An example of global classical solution for the Perona-Malik equation

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Abstract

We consider the Cauchy problem for the Perona-Malik equation

$$u_t = \mathrm{div}\left(\frac{\nabla u}{1 + |\nabla u|^2}\right)$$

in an open set $\Omega \subseteq \mathbb{R}^n$, with Neumann boundary conditions.

It is well known that in the one-dimensional case this problem does not admit any global C^1 solution if the initial condition u_0 is transcritical, namely when $|\nabla u_0(x)| - 1$ is a sign changing function in Ω.

In this paper we show that this result cannot be extended to higher dimension. We show indeed that for $n \geq 2$ the problem admits radial solutions of class $C^{2,1}$ with a transcritical initial condition.

Mathematics Subject Classification 2000 (MSC2000): 35K55, 35K65, 35B05.

Key words: Perona-Malik equation, anisotropic diffusion, forward-backward parabolic equation, degenerate parabolic equation, moving domains, overdetermined problem, subsolutions and supersolutions.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let us consider the Cauchy boundary value problem

$$u_t(x,t) - \operatorname{div}\left(\frac{\nabla u(x,t)}{1+|\nabla u(x,t)|^2}\right) = 0 \qquad \forall (x,t) \in \Omega \times [0,T), \tag{1.1}$$

$$\frac{\partial u}{\partial n}(x,t) = 0 \qquad \forall (x,t) \in \partial\Omega \times [0,T), \tag{1.2}$$

$$u(x,0) = u_0(x) \quad \forall x \in \Omega. \tag{1.3}$$

This problem was considered in [17] in the context of image denoising. In that framework u_0 is the grey level of a (noisy) picture defined in a rectangle $\Omega \subseteq \mathbb{R}^2$, and $u(x,t)$ for a given $t > 0$ should represent a restored image obtained by smoothing the regions where $|\nabla u_0| < 1$ (objects) and enhancing the regions where $|\nabla u_0| > 1$ (edges).

Equation (1.1) is the formal gradient flow of the functional

$$PM(u) := \frac{1}{2}\int_\Omega \log\left(1 + |\nabla u(x)|^2\right) dx.$$

The convex-concave behavior of the integrand makes (1.1) a forward-backward partial differential equation of parabolic type, with a forward (or subcritical) region where $|\nabla u(x,t)| < 1$, and a backward (or supercritical) region where $|\nabla u(x,t)| > 1$. If the initial condition is subcritical, namely $|\nabla u_0(x)| < 1$ for every $x \in \Omega$, then the maximum principle guarantees that the same is true for all subsequent times (see [14]). It follows that in this case equation (1.1) is always forward parabolic, and a smooth solution globally exists ($T = +\infty$).

Things are far more complicated when the initial condition is *transcritical*, namely when there are points $x \in \Omega$ where $|\nabla u_0(x)| < 1$, and points $x \in \Omega$ where $|\nabla u_0(x)| > 1$. In this case the forward-backward character of the equation makes the problem ill posed from the analytic point of view (see [15]). On the other hand, numerical computations exhibit much more stability and show that the process has the desired denoising effect on the initial condition. This is usually referred as the Perona-Malik paradox. The mathematical understanding of this phenomenology is still far away, despite of the considerable amount of interest generated by this problem in the last fifteen years (see [2, 3, 4, 5, 6, 7, 8, 12, 13, 18, 19]).

In this paper we focus on classical solutions, namely solutions which are at least of class C^1 or $C^{2,1}$. Throughout this paper $C^{2,1}$ denotes the standard parabolic space of functions with one continuous derivative with respect to time, and two continuous derivatives with respect to space variables.

As far as we know, in literature there are two main results for classical solutions, and they are both limited to the one-dimensional case. The first known result was proved in [10] and concerns local-in-time solutions. The result is that for $n = 1$ the set of initial data for which a local-in-time $C^{2,1}$ solution exists is dense in $C^1(\Omega)$. On the other hand, one cannot expect existence of local classical solutions but for a dense set of initial conditions (see [15]).

The second known result concerns global-in-time solutions, and it is a nonexistence result. It states that when $n = 1$ there does not exist any global-in-time solution of class C^1 with a transcritical initial condition. This was first proved in [14] with some technical assumptions on u_0, and then in [11] in full generality.

Many attempts have been made to extend this result in higher dimension. The main point in the one-dimensional proof is the so called *persistence of supercritical regions*, namely the fact that supercritical regions, if present for $t = 0$, cannot disappear for all subsequent times. This important qualitative property of solutions is also consistent with intuition, because it means that the process does not destroy edges.

In dimension $n = 1$ the persistence of supercritical regions follows from the fact that the L^∞ norm of $|\nabla u|$ in Ω is an increasing function of time. Unfortunately this norm is monotone for $n = 1$, but in general not for $n \geq 2$, as shown by Example 3 given in [9, Section 4]. A consequence of this example is that the proof of the nonexistence result for $n = 1$ cannot be extended to $n \geq 2$.

In this paper we show that the nonexistence result itself cannot be extended. We show indeed that for $n \geq 2$ global-in-time classical solutions *do* exist, and also the persistence of supercritical regions fails. Our main result is the following (we state it for $n = 2$, but the generalization to any $n \geq 2$ is straightforward).

Theorem 1.1 *Let $\Omega := \{x \in \mathbb{R}^2 : 1 \leq |x| \leq 5\}$ be an anulus.*

Then there exists $u \in C^{2,1}(\Omega \times [0, +\infty))$ satisfying equation (1.1) with $T = +\infty$, the Neumann boundary condition (1.2), and

$$\{x \in \Omega : |\nabla u(x, 0)| > 1\} = \{x \in \Omega : 2 < |x| < 4\}.$$

Moreover there exists $t_0 > 0$ such that $|\nabla u(x, t)| < 1$ for every $(x, t) \in \Omega \times (t_0, +\infty)$.

By the way, before Theorem 1.1 we didn't know any nontrivial example of (even local-in-time) transcritical classical solution in dimension $n \geq 2$.

In order to prove Theorem 1.1 we can limit ourselves to radial solutions. Let $r := |x|$ be a radial variable, and let us consider radial solutions $u(r, t)$. A simple computation (see [9]) shows that equation (1.1) becomes the following

$$u_t = \frac{1 - u_r^2}{(1 + u_r^2)^2} u_{rr} + \frac{1}{r} \frac{u_r}{1 + u_r^2}, \tag{1.4}$$

and the Neumann boundary condition (1.2) in the anulus becomes

$$u_r(1, t) = 0, \tag{1.5}$$

$$u_r(5, t) = 0. \tag{1.6}$$

Therefore Theorem 1.1 is a consequence of the following result.

Theorem 1.2 *There exists* $u \in C^{2,1}([1,5] \times [0,+\infty))$ *satisfying equation (1.4) in the strip* $[1,5] \times [0,+\infty)$*, the Neumann boundary conditions (1.5) and (1.6) for every* $t \geq 0$*, and the estimates*

$$0 \leq u_r(r,0) < 1 \qquad \forall r \in [1,2) \cup (4,5], \tag{1.7}$$

$$u_r(r,0) > 1 \qquad \forall r \in (2,4). \tag{1.8}$$

Moreover there exists $t_0 > 0$ *such that* $|u_r(x,t)| < 1$ *for every* $(r,t) \in [1,5] \times (t_0,+\infty)$.

We recall that equation (1.4), without the second summand in the right-hand side, is just the Perona-Malik equation in dimension one. A bureaucratic count of derivatives says that the added term is a lower order term. Nevertheless its influence on the dynamic is enormous. Due to that lower order term, the supercritical region of our radial solution disappears in a finite time, in contrast with the one dimensional case. After the extinction of its supercritical region, the solution becomes subcritical and has no more obstacles to global existence. For this reason what we need in the proof of Theorem 1.2 is to keep the solution alive and regular up to this time.

We are afraid that the possible extinction of supercritical regions makes the analytical study of the Perona-Malik equation even more difficult in dimension $n \geq 2$. We don't know whether this new phenomenon had been observed before in numerical experiments. We leave to numerical analysts any discussion about its consequences on the model and its practical applications.

We conclude with some comments on our main result.

Using an anulus instead of a ball should not be essential. This choice spares us and the reader from the technicalities due to the fact that equation (1.4) is singular for $r = 0$. The original equation however is not singular in the origin. Therefore this singularity should be easily compensated by the first Neumann boundary condition, which in a ball becomes $u_r(0,t) = 0$. For this reason it should not be difficult to find solutions with the same properties, but defined in a ball.

Concerning the nonlinearity, for simplicity we devoted this introduction to the model case of the Perona-Malik equation. On the contrary, in the following sections we work in a more general setting. What we actually do is to prove Theorem 1.1 and Theorem 1.2 for the gradient flow of any integral functional with nonconvex integrand. We refer to the beginning of section 2 for the details.

Finally, the reader could ask how special these solutions are. Theorem 1.1 indeed states the existence of just *one* such solution. What we actually construct is a *large class* of such solutions. In a word, they are not the fruit of some strange pathology, but a common feature in dimension $n \geq 2$. For example in the proof of Theorem 1.2, after choosing t_0 small enough, we have some freedom in the choice of the initial condition $u(r,0)$ in the subcritical intervals $[1,2]$ and $[4,5]$, where we only impose some inequalities and some compatibility conditions at the endpoints (see section 3.1 for the details). Our construction then completes $u(r,0)$ in the supercritical interval $[2,4]$ in such a way that the solution starting with that datum is global and becomes subcritical for $t > t_0$.

This paper is organized as follows. In section 2 we reduce the construction of the required solution to the existence of solutions in four suitable subdomains, in each one of which the equation is either (degenerate) forward parabolic or (degenerate) backward parabolic. In section 3 we approximate the degenerate problems with strictly parabolic problems depending on a small parameter $\varepsilon > 0$, and we state several ε-independent estimates. In section 4 we prove these estimates. In section 5 we pass to the limit as $\varepsilon \to 0^+$, showing that the limits are solutions of the degenerate problems. This completes the proof of Theorem 1.2, hence also of Theorem 1.1.

2 The four subproblems

Let us introduce some notations. Let $\mathcal{D} \subseteq \mathbb{R}^2$ be a compact set. The *parabolic interior* of $\mathcal{D}$ is the set $\mathrm{Int}_P(\mathcal{D})$ of points $(r,t) \in \mathcal{D}$ for which there exists $\delta > 0$ such that $[r-\delta, r+\delta] \times [t-\delta, t] \subseteq \mathcal{D}$. The *parabolic boundary* of $\mathcal{D}$ is the set $\partial_P(\mathcal{D}) := \mathcal{D} \setminus \mathrm{Int}_P(\mathcal{D})$.

Throughout this paper we assume that $\varphi \in C^\infty(\mathbb{R})$ is an even function, hence in particular

$$\varphi'(0) = \varphi'''(0) = 0. \tag{2.1}$$

Moreover we assume that

$$\varphi''(\sigma) > 0 \qquad \forall \sigma \in [0,1), \tag{2.2}$$

$$\varphi''(1) = 0, \tag{2.3}$$

$$\varphi''(\sigma) < 0 \qquad \forall \sigma \in (1,3], \tag{2.4}$$

$$\varphi'(3) > 0. \tag{2.5}$$

Of course the thresholds $\sigma = 1$ and $\sigma = 3$ can be replaced by any pair of positive numbers $\sigma_0 < \sigma_1$.

As a consequence of (2.2) and (2.3), or (2.3) and (2.4), we have that $\varphi'''(1) \leq 0$. As a consequence of (2.1) through (2.5) we have also that $\varphi'(\sigma) > 0$ for every $\sigma \in (0,3]$.

Figure 1 shows the typical behavior of $\varphi'(\sigma)$.

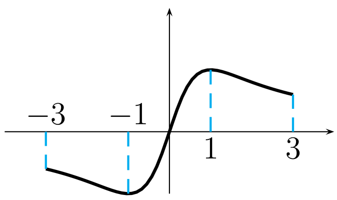

Figure 1: typical graph of φ'

We consider the following equation

$$u_t(r,t) = \varphi''(u_r(r,t))u_{rr}(r,t) + \frac{\varphi'(u_r(r,t))}{r}, \tag{2.6}$$

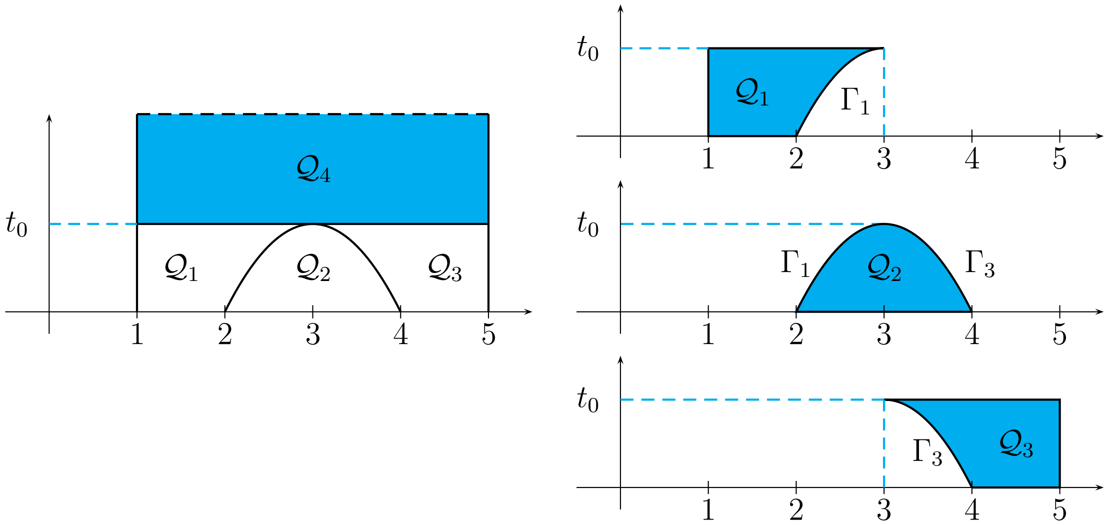

Figure 2: decomposition of $[1, 5] \times [0, +\infty)$ into four subdomains

which is the natural generalization of (1.4).

It is easy to see that equation (2.6) reduces to (1.4) when $\varphi(\sigma) = 2^{-1} \log(1 + \sigma^2)$. We believe and we hope that this generality simplifies the presentation, and shows more clearly which properties of the nonlinearity are essential in each step.

2.1 The four regions

In order to prove Theorem 1.2 we divide the strip $[1, 5] \times [0, +\infty)$ into four regions. To begin with, we fix $t_0 > 0$, and we consider the functions

$$\beta(t) := 3 - \sqrt{1 - t/t_0}, \qquad \gamma(t) := 3 + \sqrt{1 - t/t_0}, \tag{2.7}$$

defined for every $t \leq t_0$. Then we set

$$\mathcal{Q}_1 := \left\{(r, t) \in \mathbb{R}^2 : 0 \leq t \leq t_0,\ 1 \leq r \leq \beta(t)\right\}, \tag{2.8}$$

$$\mathcal{Q}_2 := \left\{(r, t) \in \mathbb{R}^2 : 0 \leq t \leq t_0,\ \beta(t) \leq r \leq \gamma(t)\right\}, \tag{2.9}$$

$$\mathcal{Q}_3 := \left\{(r, t) \in \mathbb{R}^2 : 0 \leq t \leq t_0,\ \gamma(t) \leq r \leq 5\right\}, \tag{2.10}$$

$$\mathcal{Q}_4 := [1, 5] \times [t_0, +\infty). \tag{2.11}$$

We also set

$$\Gamma_1 := \{(\beta(t), t) : t \in [0, t_0]\}, \qquad \Gamma_3 := \{(\gamma(t), t) : t \in [0, t_0]\}. \tag{2.12}$$

These sets are represented in Figure 2. We finally consider the functions

$$b(t) := \begin{cases} \min\left\{x \in \mathbb{R} : \varphi'''(1)x^2 + \beta'(t)x - \dfrac{\varphi'(1)}{\beta^2(t)} = 0\right\} & \text{if } t \in [0, t_0), \\ 0 & \text{if } t = t_0, \end{cases} \tag{2.13}$$

$$c(t) := \begin{cases} \max\left\{x \in \mathbb{R} : \varphi'''(1)x^2 + \gamma'(t)x - \dfrac{\varphi'(1)}{\gamma^2(t)} = 0\right\} & \text{if } t \in [0, t_0), \\ 0 & \text{if } t = t_0. \end{cases} \tag{2.14}$$

It is not difficult to see that $b(t)$ and $c(t)$ are well defined for $t \in [0, t_0]$ provided that t_0 is small enough. In Lemma 4.1 and Lemma 4.2 below we prove some properties of these functions, in particular their continuity. In Remark 2.5 we explain the heuristic idea behind these definitions.

In order to prove Theorem 1.2 we need a solution of (2.6) in the strip $[1, 5] \times [0, +\infty)$, satisfying the Neumann boundary conditions (1.5) and (1.6) for every $t \geq 0$, and estimates (1.7) and (1.8). Our strategy is to construct this solution by glueing together solutions of the same equation in the four subdomains. These solutions are of course required to fulfil the Neumann boundary conditions (1.5) and (1.6). In order to be glued in a $C^{2,1}$ way along the common boundaries, these solutions are also asked to satisfy the following conditions in Γ_1

$$u(\beta(t), t) = u(3, t_0) - 3 + \beta(t) - \varphi'(1) \int_t^{t_0} \frac{1}{\beta(s)}\, ds \qquad \forall t \in [0, t_0], \tag{2.15}$$

$$u_r(\beta(t), t) = 1 \qquad \forall t \in [0, t_0], \tag{2.16}$$

$$u_{rr}(\beta(t), t) = b(t) \qquad \forall t \in [0, t_0], \tag{2.17}$$

and the following conditions in Γ_3

$$u(\gamma(t), t) = u(3, t_0) - 3 + \gamma(t) - \varphi'(1) \int_t^{t_0} \frac{1}{\gamma(s)}\, ds \qquad \forall t \in [0, t_0], \tag{2.18}$$

$$u_r(\gamma(t), t) = 1 \qquad \forall t \in [0, t_0], \tag{2.19}$$

$$u_{rr}(\gamma(t), t) = c(t) \qquad \forall t \in [0, t_0]. \tag{2.20}$$

Moreover we arrange things in such a way that equation (2.6) turns out to be (degenerate) backward parabolic in $\mathcal{Q}_2$, and (degenerate) forward parabolic in the remaining three regions. Apparently all these conditions make the problem highly overdetermined. Nevertheless the degeneracy in Γ_1 and Γ_3 allows us to fulfil all conditions.

2.2 The four subproblems

Let us state the existence results in the four regions.

Theorem 2.1 (Region $\mathcal{Q}_1$) *Let $\varphi \in C^\infty(\mathbb{R})$ be a function satisfying (2.1), (2.2), and (2.3). Let us assume that $t_0 > 0$ is small enough. Let $\beta(t)$, $\mathcal{Q}_1$, Γ_1, $b(t)$ be defined by (2.7), (2.8), (2.12), and (2.13), respectively.*

Then there exists $u \in C^{2,1}(\mathcal{Q}_1)$ satisfying equation (2.6) for every $(r, t) \in \mathcal{Q}_1$, the Neumann boundary condition (1.5) for every $t \in [0, t_0]$, and the boundary conditions (2.15) through (2.17).

Moreover $0 \leq u_r(r, t) < 1$ for every $(r, t) \in \mathcal{Q}_1 \setminus \Gamma_1$.

Theorem 2.2 (Region $\mathcal{Q}_2$) *Let $\varphi \in C^\infty(\mathbb{R})$ be a function satisfying (2.3), (2.4), and (2.5). Let us assume that $t_0 > 0$ is small enough. Let $\beta(t)$, $\gamma(t)$, $\mathcal{Q}_2$, Γ_1, Γ_3, $b(t)$, $c(t)$ be defined by (2.7), (2.9), (2.12), (2.13), and (2.14), respectively.*

Then there exists $u \in C^{2,1}(\mathcal{Q}_2)$ satisfying equation (2.6) for every $(r,t) \in \mathcal{Q}_2$ and the boundary conditions (2.15) through (2.20).

Moreover $u_r(r,t) > 1$ for every $(r,t) \in \mathcal{Q}_2 \setminus (\Gamma_1 \cup \Gamma_3)$.

Theorem 2.3 (Region $\mathcal{Q}_3$) *Let $\varphi \in C^\infty(\mathbb{R})$ be a function satisfying (2.1), (2.2), and (2.3). Let us assume that $t_0 > 0$ is small enough. Let $\gamma(t)$, $\mathcal{Q}_3$, Γ_3, $c(t)$ be defined by (2.7), (2.10), (2.12), and (2.14), respectively.*

Then there exists $u \in C^{2,1}(\mathcal{Q}_3)$ satisfying equation (2.6) for every $(r,t) \in \mathcal{Q}_3$, the Neumann boundary condition (1.6) for every $t \in [0, t_0]$, and the boundary conditions (2.18) through (2.20).

Moreover $0 \leq u_r(r,t) < 1$ for every $(r,t) \in \mathcal{Q}_3 \setminus \Gamma_3$.

Theorem 2.4 (Region $\mathcal{Q}_4$) *Let $\varphi \in C^\infty(\mathbb{R})$ be a function satisfying (2.1), (2.2), and (2.3). Let $t_0 > 0$ be a real number, and let $\mathcal{Q}_4$ be defined by (2.11). Let $u_0 \in C^2([1,5])$ be a function such that*

$$u_{0r}(1) = u_{0r}(5) = 0, \tag{2.21}$$

$$0 \leq u_{0r}(r) < 1 \qquad \forall r \in [1,3) \cup (3,5]. \tag{2.22}$$

Then there exists a unique function $u \in C^{2,1}(\mathcal{Q}_4)$ satisfying equation (2.6) for every $(r,t) \in \mathcal{Q}_4$, the Neumann boundary condition (1.5) and (1.6) for every $t \geq t_0$, and the initial condition $u(r,t_0) = u_0(r)$ for every $r \in [1,5]$.

Moreover we have that $0 \leq u_r(r,t) < 1$ for every $(r,t) \in \mathcal{Q}_4 \setminus \{(3,t_0)\}$.

The idea of constructing a solution by glueing solutions in suitable subdomains has already been used in [10]. The main difference is that in [10] all subdomains are rectangles (hence with *fixed* endpoints), and the prescribed values of u, u_r, u_{rr} at the boundary do not depend on time.

Here the problem in $\mathcal{Q}_4$ is a classical initial boundary value problem in a fixed interval. On the contrary, the problems in $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$ involve *moving domains*, no initial condition, but several *time-dependent* boundary conditions in the moving endpoints.

As for initial conditions, in $\mathcal{Q}_2$ the backward character of the equation makes the solution completely determined by its values in Γ_1 and Γ_3. On the contrary, in $\mathcal{Q}_1$ and $\mathcal{Q}_3$ we have a lot of freedom in the choice of the initial condition, and for this reason there are plenty of different solutions.

Concerning the multiple boundary conditions, let us consider for example the problem in $\mathcal{Q}_1$. Once that an initial datum has been chosen, the Neumann boundary conditions (1.5) and (2.16) are enough to determine uniquely the solution. The surprising aspect is that this solution satisfies also (2.15) and (2.17), independently on the initial condition! In the following two remarks we show how the degeneracy of the equation in Γ_1 makes this possible.

Remark 2.5 Let u be a solution of equation (2.6) in $\mathcal{Q}_1$ or $\mathcal{Q}_2$, with Neumann boundary condition (2.16). If u is of class $C^{2,1}$ we can compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[u(\beta(t),t)\right] &= \beta'(t)u_r(\beta(t),t) + u_t(\beta(t),t) \\
&= \beta'(t)u_r(\beta(t),t) + \varphi''(u_r(\beta(t),t))u_{rr}(\beta(t),t) + \frac{\varphi'(u_r(\beta(t),t))}{\beta(t)}.
\end{aligned}$$

From condition (2.16) we have therefore that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[u(\beta(t),t)\right] = \beta'(t) + \frac{\varphi'(1)}{\beta(t)}.$$

Integrating this equality in $[t, t_0]$ we obtain (2.15). This is actually a proof of (2.15). The proof of (2.18) is analogous.

Remark 2.6 Let u be a solution of equation (2.6) in $\mathcal{Q}_1$ or $\mathcal{Q}_2$, with Neumann boundary condition (2.16). Let us assume that u is smooth enough, and let us compute the time derivative of (2.16). We obtain that

$$\begin{aligned}
0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[u_r(\beta(t),t)\right] &= \beta'(t)u_{rr}(\beta(t),t) + u_{rt}(\beta(t),t) \\
&= \beta'(t)u_{rr} + \varphi''(u_r)u_{rrr} + \varphi'''(u_r)u_{rr}^2 + \frac{\varphi''(u_r)u_{rr}}{\beta(t)} - \frac{\varphi'(u_r)}{\beta^2(t)},
\end{aligned}$$

where in the last line all the derivatives of u are computed in $(\beta(t), t)$. Exploiting the Neumann boundary condition (2.16) we have therefore that

$$\beta'(t)u_{rr}(\beta(t),t) + \varphi'''(1)u_{rr}^2(\beta(t),t) - \frac{\varphi'(1)}{\beta^2(t)} = 0. \tag{2.23}$$

This means that $u_{rr}(\beta(t), t)$ is a solution of the equation defining $b(t)$ in (2.13). If for $t = 0$ we have that $u_{rr}(\beta(0), 0)$ is the smallest solution of the equation, namely $b(0)$, then for all subsequent times $u_{rr}(\beta(t), t)$ remains the smallest solution, namely $b(t)$. The choice of the smallest solution is due to the fact that it tends to 0 as $t \to t_0^-$, while the other solution diverges to $+\infty$.

We point out that this simple argument is a heuristic justification of (2.17), but it is by no means a proof. In deriving (2.23) we used indeed that terms such as $\varphi''(u_r)u_{rrr}$ vanish at the moving boundary. This requires some assumption on u_{rrr}, which is beyond the $C^{2,1}$ regularity.

2.3 Proof of Theorem 1.2

Let us take $t_0 > 0$ small enough in order to apply Theorem 2.1, Theorem 2.2, and Theorem 2.3. Thus we obtain a solution of equation (2.6) in each one of the regions $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_3$. All these solutions are defined up to additive constants. We can therefore assume, without loss of generality, that they coincide in the common point $(3, t_0)$.

We claim that the three solutions glue together in order to give a solution of class $C^{2,1}$ in the rectangle $[1, 5] \times [0, t_0]$.

Let us examine indeed the solutions in $\mathcal{Q}_1$ and $\mathcal{Q}_2$. If they coincide in $(3, t_0)$, then they coincide in the whole Γ_1 because they both satisfy (2.15). Also their first and second space derivatives coincide in Γ_1 because they both satisfy (2.16) and (2.17). So these two solutions glue in a $C^{2,1}$ way.

The same is true for the solutions in $\mathcal{Q}_2$ and $\mathcal{Q}_3$. Note that in the common point $(3, t_0)$ we have that $u_{rr}(3, t_0) = b(t_0) = c(t_0) = 0$.

It remains to extend u to $[1, 5] \times [t_0, +\infty)$. To this end, we apply Theorem 2.4 using as "initial" condition the trace at $t = t_0$ of the solution we have just glued in $[1, 5] \times [0, t_0]$. From Theorem 2.1 and Theorem 2.3 it is clear that this trace satisfies assumptions (2.21) and (2.22). This completes the proof of Theorem 1.2. $\square$

3 Approximating problems

In the previous section the proof of our main result has been reduced to the proof of Theorem 2.1 through Theorem 2.4. In this section we present our approach to Theorem 2.1 and Theorem 2.2. We skip Theorem 2.3 because it is symmetric to Theorem 2.1, and we skip Theorem 2.4 because it concerns a quite standard (non overdetermined) initial boundary value problem.

The first thing to do is to choose t_0. Let us begin by considering the following constants depending only on φ

$$\gamma_0 := 3\varphi'(1) + 5, \tag{3.1}$$

$$\gamma_1 := 5\varphi'(1) + 100, \tag{3.2}$$

$$\gamma_2 := \max\left\{|\varphi'(\sigma)| + |\varphi''(\sigma)| + |\varphi'''(\sigma)| + |\varphi^{IV}(\sigma)| : \sigma \in [0, 3]\right\}. \tag{3.3}$$

Let us choose $t_0 \in (0, 1)$ satisfying the following inequalities

$$t_0 \leq \frac{1}{4\,[\varphi'(1)]^2}, \qquad t_0 \leq \frac{3}{2500\gamma_2}, \qquad t_0 \leq \frac{1}{96(\gamma_1 + 1)^4\gamma_2}, \tag{3.4}$$

$$t_0 \leq \frac{1}{(20\gamma_0^2 + 28\gamma_0 + 9)\gamma_2}, \qquad t_0 \leq \frac{1}{(12\gamma_0 + 14)\gamma_2}. \tag{3.5}$$

We stated these conditions on t_0 as they are required throughout the proofs. It is not difficult to see that the third inequality in (3.4) implies the remaining four.

3.1 The forward problem in a moving domain

Let us consider the problem in $\mathcal{Q}_1$. Let us choose $u_0 \in C^\infty([1,2])$ satisfying the compatibility conditions

$$u_{0r}(1) = 0, \qquad u_{0r}(2) = 1, \qquad u_{0rr}(2) = b(0), \tag{3.6}$$

and the following inequalities

$$0 \le u_{0r}(r) < 1 \qquad \forall r \in (1,2), \tag{3.7}$$
$$|u_{0rr}(r)| < 10 \qquad \forall r \in [1,2], \tag{3.8}$$
$$|u_{0rrr}(r)| < 10 \qquad \forall r \in [1,2]. \tag{3.9}$$

We remind that $b(0)$ depends on t_0, and for this reason the choice of u_0 depends on the choice of t_0.

Now we approximate the problem in $\mathcal{Q}_1$ with a strictly parabolic problem depending on a small parameter ε.

Theorem 3.1 *Let $\varphi \in C^\infty(\mathbb{R})$ be a function satisfying (2.1), (2.2), (2.3). Let us assume that $t_0 > 0$ satisfies (3.4) and (3.5). Let $\beta(t)$, $\mathcal{Q}_1$, $b(t)$, be defined by (2.7), (2.8), and (2.13), respectively. Let $u_0 \in C^\infty([1,2])$ be a function satisfying (3.6) through (3.9), and let $\varepsilon \in (0,1)$.*

Then there exists a unique function $u^\varepsilon \in C^{2,1}(\mathcal{Q}_1)$ satisfying equation (2.6) for every $(r,t) \in \mathcal{Q}_1$, the Neumann boundary condition (1.5) for every $t \in [0,t_0]$, the further Neumann boundary condition

$$u^\varepsilon_r(\beta(t),t) = 1 - \varepsilon \qquad \forall t \in [0,t_0], \tag{3.10}$$

and the initial condition

$$u^\varepsilon(r,0) = (1-\varepsilon)u_0(r) \qquad \forall r \in [1,2].$$

Moreover u^ε satisfies the following estimates independent on ε.

1. Maximum principle for space derivatives. *We have that*

$$0 \le u^\varepsilon_r(r,t) \le 1 - \varepsilon \qquad \forall (r,t) \in \mathcal{Q}_1. \tag{3.11}$$

2. Uniform strict parabolicity in the interior. *For every $\delta \in (0,1)$ there exists a constant $M_1 = M_1(\delta)$ such that*

$$u^\varepsilon_r(r,t) \le M_1 < 1 \qquad \forall t \in [0,t_0],\ \forall r \in [1, \beta(t) - \delta]. \tag{3.12}$$

As a consequence, there exists a constant $M_2 = M_2(\delta) > 0$ such that

$$\varphi''(u^\varepsilon_r(r,t)) \ge M_2 \qquad \forall t \in [0,t_0],\ \forall r \in [1, \beta(t) - \delta]. \tag{3.13}$$

3. Estimate on second derivatives at the fixed boundary. *We have that*

$$0 \le u^\varepsilon_{rr}(1,t) \le 100 \qquad \forall t \in [0,t_0]. \tag{3.14}$$

4. Estimate on second derivatives at the moving boundary. *We have that*

$$|u^\varepsilon_{rr}(\beta(t),t) - b(t)| \le \varepsilon \qquad \forall t \in [0,t_0]. \tag{3.15}$$

5. Global estimate on second derivatives. *There exists a constant M_3 such that*

$$|u^\varepsilon_{rr}(r,t) - b(t)| \le \varepsilon + M_3\,|r - \beta(t)| \qquad \forall (r,t) \in \mathcal{Q}_1. \tag{3.16}$$

As a consequence, there exist constants M_4 and M_5 such that

$$|u^\varepsilon_{rr}(r,t)| \le M_4 \qquad \forall (r,t) \in \mathcal{Q}_1, \tag{3.17}$$

$$|u^\varepsilon_{t}(r,t)| \le M_5 \qquad \forall (r,t) \in \mathcal{Q}_1. \tag{3.18}$$

6. Integral estimates. *There exists a constant M_6 such that*

$$\int_{\mathcal{Q}_1} \varphi''(u^\varepsilon_r(r,t))[u^\varepsilon_{rt}(r,t)]^2\,dr\,dt \le M_6. \tag{3.19}$$

In addition, for every $\delta \in (0,1)$ there exists a constant $M_7 = M_7(\delta)$ such that

$$\int_1^{\beta(t)-\delta} [u^\varepsilon_{rt}(r,t)]^2\,dr \le M_7 \qquad \forall t \in [0,t_0], \tag{3.20}$$

$$\int_1^{\beta(t)-\delta} [u^\varepsilon_{rrr}(r,t)]^2\,dr \le M_7 \qquad \forall t \in [0,t_0], \tag{3.21}$$

$$\int_0^{t_0}\int_1^{\beta(t)-\delta} [u^\varepsilon_{rrt}(r,t)]^2\,dr\,dt \le M_7. \tag{3.22}$$

3.2 The backward problem in a moving domain

The backward problem in $\mathcal{Q}_2$ can be transformed in a forward parabolic problem by reversing the time. To this end we consider the domain

$$\mathcal{T} := \left\{(r,t) \in \mathbb{R}^2 : 0 \le t \le t_0,\ \beta(t_0 - t) \le r \le \gamma(t_0 - t)\right\}. \tag{3.23}$$

The domain $\mathcal{T}$ is just $\mathcal{Q}_2$ upside-down, while its parabolic boundary $\partial_P\mathcal{T}$ is just $\Gamma_1 \cup \Gamma_3$ upside-down (see Figure 3).

With this notation proving Theorem 2.2 is equivalent to proving the following result (note in particular the minus signs in the right-hand side of (3.24)).

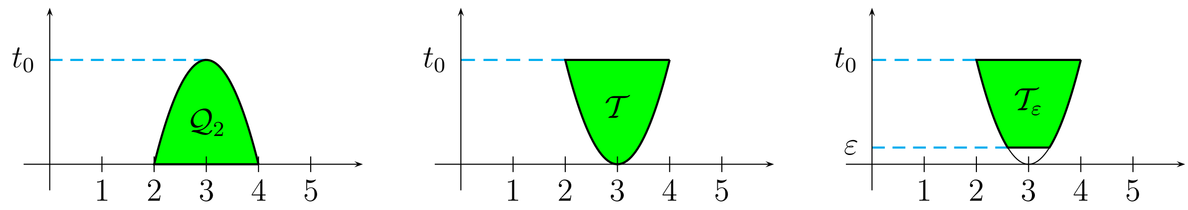

Figure 3: domains $\mathcal{Q}_2$, $\mathcal{T}$, and $\mathcal{T}_\varepsilon$

Theorem 3.2 (Region $\mathcal{T}$) *Let $\varphi \in C^\infty(\mathbb{R})$ be a function satisfying (2.3), (2.4), and (2.5). Let us assume that $t_0 > 0$ satisfies (3.4) and (3.5). Let $\beta(t)$, $\gamma(t)$, $\mathcal{T}$, $b(t)$, $c(t)$ be defined by (2.7), (3.23), (2.13), and (2.14), respectively.*

Then there exists $u \in C^{2,1}(\mathcal{T})$ satisfying equation

$$u_t(r,t) = -\varphi''(u_r(r,t))u_{rr}(r,t) - \frac{\varphi'(u_r(r,t))}{r} \tag{3.24}$$

for every $(r,t) \in \mathcal{Q}_2$, and the Neumann boundary conditions

$$u_r(\beta(t_0 - t), t) = u_r(\gamma(t_0 - t), t) = 1 \qquad \forall t \in [0, t_0].$$

Moreover we have that

$$\begin{aligned} u_r(r,t) > 1 \qquad & \forall (r,t) \in \mathrm{Int}_P(\mathcal{T}), \\ u_{rr}(\beta(t_0 - t), t) = b(t_0 - t) \qquad & \forall t \in [0, t_0], \\ u_{rr}(\gamma(t_0 - t), t) = c(t_0 - t) \qquad & \forall t \in [0, t_0]. \end{aligned}$$

We didn't state explicitly what (2.15) and (2.18) become in this new setting because, as we have seen in Remark 2.5, they easily follow from the regularity of the solution and the Neumann boundary conditions (2.16) and (2.19).

In order to approach Theorem 3.2, for every $\varepsilon \in (0, t_0)$ we set (see Figure 3)

$$\mathcal{T}_\varepsilon := \{(r,t) \in \mathcal{T} : t \geq \varepsilon\}. \tag{3.25}$$

Now we approximate the degenerate parabolic problem in $\mathcal{T}$ with a strictly parabolic problem in $\mathcal{T}_\varepsilon$. Note that in this case we prescribe an initial condition for $t = \varepsilon$. This initial condition is compatible with the Neumann boundary conditions.

Theorem 3.3 *Let φ, t_0, $\beta(t)$, $\gamma(t)$, $\mathcal{T}$, $b(t)$, $c(t)$ be as in Theorem 3.2. Let $\varepsilon \in (0, t_0)$, and let $\mathcal{T}_\varepsilon$ be defined by (3.25).*

Then there exists a unique function $u^\varepsilon \in C^{2,1}(\mathcal{T}_\varepsilon)$ satisfying equation (3.24) for every $(r,t) \in \mathcal{T}_\varepsilon$, the Neumann boundary conditions

$$u_r^\varepsilon(\beta(t_0 - t), t) = u_r^\varepsilon(\gamma(t_0 - t), t) = 1 + \varepsilon \qquad \forall t \in [\varepsilon, t_0],$$

and the initial condition

$$u^\varepsilon(r,\varepsilon) = (1+\varepsilon)r \qquad \forall r \in [\beta(t_0-\varepsilon), \gamma(t_0-\varepsilon)].$$

Moreover u^ε satisfies the following estimates independent on ε.

1. Maximum principle for space derivatives. *We have that*

$$1+\varepsilon \leq u^\varepsilon_r(r,t) \leq 3 \qquad \forall (r,t) \in \mathcal{T}_\varepsilon. \tag{3.26}$$

2. Uniform strict parabolicity in the interior. *For every compact set $K \subseteq \mathcal{T}_\varepsilon \cap \mathrm{Int}_P(\mathcal{T})$ there exists a constant M_1 (depending on K but not on ε) such that*

$$1 < M_1 \leq u^\varepsilon_r(r,t) \qquad \forall (r,t) \in K. \tag{3.27}$$

As a consequence, there exists a constant $M_2 > 0$ such that

$$-\varphi''(u^\varepsilon_r(r,t)) \geq M_2 \qquad \forall (r,t) \in K. \tag{3.28}$$

3. Estimate on second derivatives at the boundary. *We have that*

$$|u^\varepsilon_{rr}(\beta(t_0-t),t) - b(t_0-t)| \leq \sqrt{\varepsilon} \qquad \forall t \in [\varepsilon, t_0], \tag{3.29}$$

$$|u^\varepsilon_{rr}(\gamma(t_0-t),t) - c(t_0-t)| \leq \sqrt{\varepsilon} \qquad \forall t \in [\varepsilon, t_0]. \tag{3.30}$$

4. Global estimate on second derivatives. *There exists a constant M_3 such that*

$$|u^\varepsilon_{rr}(r,t) - b(t_0-t)| \leq \sqrt{\varepsilon} + M_3\,|r-\beta(t_0-t)| \qquad \forall (r,t) \in \mathcal{T}_\varepsilon, \tag{3.31}$$

$$|u^\varepsilon_{rr}(r,t) - c(t_0-t)| \leq \sqrt{\varepsilon} + M_3\,|r-\gamma(t_0-t)| \qquad \forall (r,t) \in \mathcal{T}_\varepsilon. \tag{3.32}$$

As a consequence, there exist constants M_4 and M_5 such that

$$|u^\varepsilon_{rr}(r,t)| \leq M_4 \qquad \forall (r,t) \in \mathcal{T}_\varepsilon,$$

$$|u^\varepsilon_{t}(r,t)| \leq M_5 \qquad \forall (r,t) \in \mathcal{T}_\varepsilon.$$

5. Integral estimates. *There exists a constant M_6 such that*

$$-\int_{\mathcal{T}_\varepsilon} \varphi''(u^\varepsilon_r(r,t))[u^\varepsilon_{rt}(r,t)]^2\,dr\,dt \leq M_6. \tag{3.33}$$

In addition, for every closed rectangle $[r_1,r_2]\times[t_1,t_2] \subseteq \mathcal{T}_\varepsilon \cap \mathrm{Int}_P(\mathcal{T})$ there exists a constant M_7 (depending on the rectangle, but not on ε) such that

$$\int_{r_1}^{r_2} [u^\varepsilon_{rt}(r,t)]^2\,dr \leq M_7 \qquad \forall t \in [t_1,t_2],$$

$$\int_{r_1}^{r_2} [u^\varepsilon_{rrr}(r,t)]^2\,dr \leq M_7 \qquad \forall t \in [t_1,t_2],$$

$$\int_{t_1}^{t_2}\int_{r_1}^{r_2} [u^\varepsilon_{rrt}(r,t)]^2\,dr\,dt \leq M_7.$$

4 Proof of estimates

In the following two statements we collect the properties of $b(t)$ and $c(t)$ which are needed in the main proofs. We only prove Lemma 4.1 because the proof of Lemma 4.2 is quite similar.

Lemma 4.1 *Let $t_0 > 0$, and let $\beta(t)$ be defined by (2.7). Let us assume that*

$$\frac{1}{t_0} \geq 4\sqrt{\varphi'(1)\,|\varphi'''(1)|}. \tag{4.1}$$

Then the function $b(t)$ introduced in (2.13) is well defined for every $t \in [0, t_0]$ and fulfils the following properties.

(1) We have that $b \in C^0([0, t_0]) \cap C^\infty([0, t_0))$.

(2) We have that

$$0 < -b'(t) \leq 5t_0\varphi'(1)\beta'(t) \qquad \forall t \in [0, t_0). \tag{4.2}$$

(3) The function $b(t)$ is decreasing and

$$0 \leq b(t) \leq b(0) \leq 2\varphi'(1)t_0 \qquad \forall t \in [0, t_0], \tag{4.3}$$

$$0 \leq b(t_0 - t) \leq 2\varphi'(1)\sqrt{t_0}\sqrt{t} \qquad \forall t \in [0, t_0]. \tag{4.4}$$

Proof. From (2.7) we easily deduce that

$$\beta(t) \geq 2, \qquad \beta'(t) \geq \frac{1}{2t_0}, \qquad \beta''(t) = 2t_0\,[\beta'(t)]^3 \tag{4.5}$$

for every $t \in [0, t_0]$. Moreover from assumption (4.1) we have that

$$[\beta'(t)]^2 \geq \frac{1}{4t_0^2} \geq 4\varphi'(1)|\varphi'''(1)| \qquad \forall t \in [0, t_0), \tag{4.6}$$

Let us consider the polynomial

$$p(x) := \varphi'''(1)x^2 + \beta'(t)x - \frac{\varphi'(1)}{\beta^2(t)},$$

where $t \in [0, t_0]$ is thought as a parameter. Due to our assumptions on φ we have that $\varphi'''(1) \leq 0$. Exploiting (4.6) one can therefore check that

$$p\left(\frac{\varphi'(1)}{\beta^2(t)\beta'(t)}\right) \leq 0, \qquad p\left(\frac{\varphi'(1)}{\beta'(t)}\right) \geq 0. \tag{4.7}$$

This implies that $p(x)$ has at least one real root. Moreover $p(x)$ is either a polynomial of degree one, or a polynomial of degree two representing a *concave* function. In both cases the unique or the smallest root $b(t)$ is the one lying in the interval whose endpoints appear in (4.7), hence

$$\frac{\varphi'(1)}{\beta^2(t)\beta'(t)} \leq b(t) \leq \frac{\varphi'(1)}{\beta'(t)} \qquad \forall t \in [0, t_0). \tag{4.8}$$

The regularity of $b(t)$ in $[0, t_0)$ follows from the regularity of the coefficients of the equation. The continuity up to $t = t_0$ follows from (4.8) and the fact that $\beta'(t) \to +\infty$ as $t \to t_0^-$ (we remind that we set $b(t_0) = 0$). This proves statement (1).

Let us consider now the derivative $b'(t)$. From the implicit function theorem we have that

$$-b'(t) = \frac{\beta''(t)b(t) + 2\varphi'(1)\beta'(t)[\beta(t)]^{-3}}{2\varphi'''(1)b(t) + \beta'(t)} \qquad \forall t \in [0, t_0). \tag{4.9}$$

Let us estimate the numerator of (4.9). Exploiting (4.5) and the upper bound for $b(t)$ provided by (4.8), we have that

$$\begin{aligned} 0 \ < \ \beta''(t)b(t) + 2\frac{\varphi'(1)\beta'(t)}{\beta^3(t)} \ &\leq \ 2t_0[\beta'(t)]^3\frac{\varphi'(1)}{\beta'(t)} + \frac{\varphi'(1)}{4}\beta'(t) \\ &= \ \varphi'(1)[\beta'(t)]^2\left(2t_0 + \frac{1}{4\beta'(t)}\right) \\ &\leq \ \frac{5t_0}{2}\varphi'(1)[\beta'(t)]^2. \end{aligned}$$

Let us estimate the denominator of (4.9). Exploiting once again the upper bound for $b(t)$ provided by (4.8), and inequality (4.6), we obtain that

$$2\varphi'''(1)b(t) + \beta'(t) \geq \beta'(t) - 2|\varphi'''(1)|\frac{\varphi'(1)}{\beta'(t)} = \beta'(t)\left(1 - \frac{2|\varphi'''(1)|\varphi'(1)}{[\beta'(t)]^2}\right) \geq \frac{1}{2}\beta'(t).$$

At this point (4.2) easily follows from the estimates on the numerator and the denominator of (4.9). This completes the proof of statement (2).

Let us prove now statement (3). The function $b(t)$ is decreasing because its derivative is negative. It follows that $0 = b(t_0) \leq b(t) \leq b(0)$ for every $t \in [0, t_0]$. Finally, the estimates on $b(0)$ and $b(t - t_0)$ stated in (4.3) and (4.4) follow from (4.5) and the upper bound in (4.8). $\Box$

Lemma 4.2 *Let t_0 be as in Lemma 4.1, and let $\gamma(t)$ be defined by (2.7).*

Then the function $c(t)$ introduced in (2.14) is well defined for every $t \in [0, t_0]$ and fulfils the following properties.

(1) We have that $c \in C^0([0, t_0]) \cap C^\infty([0, t_0))$.

(2) We have that (we remind that $\gamma'(t)$ is negative)

$$0 < c'(t) \leq -5t_0\varphi'(1)\gamma'(t) \qquad \forall t \in [0, t_0).$$

(3) The function $c(t)$ is increasing and

$$-2\varphi'(1)t_0 \leq c(0) \leq c(t) \leq 0 \qquad \forall t \in [0, t_0], \tag{4.10}$$

$$0 \geq c(t_0 - t) \geq -2\varphi'(1)\sqrt{t_0}\sqrt{t} \qquad \forall t \in [0, t_0]. \tag{4.11}$$

Remark 4.3 The second inequality in (3.4) implies (4.1). We can therefore apply the conclusions of Lemma 4.1 and Lemma 4.2 in the proofs of our main results. Exploiting the first and second inequality in (3.4), from statement (3) in Lemma 4.1 and Lemma 4.2 we obtain the following stronger inequalities

$$0 \leq b(t) \leq b(0) \leq 1 \qquad \forall t \in [0, t_0], \tag{4.12}$$

$$0 \leq b(t_0 - t) \leq \sqrt{t} \qquad \forall t \in [0, t_0], \tag{4.13}$$

$$-\sqrt{t} \leq c(t_0 - t) \leq 0 \qquad \forall t \in [0, t_0]. \tag{4.14}$$

We finally state a classical comparison result for fully nonlinear parabolic equations. This is the key tool in our analysis. We omit the standard proof.

Lemma 4.4 *Let $\mathcal{D} \subseteq \mathbb{R}^2$ be a compact set, and let $\psi : \mathcal{D} \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. Let us assume that*

- *ψ is nondecreasing in the last variable (degenerate ellipticity), namely*

$$\psi(r, t, p, q, s_1) \leq \psi(r, t, p, q, s_2) \qquad \forall s_1 \leq s_2,\ \forall (r, t, p, q) \in \mathcal{D} \times \mathbb{R}^2,$$

- *ψ is locally Lipschitz continuous in the third variable, namely for every $R \geq 0$ there exists a constant L such that*

$$|\psi(r, t, p_1, q, s) - \psi(r, t, p_2, q, s)| \leq L|p_1 - p_2| \qquad \forall (r, t, p_1, p_2, q, s) \in \mathcal{D} \times [0, R]^4.$$

Let u and v be two functions in $C^0(\mathcal{D}) \cap C^{2,1}(\mathrm{Int}_P(\mathcal{D}))$ such that

$$u_t \leq \psi(r, t, u, u_r, u_{rr}) \qquad \forall (r, t) \in \mathrm{Int}_P(\mathcal{D}),$$

$$v_t \geq \psi(r, t, v, v_r, v_{rr}) \qquad \forall (r, t) \in \mathrm{Int}_P(\mathcal{D}),$$

$$u(r, t) \leq v(r, t) \qquad \forall (r, t) \in \partial_P(\mathcal{D}).$$

Then $u(r, t) \leq v(r, t)$ for every $(r, t) \in \mathcal{D}$. $\square$

Remark 4.5 If we know a priori that there exist constants a and b such that $a \leq u(r, t) \leq b$ for every $(r, t) \in \mathcal{D}$, then we can weaken the assumptions of Lemma 4.4 by asking that v fulfils its differential inequality only for those $(r, t) \in \mathrm{Int}_P(\mathcal{D})$ such that $a \leq v(r, t) \leq b$. We can obtain a similar statement by swapping the role of u and v.

4.1 Proof of Theorem 3.1

Let us briefly sketch the outline of the proof. First of all we show that the initial boundary value problem has a unique solution $u^\varepsilon \in C^{2,1}(\mathcal{Q}_1) \cap C^\infty(\mathrm{Int}_P(\mathcal{Q}_1))$. Then for the sake of simplicity we set $v =: u^\varepsilon_r$, and $w := u^\varepsilon_{rr}$. These functions belong to $C^0(\mathcal{Q}_1) \cap C^\infty(\mathrm{Int}_P(\mathcal{Q}_1))$. It is easy to see that v is the solution of the following equation

$$v_t = \varphi''(v)v_{rr} + \varphi'''(v)v_r^2 + \frac{\varphi''(v)}{r}v_r - \frac{\varphi'(v)}{r^2} \qquad \forall (r,t) \in \mathrm{Int}_P(\mathcal{Q}_1), \tag{4.15}$$

with Dirichlet boundary conditions

$$v(1,t) = 0 \qquad \forall t \in [0,t_0], \tag{4.16}$$

$$v(\beta(t),t) = 1-\varepsilon \qquad \forall t \in [0,t_0], \tag{4.17}$$

and initial datum

$$v(r,0) = (1-\varepsilon)u_{0r}(r) \qquad \forall r \in [1,2]. \tag{4.18}$$

In the same way w is a solution in $\mathrm{Int}_P(\mathcal{Q}_1)$ of equation

$$\begin{aligned} w_t \;=\; & \varphi''(v)w_{rr} + 3\varphi'''(v)w_r w + \varphi^{IV}(v)w^3 \\ & + \frac{\varphi'''(v)}{r}w^2 + \frac{\varphi''(v)}{r}w_r - 2\frac{\varphi''(v)}{r^2}w + 2\frac{\varphi'(v)}{r^3}, \end{aligned} \tag{4.19}$$

where the terms in v are to be interpreted as coefficients depending on r and t. Moreover w satisfies the Dirichlet boundary conditions

$$w(1,t) = u^\varepsilon_{rr}(1,t) \qquad \forall t \in [0,t_0], \tag{4.20}$$

$$w(\beta(t),t) = u^\varepsilon_{rr}(\beta(t),t) \qquad \forall t \in [0,t_0], \tag{4.21}$$

and the initial condition

$$w(r,0) = (1-\varepsilon)u_{0rr}(r) \qquad \forall r \in [1,2]. \tag{4.22}$$

The initial data (4.18) and (4.22) can be easily estimated using (3.6) through (3.9). Indeed from Taylor's expansion we have that

$$u_{0r}(r) = u_{0r}(2) + u_{0rr}(2)(r-2) + \frac{1}{2}u_{0rrr}(\xi)(r-2)^2$$

for a suitable $\xi \in (1,2)$, hence

$$1 + b(0)(r-2) - 5(r-2)^2 \le u_{0r}(r) \le 1 + b(0)(r-2) + 5(r-2)^2 \qquad \forall r \in [1,2]. \tag{4.23}$$

Analogously we have that $u_{0rr}(r) = u_{0rr}(2) + u_{0rrr}(\xi)(r-2)$, hence

$$b(0) - 10(2-r) \le u_{0rr}(r) \le b(0) + 10(2-r) \qquad \forall r \in [1,2]. \tag{4.24}$$

Then we show that v and w satisfy five sets of inequalities in $\mathcal{Q}_1$.

- The first pair of inequalities is

$$0 \leq v(r,t) \leq 1 - \varepsilon, \tag{4.25}$$

which is equivalent to (3.11).

- The second inequality is

$$v(r,t) \leq 1 - \eta \left((r-3)^2 + \frac{t}{t_0} - 1 \right) \tag{4.26}$$

for a suitable constants $\eta > 0$. The term after η is positive in $\mathcal{Q}_1 \setminus \Gamma_1$. This implies (3.12), hence also (3.13).

- The third pair of inequalities is

$$0 \leq v(r,t) \leq \frac{20}{(1 - 800\gamma_2 t)^{1/2}} \left(1 - e^{1-r}\right), \tag{4.27}$$

where γ_2 is the constant defined in (3.3). The left-hand side and the right-hand side coincide for $r = 1$, hence $v_r(1,t)$ is bounded by their derivative computed in $r = 1$. Thus we obtain that $0 \leq v_r(1,t) \leq 20(1 - 800\gamma_2 t)^{-1/2}$. The upper bound is less than 100 due to the second inequality in (3.4). This proves (3.14).

- The fourth pair of inequalities, and probably the most delicate one, is

$$v(r,t) \geq 1 - \varepsilon - (b(t) + \varepsilon)(\beta(t) - r) - \gamma_0(\beta(t) - r)^2, \tag{4.28}$$

$$v(r,t) \leq 1 - \varepsilon - (b(t) - \varepsilon)(\beta(t) - r) + \gamma_0(\beta(t) - r)^2, \tag{4.29}$$

where γ_0 is the constant defined in (3.1). In this case $v(r,t)$ is bounded from below and from above by two functions which coincide for $r = \beta(t)$. It follows that $v_r(\beta(t),t)$ is bounded by the space derivatives of these two functions computed in $r = \beta(t)$. This implies (3.15).

- Thanks to (3.14) and (3.15) we have an estimate on the values of w at the boundary. This is the starting point to prove the fifth set of inequalities

$$b(t) - \varepsilon - \gamma_1(\beta(t) - r) \leq w(r,t) \leq b(t) + \varepsilon + \gamma_1(\beta(t) - r) \tag{4.30}$$

where γ_1 is the constant defined in (3.2). This implies (3.16), hence also (3.17) and (3.18).

All these estimates are proved using subsolutions and supersolutions. Finally the proof of (3.19) through (3.22) relies on usual energy estimates.

We are now ready to proceed with the details.

Existence and maximum principle for space derivatives Let us take a function $\varphi_\varepsilon \in C^\infty(\mathbb{R})$ which coincides with φ in the interval $[0, 1-\varepsilon]$, and such that $\varphi_\varepsilon''(\sigma) \geq \nu_\varepsilon > 0$ for every $\sigma \in \mathbb{R}$. Equation (2.6), with φ_ε instead of φ, is strictly parabolic. The initial boundary value problem in $\mathcal{Q}_1$ can be reduced to the fixed domain $[0, 1] \times [0, t_0]$ by the variable change $(s, t) \to (1 + (\beta(t) - 1)s, t)$, which of course doesn't change the strict parabolicity. By well knows results (see for example [16]) the problem admits a unique solution $u^\varepsilon \in C^{2,1}(\mathcal{Q}_1) \cap C^\infty(\mathrm{Int}_P(\mathcal{Q}_1))$. By the way, the estimates we are going to prove could be used to give a self contained proof of the existence result for ε fixed, showing in particular that the singularity of $\mathcal{Q}_1$ in $(3, t_0)$ doesn't affect the existence or the regularity of the solution.

If we show that this solution satisfies (3.11), then this same solution satisfies equation (2.6) with the original φ. Our assumptions on φ imply that $\varphi'(0) = 0$ and $\varphi'(1-\varepsilon) > 0$. It follows that $z(r, t) := 0$ is a subsolution of (4.15) through (4.18), while $z(r, t) := 1-\varepsilon$ is a supersolution of the same problem. The usual comparison principle implies (4.25).

Uniform strict parabolicity in the interior Let us choose $\eta > 0$ small enough so that

$$\eta \leq \frac{1}{8}, \qquad \eta \leq \inf_{r\in[1,2)} \frac{1 - u_{0r}(r)}{(2 - r)(4 - r)}, \qquad \eta \leq \frac{1}{9}\left(\frac{1}{t_0} + 20\gamma_2\right)^{-1} \cdot \varphi'\left(\frac{1}{2}\right). \tag{4.31}$$

Note that the infimum is positive because our assumptions on u_0 imply that the fraction is positive for every $r \in [1, 2)$ and tends to $b(0)/2 > 0$ as $r \to 2^-$.

Let $z(r, t)$ denote the right-hand side of (4.26). We claim that, for this choice of η, z is a supersolution of (4.15) through (4.18), which implies (4.26).

Boundary $r = 1$ Since $|r - 3| \leq 2$, $t \leq t_0$, and $\eta \leq 1/8$, we have that

$$z(r, t) = 1 - \eta\left((r - 3)^2 + \frac{t}{t_0} - 1\right) \geq 1 - 4\eta \geq \frac{1}{2} \qquad \forall (r, t) \in \mathcal{Q}_1. \tag{4.32}$$

In particular we have that $z(1, t) \geq 0 = v(1, t)$.

Boundary $r = \beta(t)$ In this case we have that $z(\beta(t), t) = 1 \geq 1 - \varepsilon = v(\beta(t), t)$.

Boundary $t = 0$ Thanks to the second inequality in (4.31), and the fact that $u_{0r}(r) \geq 0$ for every $r \in [1, 2]$, we have that

$$z(r, 0) = 1 - \eta\left((r - 3)^2 - 1\right) = 1 - \eta(2 - r)(4 - r) \geq$$
$$\geq u_{0r}(r) \geq (1 - \varepsilon)u_{0r}(r) = v(r, 0).$$

Differential inequality We have that

$$z_t = -\frac{\eta}{t_0}, \qquad z_r(r,t) = -2\eta(r-3), \qquad z_{rr}(r,t) = -2\eta.$$

Therefore we have to prove that

$$-\frac{\eta}{t_0} \geq -2\eta\varphi''(z) + \varphi'''(z)4\eta^2(r-3)^2 - \frac{\varphi''(z)}{r}2\eta(r-3) - \frac{\varphi'(z)}{r^2}.$$

Due to (4.32) and the properties of φ', we have that $\varphi'(z)/r^2 \geq (1/9)\varphi'(1/2)$. All the other terms are uniformly small when η is small. This shows that the differential inequality is satisfied when η is small enough, for example as soon as η fulfils the last inequality in (4.31).

Estimate on second derivatives at the fixed boundary Let us prove (4.27). We already know from (4.25) that $v(r,t) \geq 0$ in $\mathcal{Q}_1$. In order to prove the other inequality, we set for simplicity $k(t) := 20(1-800\gamma_2 t)^{-1/2}$. It is easy to check that $k(t)$ is the solution of the Cauchy problem

$$k'(t) = \gamma_2 k^3(t), \qquad k(0) = 20.$$

We claim that $z(r,t) := k(t)\left(1-e^{1-r}\right)$ is a supersolution of (4.15) through (4.18), which implies (4.27).

Boundary $r=1$ In this case we have that $z(1,t) = 0 = v(1,t)$.

Boundary $r=\beta(t)$ Since $\beta(t) \geq 2$, in this case we have that

$$z(\beta(t),t) = k(t)\left(1-e^{1-\beta(t)}\right) \geq 20(1-e^{-1}) \geq 1 \geq v(\beta(t),t).$$

Boundary $t=0$ From the assumptions on u_0 we have that

$$v(r,0) = (1-\varepsilon)u_{0r}(r) \leq u_{0r}(r) = u_{0r}(1) + (r-1)u_{0rr}(\xi) \leq 10(r-1). \qquad (4.33)$$

On the other hand, the function $1-e^{1-r}$ is concave, hence

$$z(r,0) = 20\left(1-e^{1-r}\right) \geq 20(1-e^{-1})(r-1) \geq 10(r-1). \qquad (4.34)$$

From (4.33) and (4.34) it follows that $z(r,0) \geq v(r,0)$ for every $r \in [1,2]$.

Differential inequality We have that

$$z_t(r,t) = k'(t)\left(1-e^{1-r}\right), \qquad z_r(r,t) = k(t)e^{1-r}, \qquad z_{rr}(r,t) = -k(t)e^{1-r}.$$

We have therefore to prove that

$$k'(t)\left(1-e^{1-r}\right) \geq \varphi'''(z)k^2(t)e^{2-2r} - k(t)\left(1-\frac{1}{r}\right)\varphi''(z)e^{1-r} - \frac{\varphi'(z)}{r^2}. \tag{4.35}$$

Thanks to Remark 4.5, this inequality has to be satisfied only for those values $(z,t) \in \mathcal{Q}_1$ for which $0 \leq z(r,t) \leq 1$. In this case we know that the last two terms are negative. Since $\varphi'''(0) = 0$ we have also that

$$|\varphi'''(z)| = |\varphi'''(z) - \varphi'''(0)| = \left|\varphi^{IV}(\xi)\right| z \leq \gamma_2 k(t)\left(1-e^{1-r}\right).$$

It follows that

$$\text{right-hand side of (4.35)} \leq |\varphi'''(z)|\, k^2(t) \leq \gamma_2 k^3(t)\left(1-e^{1-r}\right) = k'(t)\left(1-e^{1-r}\right),$$

which completes the proof of the differential inequality.

Estimate on second derivatives at the moving boundary – Subsolution Let us prove (4.28). To this end, we denote its right-hand side by $z(r,t)$, and we show that z is a subsolution of (4.15) through (4.18).

Boundary $r = 1$ Since $\beta(t) \geq 2$ and $\gamma_0 \geq 1$, in this case we have that

$$z(1,t) = 1 - \varepsilon - (b(t)+\varepsilon)(\beta(t)-1) - \gamma_0(\beta(t)-1)^2 \leq 1 - \gamma_0 \leq 0 = v(1,t).$$

Boundary $r = \beta(t)$ In this case we have that $z(\beta(t),t) = 1 - \varepsilon = v(\beta(t),t)$.

Boundary $t = 0$ Since $b(0) \geq 0$ and $\gamma_0 \geq 5$, from (4.23) have that

$$\begin{aligned} v(r,0) = (1-\varepsilon)u_{0r}(r) &\geq (1-\varepsilon)\left(1 - b(0)(2-r) - 5(2-r)^2\right) \geq \\ &\geq 1 - \varepsilon - (b(0)+\varepsilon)(2-r) - \gamma_0(2-r)^2 = z(r,0). \end{aligned}$$

Differential inequality Let us set for simplicity $x := \beta(t) - r$. Then we have that

$$z_r(r,t) = b(t) + \varepsilon + 2\gamma_0 x, \qquad z_{rr}(r,t) = -2\gamma_0,$$

$$z_t(r,t) = -b(t)\beta'(t) - \beta'(t)\varepsilon - (b'(t) + 2\gamma_0\beta'(t))x.$$

The differential inequality is therefore the following

$$\begin{aligned}-b(t)\beta'(t)-\beta'(t)\varepsilon-(b'(t)+2\gamma_0\beta'(t))x \quad &\leq \quad -2\gamma_0\varphi''(z)+\varphi'''(z)(b(t)+\varepsilon+2\gamma_0 x)^2\\ &\qquad +\frac{\varphi''(z)}{r}(b(t)+\varepsilon+2\gamma_0 x)-\frac{\varphi'(z)}{r^2}.\end{aligned}$$

Thanks to Remark 4.5 we can limit ourselves to prove it for all values $(z,t)\in\mathcal{Q}_1$ such that $0\leq z(r,t)\leq 1$. From the definition of $b(t)$ we have that

$$-b(t)\beta'(t)=\varphi'''(1)b^2(t)-\frac{\varphi'(1)}{\beta^2(t)}. \tag{4.36}$$

After changing the signs we have therefore to prove that

$$\begin{aligned}\beta'(t)\varepsilon+(b'(t)+2\gamma_0\beta'(t))x \quad &\geq \quad 2\gamma_0\varphi''(z)\\ &\qquad -\left[\varphi'''(z)(b(t)+\varepsilon+2\gamma_0 x)^2-\varphi'''(1)b^2(t)\right]\\ &\qquad -\frac{\varphi''(z)}{r}(b(t)+\varepsilon+2\gamma_0 x)\\ &\qquad +\left[\frac{\varphi'(z)}{r^2}-\frac{\varphi'(1)}{\beta^2(t)}\right]\\ &=: \quad I_1+I_2+I_3+I_4.\end{aligned} \tag{4.37}$$

Let us estimate the left-hand side. Exploiting (4.2), and the fact that $t_0\leq 1$ and $2\gamma_0\geq 2+5\varphi'(1)$, we have that

$$\begin{aligned}\beta'(t)\varepsilon+(b'(t)+2\gamma_0\beta'(t))x \quad &\geq \quad \beta'(t)\varepsilon+(-5\varphi'(1)t_0+2\gamma_0)\,\beta'(t)x\\ &\geq \quad \beta'(t)\varepsilon+2\beta'(t)x\\ &\geq \quad \frac{\varepsilon}{2t_0}+\frac{x}{t_0}.\end{aligned} \tag{4.38}$$

Let us estimate the four terms in the right-hand side of (4.37). From now on we exploit several times the following inequalities

$$0<\varepsilon<1, \qquad 0\leq x\leq 2, \qquad 0\leq z\leq 1, \qquad 0\leq b(t)\leq 1, \qquad 1\leq r\leq\beta(t).$$

First of all we remark that

$$|z-1|\leq\varepsilon+(b(t)+\varepsilon+\gamma_0 x)x\leq\varepsilon+(2\gamma_0+2)x.$$

As a consequence we have that

$$|\varphi''(z)|=|\varphi''(z)-\varphi''(1)|=|z-1|\cdot|\varphi'''(\xi)|\leq\gamma_2\varepsilon+\gamma_2(2\gamma_0+2)x, \tag{4.39}$$

hence

$$I_1\leq 2\gamma_0|\varphi''(z)|\leq 2\gamma_2\gamma_0\varepsilon+\gamma_2(4\gamma_0^2+4\gamma_0)x. \tag{4.40}$$

Analogously we have that

$$|\varphi'''(z) - \varphi'''(1)| = |z-1| \cdot |\varphi^{IV}(\xi)| \leq \gamma_2\varepsilon + \gamma_2(2\gamma_0 + 2)x,$$

and

$$\begin{aligned} \left|(b(t) + \varepsilon + 2\gamma_0 x)^2 - b^2(t)\right| &= (2b(t) + \varepsilon)\varepsilon + (4\gamma_0^2 x + 4b(t)\gamma_0 + 4\varepsilon\gamma_0)x \\ &\leq 3\varepsilon + (8\gamma_0^2 + 8\gamma_0)x, \end{aligned}$$

hence

$$\begin{aligned} I_2 &\leq |\varphi'''(z)| \cdot \left|(b(t) + \varepsilon + 2\gamma_0 x)^2 - b^2(t)\right| + b^2(t)\,|\varphi'''(z) - \varphi'''(1)| \\ &\leq 4\gamma_2\varepsilon + \gamma_2(8\gamma_0^2 + 10\gamma_0 + 2)x. \end{aligned} \tag{4.41}$$

Exploiting (4.39) once again, we obtain that

$$\begin{aligned} I_3 &\leq (b(t) + \varepsilon + 2\gamma_0 x)\frac{|\varphi''(z)|}{r} \\ &\leq (4\gamma_0 + 2)|\varphi''(z)| \\ &\leq \gamma_2(4\gamma_0 + 2)\varepsilon + \gamma_2(8\gamma_0^2 + 12\gamma_0 + 4)x. \end{aligned} \tag{4.42}$$

Finally we have that

$$|\varphi'(z) - \varphi'(1)| = |z-1| \cdot |\varphi''(\xi)| \leq \gamma_2\varepsilon + \gamma_2(2\gamma_0 + 2)x,$$

and

$$\left|\frac{1}{r^2} - \frac{1}{\beta^2(t)}\right| = \frac{|\beta(t) + r| \cdot |\beta(t) - r|}{r^2\beta^2(t)} \leq \frac{|\beta(t) + r|}{\beta^2(t)}x \leq x,$$

hence

$$\begin{aligned} I_4 &\leq |\varphi'(z)| \cdot \left|\frac{1}{r^2} - \frac{1}{\beta^2(t)}\right| + \frac{1}{\beta^2(t)}\,|\varphi'(z) - \varphi'(1)| \\ &\leq \gamma_2\varepsilon + \gamma_2(2\gamma_0 + 3)x. \end{aligned} \tag{4.43}$$

Summing (4.40) through (4.43), and exploiting (3.5), we obtain that

$$I_1 + I_2 + I_3 + I_4 \leq \gamma_2(6\gamma_0 + 7)\varepsilon + \gamma_2(20\gamma_0^2 + 28\gamma_0 + 9)x \leq \frac{\varepsilon}{2t_0} + \frac{x}{t_0}. \tag{4.44}$$

This estimate and (4.38) imply the differential inequality.

Estimate on second derivatives at the moving boundary – Supersolution Let us prove (4.29). To this end we show that its right-hand side, which we denote by $z(r,t)$, is a supersolution of (4.15) through (4.18).

Boundary $r = 1$ Since $\gamma_0 \geq 1 \geq b(t)$ and $\beta(t) \geq 2$, in this case we have that

$$z(1,t) = 1 - \varepsilon + (\beta(t) - 1)\left[\gamma_0(\beta(t) - 1) - b(t) + \varepsilon\right] \geq 1 - \varepsilon = v(1,t).$$

Boundary $r = \beta(t)$ As in the case of the subsolution we have that

$$z(\beta(t), t) = 1 - \varepsilon = v(\beta(t), t).$$

Boundary $t = 0$ Since $\gamma_0 \geq 5$ and $b(0) \leq 1$, from (4.23) we have that

$$v(r,0) = (1-\varepsilon)u_{0r}(r) \leq (1-\varepsilon)\left[1 + b(0)(r-2) + 5(r-2)^2\right] \leq$$
$$\leq 1 - \varepsilon + (b(0) - \varepsilon)(r-2) + \gamma_0(r-2)^2 = z(r,0).$$

Differential inequality We limit ourselves to sketch the argument, which is analogous to the case of the subsolution. After computing the derivatives and using equation (4.36), we reduce ourselves to prove that (as before $x := \beta(t) - r$)

$$\begin{aligned}
\beta'(t)\varepsilon + (2\gamma_0\beta'(t) - b'(t))x \quad \geq \quad & 2\gamma_0\varphi''(z) \\
& + \left[\varphi'''(z)(b(t) - \varepsilon - 2\gamma_0 x)^2 - \varphi'''(1)b^2(t)\right] \\
& + \frac{\varphi''(z)}{r}(b(t) - \varepsilon - 2\gamma_0 x) \\
& + \left[\frac{\varphi'(1)}{\beta^2(t)} - \frac{\varphi'(z)}{r^2}\right] \\
=: \quad & I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

Let us estimate the left-hand side. Since $b'(t) \leq 0$ and $\gamma_0 \geq 1$, we have that

$$\beta'(t)\varepsilon + (2\gamma_0\beta'(t) - b'(t))x \geq \beta'(t)\varepsilon + 2\beta'(t)x \geq \frac{\varepsilon}{2t_0} + \frac{x}{t_0}.$$

The four terms in the right-hand side can be estimated as in the case of the subsolution. In this way we obtain (4.44) once again. We conclude by using the smallness of t_0 exactly as in the case of the subsolution.

Global estimate on second derivatives – Supersolution Let us prove the upper bound in (4.30). To this end we denote the right-hand side by $z(r,t)$, and we show that it is a supersolution of (4.19) through (4.22).

Boundary $r = 1$ Since $b(t) \geq 0$, $\beta(t) \geq 2$, and $\gamma_1 \geq 100$, from (3.14) we have that

$$z(1,t) = b(t) + \varepsilon + \gamma_1(\beta(t) - 1) \geq \gamma_1 \geq 100 \geq u^{\varepsilon}_{rr}(1,t) = w(1,t).$$

Boundary $r = \beta(t)$ From (3.15) we have that

$$z(\beta(t), t) = b(t) + \varepsilon \geq u^{\varepsilon}_{rr}(\beta(t), t) = w(\beta(t), t).$$

Boundary $t = 0$ Since $\gamma_1 \geq 10$, from (4.24) we have that

$$z(r, 0) \geq b(0) + \gamma_1(2 - r) \geq b(0) + 10(2 - r) \geq (1 - \varepsilon)\,[b(0) + 10(2 - r)] \geq$$
$$\geq (1 - \varepsilon)u_{0rr}(r) = w(r, 0).$$

Differential inequality We have that

$$z_r(r, t) = -\gamma_1, \qquad z_{rr}(r, t) = 0, \qquad z_t(r, t) = b'(t) + \gamma_1\beta'(t).$$

The inequality to be satisfied in $\mathcal{Q}_1$ is therefore the following

$$b'(t) + \gamma_1\beta'(t) \geq -3\gamma_1\varphi'''(v)z + \varphi^{IV}(v)z^3 + \frac{\varphi'''(v)}{r}z^2 - \gamma_1\frac{\varphi''(v)}{r} - 2\frac{\varphi''(v)}{r^2}z + 2\frac{\varphi'(v)}{r^3},$$

where of course the terms in v are thought as coefficients.

Let us examine the left-hand side. Using (4.2) and the fact that $t_0 \leq 1$ and $\gamma_1 \geq 5\varphi'(1) + 1$, we have that

$$b'(t) + \gamma_1\beta'(t) \geq (-5\varphi'(1)t_0 + \gamma_1)\,\beta'(t) \geq \beta'(t) \geq \frac{1}{2t_0}. \tag{4.45}$$

Since $|z(r, t)| \leq 2 + 2\gamma_1$, a rough estimate of the right-hand side of the differential inequality gives that

$$\text{right-hand side} \leq \gamma_2(4\gamma_1 + 6)(2 + 2\gamma_1)^3 \leq 48\gamma_2(\gamma_1 + 1)^4.$$

Due to the last condition in (3.4), this estimate and (4.45) imply the differential inequality.

Global estimate on second derivatives – Subsolution Let us prove the lower bound in (4.30). To this end we denote the left-hand side by $z(r, t)$, and we show that it is a subsolution of (4.19) through (4.22).

Boundary $r = 1$ Since $\beta(t) - 1 \geq 1$ and $b(t) \leq 1 \leq \gamma_1$, from (3.14) we have that

$$z(1, t) = b(t) - \varepsilon - \gamma_1(\beta(t) - 1) \leq 1 - \gamma_1 \leq 0 \leq u^{\varepsilon}_{rr}(1, t) = w(1, t).$$

Boundary $r = \beta(t)$ From (3.15) we have that

$$z(\beta(t), t) = b(t) - \varepsilon \leq u^{\varepsilon}_{rr}(\beta(t), t) = w(\beta(t), t).$$

Boundary $t=0$ Since $b(0)\le 1$ and $\gamma_1\ge 10$, from (4.24) we have that

$$z(r,0)=b(0)-\varepsilon-\gamma_1(2-r)\le(1-\varepsilon)(b(0)-10(2-r))\le(1-\varepsilon)u_{0rr}(r)=w(r,0).$$

Differential inequality We have to prove that

$$b'(t)-\gamma_1\beta'(t)\le 3\gamma_1\varphi'''(v)z+\varphi^{IV}(v)z^3+\frac{\varphi'''(v)}{r}z^2+\gamma_1\frac{\varphi''(v)}{r}-2\frac{\varphi''(v)}{r^2}z+2\frac{\varphi'(v)}{r^3}.$$

Since $\gamma_1\ge 1$ and $b'(t)\le 0$, in the left-hand side we have that

$$b'(t)-\gamma_1\beta'(t)\le-\beta'(t)\le-\frac{1}{2t_0},$$

while the usual rough estimate on the right-hand side gives that

$$\text{right-hand side}\ge -48\gamma_2(\gamma_1+1)^4.$$

The conclusion follows as in the preceding case.

Integral estimates In the following estimates we introduce constants c_1, c_2, $\ldots$, all independent on ε.

Computing the time derivative and integrating by parts, we have that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_1^{\beta(t)}[u_t^\varepsilon]^2\,dr\right) &= 2\int_1^{\beta(t)}u_t^\varepsilon\cdot\frac{1}{r}\left(r\varphi'(u_r^\varepsilon)\right)_{rt}dr+\beta'(t)\left[u_t^\varepsilon(\beta(t),t)\right]^2\\ &= -2\int_1^{\beta(t)}\varphi''(u_r^\varepsilon)\left[u_{rt}^\varepsilon\right]^2\,dr+2\int_1^{\beta(t)}\frac{\varphi''(u_r^\varepsilon)}{r}u_t^\varepsilon u_{rt}^\varepsilon\,dr\\ &\quad +2\varphi''(1-\varepsilon)u_t^\varepsilon(\beta(t),t)u_{rt}^\varepsilon(\beta(t),t)+\beta'(t)\left[u_t^\varepsilon(\beta(t),t)\right]^2\\ &=: I_1+I_2+I_3+I_4. \end{aligned}\tag{4.46}$$

Note that in the integration by parts we neglected the boundary term in $r=1$ because $u_{rt}^\varepsilon(1,t)=0$ due to the Neumann boundary condition (1.5).

Let us estimate some of the terms in (4.46). From (3.18) we have that

$$2\frac{\varphi''(u_r^\varepsilon)}{r}u_t^\varepsilon u_{rt}^\varepsilon\le\varphi''(u_r^\varepsilon)\left([u_t^\varepsilon]^2+[u_{rt}^\varepsilon]^2\right)\le\varphi''(u_r^\varepsilon)\left[u_{rt}^\varepsilon\right]^2+c_1,$$

hence

$$I_2\le\int_1^{\beta(t)}\varphi''(u_r^\varepsilon)\left[u_{rt}^\varepsilon\right]^2\,dr+c_2.\tag{4.47}$$

From (3.18) we have also that

$$I_4\le c_3\beta'(t).\tag{4.48}$$

Taking the time derivative of the Neumann boundary condition (3.10) we obtain that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[u_r^\varepsilon(\beta(t),t)\right] = \beta'(t)u_{rr}^\varepsilon(\beta(t),t) + u_{rt}^\varepsilon(\beta(t),t).$$

From (3.17) we have therefore that

$$|u_{rt}^\varepsilon(\beta(t),t)| = \beta'(t)|u_{rr}^\varepsilon(\beta(t),t)| \leq c_4\beta'(t),$$

hence by (3.18)

$$I_3 \leq c_5\beta'(t). \tag{4.49}$$

Plugging (4.47), (4.48), and (4.49) in (4.46) we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_1^{\beta(t)} [u_t^\varepsilon]^2\,dr\right) \leq -\int_1^{\beta(t)} \varphi''(u_r^\varepsilon)\,[u_{rt}^\varepsilon]^2\,\,dr + c_2 + c_6\beta'(t),$$

and therefore

$$\int_0^{t_0}\int_1^{\beta(t)} \varphi''(u_r^\varepsilon)\,[u_{rt}^\varepsilon]^2\,dr\,dt \leq \int_1^2 [u_t^\varepsilon(r,0)]^2\,dr + c_2t_0 + c_6(\beta(t_0)-\beta(0)) \leq c_7.$$

This proves (3.19).

In order to prove (3.20) through (3.22), we choose a cut-off function $\rho \in C^\infty(\mathbb{R}^2)$ such that $0 \leq \rho(r,t) \leq 1$ for every $(x,t) \in \mathbb{R}^2$, and

$$\begin{array}{rcll} \rho(r,t) & = & 1 & \text{in } \{(r,t) \in \mathcal{Q}_1 : 1 \leq r \leq \beta(t)-\delta\}, \\ \rho(r,t) & = & 0 & \text{in } \{(r,t) \in \mathcal{Q}_1 : \beta(t)-\delta/2 \leq r \leq \beta(t)\}. \end{array}$$

Then we set

$$E(t) := \int_1^{\beta(t)} \rho^2[u_{rt}^\varepsilon]^2\,dr, \qquad F(t) := \int_1^{\beta(t)} \rho^2\varphi''(u_r^\varepsilon)[u_{rrt}^\varepsilon]^2\,dr,$$

$$G(t) := \int_1^{\beta(t)} \varphi''(u_r^\varepsilon)[u_{rt}^\varepsilon]^2\,dr.$$

Taking the time derivative of $E(t)$ and integrating by parts we have that

$$\begin{array}{rcl} E'(t) & = & 2\displaystyle\int_1^{\beta(t)} \rho\rho_t[u_{rt}^\varepsilon]^2\,dr + 2\int_1^{\beta(t)} \rho^2 u_{rt}^\varepsilon u_{rtt}^\varepsilon\,dr \\ & = & 2\displaystyle\int_1^{\beta(t)} \rho\rho_t[u_{rt}^\varepsilon]^2\,dr - 2\int_1^{\beta(t)} (\rho^2 u_{rt}^\varepsilon)_r u_{tt}^\varepsilon\,dr, \end{array}$$

where we neglected the boundary terms because $u_{rt} = 0$ when $r = 1$, and $\rho = 0$ when $r = \beta(t)$. Computing the derivatives in the right-hand side we end up with

$$\begin{aligned}
E'(t) &= 2\int_1^{\beta(t)} \rho\rho_t [u^\varepsilon_{rt}]^2\,dr \\
&\quad -4\int_1^{\beta(t)} \rho\rho_r \varphi'''(u^\varepsilon_r)[u^\varepsilon_{rt}]^2 u^\varepsilon_{rr}\,dr - 4\int_1^{\beta(t)} \rho\rho_r \varphi''(u^\varepsilon_r) u^\varepsilon_{rt} u^\varepsilon_{rrt}\,dr \\
&\quad -4\int_1^{\beta(t)} \rho\rho_r \frac{\varphi''(u^\varepsilon_r)}{r}[u^\varepsilon_{rt}]^2\,dr - 2\int_1^{\beta(t)} \rho^2 \varphi'''(u^\varepsilon_r) u^\varepsilon_{rt} u^\varepsilon_{rr} u^\varepsilon_{rrt}\,dr \\
&\quad -2\int_1^{\beta(t)} \rho^2 \varphi''(u^\varepsilon_r)[u^\varepsilon_{rrt}]^2\,dr - 2\int_1^{\beta(t)} \rho^2 \frac{\varphi''(u^\varepsilon_r)}{r} u^\varepsilon_{rt} u^\varepsilon_{rrt}\,dr \\
&=: I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t) + I_7(t).
\end{aligned}$$

Let us estimate separately the seven terms. From now on all constants depend on δ, but are still independent on ε.

First of all we have that $I_4(t) \leq c_8 G(t)$ and $I_6(t) = -2F(t)$.

Due to the strict parabolicity in the interior and (3.17) we have that

$$I_1(t) \leq \int_1^{\beta(t)-\delta/2} |\rho_t| \cdot [u^\varepsilon_{rt}]^2\,dr \leq c_9 \int_1^{\beta(t)} \varphi''(u^\varepsilon_r)[u^\varepsilon_{rt}]^2\,dr = c_9 G(t),$$

$$I_2(t) \leq c_{10} \int_1^{\beta(t)-\delta/2} |\varphi'''(u^\varepsilon_r)| \cdot |u^\varepsilon_{rr}| \cdot [u^\varepsilon_{rt}]^2\,dr \leq c_{11} G(t).$$

Moreover from inequality $2ab \leq \nu a^2 + \nu^{-1} b^2$ we deduce that

$$I_3(t) \leq c_{12} \int_1^{\beta(t)} \sqrt{\rho^2 \varphi''(u^\varepsilon_r)[u^\varepsilon_{rrt}]^2} \cdot \sqrt{\varphi''(u^\varepsilon_r)[u^\varepsilon_{rt}]^2}\,dr \leq \frac{1}{3}F(t) + c_{13} G(t),$$

and similarly

$$I_7(t) \leq \frac{1}{3}F(t) + c_{14} G(t).$$

In the same way, exploiting once again the strict parabolicity in the interior, and estimate (3.17), we have that

$$I_5 \leq c_{15} \int_1^{\beta(t)} \rho^2 \varphi''(u^\varepsilon_r) |u^\varepsilon_{rt}| \cdot |u^\varepsilon_{rrt}|\,dr \leq \frac{1}{3}F(t) + c_{16} G(t).$$

Putting all together we obtain that $E'(t) \leq -F(t) + c_{17} G(t)$, hence

$$E(t) + \int_0^t F(s)\,ds \leq E(0) + c_{17} \int_0^t G(s)\,ds.$$

The term $E(0)$ depends on the initial condition only. The integral of $G(t)$ is uniformly bounded because of (3.19). It follows that the left-hand side is bounded independently on ε. Thanks to the properties of $\rho(r,t)$, this proves (3.20) and (3.22). Finally, from (4.15) we have that

$$\varphi''(u_r^\varepsilon)|u_{rrr}^\varepsilon| \leq |u_{rt}^\varepsilon| + |\varphi'''(u_r^\varepsilon)| \cdot |u_{rr}^\varepsilon|^2 + \frac{\varphi''(u_r^\varepsilon)}{r}|u_{rr}^\varepsilon| + \frac{\varphi'(u_r^\varepsilon)}{r^2}.$$

From (3.17) and the uniform parabolicity in the interior it follows that

$$\int_1^{\beta(t)-\delta} |u_{rrr}^\varepsilon|^2\, dr \leq c_{18} + c_{19} \int_1^{\beta(t)-\delta} |u_{rt}^\varepsilon|^2\, dr,$$

which proves (3.21).

4.2 Proof of Theorem 3.3

Let us briefly sketch the outline of the proof, which is quite similar to the proof of Theorem 3.1. First of all we show that the initial boundary value problem has a unique solution $u^\varepsilon \in C^{2,1}(\mathcal{T}_\varepsilon) \cap C^\infty(\mathcal{T}_\varepsilon \cap \mathrm{Int}_P(\mathcal{T}))$. Then we set $v =: u_r^\varepsilon$, and $w := u_{rr}^\varepsilon$. Both v and w belong to $C^0(\mathcal{T}_\varepsilon) \cap C^\infty(\mathcal{T}_\varepsilon \cap \mathrm{Int}_P(\mathcal{T}))$. It is easy to see that v is the solution of the following equation

$$v_t = -\varphi''(v)v_{rr} - \varphi'''(v)v_r^2 - \frac{\varphi''(v)}{r}v_r + \frac{\varphi'(v)}{r^2} \qquad \forall (r,t) \in \mathrm{Int}_P(\mathcal{T}_\varepsilon), \tag{4.50}$$

with Dirichlet boundary conditions

$$v(\beta(t_0-t),t) = v(\gamma(t_0-t),t) = 1+\varepsilon \qquad \forall t \in [\varepsilon, t_0], \tag{4.51}$$

and initial datum

$$v(r,\varepsilon) = 1+\varepsilon \qquad \forall r \in [\beta(t_0-\varepsilon), \gamma(t_0-\varepsilon)]. \tag{4.52}$$

In the same way w is a solution in $\mathrm{Int}_P(\mathcal{T}_\varepsilon)$ of equation (once again the terms in v are thought as coefficients)

$$\begin{aligned} w_t \;=\; & -\varphi''(v)w_{rr} - 3\varphi'''(v)w_r w - \varphi^{IV}(v)w^3 \\ & -\frac{\varphi'''(v)}{r}w^2 - \frac{\varphi''(v)}{r}w_r + 2\frac{\varphi''(v)}{r^2}w - 2\frac{\varphi'(v)}{r^3}, \end{aligned} \tag{4.53}$$

with Dirichlet boundary conditions

$$w(\beta(t_0-t),t) = u_{rr}^\varepsilon(\beta(t_0-t),t) \qquad \forall t \in [\varepsilon, t_0], \tag{4.54}$$

$$w(\gamma(t_0 - t), t) = u^{\varepsilon}_{rr}(\gamma(t_0 - t), t) \qquad \forall t \in [\varepsilon, t_0], \tag{4.55}$$

and initial datum

$$w(r, \varepsilon) = 0 \qquad \forall r \in [\beta(t_0 - \varepsilon), \gamma(t_0 - \varepsilon)]. \tag{4.56}$$

Then we show that v and w satisfy four sets of inequalities in $\mathcal{T}_\varepsilon$.

- The first pair of inequalities is

$$1 + \varepsilon \leq v(r, t) \leq 2 + \varphi'(1)t. \tag{4.57}$$

  From the second inequality in (3.4) we have in particular that $t_0 \leq 1/\varphi'(1)$. Therefore (4.57) implies (3.26).

- The second inequality is that

$$v(r, t) \geq 1 + \eta \left(\frac{t}{t_0} - (r - 3)^2 \right) \tag{4.58}$$

  for a suitable constant $\eta > 0$. The term after η is positive in $\mathrm{Int}_P(\mathcal{T})$. This implies (3.27), hence also (3.28).

- The third pair of inequalities is

$$v(r, t) \geq 1 + \varepsilon + \left(b(t_0 - t) - \sqrt{\varepsilon}\right)(r - \beta(t_0 - t)) - \gamma_0 \left(r - \beta(t_0 - t)\right)^2, \tag{4.59}$$

$$v(r, t) \leq 1 + \varepsilon + \left(b(t_0 - t) + \sqrt{\varepsilon}\right)(r - \beta(t_0 - t)) + \gamma_0 \left(r - \beta(t_0 - t)\right)^2, \tag{4.60}$$

  where γ_0 is the constant defined in (3.1). Arguing as in the proof of Theorem 3.1, these inequalities yield (3.29).

  In an analogous way we have that

$$v(r, t) \geq 1 + \varepsilon + \left(c(t_0 - t) + \sqrt{\varepsilon}\right)(r - \gamma(t_0 - t)) - \gamma_0 \left(r - \gamma(t_0 - t)\right)^2, \tag{4.61}$$

$$v(r, t) \leq 1 + \varepsilon + \left(c(t_0 - t) - \sqrt{\varepsilon}\right)(r - \gamma(t_0 - t)) + \gamma_0 \left(r - \gamma(t_0 - t)\right)^2, \tag{4.62}$$

  which imply (3.30).

- Thanks to (3.29) and (3.30) we have an estimate on the values of w at the boundary. This is the starting point to prove the fourth set of inequalities

$$b(t_0-t) - \sqrt{\varepsilon} - \gamma_1(r - \beta(t_0-t)) \leq w(r, t) \leq b(t_0-t) + \sqrt{\varepsilon} + \gamma_1(r - \beta(t_0-t)), \tag{4.63}$$

$$c(t_0-t) - \sqrt{\varepsilon} - \gamma_1(\gamma(t_0-t) - r) \leq w(r, t) \leq c(t_0-t) + \sqrt{\varepsilon} + \gamma_1(\gamma(t_0-t) - r), \tag{4.64}$$

  where γ_1 is the constant defined in (3.2). These imply (3.31) and (3.32).

We are now ready to proceed with the details. Many steps of the proof (for example the integral estimates) are analogous to the corresponding steps in the proof of Theorem 3.1. In these cases we skip them, focussing only on what is different. We also skip the proofs of (4.61), (4.62), (4.64), which are analogous to the proofs of (4.59), (4.60), and (4.63), respectively.

Existence and maximum principle for space derivatives Let us take a function $\varphi_\varepsilon \in C^\infty(\mathbb{R})$ which coincides with φ in the interval $[1+\varepsilon, 3]$, and such that $\varphi_\varepsilon''(\sigma) \leq -\nu_\varepsilon < 0$ for every $\sigma \in \mathbb{R}$. Equation (3.24), with φ_ε instead of φ, is strictly forward parabolic. Therefore by well know arguments the initial boundary value problem admits a unique solution $u^\varepsilon \in C^{2,1}(\mathcal{T}_\varepsilon) \cap C^\infty(\mathcal{T}_\varepsilon \setminus \partial_P(\mathcal{T}))$.

It is easy to see that $z(r,t) := 1+\varepsilon$ is a subsolution of problem (4.50) through (4.52), both with φ_ε and with φ. This proves the lower bound in (3.26).

We claim that $z(r,t) := 2+\varphi'(1)t$ is a supersolution of (4.50) through (4.52). Indeed on the three sides of the parabolic boundary of $\mathcal{T}_\varepsilon$ we have that

$$z(r,t) \geq 2 \geq 1+\varepsilon = v(r,t).$$

In the interior we have that $z_r(r,t) = z_{rr}(r,t) = 0$, hence (since $1 \leq z(r,t) \leq 3$)

$$z_t(r,t) = \varphi'(1) \geq \varphi'(z(r,t)) \geq \frac{\varphi'(z(r,t))}{r},$$

which is exactly the required differential inequality (both with φ_ε and with φ).

This completes the proof of (4.57), hence of (3.26).

Uniform strict parabolicity in the interior Let us choose $\eta > 0$ small enough so that

$$\eta \leq t_0 \ (< 1), \qquad \eta \leq \left(\frac{1}{t_0} + 20\gamma_2\right)^{-1} \varphi'(3). \tag{4.65}$$

Let $z(r,t)$ denote the right-hand side of (4.58). We claim that, under these restrictions on η, z is a subsolution of (4.50) through (4.52), which implies (4.58).

Parabolic boundary When $r = \beta(t_0 - t)$ or $r = \gamma(t_0 - t)$ we have that

$$z(\beta(t_0-t),t) = z(\gamma(t_0-t),t) = 1 \leq 1+\varepsilon = v(\beta(t_0-t),t) = v(\gamma(t_0-t),t).$$

Since $\eta \leq t_0$, when $t = \varepsilon$ we have that

$$z(r,\varepsilon) = 1 + \eta\left(\frac{\varepsilon}{t_0} - (r-3)^2\right) \leq 1 + \frac{\eta\varepsilon}{t_0} \leq 1+\varepsilon = v(r,\varepsilon).$$

Differential inequality We have that

$$z_r(r,t) = -2(r-3)\eta, \qquad z_{rr}(r,t) = -2\eta, \qquad z_t(r,t) = \eta/t_0.$$

In order to show that z is a subsolution we have to prove that

$$\frac{\eta}{t_0} \leq 2\eta\varphi''(z) - 4\eta^2\varphi'''(z)(r-3)^2 + 2\eta\frac{\varphi''(z)}{r}(r-3) + \frac{\varphi'(z)}{r^2}.$$

It is easy to show that $1 \leq z(r,t) \leq 3$ for every $(r,t) \in \mathcal{T}_\varepsilon$. Therefore from the properties of φ' we have that $\varphi'(z)/r^2 \geq (1/16)\varphi'(3)$. All the other terms are uniformly small when η is small. This shows that the differential inequality is satisfied when η is small enough, for example as soon as η fulfils the last inequality in (4.65).

Estimate on second derivatives at the moving boundary – Subsolution Let us prove (4.59). To this end we denote its right-hand side by $z(r,t)$, and we show that it is a subsolution of (4.50) through (4.52).

Boundary $r = \beta(t_0 - t)$ In this case we have that

$$z(\beta(t_0-t),t) = 1+\varepsilon = v(\beta(t_0-t),t).$$

Boundary $r = \gamma(t_0 - t)$ From the explicit expressions (2.7) we have that

$$\begin{aligned} z(\gamma(t_0-t),t) &= 1+\varepsilon+\left(b(t_0-t)-\sqrt{\varepsilon}\right)\cdot 2\frac{\sqrt{t}}{\sqrt{t_0}} - 4\gamma_0\frac{t}{t_0} \\ &\leq 1+\varepsilon+2\frac{\sqrt{t}}{\sqrt{t_0}}\left(b(t_0-t)-2\gamma_0\frac{\sqrt{t}}{\sqrt{t_0}}\right). \end{aligned} \tag{4.66}$$

Exploiting estimate (4.13), and the fact that $\gamma_0 \geq 1/2$ and $t_0 \leq 1$, we have that

$$b(t_0-t) \leq \sqrt{t} \leq 2\gamma_0\sqrt{t} \leq 2\gamma_0\frac{\sqrt{t}}{\sqrt{t_0}}. \tag{4.67}$$

From (4.66) and (4.67) we conclude that $z(\gamma(t_0-t),t) \leq 1+\varepsilon = v(\gamma(t_0-t),t)$.

Boundary $t = \varepsilon$ From (4.13) we have that $b(t_0-\varepsilon) \leq \sqrt{\varepsilon}$, hence

$$\begin{aligned} z(r,\varepsilon) &= 1+\varepsilon+\left(b(t_0-\varepsilon)-\sqrt{\varepsilon}\right)(r-\beta(t_0-\varepsilon)) - \gamma_0\left(r-\beta(t_0-\varepsilon)\right)^2 \\ &\leq 1+\varepsilon = v(r,\varepsilon). \end{aligned}$$

Note that the term with $\sqrt{\varepsilon}$ instead of ε in the right-hand side of (4.59) is essential in this point of the proof.

Differential inequality Let us set for simplicity $x := r - \beta(t_0-t)$. Then we have that

$$z_r(r,t) = b(t_0-t) - \sqrt{\varepsilon} - 2\gamma_0 x, \qquad z_{rr}(r,t) = -2\gamma_0,$$
$$z_t(r,t) = b(t_0-t)\beta'(t_0-t) - \beta'(t_0-t)\sqrt{\varepsilon} - (b'(t_0-t) + 2\gamma_0\beta'(t_0-t))x.$$

Plugging these expressions in (4.50), and exploiting the equation defining $b(t_0 - t)$ as we did in (4.36), we end up with the following differential inequality

$$\begin{aligned}
\beta'(t_0 - t)\sqrt{\varepsilon} + (b'(t_0 - t) + 2\gamma_0\beta'(t_0 - t))x \ &\geq \\
\geq \ & -2\gamma_0\varphi''(z) \\
& +\varphi'''(z)\left(b(t_0 - t) - \sqrt{\varepsilon} - 2\gamma_0 x\right)^2 - \varphi'''(1)b^2(t_0 - t) \\
& +\frac{\varphi''(z)}{r}\left(b(t_0 - t) - \sqrt{\varepsilon} - 2\gamma_0 x\right) \\
& +\frac{\varphi'(1)}{\beta^2(t_0 - t)} - \frac{\varphi'(z)}{r^2}.
\end{aligned}$$

This inequality has to be proved for all $(r, t) \in \mathcal{T}_\varepsilon$ such that $1 \leq z(r, t) \leq 3$. This can be done arguing as we did in the corresponding step of the proof of Theorem 3.1 (and using from time to time that $\sqrt{\varepsilon} \geq \varepsilon$).

Estimate on second derivatives at the moving boundary – Supersolution In order to prove (4.60) it is enough to show that its right-hand side is a supersolution of (4.50) through (4.52). In this case the inequalities on the three pieces of the parabolic boundary of $\mathcal{T}_\varepsilon$ are trivial. The proof of the differential inequality is analogous to the case of the supersolution. We skip the details.

Global estimate on second derivatives – Subsolution Let $z(r, t)$ denote the left-hand side of (4.63). We claim that it is a subsolution of (4.53) through (4.56).

Boundary $r = \beta(t_0 - t)$ From (3.29) we have that

$$z(\beta(t_0 - t), t) = b(t_0 - t) - \sqrt{\varepsilon} \leq u^\varepsilon_{rr}(\beta(t_0 - t), t) = w(\beta(t_0 - t), t).$$

Boundary $r = \gamma(t_0 - t)$ Exploiting estimates (4.13), (4.14), and (3.30), and the fact that $\gamma_1 \geq 1$ and $t_0 \leq 1$, we have that

$$z(\gamma(t_0 - t), t) = b(t_0 - t) - \sqrt{\varepsilon} - 2\gamma_1\frac{\sqrt{t}}{\sqrt{t_0}} \leq \sqrt{t} - \sqrt{\varepsilon} - 2\gamma_1\frac{\sqrt{t}}{\sqrt{t_0}} \leq -\sqrt{t} - \sqrt{\varepsilon} \leq$$

$$\leq c(t_0 - t) - \sqrt{\varepsilon} \leq u^\varepsilon_{rr}(\gamma(t_0 - t), t) = w(\gamma(t_0 - t), t).$$

Boundary $t = \varepsilon$ From (4.13) we have that $b(t_0 - \varepsilon) \leq \sqrt{\varepsilon}$, hence

$$z(r, \varepsilon) = b(t_0 - \varepsilon) - \sqrt{\varepsilon} - \gamma_1(r - \beta(t_0 - \varepsilon)) \leq 0 = w(r, \varepsilon).$$

Differential inequality After computing the derivatives and changing the signs we have to prove that

$$b'(t_0-t)+\gamma_1\beta'(t_0-t) \geq -3\gamma_1\varphi'''(v)z+\varphi^{IV}(v)z^3+\frac{\varphi'''(v)}{r}z^2-\gamma_1\frac{\varphi''(v)}{r}-2\frac{\varphi''(v)}{r^2}z+2\frac{\varphi'(v)}{r^3}.$$

Note that this is exactly the same inequality satisfied by supersolutions in $\mathcal{Q}_1$ (since we reversed the time, subsolutions correspond to supersolutions and vice versa). The proof is of course completely analogous.

Global estimate on second derivatives – Supersolution Let $z(r,t)$ denote the right-hand side of (4.63). We claim that z is a supersolution of (4.53) through (4.56).

Boundary $r=\beta(t_0-t)$ From (3.29) we have that

$$z(\beta(t_0-t),t)=b(t_0-t)+\sqrt{\varepsilon}\geq u^{\varepsilon}_{rr}(\beta(t_0-t),t)=w(\beta(t_0-t),t).$$

Boundary $r=\gamma(t_0-t)$ We have that

$$z(\gamma(t_0-t),t)=b(t_0-t)+\sqrt{\varepsilon}+2\gamma_1\frac{\sqrt{t}}{\sqrt{t_0}}\geq\sqrt{\varepsilon}.$$

Moreover, from (3.30) and the fact that $c(t_0-t)\leq 0$, we have that

$$w(\gamma(t_0-t),t)=u^{\varepsilon}_{rr}(\gamma(t_0-t),t)\leq c(t_0-t)+\sqrt{\varepsilon}\leq\sqrt{\varepsilon}.$$

It follows that $z(\gamma(t_0-t),t)\geq w(\gamma(t_0-t),t)$.

Boundary $t=\varepsilon$ We have trivially that $z(r,\varepsilon)\geq 0=w(r,\varepsilon)$.

Differential inequality We have to prove the same inequality satisfied by the corresponding subsolution in $\mathcal{Q}_1$. The proof is the same.

5 Passing to the limit

The basic tool is the following compactness result. We omit the proof, for which we refer to [10, Lemma 3.1]. The key point in that (5.1) through (5.3) yield a uniform bound on the norm of f^{ε} in the Hölder space $C^{1/2,1/4}([r_1,r_2]\times[t_1,t_2])$.

Lemma 5.1 *Let $[r_1, r_2] \times [t_1, t_2]$ be a rectangle, and let $\varepsilon_0 > 0$. For every $\varepsilon \in (0, \varepsilon_0)$, let $f^\varepsilon \in C^1([r_1, r_2] \times [t_1, t_2])$. Let us assume that there exists $M \in \mathbb{R}$ such that*

$$
\begin{aligned}
|f^\varepsilon(r_1, t_1)| &\leq M \qquad \forall \varepsilon \in (0, \varepsilon_0), & (5.1)\\
\int_{r_1}^{r_2} [f_r^\varepsilon(r,t)]^2 \, dr &\leq M \qquad \forall t \in [t_1, t_2],\ \forall \varepsilon \in (0, \varepsilon_0), & (5.2)\\
\int_{t_1}^{t_2} \int_{r_1}^{r_2} [f_t^\varepsilon(r,t)]^2 \, dr \, dt &\leq M \qquad \forall \varepsilon \in (0, \varepsilon_0). & (5.3)
\end{aligned}
$$

Then the family $\{f^\varepsilon\}$ is relatively compact in $C^0([r_1, r_2] \times [t_1, t_2])$. $\Box$

5.1 Proof of Theorem 2.1

Let t_0 and u_0 be as in Theorem 3.1, and let $\{u^\varepsilon\}_{\varepsilon \in (0,1)}$ be the family of solutions produced by that theorem. We claim that u^ε uniformly converges, as $\varepsilon \to 0^+$, to a limit u, which satisfies the initial condition

$$u(r, 0) = u_0(r) \qquad \forall r \in [1, 2] \tag{5.4}$$

and all the conditions required in Theorem 2.1.

Uniqueness Let us assume that a function $u \in C^{2,1}(\mathcal{Q}_1)$ satisfies equation (2.6) for every $(r,t) \in \mathcal{Q}_1$, the Neumann boundary condition (1.5) for every $t \in [0, t_0]$, the Neumann boundary condition (2.16), and the initial condition (5.4). Let us assume also that $0 \leq u_r(r,t) \leq 1$ for every $(r,t) \in \mathcal{Q}_1$. Under this condition equation (2.6) is degenerate parabolic, hence the solution u is unique.

This shows that the limit problem provides a unique characterization of the possible limits. Therefore in what follows we can limit ourselves to show that u^ε converges to the solution of this problem up to subsequences.

Convergence of u^ε Estimates (3.11) and (3.18) provide a uniform bound on the Lipschitz constant of u^ε. Moreover the functions u^ε are equi-bounded due to the initial condition. From the classical Ascoli's Theorem it follows that (up to subsequences, which we don't relabel) u^ε uniformly converges in $\mathcal{Q}_1$ to a continuous function u, satisfying of course the initial condition (5.4).

Convergence of u_r^ε Let us consider any rectangle $[r_1, r_2] \times [t_1, t_2] \subseteq \mathcal{Q}_1 \setminus \Gamma_1$, and let us apply Lemma 5.1 in this rectangle with $f^\varepsilon = u_r^\varepsilon$. In the rectangle assumption (5.1) follows follows from (3.11), assumption (5.2) follows from (3.17), and assumption (5.3) follows from (3.19) and the strict parabolicity in the interior. We obtain that

$$u_r^\varepsilon \to u_r \qquad \text{uniformly in } [r_1, r_2] \times [t_1, t_2].$$

Taking the union over all such rectangles, we easily conclude that

$$u_r^\varepsilon(r,t) \to u_r(r,t) \qquad \forall (r,t) \in \mathcal{Q}_1 \setminus \Gamma_1.$$

We claim that u_r can be continuously extended to the whole $\mathcal{Q}_1$ by setting $u_r(r,t) = 1$ for every $(r,t) \in \Gamma_1$. In order to prove the continuity of this extension, let us consider $(r_1,t_1) = (\beta(t_1),t_1) \in \Gamma_1$ and $(r_2,t_2) \in \mathcal{Q}_1 \setminus \Gamma_1$. From (3.17) we have that

$$|u_r^\varepsilon(r,t) - (1+\varepsilon)| = |u_r^\varepsilon(r,t) - u_r^\varepsilon(\beta(t),t)| \le M_4|r - \beta(t)|.$$

Letting $\varepsilon \to 0^+$ we obtain that

$$|u_r(r,t) - 1| \le M_4|r-\beta(t)| \qquad \forall (r,t) \in \mathcal{Q}_1 \setminus \Gamma_1.$$

Therefore we have that

$$\begin{aligned} |u_r(r_2,t_2) - u_r(r_1,t_1)| &= |u_r(r_2,t_2) - 1| \\ &\le M_4|r_2 - \beta(t_2)| \\ &\le M_4\left(|r_2 - r_1| + |\beta(t_1) - \beta(t_2)|\right). \end{aligned}$$

Due to the continuity of $\beta(t)$, the right-hand side is small when (r_2,t_2) is close to (r_1,t_1). This proves the continuity of u_r up to Γ_1. Of course u_r satisfies the Neumann boundary conditions in $r=1$ and in $r=\beta(t)$.

Convergence of u_{rr}^ε Let us consider any rectangle $[r_1,r_2]\times[t_1,t_2] \subseteq \mathcal{Q}_1 \setminus \Gamma_1$, and let us apply Lemma 5.1 in this rectangle with $f^\varepsilon = u_{rr}^\varepsilon$. In the rectangle assumption (5.1) follows follows from (3.17), assumption (5.2) follows from (3.21), and assumption (5.3) follows from (3.22). We obtain that $u_{rr}^\varepsilon \to u_{rr}$ uniformly in $[r_1,r_2]\times[t_1,t_2]$.

Taking the union over all such rectangles, we easily conclude that

$$u_{rr}^\varepsilon(r,t) \to u_{rr}(r,t) \qquad \forall (r,t) \in \mathcal{Q}_1 \setminus \Gamma_1.$$

Let us extend u_{rr} to the whole $\mathcal{Q}_1$ by setting $u_{rr}(r,t) = b(t)$ for every $(r,t) \in \Gamma_1$. In order to prove the continuity of this extension, let us consider $(r_1,t_1) = (\beta(t_1),t_1) \in \Gamma_1$ and $(r_2,t_2) \in \mathcal{Q}_1 \setminus \Gamma_1$. Passing (3.16) to the limit we obtain that

$$|u_{rr}(r,t) - b(t)| \le M_3|r-\beta(t)| \qquad \forall (r,t) \in \mathcal{Q}_1 \setminus \Gamma_1,$$

hence

$$\begin{aligned} |u_{rr}(r_2,t_2) - u_{rr}(r_1,t_1)| &= |u_{rr}(r_2,t_2) - b(t_1)| \\ &\le |u_{rr}(r_2,t_2) - b(t_2)| + |b(t_2) - b(t_1)| \\ &\le M_3|r_2 - \beta(t_2)| + |b(t_2) - b(t_1)| \\ &\le M_3\left(|r_2 - r_1| + |\beta(t_2) - \beta(t_1)|\right) + |b(t_2) - b(t_1)|. \end{aligned}$$

The conclusion easily follows from the continuity of $\beta(t)$ and $b(t)$.

Convergence of u_t^ε Since u_t^ε is related to u_r^ε and u_{rr}^ε by (2.6), the convergence of u_t^ε to a continuous function in $\mathcal{Q}_1$ follows from the convergence of u_r^ε and u_{rr}^ε and the continuity of their limits. The limit of u_t^ε is of course u_t. This completes the proof that u is of class $C^{2,1}$ and solves equation (2.6).

5.2 Proof of Theorem 2.2

We have already seen in section 3.2 that Theorem 2.2 is equivalent to Theorem 3.2. In turn, the proof of Theorem 3.2 follows by passing to the limit the solutions u^ε provided by Theorem 3.3.

The argument is quite similar to the proof of Theorem 3.1. Thanks to some of the estimates of Theorem 3.3 we can apply Lemma 5.1 and pass to the limit in any rectangle $[r_1, r_2] \times [t_1, t_2] \subseteq \mathrm{Int}_P(\mathcal{T})$, where u^ε is defined for every ε small enough. Thanks to the remaining estimates of Theorem 3.3 we can show that the limit u and its derivatives u_r, u_{rr}, u_t can be continuously extended up to the boundary. We skip the details.

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