

ON THE DIRICHLET PROBLEM FOR NON-TOTALLY DEGENERATE FULLY NONLINEAR ELLIPTIC EQUATIONS

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ABSTRACT. We prove some comparison principles for viscosity solutions of fully nonlinear degenerate elliptic equations that satisfy conditions of partial non-degeneracy instead of the usual uniform ellipticity or strict monotonicity. These results are applied to the well-posedness of the Dirichlet problem under suitable conditions at the characteristic points of the boundary. The examples motivating the theory are operators of the form of sum of squares of vector fields plus a nonlinear first order Hamiltonian and the Pucci operator over the Heisenberg group.

1. Introduction. In this paper we study the comparison, uniqueness, and existence of viscosity solutions to the Dirichlet problem for some second order, fully nonlinear equations

$$F(x, u, Du, D^2u) = 0, \quad \text{in } \Omega, \quad (1)$$

where Ω is an open bounded domain of \mathbb{R}^n and $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S^n \rightarrow \mathbb{R}$ satisfies the standard continuity assumptions of [13] (S^n denotes the set of $n \times n$ symmetric matrices).

The classical comparison principle between viscosity sub- and supersolutions was obtained by Jensen [24, 25] and by Ishii-Lions [23] when either (i) F is degenerate elliptic and strictly increasing in u , or (ii) F is uniformly elliptic and nondecreasing in u .

The goal of this paper is to weaken the uniform ellipticity to various forms of partial nondegeneracy. Let us illustrate them in the case the PDE has no 0-th order terms, that is, $F = F(x, p, M)$ is independent of u . Recall that (1) is uniformly elliptic if there is a constant $\eta > 0$ such that

$$F(x, p, M + N) \leq F(x, p, M) - \eta \text{tr} N, \quad \forall N \geq 0.$$

We say the equation is *non-totally degenerate* if for some $\eta > 0$

$$F(x, p, M + rI) \leq F(x, p, M) - \eta r, \quad \forall r > 0,$$

where I is the identity matrix. This minimal nondegeneracy assumption is enough for proving the comparison principle if F is Lipschitz in p with small Lipschitz

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constant compared to η (e.g., $F = F(x, M)$). On the other hand, the smallness condition on the Lipschitz constant can be dropped if F is *nondegenerate in a fixed direction*, that is, for some $j \in \{1, \dots, n\}$,

$$F(x, p, M + rD^j) \leq F(x, p, M) - \eta r, \quad \forall r > 0,$$

where D^j is the diagonal matrix whose only non-null element is $D_{jj}^j = 1$. We also prove a comparison principle for equations of the form

$$G(x, \sigma^T(x)D(u), \sigma^T(x)D^2u\sigma(x)) = 0, \quad (2)$$

where $\sigma(\cdot)$ is an $n \times m$ matrix-valued Lipschitz function with $\text{tr}(\sigma^T\sigma) \geq \eta$ and G is uniformly elliptic on $\bar{\Omega} \times \mathbb{R}^m \times S^m$. In Sections 2 and 3 the three comparison principles are stated and proved in the general case of F depending on u in a nondecreasing way, and the nondegeneracy assumptions are made only at the points where F fails to be strictly increasing in u . The proofs are based on the observation in [23, 13] that the strict monotonicity of F with respect to u is not needed if any subsolution can be perturbed to a strict subsolution.

The nondegeneracy conditions are better understood if one checks them in the special case of quasilinear equations of the form

$$-\text{tr}(A(x)D^2u) + H(x, Du) = 0,$$

with $A(x) \geq 0$. Here uniform ellipticity means that the minimal eigenvalue of A is bounded away from 0, whereas non-total degeneracy says that the maximal eigenvalue is bounded away from 0, and non-degeneracy in a fixed direction states that, for some j , $a_{jj}(x) > 0$ for all $x \in \bar{\Omega}$.

Our motivations are the Hamilton-Jacobi-Bellman and Isaacs equations and the subelliptic nonlinear equations. In the theory of stochastic control and differential games the uniform ellipticity of the Bellman-Isaacs equation means that the whole system is disturbed by a nondegenerate noise, whereas in practice only some components are in fact affected by noise (e.g., acceleration but not velocity, see the example on Kolmogorov equations in Section 4). On the other hand, the strict monotonicity in u in the Bellman-Isaacs equation holds if the cost functional is an integral containing a discount factor with nonvanishing discount rate, which is not the case in many interesting problems, such as time-optimal control. We refer to [29, 17] for more informations on this matter.

In the theory of subelliptic equations one is given a family of smooth vector fields X_1, \dots, X_m and looks at equations of the form

$$G(x, u, D_{\mathcal{X}}u, (D_{\mathcal{X}}^2u)^*) = 0, \quad (3)$$

where $D_{\mathcal{X}}u = (X_1u, \dots, X_mu)$ is the intrinsic (or horizontal) gradient, $(D_{\mathcal{X}}^2u)_{ij} = X_i(X_ju)$ is the intrinsic Hessian, $(\cdot)^*$ denotes the symmetrized matrix, and G is uniformly elliptic on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^m \times S^m$. If we take the matrix σ whose columns are the coefficients of X_1, \dots, X_m , we see that, for any smooth u

$$D_{\mathcal{X}}u = \sigma^T(x)Du, \quad (D_{\mathcal{X}}^2u)^* = \sigma^T(x)D^2u\sigma(x) + 1\text{-st order terms},$$

and the 1-st order terms are null in some interesting cases, such as the Heisenberg group. This motivates the study of equations of the form (2). In a sequel of this paper we study the Dirichlet problem for general fully nonlinear subelliptic equations of the form (3). Here we limit ourselves to two classes of examples. The

first is the quasilinear equation

$$-\sum_{i,j=1}^m a_{ij}(x)X_iX_ju + H(x, Du) = 0 \quad (4)$$

with the matrix of the coefficients a_{ij} uniformly positive definite. The second is the fully nonlinear equation

$$\mathcal{P}^+(\sigma^T D^2 u \sigma) + H(x, Du) = 0, \quad (5)$$

where \mathcal{P}^+ is the Pucci operator on S^m (see Example 4.5 for the definition). Both of them are non-totally degenerate for any family of vector fields satisfying the classical Hörmander condition [22]. They are nondegenerate in a fixed direction if X_1, \dots, X_m are generators of a Carnot group [18]. In this last case, for instance, we get the comparison principle for both (4) and (5) if the Hamiltonian H is Lipschitz continuous in p and such that, for some modulus ω ,

$$|H(x, p) - H(y, p)| \leq \omega(|x - y|(1 + |p|)).$$

Nonlinear subelliptic equations in variational form were considered since the early 1990s [34]. The study of the fully nonlinear case began recently, see the lecture notes of Manfredi [30], the papers [7, 8, 35, 14], and the references therein.

In Sections 5, 6, and 7 we study the existence of solutions by the Perron-Wiener-Brelot method. A generalized (possibly discontinuous) viscosity solution is obtained under general conditions, whereas for the existence of a continuous solution one has to analyze the operator near the characteristic points of the boundary $\partial\Omega$. Here we give some simple results by constructing explicit barriers. Two examples where we obtain the well-posedness of the Dirichlet problem are

$$\begin{cases} -\Delta_{\mathcal{H}}u + H(x, Du) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_{\mathcal{H}}$ is the Heisenberg Laplacian, and

$$\begin{cases} \mathcal{P}^+((D_{\mathcal{H}}^2 u)^*) + H(x, Du) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $D_{\mathcal{H}}^2 u$ is the Heisenberg Hessian. In both cases H is required to satisfy the conditions above, $H(x, 0) \leq 0$, and $H(x, p) \geq -K$, whereas Ω is a domain of classical solvability for the linear problem

$$-\Delta_{\mathcal{H}}u = 0 \text{ in } \Omega, \quad u = g \text{ on } \partial\Omega, \quad g \in C(\partial\Omega),$$

e.g., any domain with C^2 boundary. The boundedness from below of H rules out equations containing linear 1-st order terms. An alternative result weakens the condition $H \geq -K$ to $H(x, p) \geq H_{hom}(x, p) - K$ with $H_{hom}(x, \cdot)$ positively 1-homogenous, by adding the following assumption at any characteristic point $z \in \partial\Omega$:

$$-\Delta_{\mathcal{H}}\Phi(z) + H_{hom}(z, D\Phi(z)) > 0,$$

where $\Phi \in C^2$ is such that $\Omega = \{x : \Phi(x) > 0\}$, $D\Phi \neq 0$ on $\partial\Omega$. The geometric meaning of this condition is illustrated in the case Ω is the Koranyi or the euclidean ball.

Comparison principles for viscosity solutions without strict monotonicity and uniform ellipticity were proved for different equations such as the infinity-Laplace equation and by different methods in [26, 6, 7, 35], see also [2] and the references

therein for first order equations. Strong maximum principles were studied in [28, 3, 4, 5].

The solvability of the Dirichlet problem and the boundary behaviour of solutions for linear subelliptic equations have a long history, see, e.g., [10, 32, 20, 19, 21, 27, 31, 33, 12]; for the semilinear case we refer to [9] and the references therein.

The paper is organized as follows. In Section 2 we list some basic assumptions. Section 3 contains the three comparison theorems. Section 4 is devoted to several examples. In Section 5 we give a general existence result by the PWB method. In Section 6 we give sufficient conditions for the continuity of the generalized solution. Section 7 contains some explicit results of well-posedness of the Dirichlet problem for quasilinear and fully nonlinear equations on the Heisenberg group.

2. Assumptions. Let us list the assumptions on F . We denote with S^n the set of the symmetric $n \times n$ matrices, with \leq the usual partial order, with I the identity matrix, and with $\text{tr}M$ the trace of a square matrix M . Throughout the paper $\Omega \subseteq \mathbb{R}^n$ is open and bounded.

$$\left\{ \begin{array}{l} F : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S^n \rightarrow \mathbb{R} \text{ is continuous;} \\ F(x, u - r, p, M + N) \leq F(x, u, p, M), \quad \forall N \geq 0, \quad \forall r \geq 0; \\ \text{for all } x, y \in \bar{\Omega}, u \in \mathbb{R}, X, Y \in S^n, \alpha > 0 \text{ such that} \\ -3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}, \\ F(y, u, \alpha(x - y), Y) - F(x, u, \alpha(x - y), X) \leq \omega(\alpha|x - y|^2 + |x - y|), \\ \text{where } \omega : [0, +\infty[\rightarrow [0, +\infty[\text{ satisfies } \omega(0+) = 0. \end{array} \right. \quad (6)$$

There exists $\eta(x) \in C(\bar{\Omega})$, $\eta \geq 0$, such that: (7)

$$\begin{aligned} F(x, u, p, M + rI) &\leq F(x, u, p, M) - \eta(x)r, \\ \forall x \in \bar{\Omega}, u \in \mathbb{R}, p \in \mathbb{R}^n, M \in S^n, r > 0. \end{aligned}$$

There exists $h(x) \in C(\bar{\Omega})$, $h \geq 0$, such that: (8)

$$\begin{aligned} F(x, u - r, p, M) &\leq F(x, u, p, M) - h(x)r, \\ \forall x \in \bar{\Omega}, u \in \mathbb{R}, p \in \mathbb{R}^n, M \in S^n, r > 0. \end{aligned}$$

There exists $K \geq \bar{\eta} > 0$ such that: $\bar{\eta} \leq h(x) + \eta(x) \leq K$, $\forall x \in \bar{\Omega}$. (9)

There exists $L \geq 0$, such that: (10)

$$\begin{aligned} |F(x, u, p + q, M) - F(x, u, p, M)| &\leq L|q|, \\ \forall x \in \bar{\Omega}, u \in \mathbb{R}, p, q \in \mathbb{R}^n, M \in S^n. \end{aligned}$$

The first assumption (6) collects the structural conditions on F that are commonly used in the viscosity theory [13].

The next condition (7) means that equation (1) is *non-totally degenerate* at the points where $\eta(x) > 0$. In the quasilinear case $F(x, u, p, M) = -\text{tr}(A(x)M) + H(x, u, p)$, $A \in S^n$ is positive semidefinite by (6), and condition (7) is equivalent to $\text{tr}A(x) \geq \eta(x)$, so at the points where $\eta(x) > 0$ there exists at least one positive eigenvalue of $A(x)$. In this case, (7) coincides with the definition of non-total degeneracy given by Bony in [10] (there exists an element a_{ij} of the matrix A such that $a_{ij}(x) \neq 0$).

The third condition (8) says that F is strictly monotone in u at the points where $h(x) > 0$. The next assumption (9) states that the operator is either non-totally degenerate or strictly monotone everywhere in $\bar{\Omega}$ and uniformly.

Finally, condition (10) is the Lipschitz continuity of F with respect to p , uniformly in the other entries.

In the next section we prove comparison results for the equation (1) under these assumptions or variants of them.

3. Comparison principles. As usual, in the theory of viscosity solutions, we say that F satisfies the comparison principle if for any $u, v : \bar{\Omega} \rightarrow \mathbb{R}$, u upper semicontinuous and viscosity subsolution of (1), v lower semicontinuous and viscosity supersolution of (1), such that $u \leq v$ on $\partial\Omega$, we have $u \leq v$ in Ω .

The first comparison result holds under the assumptions of the previous section and if F has a weak dependence on the gradient p of the solution at points where it is not strictly increasing with respect to u .

Theorem 3.1. *Assume (6), (7), (8), (9), (10) and*

$$L|x| \leq \frac{\bar{\eta}}{2}, \quad \forall x \in \Omega \text{ such that } h(x) \leq \frac{\bar{\eta}}{4}. \quad (11)$$

where $\bar{\eta}$ is the constant appearing in (9). Then the comparison principle holds.

Proof. From standard viscosity solutions theory [23, 13], we know that, under the structural conditions 6, the comparison principle holds between a supersolution v and a strict subsolution u_ϵ , i.e., an u.s.c. function in $\bar{\Omega}$ satisfying $F(x, u_\epsilon, Du_\epsilon, D^2u_\epsilon) \leq \alpha(x)$ in Ω , with $\alpha(x) \in C(\bar{\Omega})$ and $\alpha(x) < 0$. If, in addition, $u_\epsilon(x) \rightarrow u(x)$ for all x as $\epsilon \rightarrow 0$, then we obtain $u \leq v$ in Ω . Therefore we consider, for a given viscosity subsolution u ,

$$u_\epsilon(x) := u(x) + \epsilon(e^{\frac{|x|^2}{2}} - \lambda)$$

and we want to show that it is a strict subsolution for $\lambda \gg 1$, independent of $\epsilon > 0$. First we choose $\lambda \geq e^{\frac{|x|^2}{2}}$ for all $x \in \Omega$, and this implies $u_\epsilon(x) \leq u(x)$. Moreover

$$\frac{\partial u_\epsilon}{\partial x_i} = \frac{\partial u}{\partial x_i} + \epsilon x_i e^{\frac{|x|^2}{2}}, \quad \frac{\partial^2 u_\epsilon}{\partial x_i \partial x_j} = \frac{\partial u}{\partial x_i \partial x_j} + \epsilon e^{\frac{|x|^2}{2}} (\delta_{ij} + x_i x_j),$$

hence

$$D^2 u_\epsilon = D^2 u + \epsilon e^{\frac{|x|^2}{2}} (I + x \otimes x) \geq D^2 u + \epsilon e^{\frac{|x|^2}{2}} I \geq D^2 u. \quad (12)$$

(Here δ_{ij} is the Kronecker symbol, I is the identity matrix, and $(x \otimes x)_{ij} = x_i x_j$.)

Now we fix $x \in \Omega$. We have two cases:

i) $h(x) \leq \frac{\bar{\eta}}{4}$; ii) $h(x) > \frac{\bar{\eta}}{4}$.

Case i): $h(x) \leq \frac{\bar{\eta}}{4}$. From assumption (9), we have $\eta(x) \geq \frac{3}{4}\bar{\eta}$. Since $u_\epsilon \leq u$ and F is proper

$$F(x, u_\epsilon, Du_\epsilon, D^2 u_\epsilon) \leq F(x, u, Du + \epsilon x e^{\frac{|x|^2}{2}}, D^2 u + \epsilon e^{\frac{|x|^2}{2}} I).$$

Now, by (7) and (10) we have

$$F(x, u, Du + \epsilon x e^{\frac{|x|^2}{2}}, D^2 u + \epsilon e^{\frac{|x|^2}{2}} I) \leq F(x, u, Du, D^2 u) + \epsilon e^{\frac{|x|^2}{2}} (L|x| - \eta(x)).$$

By (11) $L|x| - \eta(x) \leq \frac{\bar{\eta}}{2} - \frac{3}{4}\bar{\eta}$, so u subsolution implies

$$F(x, u_\epsilon, Du_\epsilon, D^2 u_\epsilon) \leq -\epsilon e^{\frac{|x|^2}{2}} \frac{\bar{\eta}}{4} \leq -\epsilon \frac{\bar{\eta}}{4} =: \alpha(x).$$

Case ii): $h(x) > \frac{\bar{\eta}}{4}$. From (12), since F is proper:

$$F(x, u_\epsilon, Du_\epsilon, D^2u_\epsilon) \leq F(x, u_\epsilon, Du + \epsilon x e^{\frac{|x|^2}{2}}, D^2u).$$

From (8), (9), and (10)

$$\begin{aligned} F(x, u_\epsilon, Du + \epsilon x e^{\frac{|x|^2}{2}}, D^2u) &\leq F(x, u, Du, D^2u) + \epsilon[h(x)(e^{\frac{|x|^2}{2}} - \lambda) + L|x|e^{\frac{|x|^2}{2}}] \\ &\leq \epsilon[-\lambda\frac{\bar{\eta}}{4} + e^{\frac{|x|^2}{2}}(K + L|x|)] \leq -\epsilon\frac{\bar{\eta}}{4} =: \alpha(x) \end{aligned}$$

if we take $\lambda\frac{\bar{\eta}}{4} - e^{\frac{|x|^2}{2}}(K + L|x|) \geq \frac{\bar{\eta}}{4}$, i.e. $\lambda - 1 \geq \frac{4}{\bar{\eta}}e^{\frac{|x|^2}{2}}(K + L|x|)$. \square

Now we drop the condition (11) of weak dependence on Du for equations of the form

$$G(x, u, \sigma^T D(u), \sigma^T D^2u \sigma) = 0, \text{ in } \Omega, \quad (13)$$

with $\sigma(\cdot)$ a nonvanishing Lipschitz continuous $n \times m$ matrix valued function and G uniformly elliptic. More precisely, let F satisfy the basic assumptions (6) and let it be of the form

$$F(x, u, p, M) = G(x, u, \sigma^T(x)p, \sigma^T(x)M\sigma(x)), \quad (14)$$

where $G : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^m \times S^m \rightarrow \mathbb{R}$ satisfies, for some $\nu > 0$,

$$G(x, u, p, M + N) \leq G(x, u, p, M) - \nu \text{tr} N, \quad (15)$$

for all $x \in \bar{\Omega}, u \in \mathbb{R}, p \in \mathbb{R}^m, M, N \in S^m, N \geq 0$, and the matrix σ is non-totally degenerate in the sense that $\text{tr}(\sigma^T \sigma)$ does not vanish wherever F is not strictly increasing in u . In the next result we take η defined by

$$\eta(x) := \nu \text{tr}(\sigma^T \sigma)(x) \quad (16)$$

and get the non-total degeneracy condition (7):

$$\begin{aligned} F(x, u, p, M + rI) &= G(x, u, \sigma^T p, \sigma^T(M + rI)\sigma) \leq \\ &G(x, u, \sigma^T p, \sigma^T M \sigma) - r\nu \text{tr}(\sigma^T \sigma) = F(x, u, p, M) - r\eta(x). \end{aligned}$$

Theorem 3.2. *Consider the equation (13), i.e., (1) where F satisfies (6), (8), and it is of the form (14), with G uniformly elliptic (15) and $\sigma(\cdot)$ a Lipschitz continuous $n \times m$ matrix valued function. Assume that for some $L \geq 0$*

$$|G(x, u, p + q, M) - G(x, u, p, M)| \leq L|q|$$

for all entries x, u, p, q, M , and that (9) holds with η defined by (16). Then the comparison principle holds.

Proof. First of all we prove an inequality that will be useful in the sequel of the proof. For any $x, u, p, q, M, r > 0, \mu > 0$, from assumptions (6), (15), and the Lipschitz continuity of G with respect to p we have

$$\begin{aligned} &F(x, u, p + q, M + rI + r\mu x \otimes x) \\ &\leq F(x, u, p, M) + L|\sigma^T q| - r\nu \text{tr}(\sigma^T \sigma)(x) - r\mu \text{tr}(\sigma^T(x \otimes x)\sigma). \end{aligned} \quad (17)$$

Since

$$\begin{aligned} \text{tr}(\sigma^T(x \otimes x)\sigma) &= \sum_{i,j,k} \sigma_{ik} x_i x_j \sigma_{jk} = \sum_k \left(\sum_i \sigma_{ik} x_i \right) \left(\sum_j \sigma_{jk} x_j \right) \\ &= \sum_k (\sigma^T x)_k^2 = |\sigma^T x|^2, \end{aligned}$$

the inequality (17) becomes

$$\begin{aligned} & F(x, u, p + q, M + rI + r\mu x \otimes x) \\ & \leq F(x, u, p, M) + L|\sigma^T q| - \nu r\eta(x) - \nu r\mu|\sigma^T x|^2. \end{aligned} \quad (18)$$

We now prove that, if u is a viscosity subsolution, then $u_\epsilon(x) = u(x) + \epsilon(e^{\mu\frac{|x|^2}{2}} - \lambda)$ is a strict subsolution for a choice $\lambda \gg \mu \gg 1$, independent of $\epsilon > 0$. As in the proof of Theorem (3.1) we choose λ, μ such that $\lambda \geq e^{\mu\frac{|x|^2}{2}}$ for all $x \in \Omega$ and compute

$$Du_\epsilon = Du + \epsilon\mu x e^{\mu\frac{|x|^2}{2}}, \quad D^2u_\epsilon = D^2u + \epsilon\mu e^{\mu\frac{|x|^2}{2}}(I + \mu x \otimes x).$$

We consider two cases.

Case i): $h(x) \leq \frac{\bar{\eta}}{4}$. From (18) and $u_\epsilon \leq u$ we get

$$F(x, u_\epsilon, Du_\epsilon, D^2u_\epsilon) \leq F(x, u, Du, D^2u) + \epsilon\mu e^{\mu\frac{|x|^2}{2}}(L|\sigma^T x| - \eta(x) - \mu\nu|\sigma^T x|^2). \quad (19)$$

In this case, $\eta(x) \geq \frac{3\bar{\eta}}{4}$, then

$$L|\sigma^T x| - \eta(x) - \mu\nu|\sigma^T x|^2 \leq \begin{cases} -\frac{\bar{\eta}}{2}, & \text{if } |\sigma^T x| \leq \frac{\bar{\eta}}{4L}, \\ |\sigma^T x|(L - \mu\nu|\sigma^T x|), & \text{if } |\sigma^T x| > \frac{\bar{\eta}}{4L}. \end{cases}$$

Choosing now $\mu > \frac{8L^2}{\bar{\eta}\nu}$, for $|\sigma^T x| > \frac{\bar{\eta}}{4L}$, we have

$$|\sigma^T x|(L - \mu\nu|\sigma^T x|) < -L\frac{\bar{\eta}}{4L} = -\frac{\bar{\eta}}{4}.$$

Then

$$L|\sigma^T x| - \eta(x) - \mu\nu|\sigma^T x|^2 \leq -\frac{\bar{\eta}}{4}, \quad \forall x \in \Omega.$$

Therefore, plugging this inequality into (19),

$$F(x, u_\epsilon, Du_\epsilon, D^2u_\epsilon) \leq -\epsilon\mu e^{\mu\frac{|x|^2}{2}}\frac{\bar{\eta}}{4} \leq -\epsilon\mu\frac{\bar{\eta}}{4} =: \alpha(x) < 0.$$

Case ii): $h(x) > \frac{\bar{\eta}}{4}$. Then, by (8),

$$\begin{aligned} F(x, u_\epsilon, Du_\epsilon, D^2u_\epsilon) & \leq \epsilon[h(x)(e^{\mu\frac{|x|^2}{2}} - \lambda) + L\mu|\sigma^T x|e^{\mu\frac{|x|^2}{2}}] \\ & \leq \epsilon[-\lambda\frac{\bar{\eta}}{4} + e^{\mu\frac{|x|^2}{2}}(K + \mu L|\sigma^T x|)] \leq -\epsilon\mu\frac{\bar{\eta}}{4} = \alpha(x), \end{aligned}$$

if we choose

$$\lambda\frac{\bar{\eta}}{4} - e^{\mu\frac{|x|^2}{2}}(K + \mu L|\sigma^T x|) \geq \mu\frac{\bar{\eta}}{4},$$

i.e., if

$$\lambda - \mu \geq \frac{4}{\bar{\eta}}e^{\mu\frac{|x|^2}{2}}(K + \mu L|\sigma^T x|)$$

and this is possible for $\lambda \gg \mu$, because Ω is bounded. \square

Remark 3.1. From this proof we see that the comparison principle holds for F not necessarily of the form (14), provided it satisfies (18). The property (18) implies (7) by taking $q = 0$ and $\mu = 0$. Note that, the presence of the term $-\nu r\mu|\sigma^T x|^2$ in (18) allows to avoid assumption (11).

Minor modifications of the proof give the statement of Theorem 3.2 also for equations of the form $G(x, u, \sigma^T Du, \sigma^T D^2u)$.

The last result of this section is a comparison principle where we avoid condition (11) at the price of strengthening (7) to the following assumption of *non-degeneracy in a fixed direction*.

$$\text{Let } D^j \text{ be the } n \times n \text{ diagonal matrix with } D_{jj}^j = 1 \text{ and} \quad (20)$$

$$D_{ii}^j = 0, \text{ if } i \neq j, \text{ i.e. } D_{ii}^j = \delta_{ij}.$$

There exists a function $\eta(x) \in C(\bar{\Omega})$, $\eta \geq 0$, such that

$$F(x, u, p, M + rD^j) \leq F(x, u, p, M) - \eta(x)r, \quad \forall x, u, p, M, r > 0.$$

Theorem 3.3. *Assume (6), (8), (10), (20) for at least one $j \in \{1, \dots, n\}$, and (9) with η given by (20). Then the comparison principle holds.*

Proof. The proof follows the lines of the previous two, but now we take $u_\epsilon(x) := u(x) + \epsilon(e^{\mu x_j} - \lambda)$. In this case

$$\frac{\partial u_\epsilon}{\partial x_i} = \frac{\partial u}{\partial x_i} + \epsilon \mu \delta_{ij} e^{\mu x_j}, \quad D^2 u_\epsilon = D^2 u + \epsilon \mu^2 e^{\mu x_j} D^j \geq D^2 u.$$

Case i): $h(x) \leq \frac{\bar{\eta}}{4}$. From assumption (9), we have $\eta(x) \geq \frac{3\bar{\eta}}{4}$.

Since F is proper and λ is chosen such that $u_\epsilon \leq u$, by (7) and (10) we have:

$$F(x, u_\epsilon, Du_\epsilon, D^2 u_\epsilon) \leq F(x, u, Du, D^2 u) + \epsilon \mu e^{\mu x_j} (L - \eta(x)\mu).$$

Since $L - \eta(x)\mu \leq L - \mu \frac{3\bar{\eta}}{4}$,

$$F(x, u_\epsilon, Du_\epsilon, D^2 u_\epsilon) \leq -\epsilon \mu e^{\mu x_j} (\mu \frac{3\bar{\eta}}{4} - L) \leq -\epsilon \frac{\bar{\eta}}{4} =: \alpha(x)$$

for μ sufficiently large.

Case ii): $h(x) > \frac{\bar{\eta}}{4}$. From (6), (8), and (10)

$$\begin{aligned} F(x, u_\epsilon, Du_\epsilon, D^2 u_\epsilon) &\leq F(x, u, Du, D^2 u) + \epsilon [h(x)(e^{\mu x_j} - \lambda) + L\mu e^{\mu x_j}] \\ &\leq \epsilon [-\lambda \frac{\bar{\eta}}{4} + e^{\mu x_j} (K + L\mu)] \leq -\epsilon \frac{\bar{\eta}}{4} = \alpha(x), \end{aligned}$$

where the last inequality is obtained by taking $\lambda \frac{\bar{\eta}}{4} - e^{\mu x_j} (K + L\mu) \geq \frac{\bar{\eta}}{4}$, i.e., $\lambda - 1 \geq \frac{4}{\bar{\eta}} e^{\mu x_j} (K + L\mu)$. \square

4. Some examples.

Example 4.1. Quasilinear Bellman-Isaacs equations. Consider quasilinear degenerate elliptic equations of the form

$$-\text{tr}(\sigma \sigma^T(x) D^2 u) + H(x, u, Du) = 0 \quad \text{in } \Omega, \quad (21)$$

with a Hamiltonian H satisfying

$$\begin{aligned} &H : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}, \text{ continuous;} \\ &u \rightarrow H(x, u, p) \text{ nondecreasing } \forall x, p; \\ &|H(x, u, p + q) - H(x, u, p)| \leq L|q|, \quad \forall x, u, p, q; \\ &|H(x, u, \alpha(x - y)) - H(y, u, \alpha(x - y))| \leq \omega(\alpha|x - y|^2 + |x - y|), \end{aligned} \quad (22)$$

for all $\alpha > 0$ and all x, y , where ω is a modulus, as in (6). They arise in stochastic optimal control and differential games with systems whose diffusion term is not affected by the controls, see Example 4.6 below for the general case.

We will write $H = H(x, u, \sigma^T(x) Du)$ to mean that there exists $\tilde{H} : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}$ such that $H(x, u, p) = \tilde{H}(x, u, \sigma^T(x)p)$ for all x, u, p .

Corollary 4.1. *Let $\sigma(x)$ be a Lipschitz continuous $n \times m$ matrix valued function, and H satisfies (22). Then the comparison principle holds for equation (21) if either*

$$\sum_{i,k} \sigma_{ik}^2(x) > 0, \quad \forall x \in \bar{\Omega}, \quad \text{and } H = H(x, u, \sigma^T(x)Du), \quad (23)$$

or

$$\text{there exists a } j : \sum_k \sigma_{jk}^2(x) > 0, \quad \forall x \in \bar{\Omega}. \quad (24)$$

Proof. The structural condition (6) is satisfied by the Lipschitz continuity of σ and (22), in view of the Example 3.6 in [13]. If (23) holds, we observe that

$$\text{tr}(\sigma\sigma^T M) = \sum_{i,k,l} \sigma_{ik}\sigma_{lk}m_{li} = \sum_{i,k,l} \sigma_{ik}m_{il}\sigma_{lk} = \text{tr}(\sigma^T M\sigma) \quad \forall M \in S^m,$$

so we can apply Theorem 3.2. In the case of (24) we compute

$$\text{tr}(\sigma\sigma^T(x)D^j) = \sum_k \sigma_{jk}^2(x) =: \eta(x),$$

and we use Theorem 3.3 with this choice of η . \square

Remark 4.1. The hypotheses (23) and (24) of the corollary can be rewritten in terms of the eigenvalues $\{\lambda_i(x)\}$ of $\sigma(x)\sigma^T(x)$, $i = 1, \dots, n$, as follows

$$\text{either } \sum_{i=1}^n \lambda_i(x) > 0, \quad \forall x \in \bar{\Omega} \quad \text{and } H = H(x, u, \sigma^T(x)Du), \quad (25)$$

$$\text{or there exists a } j \text{ such that } \lambda_j(x) > 0, \quad \forall x \in \bar{\Omega}. \quad (26)$$

Example 4.2. Quasilinear subelliptic equations. Consider the *uniformly subelliptic* quasilinear equation

$$- \sum_{i,j=1}^m a_{ij}(x)X_iX_ju + H(x, u, Du) = 0 \quad \text{in } \Omega, \quad (27)$$

where $X_j = \sigma^j \cdot \nabla$, $j = 1, \dots, m$, are n -dimensional vector fields and the matrix $a(x) \in S^m$ of the coefficients is positive definite, i.e., for some $\nu > 0$

$$a(x) \geq \nu I \quad \forall x \in \Omega. \quad (28)$$

Corollary 4.2. *Suppose $a(\cdot)$ is Lipschitz continuous and verifies (28) and H satisfies (22). Assume the vector fields $X_j = \sigma^j \cdot \nabla$ are of class $C^{1,1}(\bar{\Omega})$ and for some coordinate axis at all points $x \in \bar{\Omega}$ at least one field does not vanish in the direction of that axis, i.e.,*

$$\exists i^* \in \{1, \dots, n\} : \sum_{k=1}^m (\sigma_{i^*k}^k(x))^2 > 0 \quad \forall x \in \bar{\Omega}. \quad (29)$$

Then the comparison principle holds for equation (27).

Proof. Let σ be the $n \times m$ matrix with columns σ^j , i.e., $\sigma_{ij} = \sigma_i^j$. We compute, for u smooth,

$$\begin{aligned} X_iX_ju &= \sum_k \sigma_k^i(\sigma^j \cdot Du)_{x_k} = \sum_{k,l} \sigma_{ki}\sigma_{lj}u_{x_lx_k} + \sum_{k,l} \sigma_{ki}(\sigma_{lj})_{x_k}u_{x_l} \\ &= (\sigma^T D^2u\sigma)_{ij} + (D\sigma^j \sigma^i) \cdot Du, \end{aligned}$$

where $D\sigma^j$ denotes the Jacobian matrix of σ^j . Therefore the PDE (27) can be rewritten as

$$-\operatorname{tr}(a\sigma^T D^2 u \sigma) - \sum_{i,j=1}^m a_{ij} (D\sigma^j \sigma^i) \cdot Du + H(x, u, Du) = 0 \quad \text{in } \Omega. \quad (30)$$

We want to apply Theorem 3.3 and we first check condition (20) for the matrix D^{i^*} , that is, $D_{lk}^{i^*} = 1$ if $l = k = i^*$ and is null in all other cases. We observe that

$$\operatorname{tr}(\sigma^T D^{i^*} \sigma) = \sum_k \sigma_{i^*k}^2 > 0 \quad \text{in } \bar{\Omega}$$

and define $\eta(x) := \nu \sum_k \sigma_{i^*k}^2$. Then, for $M \in S^n$, $r > 0$,

$$\begin{aligned} -\operatorname{tr}(a\sigma^T (M + rD^{i^*}) \sigma) + \operatorname{tr}(a\sigma^T M \sigma) &= -r \operatorname{tr}(a\sigma^T D^{i^*} \sigma) \\ &\leq -r\nu \operatorname{tr}(\sigma^T D^{i^*} \sigma) = -r\eta(x), \end{aligned}$$

where the inequality follows from (28) and $\sigma^T D^{i^*} \sigma \geq 0$.

It remains to check the structural conditions (6). The first order part of the operator in (30) clearly satisfies the assumption. Now take $\tau(x) \in S^n$ a square root of $a(x)$, i.e., $a = \tau\tau^T$, and recall that from (28) $\tau(\cdot)$ is Lipschitz continuous. By a straightforward matrix calculation, for all $M \in S^n$

$$\operatorname{tr}(\tau\tau^T \sigma^T M \sigma) = \operatorname{tr}(\sigma\tau(\sigma\tau)^T M),$$

so also the second order part of the operator in (30) verifies (6) in view of the Example 3.6 in [13]. \square

Example 4.3. Sum of squares of vector fields. Here we consider the following quasilinear perturbation of a sum of squares of vector fields

$$-\sum_{j=1}^m X_j^2 u + H(x, u, Du) = 0 \quad \text{in } \Omega. \quad (31)$$

This is clearly a special case of the last Example 4.2, it is enough to take $a(X) \equiv I$ in (27). It is also a special case of the first Example 4.1, as we show next. We write $X_j = \sigma^j \cdot \nabla$, with $\sigma^j \in C^{1,1}(\bar{\Omega}, \mathbb{R}^n)$ and take the matrix σ with $\sigma_{ij} = \sigma_i^j$. By the calculations in the previous proofs we see that (31) can be rewritten as

$$-\operatorname{tr}(\sigma\sigma^T D^2 u) - \sum_{j=1}^m (D\sigma^j \sigma^j) \cdot Du + H(x, u, Du) = 0 \quad \text{in } \Omega, \quad (32)$$

where $D\sigma^j$ denotes the Jacobian matrix of σ^j , which fits into the form (21). Therefore, if (29) and (22) hold, we have the comparison principle by either Corollary 4.1 or Corollary 4.2.

The non-degeneracy condition in a fixed direction (29) is satisfied by many interesting examples, for instance by all families X_1, \dots, X_m of generators of a Carnot group (see also the examples below). It is not satisfied, however, by all families satisfying the classical Hörmander condition (i.e., X_1, \dots, X_m and their commutators of any order span \mathbb{R}^n at each point of $\bar{\Omega}$ [22]). In this case only the non-total degeneracy condition

$$\sum_{i,k} \sigma_{ik}^2(x) > 0, \quad \text{in } \bar{\Omega} \quad (33)$$

is always true, because at no point of $\bar{\Omega}$ all vector fields can vanish. Then Corollary 4.1 gives the comparison principle under the further conditions

$$\sum_{j=1}^m D\sigma^j(x)\sigma^j(x) = 0 \quad \text{and} \quad H = H(x, u, \sigma^T(x)Du) \quad \text{in } \Omega. \quad (34)$$

Although restrictive, the former assumption is satisfied in some interesting cases, e.g., generators of Carnot groups of step 2 ([30]) and the three examples below. The latter condition is very natural for subelliptic equations since it states that H depends only on the *intrinsic (or horizontal) gradient* (X_1u, \dots, X_mu) .

Another comparison principle for (31) and (32) when the mere non-total degeneracy (33) holds is obtained from Theorem 3.1 under the additional assumption that the first order part is small with respect to the trace of $\sigma\sigma^T$, namely,

$$\max_{x \in \bar{\Omega}} |x| \left(L + \max_{x \in \bar{\Omega}} \left| \sum_{j=1}^m D\sigma^j(x)\sigma^j(x) \right| \right) \leq \min_{x \in \bar{\Omega}} \sum_{i,k} \sigma_{ik}^2(x)/2,$$

where L is the constant in (22).

We conclude this subsection with three examples that satisfy both the non-degeneracy condition in a fixed direction (29) and the first assumption in (34).

- *The Heisenberg operator.* In \mathbb{R}^3 write $x = (x_1, x_2, t)$, and take

$$\sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2x_2 & -2x_1 \end{bmatrix}. \quad (35)$$

The Heisenberg Laplacian is

$$\Delta_{\mathcal{H}}u := \text{tr}(\sigma\sigma^T(x)D^2u) = u_{x_1x_1} + u_{x_2x_2} + 4(x_1^2 + x_2^2)u_{tt} + 4x_2u_{x_1t} - 4x_1u_{x_2t}, \quad (36)$$

and the equations (31) and (32) become

$$-\Delta_{\mathcal{H}}u + H(x, u, Du) = 0. \quad (37)$$

The matrix σ satisfies (24), hence, under assumptions (22), the comparison principle holds.

- *The Grushin operator.* Here $\sigma = \begin{bmatrix} 1 & 0 \\ 0 & x_1 \end{bmatrix}$ and the equations (31) and (32) become

$$-u_{x_1x_1} - x_1^2u_{x_2x_2} + H(x, u, Du) = 0, \quad x = (x_1, x_2). \quad (38)$$

Hence (24) holds and, under assumptions (22), equation (38) satisfies the comparison principle.

- *A hypoelliptic non-Hörmander operator.* The equation

$$-u_{x_1x_1} - e^{-1/x_1^2}u_{x_2x_2} + H(x, u, Du) = 0, \quad x = (x_1, x_2), \quad (39)$$

comes from (21) by taking $\sigma = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{1}{2x_1^2}} \end{bmatrix}$. The operator in (39) does not satisfy the Hörmander condition but (24) holds.

Example 4.4. Kolmogorov type equations. Consider the following stochastic equation

$$\ddot{x} = f(x, \dot{x}, \alpha) + \frac{dw}{dt}, \quad (40)$$

where $x(t) \in \mathbb{R}^3$, $w(t)$ is a 3-dimensional standard Brownian motion, and the acceleration f depends on the control α . If we write (40) in the form:

$$\begin{cases} \dot{x} = y, \\ dy = f(x, y, \alpha)dt + dw, \end{cases}$$

we see that the 3×6 dispersion matrix σ^T is

$$\sigma^T = \begin{bmatrix} 0 & I \end{bmatrix}.$$

The Bellman equation associated to an optimal control problem of this system with cost functional $E \int_0^\tau l(x(t), y(t), \alpha(t)) dt$, where E denotes the expectation and τ the exit time of $(x(t), y(t))$ from the domain Ω , is

$$-\Delta_y u - y \cdot D_x u + \max_\alpha \{-f(x, y, \alpha) \cdot D_y u - l(x, y, \alpha)\} = 0, \quad \Omega \subset \mathbb{R}^6. \quad (41)$$

More in general, we can consider the stationary Kolmogorov-type equation:

$$-\Delta_y u - y \cdot D_x u + H(x, y, u, Du) = 0 \quad \text{in } \Omega \subset \mathbb{R}^{2m}. \quad (42)$$

Also in this case, if H satisfies (22), we have a comparison principle because (24) holds.

Note that the principal part of the equations in this example is much more degenerate than a hypoelliptic operator.

Example 4.5. Pucci-type operators for subelliptic diffusions. Fix $0 < \lambda \leq \Lambda$. For any $X \in S^m$, the Pucci's extremal operators are defined as

$$\mathcal{P}^+(X) := -\lambda \sum_{e_i > 0} e_i - \Lambda \sum_{e_i < 0} e_i,$$

$$\mathcal{P}^-(X) := -\Lambda \sum_{e_i > 0} e_i - \lambda \sum_{e_i < 0} e_i,$$

where e_i are the eigenvalues of X , or, equivalently,

$$\mathcal{P}^+(X) = \sup\{-\text{tr}(MX) : M \in S^m, \lambda I \leq M \leq \Lambda I\}, \quad (43)$$

$$\mathcal{P}^-(X) = \inf\{-\text{tr}(MX) : M \in S^m, \lambda I \leq M \leq \Lambda I\}. \quad (44)$$

We consider equations involving the Pucci operators associated to a $n \times m$ dispersion matrix $\sigma(x)$, namely,

$$\mathcal{P}^\pm(\sigma^T D^2 u \sigma) + H(x, u, Du) = 0. \quad (45)$$

They are special cases of Hamilton-Jacobi-Bellman equations (see the Example 4.6). They arise also in the study of the Pucci operator relative to a family of vector fields X_1, \dots, X_m , namely,

$$\mathcal{P}^\pm((D_{\mathcal{X}}^2 u)^*), \quad (D_{\mathcal{X}}^2 u)_{ij}^* := (X_i X_j u + X_j X_i u)/2.$$

In particular, if the vector fields generate the Heisenberg group, i.e., σ is given by (35), the horizontal Hessian is $(D_{\mathcal{X}}^2 u)^* = \sigma^T D^2 u \sigma$ for all smooth u . In the general case there is a first order correction that can be added to the term H in (45).

From Theorems 3.2 we obtain the following result.

Corollary 4.3. *Let $\sigma(\cdot)$ be a Lipschitz continuous $n \times m$ matrix-valued function and H satisfy (22). Then the comparison principle holds for equation (45) if either*

$$i) \operatorname{tr}(\sigma^T \sigma(x)) = \sum_{i,k} \sigma_{ik}^2(x) > 0, \forall x \in \bar{\Omega} \text{ and } H = H(x, u, \sigma^T(x)Du),$$

or

$$ii) \text{ there exists a } j: \sum_k \sigma_{jk}^2(x) > 0, \forall x \in \bar{\Omega}.$$

Proof. It is known that \mathcal{P}^+ is uniformly elliptic of constant λ , so in case *i*) the equation has the form (14), (15) and we can apply Theorem 3.2.

In case *ii*)

$$\mathcal{P}^+(\sigma^T(M + rD^j)\sigma) \leq \mathcal{P}^+(\sigma^T M \sigma) - \lambda r \operatorname{tr}(\sigma^T D^j \sigma),$$

and we can use Theorem 3.3 with $\eta(x) = \lambda \sum_k \sigma_{jk}^2(x)$.

It remains to check the structural condition (6). By the representation (43) it is enough to consider $\operatorname{tr}(M\sigma^T X\sigma)$ for $X \in S^n, M \in S^m, \lambda I \leq M \leq \Lambda I$. We take $\tau \in S^m$ such that $M = \tau\tau^T$ and compute $\operatorname{tr}(M\sigma^T X\sigma) = \operatorname{tr}(\sigma\tau(\sigma\tau)^T X)$, which fits into the theory of [13] because $\sigma\tau$ is Lipschitz continuous.

The proof for \mathcal{P}^- is the same. \square

As observed in the quasilinear Example 4.3, the condition on σ in *i*) is verified if the columns of σ satisfy Hörmander's condition, and in many interesting cases the stronger condition in *ii*) is also true.

Example 4.6. Fully nonlinear Bellman-Isaacs equations. We may apply the comparison theorems of Section 3 also to the Bellman-Isaacs equations arising in stochastic control and differential games with dynamics governed by the stochastic differential equation

$$dx_t = f dt + \sigma dw_t,$$

with $\sigma = \sigma(x, \alpha, \beta)$, $f = f(x, \alpha, \beta)$, where α and β are controls taking values in some compact sets A and B , respectively, and w_t is an m -dimensional Brownian motion. If we consider an integral cost functional with running cost $l(x, \alpha, \beta)$ and discount rate $c(x, \alpha, \beta)$, then the PDE associated to the value function is

$$\max_{\alpha \in A} \min_{\beta \in B} \{-\operatorname{tr}(\sigma\sigma^T D^2 u) - f \cdot Du + cu - l\} = 0. \quad (46)$$

This equation satisfies the structural assumption (6) if all data are continuous and bounded, $c \geq 0$, and σ and f are also Lipschitz in x uniformly in α, β . It satisfies the non-total degeneracy condition if

$$\eta(x) := \min_{\alpha, \beta} \sum_{j,k} \sigma_{jk}^2(x, \alpha, \beta) > 0, \forall x \in \bar{\Omega},$$

i.e., for all controls the dispersion matrix does not vanish at any point. Then Theorem 3.1 applies if the drift f is null, or small compared to η :

$$\max_{x \in \bar{\Omega}} |x| \max_{x, \alpha, \beta} |f(x, \alpha, \beta)| \leq \min_{x \in \bar{\Omega}} \eta(x)/2.$$

On the other hand, the condition of non-degeneracy in a fixed direction of Theorem 3.3 holds if there exists a j such that

$$\eta(x) := \min_{\alpha, \beta} \sum_k \sigma_{jk}^2(x, \alpha, \beta) > 0, \forall x \in \bar{\Omega},$$

that is, at least one component of the system is affected by a nondegenerate noise.

Note that the case of uncontrolled diffusion $\sigma = \sigma(x)$ leads to quasilinear equations of the form (21), whereas we obtain the subelliptic Pucci equation (45) with the operator \mathcal{P}^+ if A is a singleton, $\sigma(x, \beta) = \sigma(x)\beta$, and $B = \{\beta \in S^m : \sqrt{\lambda}I \leq \beta \leq \sqrt{\Lambda}I\}$, and with the operator \mathcal{P}^- by reversing the roles of the controls.

Example 4.7. A case of non-uniqueness. If the equation is totally degenerate there are known examples where the comparison principle is not valid. For instance, the eikonal equation $|Du| = l(x)$ in the ball $\{x \in \mathbb{R}^n : |x| < 1\}$, with $l(x) = 2|x|$, has two classical solutions null on the boundary, namely, $|x|^2 - 1$ and $1 - |x|^2$, and infinitely many viscosity solutions. The comparison principle holds if $l > 0$, see [2].

5. Existence of generalized solutions to the Dirichlet problem. In this section we consider the following Dirichlet problem

$$\begin{cases} F(x, u, Du, D^2u) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \quad (47)$$

where Ω is an open bounded domain of \mathbb{R}^n , $g(x)$ is a continuous function on $\partial\Omega$. Besides the hypotheses listed in Section 3, we assume:

$$\text{there exists } M > 0 : -M \leq F(x, 0, 0, 0) \leq M, \quad \forall x \in \bar{\Omega}. \quad (48)$$

We denote by \mathcal{S} , \mathcal{Z} the sets of subsolutions and supersolutions of (47):

$$\mathcal{S} := \{w \in BUSC(\bar{\Omega}) : w \text{ subsolution of } F = 0 \text{ in } \Omega, w \leq g, \text{ on } \partial\Omega\},$$

$$\mathcal{Z} := \{W \in BLSC(\bar{\Omega}) : W \text{ supersolution of } F = 0 \text{ in } \Omega, W \geq g, \text{ on } \partial\Omega\},$$

where $BUSC$ (resp. $BLSC$) means bounded and upper (resp. lower) semicontinuous functions. The Perron-Ishii method proposes as candidate solutions of (47)

$$\underline{u}(x) := \sup_{w \in \mathcal{S}} w(x), \quad \bar{u}(x) := \inf_{W \in \mathcal{Z}} W(x), \quad x \in \bar{\Omega}.$$

If

$$\underline{u}(x) = \bar{u}(x) =: u,$$

u is the natural generalized solution of (47). It coincides with the Perron-Wiener-Brelot solution if the equation is linear, and it is also called envelope viscosity solution ([2], [1]).

Definition 5.1. We say that w is a lower (respectively upper) barrier for problem (47) at a point $x \in \partial\Omega$ if $w \in \mathcal{S}$ (respectively $w \in \mathcal{Z}$) and

$$\lim_{y \rightarrow x} w(y) = g(x).$$

Theorem 5.1. Suppose (48) and that the hypotheses of Theorem 3.1 or 3.2 or 3.3 hold. Then

- i) \underline{u} and \bar{u} are finite and solve the PDE in (47) in the sense that their l.s.c. envelope is a supersolution and the u.s.c. envelope is a subsolution;
- ii) if for all $x \in \partial\Omega$ there is a lower (resp., upper) barrier, then $\underline{u}(x) = \bar{u}(x)$ and it is the minimal element of \mathcal{Z} (resp., the maximal element of \mathcal{S});
- iii) if for all $x \in \partial\Omega$ there is a lower and an upper barrier, then $\underline{u} = \bar{u}$ is the continuous viscosity solution of (47).

Proof. By the results of Section 3 the comparison principle between viscosity sub- and supersolutions holds. Therefore, if we prove that both sets \mathcal{S} and \mathcal{Z} are nonempty, all statements follow from standard viscosity solutions theory [13] (for *ii*) see Theorem 2 in [1]).

To prove that the sets \mathcal{S} and \mathcal{Z} are nonempty we find explicitly a subsolution and a supersolution. As in the proof of Theorem 3.1 we consider

$$w = k(e^{\frac{|x|^2}{2}} - \lambda)$$

and we want to prove that it is a subsolution to (47), for $k \geq 1$ and $\lambda \gg 1$ independent of k . First we choose $\lambda \geq (e^{\frac{|x|^2}{2}} + \max_{\partial\Omega} |g|)$, $\forall x \in \partial\Omega$, and this implies $w \leq g$ on $\partial\Omega$. Under the hypotheses of either Theorem 3.1 or 3.2

$$\begin{aligned} F(x, w, Dw, D^2w) &\leq F(x, 0, 0, 0) - ke^{\frac{|x|^2}{2}}(\eta(x) - h(x) - L|x|) - k\lambda h(x) \\ &\leq M - ke^{\frac{|x|^2}{2}}(\eta(x) - h(x) - L|x|) - k\lambda h(x). \end{aligned}$$

Then we apply the same procedure as in the proof of Theorem 3.1 to obtain that

$$-ke^{\frac{|x|^2}{2}}(\eta(x) - h(x) - L|x|) - k\lambda h(x) \leq -k\frac{\bar{\eta}}{4}, \quad (49)$$

thus, by taking $k \geq \frac{4M}{\bar{\eta}} \vee 1$, we obtain $F(x, w, Dw, D^2w) \leq M - k\frac{\bar{\eta}}{4} \leq 0$, in Ω .

Analogously, we prove that

$$W = k(\lambda - e^{\frac{|x|^2}{2}}),$$

with k and λ sufficiently large, is a supersolution:

$$\begin{aligned} F(x, W, DW, D^2W) &\geq F(x, 0, 0, 0) + ke^{\frac{|x|^2}{2}}(\eta(x) - h(x) - L|x|) + k\lambda h(x) \\ &\geq -M + ke^{\frac{|x|^2}{2}}(\eta(x) - h(x) - L|x|) + k\lambda h(x) \geq 0, \end{aligned}$$

by (49).

If, instead, the assumptions of Theorem 3.3 hold, we proceed in a similar way by taking the functions

$$w, W = \pm k(e^{\mu x_j} - \lambda),$$

and checking they are sub- and supersolutions for suitable k, μ, λ . \square

Remark 5.1. We refer to [1] for the properties of continuous dependence of the envelope viscosity solution of point *ii*) with respect to the uniform convergence of the boundary data g and of the operator F .

6. Continuity at the boundary. The theorem in the previous section provides the existence of a generalized solution when the assumptions of one of the Theorems 3.1, 3.2 or 3.3 are satisfied. Now we want to investigate if this solution is continuous up to the boundary. We first study general fully nonlinear equations and then specialize the result to the quasilinear case and to equations involving the Pucci operator. For simplicity we solve the Dirichlet problem with $g = 0$:

$$\begin{cases} F(x, u, Du, D^2u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (50)$$

The problem with a general continuous g can be treated along the same lines.

Throughout this section we assume Ω smooth, i.e. we suppose that there exists a $\Phi(x) \in C^2$ such that

$$\Omega = \{x \in \mathbb{R}^n : \Phi(x) > 0\}, \quad D\Phi(x) \neq 0 \quad \forall x \in \partial\Omega. \quad (51)$$

We denote by $n(z) = -D\Phi(z)/|D\Phi(z)|$ the outer unit normal to Ω at $z \in \partial\Omega$.

6.1. A general result.

Theorem 6.1. *Suppose $F(x, 0, 0, 0) \leq 0$ and that the hypotheses of Theorem 3.1 or 3.2 or 3.3 hold. Suppose that there exists $F_{hom} : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \times S^n \rightarrow \mathbb{R}$, continuous, positively 1-homogeneous in (u, p, X) (i.e. $F_{hom}(x, \rho u, \rho p, \rho X) = \rho F_{hom}(x, u, p, X)$, $\forall \rho > 0$) such that*

$$F(x, u, p, X) \geq F_{hom}(x, u, p, X) - M, \quad M > 0. \quad (52)$$

Assume that, for any $z \in \partial\Omega$, either

$$\sup_{\mu > 0} F_{hom}(z, 0, -n(z), X - \mu n(z) \otimes n(z)) > 0, \quad \forall X, \quad (53)$$

or

$$F_{hom}(z, 0, -n(z), X - \mu n(z) \otimes n(z)) = F_{hom}(z, 0, -n(z), X), \quad \forall X, \quad \forall \mu > 0, \quad (54)$$

$$\text{and there exists } \lambda > 0 \text{ such that } F_{hom}(z, 0, D\Phi(z), D^2\Phi(z) + \lambda I) > 0. \quad (55)$$

Then there exists a unique viscosity solution $u \in C(\bar{\Omega})$ of (47).

Remark 6.1. By analogy with the quasilinear case we call a point $z \in \partial\Omega$ *characteristic* for F if it satisfies (54) and *non-characteristic* for F if it satisfies (53), although in the fully nonlinear generality there might be points of $\partial\Omega$ that do not satisfy either condition.

Proof. From $F(x, 0, 0, 0) \leq 0$ we have that $u = 0$ is a lower barrier to problem (50). If we find an upper barrier we can conclude, by applying iii) of Theorem 5.1.

First of all we find a strict upper local barrier to the problem

$$\begin{cases} F_{hom}(x, u, Du, D^2u) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (56)$$

By *strict upper local barrier* at a point $z \in \partial\Omega$, we mean a function $W \in BLSC(\bar{B}(z, r) \cap \Omega)$, $r > 0$, $W \geq 0$, such that $F_{hom}(x, W, DW, D^2W) > 0$ in $B(z, r) \cap \Omega$, $\lim_{x \rightarrow z} W(x) = 0$ and $W(x) \geq \delta > 0$, for all $|x - z| = r$.

Let us consider

$$W(x) = 1 - e^{-\mu(\Phi(x) + \frac{\lambda}{2}|x-z|^2)}, \quad z \in \partial\Omega, \quad \mu, \lambda > 0, \quad (57)$$

where Φ is defined in (51). Then $W(z) = 0$ and $W(x) > 0$ for any $x \in \Omega$ and for any $x \in \partial\Omega$, $x \neq z$. Moreover

$$\begin{aligned} W_{x_i}(x) &= e^{-\mu(\Phi(x) + \frac{\lambda}{2}|x-z|^2)} \mu(\Phi_{x_i} + \lambda(x_i - z_i)), \\ W_{x_i x_j}(x) &= e^{-\mu(\Phi(x) + \frac{\lambda}{2}|x-z|^2)} \mu(\Phi_{x_i x_j} - \mu \Phi_{x_i} \Phi_{x_j} + \\ &\quad + \lambda \delta_{ij} - \mu \lambda \Phi_{x_j}(x_i - z_i) - \mu \lambda \Phi_{x_i}(x_j - z_j) - \mu \lambda^2 (x_i - z_i)(x_j - z_j)), \end{aligned}$$

so, in particular,

$$\begin{aligned} W_{x_i}(z) &= \mu \Phi_{x_i}(z), \\ W_{x_i x_j}(z) &= \mu \Phi_{x_i x_j}(z) - \mu^2 \Phi_{x_i} \Phi_{x_j}(z) + \mu \lambda \delta_{ij}. \end{aligned}$$

Then, since $D\Phi(z) = -n(z)|D\Phi(z)|$, we have:

$$\begin{aligned} & F_{hom}(z, 0, DW, D^2W) \\ &= F_{hom}(z, 0, \mu D\Phi(z), \mu(D^2\Phi(z) - \mu D\Phi(z) \otimes D\Phi(z) + \lambda I)) \\ &= \mu F_{hom}(z, 0, -n(z)|D\Phi(z)|, D^2\Phi(z) - \mu|D\Phi(z)|^2 n(z) \otimes n(z) + \lambda I). \end{aligned}$$

If condition (53) holds, then there exists $\mu > 0$ such that

$$F_{hom}(z, 0, DW, D^2W) > 0.$$

By continuity

$$F_{hom}(x, W, DW, D^2W) > 0$$

in $B(z, r) \cap \Omega$ for r small enough, so W is a strict upper local barrier in z to problem (56).

If, instead, conditions (54) and (55) hold, then there exists $\lambda > 0$ such that

$$F_{hom}(z, 0, DW, D^2W) = \mu F_{hom}(z, 0, D\Phi(z), D^2\Phi(z) + \lambda I) > 0,$$

i.e., also in this case W is a strict upper local barrier in z .

With W we can construct an upper barrier to problem (50) following the procedure used in [1] to prove Proposition 5:

let $w \in \mathcal{Z}$, define

$$V = \begin{cases} \min\{\rho W(x), w(x)\}, & \text{if } x \in \overline{B(z, r)} \cap \Omega, \\ w(x), & \text{otherwise.} \end{cases} \quad (58)$$

We prove that V is an upper barrier in z for ρ sufficiently large. It is obvious that $V \geq 0$ on $\partial\Omega$ and $V(z) = 0$. In $\Omega \setminus \overline{B(z, r)}$, V is a supersolution. In $\partial B(z, r) \cap \Omega$, since $W(x) \geq \delta > 0$, for all $|x - z| = r$, we can choose a ρ sufficiently large such that $V = w$, then also in this case V is a supersolution. In $B(z, r) \cap \Omega$, if we check that $\rho W(x)$ is a supersolution we have that also V is a supersolution. From assumption (52):

$$F(x, \rho W, D\rho W, D^2\rho W) \geq \rho F_{hom}(x, W, DW, D^2W) - M.$$

Since $F_{hom}(x, W, DW, D^2W) > 0$ in $B(z, r) \cap \Omega$, we can choose a ρ large enough such that $F(x, \rho W, D\rho W, D^2\rho W) \geq 0$ in $B(z, r) \cap \Omega$. \square

Remark 6.2. Assumption (52) holds for Bellman-Isaacs equations (46) with bounded running costs $l(x, \alpha, \beta)$. In this context the assumption $F(x, 0, 0, 0) \leq 0$ becomes $l \geq 0$, which is very natural in time-optimal control.

6.2. The quasilinear case. We now apply the previous general theorem to the quasilinear problem

$$\begin{cases} -\text{tr}(A(x)D^2u) + H(x, u, Du) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (59)$$

We recall that the assumptions needed to prove Corollary 4.1 and Theorem 6.1 are (51) and

$$\begin{cases} A = \sigma\sigma^T, \sigma(x) \text{ Lipschitz continuous } n \times m \text{ matrix,} \\ H : \bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}, \text{ continuous,} \\ u \rightarrow H(x, u, p) \text{ nondecreasing,} \\ |H(x, u, p + q) - H(x, u, p)| \leq L|q|, \\ |H(x, u, \alpha(x - y)) - H(y, u, \alpha(x - y))| \leq \omega(\alpha|x - y|^2 + |x - y|). \end{cases} \quad (60)$$

$$\begin{cases} H(x, u, p) \geq H_{hom}(x, u, p) - M, \\ H_{hom} \text{ continuous and positively 1-homogeneous.} \end{cases} \quad (61)$$

Corollary 6.1. *Suppose that (51), (60), (61) hold and that $H(x, 0, 0) \leq 0$. Suppose that either the hypothesis (23) or (24) of Corollary 4.1 hold. If, for any $z \in \partial\Omega$, either*

$$\sum_{ij} A_{ij}(z)n_i(z)n_j(z) > 0, \quad (62)$$

or

$$-\text{tr}(A(z)D^2\Phi(z)) + H_{hom}(z, 0, D\Phi(z)) > 0 \quad (63)$$

then there exists a unique viscosity solution $u \in C(\bar{\Omega})$ of (59).

Remark 6.3. The condition (62) says that z is a *non-characteristic point* in the classical sense of linear theory. The assumption (63) is a natural counterpart for quasilinear equation of the conditions for boundary regularity of linear equations by Fichera [15, 16, 32].

The condition (63) at the characteristic points is satisfied if Ω is convex and $H_{hom}(z, 0, -n(z)) > 0$, because Φ can be chosen concave, so $-\text{tr}(A(z)D^2\Phi(z)) \geq 0$. The condition on the first order terms can be relaxed to $H_{hom}(z, 0, -n(z)) \geq 0$ if $D^2\Phi(z) \leq -\nu I$ because $-\text{tr}(A(z)D^2\Phi(z)) \geq \nu \sum_{i,k} \sigma_{ik}^2 > 0$.

Proof. From Corollary 4.1 the comparison principle holds. We are going to check that the other conditions of Theorem 6.1 are verified. We take

$$F_{hom}(x, u, p, X) = -\text{tr}(A(x)X) + H_{hom}(x, u, p).$$

If $z \in \partial\Omega$ satisfies (62)

$$|\sigma^T(z)n(z)|^2 = \sum_{ij} A_{ij}(z)n_i(z)n_j(z) > 0.$$

Then

$$F_{hom}(z, 0, -n(z), X - \mu n(z) \otimes n(z)) = -\text{tr}(AX) + \mu |\sigma^T n|^2 + H_{hom}(z, 0, -n(z)),$$

and we can choose a sufficiently large μ such that condition (53) is satisfied.

Next we consider the case $\sum_{ij} A_{ij}(z)n_i(z)n_j(z) = |\sigma^T(z)n(z)|^2 = 0$, so (63) holds. Then

$$F_{hom}(z, 0, D\Phi(z), D^2\Phi + \lambda I) = -\text{tr}(A(z)D^2\Phi(z)) - \lambda \text{tr}A(z) + H_{hom}(z, 0, D\Phi(z)) > 0$$

for λ small enough. Therefore in this case condition (55) holds and we can apply Theorem 6.1 to complete the proof. \square

6.3. Pucci operators. Now we derive from Corollary 6.1 the existence of a continuous viscosity solution to the following nonlinear problems involving the Pucci extremal operators introduced in Section 4

$$\begin{cases} \mathcal{P}^+(\sigma^T D^2 u \sigma) + H(x, u, Du) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (64)$$

where $\mathcal{P}^+(X)$ is defined in (43).

Corollary 6.2. *Under the assumptions of Corollary 6.1 with (63) replaced by*

$$-\Lambda \text{tr}(\sigma^T(z)D^2\Phi(z)\sigma(z)) + H_{hom}(z, 0, D\Phi(z)) > 0, \quad (65)$$

there exists a unique viscosity solution $u \in C(\bar{\Omega})$ of problem (64).

Proof. We denote by e_i the eigenvalues of the matrix of X and observe that

$$\mathcal{P}^+(X) := -\lambda \sum_{e_i > 0} e_i - \Lambda \sum_{e_i < 0} e_i \geq -\Lambda \operatorname{tr}(X).$$

Then, for all X and p ,

$$\mathcal{P}^+(\sigma^T(x)X\sigma(x)) + H(x, u, p) \geq -\Lambda \operatorname{tr}(\sigma^T(x)X\sigma(x)) + H(x, u, p). \quad (66)$$

Since $\frac{H}{\Lambda}$ satisfies the hypotheses (60), (61), and $\frac{H}{\Lambda}(x, 0, 0) \leq 0$, Corollary 6.1 holds for the problem

$$\begin{cases} -\Lambda \operatorname{tr}(\sigma^T D^2 u \sigma) + H(x, u, Du) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Then the solution \bar{u} of this problem is an upper barrier for problem (64). Since the constant 0 is a lower barrier we can conclude by Theorem 5.1. \square

7. The Dirichlet problem for nonlinear equations on the Heisenberg group. In this section we give some explicit examples of Dirichlet problems where the theory of the previous sections apply and we get the existence of a continuous viscosity solution.

7.1. The quasilinear Laplace-Heisenberg equation. Let us consider the homogeneous Dirichlet problem for the quasilinear equation on the Heisenberg group introduced in Section 4, namely,

$$\begin{cases} -\Delta_{\mathcal{H}} u + H(x, u, Du) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (67)$$

First we give an existence result for problem (67) in the Koranyi's ball

$$B_{\mathcal{H}} = \{x = (x_1, x_2, t) : (x_1^2 + x_2^2)^2 + t^2 < 1\}$$

and in the Euclidean ball of \mathbb{R}^3

$$B_{\mathcal{E}} = \{x = (x_1, x_2, t) : x_1^2 + x_2^2 + t^2 < 1\}$$

that follows directly from Corollary 6.1. Then we will prove a result for much more general domains that relies on the existence theory for linear equations, under different conditions on the Hamiltonian H .

Corollary 7.1. *Let Ω be $B_{\mathcal{H}}$ or $B_{\mathcal{E}}$. Assume that $H(x, u, p)$ satisfies (60), (61), and $H(x, 0, 0) \leq 0$. Suppose that, for $z = n(z) = (0, 0, \pm 1)$,*

$$H_{\text{hom}}(z, 0, -n(z)) > 0, \quad \text{if } \Omega = B_{\mathcal{H}}, \quad (68)$$

$$H_{\text{hom}}(z, 0, -n(z)) > -2, \quad \text{if } \Omega = B_{\mathcal{E}}. \quad (69)$$

Then there exists a unique viscosity solution $u \in C(\bar{\Omega})$ of (67).

Proof. If $\Omega = B_{\mathcal{H}}$, (51) holds with

$$\Phi(x) = 1 - (x_1^2 + x_2^2)^2 - t^2.$$

For problem (67) the characteristic points of $\partial B_{\mathcal{H}}$ (i.e. the points that satisfy $\sigma^T(z)n(z) = 0$) are $(0, 0, \pm 1)$. If $z \neq (0, 0, \pm 1)$ then condition (62) holds. If $z = (0, 0, \pm 1)$

$$-\operatorname{tr}(A(z)D^2\Phi(z)) = -\Delta_{\mathcal{H}}\Phi(z) = 24(x_1^2 + x_2^2) = 0.$$

From assumption (68) we have that condition (63) holds. Therefore Corollary 6.1 gives the existence of a unique continuous viscosity solution to (67) in $\bar{B}_{\mathcal{H}}$.

If $\Omega = B_{\mathcal{E}}$ and $\Phi(x) = 1 - x_1^2 - x_2^2 - t^2$ the characteristic points are the same. Now $-\Delta_{\mathcal{H}}\Phi(z) = 4$ so condition (63) becomes (69). \square

Examples. The previous result can be applied to the linear equation

$$-\Delta_{\mathcal{H}}u + b(x) \cdot \nabla u + c(x)u = f(x), \quad (70)$$

with c and f continuous, b Lipschitz, $0 \leq f(x) \leq M$, $c(x) \geq 0$. In this case, condition (68) reads $b(z) \cdot n(z) < 0$ in $z = n(z) = (0, 0, \pm 1)$, and (69) becomes $b(z) \cdot n(z) < 2$ (which holds, e.g., if $|b(z)| < 2$).

A simple nonlinear example is

$$-\Delta_{\mathcal{H}}u + G(\nabla u) = f(x), \quad (71)$$

with f as above and $G(\cdot)$ Lipschitz continuous, 1-homogeneous, and satisfying $G(0, 0, \pm 1) > 0$ if $\Omega = B_{\mathcal{H}}$, $G(0, 0, \pm 1) > -2$ if $\Omega = B_{\mathcal{E}}$.

Remark 7.1. Both $B_{\mathcal{H}}$ and $B_{\mathcal{E}}$ are strictly convex, but $D^2\Phi(z)$ is negative definite in the euclidean case, whereas for the Koranyi ball it degenerates in the horizontal directions. For this reason in the latter case we need the more restrictive condition (68), stating the non-degeneracy of H_{hom} at the characteristic points of $B_{\mathcal{H}}$ with an appropriate sign. If the nondegeneracy has the opposite sign, i.e.,

$$H_{hom}(z, 0, -n(z)) < 0, \quad z = n(z) = (0, 0, \pm 1),$$

one expects that the generalized solution of Theorem 5.1 *ii*) does not attain the boundary data at z . This is suggested by Gaveau's theory [20, 21] in the linear case (70), and by stochastic control considerations in the case of a Hamiltonian H of Bellman type. If H_{hom} degenerates at z , i.e., $H_{hom}(z, 0, -n(z)) = 0$, the problem remains open. This is the case when H_{hom} depends only on the horizontal gradient $\sigma^T(x)Du$. The Corollary 7.2 below gives an answer under different conditions that do not include the linear case (70) with b not identically 0.

The next result is an existence theorem to problem (67) in a general bounded open set Ω (not necessarily with smooth boundary), provided that it is a *domain of solvability* for the Heisenberg Laplacian (see the precise definition below). Instead of making a non degeneracy assumption on H_{hom} at the characteristic points, we suppose that H is bounded from below.

Definition 7.1. *The bounded open set Ω is called domain of solvability for $\Delta_{\mathcal{H}}$ if the Dirichlet problem*

$$\begin{cases} \Delta_{\mathcal{H}}u = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \quad (72)$$

has a solution, continuous on $\bar{\Omega}$, for any g continuous function on $\partial\Omega$.

Remark 7.2. Ω is a domain of solvability for $\Delta_{\mathcal{H}}$ if the following conditions are satisfied.

1. A probabilistic definition of regular points of $\partial\Omega$ was introduced by Gaveau [20] and it was proved in [21] that if all points of $\partial\Omega$ are regular then Ω is a domain of solvability for $\Delta_{\mathcal{H}}$.

2. A result of Gallardo in [19] shows that, if for any $x \in \partial\Omega$ there exists an intrinsic exterior cone to Ω , then all points of $\partial\Omega$ are regular and then, from 1), Ω is a domain of solvability for $\Delta_{\mathcal{H}}$.
3. Any *smooth* domain Ω , i.e., satisfying (51) for some $\Phi \in C^2$, is a domain of solvability for $\Delta_{\mathcal{H}}$. In fact, for any point of the boundary of a smooth domain, there exists an intrinsic exterior cone [27]. Then 2) implies that all these points are regular.

The next result is obtained in a domain $\Omega \subseteq \mathbb{R}^3$ of solvability for $\Delta_{\mathcal{H}}$, and hence, from point 3 of the previous remark, it is valid for any bounded smooth domain.

Corollary 7.2. *Let Ω be a domain of solvability for $\Delta_{\mathcal{H}}$. Assume that $H(x, u, p)$ satisfies (60), $H(x, 0, 0) \leq 0$ and $H(x, u, p) \geq -K$, $K > 0$, $\forall x \in \bar{\Omega}$, $u \in \mathbb{R}$, $p \in \mathbb{R}^n$. Then there exists a unique viscosity solution $u \in C(\bar{\Omega})$ of problem (67).*

Proof. Since 0 is a lower barrier, it is enough to find an upper barrier to get the conclusion from Theorem 5.1. From the definition of domain of solvability, there exists v , continuous in $\bar{\Omega}$, solution of

$$\begin{cases} \Delta_{\mathcal{H}}v = 0, & \text{in } \Omega, \\ v = \frac{x_1^2}{2}, & \text{on } \partial\Omega. \end{cases} \quad (73)$$

Hence $w = \lambda(v - \frac{x_1^2}{2})$, $\lambda > K$ can be taken as upper barrier, because $w = 0$ on $\partial\Omega$ and $-\Delta_{\mathcal{H}}w + H(x, w, Dw) = \lambda + H(x, w, Dw) \geq \lambda - K > 0$. \square

Example. The result applies to the example (71) if $f \in C^2(\bar{\Omega})$ satisfies $0 \leq f(x) \leq M$ and G is Lipschitz, positively 1-homogeneous, and $G(p) \geq 0$ for all $p \in \mathbb{R}^n$.

7.2. Fully nonlinear equations on the Heisenberg group. The results of the previous subsection can be used to give some more explicit results on the solvability of

$$\begin{cases} \mathcal{P}^+(\sigma^T(x)D^2u\sigma(x)) + H(x, u, Du) = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (74)$$

for the Heisenberg diffusion matrix σ defined by (35) and the Pucci operator $\mathcal{P}^+(X)$ defined in (43).

Corollary 7.3. *Let $\Omega = B_{\mathcal{H}}$. Under the assumptions of Corollary 7.1 there exists an unique continuous viscosity solution of problem (74).*

Proof. Using (66) and Corollary 7.1, the solution of

$$\begin{cases} -\Lambda\Delta_{\mathcal{H}}u + H(x, u, Du) = 0, & \text{in } B_{\mathcal{H}}, \\ u = 0, & \text{on } \partial B_{\mathcal{H}}. \end{cases}$$

is an upper barrier to (74). \square

The same argument and Corollary 7.2 give the following.

Corollary 7.4. *Let Ω be a domain of solvability for $\Delta_{\mathcal{H}}$. Under assumptions of Corollary 7.2, there exists an unique continuous viscosity solution of problem (74).*

Remark 7.3. Taking $H(x, 0, 0) \geq 0$, replacing (61) with $H \leq H_{hom} + M$, with H_{hom} 1-homogeneous, $M > 0$ and replacing (68) with $H_{hom}(z, 0, n(z)) < 0$, in $z = n(z) = (0, 0, \pm 1)$, we can solve also

$$\begin{cases} \mathcal{P}^-(\sigma^T(x)D^2u\sigma(x)) + H(x, u, Du) = 0, & \text{in } B_{\mathcal{H}}, \\ u = 0, & \text{on } \partial B_{\mathcal{H}}, \end{cases} \quad (75)$$

In this case the function $u = 0$ is an upper barrier and the solution of

$$\begin{cases} -\lambda\Delta_{\mathcal{H}}u + H(x, u, Du) = 0, & \text{in } B_{\mathcal{H}}, \\ u = 0, & \text{on } \partial B_{\mathcal{H}}. \end{cases}$$

is a lower barrier.

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