Notes on the Distance Function from a Submanifold – V4

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ABSTRACT. Notes on the properties of the distance function from a submanifold of the Euclidean space and the relations with its geometry – Version IV.

Not meant for publication. Criticism and/or comments are welcome.

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1. Geometry of Submanifolds

The main objects we will consider are n–dimensional, complete submanifolds, immersed in $\mathbb{R}^{n+m}$, that is, pairs (M, φ) where M is an n–dimensional smooth manifold, compact, connected with empty boundary, and a smooth map $\varphi : M \to \mathbb{R}^{n+m}$ such that the *rank* of $d\varphi$ is everywhere equal to n.
Good references for this section are **[17, 23]** (consider also **[27, 28]**).

The manifold M gets in a natural way a metric tensor g turning it in a Riemannian manifold (M, g), by pulling back the standard scalar product of $\mathbb{R}^{n+m}$ with the immersion map φ.

Taking local coordinates around $p \in M$ given by a chart $F : \mathbb{R}^n \supset U \to M$, we identify the map φ with its expression in coordinates $\varphi \circ F : \mathbb{R}^n \supset U \to \mathbb{R}^{n+m}$, then we have local basis of T_pM and T_p^*M, respectively given by vectors $\left\{\frac{\partial}{\partial x_i}\right\}$ and covectors $\{dx_j\}$.

We will denote vectors on M by $X = X^i$, which means $X = X^i\frac{\partial}{\partial x_i}$, covectors by $Y = Y_j$, that is, $Y = Y_j dx_j$ and a general mixed tensor with $T = T^{i_1\dots i_k}_{j_1\dots j_l}$, where the indices refer to the local basis.

In all the formulas the convention to sum over repeated indices will be adopted.

The tangent space at $p \in M$ can be clearly identified with the vector subspace $d\varphi_p(T_pM)$ of $T_{\varphi(p)}\mathbb{R}^{n+m} \approx \mathbb{R}^{n+m}$. Then, we define its m–dimensional orthogonal complement N_pM to be the *normal space* to M at p. Clearly the trivial vector bundle $T\mathbb{R}^{n+m}$ decomposes as $T\mathbb{R}^{n+m} = TM \oplus^{\perp} NM$, that is, the orthogonal direct sum of the tangent bundle and the normal bundle of M.

As the metric tensor g is induced by the scalar product of $\mathbb{R}^{n+m}$, which will be denoted with $\langle \cdot \,|\, \cdot \rangle$, we have

$$g_{ij}(x) = \left\langle \frac{\partial\varphi(x)}{\partial x_i} \,\middle|\, \frac{\partial\varphi(x)}{\partial x_j} \right\rangle .$$

The metric g extends canonically to tensors as follows,

$$g(T,S) = g_{i_1s_1}\dots g_{i_ks_k}g^{j_1z_1}\dots g^{j_lz_l}T^{i_1\dots i_k}_{j_1\dots j_l}S^{s_1\dots s_k}_{z_1\dots z_l}$$

where g^{ij} is the inverse of the matrix of the coefficients g^{ij}. Then we define the norm of a tensor T as

$$|T| = \sqrt{g(T,T)}\,.$$

By means of the scalar product of $\mathbb{R}^{n+m}$ we also define a metric tensor on the normal bundle and, as above, on all the tensors acting or with values in NM.

The canonical measure induced by the metric g is given by $\mu = \sqrt{G}\,\mathcal{L}^n$ where $G = \det(g_{ij})$ and $\mathcal{L}^n$ is the standard Lebesgue measure on $\mathbb{R}^n$.

The induced covariant derivatives on (M,g) of a tangent vector field X or of a 1–form ω are given by

$$\nabla^M_i X^j = \frac{\partial}{\partial x_i}X^j + \Gamma^j_{ik}X^k \qquad \text{and} \qquad \nabla^M_i \omega_j = \frac{\partial}{\partial x_i}\omega_j - \Gamma^k_{ij}\omega_k\,,$$

where the Christoffel symbols $\Gamma = \Gamma^k_{ij}$ are expressed by the following formula,

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial}{\partial x_i}g_{jl} + \frac{\partial}{\partial x_j}g_{ij} - \frac{\partial}{\partial x_l}g_{ij}\right).$$

It is well know that, for a pair of tangent vector fields X and Y on M, we have

$$\nabla^M_X Y = \left(\nabla^{\mathbb{R}^{n+m}}_X Y\right)^M$$

where the symbol ^{M} denotes the orthogonal projection on the tangent space of M.
Here, $\nabla^{\mathbb{R}^{n+m}}_X Y$ at a point $p \in M$ denotes the covariant derivative of $\mathbb{R}^{n+m}$ acting on some local extensions of the fields X and Y in an open subset of $\mathbb{R}^{n+m}$, once considered M (actually it is sufficient only a local embedding of M around p) as a subset of $\mathbb{R}^{n+m}$. This is a well defined expression, indeed, once identified any T_pM as a vector subspace of $\mathbb{R}^{n+m}$, the extensions of the vector fields X and Y are vector fields in the ambient space $\mathbb{R}^{n+m}$ and it is easy to check that $(\nabla^{\mathbb{R}^{n+m}}_X Y)(p)$ depends only on the values of the two fields on M in the embedded neighborhood of p, by the properties of the covariant derivative.

The covariant derivative $\nabla^M T$ of a tensor $T = T^{i_1\dots i_k}_{j_1\dots j_l}$ will be denoted by $\nabla^M_s T^{i_1\dots i_k}_{j_1\dots j_l} = (\nabla^M T)^{i_1\dots i_k}_{sj_1\dots j_l}$ and with $\nabla^k T$ we will mean the k–th iterated covariant derivative.

The gradient $\nabla^M f$ of a function and the divergence $\operatorname{div} X$ of a tangent vector field are defined respectively by

$$g(\nabla^M f, v) = df_p(v) \qquad \forall v \in TM$$

and

$$\operatorname{div} X = \text{Trace}\, \nabla^M X = \nabla^M_i X^i = \frac{\partial}{\partial x_i} X^i + \Gamma^i_{ik} X^k\,.$$

The Laplacian $\Delta^M T$ of a tensor T is

$$\Delta^M T = g^{ij}\nabla^M_i \nabla^M_j T\,.$$

Using the notion of connection and covariant derivative on fiber bundles (for instance, see [**27, 28**]), one can check that the following definition is actually the covariant derivative associated to the metric g on the normal bundle of M.
For any normal vector field ν on M and a tangent vector field X, we set

$$\nabla^\perp_X \nu = \left(\nabla^{\mathbb{R}^{n+m}}_X \nu\right)^\perp$$

where the symbol $^\perp$ denotes the orthogonal projection on the normal space of M.
Then, we can consider from now on the following definition of covariant derivative of any vector field (tangent or not) Y along M as follows

$$\nabla_X Y = \nabla^M_X Y^M + \nabla^\perp_X Y^\perp = \left(\nabla^{\mathbb{R}^{n+m}}_X Y^M\right)^M + \left(\nabla^{\mathbb{R}^{n+m}}_X Y^\perp\right)^\perp\,,$$

where Y^M and $Y^\perp$ are respectively the tangent and normal components of the vector field Y.

We extend this covariant derivative also to "mixed" tensors, that is, tensors acting also on the normal bundle of M, not only on the tangent bundle.
For instance, if T "acts" on $(k+l)$–uple of vector fields along M such that the first k are tangent and the other l are normal, we have

$$\begin{aligned}\nabla_X T(X_1,\dots,X_k,\nu_1,\dots,\nu_l) &= \nabla_X (T(X_1,\dots,X_k,\nu_1,\dots,\nu_l))\\ &\quad - T(\nabla^M_X X_1,\dots,X_k,\nu_1,\dots,\nu_l) - \dots - T(X_1,\dots,\nabla^M_X X_k,\nu_1,\dots,\nu_l)\\ &\quad - T(X_1,\dots,X_k,\nabla^\perp_X \nu_1,\dots,\nu_l) - \dots - T(X_1,\dots,X_k,\nu_1,\dots,\nabla^\perp_X \nu_l)\end{aligned}$$

where ∇_X immediately after the equality "works" according to the "target" bundle of T.

Associated to the connection $\nabla^\perp$ we have also a notion of curvature, called *normal curvature*, defined in the standard way.
For a pair of tangent vector fields X, Y and any normal vector field ν, we set

$$\mathrm{R}^\perp(X,Y)\nu = \nabla^\perp_Y \nabla^\perp_X \nu - \nabla^\perp_X \nabla^\perp_Y \nu - \nabla^\perp_{[Y,X]}\nu$$

and an associated $(0,4)$–curvature tensor $\mathrm{R}^\perp(X,Y,\nu,\xi) = g(\mathrm{R}^\perp(X,Y)\nu,\xi)$ which plays the same role of the Riemann tensor in exchanging the covariant derivatives in the normal bundle.
If ξ_α is a local basis of the normal bundle (which is locally trivial) and $\nu = \nu^\alpha \xi_\alpha$, we have

$$(\nabla^\perp)^2_{ij}\nu^\alpha - (\nabla^\perp)^2_{ji}\nu^\alpha = \mathrm{R}^\perp_{ij\beta\gamma} g^{\beta\alpha}\nu^\gamma\,.$$

It is then natural to consider the following couple of tensors (their tensor nature can be easily checked).
For a pair of tangent vector fields X and Y, the form

$$\mathrm{B}(X,Y)=\left(\nabla^{\mathbb{R}^{n+m}}_X Y\right)^{\perp}$$

measures the difference between the covariant derivative of (M,g) and the one of the ambient space $\mathbb{R}^{n+m}$, indeed

$$\nabla^M_X Y=\left(\nabla^{\mathbb{R}^{n+m}}_X Y\right)^M=\nabla^{\mathbb{R}^{n+m}}_X Y-\mathrm{B}(X,Y)\,. \tag{1.1}$$

For a tangent vector field X and a normal one ν,

$$\mathrm{S}(X,\nu)=-\left(\nabla^{\mathbb{R}^{n+m}}_X \nu\right)^M$$

which clearly satisfies

$$\nabla^{\perp}_X \nu=\left(\nabla^{\mathbb{R}^{n+m}}_X \nu\right)^{\perp}=\nabla^{\mathbb{R}^{n+m}}_X \nu+\mathrm{S}(X,\nu)\,.$$

The form B is called *second fundamental form* and it is a symmetric bilinear form with values in the normal bundle NM. Its symmetry can be seen easily as the two connections have no torsion,

$$\mathrm{B}(X,Y)-\mathrm{B}(Y,X)=\nabla^M_Y X-\nabla^M_X Y-\nabla^{\mathbb{R}^{n+m}}_Y X+\nabla^{\mathbb{R}^{n+m}}_X Y=[X,Y]_{\mathbb{R}^{n+m}}-[X,Y]_M=0$$

and $d\varphi([X,Y]_M)=[d\varphi(X),d\varphi(Y)]_{\mathbb{R}^{n+m}}$.

The bilinear form S, with values in TM, can be seen as an operator $\mathrm{S}(\cdot,\nu):TM\to TM$ (for every fixed normal vector field $\nu\in NM$) called *shape operator*. Actually, S is self–adjoint and B is the associated quadratic form, if X, Y are tangent vector fields and ν is a normal one, we have

$$\begin{aligned} g(Y,\mathrm{S}(X,\nu)) &= -\,g\Big(Y,\left(\nabla^{\mathbb{R}^{n+m}}_X \nu\right)^M\Big)=-g(Y,\nabla^{\mathbb{R}^{n+m}}_X \nu)\\ &=g(\nabla^{\mathbb{R}^{n+m}}_X Y,\nu)=g\Big(\left(\nabla^{\mathbb{R}^{n+m}}_X Y\right)^{\perp},\nu\Big)\\ &=g(\mathrm{B}(X,Y),\nu)\,, \end{aligned} \tag{1.2}$$

hence, B and S can be recovered each other.
By the symmetry of B it follows that

$$g(Y,\mathrm{S}(X,\nu))=g(X,\mathrm{S}(Y,Z))$$

hence, $\mathrm{S}(\cdot,\nu)$ is self–adjoint.
Finally, it is easy to check that $|\mathrm{B}|^2=|\mathrm{S}|^2$ and also $|\nabla^k\mathrm{B}|^2=|\nabla^k\mathrm{S}|^2$ for every $k\in\mathbb{N}$.

We extend the forms B and S to any vector field along M as follows

$$\begin{aligned} \mathrm{B}(X,Y)&=\mathrm{B}(X^M,Y^M)\,,\\ \mathrm{S}(X,Y)&=\mathrm{S}(X^M,Y^{\perp})\,, \end{aligned} \tag{1.3}$$

and, for any normal vector field ν we set

$$\mathrm{B}^\nu(X,Y) = \langle \nu \,|\, \mathrm{B}(X^M, Y^M)\rangle\,,$$
$$\mathrm{S}_\nu(X) = \mathrm{S}(X^M, \nu)\,.$$

Clearly, by equation (1.2), it follows $g(Y, \mathrm{S}_\nu(X)) = \mathrm{B}^\nu(X,Y)$.

Choosing a local coordinate basis in M, we have

$$\mathrm{B}_{ij} = \mathrm{B}(\partial_{x_i}, \partial_{x_j}) = \left(\nabla^{\mathbb{R}^{n+m}}_{\partial_{x_i}} \partial_{x_j}\right)^\perp = \left(\frac{\partial}{\partial x_i}\frac{\partial \varphi}{\partial x_j}\right)^\perp = \left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right)^\perp$$

and

$$\mathrm{B}^\nu_{ij} = \left\langle \nu \,\middle|\, \frac{\partial^2 \varphi}{\partial x_i \partial x_j}\right\rangle.$$

$$(\mathrm{S}_\nu)_i = \mathrm{S}(\partial_{x_i}, \nu) = -\left(\frac{\partial \nu}{\partial x_i}\right)^M .$$

which are the more familiar definition of second fundamental form and of the shape operator.

The *mean curvature vector* H is the trace (with the induced metric) of the second fundamental form,

$$\mathrm{H} = g^{ij}\mathrm{B}_{ij}\,,$$

by this definition, clearly $\mathrm{H} \in NM$. We also define $\mathrm{H}^\nu = g^{ij}\mathrm{B}^\nu_{ij}$.

Making explicit equation (1.1) and using identity (1.2) we have the so called *Gauss–Weingarten relations*,

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \Gamma^k_{ij}\frac{\partial \varphi}{\partial x_k} + \mathrm{B}_{ij} \qquad \left(\frac{\partial \nu}{\partial x_i}\right)^M = -\mathrm{B}^\nu_{ik} g^{kj}\frac{\partial \varphi}{\partial x_j}$$

for every normal vector field ν along M.

Notice that the first relation implies

$$\Delta^M \varphi = g^{ij}\nabla^2_{ij}\varphi = g^{ij}\left(\frac{\partial^2 \varphi}{\partial x_i \partial x_j} - \Gamma^k_{ij}\frac{\partial \varphi}{\partial x_k}\right) = g^{ij}\mathrm{B}_{ij} = \mathrm{H}\,,$$

component by component.

The second fundamental form B embodies all information on the curvature properties of M, this is expressed by the following relations with the Riemann curvature tensor of (M, g),

$$\begin{aligned}
\mathrm{R}_{ijkl} &= g(\nabla^2_{ji}\partial_{x_k} - \nabla^2_{ij}\partial_{x_k}, \partial_{x_l}) = \langle \mathrm{B}_{ik} \,|\, \mathrm{B}_{jl}\rangle - \langle \mathrm{B}_{il} \,|\, \mathrm{B}_{jk}\rangle\,,\\
\mathrm{R}_{ij} &= g^{kl}\mathrm{R}_{ikjl} = \langle \mathrm{H} \,|\, \mathrm{B}_{ij}\rangle - g^{kl}\langle \mathrm{B}_{il} \,|\, \mathrm{B}_{kj}\rangle\,,\\
\mathrm{R} &= g^{ij}\mathrm{Ric}_{ij} = |\mathrm{H}|^2 - |\mathrm{B}|^2\,,
\end{aligned}$$

where the scalar products are meant in the normal space to M.

REMARK 1.1. These equations are often called *Gauss equations* by the connection with his *Theorema Egregium* about the invariance by isometry of the Gaussian curvature G of a surface in $\mathbb{R}^3$, which is actually expressed by the third equation, once we rewrite it as $\mathrm{R} = 2\mathrm{G}$.

We recall that the Gaussian curvature of a surface is the product of the principal eigenvalues of B (in codimension one, B can be seen as a *real valued* bilinear form, as we will see in a while). Equivalently, $\mathrm{G} = \det \mathrm{S}_\nu$ where ν is a local unit normal vector field.

Then, the formulas for the interchange of covariant derivatives, which involve the Riemann tensor, become

$$\nabla^M_i\nabla^M_j X^s - \nabla^M_j\nabla^M_i X^s = \mathrm{R}_{ijkl}g^{ks}X^l = \left(\langle \mathrm{B}_{ik}\,|\,\mathrm{B}_{jl}\rangle - \langle \mathrm{B}_{il}\,|\,\mathrm{B}_{jk}\rangle\right) g^{ks}X^l\,,$$

$$\nabla^M_i\nabla^M_j \omega_k - \nabla^M_j\nabla^M_i \omega_k = \mathrm{R}_{ijkl}g^{ls}\omega_s = \left(\langle \mathrm{B}_{ik}\,|\,\mathrm{B}_{jl}\rangle - \langle \mathrm{B}_{il}\,|\,\mathrm{B}_{jk}\rangle\right) g^{ls}\omega_s\,.$$

About the normal curvature, the analogous of Gauss equations are called *Ricci equations*. If ξ_α is a local basis of the normal bundle we have,

$$\mathrm{R}^\perp_{ij\alpha\beta} = -g([\mathrm{S}_\alpha,\mathrm{S}_\beta]\partial_{x_i},\partial_{x_j})$$

where S_α and S_β are respectively the operators S_{ξ_α} and S_{ξ_β} and $[\mathrm{S}_\alpha,\mathrm{S}_\beta]$ denotes the commutator operator $\mathrm{S}_\alpha\mathrm{S}_\beta - \mathrm{S}_\beta\mathrm{S}_\alpha : TM \to TM$.
Hence, the formula for the interchange of derivatives on the normal bundle become

$$\nabla^\perp_i\nabla^\perp_j \nu^\alpha - \nabla^\perp_j\nabla^\perp_i \nu^\alpha = \mathrm{R}^\perp_{ij\beta\gamma}g^{\beta\alpha}\nu^\gamma = g([\mathrm{S}_\gamma,\mathrm{S}_\beta]\partial_{x_i},\partial_{x_j})g^{\beta\alpha}\nu^\gamma\,,$$

for every normal vector field $\nu = \nu^\alpha\xi_\alpha$.

Finally, the following *Codazzi equations* hold

$$(\nabla_X\mathrm{B})(Y,Z,\nu) = (\nabla_Y\mathrm{B})(X,Z,\nu)$$

for every three tangent vector fields X, Y, Z and $\nu \in NM$.
These equation are sometimes also called *Codazzi–Mainardi equations* as Delfino Codazzi [**12**] and Gaspare Mainardi [**31**] independently derived them (actually, they were discovered earlier by Karl M. Peterson [**35**]).
They can be seen as an analogous of the II Bianchi identity satisfied by the Riemann tensor.

The importance of the Gauss, Ricci and Codazzi equations is that they are the analogous of the Frenet equations for space curves. They determine, up to isometry of the ambient space, the immersed submanifold, as it is expressed by the following fundamental theorem (first proved for surfaces in $\mathbb{R}^3$ by Pierre Ossian Bonnet [**7, 8**]), see [**6**, Chap. 2].

THEOREM 1.2. *Let (M,g) be an n–dimensional Riemannian manifold with a Riemannian vector bundle NM of rank m. Let $\nabla^\perp$ a metric connection on NM and B a symmetric bilinear form with values in NM. Define the operator $\mathrm{S}(\cdot,\nu) : TM \to TM$ by $g(Y,\mathrm{S}_\nu(X)) = \langle \nu\,|\,\mathrm{B}(X,Y)\rangle$ and suppose that the equations of Gauss, Ricci and Codazzi are satisfied by these tensors.*
Then, around any point $p \in M$ there exists an open neighborhood $U \subset M$ and an isometric immersion $\varphi : U \to \mathbb{R}^{n+m}$ such that B coincides with the second fundamental form of the immersion φ and NM is isomorphic to the normal bundle.
The immersion is unique up to an isometry of $\mathbb{R}^{n+m}$, moreover, if two immersions have the same second fundamental form and normal connection, they locally coincide up to an isometry of $\mathbb{R}^{n+m}$.

A consequence of Codazzi equation is the following computation of the difference between $\Delta\mathrm{B}$ and $\nabla^2\mathrm{H}$,

$$
\begin{aligned}
\Delta \mathrm{B}^\alpha_{ij} - \nabla^2_{ij}\mathrm{H}^\alpha &= g^{pq}\left\{\nabla^2_{pq}\mathrm{B}^\alpha_{ij} - \nabla^2_{ij}\mathrm{B}^\alpha_{pq}\right\} \\
&= g^{pq}\left\{\nabla^2_{pi}\mathrm{B}^\alpha_{qj} - \nabla^2_{ij}\mathrm{B}^\alpha_{pq}\right\} \\
&= g^{pq}\left\{\nabla^2_{ip}\mathrm{B}^\alpha_{qj} - \nabla^2_{ij}\mathrm{B}^\alpha_{pq}\right\} \\
&\quad + g^{pq}\left(\langle \mathrm{B}_{pq}\,|\,\mathrm{B}_{il}\rangle - \langle \mathrm{B}_{pl}\,|\,\mathrm{B}_{iq}\rangle\right) g^{ls}\mathrm{B}^\alpha_{sj} \\
&\quad + g^{pq}\left(\langle \mathrm{B}_{pj}\,|\,\mathrm{B}_{il}\rangle - \langle \mathrm{B}_{pl}\,|\,\mathrm{B}_{ij}\rangle\right) g^{ls}\mathrm{B}^\alpha_{sq} \\
&\quad + g^{pq}g([\mathrm{S}_\gamma,\mathrm{S}_\beta]\partial_{x_p},\partial_{x_i})g^{\beta\alpha}\mathrm{B}^\gamma_{qj} \\
&= \left(\langle \mathrm{H}\,|\,\mathrm{B}_{il}\rangle - g^{pq}\langle \mathrm{B}_{pl}\,|\,\mathrm{B}_{iq}\rangle\right) g^{ls}\mathrm{B}^\alpha_{sj} \\
&\quad + g^{pq}\left(\langle \mathrm{B}_{pj}\,|\,\mathrm{B}_{il}\rangle - \langle \mathrm{B}_{pl}\,|\,\mathrm{B}_{ij}\rangle\right) g^{ls}\mathrm{B}^\alpha_{sq} \\
&\quad + g^{pq}\left[g(\mathrm{S}_\beta(\partial_{x_p}),\mathrm{S}_\gamma(\partial_{x_i})) - g(\mathrm{S}_\beta(\partial_{x_p}),\mathrm{S}_\gamma(\partial_{x_i}))\right] g^{\beta\alpha}\mathrm{B}^\gamma_{qj} \\
&= \left(\langle \mathrm{H}\,|\,\mathrm{B}_{il}\rangle - g^{pq}\langle \mathrm{B}_{pl}\,|\,\mathrm{B}_{iq}\rangle\right) g^{ls}\mathrm{B}^\alpha_{sj} \\
&\quad + g^{pq}\left(\langle \mathrm{B}_{pj}\,|\,\mathrm{B}_{il}\rangle - \langle \mathrm{B}_{pl}\,|\,\mathrm{B}_{ij}\rangle\right) g^{ls}\mathrm{B}^\alpha_{sq} \\
&\quad + g^{pq}\left(\mathrm{B}^\beta_{pk}g^{kl}\mathrm{B}^\gamma_{il} - \mathrm{B}^\gamma_{pk}g^{kl}\mathrm{B}^\beta_{il}\right) g^{\beta\alpha}\mathrm{B}^\gamma_{qj} \\
&= \left(\langle \mathrm{H}\,|\,\mathrm{B}_{il}\rangle - g^{pq}\langle \mathrm{B}_{pl}\,|\,\mathrm{B}_{iq}\rangle\right) g^{ls}\mathrm{B}^\alpha_{sj} \\
&\quad + \left(\langle \mathrm{B}_{pj}\,|\,\mathrm{B}_{il}\rangle - \langle \mathrm{B}_{pl}\,|\,\mathrm{B}_{ij}\rangle\right) g^{pq}g^{ls}\mathrm{B}^\alpha_{sq} \\
&\quad + \langle \mathrm{B}_{il}\,|\,\mathrm{B}_{qj}\rangle g^{pq}g^{kl}\mathrm{B}^\alpha_{pk} - \langle \mathrm{B}_{pk}\,|\,\mathrm{B}_{qj}\rangle g^{pq}g^{kl}\mathrm{B}^\alpha_{il} \\
&= \langle \mathrm{H}\,|\,\mathrm{B}_{il}\rangle g^{ls}\mathrm{B}^\alpha_{sj} - \langle \mathrm{B}_{pl}\,|\,\mathrm{B}_{iq}\rangle g^{pq}g^{ls}\mathrm{B}^\alpha_{sj} - \langle \mathrm{B}_{pl}\,|\,\mathrm{B}_{jq}\rangle g^{pq}g^{ls}\mathrm{B}^\alpha_{si} \\
&\quad + \left(2\langle \mathrm{B}_{pj}\,|\,\mathrm{B}_{il}\rangle - \langle \mathrm{B}_{pl}\,|\,\mathrm{B}_{ij}\rangle\right) g^{pq}g^{ls}\mathrm{B}^\alpha_{sq}\,.
\end{aligned}
\tag{1.4}
$$

Hence, such a difference is a third order homogeneous polynomial in B.

All the relations we discussed in this section are valid in the Euclidean ambient space. When the ambient space is a general Riemannian manifolds all the formulas need a correction term due to its curvature. See [**17**, Chap. 6] *and* [**6**, Chap. 2].

1.1. The Codimension One Case. When the codimension is one, the normal space is one–dimensional, so at least locally we can define up to a sign (sometimes we will have to deal with this ambiguity) a smooth unit local normal vector field to M.
Actually, if the hypersurface M is orientable, this choice can be done globally.

In the case the hypersurface M is compact and embedded (hence, it is also orientable), we will always consider ν to be the unit *inner* normal.

The second fundamental form B then coincides with $\mathrm{B}^\nu\nu$, hence in this case we can actually consider the $\mathbb{R}$–valued bilinear form B^ν that, for sake of simplicity, we still call B, for all this section.

We will denote with H the mean curvature *function* $\mathrm{H}^\nu = g^{ij}\mathrm{B}^\nu_{ij}$ and with S the shape operator $\mathrm{S}_\nu = \mathrm{S}(\cdot,\nu) : TM \to TM$.
Notice that B, S and H are defined up to the sign of ν (with the conventional choice above, the second fundamental form of a convex hypersurface is nonnegative definite).

In the codimension one case are commonly defined the so called *principal curvatures* of M at a point p, as the eigenvalues of the form B (defined up to a sign).
The relative eigenvectors in T_pM are called *principal directions*.

In this case, many of the previous formula simplifies, as every derivative of ν must be a tangent field, hence, in particular $\nabla^\perp \nu = 0$,

$$\begin{aligned}
&\nabla^M_X Y = \nabla^{\mathbb{R}^{n+m}}_X Y - \mathrm{B}(X,Y)\nu\\
&\nabla^{\mathbb{R}^{n+m}}_X \nu = -\mathrm{S}(X)\\
&g(Y,\mathrm{S}(X)) = \mathrm{B}(X,Y)\\
&\mathrm{B}_{ij} = \Big\langle \nu \,\Big|\, \frac{\partial^2\varphi}{\partial x_i \partial x_j}\Big\rangle
\end{aligned}$$

The Gauss–Weingarten relations become

$$\frac{\partial^2\varphi}{\partial x_i\partial x_j} = \Gamma^k_{ij}\frac{\partial\varphi}{\partial x_k} + \mathrm{B}_{ij}\nu \qquad \frac{\partial\nu}{\partial x_i} = -\mathrm{B}_{ik}g^{kj}\frac{\partial\varphi}{\partial x_j}\,.$$

The Riemann curvature tensor of (M,g) is given by,

$$\begin{aligned}
\mathrm{R}_{ijkl} &= \mathrm{B}_{ik}\mathrm{B}_{jl} - \mathrm{B}_{il}\mathrm{B}_{jk}\,,\\
\mathrm{R}_{ij} &= \mathrm{H}\mathrm{B}_{ij} - g^{lk}\mathrm{B}_{il}\mathrm{B}_{kj}\,,\\
\mathrm{R} &= |\mathrm{H}|^2 - |\mathrm{B}|^2\,.
\end{aligned}$$

Notice that in these last formulas the ambiguity of the definition up to a sign of B and H vanishes.
The Ricci equations are in this case trivial, the Codazzi equations get the simple form

$$\nabla^M_i \mathrm{B}_{jk} = \nabla^M_j \mathrm{B}_{ik}$$

and imply the following *Simons' identity* **[37]**

$$\Delta^M \mathrm{B}_{ij} = \nabla^2_{ij}\mathrm{H} + \mathrm{H}\,\mathrm{B}_{il}g^{ls}\mathrm{B}_{sj} - |\mathrm{B}|^2\mathrm{B}_{ij}\,.$$

Indeed, recalling the computation (1.4), as the normal space is one–dimensional, we have

$$\begin{aligned}
\Delta^M \mathrm{B}_{ij} - \nabla^2_{ij}\mathrm{H} = &\,\mathrm{H}\mathrm{B}_{il}g^{ls}\mathrm{B}_{sj} - \mathrm{B}_{pl}\mathrm{B}_{iq}g^{pq}g^{ls}\mathrm{B}_{sj} - \mathrm{B}_{pl}\mathrm{B}_{jq}g^{pq}g^{ls}\mathrm{B}_{si}\\
&+ (2\mathrm{B}_{pj}\mathrm{B}_{il} - \mathrm{B}_{pl}\mathrm{B}_{ij})\,g^{pq}g^{ls}\mathrm{B}_{sq}\\
= &\,\mathrm{H}\mathrm{B}_{il}g^{ls}\mathrm{B}_{sj} - |\mathrm{B}|^2\mathrm{B}_{ij}\,.
\end{aligned}$$

1.2. Example 1. Curves in the Plane. Let $\gamma : (0,1) \to \mathbb{R}^2$ be a smooth curve in the plane, suppose parametrized by the arclength s.
The metric is simply by ds^2, we define the unit tangent vector $\tau = \gamma_s$ and we choose as unit normal vector $\nu = \mathrm{R}\tau$ where R is the counterclockwise rotation in $\mathbb{R}^2$.
The second fundamental form is given by

$$\mathrm{B}_{ss} = \mathrm{B}(\tau,\tau) = \Big(\nabla^{\mathbb{R}^{n+m}}_\tau \tau\Big)^\perp = (\partial_\tau \gamma_s)^\perp = \gamma^\perp_{ss} = \gamma_{ss}$$

as γ_{ss} is a normal vector.
In the case the curve is not parametrized by arclength, the metric tensor is given by $g_{ss} =$

$|\gamma_s|^2 ds^2$ and

$$\mathrm{B}_{ss} = \mathrm{B}(\tau,\tau) = \left(\nabla^{\mathbb{R}^{n+m}}_\tau \tau\right)^\perp = (\partial_\tau \gamma_s)^\perp = \gamma_{ss}^\perp = \gamma_{ss} - \frac{\langle \gamma_{ss}\,|\,\gamma_s\rangle \gamma_s}{|\gamma_s|^2}\,.$$

The mean curvature vector H is then

$$\mathrm{H} = g^{ss}\mathrm{B}_{ss} = \frac{\gamma_{ss}}{|\gamma_s|^2} - \frac{\langle \gamma_{ss}\,|\,\gamma_s\rangle \gamma_s}{|\gamma_s|^4} = \mathrm{k}\nu\,.$$

The mean curvature function k, which is defined up to the sign, is called by simplicity the *curvature* of γ.

1.3. Example 2. Curves in $\mathbb{R}^n$. Let $\gamma : (0,1) \to \mathbb{R}^n$ be a smooth curve in the space, parametrized by the arclength s.
The metric is again given by ds^2, and we still define the unit tangent vector $\tau = \gamma_s$ but now we do not have an easy way to choose a unit normal vector as in the previous situation.
The second fundamental form is given by

$$\mathrm{B}_{ss} = \mathrm{B}(\tau,\tau) = \left(\nabla^{\mathbb{R}^{n+m}}_\tau \tau\right)^\perp = (\partial_\tau \gamma_s)^\perp = \gamma_{ss}^\perp = \gamma_{ss}$$

as γ_{ss} is a normal vector. If $\gamma_{ss} \neq 0$ we define $|\gamma_{ss}| = \mathrm{k} \neq 0$ and call *unit normal of* γ the vector $\nu = \gamma_{ss}/|\gamma_{ss}|$, that is, $\gamma_{ss} = \mathrm{k}\nu$ and k is the (mean) curvature of γ which is defined up to the sign.

2. Tangential Calculus

We consider now M as an actual subset of $\mathbb{R}^{n+m}$, in order to use the coordinates of the ambient space $\mathbb{R}^{n+m}$, we can always do it at least locally as every immersion is locally an embedding. At every point $x \in M$ we have, as before, the n–dimensional tangent space $T_xM \subset \mathbb{R}^{n+m}$ with an associated linear map $P(x) : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ which is the orthogonal projection on T_xM. Then clearly, the map $(I - P(x)) : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$, where I is the identity of $\mathbb{R}^{n+m}$, is instead the orthogonal projection on the m–dimensional normal space NM at x which is the orthogonal complement of T_xM in $\mathbb{R}^{n+m}$.

In this setting, the canonical measure $\mu = \sqrt{G}\,\mathcal{L}^n$ coincides with the n–dimensional Hausdorff measure *counting multiplicities* $\widetilde{\mathcal{H}}^n \llcorner M$.
If M is actually embedded (or the self–intersections have zero measure), we have $\mu = \mathcal{H}^n \llcorner M$ with $\mathcal{H}^n$ the n–dimensional Hausdorff measure of $\mathbb{R}^{n+m}$.

We call *tangential gradient* $\nabla^M f(x)$ of a C^1 function defined in a neighborhood $U \subset \mathbb{R}^{n+m}$ of a point $x \in M$ as the projection of $\nabla^{\mathbb{R}^{n+m}} f(x)$ on T_xM.
It is easy to check that $\nabla^M f$ depends only on the restriction of f to $M \cap U$. Moreover, an extension argument shows that $\nabla^M f$ can also be defined for functions initially defined only on $M \cap U$.
If P_{ij} is the matrix of orthogonal projection $P : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ on the tangent space (here the indices refer to the coordinates of $\mathbb{R}^{n+m}$), we have $\nabla^M_i f(x) = P_{ij}(x)\nabla^j f(x)$.
Notice that $P_{ij}(x) = \nabla^M_i x_j$ for any $x \in M$.

We also define the *tangential derivative* of a vector field $Y = Y^i e_i$ in $\mathbb{R}^{n+m}$ along M, in the direction of a tangent vector $X \in T_xM$ as

$$\nabla^M_X Y(x) = \sum_{i=1}^{n+m} \langle X\,|\,\nabla^M Y^i\rangle e_i$$

where $e_1, \dots, e_{n+m}$ is the standard basis of $\mathbb{R}^{n+m}$.

In a similar way we can define the *tangential divergence* of a vector field X and the *tangential Laplacian* of a function,

$$\operatorname{div}^M X = \sum_{i=1}^{n+m} \nabla_i^M X^i \,, \qquad\qquad \Delta^M f = \operatorname{div}^M \nabla^M f$$

(here again the indices refer to the coordinates of $\mathbb{R}^{n+m}$).

By a straightforward computation one can check that all these tangential operators (if the field X is tangent to M) coincide with the intrinsic ones considering (M, g) as an abstract Riemannian manifold.

In several occasions we will consider the second fundamental form and the shape operator acting on vector fields in $\mathbb{R}^{n+m}$ as defined in formulas (1.3), that is, if $e_1, \dots, e_{n+m}$ is the standard basis of $\mathbb{R}^{n+m}$ we have

$$\mathrm{B}_{ij}^k = \langle \mathrm{B}(e_i, e_j) \,|\, e_k \rangle = \langle \mathrm{B}(e_i^M, e_j^M) \,|\, e_k^\perp \rangle \,.$$

It is then easy to see that

$$\mathrm{H}^i = \sum_{j=1}^{n+m} \mathrm{B}_{jj}^i$$

and, by means of the above tangential derivative operator, we can compute the second fundamental form as

$$\mathrm{B}(X, Y) = -\sum_{\alpha=1}^{m} \langle X | \nabla_Y^M \nu^\alpha \rangle \nu^\alpha \qquad\qquad \forall X\, Y \in T_x M \,,$$

where $\{\nu^\alpha\}$ is any local smooth orthonormal basis of the normal space to M.

For a general smooth map $\Phi : M \to \mathbb{R}^k$ we can consider the *tangential Jacobian*,

$$J^M \Phi(x) = \left[\det\left(d^M \Phi_x^* \circ d^M \Phi_x\right)\right]^{1/2}$$

where $d^M \Phi_x : T_x M \to \mathbb{R}^k$ is the linear map induced by the the tangential gradient and $\left(d^M \Phi_x\right)^* : \mathbb{R}^k \to T_x M$ is the adjoint map.

THEOREM 2.1 (Area Formula). *If Φ is a smooth injective map from M to $\mathbb{R}^k$, then we have*

$$\int_{\Phi(M)} f(y)\, d\mathcal{H}^n(y) = \int_M f(\Phi(x))\, J^M \Phi(x)\, d\mathcal{H}^n(x)$$

for every $f \in C_c^0(\mathbb{R}^k)$.

If $\{e_i\}$ is an orthonormal basis of $\mathbb{R}^{n+m}$ such that $e_1, \dots, e_n$ is a basis of $T_x M$, we can express the divergence of a tangent vector field X at the point $x \in M$ as

$$\begin{aligned}\operatorname{div} X(x) &= \sum_{i=1}^{n} g(e_i, \nabla_{e_i} X(x)) = \sum_{i=1}^{n} \langle e_i \,|\, \nabla_{e_i}^{\mathbb{R}^{n+m}} X(x) \rangle = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \langle e_i \,|\, X \rangle(x) \\ &= \sum_{i=1}^{n+m} \nabla_i^M \langle e_i \,|\, X \rangle(x) \,.\end{aligned}$$

It is not difficult to see that the last term is actually independent of the orthonormal basis $\{e_i\}$, even if $e_1, \dots, e_n$ is not a basis of T_xM. Then, we use this last expression (for any arbitrary orthonormal basis $\{e_i\}$ of $\mathbb{R}^{n+m}$) to define the tangential divergence $\operatorname{div}^M X$ of a general, not necessarily tangent, vector field X along M.
Such definition is useful in view of the following *tangential divergence formula* (see [**36**, Chap. 2, Sect. 7]),

$$\int_M \operatorname{div}^M X \, d\mu = -\int_M \langle X \,|\, \mathrm{H} \rangle \, d\mu$$

holding for every vector field X along M.
If X is a tangent vector field we recover the usual *divergence theorem*,

$$\int_M \operatorname{div} X \, d\mu = 0 \, .$$

For detailed discussions and proofs of these results we address the reader to the books of Federer [**21**] *and of Simon* [**36**].

3. Distance Functions

In all this section, $e_1, \dots, e_{n+m}$ is the canonical basis of $\mathbb{R}^{n+m}$, M is a smooth, complete, n–dimensional manifold without boundary, embedded in $\mathbb{R}^{n+m}$ and T_xM, N_xM are respectively the tangent space and the normal space to M at $x \in M \subset \mathbb{R}^{n+m}$.

The *distance function* $d^M : \mathbb{R}^{n+m} \to \mathbb{R}$ and the *squared distance function* $\eta^M : \mathbb{R}^{n+m} \to \mathbb{R}$ are respectively defined by

$$d^M(x) = \mathrm{dist}(x, M) = \min_{y \in M} |x - y| , \qquad \eta^M(x) = \frac{1}{2}[d^M(x)]^2$$

for any $x \in \mathbb{R}^{n+m}$ (we will often drop the superscript M). In this and the next sections we analyse the differentiability properties of d and η and the connection between the derivatives of these functions and the geometric properties of M.

Immediately by its definition, being the minimum of a family of Lipschitz functions with Lipschitz constant 1, the same property holds also for d (the function η is instead only locally Lipschitz). In particular, both functions are differentiable almost everywhere in $\mathbb{R}^{n+m}$, by Rademacher's theorem, moreover, at any differentiability point $x \in \mathbb{R}^{n+m}$ of d there exists a *unique* minimizing point $y \in M$ such that $d(x) = |x - y|$ and

$$\nabla d(x) = \frac{x - y}{|x - y|}$$

for such $y \in M$.
Viceversa, if the point in M of minimum distance from $x \in \mathbb{R}^{n+m} \setminus M$ is unique, the function d is differentiable at x, see Section 5.
We have also easily

$$|\nabla d(x)| = 1 \qquad \text{and} \qquad |\nabla \eta(x)|^2 = 2\eta(x)$$

at any differentiability point of d.

These properties are true even if M is merely a closed set (the relation between the regularity properties of d^M and M is analysed in **[20, 22]**, see also **[33]**) but on the second derivatives of d^M and η^M only one side estimates are available, in general. These are actually based on the convexity of the function $A^M(x) = |x|^2/2 - \eta(x)$ which can be expressed as

$$A^M(x) = \max_{y\in M} \langle x|y\rangle - \frac{1}{2}|y|^2 .$$

However, as it is natural to expect, higher regularity of M leads to higher regularity of d^M and η^M as we will see in Section 5 (see also **[4]**, for instance).

PROPOSITION 3.1. *For every point $x \in M$, there exists an open neighborhood of x in $\mathbb{R}^{n+m}$ and a constant $\sigma > 0$ such that η is smooth in the region*

$$\Omega = \{y \in U \mid d(y) < \sigma\} .$$

REMARK 3.2. If M is compact we can actually choose $U = \mathbb{R}^{n+m}$ and a uniform constant $\sigma > 0$. Moreover, since we will be mainly interested in *local* geometric properties of M and since every immersion is locally an embedding, all the differential relations that we are going to discuss hold also for submanifolds with self–intersections. We simply have to consider such local embedding in a open set of $\mathbb{R}^{n+m}$ and the distance function only from this piece of M, in a neighborhood, instead than from the whole M.

By the above discussion, in such set Ω it is defined the *projection map* $\pi^M : \Omega \to M$ associating to any point $x \in \Omega$ the unique minimizer in M of the distance from x (again we will often drop the superscript M). This minimizer point is characterized by

$$\pi^M(x) = x - d^M(x)\nabla d^M(x) = x - \nabla\eta^M(x) .$$

It should be remarked that $d(x) = \sqrt{2\eta(x)}$ is smooth on $\Omega \setminus M$ but it is not smooth up to M. In the codimension one case this difficulty can be amended by considering the *signed* distance function

$$d^*(x) = \begin{cases} d(x) & \text{if } x \notin E \\ -d(x) & \text{if } x \in E \end{cases}$$

as M is the boundary of a bounded subset E of $\mathbb{R}^{n+m}$.

In higher codimension, the function η is a good substitute of $d^*(x)$ in many situations, see **[4]** for an example of application to the motion by mean curvature.

The following result is concerned with the Hessian matrix of η.

PROPOSITION 3.3. *For any $x \in M$ the Hessian matrix $\nabla^2\eta(x)$ is the (matrix of) orthogonal projection onto the normal space N_xM.*
Moreover, for any $x \in M$, letting p to be a unit *vector orthogonal to M at x and defining*

$$\Lambda(s) = \nabla^2\eta(x + sp)$$

for any $s \in [0, \sigma]$ such that the segment $[x, x + \sigma p]$ is contained in Ω, the matrices $\Lambda(s)$ are all diagonal in a common orthonormal basis $\{e_1, \dots, e_{n+m}\}$ such that $\langle e_{n+1}, \dots, e_{n+m}\rangle = N_xM$ and, denoting by $\lambda_1(s), \dots, \lambda_{n+m}(s)$ their eigenvalues in increasing order, we have

$$\lambda_{n+1}(s) = \lambda_{n+2}(s) = \dots = \lambda_{n+m}(s) = 1 \qquad \forall s \in [0, d(x)] .$$

The remaining eigenvalues are strictly less than 1 and satisfy the ODE

$$\lambda_i'(s) = \frac{\lambda_i(s)(1 - \lambda_i(s))}{s} \qquad \forall s \in (0, d(x)]$$

for $i = 1, \dots, n$. Finally, the quotients $\lambda_i(s)/s$ are bounded in $(0, d(x)]$.

PROOF. We follow [**4**, Thm. 3.2]. Fixing $x \in M$ and representing locally M as a graph of a smooth function on the tangent space at x, it is easy to see, by an elementary geometric argument, that

$$\eta(x+y) = \frac{|Ny|^2}{2} + o(|y|^2) = \frac{1}{2}\langle Ny \,|\, y\rangle + o(|y|^2),$$

where N is the orthogonal projection on the normal space to M at the point x and $o(t)$ is a real function satisfying $|o(t)|/t \to 0$ as $t \to 0$. By differentiating twice with respect to y and evaluating at $y = 0$, we find $\eta_{ij}(x) = N_{ij}$.

Since the distance function d is smooth in $\Omega \setminus M$, differentiating the equality $|\nabla d|^2 = 1$, we get

$$d_{ij}d_j = 0\,, \qquad d_{ijk}d_j + d_{ij}d_{jk} = 0\,,$$

in $\Omega \setminus M$ and

$$\eta_j\eta_j = 2\eta\,, \qquad \eta_{ij}\eta_j = \eta_i\,, \qquad \eta_{ijk}\eta_j + \eta_{ij}\eta_{jk} = \eta_{ik}\,, \tag{3.1}$$

in the whole Ω.

Using the fact that $\nabla\eta(x+sp) = ps$ and the third identity in (3.1) we obtain,

$$\begin{aligned}\frac{d}{ds}\,\Lambda_{ij}(s) &= \frac{\partial \eta_{ij}}{\partial x_k}(x+sp)p^k \\ &= \eta_{ijk}(x+sp)\eta_k(x)/s \\ &= \frac{\Lambda_{ij}(s) - \Lambda_{ik}(s)\Lambda_{kj}(s)}{s}\end{aligned} \tag{3.2}$$

for every $s \in [0, \sigma]$.

Let $e_1, \dots, e_{n+m}$ be any basis such that $\Lambda(\sigma)$ is diagonal with associated eigenvalues $\lambda_i(\sigma)$, we consider the unique solution $\mu_i(t)$ of the ODE

$$\frac{d}{ds}\,\mu_i(s) = \frac{\mu_i(s)(1-\mu_i(s))}{s}\,, \qquad \forall s \in (0, \sigma]$$

satisfying $\mu_i(\sigma) = \lambda_i(\sigma)$, for $i = 1, \dots, n+m$.

Then the matrices

$$\widehat{\Lambda}(s) = \sum_{i=1}^{n+m} \mu_i(s) e_i \otimes e_i\,,$$

solve the differential equation (3.2) and satisfy $\widehat{\Lambda}(\sigma) = \Lambda(\sigma)$. Hence, by the uniqueness of solutions to system (3.2), we conclude $\Lambda = \widehat{\Lambda}$. Consequently the eigenvectors of $\Lambda(s)$ are equal to e_i for every $s \in (0, \sigma]$ and the eigenvalues $\lambda_i(s)$ solve,

$$\frac{d}{ds}\,\lambda_i(s) = \frac{\lambda_i(s)(1-\lambda_i(s))}{s}\,. \tag{3.3}$$

In view of the fact that $\Lambda(s)$ must converge, as $s \to 0^+$, to the matrix of orthogonal projection on the normal space to M at the point x, the conclusion of the proposition follows.

Finally, we show that the quotients $\lambda_i(s)/s$ are bounded as $s \to 0^+$, when $i = 1, \dots, n$. Solving the differential equation (3.3), we find

$$\frac{\lambda_i(s)}{s} = \frac{\lambda_i(\sigma)}{\sigma + (s-\sigma)\lambda_i(\sigma)}\,, \qquad \forall s \in (0, \sigma]\,.$$

Therefore, if $\lambda_i(\sigma) < 0$, then $\lambda_i(s) < 0$ for all s and

$$\left|\frac{\lambda_i(s)}{s}\right| \leq \left|\frac{\lambda_i(\sigma)}{\sigma}\right|, \qquad \forall s \in (0, \sigma]\,.$$

If, $\lambda_i(\sigma) > 0$ and $i = 1, \dots, n$, then $\lambda_i(s) \in [0, 1)$ for all s and

$$\left|\frac{\lambda_i(s)}{s}\right| \leq \frac{\lambda_i(\sigma)}{\sigma(1 - \lambda_i(\sigma))}, \qquad \forall s \in (0, \sigma]\,.$$

So finally, for all $s \in (0, \sigma]$ and $i = 1, \dots, n$, we have,

$$\left|\frac{\lambda_i(s)}{s}\right| \leq \max\Big\{\frac{|\lambda|}{\sigma[1 \wedge (1-\lambda)]}\,\Big|\,\lambda < 1 \text{ eigenvalue of } \nabla^2\eta(x) \text{ with } d(x) = \sigma\Big\}$$

and we are done. $\square$

As for every $x \in \Omega$ the gradient $\nabla d(x)$ is a unit vector belonging to $N_{\pi(x)}M$ and constant along the segment $\pi(x) + s(x - \pi(x))$, by using the identity

$$\nabla^2\eta = d\nabla^2 d + \nabla d \otimes \nabla d,$$

it follows that also $\nabla^2 d(\pi(x) + s(x - \pi(x)))$ is diagonal in the same basis above, diagonalizing $\nabla^2\eta(\pi(x))$. Moreover, the eigenvalue associated to the eigenvector $\nabla d(x)$ is zero, $(m-1)$ eigenvalues are equal to $1/s$ and the n remaining ones $\beta_1(s), \dots, \beta_n(s)$ are bounded and satisfy

$$\beta_i'(s) = -\beta_i^2(s) \qquad \forall s \in (0, d(x)] \tag{3.4}$$

as $\beta_i(s) = \lambda_i(s)/s$, for $i = 1, \dots, n$.

A straightforward consequence of Proposition 3.3 is the following result.

COROLLARY 3.4. *Let $x \in \Omega$ and let $\mathrm{K}_x : \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \times \mathbb{R}^{n+m} \to \mathbb{R}$ be the symmetric 3–linear form induced by $\nabla^3\eta(x)$. Then,*

$$\mathrm{K}_x(u, v, w) = 0$$

if at least two of the vectors u, v and w belong to $N_{\pi(x)}M$.

We discuss now a while the geometric meaning of the eigenvalues $\lambda_i(s)$ in Proposition 3.3. We let $x_s = x + sp$ (p is a unit vector orthogonal to T_xM) and we consider the eigenvalues $\lambda_1(s), \dots, \lambda_n(s)$ of $\nabla^2\eta(x_s)$ strictly less than 1 with $e_1, \dots, e_n$ the corresponding eigenvectors (independent of s) spanning T_xM.

PROPOSITION 3.5. *For any $i = 1, \dots, n$ we have*

$$\lim_{s \to 0^+} \frac{\lambda_i(s)}{s} = \lambda_i$$

and the values λ_i are the eigenvalues of the symmetric bilinear form

$$-\big\langle \mathrm{B}(x)(u, v)\,\big|\, p\big\rangle \qquad u, v \in T_xM$$

with associated eigenvectors $\{e_i\}$.

PROOF. By the remark following the proof of Proposition 3.3, $\lambda_i(s)/s$ are the eigenvalues $\beta_i(s)$ of $\nabla^2 d(x_s)$, then the existence of the limits is immediate as the quotients $\lambda_i(s)/s = \beta_i(s)$ are bounded and monotone, by (3.4), as $s \to 0^+$.
Let L be the affine $(n+1)$–dimensional space generated by T_xM and p, passing through x. Moreover, let $\Sigma \subset L$ be the smooth n–dimensional manifold obtained projecting $U \cap M$ on L, for a suitable neighborhood U of x, and let $\overline{\mathrm{B}}(x)$ be the second fundamental form of Σ at x, viewing Σ as a surface of codimension one in L. We denote (see Section 1.1) by $\lambda_1, \dots, \lambda_n$ the principal curvatures at x of Σ (with the orientation induced near x by p), defined as the eigenvalues of the symmetric bilinear form

$$\langle \overline{\mathrm{B}}(x)(u,v) \,|\, p\rangle \qquad\qquad u, v \in T_x\Sigma = T_xM\,.$$

Under the assumption $m = 1$, we clearly have $\Sigma = M$ and the property is a straightforward consequence of the well known formula (see for instance [**25**, Lemma 14.17])

$$\beta_i(s) = \frac{-\lambda_i}{1 - s\lambda_i} \qquad\qquad \forall s \in (0, d(x)]$$

for the eigenvalues $\beta_i(s)$ of $\nabla^2 d^\Sigma(x_s)$ corresponding to eigenvectors in L (see also [**19**]).
In the general case, we notice that, by Proposition 3.1, the function η^Σ is smooth near x and

$$\limsup_{y \to x,\, y \in L} \frac{|\eta^M(y) - \eta^\Sigma(y)|}{|y - x|^4} < +\infty \tag{3.5}$$

since Σ is obtained projecting M on the space L, containing $x + T_xM$. By this limit we infer

$$\lim_{s \to 0^+} \frac{\nabla^2\eta^M(x_s) - \nabla^2\eta^\Sigma(x_s)}{s} = 0\,.$$

As all the matrices are diagonal in the same basis, denoting by $\overline{\lambda}_i(s)$ the eigenvalues of $\nabla^2\eta^\Sigma(x_s)$ corresponding to the directions $\{e_i\}$, the quotients $\lambda_i(s)/s$ converge to the same limit of $\overline{\lambda}_i(s)/s$, that is, λ_i.

Finally, by (3.5) we have

$$\nabla^3\eta^M(x)(u,v,p) = \nabla^3\eta^\Sigma(x)(u,v,p) \qquad\qquad \forall u, v \in T_xM = T_x\Sigma\,,$$

hence, the relations in Proposition 3.9, that we will discuss in a while, yield

$$\big\langle \mathrm{B}(x)(u,v) \,|\, p\big\rangle = \big\langle \overline{\mathrm{B}}(x)(u,v) \,|\, p\big\rangle \qquad\qquad \forall u, v \in T_xM$$

as $p \in N_xM \cap N_x\Sigma$.
This shows that λ_i are the eigenvalues of $-\langle \mathrm{B}(x) \,|\, p\rangle$ and that $\{e_i\}$ are the corresponding eigenvectors. $\square$

REMARK 3.6. In particular, the sum of the eigenvalues $\beta_i(s) = \lambda_i(s)/s$ of $\nabla^2 d(x_s)$ converges as $s \to 0^+$ to the quantity $-\langle \mathrm{H}(x) \,|\, p\rangle$. This property has been used in [**4**] to extend the level set approach (see [**11, 18, 34**]) to the evolution by mean curvature of surfaces of any codimension.

For $x \in M$, we defined $P_{ij}(x)$ as the matrix of orthogonal projection $P : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ on the tangent space and we saw that $P_{ij}(x) = \nabla^M_i x_j$. Actually, by Proposition 3.3, we have

$$P_{ij}(x) = (\delta_{ij} - \eta_{ij}(x))\,,$$

since $\eta_{ij}(x)$ is the matrix of orthogonal projection on $N_x M$. Notice that such formula defining $P_{ij}(x)$ makes sense in the whole Ω, in this case, Proposition 3.3 implies

$$P(x)(T_{\pi(x)}M) = T_{\pi(x)}M, \qquad \text{and} \qquad \operatorname{Ker} P(x) = N_{\pi(x)}M\,.$$

However, we advise the reader that in general $P(x)$ is not the identity on $T_{\pi(x)}M$ ($\nabla^2 eta$ is the identity on $N_{\pi(x)}M$).

We now define the 3–tensor C with components (in the canonical basis)

$$C_{ijk}(x) = \nabla^M_i P_{jk}(x) = \nabla^M_i \nabla^M_j x_k\,,$$

which is clearly symmetric in the last two indices.
Since for any $x \in M$ the matrix $P(x)$ is the orthogonal projection on $T_x M$, we can expect that the tensor $C(x)$ (encoding the "change" in the tangent plane) contains all information on the curvature of M (see [**26, 32**]). In the following three proposition we will see that $\nabla^3\eta(x)$, the tensor $C(x)$ and the second fundamental form $\mathrm{B}(x)$ are mutually connected by simple linear relations.

PROPOSITION 3.7. *The second fundamental form tensors* $\mathrm{B}(x)$ *and the tensor* $C(x)$ *are related for any* $x \in M$ *by the identities*

$$\mathrm{B}^k_{ij}(x) = P_{is}(x) C_{jsk}(x) = P_{js}(x) C_{isk}(x)\,, \qquad C_{ijk}(x) = \mathrm{B}^k_{ij}(x) + \mathrm{B}^j_{ik}(x)\,. \tag{3.6}$$

Moreover, the mean curvature vector $\mathrm{H}(x)$ *of* M *is given by*

$$\mathrm{H}^k(x) = \sum_{s=1}^{n+m} C_{sks}(x)\,. \tag{3.7}$$

PROOF. We follow [**26**]. Let $x \in M$, $u = e_i$, $v = e_j$ and let $u' = P(x)e_i$, $v' = P(x)e_j$ be the projections of u and v on $T_x M$. We have then, at the point $x \in M$,

$$\begin{aligned}
\mathrm{B}^k_{ij} &= \frac{\partial [Pe_i]^s}{\partial v'}(\delta_{sk} - P_{sk}) = \frac{\partial P_{is}}{\partial v'}(\delta_{sk} - P_{sk}) = \nabla_l P_{is} P_{lj}(\delta_{sk} - P_{sk})\\
&= \nabla^M_j P_{is}(\delta_{sk} - P_{sk}) = \nabla^M_j P_{ik} - \nabla^M_j (P_{is}P_{sk}) + P_{is}\nabla^M_j P_{sk}\\
&= \nabla^M_j P_{ik} - \nabla^M_j P_{ik} + + P_{is}\nabla^M_j P_{sk}\\
&= P_{is}\nabla^M_j P_{sk} = P_{is} C_{jsk}
\end{aligned}$$

where we used the fact that $P^2 = P$ on M. The other relation follows by the symmetry of B.

Now we prove the second identity in (3.6). Using the first identity and the symmetry of P we get

$$\begin{aligned}
\mathrm{B}^k_{ij} + \mathrm{B}^j_{ik} &= P_{js} C_{isk} + P_{ks} C_{isj}\\
&= P_{js}\nabla^M_i P_{sk} + P_{ks}\nabla^M_i P_{sj}\\
&= \nabla^M_i (P_{js} P_{sk})\\
&= \nabla^M_i P_{jk}\\
&= C_{ijk}\,.
\end{aligned}$$

Finally, we prove (3.7),

$$\mathrm{H}^k = \mathrm{B}^k_{ii} = P_{is} C_{isk} = P_{is}\nabla^M_i P_{sk} = \nabla^M_s P_{sk} = \sum_{s=1}^{n+m} C_{sks}\,.$$

□

PROPOSITION 3.8. *The tensor $C(x)$ and $\nabla^3\eta(x)$ are related for any $x \in M$ by the identities*

$$C_{ijk}(x) = -P_{il}(x)\eta_{ljk}(x)\,, \qquad \eta_{ijk}(x) = -\frac{1}{2}\big\{C_{ijk}(x) + C_{jki}(x) + C_{kij}(x)\big\}\,. \tag{3.8}$$

PROOF. The first identity is an easy consequence of the fact that $\nabla^2\eta(x)$ is the orthogonal projection on N_xM. To prove the second one, we write (omitting the dependence on x)

$$\begin{aligned}
\eta_{ijk} = &- C_{ijk} + (\delta_{is} - P_{is})\eta_{sjk}\\
= &- C_{ijk} + (\delta_{is} - P_{is})(-C_{jsk} + (\delta_{jt} - P_{jt})\eta_{stk})\\
= &- C_{ijk} + (\delta_{is} - P_{is})(-C_{jsk} + (\delta_{jt} - P_{jt})(-C_{kst} + (\delta_{kl} - P_{kl})\eta_{stl}))\\
= &- C_{ijk} - C_{jsk}(\delta_{is} - P_{is}) - C_{kst}(\delta_{is} - P_{is})(\delta_{jt} - P_{jt})\\
&+ (\delta_{is} - P_{is})(\delta_{jt} - P_{jt})(\delta_{kl} - P_{kl})\eta_{stl}\,.
\end{aligned}$$

By Corollary 3.4, the last term is zero, so that (3.6) yields

$$\begin{aligned}
\eta_{ijk} = &- C_{ijk} - C_{jki} + C_{jsk}P_{is} - C_{kij} + C_{kit}P_{jt} + C_{ksj}P_{si} - C_{kst}P_{is}P_{jt}\\
= &- C_{ijk} - C_{jki} - C_{kij} + \mathrm{B}^k_{ij} + \mathrm{B}^j_{ki} + \mathrm{B}^i_{jk} - P_{jt}\mathrm{B}^t_{ik}\,.
\end{aligned}$$

Since $\mathrm{B}(e_i, e_k) \in N_xM$ we have $P_{jt}\mathrm{B}^t_{ik} = 0$, then exchanging the indices i and j in the above formula, averaging and using the second identity in (3.6) we eventually get

$$\begin{aligned}
\eta_{ijk} = &- C_{ijk} - C_{jki} - C_{kij} + \frac{1}{2}\big\{\mathrm{B}^k_{ij} + \mathrm{B}^j_{ki} + \mathrm{B}^i_{jk} + \mathrm{B}^k_{ji} + \mathrm{B}^i_{kj} + \mathrm{B}^j_{ik}\big\}\\
= &- \frac{1}{2}\big\{C_{ijk} + C_{jki} + C_{kij}\big\}\,.
\end{aligned}$$

□

PROPOSITION 3.9. *The second fundamental form $\mathrm{B}(x)$ and $\nabla^3\eta(x)$ are related for any $x \in M$ by the identities*

$$\mathrm{B}^k_{ij}(x) = \nabla_k(\eta_{is}\eta_{sj} - \eta_{ij})(x)\,, \qquad \eta_{ijk}(x) = -\mathrm{B}^k_{ij}(x) - \mathrm{B}^i_{jk}(x) - \mathrm{B}^j_{ki}(x)\,. \tag{3.9}$$

Moreover, the mean curvature vector $\mathrm{H}(x)$ of M is given by

$$\mathrm{H}(x) = -\Delta(\nabla\eta)(x)\,.$$

PROOF. By Using relations (3.6) and (3.8) we can write each component B^k_{ij} of the second fundamental form as a function of $\nabla^3\eta$ as follows,

$$\begin{aligned}
\mathrm{B}^k_{ij} &= P_{js}C_{iks}\\
&= -P_{js}P_{il}\eta_{lks}\\
&= -(\delta_{js} - \eta_{js})(\delta_{il} - \eta_{il})\eta_{lks}\\
&= -\eta_{ijk} + \eta_{sj}\eta_{kis} + \eta_{li}\eta_{kjl} - \eta_{js}\eta_{il}\eta_{lks}\\
&= -\eta_{ijk} + \eta_{sj}\eta_{kis} + \eta_{si}\eta_{kjs}\\
&= \nabla_k(\eta_{is}\eta_{sj} - \eta_{ij})\,.
\end{aligned} \tag{3.10}$$

Conversely, by the second identities in (3.8) and (3.6) we get

$$\begin{aligned}\eta_{ijk} &= -\frac{1}{2}\big\{C_{ijk} + C_{jki} + C_{kij}\big\}\\ &= -\frac{1}{2}\big\{\mathrm{B}^k_{ij} + \mathrm{B}^j_{ik} + \mathrm{B}^i_{jk} + \mathrm{B}^k_{ji} + \mathrm{B}^j_{ki} + \mathrm{B}^i_{kj}\big\}\\ &= -\mathrm{B}^k_{ij} + \mathrm{B}^i_{jk} + \mathrm{B}^j_{ki}\,.\end{aligned}$$

By the first formula, we have

$$\mathrm{H}^k = -\eta_{kii} + \nabla_k\Big(\sum_{i,s=1}^{n+m}\eta^2_{is}\Big)$$

for every index $k = 1,\dots,n+m$. Since $\nabla^2\eta(x)$ is symmetric, $\sum_{i,s=1}^{n+m}\eta^2_{is}(x)$ coincides with the sum of the squares of the eigenvalues of $\nabla^2\eta(x)$. By Proposition 3.3, this quantity is equal to $n + o(|x-x^0|)$ near every point $x^0\in M$, hence $\nabla_k\big(\sum_{i,s=1}^{n+m}\eta^2_{is}\big)(x)$ vanishes on M. It follows that

$$\mathrm{H}(x) = -\Delta(\nabla\eta)(x) \qquad \forall x\in M\,. \tag{3.11}$$

□

COROLLARY 3.10. *Let $x\in M$ and let $\mathrm{K}_x : \mathbb{R}^{n+m}\times\mathbb{R}^{n+m}\times\mathbb{R}^{n+m}\to\mathbb{R}$ be the symmetric 3–linear form induced by $\nabla^3\eta(x)$. Then,*

$$\mathrm{K}_x(u,v,w) = 0$$

if all the three vectors u, v and w belong to $T_{\pi(x)}M$.

PROOF. It follows by the second relation in (3.9), as the second fundamental form takes values in the normal space to M at x. □

From now on, instead of dealing with the squared distance function we will consider the function

$$A^M(x) = \frac{|x|^2 - [d^M(x)]^2}{2}\,,$$

clearly smooth as η^M in the neighborhood Ω of M. We set

$$A^M_{i_1\dots i_k}(x) = \frac{\partial^k A^M(x)}{\partial x_{i_1}\dots\partial x_{i_k}}$$

for the derivatives of A^M in Ω.
We define the k–derivative symmetric tensor $A^k(x)$ working on the k–uple of vectors $v_i\in\mathbb{R}^{n+m}$, where $v_i = v_i^j e_j$, as follows

$$A^k(x)(v_1,\dots,v_k) = A^M_{i_1\dots i_k}(x)v_1^{i_1}\dots v_k^{i_k}\,.$$

By sake of simplicity, we dropped the superscript M on A^k, by the same reason, we will also often avoid to indicate the point $x\in M$ in the sequel.

The greater convenience of A^M can be explained noticing that $\nabla^2 A^M(x)$, for $x\in M$, is the projection matrix on T_xM and this quantity often appears in the computation of tangential gradients.

We reformulate now the previous formulas in terms of A^M.

PROPOSITION 3.11. *The following properties of A^M hold,*

(a) *for any $x \in \Omega$, the vector $\nabla A^M(x)$ coincide with the projection point $\pi^M(x)$ of x on M. Moreover, $\nabla^2 A^M(x)$ is zero on $N_{\pi(x)}M$ and maps $T_{\pi(x)}M$ onto $T_{\pi(x)}M$. If $x \in M$, then $\nabla^2 A^M(x)$ is the matrix P of orthogonal projection on T_xM;*
(b) *for any $x \in \Omega$, the 3–linear form $\mathrm{K}_x : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ given by*

$$\mathrm{K}_x(u, v, w) = \sum_{i,j,k=1}^{n+m} A^M_{ijk}(x) u^i v^j w^k$$

is equal to zero if at least two of the 3 vectors u, v, w, are normal to M at $\pi(x) = \nabla A^M(x)$ or if $x \in M$ and the three vectors are all tangent;
(c) *for $x \in M$, the second fundamental form $\mathrm{B}(x)$ and the mean curvature vector $\mathrm{H}(x)$ are related to the derivatives of $A^M(x)$ by*

$$\mathrm{B}^k_{ij}(x) = A^M_{js}(x) A^M_{il}(x) A^M_{slk}(x) = \left(\delta_{kl} - A^M_{kl}(x)\right) A^M_{ijl}(x), \tag{3.12}$$

$$\mathrm{H}^k(x) = \sum_{j=1}^{n+m} A^M_{jkj}(x), \tag{3.13}$$

$$\nabla^M_i A^M_{jk}(x) = \mathrm{B}^k_{ij}(x) + \mathrm{B}^j_{ik}(x). \tag{3.14}$$

PROOF. The first statement follows by Proposition 3.3 and the second one by Corollary 3.4. The first equality in (3.12) and (3.13) follow by relations (3.11) and (3.10). The second equality in (3.12) can be obtained multiplying both sides of the second relation in (3.9) by the normal projection $(I - \nabla^2 A^M)$. Finally (3.14) is a restatement of the second equality in (3.6). □

By means of the relations in Propositions 3.7, 3.8, 3.9 we have the following estimates.

COROLLARY 3.12. *At every point of M we have,*

$$|C|^2 \leq |\nabla^3 A^M|^2 = 3|\mathrm{B}|^2 \leq 3|C|^2 .$$

PROOF. We have only to show the identity $|\nabla^3 A^M|^2 = 3|\mathrm{B}|^2$, the other inequalities are immediate as the projection P is a 1–Lipschitz map.
We compute in a orthonormal basis $\{e_i\}$ such that $\langle e_1, \dots, e_n\rangle = T_xM$, by means of the second relation in (3.9), and keeping in mind that the second fundamental form B takes

values in the normal space N_xM,

$$\begin{aligned}
|\nabla^3 A^M|^2 &= \sum_{i,j,k=1}^{n} |\eta_{ijk}|^2 \\
&= \sum_{\substack{n+1\leq i\leq n+m\\ 1\leq j,k\leq n}} |\eta_{ijk}|^2 + \sum_{\substack{n+1\leq j\leq n+m\\ 1\leq i,k\leq n}} |\eta_{ijk}|^2 + \sum_{\substack{n+1\leq k\leq n+m\\ 1\leq i,j\leq n}} |\eta_{ijk}|^2 \\
&= \sum_{\substack{n+1\leq i\leq n+m\\ 1\leq j,k\leq n}} |\mathrm{B}^k_{ij} + \mathrm{B}^i_{jk} + \mathrm{B}^j_{ki}|^2 + \sum_{\substack{n+1\leq j\leq n+m\\ 1\leq i,k\leq n}} |\mathrm{B}^k_{ij} + \mathrm{B}^i_{jk} + \mathrm{B}^j_{ki}|^2 \\
&\quad + \sum_{\substack{n+1\leq k\leq n+m\\ 1\leq i,j\leq n}} |\mathrm{B}^k_{ij} + \mathrm{B}^i_{jk} + \mathrm{B}^j_{ki}|^2 \\
&= \sum_{\substack{n+1\leq i\leq n+m\\ 1\leq j,k\leq n}} |\mathrm{B}^i_{jk}|^2 + \sum_{\substack{n+1\leq j\leq n+m\\ 1\leq i,k\leq n}} |\mathrm{B}^j_{ki}|^2 + \sum_{\substack{n+1\leq k\leq n+m\\ 1\leq i,j\leq n}} |\mathrm{B}^k_{ij}|^2 \\
&= 3 \sum_{\substack{n+1\leq i\leq n+m\\ 1\leq j,k\leq n}} |\mathrm{B}^i_{jk}|^2 \\
&= 3|\mathrm{B}|^2\,.
\end{aligned}$$

□

4. Higher Order Relations

In this section we work out some properties, about the higher derivatives of the square of the distance function from a submanifold, in particular the relations with the covariant derivatives of the second fundamental form. The main result here is a recurrence formula for A^k (Proposition 4.1), that is, the tensor of k–derivatives of the squared distance function from M, once its action is split on tangent and normal vectors. Such formula is crucial to get "structure information" and estimates on the tensors A^k (Corollary 4.3 and Proposition 4.6).

PROPOSITION 4.1. *For every $k \geq 2$ and for every $s \in \{0, \dots k\}$ there exists a family $p^{k,s}_{j_1 \dots j_{k-s}}$ of symmetric polynomial tensors of type $(s, 0)$ on M, where $j_1, \dots, j_{k-s} \in \{1, \dots, n+m\}$, which are contractions of the second fundamental form* B *and its covariant derivatives with the metric tensor g, such that*

$$A^k(X_1, \dots, X_s, N_1, \dots, N_{k-s}) = p^{k,s}_{j_1 \dots j_{k-s}}(X_1, \dots, X_s) N_1^{j_1} \dots N_{k-s}^{j_{k-s}}$$

for every s–uple of tangent vectors X_h and $(k-s)$–uple of normal vectors N_h in $\mathbb{R}^{n+m}$ (with the obvious interpretation if $s = 0$ or $s = k$, that is, for instance in this latter case the symbols indexed by $1, \dots, k-s$ are not present in the formulas).
Moreover, the tensors $p^{k,s}_{j_1 \dots j_{k-s}}$ are invariant by exchange of the j–indices and the maximum order of differentiation of B *which appears in every $p^{k,s}_{j_1 \dots j_{k-s}}$ is at most $k-3$, when $k \geq 3$. Considering the tangent plane at any point $x \in M$ also as a subset of $\mathbb{R}^{n+m}$, the polynomial tensors $p^{k,s}_{j_1 \dots j_{k-s}}$ are expressed in the coordinate basis of the Euclidean space as follows*

$$p^{k,s}_{j_1 \dots j_{k-s}}(X_1, \dots, X_s) N_1^{j_1} \dots N_{k-s}^{j_{k-s}} = p^{k,s}_{j_1 \dots j_{k-s}, i_1 \dots i_s} X_1^{i_1} \dots X_s^{i_s} N_1^{j_1} \dots N_{k-s}^{j_{k-s}}\,.$$

Then, a family of tensors satisfying the above properties can be defined recursively according to the following formulas

$$p^{2,0}_{j_1j_2} = p^{2,1}_{j_1,i_1} = 0\,, \qquad p^{2,2}_{i_1i_2} = \delta_{i_1i_2} \tag{4.1}$$

$$p^{k,0}_{j_1\dots j_k} = p^{k,1}_{j_1\dots j_{k-1},i_1} = 0 \qquad \textit{for every } k\geq 2 \tag{4.2}$$

$$\begin{aligned} p^{k+1,s}_{j_1\dots j_{k-s+1},i_0i_1\dots i_{s-1}} &= (\nabla p^{k,s-1}_{j_1\dots j_{k-s+1}})_{i_0i_1\dots i_{s-1}} \\ &\quad - \sum_{h=1}^{k-s+1} p^{k,s-1}_{j_1\dots j_{h-1}rj_{h+1}\dots j_{k-s+1},i_1\dots i_{s-1}}\mathrm{B}^{j_h}_{ri_0} \\ &\quad - \sum_{h=1}^{s-1} p^{k,s-2}_{rj_1\dots j_{k-s+1},i_1\dots i_{h-1}i_{h+1}\dots i_{s-1}}\mathrm{B}^{r}_{i_0i_h} \\ &\quad + \sum_{h=1}^{k-s+1} p^{k,s}_{j_1\dots j_{h-1}j_{h+1}\dots j_{k-s+1},i_1\dots i_{s-1}r}\mathrm{B}^{j_h}_{ri_0} \end{aligned} \qquad \textit{if } 2\leq s<k+1 \tag{4.3}$$

$$p^{k+1,k+1}_{i_0i_1\dots i_{k+1}} = \nabla p^{k,k}_{i_0i_1\dots i_k} - \sum_{h=1}^{k} p^{k,k-1}_{r,i_1\dots i_{h-1}i_{h+1}\dots i_k}\mathrm{B}^{r}_{i_0i_h}\,. \tag{4.4}$$

PROOF. If $k=2$ we have immediately

$$A^2(N_1,N_2)=0, \qquad A^2(X_1,N_1)=0, \qquad A^2(X_1,X_2)=X_1^iX_2^i=\delta_{i_1i_2}X_1^{i_1}X_2^{i_2}$$

since X_1 and X_2 are tangent and A^2 is the projection on the tangent space. Hence, formula (4.1) follows.
We argue now by induction on $k\geq 2$. When $s=0$ the value $A^k(N_1,\dots,N_k)(x)$ depends only on the function A^M on the m–dimensional normal subspace to M at x, and on this subspace A^M is identically zero, hence the first equality in (4.2) is proved.
Suppose now that $s\in\{1,\dots,k+1\}$, we extend the vectors $X_h\in T_xM$ and $N_h\in N_xM$ to a family of local vector fields, respectively tangent and normal to M, then

$$\begin{aligned} A^{k+1}(X_0,X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1}) &= \frac{\partial}{\partial X_0}\Big(A^k(X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})\Big) \\ &\quad - \sum_{h=1}^{s-1} A^k\Big(X_1,\dots X_{h-1},\frac{\partial X_h}{\partial X_0},X_{h+1},\dots,X_{s-1},N_1,\dots,N_{k-s+1}\Big) \\ &\quad - \sum_{h=1}^{k-s+1} A^k\Big(X_1,\dots,X_{s-1},N_1,\dots,\frac{\partial N_h}{\partial X_0},\dots,N_{k-s+1}\Big) \end{aligned}$$

where the last line is not present in the special case $s=k+1$ and the second line is not present if $s=1$. In this last case, we have

$$A^{k+1}(X_0,N_1,\dots,N_k) = \frac{\partial}{\partial X_0}\Big(A^k(N_1,\dots,N_k)\Big) - \sum_{h=1}^{k} A^k\Big(N_1,\dots,\frac{\partial N_h}{\partial X_0},\dots,N_k\Big) = 0$$

since the first term of the right member is zero by the first equality in (4.2) and, after decomposing $\frac{\partial N_h}{\partial X_0}$ in tangent and normal part, the tangent term is zero by induction and the normal

term is zero for (4.2) again. This shows the second equality in (4.2).
So we suppose $1 < s < k+1$, by the inductive hypothesis,

$$A^k(X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1}) = p^{k,s-1}_{j_1\cdots j_{k-s+1}}(X_1,\dots,X_{s-1})N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}$$

thus, differentiating along X_0, which is a tangent field, we obtain

$$\begin{aligned}
A^{k+1}(X_0,&X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})\\
&=\frac{\partial}{\partial X_0}\Big(p^{k,s-1}_{j_1\cdots j_{k-s+1}}(X_1,\dots,X_{s-1})N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\Big)\\
&\quad-\sum_{h=1}^{s-1}A^k\Big(X_1,\dots,\Big(\frac{\partial X_h}{\partial X_0}\Big)^M,\dots,X_{s-1},N_1,\dots,N_{k-s+1}\Big)\\
&\quad-\sum_{h=1}^{s-1}A^k\Big(X_1,\dots,\Big(\frac{\partial X_h}{\partial X_0}\Big)^\perp,\dots,X_{s-1},N_1,\dots,N_{k-s+1}\Big)\\
&\quad-\sum_{h=1}^{k-s+1}A^k\Big(X_1,\dots,X_{s-1},N_1,\dots,\Big(\frac{\partial N_h}{\partial X_0}\Big)^M,\dots,N_{k-s+1}\Big)\\
&\quad-\sum_{h=1}^{k-s+1}A^k\Big(X_1,\dots,X_{s-1},N_1,\dots,\Big(\frac{\partial N_h}{\partial X_0}\Big)^\perp,\dots,N_{k-s+1}\Big)\,.
\end{aligned}$$

We use now the symmetry of A^k and we substitute recursively $p^{k,s}$, $p^{k,s-1}$ and $p^{k,s-2}$ to A^k, according to the number of tangent vectors inside A^k,

$$\begin{aligned}
A^{k+1}(X_0,&X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})\\
&=\frac{\partial}{\partial X_0}\Big(p^{k,s-1}_{j_1\cdots j_{k-s+1}}(X_1,\dots,X_{s-1})\Big)N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad+\sum_{h=1}^{k-s+1}p^{k,s-1}_{j_1\cdots j_{k-s+1}}(X_1,\dots,X_{s-1})N_1^{j_1}\dots\frac{\partial N_h^{j_h}}{\partial X_0}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{s-1}p^{k,s-1}_{j_1\cdots j_{k-s+1}}(X_1,\dots,\nabla_{X_0}X_h,\dots,X_{s-1})N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{s-1}p^{k,s-2}_{rj_1\cdots j_{k-s+1}}(X_1,\dots,X_{h-1},X_{h+1},\dots,X_{s-1})\Big[\Big(\frac{\partial X_h}{\partial X_0}\Big)^\perp\Big]^r N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{k-s+1}p^{k,s}_{j_1\cdots j_{h-1}j_{h+1}\cdots j_{k-s+1}}\Big(X_1,\dots,X_{s-1},\Big(\frac{\partial N_h}{\partial X_0}\Big)^M\Big)N_1^{j_1}\dots N_{h-1}^{j_{h-1}}N_{h+1}^{j_{h+1}}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{k-s+1}p^{k,s-1}_{j_1\cdots j_{k-s+1}}(X_1,\dots,X_{s-1})N_1^{j_1}\dots\Big[\Big(\frac{\partial N_h}{\partial X_0}\Big)^\perp\Big]^{j_h}\dots N_{k-s+1}^{j_{k-s+1}}\,.
\end{aligned}$$

Adding the first and the third line on the right hand side we get the covariant derivative of the tensor $p^{k,s-1}_{j_1\cdots j_{k-s+1}}$ times $N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}$, adding the second and the last line we get

$$
\begin{aligned}
A^{k+1}(X_0,&X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})\\
&=\nabla p^{k,s-1}_{j_1\cdots j_{k-s+1}}(X_0,X_1,\dots,X_{s-1})N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad+\sum_{h=1}^{k-s+1}p^{k,s-1}_{j_1\cdots j_{k-s+1}}(X_1,\dots,X_{s-1})N_1^{j_1}\dots\Big[\Big(\frac{\partial N_h}{\partial X_0}\Big)^M\Big]^{j_h}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{s-1}p^{k,s-2}_{rj_1\cdots j_{k-s+1}}(X_1,\dots,X_{h-1},X_{h+1},\dots,X_{s-1})\Big[\Big(\frac{\partial X_h}{\partial X_0}\Big)^\perp\Big]^r N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{k-s+1}p^{k,s}_{j_1\cdots j_{h-1}j_{h+1}\cdots j_{k-s+1}}\Big(X_1,\dots,X_{s-1},\Big(\frac{\partial N_h}{\partial X_0}\Big)^M\Big)N_1^{j_1}\dots N_{h-1}^{j_{h-1}}N_{h+1}^{j_{h+1}}\dots N_{k-s+1}^{j_{k-s+1}}\,.
\end{aligned}
$$

Taking now into account that

$$
\Big[\Big(\frac{\partial N_h}{\partial X_0}\Big)^M\Big]^r=\Big[\Big\langle\frac{\partial N_h}{\partial X_0},\frac{\partial}{\partial x_i}\Big\rangle\frac{\partial}{\partial x_i}\Big]^r=-\Big\langle N_h,\frac{\partial}{\partial X_0}\frac{\partial}{\partial x_i}\Big\rangle\Big\langle\frac{\partial}{\partial x_i},e_r\Big\rangle=-\mathrm{B}^{j_h}_{ri_0}X_0^{i_0}N_h^{j_h}\,,
$$

where $\Big\{\frac{\partial}{\partial x_i}\Big\}_{i=1,\dots,n}$ is a basis of the tangent space of M, and

$$
\Big[\Big(\frac{\partial X_h}{\partial X_0}\Big)^\perp\Big]^r=\mathrm{B}^r_{i_0i_h}X_0^{i_0}X_h^{i_h}\,,
$$

substituting, we get

$$
\begin{aligned}
A^{k+1}(X_0,&X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})\\
&=\nabla p^{k,s-1}_{j_1\cdots j_{k-s+1}}(X_0,X_1,\dots,X_{s-1})N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{k-s+1}p^{k,s-1}_{j_1\cdots j_{h-1}rj_{h+1}\cdots j_{k-s+1}}(X_1,\dots,X_{s-1})\mathrm{B}^{j_h}_{ri_0}X_0^{i_0}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{s-1}p^{k,s-2}_{rj_1\cdots j_{k-s+1}}(X_1,\dots,X_{h-1},X_{h+1},\dots,X_{s-1})\mathrm{B}^r_{i_0i_h}X_0^{i_0}X_h^{i_h}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad+\sum_{h=1}^{k-s+1}p^{k,s}_{j_1\cdots j_{h-1}j_{h+1}\cdots j_{k-s+1}}(X_1,\dots,X_{s-1},\mathrm{B}^{j_h}_{ri_0}X_0^{i_0}e_r)N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\,.
\end{aligned}
$$

Then, expressing the tensors in coordinates, we have

$$
\begin{aligned}
A^{k+1}&(X_0,X_1,\dots,X_{s-1},N_1,\dots,N_{k-s+1})\\
&=(\nabla p^{k,s-1}_{j_1\dots j_{k-s+1}})_{i_0i_1\dots i_{s-1}}X_0^{i_0}X_1^{i_1}\dots X_{s-1}^{i_{s-1}}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{k-s+1}p^{k,s-1}_{j_1\dots j_{h-1}rj_{h+1}\dots j_{k-s+1},i_1\dots i_{s-1}}\mathrm{B}^{j_h}_{ri_0}X_0^{i_0}\dots X_{s-1}^{i_{s-1}}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad-\sum_{h=1}^{s-1}p^{k,s-2}_{rj_1\dots j_{k-s+1},i_1\dots i_{h-1}i_{h+1}\dots i_{s-1}}\mathrm{B}^{r}_{i_0i_h}X_0^{i_0}\dots X_{s-1}^{i_{s-1}}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\\
&\quad+\sum_{h=1}^{k-s+1}p^{k,s}_{j_1\dots j_{h-1}j_{h+1}\dots j_{k-s+1},i_1\dots i_{s-1}r}\mathrm{B}^{j_h}_{ri_0}X_0^{i_0}\dots X_{s-1}^{i_{s-1}}N_1^{j_1}\dots N_{k-s+1}^{j_{k-s+1}}\,,
\end{aligned}
$$

which is formula (4.3).
In the special case $s = k+1$, to get formula (4.4), we just have to repeat the computations dropping all the lines containing sums like $\sum_{h=1}^{k-s+1}$ ..., which are not present.
Finally, assuming inductively that the polynomial tensors $p^{k,s}$, $p^{k,s-1}$ and $p^{k,s-2}$ are symmetric in the j–indices and contain covariant derivatives of B only up to the order $k-3$ (when $k\geq 3$), also the claims about the symmetry and the order of the derivatives of B follow. □

EXAMPLE 4.2. We compute some $p^{k,s}$ as a consequence of this proposition.

(1) When $k = 2$ we saw that

$$p^{2,0}_{j_1j_2}=0,\qquad p^{2,1}_{j_1}=0,\qquad p^{2,2}=g\,.$$

(2) When $k = 3$ we have, by means of formulas (4.2) and (4.3),

$$
\begin{aligned}
&p^{3,0}_{j_1j_2j_3}=0,\qquad p^{3,1}_{j_1j_2}=0\\
&p^{3,2}_{j_1,i_1i_2}=p^{2,2}_{i_2r}\mathrm{B}^{j_1}_{ri_1}=\mathrm{B}^{j_1}_{i_1i_2}\\
&p^{3,3}_{i_1i_2i_3}=(\nabla p^{2,2})_{i_1i_2i_3}+p^{2,1}_{r,i_2}\mathrm{B}^{r}_{i_1i_3}+p^{2,1}_{r,i_3}\mathrm{B}^{r}_{i_1i_2}=0
\end{aligned}
$$

that is,

$$p^{3,2}_{j_1}=\mathrm{B}^{j_1}\text{ and }p^{3,3}=0\,.$$

(3) When $k = 4$ we have,

$$
\begin{aligned}
p^{4,0}_{j_1j_2j_3j_4}&=0,\qquad p^{4,1}_{j_1j_2j_3}=0\\
p^{4,2}_{j_1j_2,i_1i_2}&=p^{3,2}_{j_1,i_1r}\mathrm{B}^{j_2}_{ri_2}+p^{3,2}_{j_2,i_1r}\mathrm{B}^{j_1}_{ri_1}=\mathrm{B}^{j_1}_{i_1r}\mathrm{B}^{j_2}_{ri_2}+\mathrm{B}^{j_2}_{i_2r}\mathrm{B}^{j_1}_{ri_1}\\
p^{4,3}_{j_1,i_1i_2i_3}&=(\nabla p^{3,2}_{j_1})_{i_1i_2i_3}+p^{3,2}_{r,i_2i_3}\mathrm{B}^{j_1}_{ri_1}=(\nabla p^{3,2}_{j_1})_{i_1i_2i_3}+\mathrm{B}^{r}_{i_2i_3}\mathrm{B}^{j_1}_{ri_1}=(\nabla \mathrm{B}^{j_1})_{i_1i_2i_3}\\
&\qquad\text{since we contracted a normal vector with a tangent one,}\\
p^{4,4}_{i_1i_2i_3i_4}&=-p^{3,2}_{r}i_3i_4\mathrm{B}^{r}_{i_1i_2}-p^{3,2}_{r}i_2i_4\mathrm{B}^{r}_{i_1i_3}-p^{3,2}_{r}i_2i_3\mathrm{B}^{r}_{i_1i_4}\\
&=-\mathrm{B}^{r}_{i_3i_4}\mathrm{B}^{r}_{i_1i_2}-\mathrm{B}^{r}_{i_2i_4}\mathrm{B}^{r}_{i_1i_3}-\mathrm{B}^{r}_{i_2i_3}\mathrm{B}^{r}_{i_1i_4}\,.
\end{aligned}
$$

Proposition 4.1 allows us to write A^k in terms of the tensors $p^{k,s}$ and the projections on the tangent and normal spaces (hence contracting with the scalar product of $\mathbb{R}^{n+m}$), so we get the following corollary.

COROLLARY 4.3. *For every $k \geq 3$ the symmetric tensor A^k can be expressed as a polynomial tensor in* B *and its covariant derivatives, contracted with the scalar product of* $\mathbb{R}^{n+m}$.
The maximum order of differentiation of B *which appears in A^k is $k-3$. More precisely, the only tensors among the $p^{k,s}$ containing such highest derivative are $p^{k,k-1}_{j_1}$, given by*

$$p^{k,k-1}_{j_1} = \nabla^{k-3}\mathrm{B}^{j_1} + \mathrm{LOT}\,.$$

where we denoted with LOT (lower order terms) *a polynomial term containing only derivatives of* B *up to the order k-4.*

PROOF. Looking at the tensors with the derivative of B of maximum order among the $p^{k,s}_{j_1\dots j_{k-s}}$, by formula (4.3) and the fact that the only non zero polynomials $p^{3,s}_{j_1\dots j_{3-s},i_1\dots i_s}$ are $p^{3,2}_{j_1,i_1i_2} = \mathrm{B}^{j_1}_{i_1i_2}$ (see Example 4.2), it is clear that they come from the derivative $\nabla p^{k-1,k-2}_{j_1}$. Iterating the argument, the leading term in $p^{k,k-1}_{j_1}$ is given by $\nabla^{k-3}p^{3,2}_{j_1} = \nabla^{k-3}\mathrm{B}^{j_1}$. □

REMARK 4.4. We can see in Example 4.2 that when $k = 3$ and 4, the lower order term which appears above is zero. Actually, by a tedious computation, one can see that for $k \geq 5$ this is no more true.

COROLLARY 4.5. *For every $k \geq 3$ we have the following estimates at every point $x \in M$,*

$$C_1|\nabla^{k-3}\mathrm{B}|^2 + \mathrm{LOT}_1 \leq |A^k|^2 \leq C_2|\nabla^{k-3}\mathrm{B}|^2 + \mathrm{LOT}_2$$

where the two constants C_1 and C_2 depends only on k, n and m, and LOT_1 *and* LOT_2 *are polynomial terms containing only derivatives of* B *up to the order k-4.*
Moreover, for a couple of "universal" functions F_1 and F_2 depending only on k, n and m, we have

$$\sum_{i=3}^{k}|A^i|^2 \leq F_1\Big(\sum_{i=0}^{k-3}|\nabla^i\mathrm{B}|^2\Big)$$
$$\sum_{i=0}^{k-3}|\nabla^i\mathrm{B}|^2 \leq F_2\Big(\sum_{i=3}^{k}|A^i|^2\Big)\,.$$

PROOF. The first estimates follow by Corollary 4.3 and the structure of A^k obtained in Proposition 4.1. The second statement is obtained by such estimates, by iteration. □

The decomposition of A^k in its tangent and normal components is very useful in studying in even more detail the norm of A^k.

Fixing at a point $x \in M$ an orthonormal basis $\{e_1, \dots, e_{n+m}\}$ of $\mathbb{R}^{n+m}$ such that $\{e_1, \dots, e_n\}$ is a basis of T_xM, we have obviously

$$\begin{aligned}
|A^k|^2 &= \sum_{1\leq i_1,\dots,i_k\leq n+m}[A^k(e_{i_1},\dots,e_{i_k})]^2\\
&\geq \sum_{\substack{1\leq i_1,i_2\leq n\\ n<i_3,\dots,i_k\leq n+m}}[A^k(e_{i_1},e_{i_2},e_{i_3},\dots,e_{i_k})]^2\\
&\geq \sum_{n<j\leq n+m}\ \sum_{1\leq i_1,i_2\leq n}[A^k(e_{i_1},e_{i_2},e_j,\dots,e_j)]^2\\
&= \sum_{n<j\leq n+m}\ \sum_{1\leq i_1,i_2\leq n}[p^{k,2}_{j\dots j,i_1i_2}]^2\,,
\end{aligned}$$

that is,

$$|A^k|^2 \geq \sum_{n<j\leq n+m} |p^{k,2}_{j\dots j}|^2 \,.$$

We analyse this last term by means of formula (4.3). We have $p^{2,2} = g$ and for every $k \geq 2$,

$$p^{k+1,2}_{j\dots j,i_0 i_1} = \sum_{h=1}^{k-1} p^{k,2}_{j\dots j,i_1 r} \mathrm{B}^j_{r i_0} = (k-1)\, p^{k,2}_{j\dots j,i_1 r} \mathrm{B}^j_{r i_0} \,.$$

Then, by induction, it is easy to see that

$$p^{k,2}_{j\dots j,i_0 i_1} = (k-2)!\, \mathrm{B}^j_{i_0 r_1} \mathrm{B}^j_{r_1 r_2} \dots \mathrm{B}^j_{r_{k-3} i_1}$$

hence, as the bilinear form B^j is symmetric, denoting with λ^j_s its eigenvalues at the point $x \in M$, we conclude

$$|p^{k,2}_{j\dots j}|^2 = [(k-2)!]^2 \sum_{s=1}^{n} (\lambda^j_s)^{2(k-2)} \geq \widetilde{C} |\mathrm{B}^j|^{2k-4} \,.$$

Coming back to our estimate,

$$|A^k|^2 \geq \widetilde{C} \sum_{n<j\leq n+m} |\mathrm{B}^j|^{2k-4} \geq C \Big(\sum_{n<j\leq n+m} |\mathrm{B}^j|^2 \Big)^{k-2} = C|\mathrm{B}|^{2k-4} \,.$$

PROPOSITION 4.6. *The following estimate holds,*

$$|A^k|^2 \geq C|\mathrm{B}|^{2k-4}$$

where C is a universal constant depending only on k, n and m.

5. The Distance Function on Riemannian Manifolds

In this section we discuss more in detail some analytic properties of the distance function that we state without proof in Section 3.
We consider in full generality the distance function d^K from a closed set K of a Riemannian manifold (M, g) and we analyse the connection with the theory of *viscosity solutions* of Hamilton–Jacobi equations. Indeed, we will see that the distance function is a viscosity solution of the following Hamilton–Jacobi problem

$$\begin{cases} |\nabla u| = 1 & \text{in } M \setminus K \,, \\ u = 0 & \text{on } \partial K \end{cases}$$

and we will use the property of *semiconcavity* shared by such solutions to analyse the properties of d^K (for more details see **[33]**).

5.1. Stationary Hamilton–Jacobi Equations on Manifolds.

Let M be a smooth and connected, n–dimensional, differentiable manifold.

We consider the following Hamilton–Jacobi problem in $\Omega \subset M$,

$$\begin{cases} \mathrm{H}(x, du(x), u(x)) = 0 & \text{in } \Omega \,, \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

where $\mathrm{H} : T^*\Omega \times \mathbb{R} \to \mathbb{R}$ and T^* denotes the cotangent bundle.

DEFINITION 5.1. Given a continuous function $u : \Omega \to \mathbb{R}$ and a point $x \in M$, the *superdifferential* of u at x is the subset of $T_x^* M$ defined by

$$\partial^+ u(x) = \left\{ d\varphi(x) \,|\, \varphi \in C^1(M), \varphi(x) - u(x) = \min_M \varphi - u \right\}.$$

Similarly, the set

$$\partial^- u(x) = \left\{ d\psi(x) \,|\, \psi \in C^1(M), \psi(x) - u(x) = \max_M \psi - u \right\}$$

is called the *subdifferential* of u at x.
Notice that it is equivalent to replace the $\max$ ($\min$) on all M with the maximum (minimum) in an open neighborhood of x in M.

It is easy to see that $\partial^+ u(x)$ and $\partial^- u(x)$ are both nonempty if and only if u is differentiable at $x \in M$. In this case we have

$$\partial^+ u(x) = \partial^- u(x) = \{du(x)\}.$$

We list here without proof some of the standard properties of the sub and superdifferentials which will be needed later.

PROPOSITION 5.2. *If $\psi : N \to M$ is a map between the smooth manifolds N and M which is C^1 around $x \in N$, then*

$$\partial^+(u \circ \psi)(x) \supset \partial^+ u(\psi(x)) \circ d\psi(x) \;=\; \{v \circ d\psi(x) \,|\, v \in \partial^+ u(\psi(x))\}.$$

If ψ is a local diffeomorphism near x, the inclusion becomes an equality. An analogous statement holds for ∂^-.

PROPOSITION 5.3. *If $\theta : \mathbb{R} \to \mathbb{R}$ is a C^1 function such that $\theta'(u(x)) \geq 0$, then*

$$\partial^+(\theta \circ u)(x) \supset d\theta(u(x)) \circ \partial^+ u(x) \;=\; \{d\theta(u(x)) \circ v \,|\, v \in \partial^+ u(x)\},$$

similarly for ∂^-. If $\theta'(u(x)) > 0$ then the inclusion is an equality.

For a locally Lipschitz function u on a Riemannian manifold (M, g), $\partial^+ u(x)$ and $\partial^- u(x)$ are compact convex sets, almost everywhere coinciding with the differential of the function u, by Rademacher's theorem.
For a generic continuous function u we prove in the next proposition that $\partial^+ u(x)$ and $\partial^- u(x)$ are not empty in a dense subset.

PROPOSITION 5.4. *Let $u : \Omega \to \mathbb{R}$ be a continuous function on an open subset Ω of M. Then the subdifferential $\partial^- u(x)$ (the superdifferential $\partial^+ u(x)$) is not empty for every x in a dense subset of Ω.*

PROOF. It is always possible to endow M with a Riemannian structure giving a metric $d(\cdot\,,\cdot)$ on M which generates the same topology.
Consider a generic point $y \in \Omega$ and a geodesic ball B contained in Ω with center y. If the ball B is small enough, the function $x \mapsto d^2(x, y)$ is smooth in $\overline{B}$. Taking a large positive constant A, the function $F_A(x) = u(x) + Ad^2(x, y)$ has a local minimum at a point x_A in the interior of B. At x_A the subdifferential of the function F_A must contain the origin of $T_{x_A}^* M$, hence, being $d^2(x, y)$ differentiable in the ball B, the differential of $-d^2(x, y)$ at x_A belongs to $\partial^- u(x_A)$. As the point y and the ball B were arbitrarily chosen, the set of points where the subdifferential of u is not empty is dense in Ω.
The same argument holds for the superdifferential of u, considering the function $-u$. $\square$

Now we introduce the notion of semiconcavity which will play a central role.

DEFINITION 5.5. Given an open set $\Omega \subset \mathbb{R}^n$, a continuous function $u : \Omega \to \mathbb{R}$ is called *locally semiconcave* if, for any open convex set $\Omega' \subset \Omega$ with compact closure in Ω, there exists a constant C such that one of the following three equivalent conditions is satisfied,

(1) $\forall x, h$ with $x, x+h, x-h \in \Omega'$,
$$u(x+h) + u(x-h) - 2u(x) \leq 2C|h|^2 \,,$$
(2) $u(x) - C|x|^2$ is a concave function in Ω',
(3) $D^2 u \leq 2C \,\mathrm{Id}$ in Ω', as distributions (Id is the $n \times n$ identity matrix).

In order to give a meaning to the concept of semiconcavity when the ambient space is a differentiable manifold M, we analyse the stability of this property under composition with C^2 maps.

PROPOSITION 5.6. *Let Ω and Ω' two open subsets of $\mathbb{R}^n$. If $u : \Omega \to \mathbb{R}$ is a Lipschitz function such that $u(x) - C\,|x|^2$ is concave and $\psi : \Omega' \to \Omega$ is a C^2 function with bounded first and second derivatives, then $u \circ \psi : \Omega' \to \mathbb{R}$ is a Lipschitz function and $u \circ \psi(y) - C'|y|^2$ is concave, for a suitable constant C'.*

The proof is straightforward. Then, the following definition is well–posed.

DEFINITION 5.7. A continuous function $u : M \to \mathbb{R}$ is called *locally semiconcave* if, for any local chart $\psi : \mathbb{R}^n \to \Omega \subset M$, the function $u \circ \psi$ is locally semiconcave in $\mathbb{R}^n$.

The importance of semiconcave functions in connection with the generalized differentials is expressed by the following proposition (see **[10]**).

PROPOSITION 5.8. *Let the function $u : M \to \mathbb{R}$ be locally semiconcave, then the superdifferential $\partial^+ u$ is not empty at each point, moreover, $\partial^+ v$ is* upper semicontinuous, *namely*
$$x_k \to x, \quad v_k \to v, \quad v_k \in \partial^+ u(x_k) \quad \Longrightarrow \quad v \in \partial^+ u(x) \,.$$
In particular, if the differential du exists at every *point of $\Omega \in M$, then $u \in C^1(\Omega)$.*

Now we introduce the definition of viscosity solution.
Let Ω be an open subset of M and H, called *Hamiltonian* function, a continuous real function on $T^*\Omega \times \mathbb{R}$. We are interested in the following Hamilton–Jacobi problem

$$\mathrm{H}(x, du(x), u(x)) = 0 \qquad \text{in } \Omega \,. \tag{5.1}$$

DEFINITION 5.9. We say that a continuous function u is a *viscosity solution* of equation (5.1) if for every $x \in \Omega$,

$$\begin{cases} \mathrm{H}(x, v, u(x)) \leq 0 & \forall v \in \partial^+ u(x) \,, \\ \mathrm{H}(x, v, u(x)) \geq 0 & \forall v \in \partial^- u(x) \,. \end{cases} \tag{5.2}$$

If only the first condition is satisfied (respectively, the second) u is called a *viscosity subsolution* (respectively, a *viscosity supersolution*).

If Ω' is an open subset of another smooth differentiable manifold N and $\psi : \Omega' \to \Omega$ is a C^1 local diffeomorphism, we define the *pull–back* of the Hamiltonian function $\psi^*\mathrm{H} : T^*\Omega' \times \mathbb{R} \to \mathbb{R}$ by
$$\psi^*\mathrm{H}(y, v, r) = \mathrm{H}(\psi(y), v \circ d\psi(y)^{-1}, r) \,.$$
Taking into account Proposition 5.2, the following statement is obvious.

PROPOSITION 5.10. *If u is a viscosity solution of* $\mathrm{H} = 0$ *in* $\Omega \subset M$ *and* $\psi : \Omega' \to \Omega$ *is a* C^1 *local diffeomorphism, then* $u \circ \psi$ *is a viscosity solution of* $\psi^*\mathrm{H} = 0$ *in* $\Omega' \subset N$.

5.2. The Distance Function from a Closed Subset of a Manifold. From now on, (M, g) will be a smooth, connected and complete, Riemannian manifold without boundary, of dimension n.

We consider a closed and not empty subset K and the distance function $d^K : M \to \mathbb{R}$ from K, which is defined as the infimum of the lengths of the C^1 curves starting at x and ending at K. As M is complete, by the Theorem of Hopf–Rinow, such infimum is reached by at least one curve which will be a smooth geodesic. We will also consider the function $\eta^K = [d^K]^2/2$ as in the previous sections.

In the following we will denote the distance between two points $x, y \in M$ with $d(x, y)$ and the *exponential map* of (M, g) with $\mathrm{Exp} : TM \times \mathbb{R} \to M$. For simplicity, we will write $|v|$ for the modulus of a vector $v \in TM$, defined as $\sqrt{g(v, v)}$.

PROPOSITION 5.11. *The distance function d^K is the* unique *viscosity solution of the following Hamilton–Jacobi problem*

$$
\begin{cases} |\nabla u|^2 - 1 = 0 & \text{in } M \setminus K\,, \\ u = 0 & \text{on } K \end{cases} \tag{5.3}
$$

in the class of continuous functions bounded from below.
The function η^K is the unique *viscosity solution of*

$$
\begin{cases} |\nabla u|^2 - 2u = 0 & \text{in } M\,, \\ u = 0 & \text{on } K \end{cases} \tag{5.4}
$$

in the class of continuous functions on M such that their zero set is K.

REMARK 5.12. The restriction to lower bounded functions is necessary, $\|x\|$ and $-\|x\|$ are both viscosity solutions of Problem (5.3) with $M = \mathbb{R}^n$ and $K = \{0\}$. Moreover, the completeness of M plays an important role here, if M is the open unit ball of $\mathbb{R}^n$ the same example shows that the uniqueness does not hold.
Notice also that every function $[d^H]^2/2$ where H is a closed subset of M with $H \supset K$, is a viscosity solution of Problem (5.4), equal to zero on K.

PROOF. The quantity $d^K(x)$ is the minimum time $t \geq 0$ for any curve γ to reach a point $\gamma(t) \in K$, subject to the conditions $\gamma(0) = 0$ and $|\gamma'| \leq 1$; the function d^K is then the value function of a "minimum time problem"; this proves that d^K is also a viscosity solution of Problem (5.3), by well known results (see for example [**5**, Chap. 4, Prop. 2.3]). Then we show that the function η^K is a solution of Problem (5.4).
First of all, notice that the distance function from K is a 1–Lipschitz function, hence η^K is locally Lipschitz.
As d^K is 1–Lipschitz, at every point of K the function η^K is differentiable and its differential is zero. Hence, the definition of viscosity solution holds also for points belonging to K. In order to prove the thesis, it is then sufficient to test conditions (5.2) on the generalized differentials at the points of the open set $M \setminus K$.
Since η^K is positive in $M \setminus K$, applying Proposition 5.3 with the function $\theta(t) = \sqrt{2t}$, we see

that the function η^K is a viscosity solution of

$$g\left(\frac{\nabla u}{\sqrt{2u}},\frac{\nabla u}{\sqrt{2u}}\right)-1=0$$

in $M\setminus K$. Being there positive, it also solves

$$g(\nabla u,\nabla u)-2u=0$$

in $M\setminus K$. This fact together with the previous remark about the behavior of η^K at the points of K gives the claim.

Suppose now that u is a viscosity solution of Problem (5.3) then, u is also a solution of

$$\begin{cases}|\nabla u|-1=0 & \text{in } M\setminus K\,,\\ u=0 & \text{on } K\,.\end{cases}$$

As in the work of Kružhkov [**29**], we consider the function $v=-e^{-u}$ which, by Proposition 5.3, turns out to be a viscosity solution of

$$\begin{cases}|\nabla v|+v=0 & \text{in } M\setminus K\,,\\ v=-1 & \text{on } K\end{cases}\tag{5.5}$$

moreover, $|v|\leq e^{-\inf u}$.
We establish an uniqueness result for this last problem in the class of bounded functions v, which clearly implies the first uniqueness result. We remark that the proof is based on similar ones in [**13, 14, 24**].
We argue by contradiction, suppose that u and v are two bounded solutions of (5.5), $|u|$, $|v|\leq C$, and that at a point $\overline{x}$ we have $u(\overline{x})\geq 2\varepsilon+v(\overline{x})$ with $\varepsilon>0$.
Let $b(x,y):M\times M\to\mathbb{R}$ be a smooth function satisfying

- $b\geq 0$
- $|\nabla_x b(x,y)|$, $|\nabla_y b(x,y)|\leq 2$
- $\sup_{M\times M}|d(x,y)-b(x,y)|<\infty$

such a function can be obtained *smoothing* the distance function in $M\times M$.
We fix a point x_0 in K and we define the smooth function $B(x)=b(x,x_0)^2$. By the properties of b and the boundedness of u and v, the following function $\Psi:M\times M\to\mathbb{R}$

$$\Psi(x,y)=u(x)-v(y)-\lambda d(x,y)^2-\delta\,B(x)-\delta\,B(y)$$

has a maximum at a point $\widehat{x},\widehat{y}$ (dependent on the positive parameters δ and λ) and such maximum $\Psi(\widehat{x},\widehat{y})$ is less than $2C$. Hence, the function

$$x\mapsto[v(\widehat{y})+\lambda d(x,\widehat{y})^2+\delta\,B(x)+\delta\,B(\widehat{y})]-u(x)\tag{5.6}$$

has a minimum at $\widehat{x}$ while

$$y\mapsto[u(\widehat{x})-\lambda d(\widehat{x},y)^2-\delta\,B(\widehat{x})-\delta\,B(y)]-v(y)\tag{5.7}$$

has a maximum at $\widehat{y}$.
If $2\delta\leq\varepsilon/B(\overline{x})$ then

$$\Psi(\widehat{x},\widehat{y})\geq\Psi(\overline{x},\overline{x})\geq 2\varepsilon-2\delta B(\overline{x})\geq\varepsilon$$

hence, we get

$$\delta B(\widehat{x})+\delta B(\widehat{y})+\lambda d(\widehat{x},\widehat{y})^2+\varepsilon\leq u(\widehat{x})-v(\widehat{y})\leq 2C\,.\tag{5.8}$$

This shows that, for a fixed δ, the pair $\widehat{x}, \widehat{y}$ is contained in a bounded set and, if λ goes to $+\infty$ the distance between $\widehat{x}$ and $\widehat{y}$ goes to zero. Possibly passing to a subsequence for λ going to infinity, $\widehat{x}$ and $\widehat{y}$ converge to a common limit point z which cannot belong to K, otherwise we would get $\varepsilon \leq u(z) - v(z) = 0$, thus, for some λ large enough also $\widehat{x}$ and $\widehat{y}$ do not belong to K.
As the function $d^2(x, y)$ is smooth in $B_z \times B_z \subset M \times M$, where B_z is a small geodesic ball around z, choosing a suitable λ large enough we can differentiate the functions inside the square brackets in equations (5.6) and (5.7) obtaining

$$\widehat{v} = \delta \nabla B(\widehat{x}) + \lambda \nabla_x d^2(\widehat{x}, \widehat{y}) \ \in \partial^+ u(\widehat{x}) \,,$$
$$\widehat{w} = -\delta \nabla B(\widehat{y}) - \lambda \nabla_y d^2(\widehat{x}, \widehat{y}) \ \in \partial^- v(\widehat{y}) \,.$$

By Definition 5.9 we have that $|\widehat{v}| + u(\widehat{x}) \leq 0$ and $|\widehat{w}| + v(\widehat{y}) \geq 0$, hence

$$u(\widehat{x}) - v(\widehat{y}) + |\widehat{v}| - |\widehat{w}| \leq 0 \,.$$

Moreover,

$$\begin{aligned}
|\widehat{v}| - |\widehat{w}| &= \left|\delta \nabla B(\widehat{v}) + \lambda \nabla_x d^2(\widehat{x}, \widehat{y})\right| - \left|\delta \nabla B(\widehat{y}) + \lambda \nabla_y d^2(\widehat{x}, \widehat{y})\right| \\
&\geq \left|\lambda \nabla_x d^2(\widehat{x}, \widehat{y})\right| - \left|\lambda \nabla_y d^2(\widehat{x}, \widehat{y})\right| - |\delta \nabla B(\widehat{y})| - |\delta \nabla B(\widehat{x})| \\
&= 2\lambda d(\widehat{x}, \widehat{y}) \, |\nabla_x d(\widehat{x}, \widehat{y})| - 2\lambda d(\widehat{x}, \widehat{y}) \, |\nabla_y d(\widehat{x}, \widehat{y})| - |\delta \nabla B(\widehat{y})| - |\delta \nabla B(\widehat{x})| \\
&= 2\lambda d(\widehat{x}, \widehat{y}) - 2\lambda d(\widehat{x}, \widehat{y}) - |\delta \nabla B(\widehat{y})| - |\delta \nabla B(\widehat{x})| \\
&= - |\delta \nabla B(\widehat{y})| - |\delta \nabla B(\widehat{x})|
\end{aligned}$$

which implies,

$$u(\widehat{x}) - v(\widehat{y}) - \delta |\nabla B(\widehat{y})| - \delta |\nabla B(\widehat{x})| \leq 0 \,.$$

Finally, we have that

$$\delta |\nabla B(\widehat{x})| = 2\delta |b(\widehat{x}, x_0) \nabla b(\widehat{x}, x_0)| \leq 4\delta \sqrt{B(\widehat{x})}$$

and using the estimate $\delta B(\widehat{x}) \leq 2C$ which follows from equation (5.8),

$$\delta |\nabla B(\widehat{x})| \leq 8\sqrt{2\delta C} \leq \varepsilon/4$$

if δ was chosen small enough. Holding the same for $\widehat{y}$, we conclude that

$$u(\widehat{x}) - v(\widehat{y}) - \varepsilon/2 \leq 0$$

which is in contradiction with the fact that $u(\widehat{x}) - v(\widehat{y}) \geq \varepsilon$.

About the second uniqueness claim, if u is a continuous viscosity solution of Problem (5.4) then, by Proposition 5.4 the superdifferential of u is not empty in a dense subset of $M \setminus K$, hence, directly by the equation and by continuity, u is non negative. By the hypothesis on its zero set we conclude that u is positive in all $M \setminus K$. Composing u with the function $t \mapsto \sqrt{2t}$, we see that $\sqrt{2u}$ is a positive, continuous viscosity solution of Problem (5.3), then it must coincide with d^K, by the previous result. It follows that $u = \eta^K$. □

We now study the *singular set* of d^K,

$$Sing = \left\{ x \in M \,|\, \eta^K \text{ is not differentiable at } x \right\} \,.$$

REMARK 5.13. In this definition we used the squared distance function instead of the distance in order to avoid to consider also the points of the boundary of K, which are singular for d^K but not for η^K. It is trivial to see that outside K the distance and its square have the same regularity.

PROPOSITION 5.14. *The function d^K is locally semiconcave in $M \setminus K$.*

PROOF. The distance function d^K is a viscosity solution of $\mathrm{H} = 0$ in $M \setminus K$, where the Hamiltonian function is given by $\mathrm{H}(x, v, t) = |v|^2 - 1$. We choose a smooth local chart $\psi : \mathbb{R}^n \to \Omega \subset M$ and we define $v = d^K \circ \psi$, which is a locally Lipschitz function and, by Proposition 5.10, it is a viscosity solution of $\psi^*\mathrm{H} = 0$.
The pull–back of the Hamiltonian function on $\mathbb{R}^n$ takes the form

$$\psi^*\mathrm{H}(y, w, s) = g_{\psi(y)}(d\psi(w), d\psi(w)) - 1 = g_{ij}(y)w_i w_j - 1$$

for $(y, w, s) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and where $g_{ij}(y)$ are the components of the metric tensor of M in local coordinates.
Since the matrix $g_{ij}(y)$ is positive definite $\psi^*\mathrm{H}(y, w, s)$ is locally uniformly convex in w, hence, by Theorem 5.3 of **[30]**, it follows that $v = d^K \circ \psi$ is locally semiconcave in $\mathbb{R}^n$. Recalling Definition 5.7, this means that d^K is locally semiconcave in $M \setminus K$. □

The semiconcavity of d^K allows us to work with the superdifferentials when the gradient does not exist. Indeed, it follows that the points of $Sing$ are precisely those where the superdifferential is not a singleton and the following result is a straightforward consequence of Proposition 5.8.

PROPOSITION 5.15. *The function η^K is of class C^1 in the open set $M \setminus \overline{Sing}$ and d^K is C^1 in $M \setminus \big(K \cup \overline{Sing}\big)$.*

The semiconcavity property also gives information about the relations between the structure of the superdifferential at a point x and the set of minimal geodesics from x to K (see **[1, 2, 33]**).
The set $Ext(\partial^+\eta^K(x))$ of extremal points of the (convex) superdifferential set of η^K at x is in one–to–one correspondence with the family $\mathcal{G}(x)$ of minimal geodesics from x to K. Precisely $\mathcal{G}(x)$ is described by

$$\mathcal{G}(x) = \big\{\mathrm{Exp}(x, -v, \cdot)\,|\,[0, 1] \to M \,|\, \text{for } v \in Ext(\partial^+\eta^K(x))\big\} \,. \tag{5.9}$$

Hence, the set of points of K at minimum distance from x are given by $\mathrm{Exp}(x, -v, 1)$ for v in the set of extremal points of the superdifferential set of η^K at x. As a particular case we have that if the function η^K is differentiable at x if and only if the point of K closest to x is uniquely determined and given by $\mathrm{Exp}(x, -\nabla\eta^K(x), 1)$.

We consider now a set K which is a k–dimensional, embedded C^r submanifold of M without boundary, with $0 \leq k \leq n - 1$ (the case $k = n$ is trivial) and $r \geq 2$.

For every $p \in K$ we consider the following three subsets of T_pM,

- T_pK, the vector subspace of tangent vectors to K at p,
- $N_pK = \{w \in T_pM \,|\, g_p(w, T_pK) = 0\}$, the vector subspace of normal vectors to K at p,
- $U_pK = \{w \in N_pK \,|\, g_p(w, w) = 1\}$, the subset of unit normal vectors to K at p,

then the bundles $NK = \{(p, v) \,|\, v \in N_pK\}$ and $UK = \{(p, v) \,|\, v \in U_pK\}$ inherit the structure of TM. Being K a C^r submanifold of M, the bundles NK and UK are respectively n–dimensional and $(n-1)$–dimensional C^{r-1} submanifolds of TM.
Notice that in the special case $K = \{p\}$, we have that $NK = T_pM$ and $UK = \mathbb{S}^{n-1} \subset T_pM$.

We define the map $F : UK \times \mathbb{R}^+ \to M$ as the restriction of the exponential map of M to UK,

$$F(p, v, t) = \mathrm{Exp}(p, v, t) \qquad \forall (p, v) \in UK \text{ and } t \in \mathbb{R}^+.$$

Since UK is a C^{r-1} manifold and the exponential map of M is smooth, F and all its derivatives with respect to the variable t are of class C^{r-1}.

REMARK 5.16. If a minimal geodesic, parametrized by arc length, starts at a point $p \in M$ and arrives at a point $q \in K$, its velocity vector v at q has to belong to U_qK, otherwise the condition of minimality is easily contradicted.
Since the geodesics, parametrized by arc length, ending on K are given by the family of maps $t \mapsto F(q,v,t)$ with $(q,v) \in UK$, the distance from K of a point p is given by the formula

$$d^K(p) = \inf\{t \in \mathbb{R}^+ \mid (q,v,t) \in F^{-1}(p)\}, \tag{5.10}$$

which obviously becomes $d^K(p) = \pi_{\mathbb{R}^+}\left(F^{-1}(p)\right)$ when the counterimage is a singleton (the map $\pi_{\mathbb{R}^+}$ is the projection on the second factor of the product $UK \times \mathbb{R}^+$).
The study of the singularities of the squared distance function then reduces to the analysis of the (possibly set valued) map F^{-1}.
This problem, from the topological point of view, is naturally connected with the study of the singularities of continuous maps between Euclidean spaces. For instance, when K coincides with a single point of M the singular sets were shown to be related to the classes of singularities considered by the *Theory of Catastrophes*, see [**9**].

Let us define the C^{r-1} map $\exp : NK \to M$ by

$$\exp(p,v) = \operatorname{Exp}(p,v,1) \qquad \forall (p,v) \in NK.$$

At the points $(p,0) \in NK$ the map exp is differentiable and $d\exp(p,0)$ is invertible between $T_{(p,0)}NK$ and T_pM, indeed $T_{(p,0)}NK$ can be identified with T_pM and under such identification $d\exp(p,0)$ is the identity. Since, by hypothesis, the map exp is at least C^1, it follows that in a neighborhood of $(p,0)$ in NK the differential of exp is invertible, hence the map exp is a C^{r-1} local diffeomorphism. Holding the relation $F(p,v,t) = \exp(p,vt)$, we conclude that for small $t > 0$, the map F is a local diffeomorphism.
Being K at least C^2, by a standard result in differential geometry, there exists an open *tubular neighborhood* Ω' of K in M formed by non intersecting, minimal geodesics starting normally from K. Hence, by the previous discussion and possibly choosing a smaller tubular neighborhood Ω of K, the map F^{-1} is well defined and C^{r-1} in $\Omega \setminus K$ (see for instance, [**4**]).
Then, the gradient of η^K exists in Ω and we have, by relations (5.9) and (5.10),

$$\nabla\eta^K(p) = d^K(p)\frac{\partial F}{\partial t}(F^{-1}(p)).$$

Since $d^K = \pi_{\mathbb{R}^+}\left(F^{-1}(p)\right) \in C^{r-1}$ in Ω and the functions F, $\frac{\partial F}{\partial t}$ are of class C^{r-1}, it follows that $\nabla\eta^K$ is C^{r-1} and η^K is C^r in $\Omega \setminus K$. The same C^r regularity in $\Omega \setminus K$ follows immediately also for the distance function d^K.
Moreover, the function η^K is C^r regular also on the set K, hence in the whole neighborhood Ω, as the square regularizes the jump of the gradient in the direction normal to K, see [**3, 4**].

We summarize these results in the following proposition which has as a particular case Proposition 3.1.

PROPOSITION 5.17. *If K is a regular submanifold of class C^r, with $r \geq 2$, then there exists an open subset Λ of $UK \times \mathbb{R}^+$ with the property that if $(q,v,t) \in \Lambda$ then also $(q,v,s) \in \Lambda$ for every $0 < s < t$, and an open neighborhood Ω of K in M, such that the map $F|_\Lambda : \Lambda \to \Omega \setminus K$ is a diffeomorphism.*
Moreover,

- *for every point in Ω there is an unique point of minimum distance in K (*unique projection property in Ω*),*
- *the distance function d^K is C^r in $\Omega \setminus K$,*
- *the squared distance function η^K is C^r in Ω.*

REMARK 5.18. It can be proved that $C^{1,1}$ is the minimal regularity of K to have the unique projection property in a neighborhood, in this case also the squared distance function turns out to be of class $C^{1,1}$ (see **[20, 22]** and also **[15, 16]** for a detailed discussion of the relation between the regularity of K and of d^K.

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