A VARIATIONAL APPROACH TO THE STATIONARY SOLUTIONS OF BURGERS EQUATION

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ABSTRACT. Consider the viscous Burgers equation on a bounded interval with inhomogeneous Dirichlet boundary conditions. Following the variational framework introduced in [4], we analyze a Lyapunov functional for such equation which gives the large deviations asymptotics of a stochastic interacting particles model associated to the Burgers equation. We discuss the asymptotic behavior of this energy functional, whose minimizer is given by the unique stationary solution, as the length of the interval diverges. We focus on boundary data corresponding to a standing wave solution to the Burgers equation in the whole line. In this case, the limiting functional has in fact a one-parameter family of minimizers and we analyze the so-called development by Γ-convergence; this amounts to compute the sharp asymptotic cost corresponding to a given shift of the stationary solution.

1. Introduction

Consider the viscous Burgers equation on the interval $(a, b) \subseteq (-\infty, +\infty)$ with inhomogeneous Dirichlet boundary conditions at the endpoints

$$\begin{cases} u_t + f(u)_x = u_{xx} \,, \\ u(t,a) = u_- \,, \quad u(t,b) = u_+ \,, \end{cases} \tag{1.1}$$

where $u = u(t, x)$ satisfies $0 \le u \le 1$, the *flux* f is the function defined by $f(u) := u(1-u)$ and the boundary data satisfy $0 < u_- < u_+ < 1$. If the interval (a, b) is bounded, it is simple to show that there exists a unique stationary solution of (1.1) that can be computed explicitly. On the other hand, if we consider the case $(a, b) = (-\infty, +\infty)$ and $u_- + u_+ = 1$, there exists a *standing wave* solution of Burgers equation on the whole line. Accordingly, in this case problem (1.1) admits a one parameter family of stationary solutions $\{\bar{u}^{(z)}\,,\, z \in \mathbb{R}\}$ which is obtained by considering the translations of the standing wave $\bar{u}$ satisfying $\lim_{x\to\pm\infty} \bar{u}(x) = u_\pm$ and $\bar{u}(0) = 1/2$. If we consider the case $u_- + u_+ = 1$ in the bounded symmetric interval $(a, b) = (-\ell, \ell)$ and denote by $\bar{u}_\ell$ the unique stationary solution to (1.1), as ℓ diverges the sequence $\{\bar{u}_\ell\}$ converges to the stationary solution $\bar{u}$. We refer to [7] for a dynamical analysis of (1.1).

The main topic we here discuss is the following. Consider the case $u_- + u_+ = 1$ and fix $z \in \mathbb{R}$. If we take ℓ large we expect that there exists some function $u_\ell^{(z)}$ close to $\bar{u}^{(z)}$ such that $u_\ell^{(z)}$ is "almost" a stationary solution to (1.1) in the interval $(-\ell, \ell)$. We shall quantify being an "almost" stationary solution in terms of a

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suitable family of energy functionals measuring the probability of observing such fluctuations and compute the sharp asymptotic of the energy of $u_\ell^{(z)}$.

Since the Burgers equation is not a gradient flow, the choice of the energy functional is not trivial. Consider the problem (1.1) in the bounded interval (a,b) and denote by $V_{a,b}$, which also depends on $u_\pm$, an associated energy functional. Let us first make a short list of the properties that $V_{a,b}$ should enjoy:

(i) the unique minimizer of $V_{a,b}$ is the stationary solution to (1.1);
(ii) $V_{a,b}$ is a Lyapunov functional for the flow defined by (1.1);
(iii) as (a,b) diverges the functional $V_{a,b}$ converges to the functional $V_{-\infty,+\infty}$ associated to (1.1) in the whole line.

Of course, this list still gives a lot of freedom. However, as we next discuss, there is a natural way to meet the requirements *(i)* and *(ii)* above with some energy functional $V_{a,b}$ that has a clear interpretation in terms of large deviations, while property *(iii)* will be proven in this paper.

First we associate to (1.1) an action functional $I_{a,b}$, defined on functions depending on space and time. To this end, add an external "controlling" field $E = E(t,x)$ to obtain the perturbed equation

$$\begin{cases} v_t + f(v)_x + 2\big(\sigma(v)E\big)_x = v_{xx}, \\ v(t,a) = u_- \,, \quad v(t,b) = u_+ \,, \end{cases} \tag{1.2}$$

where σ is a given positive function which, regarding v as a density, can be interpreted as the mobility of the system. Denote by v^E the solution of this equation. The action of a path $v : (-\infty, 0] \times [a,b] \to [0,1]$ is given by

$$I_{a,b}(v) \;=\; \inf \int_{-\infty}^{0} \int_a^b \sigma(v)\, E^2 \, dx\, dt \,, \tag{1.3}$$

where the infimum is carried over all E such that $v^E = v$. Note that if v is a solution to (1.1) then $I_{a,b}(v) = 0$. Consider now the so-called *quasi-potential* [8] associated to the action functional $I_{a,b}$, that is let $V_{a,b}$ be the functional on the set of functions $u : [a,b] \to [0,1]$ defined by

$$V_{a,b}(u) \;=\; \inf \big\{ I_{a,b}(v) \,:\, v(0) = u \,,\, v(t) \to \bar u_{a,b} \text{ as } t \to -\infty \big\} \,, \tag{1.4}$$

where $\bar u_{a,b}$ is the stationary solution to (1.1). Namely, $V_{a,b}(u)$ is the minimal action to reach u starting from $\bar u_{a,b}$. Of course, $V_{a,b} \ge 0$ and $V_{a,b}(\bar u_{a,b}) = 0$; it is also simple to check that $V_{a,b}$ is a Lyapunov functional for (1.1).

The functional $V_{a,b}$ obtained by the previous general recipe depends on the choice of the mobility σ. If the boundary data are equal, $u_- = u_+ = u_\circ$, then the stationary solution is constant, $\bar u = u_\circ$. In this case, it can be shown that the quasi-potential is given by

$$V_{a,b}(u) \;=\; \int_a^b s_{u_\circ}(u)\, dx \,, \tag{1.5}$$

where $s_{u_\circ} : [0,1] \to [0,+\infty)$ is the convex function such that $s''_{u_\circ}(u) = 1/\sigma(u)$ and $s_{u_\circ}(u_\circ) = s'_{u_\circ}(u_\circ) = 0$. Referring to [2] for the proof of (1.5) in the case of periodic boundary conditions, we simply observe that the functional in (1.5) trivially satisfies the requirements *(i)*, *(ii)*, and *(iii)* above.

In the case of inhomogeneous boundary data $u_- \neq u_+$, the quasi-potential $V_{a,b}$ is in general a nonlocal functional and, as its definition involves the solution of a difficult dynamical problem, its direct analysis does not appear feasible. For the specific case of the Burgers equation here considered and the choice $\sigma(v) = v(1-v)$, in [4] it is shown that the quasi-potential $V_{a,b}$ can be expressed in terms of a much simpler variational problem which requires to optimize over functions of a single variable rather than on all paths as in (1.4).

When $\sigma(v) = v(1-v)$ the action functional $I_{a,b}$ introduced in (1.3) is the dynamical large deviations rate functional of a much studied stochastic model of interacting particles, the so-called weakly asymmetric simple exclusion process [9]. Accordingly, the quasi-potential $V_{a,b}$ describes the asymptotic behavior of the corresponding invariant measure [8]. More precisely, if we denote by $\mu^N_{a,b}$ the invariant measure of the stochastic particles model in the interval (a,b) with lattice spacing $1/N$, then as $N \to +\infty$ we have

$$\mu^N_{a,b}(\mathcal{B}) \asymp \exp\big\{ -N \inf_{u\in\mathcal{B}} V_{a,b}(u)\big\} , \tag{1.6}$$

where $\mathcal{B}$ is a measurable subset of the configuration space. In particular, the probability on the left hand side converges to one as $N \to \infty$ only if the global minimizer of $V_{a,b}$ lies in the set $\mathcal{B}$. If otherwise $\bar{u}_{a,b} \notin \mathcal{B}$ the large deviation formula (1.6) expresses the fact that the probability of $\mathcal{B}$ converges to zero exponentially fast in N with rate given by the infimum of $V_{a,b}$ on the set $\mathcal{B}$. Within this context, which takes into account the effect of fluctuations, we are thus interested not only to the global minimizer of $V_{a,b}$, but also to its minimizers in subsets of the function space. A natural question is the asymptotic behavior of the probability $\mu^N_{a,b}$ in the joint limit in which $N \to \infty$ and the interval (a,b) diverges. A simple approach to this issue, which corresponds to take first the limit $N \to \infty$ and then letting the interval (a,b) diverge, is to analyze the variational convergence of $V_{a,b}$.

As the results in [4] are the starting point of the present analysis, we briefly recall the main statement. Given $p \in [0,1]$, set $s(p) = p\log p + (1-p)\log(1-p)$. Note that $s''(p) = 1/[p(1-p)]$ so that s can be regarded as the entropy function of the homogeneous system; in the stochastic setting this function emerges naturally as the Bernoulli entropy. To the boundary data $0 < u_- < u_+ < 1$ there correspond the chemical potentials $\varphi_\pm = s'(u_\pm) = \log[u_\pm/(1-u_\pm)] \in \mathbb{R}$. Given the bounded interval $[a,b]$ define the functional $\mathcal{G}_{a,b}$ of the two variables $u = u(x)$ and $\varphi = \varphi(x)$, $x \in [a,b]$, as

$$\mathcal{G}_{a,b}(u,\varphi) \;=\; \int_a^b \big[s(u) + s(\varphi') + (1-u)\varphi - \log\big(1+e^{\varphi}\big)\big]\, dx , \tag{1.7}$$

where φ satisfies $0 \le \varphi' \le 1$ as well as $\varphi(a) = \varphi_-$ and $\varphi(b) = \varphi_+$. Optimize now in φ to get a functional $\mathcal{F}_{a,b}$ of u

$$\mathcal{F}_{a,b}(u) \;=\; \inf_{\varphi} \mathcal{G}_{a,b}(u,\varphi) . \tag{1.8}$$

In [4] it is proven that, apart an additive constant, the quasi-potential is equal to $\mathcal{F}_{a,b}$, namely

$$V_{a,b}(u) \;=\; \mathcal{F}_{a,b}(u) \;-\; \inf \mathcal{F}_{a,b} . \tag{1.9}$$

Observe that the boundary data $u_\pm$ are passed to $\mathcal{F}_{a,b}$ thought the auxiliary function φ. In particular, while the functional $\mathcal{F}_{a,b}$ is bounded on the whole $L^\infty((a,b);[0,1])$, its minimizer is smooth and satisfies the boundary conditions in (1.1).

The aim of this paper is to analyze the asymptotic behavior, in terms of Γ-convergence [5, 6], of the functionals $\mathcal{F}_{a,b}$ when the boundary data $u_\pm$ are fixed and the interval (a,b) diverges. Although the functionals are quite different, some of the arguments in the proofs of our results are similar to the ones used in the analysis of the analogous problem for the van der Waals free energy functional in a bounded interval [3].

In the case $u_- + u_+ > 1$, the stationary solution to (1.1) will essentially make the transition from u_- to u_+ close the left endpoint a; note indeed that in this case the Burgers equation on the whole line admits a travelling wave propagating towards the left. We thus set $(a,b) = (0,\ell)$ and analyze the sequence of functionals $\{\mathcal{F}_{0,\ell}\}$; we prove its Γ-convergence to a limiting functional $\mathcal{F}_{0,+\infty}$ which is basically defined as in the case of a bounded interval. In this situation, $\{\mathcal{F}_{0,\ell}\}$ has good coerciveness properties to ensure the compactness of sequences with equibounded energy. Since the unique minimizer of $\mathcal{F}_{0,+\infty}$ is given by the stationary solution $\bar{u}_{0,+\infty}$ to (1.1) in the unbounded interval $(0,+\infty)$, the minimizer of $\mathcal{F}_{0,\ell}$ converges to $\bar{u}_{0,+\infty}$. Of course, analogous results hold when $u_- + u_+ < 1$.

In contrast, the case $u_- + u_+ = 1$ is much richer. We consider the symmetric interval $(a,b) = (-\ell,\ell)$ and analyze the asymptotic behavior of the sequence of functionals $\{\mathcal{F}_{-\ell,\ell}\}$. The first result, that is the Γ-convergence to a limiting functional $\mathcal{F}_{-\infty,+\infty}$, is analogous to the previous case. However, when $u_- + u_+ = 1$, the functional $\mathcal{F}_{-\infty,+\infty}$ has a one parameter family of minimizers, given by the stationary solutions to (1.1) in the interval $(-\infty,+\infty)$. For this reason, the sequence $\{\mathcal{F}_{-\ell,\ell}\}$ is not equi-coercive: there are sequence $\{u_\ell\}$ such that $\mathcal{F}_{-\ell,\ell}(u_\ell) \to \inf \mathcal{F}_{-\infty,+\infty}$ and $z_\ell \to \infty$, where z_ℓ is the point such that $u_\ell(z_\ell) = 1/2$. We show that equi-coercivity of $\{\mathcal{F}_{-\ell,\ell}\}$ is recovered if we identify functions that differ by a translation. In particular, since modulo translations $\mathcal{F}_{-\infty,+\infty}$ has a unique minimizer, the shape of almost minimizers for $\{\mathcal{F}_{-\ell,\ell}\}$ is rigid.

Let $\bar{u}_\ell$ be the true minimizer of $\mathcal{F}_{-\ell,\ell}$. As discussed before, the sequence $\{\bar{u}_\ell\}$ converges to $\bar{u}$, the stationary solution of (1.1) in the interval $(-\infty,+\infty)$ such that $\bar{u}(0) = 1/2$. The previous statement cannot be deduced from the Γ-convergence of $\{\mathcal{F}_{-\ell,\ell}\}$. On the other hand, as it is customary in those problems having a limiting functional with plenty of minimizers, a variational description of this phenomenon is possible considering the so-called development by Γ-convergence [1]. More precisely, we introduce a rescaled excess energy $\mathcal{F}^{(1)}_{-\ell,\ell}$ by setting

$$\mathcal{F}^{(1)}_{-\ell,\ell}(u) \; = \; C(\ell)\Big[\mathcal{F}_{-\ell,\ell}(u) - \inf \mathcal{F}_{-\infty,+\infty}\Big], \tag{1.10}$$

and we look for a sequence $C(\ell) \to +\infty$ for which $\{\mathcal{F}^{(1)}_{-\ell,\ell}\}$ has a non trivial Γ-limit. Let $\alpha \in (0,1)$ be such that $u_\pm = (1 \pm \alpha)/2$. We show that the right choice of the rescaling is $C(\ell) = e^{\alpha\ell}$; this is consistent with the fact that the stationary solution $\bar{u}$ approaches the asymptotic values $u_\pm$ exponentially fast. Then, we compute the corresponding Γ-limit $\mathcal{F}^{(1)}_{-\infty,+\infty}$. Of course, $\mathcal{F}^{(1)}_{-\infty,+\infty}(u) < +\infty$ only if $u = \bar{u}^{(z)}$ for some $z \in \mathbb{R}$ and its explicit expression is given by

$$\mathcal{F}^{(1)}_{-\infty,+\infty}(u^{(z)}) \; = \; \frac{8\alpha}{1-\alpha^2}\cosh(\alpha z). \tag{1.11}$$

In particular, since $\bar{u}$ is the unique minimizer of $\mathcal{F}^{(1)}_{-\infty,+\infty}$, the variational picture in terms of development by Γ-convergence is complete. In general, the excess energy $e^{-\alpha\ell}\mathcal{F}^{(1)}_{-\infty,+\infty}(\bar{u}^{(z)})$ represents the cost for shifting by z the stationary solution $\bar{u}_\ell$.

In terms of the large deviation formula (1.6), it gives the asymptotic probability of a fluctuation close to $\bar{u}^{(z)}$ as $N \to +\infty$ and then $\ell \to +\infty$

$$\mu^N_{-\ell,\ell}(\mathcal{O}^{(z)}) \asymp \exp\Big\{ - Ne^{-\alpha\ell} \frac{8\alpha}{1-\alpha^2} \big[\cosh(\alpha z) - 1 \big] \Big\} , \tag{1.12}$$

where $\mathcal{O}^{(z)}$ is small neighborhood of $\bar{u}^{(z)}$.

We finally briefly discuss the sharp interface setting. This amounts to the change of variable $x \mapsto x/\ell$, so that one considers the Burgers equation (1.1) in the fixed interval $(-1, 1)$ with viscosity $\varepsilon = 1/\ell$. The asymptotic behavior of the energy functionals can be clearly described also in the limit $\varepsilon \to 0$, see [4]. In this setting stationary solutions to (1.1) converge to step functions. Note that fluctuations which are of order one in the unscaled variables are not seen in the sharp interface limit. In particular, the Γ-limit (1.11) of the rescaled excess energy translated into the sharp interface setting becomes degenerate, being infinite away from the minimizer. Even choosing a different rescaling in (1.10), i.e. replacing $e^{\alpha/\varepsilon}$ with $e^{\beta/\varepsilon}$, $\beta \in (0, \alpha)$, we would still get a degenerate Γ-limit. More precisely, with such a choice the Γ-limit would be zero if the interface is at distance less than $1 - \beta/\alpha$ from the origin and infinite otherwise.

2. The variational formulation

In this section we introduce precisely the variational formulation for stationary solutions to Burgers equation on bounded intervals and show uniqueness of minimizers. Fix a bounded interval $(a, b) \subset (-\infty, +\infty)$. Recalling that the flux is given by $f(u) = u(1 - u)$, the stationary solution $\bar{u}_{a,b}$ to the viscous Burgers equation (1.1) solves the boundary value problem

$$\begin{cases} u'' - [u(1-u)]' = 0 & x \in (a,b) \ , \\ u(a) = u_- \ , \quad u(b) = u_+ \ , \end{cases} \tag{2.1}$$

where $0 < u_- < u_+ < 1$. This problem admits a monotone solution, that satisfies the identity

$$u' = u(1-u) - J_{a,b} \ , \tag{2.2}$$

where the *current* $J_{a,b}$ is the constant determined by the boundary conditions, i.e. it satisfies

$$\int_{u_-}^{u_+} \frac{1}{r(1-r) - J_{a,b}} \, dr = b - a \ . \tag{2.3}$$

Observe that $J_{a,b}$ is uniquely defined by the previous condition. Moreover $J_{a,b} < \min\{u_-(1-u_-), u_+(1-u_+)\} \le 1/4$ and $J_{a,b} > 0$ as soon as $b-a > \log[u_+/(1-u_+)] - \log[u_-/(1-u_-)]$. In particular, from (2.2) and (2.3) we deduce the uniqueness of the solution for the boundary value problem (2.1). Equation (2.2) can be explicitly integrated getting

$$\bar{u}_{a,b}(x) = \tfrac{1}{2} + A_{a,b} \tanh\big[A_{a,b}(x - x_{a,b}) \big] \ , \tag{2.4}$$

where $A_{a,b} = \big(\frac{1}{4} - J_{a,b}\big)^{\frac{1}{2}}$ and $x_{a,b} \in \mathbb{R}$ is determined by imposing $\bar{u}_{a,b}(a) = u_-$. In the sequel we consider only the cases $u_- + u_+ = 1$, $(a, b) = (-\ell, \ell)$ and $u_- + u_+ \gtrless 1$, $(a, b) = \pm(0, \ell)$. The corresponding solutions to (2.1) are denoted by $\bar{u}_\ell$ and $\bar{u}^\pm_\ell$, respectively.

The stationary solution to (1.1) in the case of an unbounded interval can be described analogously. Given $0 < u_- < u_+ < 1$ such that $u_- + u_+ \gtrless 1$, we consider the boundary value problem

$$\begin{cases} u'' - [u(1-u)]' = 0 & x \in \mathbb{R}_\pm \ , \\ u(0) = u_\mp \ , \\ \lim_{x\to\pm\infty} u(x) = u_\pm \ , \end{cases} \tag{2.5}$$

whose solution $\bar{u}^\pm$ is given by

$$\bar{u}^\pm(x) = \tfrac{1}{2} + A^\pm \tanh\big[A^\pm(x - x^\pm)\big] \ , \tag{2.6}$$

where $A^\pm = \big|u_\pm - \frac{1}{2}\big|$ and $x^\pm \in \mathbb{R}$ is determined by imposing $\bar{u}^\pm(0) = u_\mp$.

Finally, given $0 < u_- < u_+ < 1$ such that $u_- + u_+ = 1$, we consider the boundary value problem

$$\begin{cases} u'' - [u(1-u)]' = 0 & x \in \mathbb{R} \ , \\ \lim_{x\to\pm\infty} u(x) = u_\pm \ , \end{cases} \tag{2.7}$$

which has a one-parameter family of solutions given by $\{\tau_z \bar{u},\, z \in \mathbb{R}\}$ where τ_z is the translation by z, i.e. $\tau_z u$ is the function defined by $(\tau_z u)(x) := u(x-z)$, and

$$\bar{u}(x) = \tfrac{1}{2} + \big(u_+ - \tfrac{1}{2}\big) \tanh\big[\big(u_+ - \tfrac{1}{2}\big)x\big] \ . \tag{2.8}$$

We now introduce precisely the variational formulation (1.7)-(1.9) proposed in [4]. Let $s : [0,1] \to \mathbb{R}$ be the convex function

$$s(u) := u \log u + (1-u)\log(1-u) \tag{2.9}$$

where, as usual, we understand that $0 \log 0 = 0$. Let also $g : [0,1] \times \mathbb{R} \times [0,1] \to \mathbb{R}$ be the continuous function

$$g(u,\varphi,p) := s(u) + s(p) + (1-u)\varphi - \log\big(1 + e^\varphi\big) \ . \tag{2.10}$$

Given $u_\pm \in (0,1)$ let

$$\varphi_\pm := s'(u_\pm) = \log \frac{u_\pm}{1-u_\pm} \tag{2.11}$$

and observe that if $u_- + u_+ = 1$ then $g(u_-,\varphi_-,0) = g(u_+,\varphi_+,0)$. We set

$$\mathcal{C}^\pm := \Big\{\varphi \in AC(\mathbb{R}_\pm) \,:\, 0 \le \varphi' \le 1 \,,\, \varphi(0) = \varphi_\mp ,\, \lim_{x\to\pm\infty} \varphi(x) = \varphi_\pm\Big\} \ ,$$

$$\mathcal{C} := \Big\{\varphi \in AC(\mathbb{R}) \,:\, 0 \le \varphi' \le 1 \,,\, \lim_{x\to-\infty} \varphi(x) = \varphi_- \,,\, \lim_{x\to+\infty} \varphi(x) = \varphi_+\Big\} \ ,$$

where AC denotes the space of absolutely continuous functions. We consider $\mathcal{C}^\pm$ and $\mathcal{C}$ endowed with the topology of uniform convergence, so that they are Polish spaces, i.e., metrizable, complete and separable. Then, we consider the spaces $L^\infty(\mathbb{R}_\pm;[0,1])$ and $L^\infty(\mathbb{R};[0,1])$ endowed with the weak* topology, and set

$$\mathcal{X}^\pm := L^\infty(\mathbb{R}_\pm;[0,1]) \times \mathcal{C}^\pm \ , \qquad \mathcal{X} := L^\infty(\mathbb{R};[0,1]) \times \mathcal{C} \ ,$$

that we consider endowed with the product topology. Observe that also $\mathcal{X}^\pm$ and $\mathcal{X}$ are Polish spaces.

If $u_- + u_+ \gtrless 1$, equivalently $\varphi_- + \varphi_+ \gtrless 0$, for each $\ell > 0$ we let $\mathcal{G}^\pm_\ell : \mathcal{X}^\pm \to (-\infty,+\infty]$ be the functional defined by

$$\mathcal{G}^\pm_\ell(u,\varphi) := \begin{cases} \displaystyle\int_{\pm(0,\ell)} \big[g(u,\varphi,\varphi) - g(u_\pm,\varphi_\pm,0)\big]\,dx & \text{if } (u,\varphi) \in B^\pm_\ell \ , \\ +\infty & \text{otherwise} \ , \end{cases} \tag{2.12}$$

where

$$B_\ell^\pm := \big\{(u,\varphi) \in \mathcal{X}^\pm :\ \varphi(x) = \varphi_\pm,\, u(x) = u_\pm \text{ for } x \in \pm[\ell,\infty)\big\}\,.$$

If $u_- + u_+ = 1$, equivalently $\varphi_- + \varphi_+ = 0$, for each $\ell > 0$ we let $\mathcal{G}_\ell : \mathcal{X} \to (-\infty,+\infty]$ be the functional defined by

$$\mathcal{G}_\ell(u,\varphi) := \begin{cases} \displaystyle\int_{-\ell}^{\ell} \big[g(u,\varphi,\varphi') - g(u_+,\varphi_+,0)\big]\, dx & \text{if } (u,\varphi) \in B_\ell\,, \\ +\infty & \text{otherwise}\,, \end{cases} \tag{2.13}$$

where

$$\begin{aligned} B_\ell :=\big\{(u,\varphi) \in \mathcal{X} :\ & \varphi(x) = \varphi_-,\, u(x) = u_- \text{ for } x \le -\ell\,, \\ & \varphi(x) = \varphi_+,\, u(x) = u_+ \text{ for } x \ge \ell\big\}\,. \end{aligned}$$

Observe that, up to an additive constant, the definition of $\mathcal{G}_\ell^\pm$ and $\mathcal{G}_\ell$ agrees with (1.7). According with (1.8), we also define the functionals $\mathcal{F}_\ell^\pm : L^\infty(\mathbb{R}_\pm;[0,1]) \to (-\infty,+\infty]$ and $\mathcal{F}_\ell : L^\infty(\mathbb{R};[0,1]) \to (-\infty,+\infty]$ by

$$\mathcal{F}_\ell^\pm(u) := \inf_{\varphi\in\mathcal{C}^\pm} \mathcal{G}_\ell^\pm(u,\varphi)\,, \qquad \mathcal{F}_\ell(u) := \inf_{\varphi\in\mathcal{C}} \mathcal{G}_\ell(u,\varphi)\,. \tag{2.14}$$

Since g is continuous and $p \mapsto g(u,\varphi,p)$ is convex, the above infima are attained as soon as $u(x) = u_\pm$ for $x \in \pm[\ell,\infty)$, respectively $u(x) = u_-$ for $x \le -\ell$ and $u(x) = u_+$ for $x \ge \ell$. As shown in [4], there are functions u for which the set of minimizers is not a singleton. Recalling (2.4), we set $\bar\varphi_\ell^\pm := s'(\bar u_\ell^\pm)$ and $\bar\varphi_\ell := s'(\bar u_\ell)$.

Proposition 2.1. *For each $\ell > 0$ the functionals $\mathcal{G}_\ell^\pm$ and $\mathcal{G}_\ell$ have a unique minimizer respectively given by $(\bar u_\ell^\pm, \bar\varphi_\ell^\pm)$ and $(\bar u_\ell, \bar\varphi_\ell)$. In particular, the unique minimizer of $\mathcal{F}_\ell^\pm$ and $\mathcal{F}_\ell$ is given by $\bar u_\ell^\pm$ and $\bar u_\ell$, respectively.*

Proof. We prove the statement only for $\mathcal{G}_\ell$. In view of the strict convexity of $[0,1] \ni u \mapsto g(u,\varphi,p)$, we can easily minimize $g(\cdot,\varphi,p)$ and the corresponding optimal u satisfies $s'(u) = \varphi$. Whence

$$\min_{u,\varphi} \mathcal{G}_\ell(u,\varphi) = \min_{\varphi} \mathcal{G}_\ell\big((s')^{-1}(\varphi),\varphi\big)\,.$$

The functional on the right hand side is clearly coercive and lower semicontinuous on $\mathcal{C}$. By the direct method of the calculus of variations, it thus admits a minimizer φ^*. A straightforward computation shows that the Euler-Lagrange equation for $\mathcal{G}_\ell\big((s')^{-1}(\varphi),\varphi\big)$ implies that $(s')^{-1}(\varphi^*)$ solves (2.1) in the interval $(-\ell,\ell)$. By the uniqueness to such problem we deduce $\varphi^* = s'(\bar u_\ell) = \bar\varphi_\ell$.

Finally, the last statement follows from the coercivity of $\mathcal{G}_\ell$. $\square$

3. Variational convergence

In this section we discuss the variational formulation on unbounded intervals. We show that the functionals in (2.12) and (2.13) are well defined also for $\ell = \infty$ and coincide with the Γ-limit of the sequences $\{\mathcal{G}_\ell^\pm\}$ and $\{\mathcal{G}_\ell\}$ as $\ell \to \infty$. In particular, this yields the stability of the boundary value problems (2.1).

We start by the following proposition which yields the basic estimates needed in sequel. Given $a_\pm \in \mathbb{R}$ we let $\vartheta_{a_-,a_+}(x) := a_- 1\!\!1_{\mathbb{R}_-}(x) + a_+ 1\!\!1_{\mathbb{R}_+}(x)$.

Proposition 3.1. *Let $u_- + u_+ \gtrless 1$. There exists a constant $C > 0$ such that for any $(u, \varphi) \in \mathcal{X}^\pm$*

$$\big\| g(u,\varphi,\varphi') - g(u_\pm,\varphi_\pm,0) \big\|_{L^1(\mathbb{R}_\pm)} \le C \left(\|u - u_\pm\|_{L^2(\mathbb{R}_\pm)} + \|\varphi - \varphi_\pm\|_{L^1(\mathbb{R}_\pm)} + 1 \right) .$$

Let otherwise $u_- + u_+ = 1$. There exists a constant $C > 0$ such that for any $(u, \varphi) \in \mathcal{X}$

$$\big\| g(u,\varphi,\varphi') - g(u_+,\varphi_+,0) \big\|_{L^1(\mathbb{R})} \le C \left(\|u - \vartheta_{u_-,u_+}\|_{L^2(\mathbb{R})} + \|\varphi - \vartheta_{\varphi_-,\varphi_+}\|_{L^1(\mathbb{R})} + 1 \right) .$$

We premise an elementary lemma.

Lemma 3.2. *Let $\varphi_+, \varphi_- \in \mathbb{R}$ be such that $\varphi_- < \varphi_+$. For each $\gamma > 0$ there exists a constant $C_\gamma(\varphi_-, \varphi_+) \in (0, +\infty)$, satisfying $C_\gamma(\varphi_-, \varphi_+) \to 0$ as $\varphi_+ - \varphi_- \to 0$, such that for any $\varphi \in \mathcal{C}^+$*

$$\int_0^{+\infty} \big| s(\varphi') \big| \, dx \le \gamma \, \|\varphi - \varphi_+\|_{L^1(\mathbb{R}_+)} + C_\gamma(\varphi_-, \varphi_+) \, .$$

Proof. Given $\delta \in (0,1)$, let $A_\delta := \big\{ x \in \mathbb{R}_+ \, : \, \varphi'(x) \in [0,\delta] \big\}$ and set $A_\delta^{\complement} := \mathbb{R}_+ \setminus A_\delta$. We write

$$\int_0^{+\infty} \big| s(\varphi') \big| \, dx = \int_{A_\delta} \big| s(\varphi') \big| \, dx + \int_{A_\delta^{\complement}} \big| s(\varphi') \big| \, dx \tag{3.1}$$

and estimate separately the two terms on the right hand side. To bound the first one, we first observe that for $p \in [0,1]$ we have $|s(p)| \le p(|\log p| + 1)$ and then use Hölder inequality as follows

$$\begin{aligned}
\int_{A_\delta} \varphi' |\log \varphi'| \, dx \; &= \; \int_{A_\delta} \big[\varphi'(x) \, (1+x) \big]^{2/3} \varphi'(x)^{1/3} (1+x)^{-2/3} |\log \varphi'(x)| \, dx \\
&\le \Big[\int_{A_\delta} \varphi'(x) \, (1+x) \, dx \Big]^{2/3} \Big[\int_{A_\delta} \varphi'(x) \, (1+x)^{-2} \big| \log \varphi'(x) \big|^3 \, dx \Big]^{1/3} \\
&\le \eta_\delta \Big[\int_{A_\delta} \varphi'(x) \, (1+x) \, dx \Big]^{2/3} \le \eta_\delta \Big[\int_0^\infty \varphi'(x) \, x \, dx \; + \; \varphi_+ - \varphi_- \Big]^{2/3} \\
&= \eta_\delta \Big[\|\varphi - \varphi_+\|_{L^1(\mathbb{R}_+)} \; + \; \varphi_+ - \varphi_- \Big]^{2/3} ,
\end{aligned}$$

where $\eta_\delta := \max_{p \in [0,\delta]} p |\log p|^3 \int_0^\infty (1+x)^{-2} \, dx$ and in the last step we used that $\|\varphi - \varphi_+\|_{L^1(\mathbb{R}_+)} = \int_0^\infty \varphi'(x) \, x \, dx$. Since $\int_{A_\delta} \varphi' \, dx \le \int_0^{+\infty} \varphi' \, dx = \varphi_+ - \varphi_-$, we get

$$\int_{A_\delta} \big| s(\varphi') \big| \, dx \; \le \; \eta_\delta \Big[\|\varphi - \varphi_+\|_{L^1(\mathbb{R}_+)} \; + \; \varphi_+ - \varphi_- \Big]^{2/3} \; + \; \varphi_+ - \varphi_- \tag{3.2}$$

To bound the second term on the right hand side of (3.1), we observe that, in view of the convexity of s, for each $p \in [\delta, 1]$ we have

$$|s(p)| \le |s(\delta)| + |s'(\delta)| (p - \delta) \le |s(\delta)| + |s'(\delta)| p \, .$$

Denoting by $\big| A_\delta^{\complement} \big|$ the Lebesgue measure of $A_\delta^{\complement}$ we then deduce

$$\int_{A_\delta^{\complement}} \big| s(\varphi'(x)) \big| \, dx \le \int_{A_\delta^{\complement}} \big[|s(\delta)| + |s'(\delta)| \, \varphi'(x) \big] \, dx \le |s(\delta)| \big| A_\delta^{\complement} \big| + |s'(\delta)| (\varphi_+ - \varphi_-) \, . \tag{3.3}$$

Moreover, we have $\left|A_\delta^{\complement}\right|^2 \le 2\delta^{-1}\|\varphi-\varphi_+\|_{L^1(\mathbb{R}_+)}$. Indeed,

$$\begin{aligned}\big\|\varphi-\varphi_+\big\|_{L^1(\mathbb{R}_+)} \;&=\; \int_0^{+\infty}\varphi'(x)\,x\,dx \;\ge\; \int_{A_\delta^{\complement}}\varphi'(x)\,x\,dx\\ &\ge\; \delta\int_{A_\delta^{\complement}} x\,dx \;\ge\; \delta\int_0^{|A_\delta^{\complement}|} x\,dx \;=\; \tfrac{\delta}{2}\left|A_\delta^{\complement}\right|^2 .\end{aligned}$$

Therefore, recalling (3.1) and noticing that $\lim_{\delta\downarrow 0}\eta_\delta = 0$, $\lim_{\delta\downarrow 0}\delta^{-1/2}s(\delta)=0$, the lemma follows easily from (3.2) and (3.3). □

Proof of Proposition 3.1. We prove the statement only for $u_-+u_+>1$. Indeed, the statement for $u_-+u_+<1$ is completely analogous, and the case $u_-+u_+=1$ follows, noticing that $g(u_+,\varphi_+,0)=g(u_-,\varphi_-,0)$, from the previous ones.

Recalling (2.10) and $\varphi_+=s'(u_+)$, we write

$$\begin{aligned}g(u,\varphi,\varphi') - g(u_+,\varphi_+,0) \;&=\; s(u)-s(u_+)-s'(u_+)(u-u_+)\\ &\quad + (1-u)(\varphi-\varphi_+) - \big[\log\big(1+e^{\varphi}\big) - \log\big(1+e^{\varphi_+}\big)\big] + s(\varphi') .\end{aligned}$$

As $\varphi\in\mathcal{C}^+$ implies $\varphi\le\varphi_+=s'(u_+)$,

$$\begin{aligned}\big|g(u,\varphi,\varphi') - g(u_+,\varphi_+,0)\big| \;&\le\; s(u)-s(u_+)-s'(u_+)(u-u_+)\\ &\quad + (1-u)(\varphi_+-\varphi) + \log\big(1+e^{\varphi_+}\big) - \log\big(1+e^{\varphi}\big) + \big|s(\varphi')\big| .\end{aligned} \tag{3.4}$$

Since s is convex and C^2 in $(0,1)$, there exists a constant $C>0$ depending only on u_+ such that

$$s(u)-s(u_+)-s'(u_+)(u-u_+) \le C(u-u_+)^2 .$$

On the other hand, we clearly have $(1-u)(\varphi_+-\varphi)\le|\varphi-\varphi_+|$. Moreover, since the real function $\varphi\mapsto\log(1+e^{\varphi})$ has Lipschitz constant one, we have

$$\log\big(1+e^{\varphi_+}\big) - \log(1+e^{\varphi}) \le |\varphi-\varphi_+| .$$

In view of (3.4), the proof is now completed by applying Lemma 3.2. □

Let

$$\mathcal{D}^\pm := \{(u,\varphi)\in\mathcal{X}^\pm \;\text{ such that }\; \|u-u_\pm\|_{L^2(\mathbb{R}_\pm)} + \|\varphi-\varphi_\pm\|_{L^1(\mathbb{R}_\pm)} < +\infty\} ,$$
$$\mathcal{D} := \{(u,\varphi)\in\mathcal{X} \;\text{ such that }\; \|u-\vartheta_{u_-,u_+}\|_{L^2(\mathbb{R})} + \|\varphi-\vartheta_{\varphi_-,\varphi_+}\|_{L^1(\mathbb{R})} < +\infty\} .$$

In view of Proposition 3.1, we can introduce the functional $\mathcal{G}^\pm : \mathcal{X}^\pm : (-\infty,+\infty]$ as follows

$$\mathcal{G}^\pm(u,\varphi) := \begin{cases}\displaystyle\int_{\mathbb{R}_\pm}\big[g(u,\varphi,\varphi') - g(u_\pm,\varphi_\pm,0)\big]\,dx & \text{if } (u,\varphi)\in\mathcal{D}^\pm ,\\ +\infty & \text{otherwise} .\end{cases} \tag{3.5}$$

Analogously, let $\mathcal{G} : \mathcal{X} : (-\infty,+\infty]$ be defined by

$$\mathcal{G}(u,\varphi) := \begin{cases}\displaystyle\int_{\mathbb{R}}\big[g(u,\varphi,\varphi') - g(u_+,\varphi_+,0)\big]\,dx & \text{if } (u,\varphi)\in\mathcal{D} ,\\ +\infty & \text{otherwise} .\end{cases} \tag{3.6}$$

Our first main result concerns the variational convergences of the sequences $\{\mathcal{G}_\ell^\pm\}$ and $\{\mathcal{G}_\ell\}$, as respectively defined in (2.12) and (2.13), as ℓ diverges. The appropriate notion is the so-called Γ-*convergence*, see [5, 6], that we next recall. Let X be a metric space and $F_n : X\to(-\infty,+\infty]$, $n\in\mathbb{N}$. The sequence of functional $\{F_n\}$ is said to Γ-converge to $F : X \to (-\infty, +\infty]$ iff the two following inequalities hold for any $x \in X$:

(i) Γ-lim inf *inequality.* For any sequence $\{x_n\} \subset X$ converging to x we have $\liminf_n F_n(x_n) \geq F(x)$.

(ii) Γ-lim sup *inequality.* There exists a sequence $\{x_n\} \subset X$ converging to x such that $\limsup_n F_n(x_n) \leq F(x)$.

We also recall that the sequence $\{F_n\}$ is *equi-coercive* iff any sequence $\{x_n\} \subset X$ such that $\limsup_n F_n(x_n) < +\infty$ is precompact. As well known [5, 6], the Γ-convergence of a sequence of equi-coercive functionals F_n implies the convergence, up to a subsequence, of their minimizers to a minimizer of the Γ-limit.

Recall that the stationary solutions to the Burgers equation $\bar{u}^{\pm}$ and $\bar{u}$ are given in (2.6) and (2.8). We set $\bar{\varphi}^{\pm} := s'(\bar{u}^{\pm})$ and $\bar{\varphi} := s'(\bar{u})$.

Theorem 3.3. *Let $0 < u_- < u_+ < 1$ be such that $u_- + u_+ \gtrless 1$.*

(i) *There exists a constant $C \in (1, +\infty)$ such that for any $(u, \varphi) \in \mathcal{D}^{\pm}$*

$$\mathcal{G}^{\pm}(u,\varphi) \leq C\left(\|u - u_{\pm}\|_{L^2(\mathbb{R}_{\pm})} + \|\varphi - \varphi_{\pm}\|_{L^1(\mathbb{R}_{\pm})}\right) + C, \tag{3.7}$$

$$\mathcal{G}^{\pm}(u,\varphi) \geq \frac{1}{C}\left(\|u - u_{\pm}\|_{L^2(\mathbb{R}_{\pm})} + \|\varphi - \varphi_{\pm}\|_{L^1(\mathbb{R}_{\pm})}\right) - C. \tag{3.8}$$

Moreover, the functional $\mathcal{G}^{\pm}$ is lower semicontinuous and coercive on $\mathcal{X}^{\pm}$. Finally, the unique minimizer of $\mathcal{G}^{\pm}$ is $(\bar{u}^{\pm}, \bar{\varphi}^{\pm})$.

(ii) *The sequence of functionals $\{\mathcal{G}^{\pm}_{\ell}\}_{\ell>0}$ is equi-coercive and Γ-converges to $\mathcal{G}^{\pm}$ as $\ell \to \infty$. In particular, $(\bar{u}^{\pm}_{\ell}, \bar{\varphi}^{\pm}_{\ell}) \to (\bar{u}^{\pm}, \bar{\varphi}^{\pm})$ as $\ell \to \infty$.*

In contrast to the previous case, in view of the translational invariance of the limiting functional $\mathcal{G}$, the sequence $\{\mathcal{G}_{\ell}\}$ is not equi-coercive. This loss of compactness takes place because the “interface” between u_- and u_+ can escape to infinity with a bounded energy cost. However, as we state below, the compactness of sequences with equibounded energy can be recovered if we identify functions modulo translations. Recall that we denote by τ_z the translation by $z \in \mathbb{R}$.

Theorem 3.4. *Let $0 < u_- < u_+ < 1$ be such that $u_- + u_+ = 1$.*

(i) *There exists a constant $C \in (1, +\infty)$ such that for any $(u, \varphi) \in \mathcal{D}$*

$$\mathcal{G}(u,\varphi) \leq C\left(\|u - \vartheta_{u_-,u_+}\|_{L^2(\mathbb{R})} + \|\varphi - \vartheta_{\varphi_-,\varphi_+}\|_{L^1(\mathbb{R})}\right) + C, \tag{3.9}$$

$$\mathcal{G}(u,\varphi) \geq \frac{1}{C}\left(\|u - \tau_z\vartheta_{u_-,u_+}\|_{L^2(\mathbb{R})} + \|\varphi - \tau_z\vartheta_{\varphi_-,\varphi_+}\|_{L^1(\mathbb{R})}\right) - C, \tag{3.10}$$

for some $z \in \mathbb{R}$ depending on (u, φ). Moreover, the functional $\mathcal{G}$ is lower semicontinuous on $\mathcal{X}$. Finally, the set of minimizers of $\mathcal{G}$ is the one-parameter family of solutions to (2.7)*.*

(ii) *Let $\{(u_{\ell}, \varphi_{\ell})\} \subset \mathcal{X}$ and assume* $\limsup_{\ell\to\infty} \mathcal{G}_{\ell}(u_{\ell}, \varphi_{\ell}) < +\infty$*. Then there exists a sequence $\{z_{\ell}\} \subset \mathbb{R}$ such that $\{(\tau_{z_{\ell}} u_{\ell}, \tau_{z_{\ell}}\varphi_{\ell})\}$ is precompact in $\mathcal{X}$. Moreover, the sequence of functionals $\{\mathcal{G}_{\ell}\}_{\ell>0}$ Γ-converges to $\mathcal{G}$ as $\ell \to \infty$. Finally, $(\bar{u}_{\ell}, \bar{\varphi}_{\ell}) \to (\bar{u}, \bar{\varphi})$ as $\ell \to \infty$.*

Remark 3.5. Recall (2.14), set $\mathcal{F}^{\pm}(u) = \inf_{\varphi} \mathcal{G}^{\pm}(u, \varphi)$ and $\mathcal{F}(u) = \inf_{\varphi} \mathcal{G}(u, \varphi)$. Theorems 3.3 and 3.4 imply the Γ-convergence of the sequences $\{\mathcal{F}^{\pm}_{\ell}\}$ and $\{\mathcal{F}_{\ell}\}$ to $\mathcal{F}^{\pm}$ and $\mathcal{F}$, respectively.

Proof of Theorem 3.3. We prove the statements only in the case $u_- + u_+ > 1$.

Proof of (i). The upper bound (3.7) is a direct consequence of Proposition 3.1. In order to prove the lower bound (3.8), we first show that there exists $C_1 = C_1(\varphi_-,\varphi_+)$ such that

$$\mathcal{G}^+(u,\varphi) \geq \frac{1}{C_1}\int_0^{+\infty} \big(\varphi_+ - \varphi\big)\, dx - C_1 \ . \tag{3.11}$$

Observe that $s' : (0,1) \to \mathbb{R}$ is given by $s'(p) = \log[p/(1-p)]$. Hence $(s')^{-1} : \mathbb{R} \to (0,1)$ is given by $(s')^{-1}(q) = e^q/(1+e^q)$. Therefore

$$f(q) := g\big((s')^{-1}(q), q, 0\big) = q - 2\log(1+e^q)$$

and in particular, $g(u_+,\varphi_+,0) = f(\varphi_+)$. By the strict convexity of $g(\cdot,q,p)$ for a fixed $(q,p) \in [\varphi_-,\varphi_+]\times[0,1]$, the infimum of $\mathcal{G}^+(\cdot,\varphi)$ for a fixed $\varphi \in \mathcal{C}^+$ is achieved when u satisfies $s'(u) = \varphi$. Hence

$$\mathcal{G}^+(u,\varphi) \geq \mathcal{G}^+\big((s')^{-1}(\varphi),\varphi\big) = \int_0^\infty \big[s(\varphi') + f(\varphi) - f(\varphi_+)\big]\, dx \ .$$

Since the real function f is concave, for any $q \in [\varphi_-,\varphi_+]$

$$f(q) - f(\varphi_+) \geq \frac{f(\varphi_-) - f(\varphi_+)}{\varphi_+ - \varphi_-}(\varphi_+ - q) =: m(\varphi_+ - q) \ . \tag{3.12}$$

It is simple to check that $m > 0$ because $\varphi_- + \varphi_+ > 0$. We thus deduce

$$\begin{aligned} \mathcal{G}^+(u,\varphi) \ &\geq \ \frac{m}{2}\int_0^\infty \big[\varphi_+ - \varphi\big]\, dx \\ &\quad + \inf\Big\{ \int_0^\infty \big[s(\psi') + \tfrac{m}{2}\big(\varphi_+ - \psi\big)\big]\, dx \ , \ \psi \in \mathcal{C}^+ \ : \ \psi - \varphi_+ \in L^1(\mathbb{R}_+)\Big\} \ . \end{aligned} \tag{3.13}$$

In view of Lemma 3.2, the infimum on the right hand side above is finite. This concludes the proof of the bound (3.11).

We next prove the L^2 bound on u. Since the right hand side of (3.13) is bounded from below, there exists a constant $C_2 = C_2(\varphi_-,\varphi_+)$ such that for any $\varphi \in \mathcal{C}^+$ $\mathcal{G}^+\big((s')^{-1}(\varphi),\varphi\big) \geq -C_2$. Therefore

$$\begin{aligned} \mathcal{G}^+(u,\varphi) \ &\geq \ \mathcal{G}^+(u,\varphi) - \mathcal{G}^+\big((s')^{-1}(\varphi),\varphi\big) - C_2 \\ &= \ \int_0^\infty \big\{ s(u) - \varphi\, u - \big[s\big((s')^{-1}(\varphi)\big) - \varphi\,(s')^{-1}(\varphi)\big]\big\}\, dx - C_2 \ . \end{aligned}$$

Since $u \in [0,1]$, $(s')^{-1}$ is locally Lipschitz on $\mathbb{R}$, $(s')^{-1}(\varphi_+) = u_+ \in (0,1)$ and s is locally Lipschitz in $(0,1)$, the L^1 bound on $\varphi - \varphi_+$ implies there exists C_3 such that

$$\mathcal{G}^+(u,\varphi) \ \geq \ \int_0^\infty \big[s(u) - s(u_+) - s'(u_+)(u-u_+)\big]\, dx - C_3\|\varphi - \varphi_+\|_{L^1(\mathbb{R}_+)} - C_2 \ .$$

The proof is completed by the uniform convexity of s on $[0,1]$.

To prove the lower semicontinuity of $\mathcal{G}^+$, given a sequence $\{(u_n,\varphi_n)\} \subset \mathcal{X}^+$ converging to (u,φ), we need to show that $\mathcal{G}^+(u,\varphi) \leq \liminf_n \mathcal{G}^+(u_n,\varphi_n)$. We can clearly assume that $\liminf_n \mathcal{G}^+(u_n,\varphi_n) < +\infty$, and therefore, by taking if necessary a subsequence, that $\{(u_n,\varphi_n)\}$ has equibounded energy. In particular, by (3.8) we deduce that $u - u_+$ belongs to $L^2(\mathbb{R}^+)$ and $\varphi - \varphi_+$ belongs to $L^1(\mathbb{R}^+)$. Again, from (3.8) we easily deduce that

$$\lim_{L\to\infty} \liminf_n \varphi_n(L) = \varphi_+ . \tag{3.14}$$

Given an interval $I \subseteq \mathbb{R}$, we introduce the localized functional $\mathcal{G}_I^+$ defined by

$$\mathcal{G}_I^+(u,\varphi) := \int_I \big[g(u,\varphi,\varphi') - g(u_+,\varphi_+,0)\big]\,dx\,. \tag{3.15}$$

By the convexity of s, for each $L > 0$ the functional $\mathcal{G}_{(0,L)}^+$ is lower semicontinuous on $\mathcal{X}^+$. Since by Proposition 3.1 $\lim_{L\to\infty} \mathcal{G}_{(L,+\infty)}^+(u,\varphi) = 0$, to complete the proof it is thus enough to show that

$$\lim_{L\to\infty} \liminf_n \mathcal{G}_{(L,+\infty)}^+(u_n,\varphi_n) \geq 0\,. \tag{3.16}$$

To this purpose, let m be as defined in (3.12). Arguing as in the proof of (3.13)

$$\begin{aligned}\mathcal{G}_{(L,+\infty)}^+(u_n,\varphi_n) &\geq \inf\Big\{ \int_L^\infty \big[s(\psi') + m\big(\varphi_+ - \psi\big)\big]\,dx\,,\ \ \psi\,:\,\psi(L) = \varphi_n(L)\Big\}\\ &= \inf\Big\{ \int_0^\infty \big[s(\psi') + m\big(\varphi_+ - \psi\big)\big]\,dx\,,\ \ \psi\,:\,\psi(0) = \varphi_n(L)\Big\}\,,\end{aligned}$$

where, of course, ψ is increasing and satisfies $\lim_{x\to+\infty}\psi(x) = \varphi_+$. In view of (3.14) and Lemma 3.2 the bound (3.16) follows.

The coercivity of $\mathcal{G}^+$ follows trivially from the equi-coercivity and Γ-convergence of the sequence $\{\mathcal{G}_\ell^+\}$ proven in item (ii) below.

Since $\mathcal{G}^\pm$ is bounded from below, coercive, and lower-semicontinuous, by arguing as in the proof of Proposition 2.1 we conclude that the unique minimizer of $\mathcal{G}^\pm$ is the solution to (2.5).

Proof of (ii). Let $\{(u_\ell,\varphi_\ell)\} \subset \mathcal{X}^+$ be a sequence such that $\mathcal{G}_\ell(u_\ell,\varphi_\ell) \leq K$ for some $K \in \mathbb{R}$; we next show that $\{(u_\ell,\varphi_\ell)\}$ is precompact. To this purpose, notice that by the very definition (2.12) of $\mathcal{G}_\ell$ we have $\mathcal{G}_\ell(u_\ell,\varphi_\ell) = \mathcal{G}(u_\ell,\varphi_\ell)$. Therefore, by the lower bound (3.8) we have

$$\|u_\ell - u_+\|_{L^2(\mathbb{R}^+)} + \|\varphi_\ell - \varphi_+\|_{L^1(\mathbb{R}^+)} \leq C(K + C). \tag{3.17}$$

Fix $\varepsilon > 0$ and let $x_{\varepsilon,\ell} := \inf\{x > 0\,,\ \varphi_\ell(x) = \varphi_+ - \varepsilon\}$. From (3.17) we deduce $\limsup_\ell x_{\varepsilon,\ell} < +\infty$. Since $\varphi_\ell' \in [0,1]$, we thus deduce the precompactness of $\{\varphi_\ell\}$ in $\mathcal{C}^+(\mathbb{R}_+)$. As $L^\infty(\mathbb{R}_+;[0,1])$ is compact with respect to the weak* topology this concludes the proof of the equi-coercivity of $\mathcal{G}_\ell$.

In order o prove the Γ-liminf inequality let $(u_\ell,\varphi_\ell) \to (u,\varphi)$, and assume without loss of generality that (u_ℓ,φ_ℓ) has equibounded energy. By the very definition (2.12) of $\mathcal{G}_\ell$ and the lower semicontinuity of $\mathcal{G}$

$$\mathcal{G}(u,\varphi) \leq \liminf_\ell \mathcal{G}(u_\ell,\varphi_\ell) = \liminf_\ell \mathcal{G}_\ell(u_\ell,\varphi_\ell).$$

To prove the Γ-limsup inequality fix $(u,\varphi) \in \mathcal{X}^+$ with finite energy. We define (u_ℓ,φ_ℓ) as follows. We set $u_\ell \equiv u$ in $[0,\ell]$, and $u_\ell \equiv u_+$ in $(\ell,+\infty)$. Moreover, we set $\varphi_\ell \equiv \varphi$ in $[0,\ell-1]$, $\varphi_\ell \equiv \varphi_+$ in $[\ell,+\infty)$, and we extend it by affine interpolation on $[\ell-1,\ell]$. A direct computation shows that (u_ℓ,φ_ℓ) is a recovery sequence for (u,φ). □

Proof of Theorem 3.4. The proof will be easily achieved by applying Theorem 3.3 and translations invariance arguments.

Proof of (i). The upper bound (3.9) on $\mathcal{G}$ is a direct consequence of Proposition 3.1. In order to prove the lower bound (3.10), let $z \in \mathbb{R}$ be such that $\varphi(z) = (\varphi_+ + \varphi_-)/2 = 0$. The lower bound then follows by applying (3.8) with $\mathbb{R}^\pm$ replaced by $\{x \leq z\}$ and $\{x \geq z\}$.

We next prove the lower semicontinuity of $\mathcal{G}$. Let $(u_n, \varphi_n) \to (u, \varphi)$; by taking, if necessary, a subsequence we assume that $\liminf_n \mathcal{G}(u_n, \varphi_n) = \lim_n \mathcal{G}(u_n, \varphi_n)$. Let $\{z_n\} \subset \mathbb{R}$ be such that $\varphi_n(z_n) = 0$. By taking, if necessary, a further subsequence we assume that $z_n \to z$ for some $z \in \mathbb{R}$. According with the notation introduced in (3.15), we write

$$\mathcal{G}(u_n, \varphi_n) = \mathcal{G}_{(-\infty, z_n)}(u_n, \varphi_n) + \mathcal{G}_{(z_n, +\infty)}(u_n, \varphi_n) \; .$$

By translation invariance the statement now follows from the lower semicontinuity of $\mathcal{G}^{\pm}$, see item (i) in Theorem 3.3.

The last statement will follow once we prove that the pair $(\bar{u}, \bar{\varphi})$ is the unique minimizer of $\mathcal{G}$ among all $(u, \varphi) \in \mathcal{X}$ satisfying $\varphi(0) = 0$. This readily follows from the uniqueness property stated in item (i) of Theorem 3.3.

Proof of (ii). Given a sequence $\{(u_\ell, \varphi_\ell)\} \subset \mathcal{X}$ such that $\mathcal{G}_\ell(u_\ell, \varphi_\ell) < +\infty$, let $\{z_\ell\} \subset \mathbb{R}$ be such that $\varphi_\ell(z_\ell) = 0$. Observe that $z_\ell \in (-\ell, \ell)$ and write

$$\begin{aligned} \mathcal{G}_\ell(u_\ell, \varphi_\ell) &= \mathcal{G}_{(-\ell, z_\ell)}(u_\ell, \varphi_\ell) + \mathcal{G}_{(z_\ell, \ell)}(u_\ell, \varphi_\ell) \\ &= \mathcal{G}_\ell^-(\tau_{-z_\ell} u_\ell, \tau_{-z_\ell} \varphi_\ell) + \mathcal{G}_\ell^+(\tau_{-z_\ell} u_\ell, \tau_{-z_\ell} \varphi_\ell) \; . \end{aligned}$$

In view of the equi-coercivity of $\mathcal{G}_\ell^{\pm}$ stated in item (ii) Theorem 3.3, the compactness property of $\{\mathcal{G}_\ell\}$ follows. Thanks again to item (ii) in Theorem 3.3, the above decomposition of the energy functionals easily yields the Γ-convergence result.

Finally, the fact that the unique minimizer $(\bar{u}_\ell, \bar{\varphi}_\ell)$ of $\mathcal{G}_\ell$ converges to $(\bar{u}, \bar{\varphi})$ in $\mathcal{X}$ follows by the Γ-convergence and the fact that $\bar{\varphi}_\ell(0) = 0$ for every ℓ. □

4. Development by Γ-convergence

In this section we analyze in more detail the functional $\mathcal{G}_\ell$ for the choice of the boundary data corresponding to a standing wave for the Burgers equation in the whole line. As we discussed previously, when $u_- + u_+ = 1$ there is a one parameter family of solutions to (2.7) given by $\tau_z \bar{u}$, $z \in \mathbb{R}$. Accordingly, the minimizers of the limiting functional $\mathcal{G}$ in (3.6) are given by $(\tau_z \bar{u}, \tau_z \bar{\varphi})$. On the other hand, recalling Theorem 3.4, the minimizer of the finite volume energy $\mathcal{G}_\ell$ is unique and converges to $(\bar{u}, \bar{\varphi})$. The purpose of this section is to provide a variational framework to select this minimizer, based again on the notion of Γ-convergence, and more precisely on the notion of development by Γ-convergence introduced in [1].

Let $\alpha \in (0, 1)$ be such that $u_\pm = (1 \pm \alpha)/2$. Recalling (2.13) and (3.6), we are interested in the asymptotic behavior of the functionals $\widehat{\mathcal{G}}_\ell : \mathcal{X} \to (-\infty, +\infty]$ defined by

$$\widehat{\mathcal{G}}_\ell(u, \varphi) := e^{\alpha \ell} \big[\mathcal{G}_\ell(u, \varphi) - \min \mathcal{G} \big] \; , \tag{4.1}$$

where the exponential rescaling has been introduced to get a non trivial limit. In particular, as stated below, the unique minimizer of the limiting functional is $(\bar{u}, \bar{\varphi})$, where $\bar{u}$ is the stationary solution satisfying $\bar{u}(0) = 1/2$, see (2.8), and $\bar{\varphi} = s'(\bar{u})$.

Theorem 4.1. *Let $\alpha \in (0,1)$ be such that $u_\pm = (1 \pm \alpha)/2$.*

(i) *Let $\{(u_\ell, \varphi_\ell)\} \subset \mathcal{X}$ be sequence such that $\limsup_{\ell \to \infty} \widehat{\mathcal{G}}_\ell(u_\ell, \varphi_\ell) < +\infty$. Then, up to a subsequence, $(u_\ell, \varphi_\ell) \to (\tau_z \bar{u}, \tau_z \bar{\varphi})$ for some $z \in \mathbb{R}$. In particular, the sequence of functionals $\{\widehat{\mathcal{G}}_\ell\}$ is equi-coercive.*

(ii) *As $\ell \to \infty$ the sequence of functionals $\{\widehat{\mathcal{G}}_\ell\}$ Γ-converges to the functional $\widehat{\mathcal{G}} : \mathcal{X} \to (-\infty, +\infty]$ defined by*

$$\widehat{\mathcal{G}}(u,\varphi) := \begin{cases} \dfrac{8\alpha}{1-\alpha^2}\cosh(\alpha z) & \textit{if } (u,\varphi) = (\tau_z\bar{u}, \tau_z\bar{\varphi}) \textit{ for some } z \in \mathbb{R}, \\ +\infty & \textit{otherwise.} \end{cases}$$

Remark 4.2. Let $\widehat{\mathcal{F}}_\ell(u) = \inf_\varphi \widehat{\mathcal{G}}_\ell(u,\varphi)$. Theorem 4.1 then imply that $\{\widehat{\mathcal{F}}_\ell\}$ is equi-coercive and Γ-converges to $\min_\varphi \widehat{\mathcal{G}}(u,\varphi)$. Note that the Γ-limit is finite only if $u = \tau_z\bar{u}$ for some $z \in \mathbb{R}$ and in such a case is given by $\widehat{\mathcal{G}}(\tau_z\bar{u}, \tau_z\bar{\varphi})$.

The proof of Theorem 4.1 is based on few computations that we collect in the following two lemmata. Recall that $\bar{u}_\ell$ is the solution to (2.1) in the symmetric interval $(-\ell,\ell)$ and $\bar{\varphi}_\ell = s'(\bar{u}_\ell)$. In particular, $\bar{u}_\ell$ solves $\bar{u}_\ell' = \bar{u}_\ell(1-\bar{u}_\ell) - J_\ell$ where $J_\ell \in \mathbb{R}$ is the constant satisfying

$$\int_{u_-}^{u_+} \frac{1}{r(1-r) - J_\ell}\, dr = 2\ell\,. \tag{4.2}$$

Analogously, $\bar{u}' = \bar{u}(1-\bar{u}) - J$ where $J = (1-\alpha^2)/4$. It is simple to check that $J_\ell \uparrow J$ as $\ell \to \infty$; in the sequel we need however the sharp asymptotic of J_ℓ.

Lemma 4.3. *Let $\alpha \in (0,1)$ be such that $u_\pm = (1\pm\alpha)/2$. Then*

$$\lim_{\ell\to\infty} e^{\alpha\ell}\big(J - J_\ell\big) = \alpha^2\,.$$

Proof. The integral on the left hand side of (4.2) can be calculated explicitly and the proof of the lemma can be achieved by few tedious computations. We give however a lighter argument below.

Let $E_\ell := J - J_\ell$, and notice that $E_\ell \downarrow 0$ as $\ell \to \infty$. By symmetry, (4.2) is equivalent to

$$\int_{1/2}^{(1+\alpha)/2} \frac{1}{r(1-r) - J + E_\ell}\, dr = \ell\,.$$

Performing the change of variables $r = (1+\alpha-s)/2$, this is further equivalent to

$$2\int_0^\alpha \frac{1}{2\alpha s - s^2 + 4E_\ell}\, ds = \ell\,, \tag{4.3}$$

where we used that $J = (1-\alpha^2)/4$. We rewrite the left hand side above as follows

$$\begin{aligned} 2\int_0^\alpha \frac{1}{2\alpha s - s^2 + 4E_\ell}\, ds \;&=\; 2\int_0^\alpha \frac{1}{2\alpha s + 4E_\ell}\, ds + R(E_\ell) \\ &=\; \frac{1}{\alpha}\log\Big[\frac{\alpha^2}{2E_\ell}\Big(1 + \frac{2E_\ell}{\alpha^2}\Big)\Big] + R(E_\ell)\,, \end{aligned} \tag{4.4}$$

where

$$R(E) := 2\int_0^\alpha \Big[\frac{1}{2\alpha s - s^2 + 4E} - \frac{1}{2\alpha s + 4E}\Big]\, ds\,.$$

It is simple to check that

$$R(0) = \lim_{E\downarrow 0} R(E) = 2\int_0^\alpha \frac{s^2}{2\alpha s(2\alpha s - s^2)}\, ds = \frac{1}{\alpha}\log 2\,.$$

From (4.3) and (4.4) we then get

$$\ell = \frac{1}{\alpha}\log\frac{\alpha^2}{E_\ell} + \frac{1}{\alpha}\log\Big(1 + \frac{2E_\ell}{\alpha^2}\Big) + R(E_\ell) - R(0)\,,$$

which, recalling $E_\ell \downarrow 0$ as $\ell \to \infty$, yields the statement. □

As follows from Theorem 3.4, $\mathcal{G}_\ell(\bar u_\ell, \bar\varphi_\ell) \to \mathcal{G}(\bar u, \bar\varphi)$ as $\ell \to \infty$. We next compute the sharp asymptotic of the energy of the minimizer $(\bar u_\ell, \bar\varphi_\ell)$.

Lemma 4.4. *Let $\alpha \in (0,1)$ be such that $u_\pm = (1 \pm \alpha)/2$. Then*

$$\lim_{\ell\to\infty} e^{\alpha\ell}\Big[\mathcal{G}_\ell(\bar u_\ell, \bar\varphi_\ell) - \mathcal{G}(\bar u, \bar\varphi)\Big] = \frac{8\alpha}{1-\alpha^2}.$$

Proof. Recall that $\bar u_\ell$ and $\bar u$ satisfy (2.2) with constants J_ℓ and J, respectively. Recall also that $\bar\varphi_\ell = s'(\bar u_\ell)$. From the very definition (2.10) of g we deduce

$$g\big(\bar u_\ell, \bar\varphi_\ell, \bar\varphi_\ell'\big) = \log J_\ell + \frac{\bar u_\ell'}{\bar u_\ell(1-\bar u_\ell)} \log \frac{\bar u_\ell(1-\bar u_\ell) - J_\ell}{J_\ell}\,, \tag{4.5}$$

both in the case $\ell \in (0,+\infty)$ and $\ell = +\infty$. Note also that $g(u_+, \varphi_+, 0) = g(u_-, \varphi_-, 0) = \log J$. From (4.5) we get

$$\begin{aligned}
&\mathcal{G}_\ell(\bar u_\ell, \bar\varphi_\ell) - \mathcal{G}(\bar u, \bar\varphi)\\
&\quad= 2\ell\,\log\frac{J_\ell}{J} + \int_{u_-}^{u_+} \frac{1}{r(1-r)}\log\Big[\frac{r(1-r)-J_\ell}{J_\ell}\,\frac{J}{r(1-r)-J}\Big]\,dr\\
&\quad= \int_{u_-}^{u_+} \frac{1}{r(1-r)}\Big[\log\frac{r(1-r)-J_\ell}{r(1-r)-J} + \frac{J_\ell}{r(1-r)-J_\ell}\log\frac{J_\ell}{J}\Big]\,dr\,,
\end{aligned} \tag{4.6}$$

where we used (4.2) in the second step.

Set $\varepsilon_\ell := 4(J - J_\ell)$ and observe that $\varepsilon_\ell \downarrow 0$ as $\ell \to \infty$. By using the symmetry of the function $r(1-r)$ with respect to $r = 1/2$ and performing the change of variable $r = (1 + \alpha - \varepsilon_\ell s)/2$ in (4.6) we deduce, recalling $J = (1-\alpha^2)/4$,

$$\begin{aligned}
\mathcal{G}_\ell(\bar u_\ell, \bar\varphi_\ell) - \mathcal{G}(\bar u, \bar\varphi) &= 4\,\varepsilon_\ell \int_0^{\alpha/\varepsilon_\ell} \frac{1}{1-\alpha^2+2\alpha\varepsilon_\ell s - \varepsilon_\ell^2 s^2}\\
&\quad\times\Big[\log\frac{1+2\alpha s-\varepsilon_\ell s^2}{2\alpha s - \varepsilon_\ell s^2} + \frac{1-\alpha^2-\varepsilon_\ell}{1+2\alpha s-\varepsilon_\ell s^2}\,\varepsilon_\ell^{-1}\log\Big(1-\frac{\varepsilon_\ell}{1-\alpha^2}\Big)\Big]\,ds\,.
\end{aligned}$$

It is now simple to check that

$$\lim_{\ell\to\infty}\varepsilon_\ell^{-1}\Big[\mathcal{G}_\ell(\bar u_\ell, \bar\varphi_\ell) - \mathcal{G}(\bar u, \bar\varphi)\Big] = \frac{4}{1-\alpha^2}\int_0^{+\infty}\Big[\log\frac{1+2\alpha s}{2\alpha s} - \frac{1}{1+2\alpha s}\Big]\,ds$$

and hence, computing the integral on the right hand side above, we get

$$\lim_{\ell\to\infty}\varepsilon_\ell^{-1}\Big[\mathcal{G}_\ell(\bar u_\ell, \bar\varphi_\ell) - \mathcal{G}(\bar u, \bar\varphi)\Big] = \frac{2}{\alpha(1-\alpha^2)}\,,$$

which, in view of Lemma 4.3 and the definition of ε_ℓ, concludes the proof. □

We have now collected all the ingredients needed to complete the proof of the development by Γ-convergence.

Proof of Theorem 4.1. Given $z \in \mathbb{R}$ and $\ell > |z|$, we introduce the function $u_\ell^{(z)} : \mathbb{R} \to (0,1)$ defined by

$$u_\ell^{(z)}(x) := \begin{cases} u_- & \text{for } x \in (-\infty, -\ell),\\ \bar u_{\ell+z}(x-z) & \text{for } x \in [-\ell, z),\\ \bar u_{\ell-z}(x-z) & \text{for } x \in [z, \ell),\\ u_+ & \text{for } x \in [\ell, +\infty). \end{cases} \tag{4.7}$$

Moreover, we set $\varphi_\ell^{(z)} := s'(u_\ell^{(z)})$. Observe that $u_\ell^{(z)}$ is a continuous piecewise smooth function and $u_\ell^{(0)} = \bar u_\ell$. Moreover, by construction we clearly have

$$\mathcal{G}_\ell(u_\ell^{(z)}, \varphi_\ell^{(z)}) = \frac{1}{2}\Big[\mathcal{G}_{\ell+z}(\bar u_{\ell+z}, \bar\varphi_{\ell+z}) + \mathcal{G}_{\ell-z}(\bar u_{\ell-z}, \bar\varphi_{\ell-z})\Big] . \tag{4.8}$$

Finally, as it is easy to verify

$$\mathcal{G}_\ell(u_\ell^{(z)}, \varphi_\ell^{(z)}) = \min\big\{\mathcal{G}_\ell(u,\varphi)\,,\ (u,\varphi)\in\mathcal{X}\ :\ \varphi(z)=0\big\} . \tag{4.9}$$

Proof of (i). Fix a sequence $\{(u_\ell,\varphi_\ell)\}\subset\mathcal{X}$ such that $\limsup_{\ell\to\infty}\widehat{\mathcal{G}}_\ell(u_\ell,\varphi_\ell) < +\infty$ and let $z_\ell\in(-\ell,\ell)$ be such that $\varphi_\ell(z_\ell)=0$. From (4.9) we deduce

$$\limsup_{\ell\to\infty} e^{\alpha\ell}\big[\mathcal{G}_\ell(u_\ell^{(z_\ell)},\varphi_\ell^{(z_\ell)}) - \mathcal{G}(\bar u,\bar\varphi)\big] \le \limsup_{\ell\to\infty}\widehat{\mathcal{G}}_\ell(u_\ell,\varphi_\ell) < +\infty .$$

By (4.8) we thus have

$$\limsup_{\ell\to\infty} e^{\alpha\ell}\big[\mathcal{G}_{\ell+z_\ell}(\bar u_{\ell+z_\ell},\bar\varphi_{\ell+z_\ell}) - \mathcal{G}(\bar u,\bar\varphi) + \mathcal{G}_{\ell-z_\ell}(\bar u_{\ell-z_\ell},\bar\varphi_{\ell-z_\ell}) - \mathcal{G}(\bar u,\bar\varphi)\big] < +\infty.$$

This bound together with Lemma 4.4 yields the uniform boundedness of the sequence $\{z_\ell\}$. Hence, by taking a subsequence, $z_\ell\to z$ for some $z\in\mathbb{R}$. From the equi-coercivity modulo translations in item (ii) of Theorem 3.4 we then deduce the precompactness of the sequence $\{(u_\ell,\varphi_\ell)\}$. Since its limit points are necessarily minimizers of $\mathcal{G}$, we get $(u_\ell,\varphi_\ell)\to(\tau_z\bar u,\tau_z\bar\varphi)$.

Proof of (ii). In order to prove the Γ-limsup inequality, let $(u,\varphi)\in\mathbb{R}$, and assume without loss of generality that $(u,\varphi) = (\tau_z\bar u,\tau_z\bar\varphi)$ for some $z\in\mathbb{R}$. We claim that $\{(u_\ell^{(z)},\varphi_\ell^{(z)})\}$, with $u_\ell^{(z)}$ as defined in (4.7) and $\varphi_\ell^{(z)} = s'(u_\ell^{(z)})$, is a recovery sequence. Indeed, from (4.8) and Lemma 4.4 it immediately follows that

$$\begin{aligned}
\lim_{\ell\to\infty}\widehat{\mathcal{G}}_\ell(u_\ell^{(z)},\varphi_\ell^{(z)}) &= \lim_{\ell\to\infty} e^{\alpha\ell}\Big[\mathcal{G}_\ell(u_\ell^{(z)},\varphi_\ell^{(z)}) - \mathcal{G}(\bar u,\bar\varphi)\Big]\\
&= \frac{1}{2}\lim_{\ell\to\infty} e^{\alpha\ell}\big[\mathcal{G}_{\ell+z}(\bar u_{\ell+z},\bar\varphi_{\ell+z}) - \mathcal{G}(\bar u,\bar\varphi) + \mathcal{G}_{\ell-z}(\bar u_{\ell-z},\bar\varphi_{\ell-z}) - \mathcal{G}(\bar u,\bar\varphi)\big]\\
&= \frac{8\alpha}{1-\alpha^2}\frac{e^{\alpha z}+e^{-\alpha z}}{2} = \frac{8\alpha}{1-\alpha^2}\cosh(\alpha z) .
\end{aligned} \tag{4.10}$$

To prove the Γ-liminf inequality, let $(u_\ell,\varphi_\ell)\to(u,\varphi)\in\mathcal{X}$. By the compactness properties stated in item (ii) of Theorem 3.4, we can assume without loss of generality that $(u,\varphi) = (\tau_z\bar u,\tau_z\bar\varphi)$ for some $z\in\mathbb{R}$. Let $z_\ell\in\varphi_\ell^{-1}(\{0\})$, $\ell>0$. Since φ_ℓ converges uniformly to φ, we have $z_\ell\to z$. Observe that (4.10) holds also if z on the left hand side is replaced by a sequence $z_\ell\to z$. Therefore, by (4.9) we deduce

$$\begin{aligned}
\liminf_{\ell\to\infty}\widehat{\mathcal{G}}_\ell(u_\ell,\varphi_\ell) &= \liminf_{\ell\to\infty} e^{\alpha\ell}\big[\mathcal{G}_\ell(u_\ell,\varphi_\ell) - \mathcal{G}(u,\varphi)\big]\\
&\ge \liminf_{\ell\to\infty} e^{\alpha\ell}\big[\mathcal{G}_\ell(u_\ell^{(z_\ell)},\varphi_\ell^{(z_\ell)}) - \mathcal{G}(u,\varphi)\big] = \frac{8\alpha}{1-\alpha^2}\cosh(\alpha z),
\end{aligned}$$

that proves the Γ-liminf inequality. □

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