Graphs of $\mathbf{W^{1,1}}$-maps with values into $\mathbf{S^1}$: relaxed energies, minimal connections and lifting

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The aim of this paper is to link the analytic results of [6] [7] [19] relative to $W^{1,1}$-mappings from B^n into S^1 to the measure theoretical geometric results in [12] [15]. The paper also contains a few remarks about mappings in $W^{1,p}$, $p \geq 2$, with values into S^2.

1 The relaxed energy of $\mathbf{W^{1,1}}$-maps

Let Ω be a simply connected, smooth, n-dimensional domain. For simplicity, we take $\Omega = B^n$, the n-dimensional unit ball, and we let $\widetilde{B}^n$ be the unit ball of radius 2, so that $B^n \subset\subset \widetilde{B}^n$. Also, let $S^1 \subset \mathbb{R}^2 \simeq \mathbb{C}$ be the unit sphere. For any non-negative integer k and for $U = \widetilde{B}^n$ or $\widetilde{B}^n \times S^1$, we will denote by $\mathcal{D}^k(U)$ the class of smooth compactly supported k-forms in U and by $\mathcal{D}_k(U)$ the usual class of k-dimensional currents in U, i.e., the dual of $\mathcal{D}^k(U)$. Moreover, $\mathcal{R}_k(U)$ denotes the subclass of k-dimensional integer multiplicity (say i.m.) rectifiable currents in U, compare [8] [10] [13]. We set

$$W^{1,1}(\widetilde{B}^n, S^1) := \{u \in W^{1,1}(\widetilde{B}^n, \mathbb{R}^2) \,:\, |u(x)| = 1 \text{ for a.e. } x \in \widetilde{B}^n\}$$

and, in the sequel, $\varphi : \widetilde{B}^n \to S^1$ being a given smooth $W^{1,1}$-map, we set

$$\begin{aligned} W^{1,1}_\varphi(\widetilde{B}^n, S^1) &:= \{u \in W^{1,1}(\widetilde{B}^n, S^1) \,:\, u = \varphi \quad \text{in} \quad \widetilde{B}^n \setminus \overline{B}^n\}\,. \\ C^1_\varphi(\widetilde{B}^n, S^1) &:= \{u \in C^1(\widetilde{B}^n, S^1) \,:\, u = \varphi \quad \text{in} \quad \widetilde{B}^n \setminus \overline{B}^n\}\,. \end{aligned}$$

Also, $\pi : B^n \times S^1 \to B^n$ and $\widehat{\pi} : B^n \times S^1 \to S^1$ will denote the projections onto the first and second factor, respectively. Finally, we denote by ω_{S^1} the *volume* 1*-form on* $S^1 \subset \mathbb{R}^2$

$$\omega_{S^1} := y^1 dy^2 - y^2 dy^1\,.$$

Graphs of $W^{1,1}$-maps. We recall from [13] that the i.m. rectifiable current $G_u \in \mathcal{R}_n(\widetilde{B}^n \times S^1)$ associated to the "graph" of a function $u \in W^{1,1}(\widetilde{B}^n, S^1)$ is defined in an *approximate sense* by

$$G_u := (Id \bowtie u)_\# [\![\, \widetilde{B}^n \,]\!]\,, \tag{1.1}$$

where $(Id \bowtie u)(x) := (x, u(x))$, u standing for the restriction of u to the set of *approximate differentiability* of u, i.e.

$$G_u(\omega) := \int_{\widetilde{B}^n} (Id \bowtie u)^\# \omega\,, \qquad \omega \in \mathcal{D}^n(\widetilde{B}^n \times S^1)\,.$$

The singular set. Following [13, Vol. II], for any $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$ we define the $(n-2)$-current $\mathbb{P}(u) \in \mathcal{D}_{n-2}(\widetilde{B}^n)$ by $2\pi \cdot \mathbb{P}(u) := -\pi_\#(\partial G_u) \llcorner \widehat{\pi}^\# \omega_{S^1}$, so that for every $\xi \in \mathcal{D}^{n-2}(\widetilde{B}^n)$

$$\mathbb{P}(u)(\xi) = -\frac{1}{2\pi} \partial G_u(\widehat{\pi}^\# \omega_{S^1} \wedge \pi^\# \xi) = \frac{1}{2\pi} \int_{\widetilde{B}^n} u^\# \omega_{S^1} \wedge d\xi\,. \tag{1.2}$$

Since $u = \varphi$ outside $\overline{B}^n$, and φ is smooth, we infer that $\mathbb{P}(u)$ is a boundaryless current supported in the closure of B^n,

$$\partial\, \mathbb{P}(u) = 0\,, \qquad \operatorname{spt} \mathbb{P}(u) \subset \overline{B}^n\,.$$

Define now the $(n-1)$-current $\mathbb{D}(u) \in \mathcal{D}_{n-1}(\widetilde{B}^n)$ by $2\pi \cdot \mathbb{D}(u) := \pi_{\#}(G_u \llcorner \widehat{\pi}^{\#}\omega_{S^1})$, so that for every $\gamma \in \mathcal{D}^{n-1}(\widetilde{B}^n)$

$$\mathbb{D}(u)(\gamma) := \frac{1}{2\pi} G_u(\widehat{\pi}^{\#}\omega_{S^1} \wedge \pi^{\#}\gamma) = \frac{1}{2\pi}\int_{\widetilde{B}^n} u^{\#}\omega_{S^1} \wedge \gamma \,.$$

Since $u = \varphi$ on $\widetilde{B}^n \setminus \overline{B}^n$, we have $\operatorname{spt}(\mathbb{D}(u) - \mathbb{D}(\varphi)) \subset \overline{B}^n$. Moreover,

$$\mathbb{P}(u) = \partial(\mathbb{D}(u) - \mathbb{D}(\varphi))\,. \tag{1.3}$$

In fact, since φ is smooth, $d\varphi^{\#}\omega_{S^1} = \varphi^{\#}d\omega_{S^1} = 0$ and hence $d(\varphi^{\#}\omega_{S^1} \wedge \xi) = -\varphi^{\#}\omega_{S^1} \wedge d\xi$, which yields

$$\begin{aligned} 2\pi\ \mathbb{P}(u)(\xi) &= \int_{\overline{B}^n} u^{\#}\omega_{S^1} \wedge d\xi + \int_{\widetilde{B}^n \setminus \overline{B}^n} \varphi^{\#}\omega_{S^1} \wedge d\xi \\ &= \int_{B^n} u^{\#}\omega_{S^1} \wedge d\xi - \int_{\widetilde{B}^n \setminus \overline{B}^n} d(\varphi^{\#}\omega_{S^1} \wedge \xi) \\ &= \int_{B^n} u^{\#}\omega_{S^1} \wedge d\xi + \int_{\partial B^n} \varphi^{\#}\omega_{S^1} \wedge \xi \end{aligned} \tag{1.4}$$

for every $\xi \in \mathcal{D}^{n-2}(\widetilde{B}^n)$. On the other hand, we compute

$$\begin{aligned} 2\pi\, \partial(\mathbb{D}(u) - \mathbb{D}(\varphi))(\xi) &:= 2\pi\,(\mathbb{D}(u) - \mathbb{D}(\varphi))(d\xi) \\ &= \int_{\overline{B}^n} u^{\#}\omega_{S^1} \wedge d\xi - \int_{\overline{B}^n} \varphi^{\#}\omega_{S^1} \wedge d\xi \\ &= \int_{B^n} u^{\#}\omega_{S^1} \wedge d\xi + \int_{B^n} d(\varphi^{\#}\omega_{S^1} \wedge \xi) \\ &= \int_{B^n} u^{\#}\omega_{S^1} \wedge d\xi + \int_{\partial B^n} \varphi^{\#}\omega_{S^1} \wedge \xi\,. \end{aligned}$$

Real and integral masses. We now recall the following

Definition 1.1 *Let $0 \le k \le n-2$. For every k-dimensional current $\Gamma \in \mathcal{D}_k(\widetilde{B}^n)$, with support $\operatorname{spt}\Gamma \subset \overline{B}^n$, we denote by*

$$m_i(\Gamma) := \inf\{\mathbf{M}(L) \mid L \in \mathcal{R}_{k+1}(\widetilde{B}^n)\,, \quad \operatorname{spt} L \subset \overline{B}^n\,, \quad \partial L = \Gamma\}$$

the integral mass *of Γ and by*

$$m_r(\Gamma) := \inf\{\mathbf{M}(D) \mid D \in \mathcal{D}_{k+1}(\widetilde{B}^n)\,, \quad \operatorname{spt} D \subset \overline{B}^n\,, \quad \partial D = \Gamma\}$$

the real mass *of Γ. Moreover, in case $m_i(\Gamma) < \infty$, we say that an i.m. rectifiable current $L \in \mathcal{R}_{k+1}(\widetilde{B}^n)$ is an* integral minimal connection *of Γ if $\partial L = \Gamma$, $\operatorname{spt} L \subset \overline{B}^n$, and $\mathbf{M}(L) = m_i(\Gamma)$.*

Of course, $m_r(\Gamma) < \infty$ if $\mathbf{M}(\Gamma) < \infty$ and $\partial\Gamma = 0$. Moreover, in the definition of integral and real mass, respectively, the infimum is actually a minimum, provided that the set on the right-hand side is non-empty. Now, if $k = n-2$ and $\Gamma = \mathbb{P}(u)$ for some $u \in W^{1,1}_{\varphi}(\widetilde{B}^n, S^1)$, by (1.3) we infer that the real mass is finite,

$$m_r(\mathbb{P}(u)) \le \mathbf{M}(\mathbb{D}(u) - \mathbb{D}(\varphi)) < \infty\,. \tag{1.5}$$

We recall that in general

$$m_r(\Gamma) \le m_i(\Gamma)\,, \tag{1.6}$$

and the strict inequality may occur if $1 \le k \le n-3$, compare [20] [27]. However, as shown by Federer [9], and by Hardt-Pitts [17], equality holds in (1.6) if Γ has dimension zero or if $k = n-2$. In particular, we obtain

$$m_r(\mathbb{P}(u)) = m_i(\mathbb{P}(u)) \qquad \forall\, u \in W^{1,1}_{\varphi}(\widetilde{B}^n, S^1) \tag{1.7}$$

and hence that *the integral minimal connection of the singularities of any map $u \in W^{1,1}_{\varphi}(\widetilde{B}^n, S^1)$ is finite* provided we are able to show that $\mathbb{P}(u)$ is an *integral flat chain*, i.e., $\mathbb{P}(u)$ is the boundary of an i.m. rectifiable current with finite mass.

THE RELAXED $W^{1,1}$-ENERGY. For $\Omega = \widetilde{B}^n$ or B^n, denote

$$\mathcal{E}_{1,1}(u,\Omega) := \int_\Omega |\nabla u|\, dx\,, \qquad u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)\,,$$

and consider the *lower semicontinuous envelope* of the functional

$$\overline{\mathcal{E}_{1,1}}(u,\Omega) := \begin{cases} \displaystyle\int_\Omega |\nabla u|\, dx & \text{if } u \in C^1_\varphi(\widetilde{B}^n, S^1) \\ +\infty & \text{elsewhere in } W^{1,1}_\varphi(\widetilde{B}^n, S^1)\,. \end{cases} \tag{1.8}$$

More precisely, we define the *relaxed* $W^{1,1}$*-energy* $u \mapsto \widetilde{\mathcal{E}_{1,1}}(u,\Omega)$ as the greatest functional on $W^{1,1}_\varphi(\widetilde{B}^n, S^1)$ which is lower than or equal to $u \mapsto \overline{\mathcal{E}_{1,1}}(u,\Omega)$ and is lower semicontinuous with respect to the strong L^1-convergence. Of course, for every $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$ we have

$$\widetilde{\mathcal{E}_{1,1}}(u,\Omega) = \inf \left\{ \liminf_{h\to\infty} \int_\Omega |\nabla u_h|\, dx \,:\, \{u_h\} \subset C^1_\varphi(\widetilde{B}^n, S^1)\,, \quad u_h \to u \quad \text{a.e.} \right\}.$$

It is well-known that

$$\mathcal{E}_{1,1}(u,\Omega) \le \widetilde{\mathcal{E}_{1,1}}(u,\Omega) \qquad \forall\, u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)\,. \tag{1.9}$$

However, since $\pi_1(S^1) \neq 0$, in general the strict inequality "$<$" may hold in (1.9), see e.g. [13].

CARTESIAN CURRENTS. In order to analyze the structure property of the relaxed $W^{1,1}$-energy, we recall from [13] some facts concerning the class of *Cartesian currents* $\text{cart}(\widetilde{B}^n \times S^1)$.

Definition 1.2 *We let*

$$\text{cart}(\widetilde{B}^n \times S^1) := \{T \in \text{cart}(\widetilde{B}^n \times \mathbb{R}^2) \mid \text{spt}\, T \subset \widetilde{B}^n \times S^1\}$$

and

$$\text{cart}_\varphi(\widetilde{B}^n \times S^1) := \{T \in \text{cart}(\widetilde{B}^n \times S^1) \mid T \llcorner (\widetilde{B}^n \setminus \overline{B}^n) \times S^1 = G_\varphi \llcorner (\widetilde{B}^n \setminus \overline{B}^n) \times S^1\}\,,$$

where $\text{cart}(\widetilde{B}^n \times \mathbb{R}^N)$ *is defined as the class of i.m. rectifiable currents* T *in* $\mathcal{R}_n(\widetilde{B}^n \times \mathbb{R}^N)$ *which have no inner boundary,* $\partial T \llcorner \widetilde{B}^n \times \mathbb{R}^N = 0$*, have finite mass,* $\mathbf{M}(T) < \infty$*, and are such that* $\|T\|_1 < \infty$*,* $\pi_\# T = [\![\widetilde{B}^n]\!]$ *and* $T^{\bar{0}0} \ge 0$*, where*

$$\|T\|_1 := \sup\{T(\varphi(x,y)|y|\, dx) \mid \varphi \in C^0_c(\widetilde{B}^n \times \mathbb{R}^N)\,,\ \|\varphi\| \le 1\}\,,$$

and $T^{\bar{0}0}$ *is the Radon measure in* $\widetilde{B}^n \times \mathbb{R}^N$ *given by*

$$T^{\bar{0}0}(\varphi(x,y)) := T(\varphi(x,y)\, dx) \qquad \forall\, \varphi \in C^0_c(\widetilde{B}^n \times \mathbb{R}^N)\,.$$

Then, to any $T \in \text{cart}_\varphi(\widetilde{B}^n \times S^1)$ we may associate a function $u_T \in BV_\varphi(\widetilde{B}^n, S^1)$, where

$$BV_\varphi(\widetilde{B}^n, S^1) := \{u \in BV(\widetilde{B}^n, \mathbb{R}^2) \,:\, |u(x)| = 1 \text{ a.e. and } u = \varphi \text{ in } \widetilde{B}^n \setminus \overline{B}^n\}\,,$$

such that

$$T(\phi(x,y)\, dx) = \int_{\widetilde{B}^n} \phi(x, u_T(x))\, dx \tag{1.10}$$

for all $\phi \in C^0(\widetilde{B}^n \times \mathbb{R}^2)$ such that $|\phi(x,y)| \le C\,(1 + |y|)$, and for any $\psi \in C^1_c(\widetilde{B}^n)$ and for $j = 1, 2$

$$(-1)^{n-i} T(\psi(x) \widehat{dx^i} \wedge dy^j) = \langle D_i u^j_T, \psi \rangle := -\int_{\widetilde{B}^n} u^j_T(x) \cdot D_i \psi(x)\, dx\,, \tag{1.11}$$

where

$$\widehat{dx^i} := dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n\,.$$

Moreover, if the corresponding BV-function u_T belongs to the Sobolev class $W^{1,1}_\varphi(\widetilde{B}^n, S^1)$, we have

$$T = G_{u_T} + L \times [\![S^1]\!] \,, \tag{1.12}$$

where $[\![S^1]\!]$ is the 1-current integration of 1-forms on S^1, with respect to the counterclockwise orientation, and L is an $(n-1)$-dimensional i.m. rectifiable current in $\mathcal{R}_{n-1}(\widetilde{B}^n)$ with *finite mass*, $\mathbf{M}(L) < \infty$, and support in $\overline{B}^n$, $\operatorname{spt} L \subset \overline{B}^n$. Now, since T satisfies the null-boundary condition

$$\partial T \llcorner \widetilde{B}^n \times S^1 = 0 \,,$$

i.e., $T(d\omega) = 0$ for all $\omega \in \mathcal{D}^{n-1}(\widetilde{B}^n \times S^1)$, if $\omega = \pi^\# \xi \wedge \widehat{\pi}^\# \omega_{S^1}$ for some $\xi \in \mathcal{D}^{n-2}(\widetilde{B}^n)$, since $d\omega = \pi^\# d\xi \wedge \widehat{\pi}^\# \omega_{S^1} = (-1)^{n-1} \widehat{\pi}^\# \omega_{S^1} \wedge \pi^\# d\xi$, whereas

$$(L \times [\![S^1]\!])(\pi^\# d\xi \wedge \widehat{\pi}^\# \omega_{S^1}) = L(d\xi) \cdot [\![S^1]\!](\omega_{S^1}) = 2\pi \, \partial L(\xi) \,,$$

we infer that

$$\partial L(\xi) = (-1)^n \frac{1}{2\pi} G_{u_T}(\widehat{\pi}^\# \omega_{S^1} \wedge \pi^\# d\xi) = (-1)^n \, \mathbb{P}(u_T)(\xi) \,.$$

In conclusion, L satisfies the boundary condition

$$\partial L = (-1)^n \ \mathbb{P}(u_T) \,. \tag{1.13}$$

On the other hand, if L is an i.m. rectifiable current in $\mathcal{R}_{n-2}(\widetilde{B}^n)$, hence with finite mass, such that $\operatorname{spt} L \subset \overline{B}^n$, satisfying (1.13), the corresponding current (1.12) belongs to the class $\operatorname{cart}_\varphi(\widetilde{B}^n \times S^1)$.

Denote now for any $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$ by

$$\mathcal{T}_u := \{ T \in \operatorname{cart}_\varphi(\widetilde{B}^n \times S^1) \mid u_T = u \} \tag{1.14}$$

the class of Cartesian currents T in $\operatorname{cart}_\varphi(\widetilde{B}^n \times S^1)$ such that the underlying BV-function u_T is equal to u. By the previous discussion we have

$$\mathcal{T}_u = \{ G_u + L \times [\![S^1]\!] \, : \, L \in \mathcal{R}_{n-1}(\widetilde{B}^n) \,, \ \operatorname{spt} L \subset \overline{B}^n , \ \partial L = (-1)^n \ \mathbb{P}(u) \} \,. \tag{1.15}$$

The density result of Bethuel. If $n \geq 2$, we denote by $R^\infty_{1,\varphi}(\widetilde{B}^n, S^1)$ the set of all the maps $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$ which are smooth except on a singular set $\Sigma(u)$ of the type

$$\Sigma(u) = \bigcup_{i=1}^r \Sigma_i \,, \qquad r \in \mathbb{N} \,, \tag{1.16}$$

where Σ_i is a smooth $(n-2)$-dimensional subset of $\overline{B}^n$ with smooth boundary, if $n \geq 3$, and Σ_i is a point if $n = 2$. The following density result appears in [4].

Theorem 1.3 *$R^\infty_{1,\varphi}(\widetilde{B}^n, S^1)$ is strongly dense in $W^{1,1}_\varphi(\widetilde{B}^n, S^1)$.*

Theorem 1.3 yields, in particular, see [11], that $\mathbb{P}(u)$ is an integral flat chain. Thus by (1.7) we readily obtain the following

Corollary 1.4 *For any $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$ the class $\mathcal{T}_u$ is non-empty.*

Moreover, the following density result holds true, see e.g. [15].

Proposition 1.5 *Let $n \geq 2$ and let $T \in \mathcal{T}_u$ satisfy* (1.12) *for some $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$. There exists a smooth sequence $\{u_h\} \subset C^1_\varphi(\widetilde{B}^n, S^1)$ such that $G_{u_h} \rightharpoonup T$ as $h \to \infty$ weakly in $\mathcal{D}_n(\widetilde{B}^n \times S^1)$, i.e., $G_{u_h}(\omega) \to T(\omega)$ for all $\omega \in \mathcal{D}^n(\widetilde{B}^n \times S^1)$, and*

$$\lim_{h\to\infty} \int_{\widetilde{B}^n} |\nabla u_h| \, dx = \mathcal{E}_{1,1}(T) := \int_{\widetilde{B}^n} |\nabla u| \, dx + 2\pi \, \mathbf{M}(L) \,.$$

We now recall that the weak convergence $G_{u_h} \rightharpoonup T$ yields the convergence of u_h to u_T weakly in the BV-sense, that the energy $T \mapsto \mathcal{E}_{1,1}(T)$ is lower semicontinuous with respect to the weak convergence as currents, and that the class $\mathrm{cart}_\varphi(\widetilde{B}^n \times S^1)$ is closed. As a consequence, compare [15], from the above density result we readily obtain the following

Proposition 1.6 *For any $n \geq 2$ and $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$ we have $\widetilde{\mathcal{E}_{1,1}}(u, \widetilde{B}^n) < \infty$. Moreover, the relaxed energy of u is given by*

$$\begin{aligned}
\widetilde{\mathcal{E}_{1,1}}(u, \widetilde{B}^n) &= \inf\{\mathcal{E}_{1,1}(T) \mid T \in \mathcal{T}_u\} \\
&= \int_{\widetilde{B}^n} |\nabla u|\, dx + 2\pi\, m_i(\mathbb{P}(u)) \\
&= \int_{\widetilde{B}^n} |\nabla u|\, dx + 2\pi\, \inf\big\{\mathbf{M}(L)\,:\, L \in \mathcal{R}_{n-1}(\widetilde{B}^n)\,,\ \mathrm{spt}\, L \subset \overline{B}^n,\ \partial L = (-1)^n\ \mathbb{P}(u)\big\}\,.
\end{aligned}$$

2 Flat norm and minimal connections

In this section we write more explicitly the action of the current $\mathbb{P}(u)$ associated to the singular set of a Sobolev map $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$, and recover in the case $n = 2$ some results from [6].

THE SINGULAR SET AS A DISTRIBUTION. We will denote by $\mathrm{Lip}(B^n, \Lambda^k T B^n)$ the class of k-forms in B^n with coefficients in $\mathrm{Lip}(B^n)$, for every $k = 0, \ldots, n$. Every $(n-2)$-form $\zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n)$ will be written as

$$\zeta = \sum_{1 \leq i < j \leq n} \zeta^{i,j}\, \widehat{dx^{i,j}}\,, \tag{2.1}$$

where $\zeta^{i,j} \in \mathrm{Lip}(B^n; \mathbb{R})$ and

$$\widehat{dx^{i,j}} := dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n\,.$$

Let $g \in W^{1,1}(B^n, \mathbb{R}^2) \cap L^\infty$. For any $i < j$ introduce the distribution $T_{i,j}(g) \in \mathcal{D}'(B^n, \mathbb{R})$ given by

$$T_{i,j}(g) := -(g \times g_{x_i})_{x_j} + (g \times g_{x_j})_{x_i} \tag{2.2}$$

where for every i

$$g \times g_{x_i} := g^1 g^2_{x_i} - g^2 g^1_{x_i}\,,$$

that is,

$$\langle T_{i,j}(g), \zeta^{i,j}\rangle = \int_{B^n} \big((g \times g_{x_i})\, \zeta^{i,j}_{x_j} - (g \times g_{x_j})\, \zeta^{i,j}_{x_i}\big)\, dx \qquad \forall\, \zeta^{i,j} \in \mathrm{Lip}(B^n, \mathbb{R})\,.$$

Definition 2.1 *Let $n \geq 2$. To any $g \in W^{1,1}(B^n, \mathbb{R}^2) \cap L^\infty$ we associate the distribution $T(g)$ of order $(n-2)$ defined by*

$$\langle T(g), \zeta\rangle := \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} \langle T_{i,j}(g), \zeta^{i,j}\rangle \qquad \forall\, \zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n)\,,$$

where ζ is decomposed as in (2.1).

Viewing the target space S^1 as a subspace of $\mathbb{C}$, from Definition 2.1 we readily obtain

Proposition 2.2 *Let $n \geq 2$. We have:*

(*a*) $T(g) = -T(\overline{g})$ *for all* $g \in W^{1,1}(B^n, \mathbb{R}^2) \cap L^\infty$;

(*b*) $T(gh) = T(g) + T(h)$ *for all* $g, h \in W^{1,1}(B^n, S^1)$.

PROOF: Property (a) is trivial, (b) follows arguing as in [6]. $\square$

For any $n \geq 2$ and $g \in W^{1,1}(B^n, \mathbb{R}^2) \cap L^\infty$, defining

$$g \times \nabla g := (g \times g_{x_1}, \dots, g \times g_{x_n}),$$

we can write

$$\langle T(g), \zeta \rangle = \int_{B^n} \left(\sum_{i=1}^n (g \times g_{x_i}) F^i(\zeta) \right) d\mathcal{H}^n \qquad \forall \, \zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n),$$

where for every fixed i

$$F^i(\zeta) := \sum_{1 \leq h < i} (-1)^{i+h} \zeta^{h,i}_{x_h} - \sum_{i < h \leq n} (-1)^{i+h} \zeta^{i,h}_{x_h},$$

ζ being decomposed as in (2.1). Therefore, setting $F(\zeta) := (F^1(\zeta), \dots, F^n(\zeta))$, we have

$$\langle T(g), \zeta \rangle = \int_{B^n} (g \times \nabla g) \cdot F(\zeta) \, dx \qquad \forall \, \zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n).$$

Notice that, if $n = 2$,

$$F(\zeta) = \nabla^\perp \zeta := (\zeta_{x_2}, -\zeta_{x_1}),$$

so that, as in [6], we have

$$\langle T(g), \zeta \rangle := \int_{B^2} (g \times \nabla g) \cdot \nabla^\perp \zeta \, dx \qquad \forall \, \zeta \in \mathrm{Lip}(B^2, \mathbb{R}).$$

THE LINK BETWEEN $T(u)$ AND $\mathbb{P}(u)$. Suppose now that $u \in W^{1,1}(B^n, S^1)$. By the dominated convergence theorem, the action of the current G_u, see (1.1), extends e.g. to forms $\omega := \widehat{\pi}^\# \omega_{S^1} \wedge \pi^\# d\zeta$, where $\zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n)$, so that

$$G_u(\widehat{\pi}^\# \omega_{S^1} \wedge \pi^\# d\zeta) = \int_{B^n} u^\# \omega_{S^1} \wedge d\zeta.$$

If ζ is given by (2.1), since for every $i < j$

$$dx^i \wedge \widehat{dx^{i,j}} = (-1)^{i-1} \, \widehat{dx^j} \quad \text{and} \quad dx^j \wedge \widehat{dx^{i,j}} = (-1)^j \, \widehat{dx^i},$$

we have

$$d\zeta = \sum_{1 \leq i < j \leq n} \left((-1)^{i-1} \zeta^{i,j}_{x_i} \, \widehat{dx^j} + (-1)^j \zeta^{i,j}_{x_j} \, \widehat{dx^i} \right).$$

Consequently, since $dx^h \wedge \widehat{dx^h} = (-1)^{h-1} dx^1 \wedge \cdots \wedge dx^n$, we have

$$\begin{aligned} u^\# \omega_{S^1} \wedge d\zeta &= (u^1 du^2 - u^2 du^1) \wedge d\zeta \\ &= \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} \left((u \times u_{x_i}) \, \zeta^{i,j}_{x_j} - (u \times u_{x_j}) \, \zeta^{i,j}_{x_i} \right) dx^1 \wedge \cdots \wedge dx^n. \end{aligned}$$

Therefore, for every $\zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n)$ satisfying (2.1) we have

$$G_u(\widehat{\pi}^\# \omega_{S^1} \wedge \pi^\# d\zeta) = \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} \langle T_{i,j}(u), \zeta^{i,j} \rangle.$$

On account of Definition 2.1, we conclude that

$$\frac{1}{2\pi} \langle T(u), \zeta \rangle = \mathbb{P}(u)(\zeta) \qquad \forall \, \zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2} T B^n). \tag{2.3}$$

THE FLAT NORM. Let $\Gamma \in \mathcal{D}_k(\widetilde{B}^n)$ with $\mathrm{spt}\, \Gamma \subset \overline{B}^n$, and suppose that Γ is the boundary of a $(k+1)$-dimensional current $D \in \mathcal{D}_{k+1}(\widetilde{B}^n)$, with $\mathrm{spt}\, D \subset \overline{B}^n$. The *flat norm* of Γ is defined by

$$F_{\overline{B}^n}(\Gamma) := \sup\{ \Gamma(\xi) \mid \xi \in \mathcal{D}^k(\widetilde{B}^n), \ \max\{\|\xi\|, \|d\xi\|\} \leq 1 \ \text{ in } \ \overline{B}^n \}.$$

Taking $k = n-2$, we now define for any $n \geq 2$

$$L(u) := F_{\overline{B}^n}(\mathbb{P}(u)) \,, \qquad u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1) \,, \tag{2.4}$$

so that for every $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$ we have

$$\begin{array}{rcl} L(u) & := & \sup\{\mathbb{P}(u)(\xi) \mid \xi \in \mathcal{D}^{n-2}(\widetilde{B}^n) \,, \ \max\{\|\xi\|, \|d\xi\|\} \leq 1 \ \text{ in } \ \overline{B}^n\} \\ & = & \dfrac{1}{2\pi} \sup\left\{ \displaystyle\int_{\widetilde{B}^n} u^\# \omega_{S^1} \wedge d\xi \mid \xi \in \mathcal{D}^{n-2}(\widetilde{B}^n) \,, \ \max\{\|\xi\|, \|d\xi\|\} \leq 1 \ \text{ in } \ \overline{B}^n \right\}. \end{array}$$

We now observe that by (1.4) and (2.3) we have

$$\mathbb{P}(u)(\xi) = \langle T(u_{|B^n}), \xi \rangle - \int_{\partial B^n} \varphi^\# \omega_{S^1} \wedge \xi \tag{2.5}$$

where $u_{|B^n}$ is the restriction of u to B^n. In order to write explicitly the boundary term, we notice that

$$\begin{array}{rcl} j > i & \Longrightarrow & dx^j \wedge \widehat{dx^{i,j}} = (-1)^j \, \widehat{dx^i} \\ h < i & \Longrightarrow & dx^h \wedge \widehat{dx^{h,i}} = (-1)^{h+1} \, \widehat{dx^i} \,. \end{array}$$

Therefore, if

$$\xi = \sum_{1 \leq i < j \leq n} \xi^{i,j} \, \widehat{dx^{i,j}} \,, \qquad \xi^{i,j} \in C^\infty_c(\widetilde{B}^n) \,,$$

we have

$$\varphi^\# \omega_{S^1} \wedge \xi = (\varphi^1 d\varphi^2 - \varphi^2 d\varphi^1) \wedge \xi = \sum_{i=1}^n A_i(x) \, \widehat{dx^i} \,,$$

where for every i

$$A_i(x) = \sum_{j>i} (-1)^j \xi^{i,j} \, (\varphi \times \varphi_{x_j}) + \sum_{h<i} (-1)^{h+1} \xi^{h,i} \, (\varphi \times \varphi_{x_h}) \,.$$

We then obtain

$$\int_{\partial B^n} \varphi^\# \omega_{S^1} \wedge \xi = \int_{\partial B^n} \sum_{i=1}^n A_i(x) \, \widehat{dx^i} = \int_{\partial B^n} \langle F, \nu \rangle \, d\mathcal{H}^{n-1} \,,$$

where ν is the outward unit normal to ∂B^n and $F(x) = (F^1, \ldots, F^n)(x)$ is the vector field of components

$$F^i(x) := (-1)^{i-1} A_i(x) = \sum_{j>i} (-1)^{i+j-1} \xi^{i,j} \, (\varphi \times \varphi_{x_j}) + \sum_{h<i} (-1)^{i+h} \xi^{h,i} \, (\varphi \times \varphi_{x_h}) \,. \tag{2.6}$$

Remark 2.3 Notice that the above boundary term depends only on the tangential components of the derivatives of φ, i.e., it can be expressed in terms of $(\varphi \times \varphi_{\tau_i}) := (\varphi^1 \varphi^2_{\tau_i} - \varphi^2 \varphi^1_{\tau_i})$, where $\tau_1, \ldots, \tau_{n-1}$ is any orthonormal frame tangent to ∂B^n such that $(\tau_1, \ldots, \tau_{n-1}, \nu)$ is a positively oriented orthonormal frame to $\mathbb{R}^n$. In particular, it is zero if e.g. the boundary datum φ is constant on ∂B^n. Moreover, in the simpler case $n = 2$ we have

$$F^1(x) = \xi \, (\varphi \times \varphi_{x_2}) \,, \qquad F^2(x) = -\xi \, (\varphi \times \varphi_{x_1}) \,,$$

so that for every $\xi \in C^\infty_c(\widetilde{B}^2)$

$$\int_{\partial B^2} \varphi^\# \omega_{S^1} \wedge \xi = \int_{\partial B^2} \xi \, D(\varphi) \cdot \nu \, d\mathcal{H}^1 \,,$$

where $D(\varphi) := (\varphi \times \varphi_{x_2}, -\varphi \times \varphi_{x_1})$. Equivalently, if (τ, ν) is positively oriented, we may write

$$\int_{\partial B^2} \varphi^\# \omega_{S^1} \wedge \xi = -\int_{\partial B^2} \xi \, (\varphi \times \varphi_\tau) \, d\mathcal{H}^1 \,.$$

In conclusion, we infer that for every $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$

$$L(u) = \frac{1}{2\pi}\,\max\Big\{ \langle T(u_{|B^n}), \zeta\rangle - \int_{\partial B^n} \langle F, \nu\rangle \, d\mathcal{H}^{n-1} \quad | \quad \zeta \in \mathrm{Lip}(\widetilde{B}^n, \Lambda^{n-2} T\widetilde{B}^n)\,, \\ \max\{\|\zeta\|_\infty, \|\nabla\zeta\|_\infty\} \le 1 \ \text{ in } \ \overline{B}^n \Big\}$$

where $F = F(\zeta, \varphi)$ is the vector field given by (2.6), with $\xi = \zeta$, and

$$\|\zeta\|_\infty := \sup_{i<j} \|\zeta^{i,j}\|_\infty\,, \qquad \|\nabla\zeta\|_\infty := \sup_{i<j} \|\nabla\zeta^{i,j}\|_\infty$$

if ζ is decomposed as in (2.1). Moreover, in the case $n = 2$ the above formula simplifies to

$$L(u) = \frac{1}{2\pi}\,\max\Big\{ \langle T(u_{|B^2}), \zeta\rangle + \int_{\partial B^2} \zeta\,(\varphi \times \varphi_\tau)\, d\mathcal{H}^1 \mid \zeta \in \mathrm{Lip}(\widetilde{B}^2)\,,\ \max\{\|\zeta\|_\infty, \|\nabla\zeta\|_\infty\} \le 1 \ \text{ in } \ \overline{B}^2 \Big\}.$$

If $g \in W^{1,1}(S^n, S^1)$, with a similar computation we obtain

$$L(g) = \frac{1}{2\pi}\,\max\Big\{ \langle T(g), \zeta\rangle \mid \zeta \in \mathrm{Lip}(S^n, \Lambda^{n-2} TS^n)\,,\ \ \|\nabla\zeta\|_\infty \le 1 \Big\}.$$

This last formula has been used in [6], in dimension $n = 2$, as definition of minimal connection of the singularities of g. Of course, the formula for $L(u)$ makes sense for any function $u \in W^{1,1}_\varphi(\widetilde{B}^n, \mathbb{R}^2) \cap L^\infty$. Moreover, by the definition of $T(u)$ we readily obtain the following

Proposition 2.4 *Let $n \ge 2$. We have:*

(c) $L(u) \le \dfrac{1}{2\pi}\left(\|u\|_{W^{1,1}(B^n)}\,\|u\|_{L^\infty(B^n)} + \|\nabla\varphi\|_{L^1(\partial B^n)}\right)$ *for all* $u \in W^{1,1}_\varphi(\widetilde{B}^n, \mathbb{R}^2) \cap L^\infty$*;*

(d) *if* $u_h, u \in W^{1,1}_\varphi(\widetilde{B}^n, \mathbb{R}^2) \cap L^\infty$ *are such that* $u_h \to u$ *in* $W^{1,1}$ *and* $\|u_h\|_\infty \le C$*, then* $L(u_h) \to L(u)$*.*

Proof: Property (c) is trivial whereas, since

$$|\langle T(u_{h|B^n}), \zeta\rangle - \langle T(u_{|B^n}), \zeta\rangle| \le C\left(\int_{B^n} |u_h|\,|\nabla(u_h - u)|\,|\nabla\zeta|\,dx + \int_{B^n} |u_h - u|\,|\nabla u|\,|\nabla\zeta|\,dx\right),$$

where $C > 0$ is an absolute constant, and the boundary term in (2.5) does not depend on u_h or u, we have

$$|L(u_h) - L(u)| \le C\left(\|u_h - u\|_{W^{1,1}(B^n)} + \|(u_h - u)\,\nabla u\|_{L^1(B^n)}\right)$$

and (d) follows from the dominated convergence theorem. $\square$

Flat norm and real mass. The following property goes back to Federer [9].

Proposition 2.5 *Let $\Gamma \in \mathcal{D}_k(\widetilde{B}^n)$ and $D \in \mathcal{D}_{k+1}(\widetilde{B}^n)$ be such that $\operatorname{spt}\Gamma \subset \overline{B}^n$, $\operatorname{spt} D \subset \overline{B}^n$, and $\partial D = \Gamma$, with $\mathbf{M}(D) < \infty$. We have*

$$F_{\overline{B}^n}(\Gamma) = m_r(\Gamma)\,.$$

Proof: Trivially

$$F_{\overline{B}^n}(\Gamma) \le \sup\{D(d\xi) \mid \xi \in \mathcal{D}^k(\widetilde{B}^n)\,,\ \|d\xi\| \le 1 \ \text{ in } \ \overline{B}^n\} \le \mathbf{M}(D)\,,$$

hence $F_{\overline{B}^n}(\Gamma) \le m_r(\Gamma)$. Conversely, consider the seminorm $\nu(\eta) := \sup_{\overline{B}^n} \|\eta\|$, for $\eta \in \mathcal{D}^{k+1}(\widetilde{B}^n)$. By means of the linear map $\xi \mapsto d\xi$ from $\mathcal{D}^k(\widetilde{B}^n)$ to $\mathcal{D}^{k+1}(\widetilde{B}^n)$, we may regard Γ as a linear functional $\widetilde{\Gamma}$ on the subspace of exact forms in $\mathcal{D}^{k+1}(\widetilde{B}^n)$, endowed with the sup-norm. Of course, $\widetilde{\Gamma}(d\xi) \le F_{\overline{B}^n}(\Gamma)\,\nu(d\xi)$ for every $\xi \in \mathcal{D}^k(\widetilde{B}^n)$. Therefore, by Hahn-Banach theorem, we can extend $\widetilde{\Gamma}$ to a linear functional S on

$\mathcal{D}^{k+1}(\widetilde{B}^n)$ such that $S = \widetilde{\Gamma}$ on exact forms in $\mathcal{D}^{k+1}(\widetilde{B}^n)$, and such that $S(\eta) \leq F_{\overline{B}^n}(\Gamma)\,\nu(\eta)$ for every $\eta \in \mathcal{D}^{k+1}(\widetilde{B}^n)$. Since $\partial S = \Gamma$ and $\operatorname{spt} S \subset \overline{B}^n$, we obtain the assertion. $\square$

Now, taking $k = n-2$, $\Gamma = \mathbb{P}(u)$ and $D = (\mathbb{D}(u) - \mathbb{D}(\varphi))$, see (1.3), Proposition 2.5 yields

$$L(u) = m_r(\mathbb{P}(u)) \qquad \forall\, u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)\,.$$

Moreover, by Definition 1.1 we have

$$m_r(\mathbb{P}(u)) \leq \mathbf{M}(\mathbb{D}(u) - \mathbb{D}(\varphi))\,.$$

Since $(\mathbb{D}(u) - \mathbb{D}(\varphi)) = (\mathbb{D}(u) - \mathbb{D}(\varphi)) \llcorner \overline{B}^n$, and by the definition of mass, for $v = u, \varphi$,

$$\begin{aligned} \mathbf{M}\big(\mathbb{D}(v) \llcorner \overline{B}^n\big) &= \frac{1}{2\pi} \sup\Big\{ \int_{\overline{B}^n} v^\# \omega_{S^1} \wedge \gamma \,:\, \gamma \in \mathcal{D}^{n-1}(B^n)\,,\ \|\gamma\| \leq 1 \Big\} \\ &\leq \frac{1}{2\pi} \int_{B^n} |\nabla v|\, dx\,, \end{aligned}$$

we conclude that

$$L(u) = m_r(\mathbb{P}(u)) \leq \frac{1}{2\pi} \left(\int_{B^n} |\nabla u|\, dx + \int_{B^n} |\nabla \varphi|\, dx \right) \qquad \forall\, u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)\,. \tag{2.7}$$

Flat norm and integral mass. By (1.7), on account of Definition 1.1, we also infer that

$$L(u) = m_i(\mathbb{P}(u)) := \inf\{\mathbf{M}(L) \,:\, L \in \mathcal{R}_{n-1}(\widetilde{B}^n)\,,\ \operatorname{spt} L \subset \overline{B}^n,\ \partial L = (-1)^n\, \mathbb{P}(u)\} \tag{2.8}$$

for every $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$. We thus obtain that for every $n \geq 2$ and $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$ *there exists an i.m. rectifiable current* $L \in \mathcal{R}_{n-1}(\widetilde{B}^n)$*, hence with finite mass, such that* $\operatorname{spt} L \subset \overline{B}^n$ *and* $(-1)^n \partial L = 2\pi\, T(u)$, compare [6, Thm. 3] for the case $n = 2$.

We notice that in [1] it is proved that the converse holds true. More precisely, for any i.m. rectifiable current $L \in \mathcal{R}_{n-1}(\widetilde{B}^n)$, with $\operatorname{spt} L \subset \overline{B}^n$, there exists a function $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$ such that $(-1)^n \partial L = 2\pi\, T(u)$.

We also recall that by the boundary rectifiability theorem, see e.g. [25, Thm. 30.3], if $L \in \mathcal{R}_{n-1}(\widetilde{B}^n)$ has boundary of finite mass, $\mathbf{M}(\partial L) < \infty$, then ∂L is an i.m. rectifiable current in $\mathcal{R}_{n-2}(\widetilde{B}^n)$. Due to (2.3) we thus obtain

Corollary 2.6 *Let* $n \geq 2$ *and* $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$*. The distribution* $T(u)$ *is a measure of finite mass if and only if the current* $\mathbb{P}(u)$ *is i.m. rectifiable in* $\mathcal{R}_{n-2}(\widetilde{B}^n)$*.*

In the case $n = 2$, this yields the representation

$$T(u) = 2\pi \sum_{h=1}^{M} (\delta_{P_h} - \delta_{N_h})\,,$$

where δ_P is the unit Dirac mass at P and the sum is finite.

Finally, as a consequence of Proposition 1.6, by (2.8) we obtain the following link between the relaxed energy $\widetilde{\mathcal{E}_{1,1}}(u)$ and the minimal connection $L(u)$.

Corollary 2.7 *Let* $n \geq 2$*. For every* $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$ *and for* $\Omega = \widetilde{B}^n$ *or* B^n *we have*

$$\widetilde{\mathcal{E}_{1,1}}(u, \Omega) - \int_\Omega |\nabla u|\, dx = 2\pi \cdot L(u)\,.$$

Therefore, by (2.7) we conclude with the following:

Corollary 2.8 *Let $n \geq 2$. For every $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$ and for $\Omega = \widetilde{B}^n$ or B^n we have*

$$\widetilde{\mathcal{E}_{1,1}}(u,\Omega) \leq \int_\Omega |\nabla u|\,dx + \int_{B^n} |\nabla u|\,dx + \int_{B^n} |\nabla \varphi|\,dx\,.$$

In particular, if φ is constant on ∂B^n we have

$$\widetilde{\mathcal{E}_{1,1}}(u,B^n) \leq 2\int_{B^n} |\nabla u|\,dx\,.$$

THE CASE WITH NO BOUNDARY DATA. In a similar way, we argue as follows. Let $\Gamma \in \mathcal{D}_k(B^n)$, and suppose that Γ is the boundary in B^n of a $(k+1)$-dimensional current $D \in \mathcal{D}_{k+1}(B^n)$, i.e., $(\partial D) \llcorner B^n = \Gamma$, with $\mathbf{M}(D) < \infty$. The *flat norm* of Γ is defined by

$$F_{B^n}(\Gamma) := \sup\{\Gamma(\xi) \mid \xi \in \mathcal{D}^k(B^n)\,,\ \|d\xi\| \leq 1\}\,.$$

Moreover, we denote respectively by

$$\begin{aligned} m_{i,B^n}(\Gamma) &:= \inf\{\mathbf{M}(L) \mid L \in \mathcal{R}_{k+1}(B^n)\,, \quad (\partial L) \llcorner B^n = \Gamma\} \\ m_{r,B^n}(\Gamma) &:= \inf\{\mathbf{M}(D) \mid D \in \mathcal{D}_{k+1}(B^n)\,, \quad (\partial D) \llcorner B^n = \Gamma\} \end{aligned}$$

the *integral* and *real mass* of Γ *in* B^n. Also, in case $m_{i,B^n}(\Gamma) < \infty$, we say that an i.m. rectifiable current $L \in \mathcal{R}_{k+1}(B^n)$ is an *integral minimal connection* of Γ *allowing connections to the boundary* if $(\partial L) \llcorner B^n = \Gamma$ and $\mathbf{M}(L) = m_{i,B^n}(\Gamma)$. Similarly to Proposition 2.5, we have

$$F_{B^n}(\Gamma) = m_{r,B^n}(\Gamma)\,.$$

Taking $k = n-2$, we now define for any $n \geq 2$

$$L(u) := F_{B^n}(\mathbb{P}(u))\,, \qquad u \in W^{1,1}(B^n, S^1)\,,$$

so that we obtain

$$L(u) = \frac{1}{2\pi} \max\{\langle T(u), \zeta\rangle \mid \zeta \in \mathrm{Lip}(B^n, \Lambda^{n-2}TB^n)\,, \quad \|\nabla\zeta\|_\infty \leq 1\}\,.$$

Setting now

$$\widetilde{\mathcal{E}_{1,1}}(u) := \inf\left\{\liminf_{h\to\infty} \int_{B^n} |\nabla u_h|\,dx \,:\, \{u_h\} \subset C^1(B^n, S^1)\,, \quad u_h \to u \quad \text{a.e.}\right\}\,,$$

we obtain for every $u \in W^{1,1}(B^n, S^1)$

$$\begin{aligned} \widetilde{\mathcal{E}_{1,1}}(u) &= \int_{B^n} |\nabla u|\,dx + 2\pi\, m_{i,B^n}(\mathbb{P}(u)) \\ &= \int_{B^n} |\nabla u|\,dx + 2\pi\,\inf\big\{\mathbf{M}(L) \,:\, L \in \mathcal{R}_{n-1}(B^n)\,,\ (\partial L) \llcorner B^n = (-1)^n\ \mathbb{P}(u)\big\}\,. \end{aligned}$$

Moreover, since

$$L(u) = m_{r,B^n}(\mathbb{P}(u)) = m_{i,B^n}(\mathbb{P}(u)) \leq \frac{1}{2\pi}\int_{B^n} |\nabla u|\,dx\,,$$

we obtain that

$$\widetilde{\mathcal{E}_{1,1}}(u) \leq 2\int_{B^n} |\nabla u|\,dx \qquad \forall\, u \in W^{1,1}(B^n, S^1)\,.$$

Finally, a statement analogous to Proposition 3.7 below holds true. In particular,

$$\begin{aligned} \widetilde{\mathcal{E}_{1,1}}(u) = \mathcal{E}_{1,1}(u) \quad &\iff \quad G_u \in \mathrm{cart}(B^n \times S^1) \iff T(u) = 0 \iff L(u) = 0 \\ &\iff \quad \text{u belongs to the strong } W^{1,1}\text{-closure of } C^\infty(B^n, S^1). \end{aligned}$$

Distributional minors. Let $G = (G^h_i)$ be a $(2 \times n)$-matrix, e.g., $G = \nabla u$ for some $u \in W^{1,1}(B^n, \mathbb{R}^2)$. For $1 \leq i < j \leq n$, we denote by $G_{i,j}$ the (2×2)-submatrix obtained by selecting the columns by i and j. We will also denote by $M_{i,j}(G)$ its determinant

$$M_{i,j}(G) := \det G_{i,j} \,.$$

We recall that the *matrix of the adjoints* of $G_{i,j}$ is defined by the formula

$$\begin{array}{ll} (\operatorname{adj} G_{i,j})^1_i := G^2_j \,, & (\operatorname{adj} G_{i,j})^1_j := -G^2_i \,, \\ (\operatorname{adj} G_{i,j})^2_i := -G^1_j \,, & (\operatorname{adj} G_{i,j})^2_j := G^1_i \,. \end{array} \tag{2.9}$$

Definition 2.9 *Let* $u \in W^{1,1}(B^n, \mathbb{R}^2) \cap L^\infty$. *The* distributional minor *of indices* $1 \leq i < j \leq n$ *of* ∇u *is defined by*

$$\operatorname{Det} \nabla_{i,j} u := \frac{1}{2} \sum_{h=1}^{2} \Big(\frac{\partial}{\partial x_i} \big(u^h(x) \, ((\operatorname{adj} \nabla u)_{i,j})^h_i \big) + \frac{\partial}{\partial x_j} \big(u^h(x) \, ((\operatorname{adj} \nabla u)_{i,j})^h_j \big) \Big) .$$

More explicitly, since $G^h_i = \nabla_i u^h = u^h_{x_i}$ if $G = \nabla u$, by (2.9) we have

$$\operatorname{Det} \nabla_{i,j} u := \frac{1}{2} \Big(\frac{\partial}{\partial x_i} \big(u^1 u^2_{x_j} - u^2 u^1_{x_j} \big) - \frac{\partial}{\partial x_j} \big(u^1 u^2_{x_i} - u^2 u^1_{x_i} \big) \Big)$$

i.e., for every $\zeta \in \operatorname{Lip}(B^n)$,

$$\langle \operatorname{Det} \nabla_{i,j} u, \zeta \rangle := -\frac{1}{2} \Big(\big(u \times u_{x_j} \big) \, D_i \zeta - \big(u \times u_{x_i} \big) \, D_j \zeta \Big) . \tag{2.10}$$

In particular, if $n = 2$ we infer that $\operatorname{Det} \nabla_{1,2} u = \operatorname{Det} \nabla u$, the *distributional determinant* of ∇u. Moreover, by (2.2) we also have that

$$T_{i,j}(u) = 2 \operatorname{Det} \nabla_{i,j} u \,.$$

Notice that if $u \in W^{1,1}(B^n, \mathbb{R}^2) \cap L^\infty$ is smooth, then $\operatorname{Det} \nabla_{i,j} u$ coincides with the pointwise determinant $M_{i,j}(\nabla u)$. In fact by (2.9) we have $\dfrac{\partial}{\partial x_i} ((\operatorname{adj} \nabla u)_{i,j})^h_i + \dfrac{\partial}{\partial x_j} ((\operatorname{adj} \nabla u)_{i,j})^h_j = 0$, so that Laplace's formulas for $h = 1, 2$ yield

$$\begin{aligned} \frac{\partial}{\partial x_i} \big(u^h(x) \, ((\operatorname{adj} \nabla u)_{i,j})^h_i \big) + \frac{\partial}{\partial x_j} \big(u^h(x) \, ((\operatorname{adj} \nabla u)_{i,j})^h_j \quad &= \quad \frac{\partial u^h}{\partial x_i} \, ((\operatorname{adj} \nabla u)_{i,j})^h_i + \frac{\partial u^h}{\partial x_j} \, ((\operatorname{adj} \nabla u)_{i,j})^h_j \\ &+ \quad u^h \left(\frac{\partial}{\partial x_i} ((\operatorname{adj} \nabla u)_{i,j})^h_i + \frac{\partial}{\partial x_j} ((\operatorname{adj} \nabla u)_{i,j})^h_j \right) \\ &= \quad M_{i,j}(\nabla u) \,. \end{aligned}$$

Of course, if $u \in \operatorname{Lip}(B^n, S^1)$, the *area formula* yields that $M_{i,j}(\nabla u) = 0$.

Let now $u \in W^{1,1}(B^n, \mathbb{R}^2) \cap L^\infty$ be such $M_{i,j}(\nabla u) \in L^1(B^n)$ for every $i < j$. Suppose in addition that the boundary of the graph ∂G_u has finite mass in $B^n \times \mathbb{R}^2$, i.e., $T(u)$ is a bounded measure, compare Corollary 2.6. With these hypotheses, in [21] it is shown that for every $i < j$ the distributional minor $\operatorname{Det} \nabla_{i,j} u$ is a signed Radon measure with finite total variation, the density of its absolute continuous part is equal to the pointwise determinant $M_{i,j}(\nabla u)$

$$\operatorname{Det} \nabla_{i,j} u = M_{i,j}(\nabla u) \cdot d\mathcal{L}^n + (\operatorname{Det} \nabla_{i,j} u)^s \,, \qquad (\operatorname{Det} \nabla_{i,j} u)^s \perp \mathcal{L}^n \,; \tag{2.11}$$

moreover, the singular part $(\operatorname{Det} \nabla_{i,j} u)^s$ is supported on a countably $\mathcal{H}^{n-2}$-rectifiable set, possibly with unbounded $\mathcal{H}^{n-2}$-measure. In particular, $(\operatorname{Det} \nabla_{i,j} u)^s$ does not contain any *Cantor type* mass and, in dimension $n = 2$, we have

$$(\operatorname{Det} \nabla u)^s = \sum_h c_h \, \delta_{x_h} \,, \qquad c_h \in \mathbb{R} \,,$$

where the sum is possibly infinite, but satisfying $\sum_h |c_h| < \infty$. Finally, if $n = 2$, notice that if the boundary ∂G_u has infinite mass, it may happen that the singular part of the distributional determinant is supported on a Cantor-type set of Hausdorff dimension $d \in]0, 2[$, compare [13] [22].

3 Lifting

In this section we extend to any dimension $n \geq 2$ a result from [6], proved in dimension $n = 2$, about the interpretation of the minimal connection $L(u)$ in terms of the L^1-*distance of the vector field* $u \times \nabla u$ *to the class of gradient maps*, where

$$u \times \nabla u := (u \times u_{x_1}, \dots, u \times u_{x_n}) . \tag{3.1}$$

More precisely, we will prove in any dimension $n \geq 2$ the following

Theorem 3.1 *For any* $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$ *we have*

$$L(u) = \frac{1}{2\pi} \min_{\psi \in BV(B^n,\mathbb{R})} |u \times \nabla u - D\psi|(B^n) .$$

In order to prove Theorem 3.1, we recall some results from [12] about the existence of a lifting of currents in $\mathrm{cart}_\varphi(\widetilde{B}^n \times S^1)$. To this aim we first recall, see [13], that the current *subgraph* of an L^1-function $\psi \in L^1(\widetilde{B}^n, \mathbb{R})$ is the $(n+1)$-dimensional current in $\mathcal{D}_{n+1}(\widetilde{B}^n \times \mathbb{R})$ defined by

$$SG_\psi(\phi(x,t) dx \wedge dt) := \int_{\widetilde{B}^n} \left(\int_0^{\psi(x)} \phi(x,t)\, dt \right) dx\,, \qquad \phi \in C^\infty_c(\widetilde{B}^n \times \mathbb{R}) . \tag{3.2}$$

Moreover, in the sequel we will denote by $i : \widetilde{B}^n \times \mathbb{R} \to \widetilde{B}^n \times S^1$ the map

$$i(x,t) := (x, \cos t, \sin t)\,,$$

and by G_{q_0} the current in $\mathcal{D}_n(\widetilde{B}^n \times S^1)$ integration over the graph of the constant map $q_0(x) \equiv (1, 0)$. In [12], see also [13, Vol. II, Sec. 6.2.2], the following is proved:

Proposition 3.2 *Let* $T \in \mathrm{cart}_\varphi(\widetilde{B}^n \times S^1)$*. The following facts hold:*

i) *There exists a current* $\Sigma \in \mathcal{D}_{n+1}(\widetilde{B}^n \times S^1)$ *such that*

$$T - G_{q_0} = (-1)^n \partial \Sigma . \tag{3.3}$$

ii) *There exists a function* $\psi \in BV(\widetilde{B}^n, \mathbb{R})$ *such that* $\Sigma = i_\# SG_\psi$*, i.e.,*

$$T - G_{q_0} = (-1)^n i_\# \partial SG_\psi . \tag{3.4}$$

In particular, $\mathbf{M}(\partial SG_\psi) = \mathbf{M}(T) + \mathcal{L}^n(\widetilde{B}^n) < \infty$.

iii) *If* $u_T \in BV_\varphi(\widetilde{B}^n, S^1)$ *is the BV-function corresponding to* T*, then*

$$u_T = e^{i\psi} \qquad \mathcal{L}^n\text{-a.e. on } \widetilde{B}^n . \tag{3.5}$$

Remark 3.3 In [15] it is shown that for every $u \in BV_\varphi(\widetilde{B}^n, S^1)$ there exists a current $T \in \mathrm{cart}_\varphi(\widetilde{B}^n \times S^1)$ such that $u_T = u$. As a consequence, from Proposition 3.2 we infer that *every BV-function* $u \in BV_\varphi(\widetilde{B}^n, S^1)$ *has a* lift ψ *in* $BV(\widetilde{B}^n, \mathbb{R})$. Notice that in general the lift of a $W^{1,1}$-function in $W^{1,1}_\varphi(\widetilde{B}^n, S^1)$ is not a Sobolev function in $W^{1,1}(\widetilde{B}^n, \mathbb{R})$, but only a BV-function. However, the existence of a lifting in the sense of (3.5) is not very useful, even if u_T belongs to $W^{1,1}_\varphi(\widetilde{B}^n, S^1)$. In fact, to recover homological and topological properties from the lifting, the right condition is (3.4). As we shall see in Proposition 3.7 below, nice properties are recovered if the lifting ψ is a Sobolev map $\psi \in W^{1,1}(\widetilde{B}^n, \mathbb{R})$.

Remark 3.4 From (3.4) it readily follows that *the area of the graph of* ψ *is equal to the mass of* T,

$$\int_{\widetilde{B}^n} \sqrt{1 + |\nabla\psi|^2}\, dx + |D^J\psi|(\widetilde{B}^n) + |D^C\psi|(\widetilde{B}^n) = \mathbf{M}(T)\,,$$

where $\nabla\psi$, $D^J\psi$ and $D^C\psi$ denote the approximate gradient, the jump part and the Cantor part of the distributional derivative of ψ, see e.g. [3] [13]. In particular, if u_T belongs to the Sobolev class $W^{1,1}_\varphi(\widetilde{B}^n, S^1)$, and $L \in \mathcal{R}_{n-1}(\widetilde{B}^n)$ is given by (1.12), we have

$$|D\psi|(\widetilde{B}^n) = \int_{\widetilde{B}^n} |\nabla u_T|\, dx + 2\pi\, \mathbf{M}(L)\,.$$

Remark 3.5 The formula (3.4) clearly yields that if $u_T \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$, the derivative $D\psi$ of the lifting ψ has a *null Cantor part*. However, in general, the lifting ψ of a function $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$ is not a Sobolev function in $W^{1,1}_\varphi(\widetilde{B}^n, \mathbb{R})$, think for instance of $n = 2$ and $u(x) := x/|x|$. However, if the graph of u has no inner boundary, i.e., if G_u belongs to $\mathrm{cart}_\varphi(\widetilde{B}^n \times S^1)$, the existence of a lifting $\psi \in W^{1,1}(\widetilde{B}^n, \mathbb{R})$ satisfying (3.4) with $T = G_u$ is provided. In fact, we have

Corollary 3.6 *Let $n \geq 2$ and let $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$. Suppose that $\partial G_u = 0$ on $\mathcal{D}^n(\widetilde{B}^n \times S^1)$, i.e., $G_u \in \mathrm{cart}_\varphi(\widetilde{B}^n \times S^1)$. There exists a function $\psi \in W^{1,1}(\widetilde{B}^n, \mathbb{R})$ such that*

$$G_u - G_{q_0} = (-1)^n i_\# \partial S G_\psi\,. \tag{3.6}$$

Moreover, $u = e^{i\psi}$ a.e. on $\widetilde{B}^n$ and

$$D\psi = \nabla\psi\, d\mathcal{L}^n = (u \times \nabla u)\, d\mathcal{L}^n\,,$$

where $u \times \nabla u$ is the L^1-vector field given by (3.1).

Proposition 3.7 *Let $n \geq 2$ and let $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$. The following facts are equivalent:*

- (a) $G_u \in \mathrm{cart}_\varphi(\widetilde{B}^n \times S^1)$;
- (b) $T(u) = 0$;
- (c) $L(u) = 0$;
- (d) *there exists a function $\psi \in W^{1,1}(\widetilde{B}^n, \mathbb{R})$ such that* (3.6) *holds and $u = e^{i\psi}$ a.e. on $\widetilde{B}^n$;*
- (e) *u belongs to the strong $W^{1,1}$-closure of smooth maps in $C^\infty_\varphi(\widetilde{B}^n, S^1)$.*

The equivalence $(b) \iff (d) \iff (e)$ was first proved in [7] for Sobolev maps in $W^{1,p}$, see also [5].

Optimal lifting. Following [6], we finally consider the energy

$$\widehat{\mathcal{E}_{1,1}}(u, \Omega) := \inf\{\, |D\psi|(\Omega) \,:\, \psi \in BV(\Omega, \mathbb{R})\,,\ u = e^{i\psi} \text{ a.e. on } \Omega\}\,,$$

where $\Omega = B^n$ or $\widetilde{B}^n$. Since Ω is simply connected, arguing exactly as in [6, Prop. 2] we obtain that

$$\widehat{\mathcal{E}_{1,1}}(u, \Omega) = \widetilde{\mathcal{E}_{1,1}}(u, \Omega) \qquad \forall\, u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)\,.$$

Therefore, by Corollary 2.7 we obtain that for every $n \geq 2$ and $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$

$$\widehat{\mathcal{E}_{1,1}}(u, \Omega) - \int_\Omega |\nabla u|\, dx = 2\pi \cdot L(u)\,.$$

Moreover, if φ is constant on ∂B^n, by Corollary 2.8 we have

$$\widehat{\mathcal{E}_{1,1}}(u, B^n) \leq 2 \int_{B^n} |\nabla u|\, dx\,.$$

In particular, since $L(u)=0 \iff T(u)=0$, by Proposition 3.7 we infer that

$$\forall\, u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)\,, \qquad \widehat{\mathcal{E}_{1,1}}(u,\Omega) = \int_\Omega |\nabla u|\,dx \qquad \iff \qquad G_u \in \mathrm{cart}_\varphi(\widetilde{B}^n \times S^1)\,.$$

Proof of Theorem 3.1: We recall that for any $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$ we have (1.15), see (1.14). This yields for the integral mass

$$m_i(u) = \inf\{\mathbf{M}(L_T) \mid G_u + L_T \times [\![S^1]\!] \in \mathcal{T}_u\}\,, \tag{3.7}$$

compare (2.8). Moreover, by Proposition 3.2, to any $T \in \mathcal{T}_u$ it corresponds a function $\psi_T \in BV(\widetilde{B}^n, \mathbb{R})$ such that

$$G_u + L_T \times [\![S^1]\!] - G_{q_0} = (-1)^n i_\# \partial S G_{\psi_T} \qquad \text{on} \quad \mathcal{D}^n(\widetilde{B}^n \times S^1)\,. \tag{3.8}$$

Let $\omega \in \mathcal{D}^n(\widetilde{B}^n \times S^1)$ be given by $\omega = \pi^\# \omega_\phi \wedge \widehat{\pi}^\# \omega_{S^1}$, where $\omega_\phi \in \mathcal{D}^{n-1}(\widetilde{B}^n)$ is given by

$$\omega_\phi := \sum_{i=1}^n (-1)^{i-1} \phi^i(x)\, \widehat{dx^i}\,, \qquad \phi = (\phi^1, \ldots, \phi^n) \in C^\infty_c(\widetilde{B}^n, \mathbb{R})\,. \tag{3.9}$$

In the sequel we omit to write the action of the projection maps π and $\widehat{\pi}$. Since

$$\begin{aligned}
\omega_\phi \wedge u^\# \omega_{S^1} &= \sum_{i=1}^n (-1)^{i-1} \phi^i \widehat{dx^i} \wedge (u^1 du^2 - u^2 du^1) \\
&= (-1)^{n-1} \sum_{i=1}^n \phi^i \cdot (u \times u_{x_i})\, dx\,,
\end{aligned}$$

we have

$$G_u(\omega_\phi \wedge \omega_{S^1}) = \int_{\widetilde{B}^n} \omega_\phi \wedge u^\# \omega_{S^1} = (-1)^{n-1} \int_{\widetilde{B}^n} \langle u \times \nabla u, \phi \rangle\, dx\,, \tag{3.10}$$

whereas

$$L_T \times [\![S^1]\!](\omega_\phi \wedge \omega_{S^1}) = L_T(\omega_\phi) \cdot [\![S^1]\!](\omega_{S^1}) = 2\pi\, L_T(\omega_\phi)$$

and

$$G_{q_0}(\omega_\phi \wedge \omega_{S^1}) = 0\,. \tag{3.11}$$

Moreover, since $d\omega_\phi = \mathrm{div}\phi\, dx$ and

$$i^\# d(\omega_\phi \wedge \omega_{S^1}) = i^\# (d\omega_\phi \wedge \omega_{S^1}) = i^\# (\mathrm{div}\phi\, dx \wedge \omega_{S^1}) = \mathrm{div}\phi\, dx \wedge dt\,,$$

on account of (3.2) we have

$$\begin{aligned}
i_\# \partial S G_{\psi_T}(\omega_\phi \wedge \omega_{S^1}) &= i_\# S G_{\psi_T}(\mathrm{div}\phi(x) dx \wedge \omega_{S^1}) \\
&= S G_{\psi_T}(\mathrm{div}\phi(x) dx \wedge dt) \\
&= \int_{\widetilde{B}^n} \mathrm{div}\phi(x)\, (\psi_T(x) - 0)\, dx = -\langle D\psi_T, \phi \rangle\,.
\end{aligned} \tag{3.12}$$

By (3.8) we have thus obtained

$$(-1)^n\, 2\pi\, L_T(\omega_\phi) = \int_{\widetilde{B}^n} \langle u \times \nabla u, \phi \rangle\, dx - \langle D\psi_T, \phi \rangle \qquad \forall\, \phi \in C^\infty_c(\widetilde{B}^n, \mathbb{R}^n)\,.$$

Moreover, since $T = G_\varphi$ on $(\widetilde{B}^n \setminus \overline{B}^n) \times S^1$, we have $D\psi_T = \varphi \times \nabla\varphi$ on $\widetilde{B}^n \setminus \overline{B}^n$ and hence

$$\mathbf{M}(L_T) := \sup\{L_T(\omega_\phi) \mid \|\phi\|_\infty \le 1\} = \frac{1}{2\pi} |u \times \nabla u - D\psi_T|(B^n)\,.$$

In conclusion, by (3.7) and (2.8) we obtain the assertion. $\square$

Proof of Corollary 3.6: By Proposition 3.2 we find the existence of a function $\psi \in BV(\widetilde{B}^n, \mathbb{R})$ such that

$$G_u - G_{q_0} = (-1)^n i_\# \partial S G_\psi ,$$

see (3.4). Taking $\omega = \pi^\# \omega_\phi \wedge \widehat{\pi}^\# \omega_{S^1} \in \mathcal{D}^n(\widetilde{B}^n \times S^1)$, where ω_ϕ is given by (3.9), using (3.10), (3.11), and (3.12) we obtain that

$$\int_{\widetilde{B}^n} \langle u \times \nabla u, \phi \rangle \, dx = \langle D\psi, \phi \rangle \qquad \forall\, \phi \in C^\infty_c(\widetilde{B}^n, \mathbb{R}^n) .$$

Therefore, since $u \times \nabla u \in L^1(\widetilde{B}^n, \mathbb{R}^n)$, we obtain $D\psi = (u \times \nabla u)\, d\mathcal{L}^n$ and hence the assertion. $\square$

Proof of Proposition 3.7: If (a) holds, then $\partial G_u = 0$, hence by (1.2) we have $\mathbb{P}(u) = 0$, which yields $T(u) = 0$, by (2.3). Conversely, if $T(u) = 0$, we have $\partial G_u = 0$, and hence $(a) \iff (b)$. The equivalence $(a) \iff (c)$ is a trivial consequence of definition (2.4). If (a) holds, we obtain (d) by Corollary 3.6. If (d) holds, and $\{\psi_h\}_h \subset C^\infty(\widetilde{B}^n, \mathbb{R})$ is such that $\psi_h \to \psi$ strongly in $W^{1,1}$, with $\varphi = e^{i\psi_h}$ on $\widetilde{B}^n \setminus \overline{B}^n$ for any h, setting $u_h := e^{i\psi_h}$ we clearly have $\{u_h\} \subset C^\infty_\varphi(\widetilde{B}^n, S^1)$ and $u_h \to u$ strongly in $W^{1,1}(\widetilde{B}^n, \mathbb{R}^2)$. Finally, if (d) holds, since by Stokes theorem $\partial G_{u_h} = 0$ on $\mathcal{D}^{n-1}(\widetilde{B}^n \times S^1)$ if u_h is smooth, and the strong $W^{1,1}$-convergence yields the weak convergence $G_{u_h} \rightharpoonup G_u$ in the sense of currents in $\mathcal{D}_n(\widetilde{B}^n \times S^1)$, we find that $\partial G_u = 0$ on $\mathcal{D}^{n-1}(\widetilde{B}^n \times S^1)$, i.e., $G_u \in \mathrm{cart}_\varphi(\widetilde{B}^n \times S^1)$. $\square$

4 Examples

The case $n = 2$. Let $a_\pm := (0, \pm 1/2)$ and consider the $W^{1,1}$-maps

$$u_+(x) := \frac{x - a_+}{|x - a_+|} , \qquad u_-(x) := \psi\Big(\frac{x - a_-}{|x - a_-|} \Big) ,$$

where $\psi : S^1 \to S^1$ is given by $\psi(y_1, y_2) := (y_1, -y_2)$. Since $u_\pm$ has degree ± 1, we may and do find a smooth $W^{1,1}$-map $\phi : (\overline{B}^2 \setminus (B(a_+, 1/4) \cup B(a_-, 1/4)) \to S^1$ satisfying

$$\phi_{|\partial B^2} \equiv (1, 0) , \quad \phi_{|\partial B(a_+, 1/4)} = u_+ , \quad \phi_{|\partial B(a_-, 1/4)} = u_- .$$

Define $u : \overline{B}^2 \to S^1$ by

$$u(x) := \begin{cases} u_+(x) & \text{if} \quad |x - a_+| < 1/4 \\ u_-(x) & \text{if} \quad |x - a_+| < 1/4 \\ \phi(x) & \text{elsewhere on } B^2 \end{cases}$$

and set $u = \varphi \equiv (1, 0)$ in $\widetilde{B}^2 \setminus B^2$, so that $u \in W^{1,2}_\varphi(\widetilde{B}^2, S^1)$.

Remark 4.1 For future use, we also may and do define ϕ so that

$$u(x_1, -x_2) = u(x_1, x_2) \qquad \forall\, (x_1, x_2) \in \overline{B}^2 .$$

Following Sec. 3.2.2 in [13, Vol. I], we have

$$\partial G_u \llcorner \widetilde{B}^2 \times S^1 = (\delta_{a_-} - \delta_{a_+}) \times [\![S^1]\!] .$$

This yields that

$$\widetilde{\mathcal{E}_{1,1}}(u, \widetilde{B}^2) = \int_{B^2} |Du| \, dx + 2\pi \, |a_+ - a_-| .$$

In fact, the current T of minimal mass in $\mathrm{cart}_\varphi(\widetilde{B}^2 \times S^1)$ satisfying $u_T = u$ is given by

$$T := G_u + [\![a_-, a_+]\!] \times [\![S^1]\!] ,$$

where $[\![a_-, a_+]\!]$ is the 1-current integration on the positively oriented segment connecting a_- to a_+, so that $\partial[\![a_-, a_+]\!] = \delta_{a_+} - \delta_{a_-}$. Moreover, see (1.13), we have

$$\mathbb{P}(u) = \delta_{a_+} - \delta_{a_-}\,, \qquad T(u) = 2\pi\,(\delta_{a_+} - \delta_{a_-})\,, \qquad L(u) = |a_+ - a_-|\,.$$

THE CASE $n \geq 3$. If $n = 3$, define $u : \overline{B}^3 \to S^1$ as the $W^{1,1}$-map given by the rotation on the x_1-axis of the map defined as in the case $n = 2$ on the 2-disk $\overline{B}^2 \simeq \overline{B}^3 \cap \{x_3 = 0\}$. By induction on the dimension n, define $u : \overline{B}^n \to S^1$ as the $W^{1,1}$-map given by the rotation on the x_1-axis of the map defined as in the case $n-1$ on the $(n-1)$-disk $\overline{B}^{n-1} \simeq \overline{B}^n \cap \{x_n = 0\}$. By Remark 4.1, in the case $n = 3$, and by the inductive argument, we infer that u is smooth outside the $(n-2)$-sphere $\Delta := \{x \in B^n \mid x_1 = 0,\, |x| = 1/2\}$. Moreover, setting again $u = \varphi \equiv (1,0)$ in $\widetilde{B}^n \setminus B^n$, this time we have

$$\partial G_u \llcorner \widetilde{B}^n \times S^1 = -[\![\Delta]\!] \times [\![S^1]\!]\,,$$

where $[\![\Delta]\!] \in \mathcal{R}_{n-2}(B^n)$ is the $(n-2)$-current integration on the $(n-2)$-sphere Δ, oriented in the natural way. Notice that $\partial[\![\Delta]\!] = 0$. Moreover, we have

$$\widetilde{\mathcal{E}_{1,1}}(u, \widetilde{B}^n) = \int_{B^n} |Du|\,dx + 2\pi\,\mathcal{H}^{n-1}(D)\,,$$

where $D := \{x \in B^n \mid x_1 = 0,\, |x| \leq 1/2\}$. In fact, the current T of minimal mass in $\mathrm{cart}_\varphi(\widetilde{B}^n \times S^1)$ satisfying $u_T = u$ is given by

$$T := G_u + [\![D]\!] \times [\![S^1]\!]\,,$$

where $[\![D]\!]$ is the $(n-1)$-current integration on the positively oriented $(n-1)$-disk D, so that $\partial[\![D]\!] = [\![\Delta]\!]$. Finally, according to (1.13), we have

$$\mathbb{P}(u) = (-1)^n[\![\Delta]\!]\,, \qquad T(u) = (-1)^n\,2\pi\,[\![\Delta]\!]\,, \qquad L(u) = \mathcal{H}^{n-1}(D)\,.$$

5 The relaxed energy of $\mathbf{W^{1,p}}$-maps into $\mathbf{S^1}$

Let $p > 1$ and $u \in W^{1,p}_\varphi(\widetilde{B}^n, S^1)$, where

$$W^{1,p}_\varphi(\widetilde{B}^n, S^1) := \{u \in W^{1,p}(\widetilde{B}^n, \mathbb{R}^2) \,:\, |u(x)| = 1 \text{ a.e. in } \widetilde{B}^n,\; u = \varphi \text{ in } \widetilde{B}^n \setminus \overline{B}^n\}\,.$$

We now briefly discuss the relaxed $W^{1,p}$-energy, defined for $u \in W^{1,p}_\varphi(\widetilde{B}^n, S^1)$ by

$$\widetilde{\mathcal{E}_{1,p}}(u) =: \inf\left\{\liminf_{h\to\infty} \int_{\widetilde{B}^n} |\nabla u_h|^p\,dx \,:\, \{u_h\} \subset C^1_\varphi(\widetilde{B}^n, S^1)\,, \quad u_h \to u \quad \text{a.e.}\right\}.$$

It is well-known that

$$p \geq 2 \quad \Rightarrow \quad \widetilde{\mathcal{E}_{1,p}}(u) = \int_{\widetilde{B}^n} |\nabla u|^p\,dx \qquad \forall\, u \in W^{1,p}_\varphi(\widetilde{B}^n, S^1)\,.$$

This property follows from standard argument for $p > n$ and by Schoen-Uhlenbeck density theorem [24] in the critical case $p = n$. Since the higher order homotopy groups of the 1-sphere are all trivial, $\pi_i(S^1) = 0$ for all $i \geq 2$, this property follows from Bethuel's theorem [4] in the case $2 \leq p < n$.

We now prove the following

Theorem 5.1 *Let $1 < p < 2$ and $u \in W^{1,p}_\varphi(\widetilde{B}^n, S^1)$. Then*

$$\widetilde{\mathcal{E}_{1,p}}(u) = \begin{cases} \displaystyle\int_{\widetilde{B}^n} |\nabla u|^p\,dx & \text{if } T(u) = 0 \\ +\infty & \text{if } T(u) \neq 0 \end{cases}$$

where $T(u)$ is given by Definition 2.1.

This answers the Open Problems 1 and 2 stated in [6].

Notice that in [7] it is proved that for $1 \leq p < 2$ a Sobolev map $u \in W^{1,p}_\varphi(\widetilde{B}^n, S^1)$ can be strongly approximated in $W^{1,p}$ by a smooth sequence in $C^1_\varphi(\widetilde{B}^n, S^1)$ if and only if $T(u) = 0$. Therefore, we obtain that for every $p > 1$ a map in $W^{1,p}_\varphi(\widetilde{B}^n, S^1)$ belongs to the sequential weak $W^{1,p}$-closure of smooth maps, i.e., $\widetilde{\mathcal{E}_{1,p}}(u) < \infty$, if and only if it belongs to the strong $W^{1,p}$-closure of smooth maps from $\widetilde{B}^n$ into S^1.

This is false in the case $p = 1$. In fact, by Proposition 1.6 we know that $\widetilde{\mathcal{E}_{1,1}}(u) < \infty$ for every $u \in W^{1,1}_\varphi(\widetilde{B}^n, S^1)$, whereas u belongs to the strong $W^{1,1}$-closure of smooth maps from $\widetilde{B}^n$ into S^1 if and only if $T(u) = 0$.

Proof of Theorem 5.1: Assume that $\widetilde{\mathcal{E}_{1,p}}(u) < \infty$ and let $\{u_h\} \subset C^1_\varphi(\widetilde{B}^n, S^1)$ be a smooth sequence satisfying $\sup_h \|u_h\|_{W^{1,p}} < \infty$ and $u_h \to u \in W^{1,p}_\varphi(\widetilde{B}^n, S^1)$ a.e.. Possibly passing to a subsequence, by [13] we infer that G_{u_h} weakly converges in $\mathcal{D}_n(\widetilde{B}^n \times S^1)$ to some current $T \in \mathrm{cart}_\varphi(\widetilde{B}^n \times S^1)$ satisfying (1.12). Let $\mathcal{L}$ be a compact subset of $\mathrm{set}(L)$, the set of points of L which have non-zero density, $\mathcal{L}$ with positive $\mathcal{H}^{n-1}$-measure. For any $x \in \mathcal{L}$ we denote by I_x the intersection with $\widetilde{B}^n$ of the straight line containing x and orthogonal to the approximate tangent $(n-1)$-space to $\mathcal{L}$ at x. Since $\mathcal{L}$ is compact, for $\mathcal{H}^{n-1}$-a.e. $x \in \mathcal{L}$, the 1-dimensional restriction of G_{u_h} to I_x is a sequence of graphs of smooth functions $u_{h|I_x}$ with equibounded $W^{1,p}$-energies. For any $x_1, x_2 \in I_x$, let $[x_1, x_2]$ denote the line segment with end points x_1, x_2. Now, in dimension one, if $p > 1$ the Hölder inequality yields

$$\int_{[x_1,x_2]} |\nabla u_{h|I_x}|\, d\mathcal{H}^1 \leq \left(\int_{[x_1,x_2]} |\nabla u_{h|I_x}|^p \, d\mathcal{H}^1 \right)^{1/p} \cdot |x_1 - x_2|^{1-1/p} \leq \widetilde{C}\, |x_1 - x_2|^{1-1/p} \,,$$

where $\widetilde{C}$ is an absolute constant, depending on the uniform upper bound for the energies $\int_{I_x} |\nabla u_{h|I_x}|^p \, d\mathcal{H}^1$. This is in contradiction to the fact that, by a slicing argument, the 1-dimensional currents $G_{u_{h|I_x}}$ have to converge "near" the point x to the graph $G_{u_{|I_x}}$ of the restriction $u_{|I_x}$ plus a vertical part of the type $\delta_x \times [\![S^1]\!]$. In conclusion, we have shown that if $T(u) \neq 0$, then $\widetilde{\mathcal{E}_{1,p}}(u) = +\infty$. The assertion follows from [7], see also [5]. $\square$

6 The relaxed energy of $\mathbf{W^{1,2}}$-maps into $\mathbf{S^2}$

In this section we collect a few remarks about the Dirichlet energy of $W^{1,2}$-maps with values into S^2, the unit sphere in $\mathbb{R}^3$. Let $\varphi : \widetilde{B}^n \to S^2$ be a given smooth $W^{1,2}$-function. For $X := W^{1,2}$, $W^{1,p}$, C^1, or BV, we set

$$X_\varphi(\widetilde{B}^n, S^2) := \{u \in X(\widetilde{B}^n, \mathbb{R}^3) \,:\, |u(x)| = 1 \text{ a.e. in } \widetilde{B}^n,\ u = \varphi \text{ in } \widetilde{B}^n \setminus \overline{B}^n\}\,.$$

Similarly to Sec. 1, if $u \in W^{1,2}_\varphi(\widetilde{B}^n, S^2)$, the i.m. rectifiable current $G_u \in \mathcal{R}_n(\widetilde{B}^n \times S^2)$ is defined in an approximate sense by (1.1). Moreover, if $n \geq 3$ we define the $(n-3)$-current $\mathbb{P}(u) \in \mathcal{D}_{n-3}(\widetilde{B}^n)$ by

$$\mathbb{P}(u)(\phi) := \frac{1}{4\pi} \partial G_u(\widehat{\pi}^\# \omega_{S^2} \wedge \pi^\# \phi) = \frac{1}{4\pi} \int_{\widetilde{B}^n} u^\# \omega_{S^2} \wedge d\phi \tag{6.1}$$

for every $\phi \in \mathcal{D}^{n-3}(\widetilde{B}^n)$, where ω_{S^2} is the *volume 2-form on* $S^2 \subset \mathbb{R}^3$,

$$\omega_{S^2} := y^1 dy^2 \wedge dy^3 + y^2 dy^3 \wedge dy^1 + y^3 dy^1 \wedge dy^2 \,,$$

and the $(n-2)$-current $\mathbb{D}(u) \in \mathcal{D}_{n-2}(\widetilde{B}^n)$ by

$$\mathbb{D}(u)(\gamma) := \frac{1}{4\pi} G_u(\widehat{\pi}^\# \omega_{S^2} \wedge \pi^\# \gamma) = \frac{1}{4\pi} \int_{\widetilde{B}^n} u^\# \omega_{S^2} \wedge \gamma$$

for every $\gamma \in \mathcal{D}^{n-2}(\widetilde{B}^n)$. Again we have that $\mathrm{spt}\, \mathbb{P}(u) \subset \overline{B}^n$, $\partial\, \mathbb{P}(u) = 0$ and (1.3) holds true. For $\Omega = \widetilde{B}^n$ or B^n, denote by

$$\mathbf{D}(u, \Omega) := \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx \,, \qquad u \in W^{1,2}_\varphi(\widetilde{B}^n, S^2)\,,$$

the *Dirichlet energy* of u and consider the *relaxed* $W^{1,2}$*-energy*

$$\widetilde{\mathbf{D}}(u,\Omega) := \inf\left\{ \liminf_{h\to\infty} \mathbf{D}(u_h,\Omega) \,:\, \{u_h\}\subset C^1_\varphi(\widetilde{B}^n,S^2)\,, \quad u_h\to u \quad \text{a.e.}\right\}.$$

Moreover, we let

$$\mathrm{cart}_\varphi(\widetilde{B}^n\times S^2) := \{T\in \mathrm{cart}(\widetilde{B}^n\times\mathbb{R}^3) \mid \mathrm{spt}\,T\subset \widetilde{B}^n\times S^2\,,\ T=G_\varphi \ \text{ in } \ (\widetilde{B}^n\setminus \overline{B}^n)\times S^2\}\,.$$

As in Sec. 1, to any $T\in \mathrm{cart}_\varphi(\widetilde{B}^n\times S^2)$ we can associate a function $u_T\in BV_\varphi(\widetilde{B}^n,S^2)$ such that (1.10) and (1.11) holds true. In particular, if the corresponding BV-function u belongs to $W^{1,2}_\varphi(\widetilde{B}^n,S^2)$, we infer that

$$T = G_u + L\times [\![\, S^2\,]\!]\,, \tag{6.2}$$

where this time $L\in\mathcal{R}_{n-2}(\widetilde{B}^n)$, with $\mathbf{M}(L)<\infty$ and $\mathrm{spt}\,L\subset\overline{B}^n$. Moreover, the null-boundary condition $\partial T=0$ applied to the forms $\omega=\pi^\#\phi\wedge\widehat{\pi}^\#\omega_{S^2}$ for some $\phi\in\mathcal{D}^{n-3}(\widetilde{B}^n)$, yields that

$$\partial L(\phi) = -\frac{1}{4\pi}G_u(\widehat{\pi}^\#\omega_{S^2}\wedge\pi^\# d\phi) = -\,\mathbb{P}(u)(\phi)\,.$$

Therefore, L satisfies the boundary condition

$$\partial L = -\,\mathbb{P}(u) \tag{6.3}$$

and, conversely, if L is an i.m. rectifiable current in $\mathcal{R}_{n-2}(\widetilde{B}^n)$ with $\mathrm{spt}\,L\subset\overline{B}^n$, hence with finite mass, satisfying (6.3), the corresponding current (6.2) belongs to the class $\mathrm{cart}_\varphi(\widetilde{B}^n\times S^2)$. Finally, denote

$$\mathcal{T}_u := \{T\in\mathrm{cart}_\varphi(\widetilde{B}^n\times S^2)\ \mid\ u_T=u\}\,,\qquad u\in W^{1,2}_\varphi(\widetilde{B}^n,S^2)\,, \tag{6.4}$$

so that we have

$$\mathcal{T}_u = \{G_u + L\times[\![\, S^2\,]\!] \,:\, L\in\mathcal{R}_{n-2}(\widetilde{B}^n)\,,\ \mathrm{spt}\,L\subset\overline{B}^n,\ \partial L = -\,\mathbb{P}(u)\}\,.$$

The following density result was proved in [14].

Theorem 6.1 *Let $n\geq 3$ and let $T\in\mathrm{cart}_\varphi(\widetilde{B}^n\times S^2)$ satisfy (6.2) for some $u\in W^{1,2}_\varphi(\widetilde{B}^n,S^2)$. There exists a smooth sequence $\{u_h\}\subset C^1_\varphi(\widetilde{B}^n,S^2)$ such that $G_{u_h}\rightharpoonup T$ as $h\to\infty$ weakly in $\mathcal{D}_n(\widetilde{B}^n\times S^2)$ and*

$$\lim_{h\to\infty}\mathbf{D}(u_h,\widetilde{B}^n) = \mathbf{D}(T) := \mathbf{D}(u,\widetilde{B}^n) + 4\pi\,\mathbf{M}(L)\,.$$

Remark 6.2 We notice that by (1.3) we have $m_r(\mathbb{P}(u))<\infty$, more precisely

$$m_r(\mathbb{P}(u)) \leq C\left(\int_{B^n}|\nabla u|^2\,dx + \int_{B^n}|\nabla\varphi|^2\,dx\right)\qquad \forall\, u\in W^{1,2}_\varphi(\widetilde{B}^n,S^2)\,,$$

where $C>0$ is an absolute constant, compare (2.7). Also, in the case of dimension $n=3$, as a consequence of [4] one sees that $\mathbb{P}(u)$ is an integral flat chain, thus by (1.7) we conclude again that *the integral mass $m_i(\mathbb{P}(u))$ is finite for every $u\in W^{1,2}_\varphi(\widetilde{B}^3,S^2)$*. This is not trivial, if $n\geq 4$. In fact, in [1] it is proved that for any i.m. rectifiable current $L\in\mathcal{R}_{n-2}(\widetilde{B}^n)$, with $\mathrm{spt}\,L\subset\overline{B}^n$, there exists a function $u\in W^{1,2}_\varphi(\widetilde{B}^n,S^2)$ such that $\partial L=-4\pi\,\mathbb{P}(u)$. Therefore, on account of the counterexamples from [20] and [27], we infer that *if $n\geq 4$, there exist Sobolev maps $u\in W^{1,2}_\varphi(\widetilde{B}^n,S^2)$ such that*

$$m_r(\mathbb{P}(u)) < m_i(\mathbb{P}(u))\,.$$

However, we have

Proposition 6.3 *For any $n\geq 3$ and $u\in W^{1,2}_\varphi(\widetilde{B}^n,S^2)$ the class $\mathcal{T}_u$ is non-empty. As a consequence, the integral mass $m_i(\mathbb{P}(u))<\infty$ and the relaxed energy $\widetilde{\mathbf{D}}(u,\Omega)<\infty$.*

PROOF: Denote by $R^\infty_{2,\varphi}(\widetilde{B}^n,S^2)$ the set of all the maps $u\in W^{1,2}_\varphi(\widetilde{B}^n,S^2)$ which are smooth except on a singular set $\Sigma(u)$ of the type (1.16), where this time Σ_i is a smooth $(n-3)$-dimensional subset of $\overline{B}^n$ with smooth boundary, if $n\geq 4$, and Σ_i is a point if $n=3$. Let $\{u_h\}\subset R^\infty_{2,\varphi}(\widetilde{B}^n,S^2)$ be such that $u_h\to u$ strongly in $W^{1,2}$, see [4]. By the "smoothness" of u_h, arguing as in [26], see [2] for the case $n=3$, we obtain that the integral mass

$$m_i(\mathbb{P}(u_h))\leq\frac{1}{4\pi}(\mathbf{D}(u_h,B^n)+\mathbf{D}(\varphi,B^n))<\infty\,.$$

Let now $T_h:=G_{u_h}+L_h\times[\![\, S^2\,]\!]$, where $L_h\in\mathcal{R}_{n-2}(\widetilde{B}^n)$ is such that $\operatorname{spt}L_h\subset\overline{B}^n$, $\partial L_h=-\,\mathbb{P}(u_h)$, and $\mathbf{M}(L_h)\leq m_i(\mathbb{P}(u_h))+1/h$. The sequence $\{T_h\}$ belongs to $\operatorname{cart}_\varphi(\widetilde{B}^n\times S^2)$ and has equibounded Dirichlet energies, $\sup_h\mathbf{D}(T_h)<\infty$. By closure-compactness, and since the Dirichlet energy $T\mapsto\mathbf{D}(T)$ is lower semicontinuous with respect to the weak convergence as currents, possibly passing to a subsequence we obtain that $T_h\rightharpoonup T$ weakly in $\mathcal{D}_n(\widetilde{B}^n\times S^2)$ to some $T\in\operatorname{cart}_\varphi(\widetilde{B}^n\times S^2)$ with $\mathbf{D}(T)<\infty$. As $u_h\to u$ strongly in $W^{1,2}$, we infer that $T\in\mathcal{T}_u$, whence $m_i(\mathbb{P}(u))<\infty$. Finally, since $\mathcal{T}_u$ is nonempty, on account of Theorem 6.1 we readily conclude that $\widetilde{\mathbf{D}}(u,\Omega)<\infty$. □

In particular, for every $n\geq 3$ the class $W^{1,2}_\varphi(\widetilde{B}^n,S^2)$ agrees with the weak sequential closure of smooth maps in $W^{1,2}_\varphi(\widetilde{B}^n,S^2)$, see [23] for a more general result.

Remark 6.4 We point out that the strong $W^{1,2}$-convergence of "smooth" sequences $\{u_h\}\subset R^\infty_{2,\varphi}(\widetilde{B}^n,S^2)$ yields that $\mathbf{M}(\mathbb{D}(u_h)-\mathbb{D}(u))\to 0$ and hence, by (1.3), that the real mass $m_r(\mathbb{P}(u_h)-\mathbb{P}(u))\to 0$. However, see Remark 6.2, differently to what happens for maps in $W^{1,1}_\varphi(\widetilde{B}^n,S^1)$, a part from the easier case $n=3$, this does not yield that the integral mass $m_i(\mathbb{P}(u_h)-\mathbb{P}(u))\to 0$. This is one of the crucial points in the proof of Theorem 6.1.

As a consequence, from the above results we readily obtain the following representation.

Proposition 6.5 *For any $n\geq 3$ and $u\in W^{1,2}_\varphi(\widetilde{B}^n,S^2)$, the relaxed energy is given by*

$$\begin{aligned}\widetilde{\mathbf{D}}(u,\Omega)&=\inf\{\mathbf{D}(T)\mid T\in\mathcal{T}_u\}\\&=\mathbf{D}(u,\Omega)+4\pi\,\inf\big\{\mathbf{M}(L)\,:\,L\in\mathcal{R}_{n-2}(\widetilde{B}^n)\,,\ \operatorname{spt}L\subset\overline{B}^n,\ \partial L=-\,\mathbb{P}(u)\big\}\\&=\mathbf{D}(u,\Omega)+4\pi\,m_i(\mathbb{P}(u))\,.\end{aligned}$$

Arguing as in [26], we finally obtain

Corollary 6.6 *For any $n\geq 3$ and $u\in W^{1,2}_\varphi(\widetilde{B}^n,S^2)$, we have*

$$\widetilde{\mathbf{D}}(u,\Omega)\leq\mathbf{D}(u,\Omega)+\mathbf{D}(u,B^n)+\mathbf{D}(\varphi,B^n)\,.$$

In particular, if φ is constant on ∂B^n we have

$$\widetilde{\mathbf{D}}(u,B^n)\leq 2\,\mathbf{D}(u,B^n)\,.$$

THE RELAXED $W^{1,p}$-ENERGY. Let us finally consider the case $p>2$, and introduce the *relaxed $W^{1,p}$-energy* of maps u in $W^{1,p}_\varphi(\widetilde{B}^n,S^2)$, given by

$$\widetilde{\mathcal{E}_{1,p}}(u):=\inf\left\{\liminf_{h\to\infty}\int_{\widetilde{B}^n}|\nabla u_h|^p\,dx\,:\,\{u_h\}\subset C^1_\varphi(\widetilde{B}^n,S^2)\,,\quad u_h\to u\quad\text{a.e.}\right\}.$$

In [16] it is shown that if p is not an integer, $p\notin\mathbb{Z}$, a map in $W^{1,p}_\varphi(\widetilde{B}^n,S^2)$ belongs to the sequential weak $W^{1,p}$-closure of smooth maps if and only if it belongs to the strong $W^{1,p}$-closure of smooth maps from $\widetilde{B}^n$ into S^2. Moreover, if $2<p<3$ and $u\in W^{1,p}_\varphi(\widetilde{B}^n,S^2)$, arguing as in Theorem 5.1 we infer that

$$\widetilde{\mathcal{E}_{1,p}}(u)=\begin{cases}\displaystyle\int_{\widetilde{B}^n}|\nabla u|^p\,dx & \text{if }\mathbb{P}(u)=0\\ +\infty & \text{if }\mathbb{P}(u)\neq 0\end{cases}$$

where $\mathbb{P}(u)$ is given by (6.1).

However, if $p = 3$, the situation is totally different from the one about the relaxed energy of Sobolev maps into S^1, see the previous section. This is due to the fact that the higher order homotopy groups of the 2-sphere are not all trivial, e.g., $\pi_3(S^2) = \mathbb{Z}$. If $h : S^3 \to S^2$ is the Hopf map, namely the one that generates the third homotopy group of S^2, and $u : B^4 \to S^2$ is given by $u(x) := h(x/|x|)$, then u belongs to $W^{1,p}(B^4, S^2)$ for every $p < 4$ and *the graph of* u *has no interior boundary*, i.e., $\partial G_u \llcorner B^4 \times S^2 = 0$. In particular, $\mathbb{P}(u) = 0$. However, the *topological singularity* at the origin is relevant, even if it cannot be treated by means of a homological theory as above, compare [18].

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