepsilon^{(3)} \circ p_\varepsilon^{(3)^{-1}} &= \mathbf{u}_\varepsilon^{(4)} \circ p_\varepsilon^{(4)^{-1}} \quad \text{in } \Omega_\varepsilon^{(3)} \cap \Omega_\varepsilon^{(4)}.
\end{aligned} \tag{2}$$

The above equations hold for the closed section but also for the open section. For the former section we have a further junction condition, hereafter called the *supplementary junction condition*:

$$\mathbf{u}_\varepsilon^{(1)} \circ p_\varepsilon^{(1)^{-1}} = \mathbf{u}_\varepsilon^{(4)} \circ p_\varepsilon^{(4)^{-1}} \quad \text{in } \Omega_\varepsilon^{(1)} \cap \Omega_\varepsilon^{(4)}. \tag{3}$$

Setting

$$\begin{aligned}
\Omega^{(1)} \cap \Omega^{(2)} &= (q_1 + b/2 - s, q_1 + b/2) \times (q_2 - h/2, q_2 - h/2 + s) \times (0, \ell),\\
\Omega^{(3)} \cap \Omega^{(2)} &= (q_1 + b/2 - s, q_1 + b/2) \times (q_2 + h/2 - s, q_2 + h/2) \times (0, \ell),\\
\Omega^{(1)} \cap \Omega^{(4)} &= (q_1 - b/2, q_1 - b/2 + s) \times (q_2 - h/2, q_2 - h/2 + s) \times (0, \ell),\\
\Omega^{(3)} \cap \Omega^{(4)} &= (q_1 - b/2, q_1 - b/2 + s) \times (q_2 + h/2 - s, q_2 + h/2) \times (0, \ell).
\end{aligned}$$

we can rewrite conditions (2) as

$$
\begin{aligned}
&\mathbf{u}^{(1)}\big(\varepsilon(z_1-q_1-\tfrac{b}{2})+q_1+\tfrac{b}{2},z_2,z_3\big)=\mathbf{u}^{(2)}\big(z_1,\varepsilon(z_2-q_2+\tfrac{h}{2})+q_2-\tfrac{h}{2},z_3\big),\\
&\mathbf{u}^{(3)}\big(\varepsilon(z_1-q_1-\tfrac{b}{2})+q_1+\tfrac{b}{2},z_2,z_3\big)=\mathbf{u}^{(2)}\big(z_1,\varepsilon(z_2-q_2-\tfrac{h}{2})+q_2+\tfrac{h}{2},z_3\big),\\
&\mathbf{u}^{(3)}\big(\varepsilon(z_1-q_1+\tfrac{b}{2})+q_1-\tfrac{b}{2},z_2,z_3\big)=\mathbf{u}^{(4)}\big(z_1,\varepsilon(z_2-q_2-\tfrac{h}{2})+q_2+\tfrac{h}{2},z_3\big),
\end{aligned}
\tag{4}
$$

which hold for $z\in\Omega^{(1)}\cap\Omega^{(2)}$, $z\in\Omega^{(3)}\cap\Omega^{(2)}$, $z\in\Omega^{(3)}\cap\Omega^{(4)}$, respectively. The supplementary junction condition (3), which we recall holds for the closed section but not for the open, can be rewritten as

$$
\mathbf{u}^{(1)}\big(\varepsilon(z_1-q_1+\frac{b}{2})+q_1-\frac{b}{2},z_2,z_3\big)=\mathbf{u}^{(4)}\big(z_1,\varepsilon(z_2-q_2+\frac{h}{2})+q_2-\frac{h}{2},z_3\big),
$$

where $z\in\Omega^{(1)}\cap\Omega^{(4)}$.

Let us consider the following 3×3 matrix valued differential operators

$$
\mathbf{H}^{(i)}_{\varepsilon}\mathbf{w}:=\left(\frac{D_1\mathbf{w}}{\varepsilon^{\alpha(i)}},\frac{D_2\mathbf{w}}{\varepsilon^{\alpha(i+1)}},D_3\mathbf{w}\right)
$$

where $D_i\mathbf{u}$ denotes the column vector of the partial derivatives of $\mathbf{w}$ with respect to y_i, and $\alpha(i)$ is the parity of i, that is $\alpha(i)=1$ if i is odd and $\alpha(i)=2$ if i is even. The above definition is motivated by the following trivial result

$$
\mathbf{H}^{(i)}_{\varepsilon}\mathbf{u}^{(i)}_{\varepsilon}=D\mathbf{u}^{\varepsilon}\circ p^{(i)}_{\varepsilon},\quad i=1,2,3,4.
$$

We also set

$$
\mathbf{E}^{(i)}_{\varepsilon}\mathbf{w}:=\mathrm{sym}(\,\mathbf{H}^{(i)}_{\varepsilon}\mathbf{w}),\quad \mathbf{W}^{(i)}_{\varepsilon}\mathbf{w}:=\mathrm{skw}(\,\mathbf{H}^{(i)}_{\varepsilon}\mathbf{w}).\tag{5}
$$

Under appropriate assumptions on the external loads, see [8], we may assume that the sequence $(\mathbf{u}^{(1)}_{\varepsilon},\mathbf{u}^{(2)}_{\varepsilon},\mathbf{u}^{(3)}_{\varepsilon},\mathbf{u}^{(4)}_{\varepsilon})\in\times_{i=1}^{4}H^1_{\#}(\Omega^{(i)};\mathbb{R}^3)$ satisfies

$$
\sum_{i=1}^{4}\|\mathbf{E}^{(i)}_{\varepsilon}\mathbf{u}^{(i)}_{\varepsilon}\|_{L^2(\Omega^{(i)};\mathbb{R}^{3\times 3})}\le C\varepsilon^2,
$$

for some constant C and every $0<\varepsilon\le 1$. Then, by using an appropriate Korn inequality, see [8], we deduce that for any sequence of positive numbers ε_n converging to 0 there exist a subsequence (not relabeled) and 4-tuples of functions $(\mathbf{v}^{(1)},\mathbf{v}^{(2)},\mathbf{v}^{(3)},\mathbf{v}^{(4)})\in\times_{i=1}^{4}H^1_{\#}(\Omega^{(i)};\mathbb{R}^3)$ and $(\vartheta^{(1)},\vartheta^{(2)},\vartheta^{(3)},\vartheta^{(4)})\in\times_{i=1}^{4}L^2(\Omega^{(i)})$ such that (as $n\to\infty$)

$$
\mathbf{u}^{(i)}_{\varepsilon_n}\rightharpoonup\mathbf{v}^{(i)},\ \text{in } H^1(\Omega^{(i)};\mathbb{R}^3),\tag{6}
$$

$$
(\mathbf{W}^{(i)}_{\varepsilon_n}\mathbf{u}^{(i)}_{\varepsilon_n})_{12}\rightharpoonup-\vartheta^{(i)}\ \text{in } L^2(\Omega^{(i)};\mathbb{R}^3).\tag{7}
$$

Moreover it can be shown that the displacements $\mathbf{v}^{(i)}$ are of Bernoulli-Navier type, that is

$$v_\alpha^{(i)} = \xi_\alpha^{(i)}(y_3),\ \alpha = 1,2, \quad v_3^{(i)} = \xi_3^{(i)}(y_3) - y_\alpha {\xi_\alpha^{(i)}}'(y_3), \tag{8}$$

for some functions

$$\xi_\alpha^{(i)} \in H^2_\#(0,\ell) := \{\xi \in H^2(0,\ell) : \xi(0) = \xi'(0) = 0\}$$

and

$$\xi_3^{(i)} \in H^1_\#(0,\ell).$$

To describe the kinematics of the beam we have, at the moment, four fields for each rectangular component of the cross section, thus a total of sixteen kinematical fields.

Lemma 1. *For both the open and closed sections, with the notation above and for almost every $y_3 \in (0,\ell)$, we have*

1. $\vartheta^{(1)} = \vartheta^{(2)} = \vartheta^{(3)} = \vartheta^{(4)} =: \vartheta$;
2. $\xi_2^{(i)}(y_3) = 0, \qquad i = 1,3$;
3. $\xi_1^{(i)}(y_3) = 0, \qquad i = 2,4$;
4. $\xi_3^{(1)}(y_3) - (q_1 + b/2){\xi_1^{(1)}}'(y_3) = \xi_3^{(2)}(y_3) - (q_2 - h/2){\xi_2^{(2)}}'(y_3)$;
5. $\xi_3^{(3)}(y_3) - (q_1 + b/2){\xi_1^{(3)}}'(y_3) = \xi_3^{(2)}(y_3) - (q_2 + h/2){\xi_2^{(2)}}'(y_3)$;
6. $\xi_3^{(3)}(y_3) - (q_1 - b/2){\xi_1^{(3)}}'(y_3) = \xi_3^{(4)}(y_3) - (q_2 + h/2){\xi_2^{(4)}}'(y_3)$;
7. $\xi_1^{(1)}(y_3) - \xi_1^{(3)}(y_3) = h\vartheta(y_3)$;
8. $\xi_2^{(2)}(y_3) - \xi_2^{(4)}(y_3) = b\vartheta(y_3)$.

Moreover $\vartheta \in H^2_\#(0,\ell)$.

For brevity we omit the proof, see [8], we simply mention that they can be obtained by appropriately taking the limit of (4). The above "junction conditions" considerably reduce the number of independent kinematical variables. The first result of the Lemma states that the rotations of the four rectangles comprising the cross section are the same, the second and the third instead show that some of the kinematical variables have no importance. The fourth, fifth and sixth simply say that the longitudinal displacement on three of the four overlapping regions in the closed section (the fourth is taken into account by the supplementary junction condition) and on all overlapping regions in the open section are the same. Finally the last two conditions put into relation the displacements on opposite rectangles with the rotation of the section.

Lemma 2. *For the closed section thin walled beam we also have that*

1. $\xi_3^{(1)}(y_3) - (q_1 - b/2){\xi_1^{(1)}}'(y_3) = \xi_3^{(4)}(y_3) - (q_2 - h/2){\xi_2^{(4)}}'(y_3)$,

for almost every $y_3 \in (0,\ell)$.

This last Lemma, which holds only for the closed section, is deduced from the supplementary junction condition.

3 Kinematics of the open cross section thin-walled beam

In this short section we show that, by taking into account Lemma 1, the sixteen kinematical variables reduce to only four. We define ϑ as in Lemma 1, and we set

$$\eta_1 := \xi_1^{(1)} + (q_2 - c_2 - \frac{h}{2})\vartheta, \quad \eta_2 := \xi_2^{(2)} - (q_1 - c_1 + \frac{b}{2})\vartheta,$$

$$\eta_3 := \xi_3^{(1)} + (q_2 - \frac{h}{2})\xi_2^{(2)\prime} - K\vartheta',$$

where $(c_1, c_2) \in \mathbb{R}^2$ and $K \in \mathbb{R}$. We then find

$$\begin{aligned}
\xi_1^{(1)} &= \eta_1 - (q_2 - c_2 - \frac{h}{2})\vartheta, & \xi_2^{(1)} &= 0,\\
\xi_1^{(2)} &= 0, & \xi_2^{(2)} &= \eta_2 + (q_1 - c_1 + \tfrac{b}{2})\vartheta,\\
\xi_1^{(3)} &= \eta_1 - (q_2 - c_2 + \frac{h}{2})\vartheta, & \xi_2^{(3)} &= 0,\\
\xi_1^{(4)} &= 0, & \xi_2^{(4)} &= \eta_2 + (q_1 - c_1 - \tfrac{b}{2})\vartheta,
\end{aligned}$$

and

$$\begin{aligned}
\xi_3^{(1)} &= \eta_3 - (q_2 - \frac{h}{2})\eta_2' - (q_2 - \frac{h}{2})(q_1 - c_1 + \frac{b}{2})\vartheta' + K\vartheta',\\
\xi_3^{(2)} &= \eta_3 - (q_1 + \frac{b}{2})\eta_1' + (q_1 + \frac{b}{2})(q_2 - c_2 - \frac{h}{2})\vartheta' + K\vartheta',\\
\xi_3^{(3)} &= \eta_3 - (q_2 + \frac{h}{2})\eta_2' - [(q_1 + \frac{h}{2})(q_1 - c_1 + \frac{b}{2}) + h(q_1 + \frac{b}{2}) - K]\vartheta',\\
\xi_3^{(4)} &= \eta_3 - (q_1 - \frac{b}{2})\eta_1' - [h(q_1 + \frac{b}{2}) + b(q_2 + \frac{h}{2}) - (q_1 - \frac{b}{2})(q_2 - c_2 + \frac{h}{2}) - K]\vartheta'.
\end{aligned}$$

Substituting these quantities in (8) we find the following displacements

$$\begin{aligned}
v_1^{(1)} &= \eta_1 - (q_2 - c_2 - \frac{h}{2})\vartheta, & v_2^{(1)} &= 0,\\
v_3^{(1)} &= \eta_3 - y_1\eta_1' - (q_2 - \frac{h}{2})\eta_2' + \psi^{(1)}(y_1)\vartheta', & &\\
v_1^{(2)} &= 0, & v_2^{(2)} &= \eta_2 + (q_1 - c_1 + \tfrac{b}{2})\vartheta,\\
v_3^{(2)} &= \eta_3 - (q_1 + \frac{b}{2})\eta_1' - y_2\eta_2' + \psi^{(2)}(y_2)\vartheta', & &\\
v_1^{(3)} &= \eta_1 - (q_2 - c_2 + \frac{h}{2})\vartheta, & v_2^{(3)} &= 0,\\
v_3^{(3)} &= \eta_3 - y_1\eta_1' - (q_2 + \frac{h}{2})\eta_2' + \psi^{(3)}(y_1)\vartheta', & &
\end{aligned}$$

$$v_1^{(4)} = 0, \qquad v_2^{(4)} = \eta_2 + (q_1 - c_1 - \tfrac{b}{2})\vartheta,$$

$$v_3^{(4)} = \eta_3 - (q_1 - \frac{b}{2})\eta_1' - y_2\eta_2' + \psi^{(4)}(y_2)\vartheta'.$$

where the so-called "sector coordinates"

$$\begin{aligned}
\psi^{(1)}(y_1) &:= y_1(q_2 - c_2 - \frac{h}{2}) - (q_2 - \frac{h}{2})(q_1 - c_1 + \frac{b}{2}) + K,\\
\psi^{(2)}(y_2) &:= -y_2(q_1 - c_1 + \frac{b}{2}) + (q_1 + \frac{b}{2})(q_2 - c_2 - \frac{h}{2}) + K,\\
\psi^{(3)}(y_1) &:= y_1(q_2 - c_2 + \frac{h}{2}) - (q_1 + \frac{h}{2})(q_1 - c_1 + \frac{b}{2}) - h(q_1 + \frac{b}{2}) + K,\\
\psi^{(4)}(y_2) &:= -y_2(q_1 - c_1 - \frac{b}{2}) - h(q_1 + \frac{b}{2}) - b(q_2 + \frac{h}{2})\\
&\qquad + (q_1 - \frac{b}{2})(q_2 - c_2 + \frac{h}{2}) + K,
\end{aligned}$$

are defined up to an additive constant K.

We notice that in the plane of the section, plane *1-2* , the motion is described by a translation (η_1, η_2) and a rotation ϑ about the point of coordinates (c_1, c_2). In direction *3* instead the motion is a Bernoulli-Navier displacement plus a quantity proportional to ϑ'.

4 On the kinematics of the closed cross section thin walled beam

In the case of the closed section not only Lemma 1 holds but we also have the supplementary junction condition which is taken into account in Lemma 2. Thus the results of the previous section still hold. We now study the consequences of Lemma 2.

Subtracting equation *5.* from equation *4.* of Lemma 1 we find

$$\xi_3^{(1)} - \xi_3^{(3)} - (q_1 + b/2)(\xi_1^{(1)\prime} - \xi_1^{(3)\prime}) = h\xi_2^{(2)\prime}, \qquad (9)$$

and subtracting equation *6.* of Lemma 1 from equation *1.* of Lemma 2 we deduce

$$\xi_3^{(1)} - \xi_3^{(3)} - (q_1 + b/2)(\xi_1^{(1)\prime} - \xi_1^{(3)\prime}) = h\xi_2^{(4)\prime}. \qquad (10)$$

Hence, taking the difference of (9) and (10) we obtain

$$b(\xi_1^{(1)\prime} - \xi_1^{(3)\prime}) = h(\xi_2^{(4)\prime} - \xi_2^{(2)\prime}),$$

which leads, after taking into account equations *7.* and *8.* of Lemma 1 to

$$bh\vartheta' = -bh\vartheta'.$$

Thus $\vartheta' = 0$, and since $\vartheta(0) = 0$ we have $\vartheta = 0$. Hence, the results of the previous section hold with $\vartheta = 0$.

We interpret the result $\vartheta = 0$ not as a statement that the section does not rotate about the axis of the beam, but instead as a sign that for a closed section thin-walled beam one should not look, as for the open sections, at the sequence $(\mathbf{W}^{(i)}_{\varepsilon_n}\mathbf{u}^{(i)}_{\varepsilon_n})_{12}$ to deduce the twist of the section. This sequence delivers a trivial result, $\vartheta = 0$, and to deduce the twist one should more likely consider the sequence generated by $(\mathbf{W}^{(i)}_{\varepsilon_n}\mathbf{u}^{(i)}_{\varepsilon_n})_{12}$ divided by some power of ε_n. To find the right power it is necessary to deduce an *ad hoc* Korn inequality for closed thin walled sections beams. This will be the aim of a future work.

References

1. E. Acerbi, G. Buttazzo, and D. Percivale, A variational definition of the strain energy for an elastic string, *J. Elasticity*, Vol. 25, pp.137–148 (1991).
2. G. Anzellotti, S. Baldo, and D. Percivale, Dimension reduction in variational problems, asymptotic development in Γ-convergence and thin structures in elasticity, *Asymptot. Anal.* **9**(1) (1994), 61–100.
3. F. Bourquin, P.G. Ciarlet, G. Geymonat, and A. Raoult, Gamma-convergence et analyse asymptotique des plaques minces, *C.R. Acad. Sci. Paris, t.* **315**, Série I, (1992), 1017–1024.
4. A. Braides, *Γ-convergence for beginners*, Oxford Lecture Series in Mathematics and its Applications, 22. Oxford University Press, Oxford, 2002.
5. G. Dal Maso, *An introduction to Γ-convergence*, Birkhäuser, Boston, 1993.
6. E. De Giorgi, and T. Franzoni, Su un tipo di convergenza variazionale, *Rend. Sem. Mat. Brescia* **3** (1979), 63–101.
7. L. Freddi, A. Morassi and R. Paroni, Thin-walled beams: the case of the rectangular cross-section, *J. Elasticity* **76** (2004), 45–66.
8. L. Freddi, A. Morassi and R. Paroni, Thin-walled beams: a derivation of Vlassov theory via Γ–convergence, *J. Elasticity* **86** (2007), 263–296.
9. H. Le Dret and A. Raoult, The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity, *J. Math. Pures Appl.*, Vol. 74, pp.549–578 (1995).
10. M.G. Mora and S. Müller, A nonlinear model for inextensible rods as a low energy Γ-limit of three-dimensional nonlinear elasticity, *Ann. I. H. Poincaré*, Vol. 21, pp.271–293 (2004).
11. A. Morassi, Torsion of thin tubes: a justification of some classical results, *J. Elasticity* **39** (1995), 213–227.
12. A. Morassi, Torsion of thin tubes with multicell cross-section, *Meccanica* **34** (1999), 115–132.
13. A. Morassi, An asymptotic analysis of the flexure problem for thin tubes, *Math. Mech. Solids* **4** (1999), 357–390.
14. D. Percivale, Thin elastic beams: the variational approach to St. Venant's problem, *Asymptot. Anal.* **20** (1999), 39–59.

15. J.M. Rodríguez, and J.M. Viaño, Asymptotic derivation of a general linear model for thin-walled elastic rods, *Comput. Methods Appl. Mech. Engrg.* **147** (1997), 287–321.
16. J.M. Rodríguez, and J.M. Viaño, Asymptotic analysis of Poisson's equation in a thin domain and its application to thin-walled elastic beams and tubes, *Math. Methods Appl. Sci.* **21** (1998), 187–226.
17. B.Z. Vlassov, *Pièces Longues en Voiles Minces*, Éditions Eyrolles, Paris, 1962.