Rectifiability and perimeter in the Heisenberg group

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Abstract. In this paper, we fully extend to the Heisenberg group endowed with its intrinsic Carnot-Carathéodory metric and perimeter the classical De Giorgi's rectifiability divergence theorems.

Key words. Heisenberg group – Carnot-Carathéodory metric – rectifiability – divergence theorem

1. Introduction

A classical result due to E. De Giorgi ([16], [17]) states that, if $E \subset \mathbb{R}^d$ is a set of locally finite perimeter (a Caccioppoli set, according to De Giorgi's terminology) then the associated perimeter measure $|\partial E|$ is concentrated on a portion of the boundary, the so-called reduced boundary $\partial^* E \subset \partial E$ and that $\partial^* E$ is $\mathcal{H}^{d-1}$-countably rectifiable. This means that $\partial^* E$, up to a set of $(d-1)$-Hausdorff measure zero, is a countable union of compact subsets of C^1 submanifolds. We remind that the perimeter of E is defined as the total variation of its characteristic function $\mathbf{1}_E$, i.e., if Ω is any open subset of $\mathbb{R}^d$

$$|\partial E|(\Omega) := \sup \left\{ \int_E \operatorname{div} \psi(x)\ dx : \psi \in \mathbf{C}_0^1(\Omega; \mathbb{R}^d), |\psi(x)| \le 1 \right\},$$

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Through Riesz' representation theorem, if E is a Caccioppoli set then $|\partial E|$ defines a Radon measure in $\mathbb{R}^d$, the so-called perimeter measure, that, when the set E is sufficiently regular, coincides with the $(d-1)$-dimensional Hausdorff measure concentrated on the boundary of E. However, the notion of perimeter is much more natural in the context of geometric problems in Calculus of Variations, since the perimeter, unlike the Hausdorff measure, is lower semicontinuous with respect to the convergence of sets meant as the L^1- convergence of their characteristic functions, (see, e.g.,[18] and [37]).

In this paper we prove an exact counterpart of De Giorgi's result in the setting of the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ endowed with a left-invariant metric d that is equivalent to its Carnot-Carathéodory metric (see Definition 2 for a precise definition of such a metric). Throughout this paper, we shall denote the points of $\mathbb{H}^n$ by $P = [z,t] = [x+iy,t]$, $z \in \mathbb{C}^n$, $x,y \in \mathbb{R}^n$, $t \in \mathbb{R}$. If $P = [z,t]$, $Q = [\zeta,\tau] \in \mathbb{H}^n$ and $r > 0$, following the notations of [57], where the reader can find an exhaustive introduction to the Heisenberg group, we define the group operation

$$P \cdot Q := [z + \zeta, t + \tau + 2\Im m(z\bar{\zeta})]$$

and a family of non isotropic dilations

$$\delta_r(P) := [rz, r^2 t].$$

Moreover, $\mathbb{H}^n$ can be endowed with the homogeneous norm $\|P\|_\infty := \max\{|z|, |t|^{1/2}\}$ and the distance d we shall deal with will be defined as

$$d(P,Q) := \|P^{-1} \cdot Q\|_\infty.$$

We shall denote $U(P,r)$ and $B(P,r)$, respectively, the open and the closed ball with center P and radius r with respect to the distance d.

It is known that the topological dimension of $\mathbb{H}^n$ is $2n+1$, since as a smooth manifold it coincides with $\mathbb{R}^{2n+1}$, whereas the Hausdorff dimension of $(\mathbb{H}^n, d)$ is $Q = 2n+2$. This phenomenon is already evident from the intrinsic isoperimetric inequality in $\mathbb{H}^n$ proved first by P. Pansu (see [50] and [49]), and then in a different form but in the general framework of Carnot-Carathéodory spaces by several authors (see, e.g.,[58], [59], [13], [26], [29], [10], [35], [42] and for a general discussion on the geometry of Carnot-Carathéodory spaces see also [38] and [6]).

Since many equivalent distances can be defined in $\mathbb{H}^n$, it will be worthwhile to spend later few words in order to justify our choice, but first of all we want to stress now that our interest in extending De Giorgi's theorem in the setting of the Heisenberg group is primarily motivated by the fact that $\mathbb{H}^n$ provides the simplest example of a metric space that is not Euclidean, even locally, but is still endowed with a sufficiently rich underlying structure, due to the existence of intrinsic families of translations and dilations. Indeed, the geometry of $\mathbb{H}^n$ is not Euclidean at every scale, since it was proved by S. Semmes ([55]) that there are no bilitschitz maps from $\mathbb{H}^n$ to an Euclidean space. This fact relies on deep interlacing algebric and metric properties related to the non-commutativity of $\mathbb{H}^n$ through a Rademacher's type theorem due to P. Pansu ([48]). In a sense, our interest can be viewed in the framework of the general project meant to develope geometric measure theory in the setting of metric spaces. Such a project, already embryonally contained in Federer's book [24], has been explicitly formulated and carried on in the last few years by De Giorgi [19], [20], [21], Preiss and Tisěr [53], Kirchheim [40], David & Semmes [15] and Ambrosio & Kirchheim [3], [4]. Moreover, we notice that there might be another path leading naturally to the study of the structure of sets of finite perimeter in the Heisenberg group, starting from the study of regularity of minimal surfaces in Carnot-Carathéodory spaces, minimal surfaces whose existence was proved in [35], where the problem we are addressing here is already mentioned.

It is well known that the Lie algebra of left invariant vector fields in $\mathbb{H}^n$ is given by

$$\begin{aligned} X_j &= \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad j = 1, \dots, n \\ Y_j &= \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \qquad j = 1, \dots, n \\ T &= \frac{\partial}{\partial t}, \end{aligned}$$

the only non-trivial commutator relations being

$$[X_j, Y_j] = -4T, \qquad j = 1, \dots, n.$$

So the vector fields $X_1, \dots, X_n, Y_1 \dots, Y_n$ satisfy the so called Hörmander condition (see [39]). Throughout this paper, we shall identify vector fields and associated first order differential operators; thus the vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n$ define a vector bundle on $\mathbb{H}^n$, the so called horizontal vector bundle $\mathrm{H}\mathbb{H}^n$ according to the notation of Gromov, (see [38] and [41]), that is a vector subbundle of $\mathrm{T}\mathbb{H}^n$, the tangent vector bundle of $\mathbb{H}^n$. Since each fiber of $\mathrm{H}\mathbb{H}^n$ can be canonically identified with a vector subspace of $\mathbb{R}^{2n+1}$, each section ϕ of $\mathrm{H}\mathbb{H}^n$ can be identified with a map $\phi : \mathbb{H}^n \to \mathbb{R}^{2n+1}$. At each point $P \in \mathbb{H}^n$ the horizontal fiber is indicated as $\mathrm{H}\mathbb{H}^n_P$ and each fiber can be endowed with the scalar product $\langle \cdot, \cdot \rangle_P$ and the associated norm $| \cdot |_P$ that make the vector fields

$X_1, \dots, X_n, Y_1, \dots, Y_n$ orthonormal. Hence we shall also identify a section of $\mathrm{H}\mathbb{H}^n$ with its canonical coordinates with respect to this moving frame. In this way, a section ϕ will be identified with a function $\phi = (\phi_1, \dots, \phi_{2n}) : \mathbb{H}^n \to \mathbb{R}^{2n}$. Thus we can define the $\mathbb{H}$-divergence of a section ϕ as $\mathrm{div}_{\mathbb{H}}\, \phi = \sum_{j=1}^n (X_j \phi_j + Y_j \phi_{n+j})$. Analogously, if f is a real function defined in an open subset of $\mathbb{H}^n$, its $\mathbb{H}$-gradient is the section of $\mathrm{H}\mathbb{H}^n$ defined by $\nabla_{\mathbb{H}} f = (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f)$.

From now on, if Ω is an open subset of $\mathbb{H}^n$, we shall denote by $\mathbf{C}^k(\Omega, \mathrm{H}\mathbb{H}^n)$ the set of all $\mathbf{C}^k$-sections of $\mathrm{H}\mathbb{H}^n$; $\mathbf{C}^k_0(\Omega, \mathrm{H}\mathbb{H}^n)$ will denote the set of compactly supported $\mathbf{C}^k$-sections of $\mathrm{H}\mathbb{H}^n$.

To introduce our results, let us start by recalling some related notions already existing in the literature.

There are several ways of defining bounded variation functions associated with a vector subbundle of $\mathrm{T}\mathbb{R}^d$ generated by a family of vector fields; these definitions have been proposed independently in the last few years by different authors (see [7], [9], [35], [30]). All these definitions are in fact equivalent, as it is proved in [30]: see in particular the beginning of Section 2 in [30] for a discussion. Following one of these definitions (see Definition 4 below), we shall say that $E \subset \mathbb{H}^n$ has locally finite $\mathbb{H}^n$-perimeter (or, following De Giorgi, E is a $\mathbb{H}$-Caccioppoli set) if for any bounded open set $\Omega \subseteq \mathbb{H}^n$

$$|\partial E|_{\mathbb{H}}(\Omega) := \sup \left\{ \int_E \mathrm{div}_{\mathbb{H}}\, \phi \, dh : \phi \in \mathbf{C}^1_0(\Omega, \mathrm{H}\mathbb{H}^n),\ |\phi(P)|_P \le 1 \right\} < \infty,$$

where dh is the $2n+1$ Lebesgue measure in $\mathbb{R}^{2n+1}$. In such a way, $|\partial E|_{\mathbb{H}}$ defines a Radon measure in $\mathbb{H}^n$. If ∂E is a regular manifold with outward unit normal n, then

$$|\partial E|_{\mathbb{H}} = \left(\sum_{j=1}^n \langle X_j, n \rangle^2 + \langle Y_j, n \rangle^2 \right)^{1/2} \mathcal{H}^{2n} \llcorner \partial E,$$

where $\mathcal{H}^s$ (respectively $\mathcal{S}^s$) denotes the Euclidean s-dimensional Hausdorff measure (respectively the spherical Euclidean s-dimensional Hausdorff measure), see [10] and [30].

Now, Riesz' representation theorem yields the existence of a $|\partial E|_{\mathbb{H}}$-measurable section ν_E of $\mathrm{H}\mathbb{H}^n$ such that $|\nu_E(P)|_P = 1$ for $|\partial E|_{\mathbb{H}}$-a.e. P and for all $\phi \in \mathbf{C}^\infty_0(\mathbb{H}^n, \mathrm{H}\mathbb{H}^n)$ we have

$$-\int_E \mathrm{div}_{\mathbb{H}}\, \phi \, dh = \int_{\mathbb{H}^n} \langle \nu_E, \phi \rangle \, d|\partial E|_{\mathbb{H}}.$$

We shall call ν_E a *generalized inward normal.* Then, following De Giorgi's argument (see [17] and, for instance, [37], [23]), we can define the $\mathbb{H}$-reduced boundary $\partial^*_{\mathbb{H}} E$ as follows we say that $P \in \partial^*_{\mathbb{H}} E$ if and only if

$$
\begin{aligned}
&|\partial E|_{\mathbb{H}}(U(P,r)) > 0 \qquad \text{for any } r > 0;\\
&\text{there exists} \quad \lim_{r\to 0} \fint_{U(P,r)} \nu_E d|\partial E|_{\mathbb{H}};\\
&|\lim_{r\to 0} \fint_{U(P,r)} \nu_E d|\partial E|_{\mathbb{H}}| = 1.
\end{aligned}
$$

The limit in the second line must be understood as a convergence of the averages of the coordinates of ν_E with respect to the moving base of the fibers.

Observe also that the definition of $\partial^*_{\mathbb{H}} E$ depends only on E and not on the particular representative ν_E chosen, because only integrals of it are involved. Finally, because an appropriate differentiation theorem holds, (see Lemma 5 and the discussion at the end of this introduction), we have

$$
\nu_E(P) = \lim_{r\to 0} \fint_{U(P,r)} \nu_E d|\partial E|_{\mathbb{H}} \quad ; \quad |\nu_E(P)|_P = 1, \quad \forall P \in \partial^*_{\mathbb{H}} E.
$$

We are now able to state our main theorem and to give a sketch of the proof. However, we want to precede the statements by some comments on the correct position of the problem, enlightening the features we consider the most relevant in the perspective of the general program we mentioned at the beginning, i.e. a geometric measure theory in metric spaces. As for the proof, our path will follow closely the original scheme of De Giorgi (see also for instance [37], [60], [23]). We shall stress step by step when some *ad hoc* argument or a new technicality will be in order to bypass specific difficulties due to the geometry we are dealing with.

The first key point we want to stress here deals with the meaning of rectifiability in $\mathbb{H}^n$. Basically, in the Euclidean framework, a set $F \subset \mathbb{R}^d$ is (countably) $(d-1)$-rectifiable (from now on, we shall say only 'rectifiable') if, roughly speaking, it is, up to a $\mathcal{H}^{d-1}$-negligeable set, a countable union of compact subsets K_j of good hypersurfaces (i.e. Lipschitz or continously differentiable hypersurfaces). Looking for a similar statement in the setting of the Heisenberg group (or, in general, of a metric space), we must ask preliminarily what are the good hypersurfaces in $\mathbb{H}^n$. In fact, there is a classical notion of rectifiability in a metric space that goes back to Federer (see [24] 3.2.14) that has been recently used by Ambrosio & Kirchheim (see [3], [4]) in the contest of a theory of currents in metric spaces (as for the rectifiability in metric spaces see for instance [44], [34], [40], [45], [53] and also the monografy [43] and the references therein). According to this notion, a good hypersurface in a metric space should be the image of an open subset of an Euclidean space via a Lipschitz map. Unfortunately, such a notion does not fit the geometry of the Heisenberg group, that indeed would be, according with this definition, purely unrectifiable (see [3]). On the other hand, in the Euclidean setting $\mathbb{R}^d$, a $\mathbf{C}^1$ hypersurface can be equivalently viewed as the (local) set of zeros of a function $f : \mathbb{R}^d \to \mathbb{R}$ with non-vanishing gradient. Such a notion can be easily transposed to the Heisenberg group, since there is an intrinsic notion of $\mathbf{C}^1_{\mathbb{H}}$ functions: we can say that a continuous real function f on $\mathbb{H}^n$ belongs to $\mathbf{C}^1_{\mathbb{H}}$ if $\nabla_{\mathbb{H}} f$ (in the sense of distributions) is a continuous vector-valued function. Thus, a $\mathbb{H}$-regular hypersurface Σ will be locally defined as the set of points $P \in \mathbb{H}^n$ such that $f(P) = 0$, provided that $\nabla_{\mathbb{H}} f \neq 0$ on Σ (see Definition 12). Few comments are now in order, since this definition is one of the crucial points of this paper.

First of all, we point out that the class of $\mathbb{H}$-regular hypersurfaces is different from the class of Euclidean regular hypersurfaces, in the sense that there are $\mathbb{H}$-regular hypersurfaces that are not Euclidean continuously differentiable submanifolds, and conversely there are continuously differentiable $2n$-submanifolds in $\mathbb{R}^{2n+1}$ that are not $\mathbb{H}$-regular hypersurfaces (see Remarks 7 and 2). We notice that Euclidean continuously differentiable $2n$-manifolds are $\mathbb{H}$-regular hypersurfaces provided they do not contain characteristic points, i.e. points P's such that the Euclidean tangent space at P coincides with the orizontal fiber $\mathrm{H}\mathbb{H}^n_P$ at P. Frobenius theorem yields that, for a general smooth manifold, the set of characteristic points has empty interior; in fact there are few characteristic points ([33], [22], [5]). On the other hand the boundary of a smooth bounded set with trivial topology in $\mathbb{H}^n$ does always contain characteristic points. The fact that these points should not be allowed in $\mathbb{H}$-regular submanifolds is not surprising: for example it is already well known from the theory of subelliptic pde's that characteristic points of the boundary can behave like cusps for Laplace operator.

Another important point supporting our choice of $\mathbb{H}$-regular hypersurfaces is the fact that this definition yields an Implicit Function Theorem, so that a $\mathbb{H}$-regular manifold Σ has a local parametrization $\Phi : \mathbb{R}^{2n} \to \mathbb{H}^n$. In general, such a parametrization is not continuosly differentiable, but nevertheless through Φ we can write explicitly the perimeter in local coordinates. For the sake of simplicity we restrict ourselves to the case $n = 1$. Thus the Implicit Function Theorem, reads as follows (Theorem 8).

Theorem (Implicit Function Theorem). *Let Ω be an open set in $\mathbb{H}^1$, $0 \in \Omega$, and let $f \in \mathbf{C}^1_{\mathbb{H}}(\Omega)$ be such that $|Xf(0)| = X_1 f(0) > 0$, $f(0) = 0$. Then, if we put*

$$E = \{[z,t] \in \Omega : f([z,t]) < 0\}, \quad \Gamma = \{[z,t] \in \Omega : f([z,t]) = 0\},$$

there exists a neighborhood $\mathcal{U} \ni 0$, $\mathcal{U} = \{[x+iy,t] : |x| < h, |y| < \delta, t = 2xy + \tau, |\tau| < \delta^2\}$, *such that*

$$E \text{ has finite } \mathbb{H}\text{-perimeter in } \mathcal{U}; \tag{i}$$

$$\partial E \cap \mathcal{U} = \partial^*_{\mathbb{H}} E \cap \mathcal{U} = \Gamma \cap \mathcal{U}; \tag{ii}$$

$$\nu_E(P) = -\nabla_{\mathbb{H}} f(P)/|\nabla_{\mathbb{H}} f(P)|_P \text{ for all } P \in \Gamma \cap \mathcal{U}. \tag{iii}$$

If we put now $I = [-\delta, \delta] \times [-\delta^2, \delta^2]$, $J = [-h, h]$, *then there exists a unique continuous function*

$$\phi = \phi(\eta, \tau) : I \to J$$

such that the following parametrization of Γ *and integral representation of the perimeter hold*

$$\Gamma \cap \bar{\mathcal{U}} = \{[\phi(\eta,\tau) + i\eta, 2\phi(\eta,\tau)\eta + \tau] \in \bar{\mathcal{U}}; (\eta,\tau) \in I\}; \tag{iv}$$

$$|\partial E|_{\mathbb{H}}(\mathcal{U}) = \int_I \frac{|\nabla_{\mathbb{H}} f|}{Xf}(\Phi(\eta,\tau))\, d\eta\, d\tau, \tag{v}$$

where $\Phi(\eta,\tau) = (\phi(\eta,\tau), \eta, 2\phi(\eta,\tau)\eta + \tau)$.

Having in mind the Euclidean case, an intriguing problem arises: we do not know if Federer's definition of rectifiability can be recovered replacing the Euclidean space by a different 'model space' for hypersurfaces in $\mathbb{H}^n$, in the sense that all $\mathbb{H}$-regular hypersurfaces should be viewed as the image of a suitable fixed metric space through Lipschitz maps. Notice however that a $\mathbb{H}$-regular hypersurface contains very few $\mathbb{H}$-rectifiable curves; in particular we cannot define a geodesic distance on a $\mathbb{H}$-rectifiable hypersurface. A possible solution of this problem has been announced by S.Pauls (see [51]).

There is a final – somehow indirect – argument supporting our choice, that we would like to mention, since it is closely related to our blow-up argument in De Giorgi's proof. For the sake of simplicity, take $n = 1$, and in $\mathbb{H}^1$ consider the plane $\{t = 0\}$ and the 'vertical' plane $\{ax + by = 0\}$. A straightforward computation shows that $\{t = 0\}$ is not a $\mathbb{H}$-regular hypersurface, whereas $\{ax+by = 0\}$ is a $\mathbb{H}$-regular hypersurface, provided $a^2 + b^2 \neq 0$. This means that $\{t = 0\}$ is not a true intrinsic plane. This is perfectly coherent with the group structure of $\mathbb{H}^1$, since the sets $\{ax + by = 0\}$ are maximal proper *subgroups* of $\mathbb{H}^1$, unlike $\{t = 0\}$ that is not invariant under group translations, as we should expect from a plane. On the other hand, if we take a $\mathbb{H}$-regular hypersurface Σ such that $0 \in \Sigma$, and we perform a blow-up of Σ by means of the Heisenberg dilations δ_r, then the limit is a 'vertical' plane of the form $\{ax + by = 0\}$ that we can call consistently the *tangent group* $T^g_{\mathbb{H}}\Sigma(0)$ to Σ at the origin. More precisely we give the following definition

Definition (Tangent Plane). *If $\Sigma = \{P : f(P) = 0\} \subset \mathbb{H}^n$ is a $\mathbb{H}$-regular hypersurface we define the tangent group $T^g_{\mathbb{H}}\Sigma(P)$ to Σ at P as follows*

$$T^g_{\mathbb{H}}\Sigma(P) := \{[x+iy,t] \in \mathbb{H}^n : \sum_{j=1}^{n} X_j f(P)x_j + Y_j f(P)y_j = 0\}.$$

$T^g_{\mathbb{H}}\Sigma(P)$ is a proper subgroup of $\mathbb{H}^n$. We can define the tangent plane to Σ at P as

$$T_{\mathbb{H}}\Sigma(P) := P \cdot T^g_{\mathbb{H}}\Sigma(P).$$

Observe that this is a good definition. Indeed the tangent plane does not depend on the particular function f defining the surface Σ because of point (iii) of Implicit Function Theorem.

A second delicate point appears when we want to prove that the perimeter measure is concentrated on the $\mathbb{H}$-reduced boundary. In the Euclidean setting, this follows straightforwardly by Lebesgue-Besicovitch differentiation theorem for general Radon measures, whose proof in turn relies on Besicovitch covering lemma for Euclidean balls. Unfortunately, the last result fails to hold in $\mathbb{H}^n$ for the metric d or even for the metric associated with the norm $\|P\|_4 := (|z|^4 + t^2)^{1/4}$ ([41], [54]). It is a completely open problem whether the differentiation theorem still holds in $\mathbb{H}^n$ for general Radon measures. Nevertheless it can be found in Federer's book that an asymptotic doubling condition on the measure under scrutiny is sufficient to yield a differentiation result for it. Luckely, such an asymptotic estimate has been recently proved by Ambrosio ([2]) for perimeter type measures on Ahlfors metric spaces.

We can state now our main result. Denote as $\mathcal{H}^s_d$ (and respectively $\mathcal{S}^s_d$) the s-dimensional Hausdorff measure (respectively the spherical Hausdorff measure) constructed, according to Federer definition, using the distance d. Then

Theorem (Main Theorem). *If $E \subseteq \mathbb{H}^n$ is a $\mathbb{H}$-Caccioppoli set, if $Q = 2n+2$ is the homogeneus dimension of $\mathbb{H}^n$, then $\partial^*_{\mathbb{H}}E$ is $\mathbb{H}$-rectifiable, that is*

$$\partial^*_{\mathbb{H}}E = N \cup \bigcup_{h=1}^{\infty} K_h, \tag{i}$$

where $\mathcal{H}^{Q-1}_d(N) = 0$ and K_h is a compact subset of a $\mathbb{H}$-regular hypersurface S_h;

$$\nu_E(P) \in \mathrm{H}\mathbb{H}^n_P \text{ and } \langle \nu_E(P), v\rangle_P = 0 \qquad \forall v \in T_{\mathbb{H}}S_h(P), \text{ and } \forall P \in K_h, \tag{ii}$$

we say, in this case, that $\nu_E(P)$ is the $\mathbb{H}$-normal to S_h at P;

$$|\partial E|_{\mathbb{H}} = \frac{2\omega_{2n-1}}{\omega_{2n+1}} \, \mathcal{S}_d^{Q-1} \llcorner \partial_{\mathbb{H}}^* E. \tag{iii}$$

As far as we know it is an open problem whether the intrinsic spherical Hausdorff measure $\mathcal{S}_d^{Q-1}$ can be replaced by a geometric constant times the intrinsic Hausdorff measure $\mathcal{H}_d^{Q-1}$.

The above main theorem is a relevant improvement of our previous result stated in [32] where (iii) was proved under a supplementary assumption on the oscillation of the generalized normal.

It is well known (see [24], [23] and [60]) that in many applications the so called *measure theoretic boundary* is much easier to deal with than the reduced boundary, hence we introduce an intrinsic measure theoretic boundary $\partial_{*,\mathbb{H}} E \supseteq \partial_{\mathbb{H}}^* E$ and we also prove

$$|\partial E|_{\mathbb{H}} = \frac{2\omega_{2n-1}}{\omega_{2n+1}} \, \mathcal{S}_d^{Q-1} \llcorner \partial_{*,\mathbb{H}} E. \tag{iv}$$

Finally, an intrinsic divergence theorem holds.

Theorem (Divergence Theorem). *Assume that $E \subseteq \mathbb{H}^n$ is a $\mathbb{H}$-Caccioppoli set, $Q = 2n+2$. Then for all compactly supported sections ϕ of $\mathrm{H}\mathbb{H}^n$, $\phi \in \mathbf{C}_0^1(\mathbb{H}^n, \mathrm{H}\mathbb{H}^n)$, we have*

$$-\int_E div_{\mathbb{H}} \phi \, dh = \frac{2\omega_{2n-1}}{\omega_{2n+1}} \int_{\partial_{*,\mathbb{H}} E} \langle \nu_E, \phi \rangle \, d\mathcal{S}_d^{Q-1}.$$

As we have already said above, the proof of the rectifiability of the $\mathbb{H}$-reduced boundary $\partial_{\mathbb{H}}^* E$ follows De Giorgi's original proof, by performing a blow-up of the set E around a point $P \in \partial_{\mathbb{H}}^* E$ with the group dilations δ_r and by showing that the dilated sets converge to a half-space whose boundary is a lateral of a tangent group.

Similar blow-up procedures, dealing with nonlinear pde's associated with the Kohn Laplacian in $\mathbb{H}^n$, appear in [11] and [12]; also in these papers the phenomenon of the 'bad behavior' of characteristic points appears. In particular, since the authors work with regular Euclidean submanifolds, the concept of $\mathbb{H}$-flat manifold is introduced.

The proof of our result relies on the coarea formula for the Heisenberg group (Proposition 7), and on a compactness argument ([35]); it is crucial that, using divergence theorem, we can transfer all computations to the boundaries of our metric balls, that are fairly simple Euclidean Lipschitz domains. This is basically the reason that lead us to the choice of the metric d instead of other metrics, as for instance Carnot-Carathéodory metric.

Another point that requires an *ad hoc* argument is the proof of the fact that the limit set of the blow-up procedure is a Heisenberg plane. This result is obtained relying on the fact that each couple of points in $\mathbb{H}^n$ can be joined by piecewise integral curves of the left invariant vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n$. This is well known (in fact, it follows from Chow's theorem since these vector fields generate the Lie algebra of left invariant vector fields), and it is the base of classical Hörmander's and Bony's results on hypoellipticity and maximum principle for sum-of-squares operators, as well as of the theory of Carnot-Carathéodory metrics.

Eventually, let us give a short abstract of the paper. In Section 2 we establish our notations and we recall more or less known results; in Section 3 we prove the intrinsic Coarea Formula and other preliminary results; in Section 4 we perform the blow-up at a point of the $\mathbb{H}$-reduced boundary and we prove that the limit set is a half-space; in Section 5 we recall basic results on differentiability in $\mathbb{H}^n$, essentially due to P.Pansu; in Section 6 we introduce our definition of $\mathbb{H}$-regular hypersurfaces, together with the associated Implicit Function Theorem. Moreover, a Whitney type theorem in $\mathbb{H}^n$ is proved; finally, Section 7 contains the Main Theorem, together with its proof and some applications.

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2. Notations

In this paper we indicate by $\mathbb{H}^n$ the n-dimensional Heisenberg group $\mathbb{H}^n \simeq \mathbb{C}^n \times \mathbb{R} \simeq \mathbb{R}^{2n+1}$. The points in $\mathbb{H}^n$ will be denoted as $P = [z,t] = [x+iy,t] = [p_1, p_2, \dots, p_{2n}, p_{2n+1}]$. If $P = [z,t]$, $Q = [\zeta, \tau] \in \mathbb{H}^n$ and $r > 0$, following the notations of [57], we define the group operation

$$P \cdot Q = [z + \zeta, t + \tau + 2\Im m(z\bar{\zeta})]$$

and the family of non isotropic dilations δ_r defined as

$$\delta_r(P) := [rz, r^2 t].$$

It is also useful to consider the group translations $\tau_P : \mathbb{H}^n \to \mathbb{H}^n$ defined as

$$Q \mapsto \tau_P(Q) := P \cdot Q$$

for any fixed $P \in \mathbb{H}^n$. We denote as $P^{-1} := [-z, -t]$ the inverse of P and as 0 the origin of $\mathbb{R}^{2n+1}$. We shall endow $\mathbb{H}^n$ with the homogeneous norm $\|P\|_\infty := \max\{|z|, |t|^{1/2}\}$ and with the distance, associated with the norm,

$$d(P, Q) := \|P^{-1} \cdot Q\|_\infty. \tag{1}$$

We explicitely observe that

Proposition 1. *The function d defined by (1) is a distance in $\mathbb{H}^n$ and the usual properties related with translations and dilations hold, i.e. for all P, Q, $Q' \in \mathbb{H}^n$ and for all $r > 0$*

$$d(\tau_P Q, \tau_P Q') = d(Q, Q') \quad \textit{and} \quad d(\delta_r Q, \delta_r Q') = r\, d(Q, Q'). \tag{2}$$

In addition, for any bounded subset Ω of $\mathbb{H}^n$ there exist positive constants $c_1(\Omega)$, $c_2(\Omega)$ such that

$$c_1(\Omega)|P - Q|_{\mathbb{R}^{2n+1}} \le d(P, Q) \le c_2(\Omega)|P - Q|^{1/2}_{\mathbb{R}^{2n+1}}$$

for $P, Q \in \Omega$. In particular, the topologies defined by d and by the Euclidean distance coincide on $\mathbb{H}^n$.

Remark 1. We stress that, because the topologies defined by d and by the Euclidean distance coincide, the topological dimension of $\mathbb{H}^n$ is $2n + 1$. On the contrary the Hausdorff dimension of $(\mathbb{H}^n, d)$ is $Q = 2n + 2$.

From now on, $U(P, r)$ and $B(P, r)$ will be, respectively, the open and closed ball with centre P and radius r with respect to the distance d. We notice explicitly that $U(P, r)$ is an Euclidean Lipschitz domain in $\mathbb{R}^{2n+1}$.

There is a natural measure dh on $\mathbb{H}^n$ which is given by the Lebesgue measure $d\mathcal{L}^{2n+1} = dz\, dt$ on $\mathbb{C}^n \times \mathbb{R}$. The measure dh is left (and right) invariant and it is the Haar measure of the group. If $E \subset \mathbb{H}^n$ then $|E|$ is its Lebesgue measure and ω_k will denote the k-dimensional Lebesgue measure of the unit Euclidean ball in $\mathbb{R}^k$.

Definition 1. *(see [24]) We denote by $\mathcal{H}^m$ the m-dimensional Hausdorff measure obtained from the Euclidean distance in $\mathbb{R}^{2n+1} \simeq \mathbb{H}^n$, and by $\mathcal{H}^m_d$ the m-dimensional Hausdorff measure obtained from the distance d in $\mathbb{H}^n$. Analogously, $\mathcal{S}^m$ and $\mathcal{S}^m_d$ will denote the corresponding spherical measures.*

Translation invariance and homogeneity under dilations of Hausdorff measures follow as usual from (2), more precisely we have

Proposition 2. *Let $A \subseteq \mathbb{H}^n$, $P \in \mathbb{H}^n$ and $m,r \in [0,\infty)$. Then*

$$\mathcal{H}_d^m(\tau_P A) = \mathcal{H}_d^m(A)$$
$$\mathcal{H}_d^m(\delta_r(A)) = r^m \mathcal{H}_d^m(A).$$

The Lie algebra of the left invariant vector fields of $\mathbb{H}^n$ is generated by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t} \qquad T = \frac{\partial}{\partial t},$$

for $j = 1, \ldots, n$ and the only non-trivial commutator relations are

$$[X_j, Y_j] = -4T, \qquad \text{for } j = 1, \ldots, n.$$

In the following we shall identify the vector fields and the associated first order differential operators. The vector fields $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ define a vector bundle on $\mathbb{H}^n$ (the horizontal vector bundle $H\mathbb{H}^n$) that can be canonically identified with a vector subbundle of the tangent vector bundle of $\mathbb{R}^{2n+1}$. Since each fiber of $H\mathbb{H}^n$ can be canonically identified with a vector subspace of $\mathbb{R}^{2n+1}$, each section ϕ of $H\mathbb{H}^n$ can be identified with a map $\phi : \mathbb{H}^n \to \mathbb{R}^{2n+1}$. At each point $P \in \mathbb{H}^n$ the horizontal fiber is indicated as $H\mathbb{H}^n_P$ and each fiber can be endowed with the scalar product $\langle \cdot, \cdot \rangle_P$ and the norm $|\cdot|_P$ that make the vector fields $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ orthonormal. Hence we shall also identify a section of $H\mathbb{H}^n$ with its canonical coordinates with respect to this moving frame. In this way, a section ϕ will be identified with a function $\phi = (\phi_1, \ldots, \phi_{2n}) : \mathbb{H}^n \to \mathbb{R}^{2n}$. As it is common in Riemannian geometry, when dealing with two sections ϕ and ψ whose argument is not explicitely written, we shall drop the index P in the scalar product writing $\langle \psi, \phi \rangle$ for $\langle \psi(P), \phi(P) \rangle_P$. The same convention shall be adopted for the norm.

For sake of completness, let us recall here the definition of the Carnot–Carathéodory metric associated with $X_1, \ldots, X_n, Y_1, \ldots, Y_n$. In fact, this definition has been developed in a much larger setting (see, e.g., [25], [47]).

Definition 2. *We say that an absolutely continuous curve $\gamma : [0,T] \to \mathbb{H}^n$ is a sub-unit curve with respect to $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ if there exist real measurable functions $a_1(s), \ldots, a_{2n}(s)$, $s \in [0,T]$ such that $\sum_j a_j^2 \le 1$ and*

$$\dot\gamma(s) = \sum_{j=1}^n a_j(s) X_j(\gamma(s)) + \sum_{j=1}^n a_{j+n}(s) Y_j(\gamma(s)), \qquad \textit{for a.e. } s \in [0,T].$$

If $P_1, P_2 \in \mathbb{H}^n$, their Carnot–Carathéodory distance $d_C(P_1, P_2)$ is

$$d_C(P_1,P_2) = \inf \{T > 0 : \textit{ there is a subunit curve } \gamma : [0,T] \to \mathbb{H}^n, \\ \gamma(0) = P_1, \gamma(T) = P_2\}.$$

Notice that the above set of curves joining P_1 and P_2 is not empty, by Chow's theorem, for the rank of the Lie algebra generated by $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ is $2n+1$, and hence d_C is a distance on $\mathbb{H}^n$. We shall denote respectively by $U_C(P,r)$ and $B_C(P,r)$ open and closed balls for d_C.

Remark 2. Alternatively, sub-unit curves can be defined as absolutely continuous functions γ such that $\dot\gamma$ is a measurable section of $\mathrm{H}\mathbb{H}^n$ such that $|\dot\gamma(s)|_{\gamma(s)} \le 1$ for a.e. s.

The following result is well known: see, for instance, [6], [59].

Proposition 3. *The Carnot–Carathéodory distance d_C is (globally) equivalent to the distance d defined in (1).*

If Ω is an open subset of $\mathbb{H}^n$ and $k \ge 0$ is a non negative integer, the symbols $\mathbf{C}^k(\Omega)$, $\mathbf{C}^\infty(\Omega)$ indicate the usual (Euclidean) spaces of real valued continuously differentiable functions. We denote by $\mathbf{C}^k(\Omega, \mathrm{H}\mathbb{H}^n)$ the set of all C^k-sections of $\mathrm{H}\mathbb{H}^n$ where the C^k regularity is understood as regularity between smooth manifolds. The notions of $\mathbf{C}_0^k(\Omega, \mathrm{H}\mathbb{H}^n)$, $\mathbf{C}^\infty(\Omega, \mathrm{H}\mathbb{H}^n)$ and $\mathbf{C}_0^\infty(\Omega, \mathrm{H}\mathbb{H}^n)$ are defined analogously. Observe that in the fourth section different and more intrinsic spaces of differentiable functions, the spaces $\mathbf{C}_{\mathbb{H}}^k(\Omega)$, are introduced.

The similarity among some statements in $\mathbb{H}^n$ with others in $\mathbb{R}^{2n+1}$ is clear using intrinsic notions of gradient for functions $\mathbb{H}^n \to \mathbb{R}$ and of divergence for sections of $\mathrm{H}\mathbb{H}^n$.

Definition 3. *If $\Omega \subseteq \mathbb{H}^n$ is open , $\phi = (\phi_1, \dots, \phi_{2n}) \in \mathbf{C}^1(\Omega, \mathrm{H}\mathbb{H}^n)$ and $f \in \mathbf{C}^1(\Omega)$, define*

$$\nabla_{\mathbb{H}} f := (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f) \tag{3}$$

and

$$div_{\mathbb{H}} \phi := \sum_{j=1}^n X_j \phi_j + Y_j \phi_{n+j}. \tag{4}$$

Alternatively $\nabla_{\mathbb{H}} f$ can be defined as the section of $\mathrm{H}\mathbb{H}^n$

$$\nabla_{\mathbb{H}} f := \sum_{j=1}^n (X_j f) X_j + (Y_j f) Y_j$$

whose canonical coordinates are $(X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f)$.

This is consistent with the identification we mentioned of sections and their coordinates.

Finally, through this paper we shall use the following notation: $C = C([z,t])$ is the $2n \times (2n+1)$ matrix whose rows are the components of the vectors $X_1, \dots, X_n, Y_1, \dots, Y_n$, that is

$$C(P) := \begin{bmatrix} X_1(P) \\ \vdots \\ X_n(P) \\ Y_1(P) \\ \vdots \\ Y_n(P) \end{bmatrix} = \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 & 2y_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 & 2y_n \\ 0 & \dots & 0 & 1 & \dots & 0 & -2x_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 1 & -2x_n \end{bmatrix} \tag{5}$$

We point out that $\nabla_{\mathbb{H}} = C\nabla$, where ∇ is the Euclidean gradient, and that the identification of a vector $\xi = (\xi_1, \dots, \xi_{2n}) \in H\mathbb{H}^n_P$ with a vector $\tilde{\xi} \in \mathbb{R}^{2n+1}$ through the corresponding imbedding of the horizontal fibers can be expressed by using C since $\tilde{\xi} = {}^tC(P)\xi$.

In $\mathbb{H}^n$ there is a natural definition of bounded variation functions and of finite perimeter sets (see [30], [35], [9]).

Definition 4. *We say that $f : \Omega \to \mathbb{R}$ is of bounded $\mathbb{H}$-variation in an open set $\Omega \subset \mathbb{H}^n$, ($f \in BV_{\mathbb{H}}(\Omega)$), if $f \in L^1(\Omega)$ and if*

$$\int_{\Omega} d|\nabla_{\mathbb{H}} f| := \sup\left\{ \int_{\Omega} f \, div_{\mathbb{H}} \phi \, dh \right\} < +\infty, \tag{6}$$

where the supremum is taken over all $\phi \in \mathbf{C}^1_0(\Omega, H\mathbb{H}^n)$, $|\phi| \le 1$. Analogously the space $BV_{\mathbb{H},\mathrm{loc}}(\Omega)$ is defined in the usual way.

Observe that (6) gives back the pointwise definition (3) when applied to $\mathbf{C}^1$ functions. The following compactness result for $BV_{\mathbb{H}}(\Omega)$ is basically proved in [35].

Theorem 1. *$BV_{\mathbb{H},\mathrm{loc}}(\mathbb{H}^n)$ is compactly embedded in $L^p_{\mathrm{loc}}(\mathbb{H}^n)$ for $1 \le p < \frac{Q}{Q-1}$ where as usual (see e.g. [57]) $Q := 2n+2$ indicates the homogeneous dimension of $\mathbb{H}^n$.*

Proof. The statement follows through a diagonal procedure from Theorem 1.35 in [35] exhausting $\mathbb{H}^n$ with a family of Carnot-Carathéodory balls $U_C(0,k)$, with $k \in \mathbb{N}$. Indeed, following the terminology of [35], Carnot-Carathéodory balls are (PS)-domains.

□

Definition 5. *[$\mathbb{H}$-Caccioppoli sets] We say that $E \subset \mathbb{H}^n$ is a locally finite $\mathbb{H}$-perimeter set (or a $\mathbb{H}$-Caccioppoli set) if $\mathbf{1}_E \in BV_{\mathbb{H},\mathrm{loc}}(\mathbb{H}^n)$, where we indicate as $\mathbf{1}_E$ the characteristic function of the set E. In this case, the measure $|\nabla_{\mathbb{H}} \mathbf{1}_E|$ will be called $\mathbb{H}$-perimeter of E and will be denoted by $|\partial E|_{\mathbb{H}}$.*

The following result is a straightforward consequence of the definition (see e.g. [30]):

Proposition 4. *The perimeter measure is lower semicontinuous with respect to the L^1 convergence of the characteristic functions of the sets.*

Remark 3. The $\mathbb{H}$-perimeter is invariant under group translations, indeed $\operatorname{div}_{\mathbb{H}}$ is invariant under group translations and the Jacobian determinant of τ_P as a map $\mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ is 1. Moreover $|\partial(\delta_\lambda E)|_{\mathbb{H}}(U(0,L)) = \lambda^{1-Q}|\partial E|_{\mathbb{H}}(U(0,\lambda L))$. Finally, if f is a bounded variation function in a bounded open set $\Omega \subset \mathbb{H}^n \simeq \mathbb{R}^{2n+1}$ in the Euclidean sense (see, e.g., [23]), then $f \in BV_{\mathbb{H}}(\Omega)$, and $|\nabla_{\mathbb{H}} f| \le c(\Omega)|\nabla f|$, since the left invariant vector fields are locally bounded.

Proposition 5. *If E is a smooth domain, then $|\partial E|_{\mathbb{H}} = |Cn|\ \mathcal{H}^{2n} \llcorner \partial E$, where n is the Euclidean outward normal to ∂E ([10], p.211; [30], Remark 2.3.3). By a standard approximation argument, the same representation holds for Lipschitz graphs (Lipschitz in the Euclidean sense).*

We shall use this fact when dealing with our metric balls $U(P,r)$. In particular, we have:

Proposition 6. *Any Euclidean Caccioppoli set in $\mathbb{H}^n \simeq \mathbb{R}^{2n+1}$ is a $\mathbb{H}$-Caccioppoli set and the $\mathbb{H}$-perimeter measure $|\partial E|_{\mathbb{H}}$ is absolutely continuous with respect to the standard Euclidean perimeter measure $|\partial E|$.*

It is easy to show that the inclusion in Proposition 6 is strict, in the sense that there exist $\mathbb{H}$-Caccioppoli sets that are not Caccioppoli sets in $\mathbb{R}^{2n+1}$. Indeed consider the following example

Example 1. If $k \in \mathbb{N}$, put $E_k = \{P \in \mathbb{H}^1 : r_{2k+1} \le \|P\|_\infty \le r_{2k}\}$, where $(r_k)_{k\in\mathbb{N}}$ is a strictly decreasing sequence of positive real numbers such that $\displaystyle\sum_{k\in N} r_k^2 = \infty$ and $\displaystyle\sum_{k\in N} r_k^3 < \infty$, and define $E = \displaystyle\bigcup_{k=1}^{\infty} E_k$. Then E has finite $\mathbb{H}$-perimeter but it has not locally finite Euclidean perimeter in $\mathbb{R}^3$.

Proof. Take any open neighborhood $\mathcal{O}$ of the origin; then there is k_0 sufficiently large such that

$$|\partial E|(\mathcal{O}) \ge \sum_{k\ge k_0} \mathcal{H}^2(\partial E_k),$$

but $\mathcal{H}^2(\partial E_k) = 2\pi\left(r_{2k}^2 + 2r_{2k}^3\right) + 2\pi\left(r_{2k+1}^2 + 2r_{2k+1}^3\right)$ hence $|\partial E|(\mathcal{O}) = \infty$.

On the other hand, by the explicit form of the $\mathbb{H}$-perimeter for Lipschitz domains (see Proposition 5) we need only to show that

$$\sum_{k=1}^{\infty} \int_{\partial E_k} |Cn| \, d\mathcal{H}^2 < \infty, \tag{7}$$

where n denotes the unit (Euclidean) outward normal vector field to ∂E_k. Now, an easy explicit calculation of $|Cn|$ shows that

$$\int_{\{\|[z,t]\|_\infty = r_j\}} |Cn| \, d\mathcal{H}^2 \quad = 4 \int_{\{|z| \le r_j\}} |z| \, dz + \int_{\{|z| = r_j, |t| \le r_j^2\}} d\mathcal{H}^2 \simeq r_j^3,$$

for $j = 0, 1, \dots$, so that (7) follows from the assumptions on the sequence r_k.

□

For each $f \in BV_{\mathbb{H}}(\Omega)$, the vector-valued distribution

$$(X, Y)f := (X_1 f, \dots, X_n f, Y_1 f, \dots, Y_n f),$$

that acts from $\mathbf{C}_0^\infty(\Omega, H\mathbb{H}^n)$ to $\mathbb{R}^{2n}$ as

$$((X, Y)f)(\psi) := -\int f \operatorname{div}_{\mathbb{H}} \psi \, dh$$

can be extended, as a vector-valued functional, on all of $\mathbf{C}_0^0(\Omega, H\mathbb{H}^n)$. Once more, we call $\nabla_{\mathbb{H}} f$ such an extension. By Riesz' representation theorem the functional $\psi \mapsto \nabla_{\mathbb{H}} f(\psi)$ can be represented as

$$\nabla_{\mathbb{H}} f(\psi) = \int \langle \sigma_f, \psi \rangle \, d|\nabla_{\mathbb{H}} f| \tag{8}$$

for all sections $\psi \in \mathbf{C}_0^0(\Omega; H\mathbb{H}^n)$ where the positive Radon measure $|\nabla_{\mathbb{H}} f|$ is defined in (6) and σ_f is a $|\nabla_{\mathbb{H}} f|$-measurable section of $H\mathbb{H}^n$ with $|\sigma_f(P)|_P = 1$ for $|\nabla_{\mathbb{H}} f|$-a.e. $P \in \mathbb{H}^n$.

In the special case when $f = \mathbf{1}_E$ and E is a $\mathbb{H}$-Caccioppoli set, the section $\sigma_{\mathbf{1}_E}$ can be interpreted as a generalized inward normal to the set E and shall be indicated in this paper as ν_E. Hence (8) can be restated as follows ([30]).

Theorem 2. *There exists a $|\partial E|_{\mathbb{H}}$-measurable section ν_E of $H\mathbb{H}^n$ such that*

$$-\int_E div_{\mathbb{H}} \phi \, dh = \int_{\mathbb{H}^n} \langle \nu_E, \phi \rangle \, d|\partial E|_{\mathbb{H}} \qquad \forall \phi \in \mathbf{C}_0^\infty(\Omega; H\mathbb{H}^n);$$

$$|\nu_E(P)|_P = 1 \qquad for\ |\partial E|_{\mathbb{H}} - a.e.\ P \in \mathbb{H}^n.$$

Here, the measurability of ν_E is meant in the sense that its coordinates $\nu_1, \dots, \nu_{2n}$ are $|\partial E|_{\mathbb{H}}$-measurable functions.

We introduce now one of the basic concepts of our paper. On the other hand, because of Theorem 2, it is a straightforward generalization of the definition given by De Giorgi in his celebrated 1955 paper [17] (see also, e.g., [24] and [23]).

Definition 6. *[Reduced boundary] Let E be a $\mathbb{H}$-Caccioppoli set; we shall say that $P \in \partial^*_{\mathbb{H}} E$, the $\mathbb{H}$-reduced boundary of E, if*

$$|\partial E|_{\mathbb{H}}(U(P,r)) > 0 \qquad \text{for any } r > 0; \tag{i}$$

$$\text{there exists} \quad \lim_{r\to 0} \fint_{U(P,r)} \nu_E \, d|\partial E|_{\mathbb{H}}; \tag{ii}$$

$$|\lim_{r\to 0} \fint_{U(P,r)} \nu_E \, d|\partial E|_{\mathbb{H}}| = 1. \tag{iii}$$

Observe that the definition of $\partial^*_{\mathbb{H}} E$ depends only on E and not on the particular representative ν_E since only integrals of it are involved. Moreover, because of differentiation Lemma 5 the perimeter measure is concentrated on the reduced boundary and

$$\nu_E(P) = \lim_{r\to 0} \fint_{U(P,r)} \nu_E \, d|\partial E|_{\mathbb{H}} \tag{9}$$

and

$$|\nu_E(P)|_P = 1 \quad \text{for } |\partial E|_{\mathbb{H}}\text{-a.e. } P \in \mathbb{H}^n \tag{10}$$

Hence possibly re-defining ν_E in a $|\partial E|_{\mathbb{H}}$-negligible set, we can assume that ν_E is the limit of the averages at any point P in the reduced boundary.

The limits in Definition (6) and in (9) can be understood as a convergence of the averages of the coordinates of ν_E with respect to the moving base of the fibers; analogously the first equality in (9) is understood as an identity of the corresponding canonical coordinates. The same formulas can be equivalently understood in $\mathbb{R}^{2n+1}$, given the natural identification of a section of $H\mathbb{H}^n$ with a map from $\mathbb{R}^{2n+1}$ to $\mathbb{R}^{2n+1}$. Once more, as after Definition 5, we observe that the $\mathbb{H}$-reduced boundary of a set is invariant under group translations, i.e.

$$P_0 \in \partial^*_{\mathbb{H}} E \quad \text{if and only if} \quad \tau_P(P_0) \in \partial^*_{\mathbb{H}}(\tau_P E)$$

and also

$$\nu_E(P_0) = \nu_{\tau_P E}(\tau_P(P_0)).$$

Theorem 3. *Denote by $Q = 2n+2$ the so-called homogeneous dimension in $\mathbb{H}^n$. Then*

$$\mathcal{L}^{2n+1} = \frac{2\omega_{2n}}{\omega_{2n+2}} \mathcal{S}_d^Q = \frac{2\omega_{2n}}{\mathcal{H}_d^Q(B(0,1))} \mathcal{H}_d^Q. \tag{11}$$

In particular (as proved in [46] and [50]) the Hausdorff dimension of $\mathbb{H}^n$ as a metric space is Q.

Proof of Theorem 3. The first equality is known ([50],[46]). Then, from Haar theorem (see [24] Theorem 2.7.7) $\mathcal{H}_d^Q$ is a constant multiple of $\mathcal{S}_d^Q$ and the proof is completed.

□

Definition 7. *Let $[z,t], P_0 \in \mathbb{H}^n$ be given. We set*

$$\pi_{P_0}([z,t]) = \sum_{j=1}^n x_j X_j(P_0) + \sum_{j=1}^n y_j Y_j(P_0).$$

The map $P_0 \to \pi_{P_0}([z,t])$ is a smooth section of $\mathrm{H}\mathbb{H}^n$.

3. Coarea formula and asymptotic doubling property for the perimeter

The following coarea formula (see e.g. [23] for the corresponding Euclidean statement) was proved in [30] when f is the characteristic function of an open set.

Proposition 7. [Coarea formula] *Let $f \in L^1(\Omega)$; then*

$$\int_\Omega f(P)|\nabla_{\mathbb{H}} \|P\|_\infty|\, dh = \int_{-\infty}^{+\infty} ds \left(\int_{\partial U(0,s)\cap\Omega} f(P)|Cn_s(P)| d\mathcal{H}^{2n} \right), \tag{12}$$

where $n_s(P)$ is for $\mathcal{H}^{2n}$-a.e. $P \in \partial U(0,s)$ the usual Euclidean outward unit normal to $\partial U(0,s)$, and $|\nabla_{\mathbb{H}}\|P\|_\infty|$ denotes the Euclidean norm in $\mathbb{R}^{2n}$ of the vector $\nabla_{\mathbb{H}}\|P\|_\infty$ that is well defined since $\|P\|_\infty$ is Lipschitz continuous in $\mathbb{R}^{2n+1}$. In particular, for L^1-a.e. $r > 0$,

$$\frac{d}{dr} \int_{U(0,r)} f(P)|\nabla_{\mathbb{H}}\|P\|_\infty|\, dh = \int_{\partial U(0,r)} f(P)|Cn_r(P)|\, d\mathcal{H}^{2n}. \tag{13}$$

Proof. Let $P = [z,t]$ and remember that $\nabla_{\mathbb{H}} = C\nabla$ where C is defined in (5). Then since

$$\nabla \|P\|_\infty = \begin{cases} [z/|z|, 0] & \text{if } \sqrt{|t|} < |z| \\ [0, \operatorname{sgn} t/2\sqrt{|t|}] & \text{if } \sqrt{|t|} > |z| \end{cases} \tag{14}$$

we obtain that

$$|\nabla_{\mathbb{H}} \|P\|_\infty| = \begin{cases} 1 & \text{if } \sqrt{|t|} < |z| \\ |z|/\sqrt{|t|} & \text{if } \sqrt{|t|} > |z|. \end{cases} \tag{15}$$

In particular, the integral in left-hand-side of (12) is well defined. Now, by Theorem 2.3.5 in [30] and by Proposition 5, if A is an open subset of $\mathbb{H}^n$, then

$$\begin{aligned} \int_A |\nabla_{\mathbb{H}} \|P\|_\infty| \; dh &= \int_0^\infty |\partial E_s|_{\mathbb{H}}(A) \; ds \\ &= \int_0^\infty ds \left(\int_{A \cap \partial U(0,s)} |C n_s| \; d\mathcal{H}^{2n} \right), \end{aligned} \tag{16}$$

where $E_s = \{P = [z,t] : \|P\|_\infty > s\}$. To obtain (16) for a general Lebesgue measurable set B, just observe that the two (outer) measures μ and ν defined as

$$\mu(B) = \int_B |\nabla_{\mathbb{H}} \|P\|_\infty| \; dh$$

and

$$\nu(B) = \int_0^\infty ds \left(\int_{B \cap \partial U(0,s)} |C n_s| \; d\mathcal{H}^{2n} \right),$$

are Radon measures and then from (16) we have

$$\begin{aligned} \mu(B) &= \inf\{\mu(U) : U \text{ open}, \; B \subset U\} \\ &= \inf\{\nu(U) : U \text{ open}, \; B \subset U\} = \nu(B). \end{aligned}$$

To prove that ν is a Radon measure (the analogous statement for μ is straightforward) just observe that ν is finite on any compact set K, since, if A_K is a bounded open neighborhood of K, it follows from (16) with $A = A_K$ that

$$\nu(K) \le \nu(A_K) = \int_{A_K} \left|\nabla_{\mathbb{H}} \|P\|_\infty\right| \; dh < \infty$$

and that ν is a regular Borel measure, since $\mathcal{H}^{2n}$ is an (outer) regular Borel measure.

To conclude it is enough to check (12) when f is any simple measurable function as $f = \sum_{i=1}^{k} \alpha_i \mathbf{1}_{A_i}$ with A_i Lebesgue measurable and $\alpha_i \in \mathbb{R}$. This is trivial, indeed

$$\begin{aligned}
\int_{\mathbb{H}^n} f(P)|\nabla_{\mathbb{H}}\|P\|_\infty|\, dh &= \sum_{i=1}^{k} \alpha_i \int_{A_i} |\nabla_{\mathbb{H}}\|P\|_\infty|\, dh \\
&= \sum_{i=1}^{k} \alpha_i \int_0^{+\infty} ds \int_{\partial U(0,s)\cap A_i} |Cn_s(P)| d\mathcal{H}^{2n} \\
&= \int_0^{+\infty} ds \int_{\partial U(0,s)\cap A_i} \sum_{i=1}^{k} \alpha_i |Cn_s(P)|\, d\mathcal{H}^{2n} \\
&= \int_0^{+\infty} ds \int_{\partial U(0,s)} f(P)|Cn_s(P)| d\mathcal{H}^{2n}.
\end{aligned}$$

□

From Proposition 7 we obtain the following formula of integration by parts (see [23], Lemma 1, Section 5.7).

Corollary 1. *Let E be a $\mathbb{H}$-Caccioppoli set in $\mathbb{H}^n$; then for every $\phi \in \mathbf{C}_0^1(\mathbb{H}^n, \mathrm{H}\mathbb{H}^n)$ we have:*

$$\begin{aligned}
\int_{E\cap U(0,r)} div_{\mathbb{H}} \phi\, dh &= \int_{U(0,r)} \langle \phi, \nu_E\rangle\, d|\partial E|_{\mathbb{H}} + \int_E \langle \phi, \nu_{U(0,r)}\rangle\, d|\partial U(0,r)|_{\mathbb{H}} \\
&= \int_{U(0,r)} \langle \phi, \nu_E\rangle\, d|\partial E|_{\mathbb{H}} + \int_{E\cap\partial U(0,r)} \langle \phi, Cn_r\rangle\, d\mathcal{H}^{2n},
\end{aligned} \tag{17}$$

for L^1-a.e. $r > 0$, where $n_r(P)$ is defined in Proposition 7.

Remark 4. The last equality is an application of Remark 5. We point out that there is some kind of ambiguity in the last formula of (17), since in the first integral ϕ is a section of the horizontal bundle, whereas in the last one ϕ is identified with a function taking values in $\mathbb{R}^{2n}$ through its coordinates with respect to the canonical moving frame.

Proof of Corollary 1. If $g : \mathbb{H}^n \to \mathbb{R}$ is any Euclidean Lipschitz function, then, with our usual identification, it follows from formula (2.10) in [30], that

$$\begin{aligned}
\int_{\mathbb{H}^n} \langle g\phi, \nu_E\rangle\, d|\partial E|_{\mathbb{H}} &= \int_E \mathrm{div}_{\mathbb{H}}\,(g\phi)\, dh = \int_E \mathrm{div}\,(g\, {}^tC\phi)\, dh \qquad (18) \\
&= \int_E g\, \mathrm{div}\,({}^tC\phi)\, dh + \int_E \langle C\nabla g, \phi\rangle\, dh.
\end{aligned}$$

Choose now $g(P) = g_\epsilon(\|P\|_\infty)$ where

$$g_\epsilon(s) = \begin{cases} 1 & \text{if } \ 0 \le s \le r, \\ (r-s+\epsilon)/\epsilon & \text{if } r \le s \le r+\epsilon, \\ 0 & \text{if } \ s \ge r+\epsilon \end{cases}$$

Thus, from (18), we get

$$\int_{\mathbb{H}^n} \langle g_\epsilon \phi, \nu_E \rangle \, d|\partial E|_{\mathbb{H}} = \int_E g_\epsilon \mathrm{div}_{\mathbb{H}} \phi \, dh + \int_E \langle \nabla_{\mathbb{H}} g_\epsilon, \phi \rangle \, dh.$$

Now, as $\epsilon \to 0$,

$$\int_{\mathbb{H}^n} \langle g_\epsilon \phi, \nu_E \rangle \, d|\partial E|_{\mathbb{H}} \to \int_{B(0,r)} \langle \phi, \nu_E \rangle \, d|\partial E|_{\mathbb{H}}$$

and

$$\int_E g_\epsilon \mathrm{div}_{\mathbb{H}} \phi \, dh \to \int_{E \cap B(0,r)} \mathrm{div}_{\mathbb{H}} \phi \, dh.$$

Thus, we have only to consider the limit as $\epsilon \to 0$ of

$$\int_E \langle \nabla_{\mathbb{H}} g_\epsilon, \phi \rangle \, dh = -\frac{1}{\epsilon} \int_{E \cap \{r < \|P\|_\infty < r+\epsilon\}} \langle \nabla_{\mathbb{H}} \|P\|_\infty, \phi \rangle \, dh. \tag{19}$$

Write now

$$u(P) = \begin{cases} \mathbf{1}_E(P) \langle \frac{\nabla_{\mathbb{H}} \|P\|_\infty}{|\nabla_{\mathbb{H}} \|P\|_\infty|}, \phi \rangle & \text{if } \ |\nabla_{\mathbb{H}} \|P\|_\infty| \ne 0, \\ 0 & \text{otherwise} \end{cases}$$

and notice that $u \in L^1(\mathbb{H}^n)$. By applying (13), it follows from (19) that

$$\int_E \langle \nabla_{\mathbb{H}} g_\epsilon, \phi \rangle \, dh \to -\int_{E \cap \partial U(0,r)} \langle \frac{\nabla_{\mathbb{H}} \|P\|_\infty}{|\nabla_{\mathbb{H}} \|P\|_\infty|}, \phi \rangle |Cn_r| \, d\mathcal{H}^{2n}$$

as $\epsilon \to 0$ and for L^1-a.e. $r > 0$. Therefore, the proof will be achieved simply by showing that

$$\langle \frac{\nabla_{\mathbb{H}} \|P\|_\infty}{|\nabla_{\mathbb{H}} \|P\|_\infty|}, \phi \rangle |Cn_r| = \langle Cn_r, \phi \rangle \tag{20}$$

on $\partial U(0,r)$. This is an elementary direct computation. Indeed, we have $\partial U(0,r) = \cup_{i=1}^3 \mathit{\Gamma}_i$, where $\mathit{\Gamma}_1 = \{[z,t] : |z| \le r, t = r^2\}$, $\mathit{\Gamma}_2 = \{[z,t] : |z| \le r, t = -r^2\}$, $\mathit{\Gamma}_3 = \{[z,t] : |z| = r, t \le r^2\}$, and $n_r \equiv [0,1]$ on $\mathit{\Gamma}_1$, $n_r \equiv [0,-1]$ on $\mathit{\Gamma}_2$, $n_r \equiv [z/r, 0]$ on $\mathit{\Gamma}_3$, so that

$$|Cn_r(P)| \equiv |Cn_r([z,t])| = \begin{cases} 2|z| & \text{on } \mathit{\Gamma}_1 \cup \mathit{\Gamma}_2, \\ 1 & \text{on } \ \mathit{\Gamma}_3. \end{cases}$$

Moreover, as computed in (14) and in (15),

$$|\nabla_{\mathbb{H}}\|P\|_\infty| \equiv |\nabla_{\mathbb{H}}\|[z,t]\|_\infty| = \begin{cases} |z|/r \text{ on } \Gamma_1 \cup \Gamma_2, \\ 1 \quad \text{on} \quad \Gamma_3, \end{cases}$$

and

$$\nabla_{\mathbb{H}}\|P\|_\infty \equiv \nabla_{\mathbb{H}}\|[x+iy,t]\|_\infty = \begin{cases} \operatorname{sign} t(y,-x)/r \text{ on } \Gamma_1 \cup \Gamma_2, \\ \quad z/r \qquad \text{on} \quad \Gamma_3. \end{cases}$$

Therefore (20) holds and the proof is completed.

□

From Corollary 1 and (13), the following local estimate for the $\mathbb{H}$-perimeter follows (see [1], Proposizione 3.2.1).

Corollary 2. *Let E be a $\mathbb{H}$-Caccioppoli set, and define for $r > 0$*

$$m(r) := \int_{E\cap U(0,r)} |\nabla_{\mathbb{H}}\|P\|_\infty|\, dh. \tag{21}$$

Then

$$|\partial(E\cap U(0,r))|_{\mathbb{H}}(\mathbb{H}^n) \le |\partial E|_{\mathbb{H}}(U(0,r)) + m'(r) \qquad \text{for } L^1\text{-a.e. } r > 0. \tag{22}$$

Proof. In (17) take the supremum with respect to all ϕ with $|\phi(P)|_P \le 1$ for all $P \in \mathbb{H}^n$ to get

$$|\partial E\cap U(0,r)|_{\mathbb{H}}(\mathbb{H}^n) \le |\partial E|_{\mathbb{H}}(U(0,r)) + \int_{\partial U(0,r)} \mathbf{1}_E |Cn_r|\, d\mathcal{H}^{2n},$$

and hence (22) follows using (13) in Proposition 7 with $f = \mathbf{1}_E$.

□

The following isoperimetric inequalities are proved in [35].

Theorem 4. *There is a constant $c > 0$ independent on $r > 0$ such that for any $\mathbb{H}$-Caccioppoli set $E \subset \mathbb{H}^n, \forall P \in \mathbb{H}^n$, $\forall r > 0$*

$$\min\{|E\cap U_C(P,r)|, |E^c\cap U_C(P,r)|\}^{\frac{Q-1}{Q}} \le c\,|\partial E|_{\mathbb{H}}(U_C(P,r)) \tag{23}$$

and

$$\min\{|E|,|E^c|\}^{\frac{Q-1}{Q}} \le c\,|\partial E|_{\mathbb{H}}(\mathbb{H}^n). \tag{24}$$

The last crucial tool we shall use is an asymptotic doubling estimate for perimeter type measures (see [2]) that in our context reads as:

Theorem 5. *Let E be a $\mathbb{H}$-Caccioppoli set. Then there are absolute constants $\tau > 0$ and $k > 0$ such that for $|\partial E|_{\mathbb{H}}$- a.e. $P \in \mathbb{H}^n$*

$$\tau < \liminf_{r \to 0} \frac{|\partial E|_{\mathbb{H}}(U(P,r))}{r^{Q-1}} \leq \limsup_{r \to 0} \frac{|\partial E|_{\mathbb{H}}(U(P,r))}{r^{Q-1}} < +\infty \tag{25}$$

and

$$\liminf_{r \to 0} \ \min \left\{ \frac{|E \cap U(P,r)|}{|U(P,r)|}, \frac{|E^c \cap U(P,r)|}{|U(P,r)|} \right\} \geq k. \tag{26}$$

4. Blow-up at a point of the $\mathbb{H}$-reduced boundary

Our main result in this section states that $\mathbb{H}$-Caccioppoli sets have an "approximate tangent plane" at each point of their reduced boundary. The precise meaning of this statement is the content of the following Blow-up Theorem.

For any set $E \subset \mathbb{H}^n$, $P_0 \in \mathbb{H}^n$ and $r > 0$ we consider the translated and dilated sets E_{r,P_0} defined as

$$E_{r,P_0} = \{P : P_0 \cdot \delta_r(P) \in E\} = \delta_{\frac{1}{r}}(\tau_{P_0^{-1}}(E)).$$

If P_0 is fixed and there is no ambiguity, we shall write simply E_r, and in addition we set $E_{P_0} = E_{1,P_0}$. Moreover if $v \in \mathrm{H}\mathbb{H}^n_{P_0}$ we consider the halfspaces $S^+_{\mathbb{H}}(v)$ and $S^-_{\mathbb{H}}(v)$ defined as

$$\begin{aligned} S^+_{\mathbb{H}}(v) &:= \{P : \langle \pi_{P_0} P, v \rangle_{P_0} \geq 0\} \\ S^-_{\mathbb{H}}(v) &:= \{P : \langle \pi_{P_0} P, v \rangle_{P_0} \leq 0\}. \end{aligned} \tag{27}$$

The common topological boundary $T^g_{\mathbb{H}}(v)$ of $S^+_{\mathbb{H}}(v)$ and of $S^-_{\mathbb{H}}(v)$ is the subgroup of $\mathbb{H}^n$

$$T^g_{\mathbb{H}}(v) := \{P : \langle \pi_{P_0} P, v \rangle_{P_0} = 0\}.$$

Theorem 6 (Blow-up Theorem). *If E is a $\mathbb{H}$-Caccioppoli set, $P_0 \in \partial^*_{\mathbb{H}} E$ and $\nu_E(P_0) \in \mathrm{H}\mathbb{H}^n_{P_0}$ is the inward normal as defined in Theorem 2 then*

$$\lim_{r \to 0} \mathbf{1}_{E_{r,P_0}} = \mathbf{1}_{S^+_{\mathbb{H}}(\nu_E(P_0))} \qquad \text{in } L^1_{loc}(\mathbb{H}^n) \tag{28}$$

and for all $R > 0$

$$\begin{aligned} \lim_{r \to 0} |\partial E_{r,P_0}|_{\mathbb{H}}(U(0,R)) &= |\partial S^+_{\mathbb{H}}(\nu_E(P_0))|_{\mathbb{H}}(U(0,R)) \\ &= 2\omega_{2n-1} R^{2n+1}. \end{aligned} \tag{29}$$

Proof. Without loss of generality, it is possible to assume that $P_0 = 0 \in \partial^*_{\mathbb{H}} E$.
To prove the theorem it is enough to show that, given any sequence $\{r_k\}_k$ with $r_k \to 0$, there exists a subsequence $\{s_j\}_j \subset \{r_k\}_k$ such that

$$\lim_{j\to+\infty} \mathbf{1}_{E_{s_j,P_0}} = \mathbf{1}_{S^+_{\mathbb{H}}(\nu_E(P_0))}$$

in $L^1_{\mathrm{loc}}(\mathbb{H}^n)$ and that $\forall R > 0$

$$\lim_{j\to+\infty} |\partial E_{s_j,P_0}|_{\mathbb{H}}(U(0,R)) = 2\omega_{2n-1} R^{2n+1}.$$

The proof of these facts is divided in a number of steps.
Claim 1. *There is a $\mathbb{H}$-Caccioppoli set F such that*

$$\lim_{j\to+\infty} \mathbf{1}_{E_{s_j,P_0}} = \mathbf{1}_F. \tag{30}$$

Proof of Claim 1: Fix $R > 0$, then for every $\phi \in \mathbf{C}^1_0(U(0,R), H\mathbb{H}^n)$ with $|\phi(P)|_P \le 1$ we have

$$\int_{E_r} \mathrm{div}_{\mathbb{H}}\, \phi \, dh \;=\; \frac{1}{r^{2n+2}} \int_E \mathrm{div}_{\mathbb{H}}\, \phi \circ \delta_{\frac{1}{r}}\, dh \;=\; \frac{1}{r^{2n+1}} \int_E \mathrm{div}_{\mathbb{H}}\, (\phi \circ \delta_{\frac{1}{r}})\, dh.$$

ial Observe that $\phi\circ\delta_{\frac{1}{r}} \in \mathbf{C}^1_0(U(0,rR), H\mathbb{H}^n)$, hence by definition it follows

$$|\partial E_r|_{\mathbb{H}}(U(0,R)) \le \frac{|\partial E|_{\mathbb{H}}(U(0,rR))}{r^{2n+1}}. \tag{31}$$

Because of the following two lemmas the right hand side is bounded by a constant independent on r.

Lemma 1. *If E is a $\mathbb{H}$-Caccioppoli set and $0 \in \partial^*_{\mathbb{H}} E$, then for any $c > 1$ there is $r_0 > 0$ such that*

$$|\partial E|_{\mathbb{H}}(U(0,r)) \le c \int_{\partial U(0,r)} |Cn_r| d\mathcal{H}^{2n}, \qquad \forall r \in (0, r_0). \tag{32}$$

Proof. Let $\phi \in \mathbf{C}^1_0(\mathbb{H}^n, \mathrm{H}\mathbb{H}^n)$ be such that $\phi(P) = \nu_E(0)$ for $P \in U(0,r)$ where the identity means equality of the corresponding coordinates in the canonical moving base. Then $\mathrm{div}_{\mathbb{H}}\,(\phi) = 0$ in $U(0,r)$ and Corollary 1 yields

$$\int_{U(0,r)} \langle \phi, \nu_E \rangle \; d|\partial E|_{\mathbb{H}} = -\int_{E\cap\partial U(0,r)} \langle \phi, Cn_r \rangle \; d\mathcal{H}^{2n}.$$

Then from the definition of $0 \in \partial^*_{\mathbb{H}} E$ we get

$$|\partial E|_{\mathbb{H}}(U(0,r)) = (1+o(1)) \int_{U(0,r)} \langle \phi, \nu_E \rangle \, d|\partial E|_{\mathbb{H}}$$
$$\leq (1+o(1)) \int_{\partial U(0,r)} |Cn_r| \, d\mathcal{H}^{2n}.$$

□

Lemma 2. *For any positive r*

$$\int_{\partial U(0,r)} |Cn_r| \, d\mathcal{H}^{2n} = 2\omega_{2n} \frac{4n^2+1}{2n+1} \, r^{2n+1}. \tag{33}$$

Proof. It is a direct computation. On each of the two bases of $\partial U(0,r)$ i.e. on $\{[z,t] \in \mathbb{H}^n : |z| \leq r^2, t = \pm r^2\}$ we have $|Cn_r|([z,t]) = 2|z|$ hence $\int_{\text{base}} |Cn_r| \, d\mathcal{H}^{2n} = \frac{2\omega_{2n}}{2n+1} \, r^{2n+1}$. On the lateral surface $\{[z,t] \in \mathbb{H}^n : |z| = r^2, |t| \leq r^2\}$ we have $|Cn_r| = 1$ hence $\int_{\text{lat.surf.}} |Cn_r| \, d\mathcal{H}^{2n} = 2(2n-1)\omega_{2n} r^{2n+1}$.

□

From (31), (32) and (33) it follows that for all $R > 0$ and $r > 0$

$$|\partial E_r|_{\mathbb{H}}(U(0,R)) \leq c_R$$

and

$$\|\mathbf{1}_{E_r}\|_{L^1(U(0,R))} = |E_r \cap U(0,R)| \leq |U(0,R)|.$$

By the compactness theorem 1, there exist a sequence $s_j \to 0$ and a function $f \in BV_{\mathbb{H},\mathrm{loc}}(\mathbb{H}^n)$ such that

$$\mathbf{1}_{E_{s_j}} \to f \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{H}^n).$$

We may assume that the above sequence converges pointwise a.e. to f, and therefore $f = \mathbf{1}_F$ for a suitable measurable $F \subset \mathbb{H}^n$. By the semicontinuity of the variation (see [30], [35]),

$$|\partial F|_{\mathbb{H}}(U(0,R)) \leq \liminf_{j\to\infty} |\partial E_{s_j}|_{\mathbb{H}}(U(0,R)) \leq c_R < \infty$$

for every $R > 0$. Hence F is a $\mathbb{H}$-Caccioppoli set, i.e. $\mathbf{1}_F \in BV_{\mathbb{H},\mathrm{loc}}(\mathbb{H}^n)$. In particular, by Theorem 2, for each $\phi \in \mathbf{C}^1_0(\mathbb{H}^n, \mathrm{H}\mathbb{H}^n)$

$$-\int_F \mathrm{div}_{\mathbb{H}} \, \phi \, dh = \int_{\mathbb{H}^n} \langle \nu_F, \phi \rangle_P \, d|\partial F|_{\mathbb{H}}, \tag{34}$$

where ν_F defined in (2) is $|\partial F|_{\mathbb{H}}$-measurable, $|\nu_F|_P = 1$ for $|\partial F|_{\mathbb{H}}$-a.e. P. This concludes the proof of Claim 1.

In the sequel we show that $F = S^+_{\mathbb{H}}(\nu_E(0))$, in particular that F is independent on the sequence $\{s_j\}_j$.

Claim 2.

$$\nu_F(P) = \nu_E(0) \qquad \textit{for } |\partial F|_{\mathbb{H}} - \textit{a.e. } P$$

where the identity means equality of the coordinates with respect to the canonical moving base.

Proof of Claim 2. Because $\mathbf{1}_{E_{s_j}} \to \mathbf{1}_F$ in $L^1_{\text{loc}}(\mathbb{H}^n)$ as $j \to +\infty$, we have

$$\lim_{j\to+\infty} \int_{E_{s_j}} \operatorname{div}_{\mathbb{H}} \phi \, dh = \int_F \operatorname{div}_{\mathbb{H}} \phi \, dh$$

for any $\phi \in \mathbf{C}^1_0(\mathbb{H}^n, \mathrm{H}\mathbb{H}^n)$ and, since both E_{s_j} and F are $\mathbb{H}$-Caccioppoli sets,

$$\lim_{j\to+\infty} \int_{\mathbb{H}^n} \langle \phi, \nu_{E_{s_j}} \rangle \, d|\partial E_{s_j}|_{\mathbb{H}} = \int_{\mathbb{H}^n} \langle \phi, \nu_F \rangle \, d|\partial F|_{\mathbb{H}}$$

that is

$$\nu_{E_{s_j}} |\partial E_{s_j}|_{\mathbb{H}} \rightharpoonup \nu_F |\partial F|_{\mathbb{H}} \qquad \text{as } j \to +\infty,$$

weakly as vector valued Radon measures. It follows that for a.e. $R > 0$, (precisely for all those R such that $|\partial F|_{\mathbb{H}}(\partial U(0,R)) = 0$)

$$\lim_{j\to+\infty} \int_{U(0,R)} \nu_{E_{s_j}} \, d|\partial E_{s_j}|_{\mathbb{H}} = \int_{U(0,R)} \nu_F \, d|\partial F|_{\mathbb{H}}. \tag{35}$$

Moreover, as computed before,

$$|\partial E_{s_j}|_{\mathbb{H}}(U(0,R)) = \frac{1}{s_j^{2n+1}} \, |\partial E|_{\mathbb{H}}(U(0, s_j R)) \tag{36}$$

and, again for all $\phi \in \mathbf{C}^1_0(\mathbb{H}^n, \mathrm{H}\mathbb{H}^n)$,

$$\int_{U(0,R)} \langle \phi, \nu_{E_{s_j}} \rangle \, d|\partial E_{s_j}|_{\mathbb{H}} = \frac{1}{s_j^{2n+1}} \int_{U(0,s_j R)} \langle \phi \circ \delta_{\frac{1}{s_j}}, \nu_E \rangle \, d|\partial E|_{\mathbb{H}}$$

which yields, with a limiting procedure in ϕ,

$$\int_{U(0,R)} \nu_{E_{s_j}} \, d|\partial E_{s_j}|_{\mathbb{H}} = \frac{1}{s_j^{2n+1}} \int_{U(0,s_j R)} \nu_E \, d|\partial E|_{\mathbb{H}}. \tag{37}$$

Hence from (36) and (37) we have

$$\fint_{U(0,R)} \nu_{E_{s_j}}\ d|\partial E_{s_j}|_{\mathbb{H}} = \fint_{U(0,s_jR)} \nu_E\ d|\partial E|_{\mathbb{H}}.$$

The right hand side has limit $\nu_E(0)$, because $0 \in \partial^*_{\mathbb{H}} E$; hence

$$\int_{U(0,R)} \nu_{E_{s_j}}\ d|\partial E_{s_j}|_{\mathbb{H}} = (1+o(1))\ \nu_E(0)|\partial E_{s_j}|_{\mathbb{H}}(U(0,R))$$

and

$$\int_{U(0,R)} \langle \nu_E(0), \nu_{E_{s_j}} \rangle\ d|\partial E_{s_j}|_{\mathbb{H}} = (1+o(1))\ |\partial E_{s_j}|_{\mathbb{H}}(U(0,R)). \tag{38}$$

Finally, by the semicontinuity of the perimeter,

$$|\partial F|_{\mathbb{H}}(U(0,R)) \le \liminf_{j\to+\infty} |\partial E_{s_j}|_{\mathbb{H}}(U(0,R))$$

and from (38)

$$= \liminf_{j\to+\infty} \int_{U(0,R)} \langle \nu_E(0), \nu_{E_{s_j}}(P) \rangle\ d|\partial E_{s_j}|_{\mathbb{H}}(P)$$

from (35)

$$= \int_{U(0,R)} \langle \nu_E(0), \nu_F(P) \rangle\ d|\partial F|_{\mathbb{H}}(P).$$

This concludes the proof of Claim 2 because $|\langle \nu_E(0), \nu_F(P)\rangle| \le 1$, while $|\langle \nu_E(0), \nu_F(P)\rangle| = 1$ precisely when $\nu_F(P) = \nu_E(0)$.

Claim 3. *There is $\tilde{Q} \in \mathbb{H}^n$ such that*

$$F = \tau_{\tilde{Q}}(S^+_{\mathbb{H}}(\nu_E(0)))$$

Proof of Claim 3. Take $\zeta \in \mathbb{R}^{2n}$ and consider both the differential operator (or constant vector field)

$$L_\zeta := \sum_{j=1}^{n} \zeta_j X_j + \zeta_{n+j} Y_j$$

and the test function $\phi \in \mathbf{C}_0^1(\mathbb{H}^n, \mathrm{H}\mathbb{H}^n)$ whose canonical coordinates are $\psi\,\zeta$ with $\psi \in \mathbf{C}_0^1(\mathbb{H}^n, \mathbb{R})$. Then

$$\int_{\mathbb{H}^n} \mathbf{1}_F L_\zeta \psi \, dh = \int_{\mathbb{H}^n} \mathbf{1}_F \mathrm{div}_{\mathbb{H}}\, \phi \, dh$$
$$= - \int_{\mathbb{H}^n} \langle \phi, \nu_F \rangle \, d|\partial F|_{\mathbb{H}} = - \int_{\mathbb{H}^n} \psi \langle \zeta, \nu_F \rangle \, d|\partial F|_{\mathbb{H}}$$

Because $\nu_F(P) = \nu_E(0)$ for all P, this yields that (in the sense of distributions)

$$\begin{cases} L_\zeta \mathbf{1}_F = 0 & \text{if} \quad \langle \zeta, \nu_E(0) \rangle_{\mathbb{R}^{2n}} = 0 \\ L_\zeta \mathbf{1}_F \geq 0 & \text{if} \quad \zeta = \nu_E(0). \end{cases}$$

It follows that $\mathbf{1}_F$ is constant along the integral curves of L_ζ when ζ is orthogonal to ν_F while it is increasing along the integral curves of L_ζ when $\zeta = \nu_F$. From this fact and from the following Lemma Claim 3 is completely proved.

Lemma 3. *Assume that $g : \mathbb{H}^n \to \mathbb{R}$ is constant along the integral lines of any vector field orthogonal to $\nu_F \in \mathrm{H}\mathbb{H}^n_0$ and increasing along the integral lines of ν_F. Then the level lines of g are "vertical hyperplanes orthogonal to ν_F" that is sets that are group translations of*

$$\partial S^+_{\mathbb{H}}(F, 0) = \{P : \langle \pi_0 P, \nu_F \rangle = 0\}.$$

Proof. Suppose first $n = 1$; in this case, rotations of $\mathbb{R}^3$ leaving the t-axis fixed are both isometries and automorphisms of the group (see [57], Chapter XII, 7.5) and hence we can assume without loss of generality that $\nu_F = X_1(0)$. Then $\forall P, Q \in \mathbb{H}^1$, $P = [p_1, p_2, p_3]$ and $Q = [q_1, q_2, q_3]$

$$p_1 > q_1 \Longrightarrow g(P) \geq g(Q). \tag{39}$$

To prove (39), it is enough to show that following the integral lines of Y_1 with positive or negative direction and the integral lines of X_1 only in the positive direction it is possible to reach Q from P. From the invariance of the vector fields under group translations it is enough to show that it is possible to connect the origin with any point $[\eta, r_2, r_3]$ with $\eta > 0$. This fact can be verified directly and elementarily: first follow Y_1 for a time $s_1 = r_2/2 + r_3/4\eta$ to reach $[0, s_1, 0]$, then X_1 for a time $s_2 = \eta > 0$ to $[\eta, s_1, r_2\eta + r_3/2]$ and, finally, Y_1 once more for a time $s_3 = r_2/2 - r_3/4\eta$ to reach $[\eta, r_2, r_3]$. To complete the proof, $\forall t \in \mathbb{R}$ let

$$l(t) = \inf\{\eta : \text{there is } R = [\eta, r_2, r_3] \text{ such that } g(R) \geq t\}$$

then from (39) and the definition of $l(t)$

$$g(P) \geq t \quad \forall P \text{ with } p_1 > l(t) \qquad \text{and} \qquad g(P) < t \quad \forall P \text{ with } p_1 < l(t)$$

hence for the level set $E_t := \{P : g(P) \geq t\}$ we have

$$\{P : p_1 > l(t)\} \subset E_t \subset \{P : p_1 \geq l(t)\}.$$

Suppose now $n \geq 2$; in this case it is possible to reduce ourselves to an Euclidean argument. Let us point preliminarily that there exist two vectors $u, v \in \mathbb{R}^{2n} \equiv \mathrm{H}\mathbb{H}^n_0$ both orthogonal to ν_F (in the Euclidean sense) such that $\omega(u,v) \neq 0$, where ω is the canonical symplectic form on $\mathbb{R}^{2n}$. Indeed, let W be the Euclidean orthogonal complement of ν_F in $\mathbb{R}^{2n}$, so that $\dim W = 2n-1$. On the other hand (see e.g. [8] p. 18),

$$\dim \{u \in \mathbb{R}^{2n} \, : \, \omega(v,u) = 0 \text{ for all } v \in W\} = 1.$$

Thus, our statement follows since $\dim W > 1$. Since

$$[L_u, L_v] = -4 \, \omega(u,v) \, T,$$

then $\mathbf{1}_F$ is independent of the variable t, and thus the statement is reduced to the corresponding Euclidean one in $\mathbb{R}^{2n}$. □

Finally we prove that the point $\tilde{Q}$ in Claim 3 can be chosen to be the origin.

Claim 4.

$$F = S^+_{\mathbb{H}}(\nu_E(0))$$

Proof of Claim 4. It is enough to prove that $\langle \pi_0 \tilde{Q}, \nu_E(0) \rangle_0 = 0$ because this yields that $\tau_{\tilde{Q}} S^+_{\mathbb{H}}(\nu_E(0)) = S^+_{\mathbb{H}}(\nu_E(0))$.

First we show that since $0 \in \partial^*_{\mathbb{H}} E$ then there is $k = k(n) > 0$ such that

$$\liminf_{r \to 0} \, \min \left\{ \frac{|E \cap U(0,r)|}{|U(0,r)|}, \frac{|E^c \cap U(0,r)|}{|U(0,r)|} \right\} = k. \tag{40}$$

Indeed, from the definition of $0 \in \partial^*_{\mathbb{H}} E$

$$\lim_{r \to 0} \left| \fint_{U(0,r)} \nu_E \, d|\partial E|_{\mathbb{H}} \right| = 1, \tag{41}$$

hence there exist $r_1 > 0$ such that

$$|\partial E|_{\mathbb{H}}(U(0,r)) \leq 2 \left| \int_{U(0,r)} \nu_E \, d|\partial E|_{\mathbb{H}} \right|, \qquad \text{for all } r \in (0, r_1). \tag{42}$$

From Corollary 1 it follows that

$$\int_{U(0,r)} \nu_E \, d|\partial E|_{\mathbb{H}} + \int_{E\cap\partial U(0,r)} C n_r \, d\mathcal{H}^{2n} = 0, \qquad \text{for } L^1\text{-a.e. } r > 0, \quad (43)$$

where n_r is the euclidean norm to $\partial U(0,r)$. From (42), (43) and Lemma 2 it follows that for L^1-a.e. $r > 0$

$$|\partial E|_{\mathbb{H}}(U(0,r)) \leq 2 \int_{E\cap\partial U(0,r)} |C n_r| \, d\mathcal{H}^{2n} = c_1 r^{2n+1}. \qquad (44)$$

Inequality (44) actually holds for all $r \in (0,r_1)$. Indeed, for each $r \in (0,r_1)$, take a sequence $\{r_h\}_h \subset (0,r_1)$, $r_h \downarrow r$, with the property that (44) holds for each r_h. Then the by the semicontinuity of the perimeter

$$\begin{aligned}|\partial E|_{\mathbb{H}}(U(0,r)) &= |\partial(E\cap U(0,r))|_{\mathbb{H}}(U(0,r)) \\ &\leq \liminf_{h\to+\infty} |\partial(E\cap U(0,r_h))|_{\mathbb{H}}(U(0,r)) \\ &\leq \liminf_{h\to+\infty} |\partial(E\cap U(0,r_h))|_{\mathbb{H}}(U(0,r_h)) \leq c_2 r^{2n+1}.\end{aligned}$$

As in Corollary 2 we set $m(r) := \int_{E\cap U(0,r)} |\nabla_{\mathbb{H}}(\|P\|_\infty)| \, dh$, then from Proposition 7 the derivative of m exists and

$$m'(r) = \int_{\partial U(0,r)} \mathbf{1}_E |C n_r| \, d\mathcal{H}^{2n}, \qquad \text{for } L^1\text{-a.e. } r > 0. \qquad (45)$$

The global isoperimetric inequality (24) applied to $E\cap U(0,r)$ gives

$$|E\cap U(0,r)|^{\frac{2n+1}{2n+2}} \leq c_3 |\partial(E\cap U(0,r))|_{\mathbb{H}}(\mathbb{H}^n)$$

from (22) it follows

$$\leq c_4(|\partial E|_{\mathbb{H}}(U(0,r)) + m'(r))$$

and from (44) and (45) we have

$$\leq c_5 m'(r).$$

Hence we have obtained the differential inequality

$$m(r)^{\frac{2n+1}{2n+2}} \leq c_4 m'(r), \qquad \text{for } L^1\text{-a.e. } r \in (0,r_1),$$

that in turn yields

$$m(r) \geq c_6 r^{2n+2}, \qquad \forall r \in (0,r_1).$$

Thus

$$\liminf_{r\to 0} \frac{|E\cap U(0,r)|}{|U(0,r)|} = k(n) > 0.$$

The same argument with E^c substituting E gives

$$\liminf_{r\to 0} \frac{|E^c\cap U(0,r)|}{|U(0,r)|} = k(n) > 0$$

and the proof of (40) is complete.

To conclude the proof of Claim 4, assume, by contradiction, that $\langle \pi_0\tilde{Q}, \nu_E(0)\rangle_0 > 0$ then, if $0 < \bar{r} < \langle \pi_0\tilde{Q}, \nu_E(0)\rangle_0$, clearly $U(0,\bar{r}) \cap \tau_{\tilde{Q}}(S^+_{\mathbb{H}}(\nu_E(0))) = \emptyset$. On the contrary, from Claim 2, Claim 3 and (40) it follows

$$|U(0,\bar{r}) \cap \tau_{\tilde{Q}}(S^+_{\mathbb{H}}(\nu_E(0)))| = \lim_{j\to+\infty} |U(0,\bar{r}) \cap E_{s_j}| \geq k\,|U(0,\bar{r})| > 0.$$

If we assume $\langle \pi_0\tilde{Q}, \nu_E(0)\rangle_0 < 0$ the contradiction is obtained working with E^c. This concludes the proof of Claim 4 that is of (28).

The second and last statement of the Blow-up Theorem is point (iii) of the following Lemma.

Lemma 4. *Let $P \in \partial^*_{\mathbb{H}} E$. Then*

$$\lim_{r\to 0} \frac{|U(P,r)\cap E\cap \tau_P(S^-_{\mathbb{H}}(\nu_E(P)))|}{|U(P,r)|} = 0 \qquad (i)$$

$$\lim_{r\to 0} \frac{|(U(P,r)\setminus E)\cap \tau_P(S^+_{\mathbb{H}}(\nu_E(P)))|}{|U(P,r)|} = 0 \qquad (ii)$$

$$\lim_{r\to 0} \frac{|\partial E|_{\mathbb{H}}(U(P,r))}{r^{2n+1}} = 2\omega_{2n-1}. \qquad (iii)$$

Observe that the open balls $U(P,r)$ can be substituted by the closed balls $B(P,r)$.

Proof. By translation, without loss of generality we may assume $P = 0$. By rescaling

$$\begin{aligned}\lim_{r\to 0} \frac{|U(0,r)\cap E\cap S^-_{\mathbb{H}}(\nu_E(0))|}{|U(0,r)|} &= \lim_{r\to 0} \int_{\mathbb{H}^n} \mathbf{1}_{U(0,1)} \mathbf{1}_{E_r} \mathbf{1}_{S^-_{\mathbb{H}}(\nu_E(0))}\, dh \\ &= \int_{\mathbb{H}^n} \mathbf{1}_{U(0,1)} \mathbf{1}_{S^+_{\mathbb{H}}(\nu_E(0))} \mathbf{1}_{S^-_{\mathbb{H}}(\nu_E(0))}\, dh = 0.\end{aligned}$$

This proves (i) and also (ii) replacing E with E^c. To prove (iii) observe that from the homogeneity of the perimeter (see Remark 3) we get

$$\lim_{r\to 0}\frac{|\partial E|_{\mathbb{H}}(U(0,r))}{r^{2n+1}} = \lim_{r\to 0}|\partial E_r|_{\mathbb{H}}(U(0,1)).$$

Following the same computations as in Claim 2

$$\nu_{E_r}\, d|\partial E_r|_{\mathbb{H}} \rightharpoonup \nu_E(0)\, d|\partial S^+_{\mathbb{H}}(\nu_E(0))|_{\mathbb{H}} = \nu_E(0)\, d(\mathcal{H}^{2n} \llcorner \partial S^+_{\mathbb{H}}).$$

The last equality follows from Proposition 5. Now, because

$$|\partial S^+_{\mathbb{H}}(\nu_E(0))|_{\mathbb{H}}(\partial U(0,1)) = 0,$$

it follows that

$$\lim_{r\to 0}\int_{U(0,1)} \nu_{E_r}\, d|\partial E_r|_{\mathbb{H}} = \nu_E(0)\int_{U(0,1)} d|\partial S^+_{\mathbb{H}}(\nu_E(0))|_{\mathbb{H}}. \tag{46}$$

Finally, since $0 \in \partial^* E$, we have also that for $r \to 0$

$$\int_{U(0,1)} \nu_{E_r}\, d|\partial E_r|_{\mathbb{H}} = (1+o(1))\ \nu_E(0)\ |\partial E_r|_{\mathbb{H}}(U(0,1))$$

that together with (46) gives

$$\begin{aligned}\lim_{r\to 0}|\partial E_r|_{\mathbb{H}}(U(0,1)) &= |\partial S^+_{\mathbb{H}}(\nu_E(0))|_{\mathbb{H}}(U(0,1)) \\ &= \mathcal{H}^{2n}(\partial S^+_{\mathbb{H}} \cap U(0,1)) = 2\omega_{2n-1}.\end{aligned}$$

□

5. Calculus in the Heisenberg group

Let us give now some elementary definitions and results concerning intrinsic differentiability in the Heisenberg group. These results are basically due to P. Pansu ([48]), or are inspirated by his ideas.

Definition 8. *We shall say that a map L from $\mathbb{H}^n$ to $\mathbb{R}$ is $\mathbb{H}$-linear if it is a homomorphism and if it is positively homogeneous of degree 1 with respect to the dilations of $\mathbb{H}^n$.*

Definition 9. *Let Ω be an open set in $\mathbb{H}^n$. We shall say $f : \Omega \to \mathbb{R}$ is Pansu-differentiable (differentiable in the sense of Pansu: see [48] and [41]) at P_0 if there exists a $\mathbb{H}$-linear map L from $\mathbb{H}^n$ to $\mathbb{R}$ such that*

$$\lim_{P\to P_0}\frac{f(P)-f(P_0)-L(P_0^{-1}\cdot P)}{d(P,P_0)} = 0.$$

Remark 5. The above definition is equivalent to the following one: there exists a homomorphism L from $\mathbb{H}^n$ to $\mathbb{R}$ such that

$$\lim_{\lambda\to 0+} \frac{f(\tau_{P_0}(\delta_\lambda v)) - f(P_0)}{\lambda} = L(v)$$

uniformly with respect to v belonging to a compact set in $\mathbb{H}^n$. In particular, L is unique and we shall write $L = d_{\mathbb{H}} f(P_0)$.

Proposition 8. *A map L from $\mathbb{H}^n$ to $\mathbb{R}$ is $\mathbb{H}$-linear if and only if there exists $(a,b) \in \mathbb{R}^{2n}$ such that, if $v = [x+iy,t] \in \mathbb{H}^n$, then $L(v) = \langle (a,b),(x,y)\rangle_{\mathbb{R}^{2n}}$.*

Proof. It is clear that any function L of the form $L(v) = \langle (a,b),(x,y)\rangle_{\mathbb{R}^{2n}}$ is $\mathbb{H}$-linear. Let now L be $\mathbb{H}$-linear; for any $t\in\mathbb{R}$ we have:

$$2L([0,t]) = L([0,t]\cdot[0,t]) = L([0,2t]) = \sqrt{2}L([0,t]),$$

and hence $L([0,t]) = 0$. Thus $L([z,t]) = L([z,0]) + L([0,t]) = L([z,0])$. By our hypotheses, the map $z \to L([z,0])$ is additive and positively homogeneous and hence is linear. Thus the statement follows from the usual representation of linear functionals in the Euclidean space.

□

Definition 10. *With the notations of Definition 9, we shall say that f is differentiable along X_j (Y_j) at P_0 if the map $\lambda \mapsto f(\tau_{P_0}(\delta_\lambda e_j))$ (respectively: $\lambda \mapsto f(\tau_{P_0}(\delta_\lambda e_{j+n}))$) is differentiable at $\lambda = 0$, where e_k is the k-th vector of the canonical basis of $\mathbb{R}^{2n+1}$.*

Clearly if $f \in \mathbf{C}^1(\Omega)$ then f is differentiable along X_j and Y_j at all points of Ω. Hence if, given f differentiable along X_j and Y_j at P_0 for $j = 1,\dots,n$, we define

$$\nabla_{\mathbb{H}} f = \sum_{j=1}^n (X_j f)X_j + \sum_{j=1}^n (Y_j f)Y_j \tag{47}$$

then this definition naturally extends the one given in (3) of Definition 3.

Proposition 9. *With the notations of Definition 9 and Proposition 8, if f is Pansu-differentiable at P_0, then it is differentiable along X_j and Y_j at P_0 for $j = 1,\dots,n$, and*

$$d_{\mathbb{H}} f(P_0)(v) = \langle \nabla_{\mathbb{H}} f, \pi_{P_0}(v)\rangle_{P_0}. \tag{48}$$

Proof. By definition, if $j = 1, \dots, n$, then

$$d_{\mathbb{H}} f(P_0)(e_j) = X_j f(P_0) \text{ and } d_{\mathbb{H}} f(P_0)(e_{n+j}) = Y_j f(P_0).$$

Then, by Proposition 8, $d_{\mathbb{H}} f(P_0)([z,t]) = \langle (a,b),(x,y)\rangle_{\mathbb{R}^{2n}}$, where $(a,b) = \nabla_{\mathbb{H}} f(P_0)$, and the assertion follows.

□

Definition 11. *If $\Omega \subset \mathbb{H}^n$, we shall denote by $\mathbf{C}^1_{\mathbb{H}}(\Omega)$ the set of continuous real functions in Ω such that $\nabla_{\mathbb{H}} f$ is continuous in Ω. Moreover, we shall denote by $\mathbf{C}^1_{\mathbb{H}}(\Omega, \mathrm{H}\mathbb{H}^n)$ the set of all sections ϕ of $\mathrm{H}\mathbb{H}^n$ whose canonical coordinates $\phi_j \in \mathbf{C}^1_{\mathbb{H}}(\Omega)$ for $j = 1, \dots, 2n$.*

The following proposition can be proved via an approximation argument.

Proposition 10. *With the notations of Definition 11, a continuous function belongs to $\mathbf{C}^1_{\mathbb{H}}(\Omega)$ if and only if its distributional derivatives $X_j f$, $Y_j f$ are continuous in Ω for $j = 1, \dots, n$.*

Proof. Suppose $f \in \mathbf{C}^1_{\mathbb{H}}(\Omega)$; arguing e.g. as in [31], if J_ϵ is an usual Friedrichs' mollifier, we can show that $f * J_\epsilon$ converges uniformly on any compact set to f, and $X_j(f * J_\epsilon)$ converges locally to $X_j f$ in any L^p, $j = 1, \dots, n$, so that $X_j f$ coincide with $X_j f$ in the sense of distributions, $j = 1, \dots, n$. The same arguments can be applied to the Y_j's. On the other hand, if $X_j f$ and $Y_j f$ (in the sense of distributions) are continuous, it is known ([36], [28]) that f is Lipschitz continuous with respect to d and hence, differentiable a.e. along X_j and Y_j, $j = 1, \dots, n$, by the Rademacher type theorem in $\mathbb{H}^n$ due to Pansu [48] and Proposition 9.

□

Remark 6. $\mathbf{C}^1(\Omega) \subset \mathbf{C}^1_{\mathbb{H}}(\Omega)$, and the inclusion is strict. Indeed, consider the following example: choose $n = 1$, and $f([z,t]) := x - g(y, 2xy + t)$, where

$$g(p,q) := \frac{|p|^\alpha q}{p^4 + q^2} \quad \text{if} \quad (p,q) \neq (0,0) \quad \text{and} \quad g(0,0) = 0.$$

Then $f \in \mathbf{C}^1_{\mathbb{H}}(\Omega)$ if $3 < \alpha < 4$, but f is not locally Lipschitz continuous with respect to the Euclidean metric of $\mathbb{R}^3$.

We shall discuss again this example at the end of next section in the context of regularity of implicitely defined functions

Finally we state a typical elementary calculus result. Its proof inside the Heisenberg group, is as elementary as it is in $\mathbb{R}^n$.

Theorem 7. *If $f \in \mathbf{C}^1_{\mathbb{H}}(\Omega)$ then f is Pansu-differentiable at any point $P_0 \in \Omega$.*

Proof. By the connectivity property of the vector fields we get that there are geometric constants c and c_n such that, for any $g \in \mathbf{C}^1_{\mathbb{H}}(\Omega)$ and for any P and $P_0 \in \Omega$, we have

$$|g(P) - g(P_0)| \le 2cr\|\nabla_{\mathbb{H}} g\|_{L^\infty(U_C(P_0,c_n r))}.$$

Then take

$$g(P) := f(P) - f(P_0) - d_{\mathbb{H}} f(P_0)(P - P_0),$$

and the theorem follows.

□

6. $\mathbb{H}$-rectifiable sets, Implicit function theorem and Whitney type theorem

Definition 12. *We shall say that $S \subset \mathbb{H}^n$ is a $\mathbb{H}$-regular hypersurface if for every $P \in S$ there exist an open ball $U(P,r)$ and a function $f \in \mathbf{C}^1_{\mathbb{H}}(U(P,r))$ such that*

$$S \cap U(P,r) = \{Q \in U(P,r) : f(Q) = 0\}; \tag{i}$$

$$\nabla_{\mathbb{H}} f(P) \neq 0. \tag{ii}$$

Remark 7. We point out that the classes of Euclidean regular hypersurfaces and of $\mathbb{H}$-regular hypersurfaces are disjoint. On one side, in $\mathbb{R}^3 \simeq \mathbb{H}^1$, the Euclidean plane $\{(x,y,t) : t = 0\}$ is a smooth Euclidean submanifold but is not $\mathbb{H}$-regular at the origin. On the other side, a routine computation shows that $\Gamma := \{(x,y,t) : g(x,y,t) \equiv x - \sqrt{x^4 + y^4 + t^2} = 0\}$ is a $\mathbb{H}$-regular hypersurface since $g \in \mathbf{C}^1_{\mathbb{H}}$ and $Xg(0,0,0) = 1$ but Γ is not an Euclidean $\mathbf{C}^1$-submanifold at the origin. Obviously regular Euclidean hypersurfaces whose points are never characteristic (a point P is said to be characteristic for an hypersurface Σ if the Euclidean tangent plane to Σ at P coincides with the horizontal fiber $\mathrm{H}\mathbb{H}^n_P$) are $\mathbb{H}$-regular.

Definition 13. *If $S \subset \mathbb{H}^n$ is a $\mathbb{H}$-regular hypersurface and $P \in S$, we define the tangent group $T^g_{\mathbb{H}} S(P)$ to S at P as follows*

$$T^g_{\mathbb{H}} S(P) := \{Q : \langle \nabla_{\mathbb{H}}(f \circ \tau_P)(0), \pi_0(Q)\rangle_0 = 0\}.$$

By (ii) of Definition 12, $T^g_{\mathbb{H}} S(P)$ is a proper subgroup of $\mathbb{H}^n$. Then the tangent plane to S at P is the lateral

$$T_{\mathbb{H}} S(P) := P \cdot T^g_{\mathbb{H}} S(P).$$

Once more, observe that this is a good definition. Indeed the tangent plane does not depend on the particular function f defining the surface S because of point (iii) of Theorem 8.

Definition 14. *We shall say that $\varGamma \subset \mathbb{H}^n$ is ($(Q-1)$-dimensional) $\mathbb{H}$-rectifiable if there exists a sequence of $\mathbb{H}$-regular hypersurfaces $(S_j)_{j\in\mathbb{N}}$ such that*

$$\mathcal{H}_d^{Q-1}\left(\varGamma \setminus \bigcup_{j\in\mathbb{N}} S_j\right) = 0. \tag{49}$$

Theorem 8 (Implicit Function Theorem). *Let $\varOmega$ be an open set in $\mathbb{H}^n$, $0 \in \varOmega$, and let $f \in \mathbf{C}^1_{\mathbb{H}}(\varOmega)$ be such that $X_1 f(0) > 0$, $f(0) = 0$. Then, if*

$$E = \{[z,t] \in \varOmega : f([z,t]) < 0\}, \quad \varGamma = \{[z,t] \in \varOmega : f([z,t]) = 0\},$$

there exists a neighborhood $\mathcal{U} \ni 0$,

$$\begin{aligned} \mathcal{U} = \{[x+iy,t] : &|x_1| < h, |y_1| < \delta, |x_j| < \delta, |y_j| < \delta, j = 2,\dots,n, \\ &|t - 2x_1y_1| = |\tau| < \delta^2\}, \end{aligned}$$

such that

$$E \text{ has finite } \mathbb{H}\text{-perimeter in } \mathcal{U}; \tag{i}$$

$$\partial E \cap \mathcal{U} = \partial^*_{\mathbb{H}} E \cap \mathcal{U} = \varGamma \cap \mathcal{U}; \tag{ii}$$

$$\nu_E(P) = -\nabla_{\mathbb{H}} f(P)/|\nabla_{\mathbb{H}} f(P)|_P \qquad \text{for all } P \in \varGamma \cap \mathcal{U}. \tag{iii}$$

If we put now $I = [-\delta,\delta]^{2n-1} \times [-\delta^2,\delta^2]$, $J = [-h,h]$, $\hat{\xi}_1 = (\xi_2,\dots,\xi_n)$, then there exists a unique function

$$\phi = \phi(\hat{\xi}_1, \eta, \tau) : I \to J$$

such that the following parametrization holds

$$\begin{aligned} \varGamma \cap \bar{\mathcal{U}} = \{[z,t] \in \bar{\mathcal{U}} : &[x_1, \hat{x}_1, y, t] \\ &= [\phi(\hat{\xi}_1,\eta,\tau), \hat{\xi}_1, \eta, 2\phi(\hat{\xi}_1,\eta,\tau)\eta_1 + \tau]; (\hat{\xi}_1,\eta,\tau) \in I\}; \end{aligned} \tag{iv}$$

$$\phi \text{ is continuous;} \tag{v}$$

the $\mathbb{H}$-perimeter has an integral representation

$$|\partial E|_{\mathbb{H}}(\mathcal{U}) = \int_I \frac{|\nabla_{\mathbb{H}} f|}{X_1 f}(\phi, \hat{\xi}_1, \eta, 2\phi\eta_1 + \tau)\, d\hat{\xi}_1\, d\eta\, d\tau, \tag{vi}$$

where $\phi = \phi(\hat{\xi}_1, \eta, \tau)$.

Remark 8. The example in 7 shows that we cannot expect in general that the implicitely defined function ϕ is an Euclidean Lipschitz function. Detailed computations can be found in Example 3.

Proof of Theorem 8.
(Step 1) Existence of the implicitely defined function ϕ: Choose δ so that $X_1 f > 0$ in $\bar{\mathcal{U}}$, and let J_ϵ be an usual Friedrichs' mollifier. If we put $f_\epsilon = f * J_\epsilon$, then $f_\epsilon \to f$ as $\epsilon \to 0$ uniformly on $\bar{\mathcal{U}}$, because of the continuity of f. Analogously, $(X_j f) * J_\epsilon \to X_j f$ and $(Y_j f) * J_\epsilon \to Y_j f$ as $\epsilon \to 0$ uniformly on $\bar{\mathcal{U}}$ for $j = 1, \ldots, n$. We follow now the same argument as in p. 90 and 91 of [31] and we have

$$X_j f_\epsilon = (X_j f) * J_\epsilon - ((X_j f) * J_\epsilon - X_j f_\epsilon)$$

for $j = 1, \ldots, n$. To prove that $(X_j f) * J_\epsilon - X_j f_\epsilon \to 0$ as $\epsilon \to 0$ uniformly on $\bar{\mathcal{U}}$ observe that

$$\begin{aligned}
&|((X_j f) * J_\epsilon - X_j f_\epsilon)([z,t])| \\
&= \left| \int_{|(\zeta - z, \tau - t)| < \epsilon} (x_j - \xi_j) f([\zeta, \tau]) \partial_{2n+1} J_\epsilon(\zeta - z, \tau - t)\, d\zeta\, d\tau \right| \\
&\le \int_{|(\zeta - z, \tau - t)| < \epsilon} |x_j - \xi_j|\, |f([\zeta, \tau]) - f([z,t])|\, |\partial_{2n+1} J_\epsilon(\zeta - z, \tau - t)|\, d\zeta\, d\tau \\
&+ \left| f([z,t]) \int_{|(\zeta - z, \tau - t)| < \epsilon} (x_j - \xi_j) \partial_{2n+1} J_\epsilon(\zeta - z, \tau - t)\, d\zeta\, d\tau \right| = I_1 + I_2
\end{aligned}$$

where $\zeta = (\xi + i\eta)$. As in [30], I_2 vanishes, and, if we denote by ω the modulus of continuity of f, I_1 can be bounded by

$$C\epsilon^{n+1} \omega(\epsilon) \sup_{|P| \le 1} |\partial_{2n+1} J_\epsilon(P)| \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Since the same argument can be carried out for Y_j, we obtain

$$X_j f_\epsilon \to X_j f \quad , \quad Y_j f_\epsilon \to Y_j f \quad \text{as } \epsilon \to 0 \tag{50}$$

uniformly on $\bar{\mathcal{U}}$ for $j = 1, \ldots, n$. In particular, if $(\hat{\xi}_1, \eta, \tau) \in I$,

$$\frac{\partial}{\partial s} f_\epsilon(s, \hat{\xi}_1, \eta, 2s\eta_1 + \tau) = (X_1 f_\epsilon)(s, \hat{\xi}_1, \eta, 2s\eta_1 + \tau)$$

converges, uniformly with respect to s, to $(X_1 f)(s, \hat{\xi}_1, \eta, 2s\eta_1 + \tau)$. Hence the map

$$s \mapsto f(s, \hat{\xi}_1, \eta, 2s\eta_1 + \tau)$$

is differentiable for all s, $|s| \le h$ and

$$\frac{\partial}{\partial s} f(s, \hat{\xi}_1, \eta, 2s\eta_1 + \tau) = (X_1 f)(s, \hat{\xi}_1, \eta, 2s\eta_1 + \tau) > 0$$

when $|\hat{\xi}_1| \le \delta$, $|\eta| \le \delta$, $|\tau| \le \delta^2$. Since f vanishes for $s = 0$, $\hat{\xi}_1 = 0$, $\eta = 0$, $\tau = 0$, then $f(h, 0, 0, 0) > 0 > f(-h, 0, 0, 0)$ and by continuity,

$$f(h, \hat{\xi}_1, \eta, 2h\eta_1 + \tau) > 0 > f(-h, \hat{\xi}_1, \eta, -2h\eta_1 + \tau)$$

when $|\hat{\xi}_1| \le \delta$, $|\eta| \le \delta$, $|\tau| \le \delta^2$ if δ is small enough. Now the existence of ϕ such that (iv) holds follows by a continuity argument.

(Step 2) ϕ is continuous: It is enough to show that, if $(\hat{\xi}_1, \eta, \tau) \in I$, and $(\hat{\xi}_{k1}, \eta_k, \tau_k) \to (\hat{\xi}_1, \eta, \tau)$, then there exists a subsequence such that $\phi(\widehat{\xi_{k_j 1}}, \eta_{k_j}, \tau_{k_j}) \to \phi(\hat{\xi}_1, \eta, \tau)$. In fact, by the compactness of J, we can extract a subsequence $\phi(\widehat{\xi_{k_j 1}}, \eta_{k_j}, \tau_{k_j})$ converging to $\phi_0 \in J$. By the continuity of f

$$\begin{aligned} 0 = & f(\phi(\widehat{\xi_{k_j 1}}, \eta_{k_j}, \tau_{k_j}), \widehat{\xi_{k_j 1}}, \eta_{k_j}, 2\eta_1 \phi(\hat{\xi}_{k_j 1}, \eta_{k_j}, \tau_{k_j}) + \tau_{k_j}) \\ & \longrightarrow f(\phi_0, \hat{\xi}_1, \eta, 2\eta\phi_0 + \tau), \end{aligned}$$

and then $\phi_0 = \phi(\hat{\xi}_1, \eta, \tau)$, by (iv). This proves (v).

(Step 3) $\mathcal{U} \cap \partial E = \mathcal{U} \cap \Gamma$: By the continuity of f, clearly $\partial E \subseteq \Gamma$. On the other hand, let $(x, y, t) \in \Gamma \cap \mathcal{U}$ be given; then $t = 2x_1 y_1 + \tau$ for some τ, $|\tau| < \delta^2$. As we proved above, the function $s \to f(s, \hat{x}_1, y, 2sy_1 + \tau)$ is strictly increasing and vanishes for $s = x_1$. Then there is a sequence $s_k \to x_1$ such that the points $(s_k, \hat{x}_1, y, 2s_k y_1 + \tau) \in E$ and converge to (x, y, t), so that $\Gamma \subseteq \partial E$. This proves the second part of (ii).

Remark 9.

$$\partial E \cap \mathcal{U} = \bigcap_{n=1}^{\infty} \tilde{\Gamma}_n \qquad \text{where}$$

$\tilde{\Gamma}_n := \{(x, y, t) \in \mathcal{U} : [x_1, \hat{x}_1, y, t] = [s, \hat{\xi}_1, \eta, 2s\eta_1 + \tau],\ (\hat{\xi}_1, \eta, \tau) \in I, |\phi(\hat{\xi}_1, \eta, \tau) - s| < \frac{1}{n}\}$.

Since

$$\left| \det \frac{\partial(x, y, t)}{\partial(s, \hat{\xi}_1, \eta, \tau)} \right| \equiv 1,$$

then $|\tilde{\Gamma}_n| = \frac{2}{n}|I|$, and hence $|\partial E \cap \mathcal{U}| = 0$.

(Step 4) E has finite $\mathbb{H}$-perimeter in $\mathcal{U}$: The proof of this step is based on the construction of a sequence $\{h_\epsilon\}$, bounded in $BV_{\mathbb{H}}(\mathcal{U})$ and convergent to $\mathbf{1}_E$ in $L^1(\mathcal{U})$.
Let us consider again the approximations f_ϵ we introduced at the beginning of the proof, and consider the functions g and $g_\epsilon : J \times I \to \mathbb{R}$ defined as

$$g(\xi,\eta,\tau) := f(\xi,\eta,2\xi_1\eta_1+\tau) \quad \text{and} \quad g_\epsilon(\xi,\eta,\tau) := f_\epsilon(\xi,\eta,2\xi_1\eta_1+\tau).$$

As we showed above, $\dfrac{\partial g_\epsilon}{\partial \xi_1}(\xi,\eta,\tau)$ converges uniformly to $X_1 f(\xi,\eta,2\xi_1\eta_1+\tau)$ on $J\times I$, so that there are $\mu>0$, $\epsilon_0>0$ such that $\dfrac{\partial g_\epsilon}{\partial \xi_1} \geq \mu > 0$ on $J \times I$ for all $0<\epsilon<\epsilon_0$. Thus we can apply classical implicit function theorem in $J\times I$ to obtain the existence of a smooth function $\phi_\epsilon : I \to J$ such that $g_\epsilon(\phi_\epsilon(\hat{\xi}_1,\eta,\tau),\hat{\xi}_1,\eta,\tau) \equiv 0$. Note that the choice of δ and h, defining I and J, depends on μ and not on ϵ: indeed $g_\epsilon(h,0,0,0) \geq h\mu > 0$ and $g_\epsilon(-h,0,0,0) \leq -h\mu < 0$ for all $\epsilon \in (0,\epsilon_0)$ and we have only to show that (for instance) there is $\delta_1>0$ independent of ϵ such that $g_\epsilon(h,\hat{\xi}_1,\eta,\tau) > 0$ for $(\hat{\xi}_1,\eta,\tau) \in I$. Since $g_\epsilon \to g$ uniformly then $\sup|g_\epsilon - g| < h\mu/3$ if $\epsilon < \epsilon_1(h,\mu)$; $|g(h,\hat{\xi}_1,\eta,\tau) - g(h,0,0,0)| < h\mu/3$ if $\delta < \delta_0(h,\mu)$. Thus $|g_\epsilon(h,\hat{\xi}_1,\eta,\tau) - g_\epsilon(h,0,0,0)| < h\mu$ for $\epsilon<\epsilon_1$, $\delta<\delta_1$, so that $g_\epsilon(h,\hat{\xi}_1,\eta,\tau) > g_\epsilon(h,0,0,0) - |g_\epsilon(h,\hat{\xi}_1,\eta,\tau) - g_\epsilon(h,0,0,0)| > 0$.

Then define $h_\epsilon : \mathcal{U} \to \mathbb{R}$ and $E_\epsilon \subset \mathbb{H}^n$ as

$$h_\epsilon(x,y,t) = H(\phi_\epsilon(\hat{x}_1,y,t-2x_1y_1) - x_1)$$

where $H(s)=1$ for $s>0$ and $H(s)=0$ for $s\leq 0$, and

$$E_\epsilon = \{(x,y,t) : x_1 < \phi_\epsilon(\hat{x}_1,y,t-2x_1y_1)\}.$$

Then

$$h_\epsilon = \mathbf{1}_{E_\epsilon} \to \mathbf{1}_E \quad \text{in } L^1(\mathcal{U}), \qquad \text{as } \epsilon \to 0. \tag{51}$$

Indeed, by dominate convergence theorem, we need only to prove that $h_\epsilon \to \mathbf{1}_E$ a.e. To this end, let us prove first that $\phi_\epsilon \to \phi$ uniformly on I. By contradiction, suppose there exist $\sigma>0$, $\epsilon_k \to 0$, $(\hat{\xi}_{k1},\eta_k,\tau_k) \in I$ such that

$$|\phi_{\epsilon_k}(\hat{\xi}_{k1},\eta_k,\tau_k) - \phi(\hat{\xi}_{k1},\eta_k,\tau_k)| \geq \sigma.$$

Without loss of generality we may assume that

$$(\hat{\xi}_{k1},\eta_k,\tau_k) \to (\hat{\xi}_1,\eta,\tau) \quad \text{and} \quad \phi_{\epsilon_k}(\hat{\xi}_{k1},\eta_k,\tau_k) \to \phi_0$$

as $k \to \infty$, so that $|\phi_0 - \phi(\hat{\xi}_1, \eta, \tau)| \geq \sigma$. On the other hand

$$\begin{aligned} 0 =& f_{\epsilon_k}(\phi_{\epsilon_k}(\hat{\xi}_{k1}, \eta_k, \tau_k), \hat{\xi}_{k1}, \eta_k, 2\phi_{\epsilon_k}(\hat{\xi}_{k1}, \eta_k, \tau_k)\eta_{k1} + \tau_k) \\ & \longrightarrow f(\phi_0, \hat{\xi}_1, \eta, 2\phi_0\eta_1 + \tau), \end{aligned}$$

since $f_{\epsilon_k} \to f$ uniformly on $\bar{\mathcal{U}}$, and hence $\phi_0 = \phi(\hat{\xi}_1, \eta, \tau)$, by (iv), a contradiction.

Take now $(x, y, t) \in E \cap \mathcal{U}$; by definition of $\mathcal{U}$, $t = 2x_1y_1 + \tau$, with $|\tau| < \delta^2$. On the other hand, by definition of ϕ,

$$\begin{gathered} f(\phi(\hat{x}_1, y, \tau), \hat{x}_1, y, 2\phi(\hat{x}_1, y, \tau)y_1 + \tau) = 0 \\ f(x, y, t) = f(x_1, \hat{x}_1, y, 2x_1y_1 + \tau) < 0, \end{gathered}$$

and hence $\phi(\hat{x}_1, y, \tau) > x_1$, since the map

$$s \mapsto f(s, \hat{x}_1, y, 2sy_1 + \tau)$$

is strictly increasing on $[-h, h]$. Therefore $\phi_\epsilon(\hat{x}_1, y, t - 2x_1y_1) > x_1$ if ϵ is close to zero, because of the uniform convergence of ϕ_ϵ to ϕ; this implies that $h_\epsilon(x, y, t) = 1 \to 1 = \mathbf{1}_E(x, y, t)$ as $\epsilon \to 0$. The same argument can be applied if $f(x, y, t) > 0$, and hence a.e. since $|\Gamma \cap \mathcal{U}| = 0$.

We prove now that the h_ϵ's have equibounded variations.
Indeed, if $\psi \in \mathbf{C}_0^\infty(\mathcal{U}, \mathrm{H}\mathbb{H}^n)$, $|\psi(P)|_P \leq 1$ for all $P \in \mathcal{U}$, then

$$\begin{gathered} \int_{\mathcal{U}} h_\epsilon \mathrm{div}_{\mathbb{H}}\, \psi\, dxdydt = \int_{\mathcal{U} \cap E_\epsilon} \mathrm{div}_{\mathbb{H}}\, \psi\, dxdydt \\ \int_{\mathcal{U} \cap \partial E_\epsilon} \langle \psi, Cn \rangle_{\mathbb{R}^{2n}}\, d\mathcal{H}^{2n} \leq \int_{\mathcal{U} \cap \partial E_\epsilon} |Cn|_{\mathbb{R}^{2n}}\, d\mathcal{H}^{2n}. \end{gathered}$$

On the other hand, a parameterization of $\mathcal{U} \cap \partial E_\epsilon$ is given by $\Phi_\epsilon : I \to \mathbb{R}$ defined as

$$\Phi_\epsilon(\hat{\xi}_1, \eta, \tau) := (\phi_\epsilon(\hat{\xi}_1, \eta, \tau), \hat{\xi}_1, \eta, 2\eta_1\phi_\epsilon(\hat{\xi}_1, \eta, \tau) + \tau).$$

Hence

$$\begin{aligned} & \int_{\mathcal{U}} h_\epsilon \mathrm{div}_{\mathbb{H}}\, \psi\, dxdydt \\ & \quad \leq \int_I \left| C(\Phi_\epsilon)\, \frac{\partial \Phi_\epsilon}{\partial \xi_2} \wedge \cdots \wedge \frac{\partial \Phi_\epsilon}{\partial \xi_n} \wedge \frac{\partial \Phi_\epsilon}{\partial \eta_1} \wedge \cdots \wedge \frac{\partial \Phi_\epsilon}{\partial \eta_n} \wedge \frac{\partial \Phi_\epsilon}{\partial \tau} \right| d\hat{\xi}_1 d\eta d\tau. \end{aligned}$$

The ℓ-th component J_ℓ of $\frac{\partial \Phi_\epsilon}{\partial \xi_2} \wedge \cdots \wedge \frac{\partial \Phi_\epsilon}{\partial \xi_n} \wedge \frac{\partial \Phi_\epsilon}{\partial \eta_1} \wedge \cdots \wedge \frac{\partial \Phi_\epsilon}{\partial \eta_n} \wedge \frac{\partial \Phi_\epsilon}{\partial \tau}$ is $(-1)^{\ell+1}$ times the minor we obtain eliminating the ℓ-th column from ${}^t\mathcal{J}_{\Phi_\epsilon}$, the transposed of the Jacobian matrix of Φ_ϵ. If we denote by e_j^{2n} the j-vector of the canonical basis of $\mathbb{R}^{2n}$, we have

$$
{}^t\mathcal{J}_{\Phi_\epsilon} = \begin{bmatrix}
\frac{\partial\phi_\epsilon}{\partial\xi_2} & 1 \dots 0 & 0 \dots 0 & 2\eta_1\frac{\partial\phi_\epsilon}{\partial\xi_2} \\
\frac{\partial\phi_\epsilon}{\partial\xi_3} & 0 \dots 0 & 0 \dots 0 & 2\eta_1\frac{\partial\phi_\epsilon}{\partial\xi_3} \\
. & \dots . & \dots . & \dots \\
\frac{\partial\phi_\epsilon}{\partial\xi_n} & 0 \dots 1 & 0 \dots 0 & 2\eta_1\frac{\partial\phi_\epsilon}{\partial\xi_n} \\
\frac{\partial\phi_\epsilon}{\partial\eta_1} & 0 \dots 0 & 1 \dots 0 & 2\eta_1\frac{\partial\phi_\epsilon}{\partial\eta_1} + 2\phi_\epsilon \\
. & \dots . & \dots . & \dots \\
\frac{\partial\phi_\epsilon}{\partial\eta_n} & 0 \dots 0 & 0 \dots 1 & 2\eta_1\frac{\partial\phi_\epsilon}{\partial\eta_n} \\
\frac{\partial\phi_\epsilon}{\partial\tau} & 0 \dots 0 & 0 \dots 0 & 2\eta_1\frac{\partial\phi_\epsilon}{\partial\tau} + 1
\end{bmatrix}
$$

$$
= \left[\nabla\phi_\epsilon, e_1^{2n}, \dots, e_{2n-1}^{2n}, e_{2n}^{2n} + 2\eta_1\nabla\phi_\epsilon + 2\phi_\epsilon e_n^{2n}\right].
$$

Thus

$$
J_1 = \det\left[e_1^{2n}, \dots, e_{2n-1}^{2n}, e_{2n}^{2n} + 2\eta_1\nabla\phi_\epsilon + 2\phi_\epsilon e_n^{2n}\right] = 2\eta_1\frac{\partial\phi_\epsilon}{\partial\tau} + 1.
$$

If $1 < \ell < 2n+1$, remember that the ℓ-th column of ${}^t\mathcal{J}_{\Phi_\epsilon}$ is $e_{\ell-1}^{2n}$, and thus

$$
\begin{aligned}
(-1)^{\ell+1}J_\ell &= \det\left[\nabla\phi_\epsilon, e_1^{2n}, \dots, e_{\ell-2}^{2n}, e_\ell^{2n}, \dots, e_{2n}^{2n}\right] \\
&\quad + 2\eta_1 \det\left[\nabla\phi_\epsilon, e_1^{2n}, \dots, e_{\ell-2}^{2n}, e_\ell^{2n}, \dots, \nabla\phi_\epsilon\right] \\
&\quad + 2\phi_\epsilon \det\left[\nabla\phi_\epsilon, e_1^{2n}, \dots, e_{\ell-2}^{2n}, e_\ell^{2n}, \dots, e_n^{2n}\right] \\
&= J_\ell^1 + J_\ell^2 + J_\ell^3.
\end{aligned}
$$

Obviously, $J_\ell^2 = 0$. To evaluate J_ℓ^1, write

$$
\nabla\phi_\epsilon = \sum_{j=2}^n \frac{\partial\phi_\epsilon}{\partial\xi_j} e_{j-1}^{2n} + \sum_{j=1}^n \frac{\partial\phi_\epsilon}{\partial\eta_j} e_{n+j-1}^{2n} + \frac{\partial\phi_\epsilon}{\partial\tau} e_{2n}^{2n},
$$

so that

$$
\begin{aligned}
J_\ell^1 &= \sum_{j=2}^n \frac{\partial\phi_\epsilon}{\partial\xi_j} \det\left[e_{j-1}^{2n}, e_1^{2n}, \dots, e_{\ell-2}^{2n}, e_\ell^{2n}, \dots, e_{2n}^{2n}\right] \\
&\quad + \sum_{j=1}^n \frac{\partial\phi_\epsilon}{\partial\eta_j} \det\left[e_{n+j-1}^{2n}, e_1^{2n}, \dots, e_{\ell-2}^{2n}, e_\ell^{2n}, \dots, e_{2n}^{2n}\right] \\
&\quad + \frac{\partial\phi_\epsilon}{\partial\tau} \det\left[e_{2n}^{2n}, e_1^{2n}, \dots, e_{\ell-2}^{2n}, e_\ell^{2n}, \dots, e_{2n}^{2n}\right].
\end{aligned}
$$

The last term above is clearly zero (first and last columns are equal). Analogously, all terms in the first sum with $j \neq \ell$ vanish, as well as all terms in the second sum with $n+j \neq \ell$. Thus, if $1 < \ell \leq n$, all terms in the second sum vanish since $n+j \neq \ell$ and

$$J_\ell^1 = (-1)^\ell \frac{\partial \phi_\epsilon}{\partial \xi_\ell},$$

whereas, if $n < \ell \leq 2n$, all terms in the first sum vanish since $j \neq \ell$ and

$$J_\ell^1 = (-1)^\ell \frac{\partial \phi_\epsilon}{\partial \eta_{\ell-n}}.$$

Consider now J_ℓ^3. If $n \neq \ell - 1$, then $J_\ell^3 = 0$, since the column e_n^{2n} appears twice (indeed the determinant in J_ℓ^3 contains all vectors e_j^{2n} with $1 \leq j \leq 2n-1$, $j \neq \ell - 1$, and in addition $n \leq 2n-1$). If $\ell = n+1$, then

$$J_{n+1}^3 = (-1)^{n-1} 2\phi_\epsilon \det\left[\nabla\phi_\epsilon, e_1^{2n}, \dots, e_{2n}^{2n-1}\right].$$

We can argue now as in the estimate of J_ℓ^1, writing

$$\begin{aligned}
\det\left[\nabla\phi_\epsilon, e_1^{2n}, \dots, e_{2n}^{2n-1}\right] &= \sum_{j=2}^{n} \frac{\partial \phi_\epsilon}{\partial \xi_j} \det\left[e_{j-1}^{2n}, e_1^{2n}, \dots, e_{2n-1}^{2n}\right] \\
&+ \sum_{j=1}^{n} \frac{\partial \phi_\epsilon}{\partial \eta_j} \det\left[e_{n+j-1}^{2n}, e_1^{2n}, \dots, e_{2n-1}^{2n}\right] \\
&+ \frac{\partial \phi_\epsilon}{\partial \tau} \det\left[e_{2n}^{2n}, e_1^{2n}, \dots, e_{2n-1}^{2n}\right].
\end{aligned}$$

Since in the first sum $j - 1 \leq n - 1 < 2n - 1$, and in the second one $n+j-1 \leq 2n-1$, then in each determinant of the first sum (of the second sum) the vector e_j^{2n} (the vector e_{n+j-1}^{2n}, respectively) appears twice, so that the two sums vanish. On the other hand, the last determinant equals $(-1)^{2n-1} = -1$, so that

$$J_{n+1}^3 = (-1)^n \, 2\phi_\epsilon \frac{\partial \phi_\epsilon}{\partial \tau}.$$

Thus

$$\begin{aligned}
J_\ell &= -\frac{\partial \phi_\epsilon}{\partial \xi_\ell} && \text{if } 2 \leq \ell \leq n, \\
J_{n+1} &= -\frac{\partial \phi_\epsilon}{\partial \eta_1} + 2\phi_\epsilon \frac{\partial \phi_\epsilon}{\partial \tau} && \text{if } \ell = n+1, \\
J_\ell &= -\frac{\partial \phi_\epsilon}{\partial \eta_{\ell+1-n}} && \text{if } n+1 < \ell \leq 2n.
\end{aligned}$$

Finally, if $\ell = 2n+1$, arguing as in the estimate of J_ℓ^1, we have:

$$\begin{aligned} J_\ell &= \sum_{j=2}^{n} \frac{\partial \phi_\epsilon}{\partial \xi_j} \det\left[e_{j-1}^{2n}, e_1^{2n}, \dots, e_{2n-1}^{2n}\right] \\ &+ \sum_{j=1}^{n} \frac{\partial \phi_\epsilon}{\partial \eta_j} \det\left[e_{n+j-1}^{2n}, e_1^{2n}, \dots, e_{2n-1}^{2n}\right] \\ &+ \frac{\partial \phi_\epsilon}{\partial \tau} \det\left[e_{2n}^{2n}, e_1^{2n}, \dots, e_{2n-1}^{2n}\right] = -\frac{\partial \phi_\epsilon}{\partial \tau}. \end{aligned}$$

Thus, eventually,

$$\begin{aligned} &\frac{\partial \Phi_\epsilon}{\partial \xi_2} \wedge \cdots \wedge \frac{\partial \Phi_\epsilon}{\partial \tau} \\ &= \left(2\eta_1 \frac{\partial \phi_\epsilon}{\partial \tau} + 1, -\frac{\partial \phi_\epsilon}{\partial \xi_2}, \dots, -\frac{\partial \phi_\epsilon}{\partial \xi_n}, -\frac{\partial \phi_\epsilon}{\partial \eta_1} + 2\phi_\epsilon \frac{\partial \Phi_\epsilon}{\partial \tau}, -\frac{\partial \phi_\epsilon}{\partial \eta_2}, \dots, -\frac{\partial \Phi_\epsilon}{\partial \tau}\right), \end{aligned}$$

and, if $C = C(\phi_\epsilon(\hat{\xi}_1, \eta, \tau), \hat{\xi}_1, \eta, 2\eta_1 \phi_\epsilon(\hat{\xi}_1, \eta, \tau) + \tau)$, it follows

$$\begin{aligned} &\left| C\ \frac{\partial \Phi_\epsilon}{\partial \xi_2} \wedge \cdots \wedge \frac{\partial \Phi_\epsilon}{\partial \tau} \right|^2 \\ &\quad = \sum_{j=1}^{n} \langle X_j, \frac{\partial \Phi_\epsilon}{\partial \xi_2} \wedge \cdots \wedge \frac{\partial \Phi_\epsilon}{\partial \tau} \rangle^2 + \sum_{j=1}^{n} \langle Y_j, \frac{\partial \Phi_\epsilon}{\partial \xi_2} \wedge \cdots \wedge \frac{\partial \Phi_\epsilon}{\partial \tau} \rangle^2 \\ &\quad = \left(2\eta_1 \frac{\partial \phi_\epsilon}{\partial \tau} + 1 - 2\eta_1 \frac{\partial \phi_\epsilon}{\partial \tau}\right)^2 + \sum_{j=2}^{n} \left(-\frac{\partial \phi_\epsilon}{\partial \xi_j} - 2\eta_j \frac{\partial \phi_\epsilon}{\partial \tau}\right)^2 \\ &\qquad + \left(-\frac{\partial \phi_\epsilon}{\partial \eta_1} + 2\phi_\epsilon \frac{\partial \phi_\epsilon}{\partial \tau} + 2\phi_\epsilon \frac{\partial \phi_\epsilon}{\partial \tau}\right)^2 + \sum_{j=2}^{n} \left(-\frac{\partial \phi_\epsilon}{\partial \eta_j} + 2\eta_j \frac{\partial \phi_\epsilon}{\partial \tau}\right)^2 \\ &\quad = 1 + \sum_{j=2}^{n} \left(\frac{\partial \phi_\epsilon}{\partial \xi_j} + 2\eta_j \frac{\partial \phi_\epsilon}{\partial \tau}\right)^2 + \left(\frac{\partial \phi_\epsilon}{\partial \eta_1} - 4\phi_\epsilon \frac{\partial \phi_\epsilon}{\partial \tau}\right)^2 \\ &\qquad + \sum_{j=2}^{n} \left(\frac{\partial \phi_\epsilon}{\partial \eta_j} - 2\eta_j \frac{\partial \phi_\epsilon}{\partial \tau}\right)^2. \end{aligned}$$

Now, by classical implicit function theorem, we have:

$$\begin{aligned} &\frac{\partial \phi_\epsilon}{\partial \xi_j} = -\frac{\partial f_\epsilon / \partial x_j}{X_1 f_\epsilon}, \qquad \frac{\partial \phi_\epsilon}{\partial \eta_j} = -\frac{\partial f_\epsilon / \partial y_j}{X_1 f_\epsilon}, \qquad j = 2, \dots, n; \\ &\frac{\partial \phi_\epsilon}{\partial \eta_1} = -\frac{\partial f_\epsilon / \partial y_1 + 2\phi_\epsilon \partial f_\epsilon / \partial t}{X_1 f_\epsilon}; \qquad \frac{\partial \phi_\epsilon}{\partial \tau} = -\frac{\partial f_\epsilon / \partial t}{X_1 f_\epsilon}. \end{aligned}$$

Thus

$$\begin{aligned}
&\left|\frac{\partial \Phi_\epsilon}{\partial \xi_2}\wedge\cdots\wedge\frac{\partial \Phi_\epsilon}{\partial \tau}\right|^2\\
&= 1+\sum_{j=1}^n (X_j f_\epsilon)^2 (X_1 f_\epsilon)^{-2}+\sum_{j=1}^n (Y_j f_\epsilon)^2 (X_1 f_\epsilon)^{-2}\\
&= \left|\nabla_{\mathbb{H}} f_\epsilon(\Phi_\epsilon(\hat{\xi}_1,\eta,\tau))\right|^2_{\Phi_\epsilon(\hat{\xi}_1,\eta,\tau)}\cdot\left|X_1 f_\epsilon(\Phi_\epsilon(\hat{\xi}_1,\eta,\tau))\right|^{-2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{\mathcal{U}} h_\epsilon \operatorname{div}_{\mathbb{H}}\psi\,dxdydt &\le \int_I \frac{\left|\nabla_{\mathbb{H}} f_\epsilon(\Phi_\epsilon(\hat{\xi}_1,\eta,\tau))\right|^2_{\Phi_\epsilon(\hat{\xi}_1,\eta,\tau)}}{\left|X_1 f_\epsilon(\Phi_\epsilon(\hat{\xi}_1,\eta,\tau))\right|^2}\,d\hat{\xi}_1 d\eta d\tau\\
&\le c|I|,
\end{aligned}$$

since the point $\Phi_\epsilon(\hat{\xi}_1,\eta,\tau)$ belongs to $\bar{\mathcal{U}}$, $X_j f_\epsilon$ and $Y_j f_\epsilon$ converge uniformly on $\bar{\mathcal{U}}$ to $X_j f$ and $Y_j f$ respectively for $j=1,\dots,n$ and hence they are bounded on $\bar{\mathcal{U}}$.

Thus, the functions h_ϵ have equibounded variations and, from (51), $\mathbf{1}_E\in BV_{\mathbb{H}}(\mathcal{U})$.

(Step 5) The area formula for $|\partial E|_{\mathbb{H}}(\mathcal{U})$: For all $\psi\in\mathbf{C}_0^\infty(\mathcal{U},\mathrm{H}\mathbb{H}^n)$, $|\psi(P)|_P\le 1$,

$$\begin{aligned}
\int_{\mathcal{U}} \mathbf{1}_E \operatorname{div}_{\mathbb{H}}\psi\,dxdydt &= \lim_{\epsilon\to 0}\int_{\mathcal{U}} h_\epsilon \operatorname{div}_{\mathbb{H}}\psi\,dxdydt\\
&= \lim_{\epsilon\to 0}\int_I \langle \psi(\Phi_\epsilon(\hat{\xi}_1,\eta,\tau)), C\,\frac{\partial \Phi_\epsilon}{\partial \xi_2}\wedge\cdots\wedge\frac{\partial \Phi_\epsilon}{\partial \tau}\rangle_{\mathbb{R}^{2n}}\,d\hat{\xi}_1 d\eta d\tau \qquad (52)\\
&= \int_I \frac{\langle \psi(\Phi(\hat{\xi}_1,\eta,\tau)),\nabla_{\mathbb{H}} f(\Phi(\hat{\xi}_1,\eta,\tau))\rangle_{\Phi(\hat{\xi}_1,\eta,\tau)}}{|X_1 f(\Phi(\hat{\xi}_1,\eta,\tau))|}\,d\hat{\xi}_1 d\eta d\tau,
\end{aligned}$$

by dominate convergence theorem, since $X_j f_\epsilon\to X_j f$ and $Y_j f_\epsilon\to Y_j f$, uniformly on $\bar{\mathcal{U}}$, then $(X_j f_\epsilon)\circ\Phi_\epsilon\to(X_j f)\circ\Phi$ and $(Y_j f_\epsilon)\circ\Phi_\epsilon\to(Y_j f)\circ\Phi$ pointwise, for $j=1,\dots,n$; moreover $|(X_j f_\epsilon)\circ\Phi_\epsilon|\le c$ on I for $j=2,\dots,n$, $|(Y_j f_\epsilon)\circ\Phi_\epsilon|\le c$ on I for $j=1,\dots,n$ and $(X_1 f_\epsilon)\circ\Phi_\epsilon\ge c>0$.

Taking now the supremum with respect to ψ the proof of (vi) is complete.

(Step 6) $\nu_E(P)=\dfrac{\nabla_{\mathbb{H}} f(P)}{|\nabla_{\mathbb{H}} f(P)|}$: Since $\mathbf{1}_E\in BV_{\mathbb{H}}(\mathcal{U})$, then $\nabla_{\mathbb{H}}\mathbf{1}_E$ is a vector valued Radon measure (see [30] or the observations before Theorem

2). Moreover, for all $\psi \in \mathbf{C}_0^0(\mathcal{U}, \mathrm{H}\mathbb{H}^n)$, by (8) and arguing as above, we get

$$-\int_{\mathcal{U}} \psi d\nabla_{\mathbb{H}} \mathbf{1}_E = \int_I \frac{\langle \psi(\Phi(\hat{\xi}_1, \eta, \tau)), \nabla_{\mathbb{H}} f(\Phi(\hat{\xi}_1, \eta, \tau))\rangle_{\Phi(\hat{\xi}_1,\eta,\tau)}}{|X_1 f(\Phi(\hat{\xi}_1, \eta, \tau))|}\, d\hat{\xi}_1 d\eta d\tau. \tag{53}$$

Taking the supremum on all $\psi \in \mathbf{C}_0^0(\mathcal{U}, \mathrm{H}\mathbb{H}^n)$, $|\psi(P)|_P \le 1$, $\operatorname{supp}\psi \subseteq \mathcal{O} \subseteq \mathcal{U}$, where $\mathcal{O}$ is an open set, we obtain

$$|\partial E|_{\mathbb{H}}(\mathcal{O}) = \int_{\Phi^{-1}(\mathcal{O})} \frac{|\nabla_{\mathbb{H}} f|}{|X_1 f|} \circ \Phi \; d\hat{\xi}_1 d\eta d\tau. \tag{54}$$

Thus, with the notations of [43], Definition 1.17, $|\partial E|_{\mathbb{H}}$ is the image of the measure

$$d\mu = \frac{|\nabla_{\mathbb{H}} f|}{|X_1 f|} \circ \Phi \; d\hat{\xi}_1 d\eta d\tau$$

in $\mathbb{R}^{2n}$, given that equality (54) still holds for any Borel set $\mathcal{O}$ because both measures are Radon measures ([43], Theorem 1.18). Hence, ([43], Theorem 1.19), for all $\psi \in \mathbf{C}_0^0(\mathcal{U}, \mathrm{H}\mathbb{H}^n)$

$$\int_{\mathcal{U}} \langle \psi(P), \nu_E(P)\rangle_P d|\partial E|_{\mathbb{H}} = \int_I \langle \psi \circ \Phi, \nu_E \circ \Phi\rangle_\Phi \frac{|\nabla_{\mathbb{H}} f|}{|X_1 f|} \circ \Phi \; d\hat{\xi}_1 d\eta d\tau.$$

We notice that Theorem 1.19 in [43] would require $\langle \psi(P), \nu_E(P)\rangle_P$ to be a Borel function. Nevertheless, the result is still true since $\langle \psi(P), \nu_E(P)\rangle_P$ is $|\partial E|_{\mathbb{H}}$-measurable, thanks to [24], 2.3.6. Thus, from (52),(53) and the arbitrariness of the choice of ψ,

$$\nu_E \circ \Phi = \frac{\nabla_{\mathbb{H}} f}{|\nabla_{\mathbb{H}} f|} \circ \Phi, \qquad L^{2n}\text{-a.e. in } I$$

and, because of (54),

$$\nu_E = \frac{\nabla_{\mathbb{H}} f}{|\nabla_{\mathbb{H}} f|}, \qquad |\partial E|_{\mathbb{H}}\text{-a.e. on } \Gamma \cap \mathcal{U}.$$

Thus (iii) is proved. In particular, ν_E can be identified with a continuous function and hence $\Gamma \cap \mathcal{U} = \partial_{\mathbb{H}}^* E \cap \mathcal{U}$, so proving the last part of (ii).

□

The following examples are intended to show the possible lack of regularity of functions in $\mathbf{C}_{\mathbb{H}}^1(\mathbb{R}^3)$ and of implicitely defined functions.

Example 2. Let $f : \mathbb{H}^1 \to \mathbb{R}$ be defined as

$$f(x,y,t) := x - g(y, t + 2xy) \quad \text{if} \quad (x,y,t) \in \mathbb{H}^1 \equiv \mathbb{R}^3,$$

where, for any $\alpha > 0$, $g \equiv g_\alpha : \mathbb{R}^2 \to \mathbb{R}$ is

$$g(a,b) := \begin{cases} \frac{|a|^\alpha b}{a^4 + b^2} & \text{if } (a,b) \neq (0,0) \\ 0 & \text{if } (a,b) = (0,0), \end{cases}$$

then

$$f \in \mathbf{C}^1_{\mathbb{H}}(\mathbb{R}^3) \cap \mathbf{C}^{0,4-\alpha}_{loc}(\mathbb{R}^3) \text{ but } f \notin \mathrm{Lip}_{loc}(\mathbb{R}^3), \quad \text{if} \quad 10/3 < \alpha < 4.$$

Proof. Indeed, if $\alpha > 2$, $g \in \mathbf{C}^0(\mathbb{R}^2) \cap \mathbf{C}^1(\mathbb{R}^2 \setminus \{(0,0)\})$ whence $f \in \mathbf{C}^0(\mathbb{R}^3) \cap \mathbf{C}^1(\mathbb{R}^3 \setminus \{(x,0,0) : x \in \mathbb{R}\})$ and

$$\nabla_{\mathbb{H}} f(x,y,t) = (1 - 4y\frac{\partial g}{\partial b}(y, t + 2xy), -\frac{\partial g}{\partial a}(y, t + 2xy))$$

if $(y,t) \neq (0,0)$, and

$$\nabla_{\mathbb{H}} f(x,y,t) = (1,0)$$

if $(y,t) = (0,0)$. On the other hand, if $\alpha > 3$, Xf and Yf are continuous functions on $\mathbb{R}^3$ and hence $f \in \mathbf{C}^1_{\mathbb{H}}(\mathbb{R}^3)$.

To prove that $f \notin \mathrm{Lip}_{loc}(\mathbb{R}^3)$ for $3 < \alpha < 4$ it is sufficient to notice that for any $r > 0$

$$\sup_{\{(0,y,t) \in B(0,r), t \neq 0\}} \frac{|f(0,y,t) - f(0,y,0)|}{|t|} = \infty.$$

Moreover it easy to see that $g \in \mathbf{C}^{0,\beta}_{loc}(\mathbb{R}^2)$ if $\beta = 4 - \alpha$ and $\frac{10}{3} < \alpha < 4$. Hence $f \in \mathbf{C}^{0,\beta}_{loc}(\mathbb{R}^3)$. □

Example 3. Let f and α as in the previous example and $E := \{(x,y,t) \in \mathbb{R}^3 : f(x,y,t) < 0\}$, $\Gamma := \{(x,y,t) \in \mathbb{R}^3 : f(x,y,t) = 0\}$. Observe that, as $f \in \mathbf{C}^1_{\mathbb{H}}(\mathbb{R}^3)$, $0 \in \Gamma$ and $Xf(0) = 1$, we can apply the Implicit Function Theorem (i.e. Theorem 8) to get the existence of a neighborhood $\mathcal{U}$ of $0 \in \mathbb{R}^3$, $\mathcal{U} = \{(x,y,t) \in \mathbb{R}^3 : |x| < h, |y| < \delta, t = 2xy + \tau \text{ with } \tau < \delta^2\}$ and of a (unique) continuous function $\phi : I := [-\delta, \delta] \times [-\delta^2, \delta^2] \to J := [-h, h]$ parametrizing the $\mathbb{H}$- regular surface $S := \bar{\mathcal{U}} \cap \Gamma$

$$S = \{(x,y,t) : x = \phi(\eta,\tau),\ y = \eta,\ t = 2\phi(\eta,\tau)\eta + \tau \text{ with } (\eta,\tau) \in I\} ,$$

then

$$\phi \notin \mathrm{Lip}(I) .$$

Proof. Because $(0,0,t),(0,y,0) \in S$ for $|t| < \delta^2$ and for $|y| < \delta$

$$\phi(\eta,0) = \phi(0,\tau) = 0, \qquad \forall \eta \in [-\delta,\delta] \text{ and } \forall \tau \in [-\delta^2,\delta^2]. \tag{55}$$

From the definition of S and (55)

$$\frac{\phi(\eta,\tau)-\phi(\eta,0)}{\tau} = |\eta|^\alpha \left(4\frac{\phi(\eta,\tau)-\phi(\eta,0)}{\tau}\eta + 1\right) \frac{1}{\eta^4 + (4\phi(\eta,\tau)\eta+\tau)^2}$$

for every $\eta \in [-\delta,\delta]$, $\tau \in [-\delta^2,\delta^2] \setminus \{0\}$. Then, defining

$$\psi(\eta) := \liminf_{\tau\to 0} \frac{\phi(\eta,\tau)-\phi(\eta,0)}{\tau},$$

we have

$$\psi(\eta) = |\eta|^{\alpha-4}\left(4\psi(\eta)\eta + 1\right), \qquad \forall \eta \in [-\delta,\delta] \setminus \{0\}.$$

Hence

$$\liminf_{\tau\to 0} \frac{\phi(\eta,\tau)-\phi(\eta,0)}{\tau} = \frac{|\eta|^{\alpha-4}}{1-4|\eta|^{\alpha-4}\eta}, \qquad \forall \eta \in [-\delta,\delta] \setminus \{0\}$$

that is unbounded for η small because $3 < \alpha < 4$. □

Theorem 9 (Whitney Extension Theorem). *Let $F \subset \mathbb{H}^n$ be a closed set, and let $f : F \to \mathbb{R}$, $k : F \to H\mathbb{H}^n$ be continuous functions. We set*

$$R(Q,P) := \frac{f(Q) - f(P) - \langle k(P), \pi_P(P^{-1}\cdot Q)\rangle_P}{d(P,Q)},$$

and, if $K \subset F$ is a compact set,

$$\rho_K(\delta) := \sup\{|R(Q,P)| : \quad P,Q \in K,\ 0 < d(P,Q) < \delta\}.$$

If $\rho_K(\delta) \to 0$ as $\delta \to 0$ for every compact set $K \subset F$, then there exist $\tilde f : \mathbb{H}^n \to \mathbb{R}$, $\tilde f \in \mathbf{C}^1_{\mathbb{H}}(\mathbb{H}^n)$ such that

$$\tilde f_{|F} \equiv f, \quad \nabla_{\mathbb{H}} \tilde f_{|F} \equiv k.$$

Proof. The proof follows closely the one in Euclidean spaces as can be found e.g. in Section 6.5 of [23] and also in [60] . We briefly indicate the main points of the argument.

For any $P \in \mathbb{H}^n$ let $r(P) := \frac{1}{20}\min\{1, dist(P,F)\}$ where, clearly, $dist(P,F) := \inf\{d(P,Q) : \ Q \in F\}$. Vitali covering theorem yields the existence of a countable set $S \in \mathbb{H}^n \setminus F$ such that

$$U := \mathbb{H}^n \setminus F = \bigcup_{P\in S} U(P, 5r(P))$$

and such that the balls $U(P, r(P))$ are pairwise disjointed. Fix $Q \in U$ and define

$$S_Q := \{P \in S : \ U(Q, 10r(Q)) \cap U(P, 10r(P)) \neq \emptyset\}.$$

Then

$$\mathrm{Card}(S_Q) \leq (129)^{2n+2} \quad \text{and} \quad \frac{1}{3} \leq \frac{r(Q)}{r(P)} \leq 3, \quad \forall P \in S_Q. \tag{56}$$

The proof of (56) depends only on the triangular inequality for the distance and on the estimate on the volume of balls in terms of their radius and can be carried out repeating verbatim the arguments in [23].

Now let $\mu : \mathbb{R} \to \mathbb{R}$ be a smooth non increasing function such that

$$\mu(t) := \begin{cases} 1 \text{ for } t \leq 1 \\ 0 \text{ for } t \geq 2^{3/4}. \end{cases}$$

Then for any $P \in S$ define

$$g_P(Q) := \mu\left(\frac{\delta(P,Q)}{5r(P)}\right) \qquad \forall Q \in \mathbb{H}^n \tag{57}$$

where $\delta(P,Q)$ is the regularized distance defined as $\delta(P,Q) := \hat{\delta}(P^{-1} \cdot Q)$ and $\hat{\delta}(P) = \hat{\delta}([x+iy,t]) := ((|x|^2 + |y|^2)^2 + |t|^2)^{1/4}$. Since

$$d(P,Q) \leq \delta(P,Q) \leq 2^{1/4} d(P,Q)$$

it follows that $g_P \in \mathbf{C}^\infty(\mathbb{H}^n)$, $0 \leq g_P(Q) \leq 1$ and that

$$g_P(Q) = \begin{cases} 1 \text{ for } Q \in U(P, 5r(P)) \\ 0 \text{ for } Q \in \mathbb{H}^n \setminus U(P, 10r(P)). \end{cases}$$

Moreover there are constants c and c_1 such that

$$|X_j g_P(Q)| \leq \frac{c}{r(P)} \leq \frac{c_1}{r(Q)} \qquad j = 1, \ldots, n \tag{58}$$

$$|Y_j g_P(Q)| \leq \frac{c}{r(P)} \leq \frac{c_1}{r(Q)} \qquad j = 1, \ldots, n; \tag{59}$$

finally

$$g_P(Q) = 0 \quad \text{on} \quad U(Q, 10r(Q)) \quad \text{if} \quad P \notin S_Q. \tag{60}$$

Now define

$$\sigma(Q) := \sum_{P \in S} g_P(Q) \qquad \forall Q \in \mathbb{H}^n;$$

because of (60)

$$\sigma(R) = \sum_{P \in S_Q} g_P(R) \quad \text{if} \quad R \in U(Q, 10r(Q)).$$

Since the cardinality of S_Q is finite and because of (58), it follows that $\sigma \in \mathbf{C}^\infty(\mathbb{H}^n \setminus F)$, $\sigma \geq 1$ on $\mathbb{H}^n \setminus F$ and there is a constant c_2 such that

$$|X_j \sigma(P)| \leq \frac{c_2}{r(P)}, \qquad |Y_j \sigma(P)| \leq \frac{c_2}{r(P)}$$

for all $P \in \mathbb{H}^n \setminus F$, $j = 1, \ldots, n$. Finally define the partition of unity subordinate to the covering $\{U(P, r(P))\}$ as

$$v_P(Q) := \frac{g_P(Q)}{\sigma(Q)}$$

and the extended function $\tilde{f}$ as

$$\tilde{f}(Q) := \begin{cases} f(Q) & \text{if } Q \in F \\ \sum_{P \in S} v_P(Q)[f(\hat{P}) + \langle k(\hat{P}), \hat{P}^{-1} \cdot Q \rangle_{\hat{P}}] & \text{if } Q \in \mathbb{H}^n \setminus F \end{cases}$$

where $\hat{P} \in F$ is such that $dist(P, F) = \delta(P, \hat{P})$. Now the proof that $\tilde{f} \in \mathbf{C}^1_{\mathbb{H}}(\mathbb{H}^n)$ and that

$$X_j \hat{f}(Q) = k_j(Q), \qquad Y_j \hat{f}(Q) = k_{n+j}(Q)$$

($j = 1, \ldots, n$, $Q \in F$) follows almost verbatim the proof in ([23]).

□

7. Main result

The following theorem on the structure of the boundary of $\mathbb{H}$-Caccioppoli sets together with the generalized divergence formula proved in the Corollary 3 are two of the main results of this paper and were also some of its motivations. We keep using the notations of Section 2.

Theorem 10 (Structure of $\mathbb{H}$-Caccioppoli sets). *If $E \subseteq \mathbb{H}^n$ is a $\mathbb{H}$-Caccioppoli set then*

$$\partial^*_{\mathbb{H}} E \text{ is } \mathbb{H}\text{-rectifiable,} \tag{i}$$

that is $\partial^*_{\mathbb{H}} E = N \cup \bigcup_{h=1}^\infty K_h$, where $\mathcal{H}_d^{2n+1}(N) = 0$ and K_h is a compact subset of a $\mathbb{H}$-regular hypersurface S_h;

$$\nu_E(P) \text{ is } \mathbb{H}\text{-normal to } S_h \text{ at } P, \qquad \forall P \in K_h, \tag{ii}$$

that is $\nu_E(P) \in \mathrm{H}\mathbb{H}^n_P$ *and* $\langle \nu_E(P), v\rangle_P = 0$ *for all* $v \in T_{\mathbb{H}} S_h(P)$*;*

$$|\partial E|_{\mathbb{H}} = \frac{2\omega_{2n-1}}{\omega_{2n+1}}\, \mathcal{S}_d^{2n+1} \llcorner \partial^*_{\mathbb{H}} E. \tag{iii}$$

Remark 10. Statements (i) and (ii) in Theorem 10 are verbatim restatements of the classical results proved by De Giorgi in [16] and [17]. Their proofs can be carried out, given what we proved in the previous sections of the present paper, following an already well established path (see, e.g., [37] and [23]). On the contrary, the third statement is slightly different: in it there is the spherical Hausdorff measure were usually Hausdorff measure appears. Indeed we don't know the exact relation between lower dimensional (intrinsic) Hausdorff measures and (intrinsic) spherical Hausdorff measures.

Lemma 5. *Assume* E *is a* $\mathbb{H}$*-Caccioppoli set, then*

$$\lim_{r\to 0} \fint_{U(P,r)} \nu_E \, d|\partial E|_{\mathbb{H}} = \nu_E(P), \qquad \textit{for } |\partial E|_{\mathbb{H}}\textit{-a.e. } P,$$

that is $|\partial E|_{\mathbb{H}}$*-a.e.* $P \in \mathbb{H}^n$ *belongs to the reduced boundary* $\partial^*_{\mathbb{H}} E$*.*

Proof. The proof depends crucially on Theorem 5 due to Ambrosio and on Federer's differentiation results for measures of doubling type. Indeed from (25) it follows the asymptotic doubling estimate

$$\limsup_{r\to 0} \frac{|\partial E|_{\mathbb{H}}(B(P,2r))}{|\partial E|_{\mathbb{H}}(B(P,r))} < \infty \qquad \text{for } |\partial E|_{\mathbb{H}}\text{-a.e. } P \in \mathbb{H}^n\,. \tag{61}$$

Now, (61) given, we apply 2.8.17 in [24] with $X := \mathbb{H}^n$, $V(X) := \{B(P,r) : P \in X,\, r > 0\}$, $\delta(B(P,r)) := r/4$, $\tau := 2$, $\Phi := |\partial E|_{\mathbb{H}}$ and

$$\tilde{B}(P,r) := \bigcup \{B(Q,\rho) : \rho < r,\, B(Q,\rho) \cap B(P,r) \neq \emptyset\} \subseteq B(P,2r),$$

getting that $V(\mathbb{H}^n)$ is a $|\partial E|_{\mathbb{H}}$-Vitali relation (see [24], 2.8.16) . Then, from 2.9.8 in [24] with $f = \nu_E$, we get

$$\lim_{r\to 0} \fint_{B(P,r)} \nu_E \, d|\partial E|_{\mathbb{H}} = \nu_E(P), \qquad \text{for } |\partial E|_{\mathbb{H}}\text{-a.e. } P \in \mathbb{H}^n. \tag{62}$$

Finally, by usual arguments of measure theory, one can replace the closed balls $B(P,r)$ in (62) with the open ones $U(P,r)$.

□

Proof of Theorem 10. First, analogously to Lemma 3 in Section 5.7.3 of [23], from (iii) of Lemma 4 and from Vitali covering theorem we get the existence of a constant c such that

$$\mathcal{H}_d^{Q-1}(B) \leq c|\partial E|_{\mathbb{H}}(B), \qquad \text{for all } B \subseteq \partial^*_{\mathbb{H}} E. \tag{63}$$

Following the argomentation in the proof of Theorem 2, Section 5.7.3 of [23], by Lusin's and Egoroff theorems, from Proposition 4, we get that $\partial^*_{\mathbb{H}} E = N \cup \bigcup_{h=1}^{\infty} K_h$, where $|\partial E|_{\mathbb{H}}(N) = 0$ and the K_h's are compact sets such that $\nu_{E_{|K_h}}$ is continuous. Thus the first two parts of the theorem will be proved by showing that each K_h is contained in a $\mathbb{H}$-regular hypersurface S_h whose normal coincides on K_h with the generalized normal defined in (2). This can be obtained repeating almost verbatim the arguments of [23] since

$$\lim_{r\to 0} \frac{|(U(P,r)\cap E)\cap \tau_P(S^+_{\mathbb{H}}(E,P))|}{|U(P,r)|} = 0$$
$$\lim_{r\to 0} \frac{|(U(P,r)\setminus E)\cap \tau_P(S^-_{\mathbb{H}}(E,P))|}{|U(P,r)|} = 0$$

uniformly for $P \in K_h$ for any h. In particular if $\epsilon > 0$ and $r < \delta(\epsilon, h)$ we have

$$\frac{|U(P,r)\cap E\cap \tau_P(S^+_{\mathbb{H}}(E,P))|}{|U(P,r)|} \le \frac{\epsilon^{2n+2}}{2^{2n+4}}$$
$$\frac{|U(P,r)\cap E\cap \tau_P(S^-_{\mathbb{H}}(E,P))|}{|U(P,r)|} \ge \left(\frac{1}{2} - \frac{\epsilon^{2n+2}}{2^{2n+4}}\right).$$

Similar inequalities hold with E^c instead of E. These all together imply that

$$\varrho_h(\delta) \to 0 \quad \text{as } \delta \to 0,$$

where

$$\varrho_h(\delta) := \sup\left\{ \frac{|\langle \nu_E(P), \pi_P(P^{-1}\cdot Q\rangle_P|}{\|P^{-1}\cdot Q\|_\infty} : \quad P, Q \in K_h, \|P^{-1}\cdot Q\|_\infty < \delta \right\}.$$

Now the existence of the regular surfaces S_h is a consequence of Whitney extension theorem.

Let us prove (iii). From Lemma 5 we have

$$|\partial E|_{\mathbb{H}} = |\partial E|_{\mathbb{H}} \llcorner \partial^*_{\mathbb{H}} E. \tag{64}$$

Define $\mu := \frac{\omega_{2n+1}}{2\omega_{2n-1}} |\partial E|_{\mathbb{H}}$. μ is a Radon measure and from Lemma 4 (iii), we get

$$\Theta^{*\,2n+1}(\mu, P) := \limsup_{r\to 0} \frac{\mu(B(P,r))}{\omega_{2n+1} r^{2n+1}} = 1, \qquad \text{for all } P \in A. \tag{65}$$

From classical results of measure theory (see e.g. theorems 2.10.17(2) and 2.10.19(3) in Federer's book [24]), (65) implies

$$\mu \equiv \frac{\omega_{2n+1}}{2\omega_{2n-1}}|\partial E|_{\mathbb{H}} = \frac{\omega_{2n+1}}{2\omega_{2n-1}}\mathcal{S}_d^{2n+1} \llcorner \partial^*_{\mathbb{H}} E. \tag{66}$$

Note that in obtaining (66) we used that $\mathrm{diam} B(P,r) = 2r$.

□

As it is usual in the literature (see e.g. [23] and [60]), we define also the measure theoretic boundary $\partial_{*,\mathbb{H}} E$ of a set $E \subset \mathbb{H}^n$. We observe that, as in the Euclidean setting, $\partial_{*,\mathbb{H}} E \supset \partial^*_{\mathbb{H}} E$ and it is an $\mathbb{H}$-rectifiable set.

Definition 15. *Let $E \subset \mathbb{H}^n$ be a measurable set, we say that $P \in \partial_{*,\mathbb{H}} E$, the measure theoretic boundary of E, if*

$$\limsup_{r\to 0^+} \frac{|E \cap U(P,r)|}{|U(P,r)|} > 0 \qquad \textit{and} \qquad \limsup_{r\to 0^+} \frac{|E^c \cap U(P,r)|}{|U(P,r)|} > 0.$$

Observe that two equivalent but different doubling distances yield the same measure theoretic boundary of E while the respectively induced reduced boundaries can be different.

Lemma 6. *Let E be an $\mathbb{H}$-Caccioppoli set, then*

$$\partial^*_{\mathbb{H}} E \subseteq \partial_{*,\mathbb{H}} E \subseteq \partial E. \tag{i}$$

$$\Theta^{*\,2n+1}(|\partial E|_{\mathbb{H}}, P) = 0 \qquad \textit{for } \mathcal{S}_d^{2n+1}\textit{-a.e. } P \in \mathbb{H}^n \setminus \partial^*_{\mathbb{H}} E, \tag{ii}$$

where $\Theta^{*\,2n+1}(|\partial E|_{\mathbb{H}}, P)$ is defined as in (65);

$$\mathcal{S}_d^{2n+1}(\partial_{*,\mathbb{H}} E \setminus \partial^*_{\mathbb{H}} E) = 0. \tag{iii}$$

Proof. The argument used inside the proof of Claim 4 of the Blow Up Theorem to obtain (40) yields that $\partial^*_{\mathbb{H}} E \subset \partial_{*,\mathbb{H}} E$. Hence (i) is proved. To prove (ii) define for any $\epsilon > 0$

$$A_\epsilon := (\mathbb{H}^n \setminus \partial^*_{\mathbb{H}} E) \cap \left\{ P \in \mathbb{H}^n \,:\, \Theta^{*\,2n+1}(|\partial E|_{\mathbb{H}}, P) > \epsilon \right\}.$$

Then, using once more Theorem 2.10.19(3) of [24], it follows that

$$|\partial E|_{\mathbb{H}}(A_\epsilon) \geq \epsilon\, \mathcal{S}_d^{2n+1}(A_\epsilon).$$

Because of Lemma 5 $|\partial E|_{\mathbb{H}}(\mathbb{H}^n \setminus \partial^*_{\mathbb{H}} E) = 0$ and then

$$\mathcal{S}_d^{2n+1}(A_\epsilon) = 0 \qquad \forall \epsilon > 0.$$

Then by an exhausting argument we get the thesis.

To prove (iii), thanks to (ii), it is enough to prove that

$$\Theta^{*\,2n+1}(|\partial E|_{\mathbb{H}}, P) > 0 \qquad \forall P \in \partial_{*,\mathbb{H}} E. \tag{67}$$

From $P \in \partial_{*,E}$, with elementary considerations, one gets the existence of $\delta > 0$ and of a sequence $\{r_i\}_i$, $r_i \to 0$ for $i \to +\infty$, such that

$$\limsup_{i\to+\infty} \frac{|E \cap U_C(P, r_i)|}{|U_C(P, r_i)|} > \delta \qquad \text{and} \qquad \limsup_{i\to+\infty} \frac{|E^c \cap U_C(P, r_i)|}{|U_C(P, r_i)|} > \delta.$$

Then the relative isoperimetric inequality (23) yields

$$\delta^{\frac{2n+1}{2n+2}} |U_C(P, r_i)|^{\frac{2n+1}{2n+2}} \le c\, |\partial E|_{\mathbb{H}}(U_C(P, r_i)), \qquad \forall i \in \mathbb{N}$$

and the same for E^c. These two together imply (67).

□

From Theorem 10 and Lemma 6 we get a sharper characterization of the $\mathbb{H}$-perimeter measure in terms of the $2n+1$-dimensional spherical Hausdorff measure restricted to the measure theoretic boundary instead of the reduced boundary. This may be of some interest because sometimes it is easier to work with the first one than with the second.

Corollary 3. [Main Corollary] *Let $E \subset \mathbb{R}^{2n+1} \equiv \mathbb{H}^n$. If E is a $\mathbb{H}$-Caccioppoli set then*

$$|\partial E|_{\mathbb{H}} = \frac{2\omega_{2n-1}}{\omega_{2n+1}}\, \mathcal{S}_d^{2n+1} \llcorner \partial_{*,\mathbb{H}} E, \tag{i}$$

and the following version of the divergence theorem holds

$$-\int_E div_{\mathbb{H}}\phi\, dh = \frac{2\omega_{2n-1}}{\omega_{2n+1}} \int_{\partial_{*,\mathbb{H}}E} \langle \nu_E, \phi\rangle\, d\mathcal{S}_d^{2n+1}, \qquad \forall \phi \in \mathbf{C}_0^1(\mathbb{H}^n, \mathrm{H}\mathbb{H}^n). \tag{ii}$$

An even sharper statement holds for euclidean $\mathbf{C}^1$ domains. Indeed Z. Balogh, in [5], has proved that the set $C(S)$ of the characteristic points of an euclidean $\mathbf{C}^1$ surface $S \subset \mathbb{R}^{2n+1} \equiv \mathbb{H}^n$ has $\mathcal{H}_d^{2n+1}(C(S)) = 0$ and hence also $\mathcal{S}_d^{2n+1}(C(S)) = 0$. The authors are grateful to professor Balogh for showing them his still unpublished paper. Now, because non characteristic points of a $\mathbf{C}^1$ boundary ∂E belong to the reduced boundary $\partial_{\mathbb{H}}^* E$, it follows that $\mathcal{S}_d^{2n+1}(\partial E \setminus \partial_{\mathbb{H}}^* E) = 0$. These facts and Proposition 5 prove the following

Corollary 4. *If E has boundary of class $\mathbf{C}^1$ (and hence E is a $\mathbb{H}$-Caccioppoli set), then*

$$|\partial E|_{\mathbb{H}} = \frac{2\omega_{2n-1}}{\omega_{2n+1}}\, \mathcal{S}_d^{2n+1} \llcorner \partial E = |Cn_E|\, \mathcal{H}^{2n} \llcorner \partial E, \tag{i}$$

where n_E denotes the Euclidean outward normal to ∂E and C is defined in (5). Again a version of the divergence theorem holds: if $\phi \in \mathbf{C}_0^1(\mathbb{H}^n, \mathrm{H}\mathbb{H}^n)$ then

$$- \int_E div_{\mathbb{H}} \phi \, dh = \frac{2\omega_{2n-1}}{\omega_{2n+1}} \int_{\partial E} \langle \nu_E, \phi \rangle \, d\mathcal{S}_d^{2n+1}. \qquad (ii)$$

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