# Flattening results for elliptic PDEs in unbounded domains with applications to overdetermined problems

Alberto Farina & Enrico Valdinoci

February 16, 2009

**Abstract**

Semilinear elliptic PDEs are dealt with, either in the halfspace or in overdetermined settings.

We obtain several new results and also give new proofs of celebrated theorems by exploiting some geometric analysis of level sets, a geometric inequality and a pointwise gradient estimate.

## 1 Introduction

In this paper, we will consider several geometric features of the solutions of some semilinear elliptic PDEs.

The boundary value problems we consider are of the type

$$\begin{cases} \begin{array}{c} \Delta u + f(u) = 0 \\ u > 0 \end{array} & \text{in } \Omega, \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega$ is either a halfspace or an unbounded epigraph, in which case a second boundary condition $\partial_\nu u =$ const on $\partial\Omega$ is imposed.

We provide a geometric formula for monotone solutions in domains with boundary, a pointwise gradient estimate for the halfspace, several rigidity results for unbounded domains and an application to overdetermined problems inspired by Schiffer's conjecture.

The main motivation for our results arises from the following natural question. If $\Omega$ is an unbounded "nice" domain in $\mathbb{R}^n$ and $u$ is a solution of the problem

$$\begin{cases} \begin{array}{c} \Delta u + f(u) = 0 \\ u > 0 \end{array} & \text{in } \Omega, \\ \\ \begin{array}{c} u = 0 \\ \partial_\nu u = c \end{array} & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

are there "natural" geometric assumptions under which one can conclude that $\Omega$ is necessarily a halfspace and $u$ is necessarily a function of only one variable?

Note that problem (1.1) is "overdetermined", that is both Dirichlet and Neumann boundary conditions are prescribed (as usual, in (1.1), $\nu$ denotes the exterior normal of $\partial\Omega$ and $c$ is a constant). Also, the PDE in (1.1) is elliptic and semilinear.

To the best of our knowledge, there are no results in the literature for *unbounded* domains $\Omega$ which answer the above natural question. One of the main scopes of this paper is thus to develop a geometric approach for problem (1.1), which will allow us to obtain new results in that direction and also to give new proofs, under weaker assumptions, of some previous related results of [BCN97b, AAC01]. Our study will be possible via a suitable weighted Poincaré-type inequality, inspired by the one of [SZ98a, SZ98b]: these weights are related to the curvatures of $u$ and they will encode valuable information on the symmetry of both the solution $u$ and of the domain $\Omega$.

Following are the details of the assumptions we take and of the results we obtain.

## 1.1 A general geometric formula

Given a smooth function $v$, one may consider the level sets $\{v = c\}$.

In the vicinity of $\{\nabla v \neq 0\}$, these level sets are smooth manifolds, so one can introduce the principal curvatures

$$\kappa_1, \dots, \kappa_{n-1}$$

at any point of such manifolds.

We set

$$\mathcal{K} := \sqrt{\kappa_1^2 + \cdots + \kappa_{n-1}^2}.$$

Also, it is customary to consider the tangential gradient along level sets of $v$ at these points, that is

$$\nabla_T g := \nabla g - \left( \nabla g \cdot \frac{\nabla v}{|\nabla v|} \right) \frac{\nabla v}{|\nabla v|}.$$

Using this notation, we provide here the following general geometric result for solutions of semilinear elliptic PDEs:

**Theorem 1.1.** *Let $\Omega$ be an open subset of $\mathbb{R}^n$ with $C^3$ boundary and let $\nu$ be its exterior normal.*

*Suppose that $u \in C^2(\overline{\Omega})$ satisfies*

$$\Delta u(x) + f(u(x)) = 0,$$

*with $f$ locally Lipschitz, and*

$$\partial_n u(x) > 0$$

*for any $x \in \Omega$.*

*Assume also that*

$$u \text{ is constant on } \partial\Omega. \tag{1.2}$$

*Then,*

$$\int_\Omega \left[ |\nabla \varphi|^2 - f'(u) \frac{\partial_n u \varphi^2}{\epsilon + \partial_n u} \right] - \int_{\partial\Omega} \frac{\varphi^2}{\epsilon + \partial_n u} \nabla \partial_n u \cdot \nu \geq 0, \tag{1.3}$$

*for any $\varphi \in W_0^{1,\infty}(K)$, for any compact set $K \subset \mathbb{R}^n$ and for any $\epsilon > 0$.*

*Moreover,*

$$\int_\Omega \Big(|\nabla u|^2\mathcal{K}^2 + |\nabla_T|\nabla u||^2\Big)\varphi^2 \\ + \limsup_{\epsilon\to 0^+}\int_{\partial\Omega}\frac{\varphi^2}{\epsilon+\partial_n u}\Big(|\nabla u|^2\partial^2_{n,\nu}u - \partial^2_{i,\nu}u\partial_i u\partial_n u\Big) \\ \le \int_\Omega |\nabla u|^2|\nabla\varphi|^2 \tag{1.4}$$

*for any* $\varphi\in W^{1,\infty}_0(K)$*, for any compact set* $K\subset\mathbb{R}^n$*.*
*If, in addition, either* $\Omega=\mathbb{R}^n_+$ *or* $\partial_\nu u$ *is constant on* $\partial\Omega$*, then*

$$\int_\Omega \Big(|\nabla u|^2\mathcal{K}^2 + |\nabla_T|\nabla u||^2\Big)\varphi^2 \le \int_\Omega |\nabla u|^2|\nabla\varphi|^2 \tag{1.5}$$

*for any* $\varphi\in W^{1,\infty}_0(K)$*, for any compact set* $K\subset\mathbb{R}^n$*.*

The above result may be seen as a generalization of the geometric formula of [SZ98a, SZ98b]. Notice, in particular, that (1.3) is a more general version of the classical stability condition (see, for instance, [AAC01, FSV08]), while (1.4) is a Poincaré-type inequality, in which a suitably weighted $L^2$-norm of any test function is controlled by a suitably weighted $L^2$-norm of the gradient. Remarkably, the weights have a nice geometric meaning, which makes inequality (1.4) very feasible for applications, as we will see in what follows.

Moreover, such Poincaré-type inequalities may be seen as extensions of the stability condition for minimal surfaces, in which the $L^2$-norm of any test function, weighted by the curvature, is controlled by the $L^2$-norm of the gradient (see, for instance, formula (10.20) in [Giu84]).

Differently from [SZ98a, SZ98b], here we will exploit (1.4) by keeping track of some relevant boundary terms, by allowing the supports our test functions to meet the boundary of the domain (notice, indeed that, in the statement of Theorem 1.1, we allow that $K\cap\partial\Omega\neq\emptyset$).

We will use Theorem 1.1 in concrete cases. In particular, we will give new proofs, with different methods and under weaker assumptions, of some results first proven in [BCN97b, AAC01] and we will show new symmetry and rigidity properties in halfspaces and overdetermined problems.

The results obtained will be different, according to the dimension of the space, to the shape of the domain, to the properties of the nonlinearity and to the boundary conditions.

## 1.2 The case of dimension $n=2$

First, we provide new, and simpler, proofs of a rigidity result of [BCN97b].
For this, we use the standard notation $\mathbb{R}^n_+ := \mathbb{R}^{n-1}\times(0,+\infty)$.

**Theorem 1.2.** *Let* $\Omega$ *be an open, connected subset of* $\mathbb{R}^2$ *with* $C^3$ *boundary. Let* $f$ *be a locally Lipschitz function. Suppose that* $u\in C^2(\overline{\Omega})$*, with* $|\nabla u|\in L^\infty(\Omega)$*, satisfies*

$$\Delta u(x)+f(u(x))=0, \qquad \textit{and} \qquad \partial_2 u(x)>0$$

*for any* $x\in\Omega$*, with* $u=0$ *on* $\partial\Omega$*.*

*Assume also that either $\Omega$ is a halfplane or that $\partial_\nu u$ is constant on $\partial\Omega$.*

*Then, $\Omega$ must be a halfplane and there exist $\omega \in \mathrm{S}^1$ and $u_o : \mathbb{R} \to (0, +\infty)$ in such a way that $\omega$ is normal to $\partial\Omega$ and $u(x) = u_o(\omega \cdot x)$ for any $x \in \Omega$.*

**Theorem 1.3.** *Let $f$ be a locally Lipschitz function. Suppose that $u \in C^2(\overline{\mathbb{R}^2_+})$, with $|\nabla u| \in L^\infty(\mathbb{R}^2_+)$, satisfies*

$$\Delta u(x) + f(u(x)) = 0 \qquad \text{and} \qquad u(x) > 0$$

*for any $x \in \mathbb{R}^2_+$, with*

$$u(x_1, 0) = 0 \quad \text{for any } x_1 \in \mathbb{R}.$$

*Then, there exists $u_o : (0, +\infty) \to (0, +\infty)$ in such a way that*

$$u(x_1, x_2) = u_o(x_2) \quad \text{for any } (x_1, x_2) \in \mathbb{R}^2_+. \tag{1.6}$$

We observe that when $|\nabla u|$ is not bounded Theorems 1.2 and 1.3 do not hold (a counterexample is $u(x_1, x_2) = x_2 e^{x_1}$).

Theorem 1.3 was first proven in [BCN97b] (see in particular Theorem 1.5 there). The proof we give here is new, somewhat simpler, and it relies on geometric inequalities.

In fact, a slight improvement of Theorem 1.5 of [BCN97b] is given by Theorem 1.3 here, since the solution was assumed to be bounded in [BCN97b], while the weaker condition of having a bounded gradient is enough for our Theorem 1.3 (indeed, we recall that bounded solutions of semilinear PDEs have a bounded gradient, due to standard elliptic estimates, see [GT83], but the converse is not necessarily true, as the linear functions show).

## 1.3 A pointwise gradient bound in the halfspace

In order to deal with semilinear PDEs in $\mathbb{R}^n_+$, the following pointwise gradient estimate turns out to be quite useful:

**Theorem 1.4.** *Let $n \geq 1$. Let $F \in C^{1,1}_{\mathrm{loc}}(\mathbb{R})$, with*

$$F'(0) \geq 0. \tag{1.7}$$

*Let $u \in C^2(\overline{\mathbb{R}^n_+}) \cap L^\infty(\mathbb{R}^n_+)$ be a solution of $\Delta u + F'(u) = 0$ in $\mathbb{R}^n_+$, with $u \geq 0$ in $\mathbb{R}^n_+$ and $u = 0$ on $\partial\mathbb{R}^n_+$.*

*Then,*

$$\frac{1}{2}|\nabla u(x)|^2 \leq \sup_{r \in [0, \|u\|_{L^\infty(\mathbb{R}^n_+)}]} F(r) - F(u(x)), \tag{1.8}$$

*for any $x \in \mathbb{R}^n_+$.*

*Also,*

$$\sup_{r \in [0, \|u\|_{L^\infty(\mathbb{R}^n_+)}]} F(r) \;=\; F\big(\|u\|_{L^\infty(\mathbb{R}^n_+)}\big) \tag{1.9}$$

*and, if $u$ does not vanish identically, then*

$$\sup_{r \in [0, \|u\|_{L^\infty(\mathbb{R}^n_+)}]} F(r) \;>\; F(t) \tag{1.10}$$

*for any $t \in \big[0, \|u\|_{L^\infty(\mathbb{R}^n_+)}\big)$.*

The pointwise estimate of Theorem 1.4 may be seen as an extension of the one obtained in [Mod85], where a similar result was proven in the case of solutions in the entire space $\mathbb{R}^n$.

An important strengthening of the work of [Mod85] was also performed in [CGS94], where singular and degenerate operators were considered (still in the whole space $\mathbb{R}^n$).

With respect to [Mod85, CGS94], here we need to take into account the presence of the boundary terms on $\partial\mathbb{R}^n$, and this will provide several technical complications in the proof.

For our paper, Theorem 1.4 will play a crucial role in the proof of the subsequent Theorem 1.5.

### 1.4 The case of the halfspace in dimension $n = 3$

Under suitable conditions, it is possible to extend Theorem 1.3 to higher dimension. We start with $n = 3$, in which we give a new proof of the following result, which was first proved in [BCN97b], under a stronger assumption on the regularity of the nonlinearity:

**Theorem 1.5.** *Let $f$ be locally Lipschitz.*
*Suppose that $u \in C^2(\overline{\mathbb{R}^3_+}) \cap L^\infty(\mathbb{R}^3_+)$ satisfies*

$$\Delta u(x) + f(u(x)) = 0 \qquad \textit{and} \qquad u(x) > 0$$

*for any $x \in \mathbb{R}^3_+$, with*

$$u(x', 0) = 0 \quad \textit{for any } x' \in \mathbb{R}^2.$$

*Assume that $f(0) \geq 0$.*
*Then, there exists $u_o : (0, +\infty) \to (0, +\infty)$ in such a way that*

$$u(x', x_3) = u_o(x_3) \quad \textit{for any } (x', x_3) \in \mathbb{R}^3_+.$$

We remark that when $u$ is not bounded Theorem 1.5 does not hold (a counterexample for Theorem 1.5 would be $u(x_1, x_2, x_3) = x_3 e^{x_1}$). For results related to our Theorem 1.5 see also Theorem 1.5 in [BCN97b] and Corollary 1.3 in [FSV08].

Theorem 1.5 was first proven in [BCN97b]. Again, the proof we give here is new and it does not use the technology of [BCN93, BCN97b].

Also, differently from [BCN97b], Theorem 1.5 here above does not require $f$ to be $C^1(\mathbb{R})$, and this does not seem to be possible with the techniques of [BCN97b].

We would like to recall that the extension to locally Lipschitz nonlinearities is not of merely academic interest, since several physical applications deal with locally Lipschitz forces (for instance, suspension bridges, see [FV08]).

### 1.5 The case of the overdetermined epigraph in dimension $n = 2$, $3$

We now consider the epigraph case, namely the case in which our domain may be written as

$$\{x_n = \Gamma(x'),\ \ x' \in \mathbb{R}^{n-1}\}.$$

It is customary to say that the epigraph given by $\Gamma$ is coercive if

$$\lim_{|x'| \to +\infty} \Gamma(x') = +\infty.$$

With this notation, we obtain the following two results, which, to the best of our knowledge, are new:

**Theorem 1.6.** *Let $f$ be locally Lipschitz and suppose that*

- *either $n = 2$*
- *or $n = 3$ and $f(r) \geq 0$ for any $r \geq 0$.*

*Let $\Omega$ be an open subset of $\mathbb{R}^n$ with $C^3$ boundary.*
*Suppose that $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$ satisfies*

$$\Delta u(x) + f(u(x)) = 0 \quad \text{and} \quad u(x) > 0$$

*for any $x \in \Omega$, with*

$$u(x) = 0 \quad \text{for any } x \in \partial\Omega.$$

*Assume also that $\partial_\nu u$ is constant on $\partial\Omega$.*
*Then, $\Omega$ cannot be a uniformly Lipschitz coercive epigraph.*

**Theorem 1.7.** *Let $n = 2, 3$.*
*Let $\Omega$ be an open epigraph of $\mathbb{R}^n$ with $C^3$ and uniformly Lipschitz boundary.*
*Suppose that $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$ satisfies*

$$\Delta u + u - u^3 = 0 \quad \text{and} \quad u > 0 \tag{1.11}$$

*in $\Omega$, with*

$$u(x) = 0 \quad \text{for any } x \in \partial\Omega.$$

*Assume also that $\partial_\nu u$ is constant on $\partial\Omega$.*
*Then, we have that $\Omega = \mathbb{R}^n_+$ up to isometry and*

$$u(x_1, \dots, x_n) = \tanh\left(\frac{\sqrt{2}\, x_n}{2}\right) \quad \text{for any } (x_1, \dots, x_n) \in \mathbb{R}^n_+.$$

We observe that (1.11) is the classical Allen-Cahn equation. In fact, Theorem 1.7 will be, for us, the consequence of the following, more general, result:

**Theorem 1.8.** *Let $n = 2, 3$ and $f$ be locally Lipschitz.*
*Let $\Omega$ be an open epigraph of $\mathbb{R}^n$ with $C^3$ and uniformly Lipschitz boundary.*
*Suppose that $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$ satisfies*

$$\Delta u(x) + f(u(x)) = 0 \quad \text{and} \quad u(x) > 0$$

*for any $x \in \Omega$, with*

$$u(x) = 0 \quad \text{for any } x \in \partial\Omega.$$

*Assume also that $\partial_\nu u$ is constant on $\partial\Omega$ and that*

$$\begin{aligned} \text{there exists } \delta_3 > \delta_2 > \delta_1 > 0 \text{ in such a way that} \\ f(t) > \delta_1 t \text{ for any } t \in (0, \delta_1), \\ f \text{ is nonincreasing on } (\delta_2, \delta_3), \\ f > 0 \text{ on } (0, \delta_3) \\ \text{and } f \le 0 \text{ on } [\delta_3, +\infty). \end{aligned} \tag{1.12}$$

*Then, we have that $\Omega = \mathbb{R}^n_+$ up to isometry and that there exists $u_o : (0, +\infty) \to (0, +\infty)$ in such a way that*

$$u(x_1, \ldots, x_n) = u_o(x_n) \quad \text{for any } (x_1, \ldots, x_n) \in \mathbb{R}^n_+.$$

The question on whether or not similar results hold in higher dimension remains open.

We observe that Theorem 1.8 should be compared with Theorem 7.1 in [BCN97a]. Indeed, the result of Theorem 7.1 in [BCN97a] is similar to the one of Theorem 1.8 here and it even holds in any dimension, but there an additional flatness condition of the graph at infinity is needed (see (7.2) in [BCN97a]). The use of such additional flatness was actually crucial in [BCN97a] for using the sliding method: it is therefore the use of a new geometric tool that allows us to prove our Theorem 1.8 without any additional flatness assumption.

### 1.6 Overdetermined problems in dimension $n = 2, 3$: a version of Schiffer's conjecture in unbounded domains

We give the following result for overdetermined problems:

**Theorem 1.9.** *Let $\lambda$, $c \in \mathbb{R}$.*
*Let $\Omega$ be a $C^3$ and uniformly Lipschitz epigraph of $\mathbb{R}^n$, with $n = 2, 3$.*
*Then, there exists no solution $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$ of*

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = c & \text{on } \partial\Omega. \end{cases} \tag{1.13}$$

We wonder whether analogous results hold in higher dimension.

**Remark 1.10.** Theorem 1.9 may be related to a version for unbounded domains of Schiffer's conjecture (see, for instance, [Ser71, Wei71, Yau82, GS93, Ebe93, WCG95]).

Indeed, Schiffer's conjecture consists of the question whether or not there exists $u \in C^2(\overline{\Omega})$ solving

$$\begin{cases} \Delta u + \lambda u = 0 & \text{in } \Omega, \\ u = b & \text{on } \partial\Omega, \\ \partial_\nu u = c & \text{on } \partial\Omega \end{cases} \tag{1.14}$$

for some $\lambda$, $b$ and $c \in \mathbb{R}$, then $\Omega$ is a ball.

While the original Schiffer's conjecture deals with a bounded $\Omega$ in (1.14), Theorem 1.9 considers an analogous problem in unbounded domains. As far as we know, Theorem 1.9 is the first attempt to look for phenomena of the type of the Schiffer's conjecture in unbounded domains.

In fact (see Section 11 below), the proof of Theorem 1.9 consists in showing that if a solution of (1.13) existed, then $\lambda$ would be positive and $\Omega$ would be a halfspace.

But then (recall Theorem 1.5), $u$ needs to be a function of only one variable. Thus, by solving the ODE associated to (1.13), one would easily obtain a contradiction with the fact that $u$ is positive.

## 1.7 The case of the halfspace in dimensions $n = 4$, $5$

Exploiting some parabolicity estimates of [DF08] and Theorem 1.4 of [BCN97b], we are also in the position of extending Theorem 1.5 to the case $n = 4$, $5$.

A similar result for $n = 4$ was also given in [AAC01], with different methods. To the best of our knowledge, the case $n = 5$ was not considered in any previous literature and new ingredients were needed to deal with it.

**Theorem 1.11.** *Let $n = 4$, $5$. Let $f \in C^1(\mathbb{R})$. Suppose that $u \in C^2(\overline{\mathbb{R}^n_+}) \cap L^\infty(\mathbb{R}^n_+)$ satisfies*

$$\Delta u(x) + f(u(x)) = 0 \qquad \text{and} \qquad u(x) > 0$$

*for any $x \in \mathbb{R}^n_+$, with*

$$u(x', 0) = 0 \quad \text{for any } x' \in \mathbb{R}^4.$$

*Assume*

- *either that $f(r) \geq 0$ for any $r \geq 0$,*
- *or that there exists $\zeta > 0$ in such a way that $f(r) \geq 0$ for any $r \in [0, \zeta]$ and $f(r) \leq 0$ for any $r \in [\zeta, +\infty)$.*

*Then, there exists $u_o : (0, +\infty) \to (0, +\infty)$ in such a way that*

$$u(x', x_n) = u_o(x_n) \quad \text{for any } (x', x_n) \in \mathbb{R}^n_+.$$

What happens in dimension $n \geq 6$ is, for what we know, an open question.

We also stress that we cannot prove Theorem 1.11 when $f$ is only locally Lipschitz, since we need to take limits of a suitable stability condition in (12.2) below, for which the continuity of $f'$ plays a crucial role.

## 1.8 Organization of the paper

The rest of this paper is organized as follows. Section 2 contains some elementary level set analysis, an easy parabolicity estimate and a classical consequence of the Maximum Principle.

Section 3 is inspired by some computations in [SZ98a, SZ98b, FSV08] and it develops the estimates necessary for proving Theorem 1.1. The proof of Theorem 1.1 is then finished in Section 4.

Some preliminary rigidity results are collected in Section 5.

Theorems 1.2 and 1.3 are then proved in Sections 6 and 7.

In Section 8 we prove Theorem 1.4.

Some energy estimates, inspired by [AAC01], are collected in Section 9.

The remaining main results are then proved in Sections 10–12.

# 2 Preliminaries

We collect here some elementary observations, to be used in the proofs of the main results.

## 2.1 Level set analysis

This part is a local version of a global geometric analysis performed in [FSV08] under greater generality (we give here full details for the reader's convenience).

In the forthcoming arguments, $U$ will denote an open subset of $\mathbb{R}^n$ and we will take $v \in C^2(U) \cap C(\overline{U})$.

Given $\bar{x} \in U$, we denote the level set of $v$ through $\bar{x}$ by

$$L_{v,\bar{x}} := \{x \in U \text{ s.t. } v(x) = v(\bar{x})\}.$$

We will suppose in what follows that $\nabla v(x) \neq 0$ for any $x \in U$. In particular, $L_{v,\bar{x}}$ is a smooth hypersurface.

**Lemma 2.1.** *Let $M$ be a connected component of $L_{v,x}$.*

*Suppose that $M \neq \emptyset$ and $M$ is contained in a hyperplane $\pi$. Then, $M$ agrees with a connected component of $\pi \cap U$.*

*Proof.* We show that

$$M \text{ is open in the topology of } \pi \cap U. \tag{2.1}$$

For this, let $z \in M$: we show that there exists an open set $O \subset \mathbb{R}^n$ containing $z$ and such that $O \cap (\pi \cap U) \subseteq M$.

To check this, we observe that since $M$ is a smooth connected hypersurface lying in $\pi$, there exists an open set $O \subset U$, for which $z \in O$ and $O \cap M = O \cap \pi$.

As a consequence,

$$O \cap (\pi \cap U) = O \cap \pi = O \cap M \subseteq M,$$

proving (2.1).

Also, $M$ is obviously closed in $U$, since so is $L_{v,\bar{x}}$, and this, together with (2.1), gives the desired claim. ☐

**Lemma 2.2.** *Suppose that a non-empty connected component $\bar{L}$ of $L_{v,x}$ has zero principal curvatures at all points.*

*Then, there exists a hyperplane $\pi$ is such a way that $\bar{L}$ agrees with a connected component of $\pi \cap U$.*

*Proof.* We use an elementary differential geometry argument (see, for instance, page 311 in [Ser94]). Since the principal curvatures vanish identically, the normal of $\bar{L}$ is constant, thence $\bar{L}$ is contained in a hyperplane.

Then, the claim follows from Lemma 2.1. ☐

Following is the flatness result for domain that we will use in the sequel:

**Lemma 2.3.** *Let $v > 0$ in $U$ and $v = 0$ on $\partial U$.*

*Suppose that any connected component of $L_{v,x}$ has zero principal curvatures at all points.*

*Then, there exists $v_o : \mathbb{R} \to \mathbb{R}$ and $\omega \in S^{n-1}$ in such a way that any connected component of $U$ is a slab of the form*

$$\big\{x \in \mathbb{R}^n \ \ s.t. \ \ \omega \cdot x \in (a, b)\big\},$$

*for suitable $a$, $b \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$, and $v(x) = v_o(\omega \cdot x)$, for any $x \in U$.*

*Proof.* Each connected component $\bar{L}$ of $L_{v,x}$ agrees with a connected component $\pi_{\bar{L}} \cap U$, for some hyperplane $\pi_{\bar{L}}$, due to Lemma 2.2.

Moreover, since $v > 0$ in the connected component of $L_{v,\bar{x}}$, such a connected component cannot touch $\partial U$.

Consequently, the connected components of any level set of $v$ are all portions of hyperplanes that do not meet $\partial U$ and therefore they must all be parallel hyperplanes. □

### 2.2 A consequence of the Maximum Principle

**Lemma 2.4.** *Let $u \in C^2(\overline{\mathbb{R}^n_+}) \cap L^\infty(\mathbb{R}^n_+)$ be a solution of $\Delta u + f(u) = 0$ in $\mathbb{R}^n_+$, with $f$ locally Lipschitz and $u(x', 0) = 0$ for any $x' \in \mathbb{R}^{n-1}$.*

*Assume that there exists $\zeta > 0$ in such a way that $f(r) \geq 0$ for any $r \in [0, \zeta]$ and $f(r) \leq 0$ for any $r \in [\zeta, +\infty)$.*

*Then, $u(x) \leq \zeta$ for any $x \in \mathbb{R}^n_+$.*

*Proof.* Suppose, by contradiction that $u(x_o) > \zeta$ for some $x_o \in \mathbb{R}^n_+$. Then, there exists an open connected neighborhood $D \subset \mathbb{R}^n_+$ of $x_o$ for which $u(x) > \zeta$ for any $x \in D$ and $u(x) = \zeta$ for any $x \in \partial D$.

Let $z(x) := \zeta - u(x)$. Then, $z < 0$ in $D$ and

$$\|z\|_{L^\infty(D)} \leq \zeta + \|u\|_{L^\infty(\mathbb{R}^n_+)}.$$

Also,

$$\Delta z(x) = -\Delta u(x) = f(\zeta - z(x)) \leq 0$$

for any $x \in D$, while $z = 0$ on $\partial D$.

As a consequence of this and of the Maximum Principle (see the version of it given in Lemma 2.1 of [BCN97a]), we would get that $z \geq 0$ in $D$, which is a contradiction. □

## 3 A Poincaré-type inequality

Following is a modification of a standard result relating monotone and stable solutions (see [AAC01, FSV08]). Differently from the previous literature, boundary terms will be taken into account here.

**Lemma 3.1.** *Let $\Omega$ be an open subset of $\mathbb{R}^n$ with $C^3$ boundary and let $\nu$ be its exterior normal.*
*Let $f$ be a locally Lipschitz function.*
*Suppose that $u \in C^2(\overline{\Omega})$ satisfies*

$$\Delta u(x) + f(u(x)) = 0 \tag{3.1}$$

*and*

$$\partial_n u(x) > 0 \tag{3.2}$$

*for any $x \in \Omega$.*
*Suppose also that*

$$u \text{ is constant on } \partial\Omega. \tag{3.3}$$

*Then,*

$$\int_\Omega \Big[ |\nabla\phi|^2 - f'(u)\frac{\partial_n u \phi^2}{\epsilon + \partial_n u} \Big] - \int_{\partial\Omega} \frac{\phi^2}{\epsilon + \partial_n u} \nabla \partial_n u \cdot \nu \geq 0, \tag{3.4}$$

*for any $\phi \in W^{1,\infty}_0(K)$, for any compact set $K \subset \mathbb{R}^n$ and for any $\epsilon > 0$.*

*Proof.* We remark that

$$u \in W^{3,2}\big(B_r(p) \cap \Omega\big) \text{ for any } r > 0 \text{ and any } p \in \mathbb{R}^n, \tag{3.5}$$

thanks to (3.3) here above and Theorem 8.13 in [GT83].
Now, let $K \subset \mathbb{R}^n$ be compact and $\phi \in W^{1,\infty}_0(K)$. Note that the map $x \mapsto f(u(x))$ is locally Lipschitz, so, from (3.1),

$$\Delta \partial_n u(x) + f'(u(x))\partial_n u(x) = 0 \tag{3.6}$$

for almost any $x \in \Omega$. We now fix $\epsilon > 0$ and we use (3.2), (3.5) and (3.6) to get

$$\begin{aligned}
0 &= -\int_\Omega \Big( \Delta\partial_n u + f'(u)\partial_n u \Big) \frac{\phi^2}{\epsilon + \partial_n u} \\
&= \int_\Omega \nabla\partial_n u \cdot \nabla \frac{\phi^2}{\epsilon + \partial_n u} - f'(u)\frac{\partial_n u \phi^2}{\epsilon + \partial_n u} - \int_{\partial\Omega} \frac{\phi^2}{\epsilon + \partial_n u} \nabla\partial_n u \cdot \nu \\
&= \int_\Omega 2\phi \nabla\partial_n u \cdot \frac{\nabla\phi}{\epsilon + \partial_n u} - \frac{|\nabla\partial_n u|^2 \phi^2}{(\epsilon + \partial_n u)^2} - f'(u)\frac{\partial_n u \phi^2}{\epsilon + \partial_n u} - \int_{\partial\Omega} \frac{\phi^2}{\epsilon + \partial_n u}\nabla\partial_n u \cdot \nu.
\end{aligned}$$

Cauchy Inequality then yields (3.4). □

Next result is a variation of a geometric formula of [SZ98a, SZ98b]. Differently from [SZ98a, SZ98b], we will keep track of the boundary terms.

**Proposition 3.2.** *Let $\Omega$ be an open subset of $\mathbb{R}^n$ with $C^3$ boundary and let $\nu$ be its exterior normal.*
*Suppose that $u \in C^2(\overline{\Omega})$ satisfies* (3.1), (3.2) *and* (3.3)*, and that $f$ is locally Lipschitz.*

*Then,*

$$\begin{aligned}
&\int_\Omega \Big(|\nabla u|^2\mathcal{K}^2 + |\nabla_T|\nabla u||^2\Big)\varphi^2 \\
&\qquad + \limsup_{\epsilon\to 0^+}\int_{\partial\Omega}\frac{\varphi^2}{\epsilon+\partial_n u}\Big(|\nabla u|^2\partial^2_{n,\nu}u - \partial^2_{i,\nu}u\partial_i u\partial_n u\Big) \\
&\qquad\quad \le \int_\Omega |\nabla u|^2|\nabla\varphi|^2
\end{aligned} \tag{3.7}$$

*for any* $\varphi \in W_0^{1,\infty}(K)$*, for any compact set* $K \subset \mathbb{R}^n$*.*

*Proof.* Given a matrix $M \in \mathrm{Mat}(n\times n)$, we define

$$|M| := \sqrt{\sum_{1\le i,j\le n} M_{i,j}^2}\,. \tag{3.8}$$

Since the map $x \mapsto f(u(x))$ is locally Lipschitz, by differentiating (3.1) we obtain, almost everywhere,

$$\Delta(\partial_i u) + f'(u)\partial_i u = 0.$$

Therefore, for any test function $\varphi$, we have that

$$\begin{aligned}
&\int_\Omega f'(u)\,|\nabla u|^2\frac{\partial_n u\varphi^2}{\epsilon+\partial_n u} = \int_\Omega f'(u)\,\partial_i u\partial_i u\frac{\partial_n u\varphi^2}{\epsilon+\partial_n u} \\
&= -\int_\Omega \Delta(\partial_i u)\partial_i u\frac{\partial_n u\varphi^2}{\epsilon+\partial_n u} \\
&\quad = \int_\Omega \nabla(\partial_i u)\cdot\nabla\left(\partial_i u\frac{\partial_n u\varphi^2}{\epsilon+\partial_n u}\right) - \int_{\partial\Omega}\partial_\nu(\partial_i u)\partial_i u\frac{\partial_n u\varphi^2}{\epsilon+\partial_n u} \\
&\quad\quad = \int_\Omega |D^2u|^2\frac{\partial_n u\varphi^2}{\epsilon+\partial_n u} + \int_\Omega \partial_i u\,\nabla(\partial_i u)\cdot\nabla\left(\frac{\partial_n u\varphi^2}{\epsilon+\partial_n u}\right) \\
&\quad\quad\quad - \int_{\partial\Omega}\partial^2_{i,\nu}u\partial_i u\frac{\partial_n u\varphi^2}{\epsilon+\partial_n u},
\end{aligned} \tag{3.9}$$

where (3.5), (3.8) and the standard repeated index summation have been used.

We now make use of Lemma 3.1 by choosing $\phi := |\nabla u|\varphi$. Then, (3.4) and (3.9) imply that

$$
\begin{aligned}
&\int_{\partial\Omega} \frac{|\nabla u|^2\varphi^2}{\epsilon+\partial_n u}\partial^2_{n,\nu}u \\
&\quad\le \int_\Omega |\nabla|\nabla u||^2\varphi^2 + |\nabla u|^2|\nabla\varphi|^2 + 2|\nabla u|\varphi\nabla|\nabla u|\cdot\nabla\varphi - f'(u)|\nabla u|^2\frac{\partial_n u\varphi^2}{\epsilon+\partial_n u} \\
&\quad= \int_\Omega |\nabla|\nabla u||^2\varphi^2 + |\nabla u|^2|\nabla\varphi|^2 + \frac{1}{2}\nabla|\nabla u|^2\cdot\nabla\varphi^2 \\
&\qquad - \int_\Omega \left[|D^2u|^2\frac{\partial_n u\varphi^2}{\epsilon+\partial_n u} + \partial_i u\,\nabla(\partial_i u)\cdot\nabla\left(\frac{\partial_n u\varphi^2}{\epsilon+\partial_n u}\right)\right] + \int_{\partial\Omega}\partial^2_{i,\nu}u\partial_i u\frac{\partial_n u\varphi^2}{\epsilon+\partial_n u} \qquad (3.10)\\
&\quad= \int_\Omega |\nabla|\nabla u||^2\varphi^2 + |\nabla u|^2|\nabla\varphi|^2 + \frac{1}{2}\nabla|\nabla u|^2\cdot\nabla\varphi^2 \\
&\qquad - \int_\Omega |D^2u|^2\frac{\partial_n u\varphi^2}{\epsilon+\partial_n u} + \int_{\partial\Omega}\partial^2_{i,\nu}u\partial_i u\frac{\partial_n u\varphi^2}{\epsilon+\partial_n u} \\
&\qquad - \frac{1}{2}\int_\Omega\left[\nabla|\nabla u|^2\cdot\nabla\varphi^2\frac{\partial_n u}{\epsilon+\partial_n u} + \varphi^2\nabla|\nabla u|^2\cdot\nabla\frac{\partial_n u}{\epsilon+\partial_n u}\right].
\end{aligned}
$$

By rearranging the terms of (3.10), and by employing (3.2) and the Dominated Convergence Theorem, we thus obtain that

$$
\begin{aligned}
&\limsup_{\epsilon\to 0^+}\int_{\partial\Omega}\frac{|\nabla u|^2\varphi^2}{\epsilon+\partial_n u}\partial^2_{n,\nu}u - \partial^2_{i,\nu}u\partial_i u\frac{\partial_n u\varphi^2}{\epsilon+\partial_n u} \\
&\quad\le \int_\Omega |\nabla|\nabla u||^2\varphi^2 + |\nabla u|^2|\nabla\varphi|^2 + \frac{1}{2}\nabla|\nabla u|^2\cdot\nabla\varphi^2 \\
&\qquad - \liminf_{\epsilon\to 0^+}\int_\Omega\left[|D^2u|^2\frac{\partial_n u\varphi^2}{\epsilon+\partial_n u} + \frac{1}{2}\nabla|\nabla u|^2\cdot\nabla\varphi^2\frac{\partial_n u}{\epsilon+\partial_n u} + \frac{1}{2}\varphi^2\nabla|\nabla u|^2\cdot\nabla\frac{\partial_n u}{\epsilon+\partial_n u}\right] \\
&\quad= \int_\Omega |\nabla|\nabla u||^2\varphi^2 + |\nabla u|^2|\nabla\varphi|^2 - |D^2u|^2\varphi^2.
\end{aligned}
$$

This gives (3.7), thanks to the following differential geometry formula (see [SZ98a, SZ98b]), which holds on $\{\nabla u \neq 0\}$:

$$|D^2u|^2 - |\nabla|\nabla u||^2 = |\nabla u|^2\mathcal{K}^2 + |\nabla_T|\nabla u||^2. \qquad \square$$

**Remark 3.3.** It is worth to note that condition (3.3) was used in Lemma 3.1 and Proposition 3.2 only to obtain (3.5). Accordingly, the results of Lemma 3.1 and Proposition 3.2 are still valid if (3.3) is dropped, but (3.5) holds true.

**Corollary 3.4.** *Let $\Omega$ be an open subset of $\mathbb{R}^n$ with $C^3$ boundary and let $\nu$ be its exterior normal.*

*Suppose that $u \in C^2(\overline{\Omega})$ satisfies* (3.1) *and* (3.2) *with $f$ locally Lipschitz.*

*Let*

$$u \text{ be constant on } \partial\Omega. \qquad (3.11)$$

*Assume also that either*

$$\Omega = \mathbb{R}^n_+ \qquad (3.12)$$

*or*

$$\partial_\nu u \text{ is constant on } \partial\Omega. \qquad (3.13)$$

*Then*
$$\int_\Omega \Big( |\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2 \Big) \varphi^2 \le \int_\Omega |\nabla u|^2 |\nabla \varphi|^2 \tag{3.14}$$
*for any* $\varphi \in W_0^{1,\infty}(K)$*, for any compact* $K \subset \mathbb{R}^n$*.*

*Proof.* We claim that, on $\partial\Omega$,
$$|\nabla u|^2 \partial^2_{n,\nu} u = \partial^2_{i,\nu} u \partial_i u \partial_n u. \tag{3.15}$$
Indeed, if (3.12) holds, we have that $\partial_i u = 0$ on $\partial\Omega$ for any $i = 1, \ldots, n-1$, due to (3.11), and this plainly gives (3.15).

If, on the other hand, (3.13) holds, we have that at any point of $\partial\Omega$
$$\nabla(\partial_\nu u) = \pm |\nabla(\partial_\nu u)| \nu$$
and so
$$\nabla(\partial_\nu u) \cdot \nu = \pm |\nabla(\partial_\nu u)|.$$
Therefore, on $\partial\Omega$,
$$\begin{aligned} |\nabla u|^2 \partial^2_{n,\nu} u &= |\nabla u|^2 \nabla(\partial_\nu u) \cdot e_n \\ &= \pm |\nabla u|^2 |\nabla(\partial_\nu u)| \big(\nu \cdot e_n\big) = |\nabla u|^2 \big(\nabla(\partial_\nu u) \cdot \nu\big) \big(\nu \cdot e_n\big). \end{aligned} \tag{3.16}$$
Analogously, at any point of $\partial\Omega$,
$$\nabla u = \pm |\nabla u| \, \nu$$
due to (3.11).

Consequently, on $\partial\Omega$,
$$\partial^2_{i,\nu} u \partial_i u \partial_n u = \big(\nabla(\partial_\nu u) \cdot \nabla u\big) \big(\nabla u \cdot e_n\big) = |\nabla u|^2 \big(\nabla(\partial_\nu u) \cdot \nu\big) \big(\nu \cdot e_n\big).$$
This identity and (3.16) show that (3.15) holds in this case too.

Then, (3.14) follows from (3.7) and (3.15). □

We observe that the geometric estimate of [SZ98a, SZ98b] is thus fully recovered in the framework given by (3.11) and either (3.12) or (3.13), thanks to (3.14).

Differently from [SZ98a, SZ98b], however, formula (3.14) holds for test functions whose support may cross the boundary (i.e., in the notation of Corollary 3.4, we may have that $K \cap \partial\Omega \neq \emptyset$).

# 4 Proof of Theorem 1.1

This follows by gathering the results of Lemma 3.1, Proposition 3.2 and Corollary 3.4. □

# 5 Some symmetry results

**Lemma 5.1.** *Suppose that $\Omega$ is an open, connected subset of $\mathbb{R}^2$ and that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies* (3.14)*. Assume also that $|\nabla u| \in L^\infty(\Omega)$.*
*Then, up to isometry, $\Omega$ is a slab of the form*

$$\{(x_1, x_2) \in \mathbb{R}^2 \ \ s.t. \ \ x_2 \in (a, b)\},$$

*for suitable $a, b \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$.*
*Moreover, there exists $u_o : (0, +\infty) \to (0, +\infty)$ in such a way that*

$$u(x_1, x_2) = u_o(x_2) \quad \textit{for any } (x_1, x_2) \in \Omega.$$

*Proof.* Given $R \geq 1$, we set

$$\varphi_R(x) := \chi_{B_{\sqrt{R}}}(x) + \frac{2\ln(R/|x|)}{\ln R} \chi_{B_R \setminus B_{\sqrt{R}}}(x). \tag{5.1}$$

We plug $\varphi_R$ inside (3.14), to obtain

$$\int_{\Omega \cap B_{\sqrt{R}}} \Big( |\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2 \Big) \leq \frac{C}{(\ln R)^2} \int_{B_R \setminus B_{\sqrt{R}}} \frac{1}{|x|^2} \leq \frac{C'}{\log R}, \tag{5.2}$$

for appropriate $C$, $C' > 0$, thanks to the fact that $|\nabla u|$ is bounded.
By taking $R$ arbitrarily large, we deduce that both $\mathcal{K}$ and $|\nabla_T |\nabla u||$ vanish identically in $\Omega$.
Consequently, the desired claim follows from Lemma 2.3. □

**Remark 5.2.** It is worth noting that Lemma 5.1 deeply uses the fact of working in dimension 2 in the last estimate in (5.2). Analogously, the choice of the capacitary function in (5.1) is motivated by the fundamental solution in $\mathbb{R}^2$ and it would not work, in general, in $\mathbb{R}^n$.

**Corollary 5.3.** *Let $\Omega$ be an open, connected subset of $\mathbb{R}^2$ with $C^3$ boundary.*
*Suppose that $u \in C^2(\overline{\Omega})$ satisfies* (3.1) *and* (3.2)*, that $|\nabla u| \in L^\infty(\Omega)$ and that*

$$\limsup_{\epsilon \to 0^+} \int_{\partial\Omega} \frac{\varphi^2}{\epsilon + \partial_n u} \Big( |\nabla u|^2 \partial^2_{n,\nu} u - \partial^2_{i,\nu} u \partial_i u \partial_n u \Big) \geq 0. \tag{5.3}$$

*Then, up to isometry, $\Omega$ is a slab of the form*

$$\{(x_1, x_2) \in \mathbb{R}^2 \ \ s.t. \ \ x_2 \in (a, b)\},$$

*for suitable $a, b \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$.*
*Also, there exists $u_o : (0, +\infty) \to (0, +\infty)$ in such a way that*

$$u(x_1, x_2) = u_o(x_2) \quad \textit{for any } (x_1, x_2) \in \Omega.$$

*Proof.* We make use of Proposition 3.2 to attain that (3.7) holds true. Then, from (5.3), we conclude that (3.14) holds true as well.
The claim thus follows from Lemma 5.1 and the strict monotonicity of $u$ in the $n$th direction. □

## 6 Proof of Theorem 1.2

The assumptions of Corollary 3.4 are implied by the ones of Theorem 1.2 and therefore (3.14) holds true.

Thence Theorem 1.2 follows from Lemma 5.1. □

## 7 Proof of Theorem 1.3

By Theorem 1.1' in [BCN97b], we have that

$$\partial_2 u > 0 \text{ in } \mathbb{R}^2_+. \tag{7.1}$$

Thence, (1.6) follows from Theorem 1.2. □

## 8 Proof of Theorem 1.4

We first prove (1.8). Since this proof will be quite long, several intermediate claims will be emphasized in display.

The technique for proving (1.8) is inspired by [CGS94].

Differently from [CGS94], due to the presence of the boundary, several technicals details are needed here.

We define

$$G(t) := \sup_{r\in[0,\|u\|_{L^\infty(\mathbb{R}^n_+)}]} F(r) - F(t).$$

Note that

$$G(t) \geq 0 \tag{8.1}$$

for any $t \in [0, \|u\|_{L^\infty(\mathbb{R}^n_+)}]$.

Also, given $v \in C^2(\mathbb{R}^n_+)$ and $x \in \mathbb{R}^n_+$, following [CGS94], we define

$$P(v,x) := |\nabla v(x)|^2 - 2G(v(x)). \tag{8.2}$$

Given $\ell \in \mathbb{R}$, we also set

$$\begin{aligned} \mathfrak{F}_\ell \;\; := \;\; & \Big\{ v \in C^2(\mathbb{R}^{n-1}\times[\ell,+\infty)) \text{ solutions of } \Delta v = G'(v) \text{ in } \mathbb{R}^{n-1}\times(\ell,+\infty), \\ & \text{with } 0 \leq v \leq \|u\|_{L^\infty(\mathbb{R}^n_+)} \text{ and } v(x',\ell) = 0 \text{ for any } x' \in \mathbb{R}^{n-1} \Big\}. \end{aligned}$$

Given any domain $\Omega \subseteq \mathbb{R}^n$ and $v \in C^2(\Omega)$ satisfying $\Delta v = G'(v)$ in $\Omega$, we recall (see formula (2.7) of [CGS94]) that[1]

$$|\nabla v(x)|^2 \Delta P(v,x) - 2G'(v(x))\nabla v(x)\cdot\nabla P(v,x) \geq \frac{|\nabla P(v,x)|^2}{2} \tag{8.3}$$

[1] A caveat has to be taken into account in order to use here the computation of [CGS94]. Namely, $G$ is here only $C^{1,1}_{\rm loc}(\mathbb{R})$, so, in principle, the computation of [CGS94] has to be understood in the weak distributional sense. See also Lemma 4.11 in [FSV08] for further comments.

at any point $x \in \Omega$ where $\nabla v(x) \neq 0$.

We consider

$$P_o := \sup_{\substack{\ell \in \mathbb{R},\, v \in \mathfrak{F}_\ell \\ x \in \mathbb{R}^{n-1} \times (\ell, +\infty)}} P(v, x).$$

We observe that the above sup is finite, thanks to standard elliptic estimates.

We claim that

$$P_o \leq 0. \tag{8.4}$$

The proof of (8.4) is quite long and it will be finished after (8.34).

To prove (8.4), we argue by contradiction. We suppose that

$$P_o > 0 \tag{8.5}$$

and we take $\ell_k \in \mathbb{R}$, $v_k \in \mathfrak{F}_{\ell_k}$ and $x_k \in \mathbb{R}^n \times (\ell_k, +\infty)$ in such a way that

$$P_o - \frac{1}{k} \leq P(v_k, x_k).$$

We define

$$\ell'_k := \ell_k - (x_k)_n \quad \text{ and } \quad u_k(x) := v_k(x + x_k).$$

Note that

$$\ell'_k < 0, \tag{8.6}$$

since $x_k \in \mathbb{R}^{n-1} \times (\ell_k, +\infty)$, that $u_k \in C^2(\mathbb{R}^{n-1} \times [\ell'_k, +\infty))$ is a solution of $\Delta u_k = G'(u_k)$ in $\mathbb{R}^{n-1} \times (\ell'_k, +\infty)$, with $0 \leq u_k \leq \|u\|_{L^\infty(\mathbb{R}^n)}$, that $u_k(x', \ell'_k) = 0$ for any $x' \in \mathbb{R}^{n-1}$ and that

$$P_o - \frac{1}{k} \leq P(u_k, 0) \leq P_o. \tag{8.7}$$

By elliptic regularity (see, e.g., page 311 of [GT83]),

$$\|u_k\|_{C^{2,1/2}(\mathbb{R}^n_+ \times [\ell'_k, +\infty))} \leq C, \tag{8.8}$$

for a suitable $C > 0$, possibly depending on $\|u\|_{L^\infty(\mathbb{R}^n_+)}$.

We claim that

$$\ell'_k \text{ is bounded.} \tag{8.9}$$

To prove (8.9), we suppose the converse. Then, by (8.6), we have that $\ell'_k \to -\infty$, up to subsequence.

Therefore, by (8.8), $u_k$ converges, up to subsequence, in $C^2_{\rm loc}(\mathbb{R}^n)$ to some $u_\infty \in C^2(\mathbb{R}^n)$ which solves

$$\Delta u_\infty = G'(u_\infty) \text{ in } \mathbb{R}^n. \tag{8.10}$$

By taking the limit in (8.7), we also have that $P_o = P(u_\infty, 0)$, and so, by (8.5), that

$$|\nabla u_\infty(0)|^2 - 2G(u_\infty(0)) > 0.$$

This and (8.10) are in contradiction with Lemma 4.11 in [FSV08], so (8.9) is proved.

In light of (8.9), up to subsequence, we may and do suppose that

$$\ell'_k \to \ell', \text{ for some } \ell' \in \mathbb{R}. \tag{8.11}$$

In fact, from (8.6),

$$\ell' \leq 0. \tag{8.12}$$

We now take $\tilde{u}_k \in C^{2,1/2}(\mathbb{R}^n)$ to be an extension of $u_k$, that is $\tilde{u}_k = u_k$ on $\mathbb{R}^{n-1} \times [\ell'_k, +\infty)$, for which

$$\|\tilde{u}_k\|_{C^{2,1/2}(\mathbb{R}^n)} \leq C. \tag{8.13}$$

We recall that such an extension is possible, due to (8.8) and Lemma 6.37 of [GT83].

Consequently, we have that $\tilde{u}_k$ converges, up to subsequence, in $C^2_{\mathrm{loc}}(\mathbb{R}^n)$ to some $u_\infty \in C^2(\mathbb{R}^n)$, with

$$\|u_\infty\|_{C^2(\mathbb{R}^n)} \leq C. \tag{8.14}$$

By (8.11) and (8.14), we have that $u_\infty$ is a solution of

$$\Delta u_\infty = G'(u_\infty) \text{ in } \mathbb{R}^{n-1} \times [\ell', +\infty). \tag{8.15}$$

Then, for any $\epsilon > 0$, there exists $k_\epsilon$ such that, if $k \geq k_\epsilon$,

$$|u_\infty(x', \ell')| \leq |u_k(x', \ell')| + \epsilon,$$

and so, recalling (8.8) once more,

$$|u_\infty(x', \ell')| \leq |u_k(x', \ell'_k)| + C|\ell' - \ell'_k| + \epsilon = C|\ell' - \ell'_k| + \epsilon.$$

As a consequence, taking $k$ arbitrarily large,

$$|u_\infty(x', \ell')| \leq \epsilon.$$

Then, taking $\epsilon$ as small as we wish, we deduce that

$$u_\infty(x', \ell') = 0 \qquad \text{for any } \; x' \in \mathbb{R}^{n-1}. \tag{8.16}$$

Also,

$$u_\infty(x) = \lim_{k \to +\infty} u_k(x) \in \Big[0, \|u\|_{L^\infty(\mathbb{R}^n_+)}\Big] \tag{8.17}$$

so $u_\infty \in \mathfrak{F}_{\ell'}$, and, passing to the limit in (8.7) and recalling (8.5),

$$|\nabla u_\infty(0)|^2 - 2G(u_\infty(0)) = P(u_\infty, 0) = P_o > 0. \tag{8.18}$$

In particular,

$$u_\infty \text{ cannot be constant,} \tag{8.19}$$

otherwise, by (8.16), it would vanish identically and so, from (8.1), (8.17) and (8.18),

$$0 \geq 0 - 2G(0) = |\nabla u_\infty(0)|^2 - 2G(u_\infty(0)) > 0.$$

This contradiction proves (8.19).

By (1.7), (8.15), (8.16), (8.17), (8.19) and Hopf Principle (see Lemma 3.4 in [GT83]), we thus obtain that

$$\partial_n u_\infty(x', \ell') > 0 \qquad \text{for any } \; x' \in \mathbb{R}^{n-1}. \tag{8.20}$$

Furthermore, we have that

$$\inf_{\mathbb{R}^{n-1} \times (\ell', +\infty)} |\nabla u_\infty| = 0. \tag{8.21}$$

To check this, we argue by contradiction, supposing that

$$\inf_{\mathbb{R}^{n-1}\times(\ell',+\infty)} |\nabla u_\infty| \geq c > 0, \tag{8.22}$$

and we consider the solution $\gamma \in C^1(\mathbb{R}, \mathbb{R}^{n-1} \times (\ell', +\infty))$ of the ODE

$$\begin{cases} \gamma'(t) = \dfrac{\nabla u_\infty(\gamma(t))}{|\nabla u_\infty(\gamma(t))|} \\ \\ \gamma(0) = (0, \dots, 0, \ell' + 1). \end{cases}$$

Note that $\gamma$ is globally defined, due to (8.22) and it does not hit the boundary because of (8.20).

Consequently, utilizing (8.17) and (8.22), for any $t > 0$,

$$\begin{aligned} 2\|u\|_{L^\infty(\mathbb{R}^n_+)} &\geq u_\infty(\gamma(t)) - u_\infty(\gamma(0)) \\ &= \int_0^t \nabla u_\infty(\gamma(s)) \cdot \gamma'(s)\, ds \\ &= \int_0^t |\nabla u_\infty(\gamma(s))|\, ds \\ &\geq ct. \end{aligned}$$

The latter gives a contradiction for large $t$ and it thus proves (8.21).

Now, we claim that

$$\text{if } P(u_\infty, y) = P_o \text{ for some } y \in \mathbb{R}^{n-1} \times [\ell', +\infty), \text{ then } y_n = \ell'. \tag{8.23}$$

To prove this, we argue again by contradiction, and we suppose that $y$ lies in $\mathbb{R}^{n-1} \times (\ell', +\infty)$. Then,

$$\text{the set } U := \Big\{ x \in \mathbb{R}^{n-1} \times (\ell', +\infty) \text{ s.t. } P(u_\infty, x) = P_o \Big\} \text{ is non-empty.} \tag{8.24}$$

We claim that

$$\text{if } x \in U, \text{ then } |\nabla u_\infty(x)| > 0. \tag{8.25}$$

To prove this, we take $x \in U$ and we deduce from (8.1), (8.5) and (8.17) that

$$0 < P_o = P(u_\infty, x) = |\nabla u_\infty(x)|^2 - G(v(x)) \leq |\nabla u_\infty(x)|^2,$$

proving (8.25).

Moreover, since $u_\infty \in C^2(\mathbb{R}^n)$, we have that

$$U \text{ is closed in } \mathbb{R}^{n-1} \times (\ell', +\infty). \tag{8.26}$$

We plan to prove that

$$U \text{ is also open.} \tag{8.27}$$

For this, we take $x$ in $U$ and employ (8.25) to deduce that

$$\inf_{B_{r_x}(x)} |\nabla u_\infty| > 0$$

for some small $r_x > 0$. This, (8.3) and the Strong Maximum Principle (see Theorem 8.19 of [GT83]) imply that $P(y, u_\infty) = P_o$ for any $y \in B_{r_x}(x)$.

This proves (8.27).

From (8.24), (8.26) and (8.27), we conclude that

$$U = \mathbb{R}^{n-1} \times (\ell', +\infty). \tag{8.28}$$

We now recall (8.21) and we take $x_j \in \mathbb{R}^{n-1} \times (\ell', +\infty)$ in such a way that

$$\lim_{j \to +\infty} |\nabla u_\infty(x_j)| = 0. \tag{8.29}$$

Then, from (8.1), and (8.28)

$$P_o = P(u_\infty, x_j) = |\nabla u_\infty(x_j)|^2 - 2G(u_\infty(x_j)) \leq |\nabla u_\infty(x_j)|^2$$

for any $j \in \mathbb{N}$.

Therefore, by (8.29), we obtain $P_o \leq 0$, in contradiction with (8.5).

This proves (8.23).

What is more, from (8.18) and (8.23), we deduce that

$$\ell' = 0.$$

Then,

$$P(u_\infty, 0) = P_o \geq P(u_\infty, x) \text{ for any } x \in \mathbb{R}^{n-1} \times (\ell', +\infty) = \mathbb{R}^n_+. \tag{8.30}$$

Note that

$$P(u_\infty, x) < P_o \text{ for any } x \in \mathbb{R}^{n-1} \times (\ell', +\infty), \tag{8.31}$$

thanks to (8.23).

Consequently, (8.3), (8.20), (8.30), (8.31) and Hopf Principle (see Theorem 5.5.1 on page 120 of [PS07]) give that

$$\partial_n P(u_\infty, 0) < 0. \tag{8.32}$$

On the other hand, from (8.2),

$$\partial_n P(u_\infty, x) = \nabla u_\infty(x) \cdot \nabla(\partial_n u_\infty(x)) - G'(u_\infty(x)) \partial_n u_\infty(x)$$

for any $x \in \mathbb{R}^n_+$ and so, from (8.14) and (8.16),

$$\partial_n P(u_\infty, 0) = \partial_n u_\infty(0) \Big( \partial^2_{n,n} u_\infty(0) - G'(0) \Big). \tag{8.33}$$

Moreover, by (8.15) and (8.16),

$$G'(0) = \Delta u_\infty(0) = \partial^2_{n,n} u_\infty(0) \tag{8.34}$$

and so (8.33) gives that $\partial_n P(u_\infty, 0) = 0$.

Since this is in contradiction with (8.32), the proof of (8.4) is now finished.

Now, from (8.4),

$$0 \geq P_o \geq \sup_{\substack{v \in \mathfrak{F}_0 \\ x \in \mathbb{R}^{n-1} \times (0, +\infty)}} P(v, x) \geq P(u, x),$$

for any $x \in \mathbb{R}^n_+$.

This and (8.2) imply (1.8).

We now prove (1.9). For this, we first prove that

$$\sup_{r\in[0,\|u\|_{L^\infty(\mathbb{R}^n_+)}]} F(r) = \max\Big\{F(0),\ F(\|u\|_{L^\infty(\mathbb{R}^n_+)})\Big\}. \tag{8.35}$$

To prove (8.35), we may restrict ourselves to the case in which $u$ is not constant (otherwise (8.35) is obvious). In such case, Theorem 1.1 in [BCN97b] states that

$$\partial_n u > 0 \text{ in } \mathbb{R}^n_+. \tag{8.36}$$

For proving (8.35), let us argue by contradiction, and suppose that

$$\sup_{r\in[0,\|u\|_{L^\infty(\mathbb{R}^n_+)}]} F(r) = F(\theta),$$

for some $\theta \in \Big(0, \|u\|_{L^\infty(\mathbb{R}^n_+)}\Big)$. Then, there must exists $x_+ \in \mathbb{R}^n_+$ for which $\theta = u(x_+)$ and so

$$\sup_{r\in[0,\|u\|_{L^\infty(\mathbb{R}^n_+)}]} F(r) = F(u(x_+)).$$

Therefore, by (1.8),

$$\frac{1}{2}|\nabla u(x_+)|^2 \le \sup_{r\in[0,\|u\|_{L^\infty(\mathbb{R}^n_+)}]} F(r) - F(u(x_+)) = 0.$$

Since this is in contradiction with (8.36), we have concluded the proof of (8.35).

We now complete the proof of (1.9) by arguing again by contradiction. Indeed, if (1.9) did not hold, we would have from (1.8) and (8.35) that

$$\frac{1}{2}|\nabla u(x_*)|^2 \le \sup_{r\in[0,\|u\|_{L^\infty(\mathbb{R}^n_+)}]} F(r) - F(u(x_*)) = \sup_{r\in[0,\|u\|_{L^\infty(\mathbb{R}^n_+)}]} F(r) - F(0) = 0,$$

for any $x^* \in \partial\mathbb{R}^n_+$.

Since the latter is in contradiction with Hopf Principle (see Lemma 3.4 in [GT83]), the proof of (1.9) is completed.

We now prove (1.10). For this, we make the above arguments more precise. Suppose, by contradiction, that $u$ is not identically zero and that

$$\sup_{r\in[0,\|u\|_{L^\infty(\mathbb{R}^n_+)}]} F(r) \;=\; F(\bar t)$$

for some $\bar t \in \big[0, \|u\|_{L^\infty(\mathbb{R}^n_+)}\big)$. Then, by the continuity of $u$, there exists $\bar x \in \overline{\mathbb{R}^n_+}$ for which $u(\bar x) = \bar t$ and therefore (1.8) gives

$$\frac{1}{2}|\nabla u(\bar x)|^2 \le \sup_{r\in[0,\|u\|_{L^\infty(\mathbb{R}^n_+)}]} F(r) - F(u(\bar x)) = \sup_{r\in[0,\|u\|_{L^\infty(\mathbb{R}^n_+)}]} F(r) - F(\bar t) = 0.$$

That is, $\partial_n u(\bar x) = 0$. Then, (8.36) says that $\bar x$ cannot belong to the interior of $\mathbb{R}^n_+$. But Hopf Principle says that $\bar x$ cannot lie on $\partial\mathbb{R}^n_+$ either, and this contradiction proves (1.10).

The proof of Theorem 1.4 is thus completed. $\square$

# 9 Energy estimates

In this section, $\Omega_o$ will denote a $C^3$ and uniformly Lipschitz epigraph of $\mathbb{R}^n$. In particular, we have that there exists $C_o > 0$ in such a way that

$$\int_{B_R(x)\cap\partial\Omega_o} d\sigma \le C_o R^{n-1} \tag{9.1}$$

for any $x \in \mathbb{R}^n$ and $R \ge 0$, where "$d\sigma$" denotes the surface element.
Let $f \in C^1(\mathbb{R})$ and

$$F(r) := \int_0^r f(\tau)\, d\tau. \tag{9.2}$$

For $R \ge 0$, we set $B_R^+ := B_R \cap \Omega_o$. Given $c \in \mathbb{R}$ and $v \in C^1(B_R^+)$, we also define

$$\mathcal{E}_{R,c}(v) := \int_{B_R^+} \frac{|\nabla v|^2}{2} - F(v) + c\, dx. \tag{9.3}$$

Also, given $v : \Omega_o \to \mathbb{R}$ and $t \ge 0$, we define

$$v^t(x', x_n) := v(x', x_n + t) \qquad \text{for any } (x', x_n) \in \Omega_o.$$

Note that the domain of $v^t$ does contain $\Omega_o$ since $t \ge 0$ and $\Omega_o$ is an epigraph.

**Lemma 9.1.** *Fix $c \in \mathbb{R}$. Let $v \in C^2(\overline{\Omega}_o)$. Suppose that*

$$\Delta v(x) + f(v(x)) = 0 \tag{9.4}$$

*and*

$$\partial_n v(x) > 0 \tag{9.5}$$

*for any $x \in \Omega_o$.*
*Assume also that there exists $M \ge 0$ in such a way that*

$$\sup_{x\in\Omega_o} |v(x)| + |\nabla v(x)| \le M. \tag{9.6}$$

*Then, there exists $C > 0$, possibly depending on $M$ but not on $c$, in such a way that*

$$\mathcal{E}_{R,c}(v) \le \mathcal{E}_{R,c}(v^t) + CR^{n-1}$$

*for any $t$, $R \ge 0$.*

*Proof.* We argue as in [AAC01]:

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_{R,c}(v^t) &= \int_{B_R^+} \nabla v^t \cdot \nabla(\partial_t v^t) - f(v^t)\partial_t v^t \\ &= \int_{\partial B_R^+} \partial_t v^t \nabla v^t \cdot \nu \\ &\ge -M \int_{\partial B_R^+} \partial_t v^t \, d\sigma, \end{aligned}$$

thanks to (9.4), (9.5) and (9.6).

Therefore, for any $T \geq 0$, by using Fubini's Theorem and (9.6) once more, we conclude that

$$\begin{aligned} \mathcal{E}_{R,c}(v^T) - \mathcal{E}_{R,c}(v) &= \int_0^T \frac{d}{dt}\mathcal{E}_{R,c}(v^t)\,dt \\ &\geq -M \int_{\partial B_R^+} \int_0^T \partial_t v^t\,dt\,d\sigma \\ &= -M \int_{\partial B_R^+} (v^T - v)\,d\sigma \\ &\geq -2M^2 \int_{\partial B_R^+} d\sigma, \end{aligned}$$

which gives the desired claim via (9.1). □

**Lemma 9.2.** *Let $u \in C^2(\Omega_o) \cap L^\infty(\Omega_o)$ be a solution of $\Delta u + f(u) = 0$ in $\Omega_o$, with $\partial_n u > 0$ in $\Omega_o$ and $f$ locally Lipschitz.*

*Then, the following limit exists*

$$u_\infty(x') := \lim_{x_n \to +\infty} u(x', x_n) \qquad \text{for any } x' \in \mathbb{R}^{n-1}. \tag{9.7}$$

*and the convergence is in $C^2_{\rm loc}(\mathbb{R}^{n-1})$.*

*Furthermore,*

$$\Delta u_\infty(x') + f(u_\infty(x')) = 0 \quad \text{ for any } x' \in \mathbb{R}^{n-1}. \tag{9.8}$$

*Also, if $\kappa_1^\infty, \dots, \kappa_{n-2}^\infty$ are the mean curvatures of the level sets of $u_\infty$ at points where $\nabla u_\infty \neq 0$ and $\mathcal{K}_\infty := \sqrt{(\kappa_1^\infty)^2 + \cdots + (\kappa_{n-2}^\infty)^2}$, we have that*

$$\int_{\mathbb{R}^{n-1}\cap\{\nabla u_\infty \neq 0\}} \Big(|\nabla u_\infty|^2 \mathcal{K}_\infty^2 + |\nabla_T |\nabla u_\infty||^2\Big)\varphi^2 \leq \int_{\mathbb{R}^{n-1}} |\nabla u_\infty|^2 |\nabla \varphi|^2, \tag{9.9}$$

*for any $\varphi \in W_0^{1,\infty}(\mathbb{R}^{n-1})$.*

*Proof.* We use the monotonicity and the boundedness of $u$ and to obtain that the limit in (9.7) exists, and the convergence is in $C^2_{\rm loc}(\mathbb{R}^{n-1})$, due to standard elliptic estimates (see, e.g., [GT83]).

This also implies (9.8).

We now prove (9.9), using some arguments of [FSV08].

For this, we use (1.4) to see that

$$\int_{\mathbb{R}^n} \Big(|\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2\Big)\phi^2 \leq \int_{\mathbb{R}^n} |\nabla u|^2 |\nabla \phi|^2 \tag{9.10}$$

for any $\phi \in W_0^{1,\infty}(\Omega_o)$ (note that $\phi$ in (9.10) is supported inside $\Omega_o$, so no boundary term arises).

As usual, the quantity $\mathcal{K}$ in (1.4) is obtained by the mean curvatures of the level sets of $u$. Thus,

$$|\nabla u_\infty(x')|^2 \mathcal{K}_\infty^2(x') = \lim_{x_n \to +\infty} |\nabla u(x', x_n)|^2 \mathcal{K}^2(x', x_n) \tag{9.11}$$

for any $x' \in \{\nabla u_\infty \neq 0\}$, and the above convergence is locally uniform.
Now, let $\varphi \in W_0^{1,\infty}(\mathbb{R}^{n-1})$. Let $S$ be the support of $\varphi$ and let

$$s := \sup_{\substack{(x',x_n)\in\partial\Omega_o\\ x'\in S}} x_n \in \mathbb{R}. \tag{9.12}$$

Let $\tau \in C_0^\infty([0,1])$ with

$$\int_{\mathbb{R}} \tau^2(t)\, dt = 1. \tag{9.13}$$

Fix also $\mu \in (0,1)$ and let $\eta(t) := \sqrt{\mu}\,\tau\big(\mu(t-s-1)\big)$.
Note that, by (9.13),

$$\int_{\mathbb{R}} \eta^2(t)\, dt = 1. \tag{9.14}$$

We now define $\phi(x',x_n) := \varphi(x')\eta(x_n)$.
We claim that

$$\text{for any } t \geq 0,\ \phi(x',x_n-t) \text{ vanishes when } (x',x_n) \text{ is outside } \Omega_o. \tag{9.15}$$

To check this, we argue by contradiction and suppose that $\phi(\bar{x}',\bar{x}_n-\bar{t}) \neq 0$, with $(\bar{x}',\bar{x}_n) \in \mathbb{R}^n \setminus \Omega_o$ and $\bar{t} \geq 0$.
Then, $\bar{x}' \in S$ and so, recalling (9.12),

$$s \geq \bar{x}_n. \tag{9.16}$$

Also, $\bar{x}_n - \bar{t}$ must lie in the support of $\eta$, that is $\mu(\bar{x}_n - \bar{t} - s - 1)$ lies in the support of $\tau$, which, in turn, is in $[0,1]$. This gives that

$$\bar{x}_n \geq \bar{t} + s + 1 \geq s + 1\,.$$

The latter estimate is in contradiction with (9.16) and therefore this proves (9.15).
Due to (9.15), we can plug the map $(x',x_n) \mapsto \phi(x',x_n+t)$ inside (9.10), for any $t \geq 0$. Accordingly, recalling also (9.11) and (9.14),

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}\cap\{\nabla u_\infty\neq 0\}} \Big(|\nabla u_\infty|^2\mathcal{K}_\infty^2 + |\nabla_T|\nabla u_\infty||^2\Big)\phi^2 \\
=\ & \int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}\cap\{\nabla u_\infty\neq 0\}} \Big(|\nabla u_\infty(x')|^2\mathcal{K}_\infty^2(x') + |\nabla_T|\nabla u_\infty||^2(x')\Big)\varphi^2(x')\eta^2(x_n)\, dx'\, dx_n \\
=\ & \lim_{t\to+\infty}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}\cap\{\nabla u_\infty\neq 0\}} \Big(|\nabla u(x',x_n+t)|^2\mathcal{K}^2(x',x_n+t) + |\nabla_T|\nabla u||^2(x',x_n+t)\Big) \\
&\qquad\cdot\phi^2(x',x_n)\, dx'\, dx_n \\
=\ & \lim_{t\to+\infty}\int_{\mathbb{R}}\int_{\mathbb{R}^{n-1}\cap\{\nabla u_\infty\neq 0\}} \Big(|\nabla u(x',x_n)|^2\mathcal{K}^2(x',x_n) + |\nabla_T|\nabla u||^2(x',x_n)\Big) \\
&\qquad\cdot\phi^2(x',x_n-t)\, dx'\, dx_n \\
\leq\ & \lim_{t\to+\infty}\int_{\mathbb{R}^n} |\nabla u(x',x_n)|^2|\nabla\phi(x',x_n-t)|^2\, d(x',x_n).
\end{aligned}$$

Therefore, by Cauchy Inequality, and using (9.14) once more,

$$
\begin{aligned}
&\int_{\mathbb{R}^{n-1}\cap\{\nabla u_\infty\neq 0\}} \Big(|\nabla u_\infty|^2\mathcal{K}_\infty^2+|\nabla_T|\nabla u_\infty||^2\Big)\phi^2 \\
\leq\ & \lim_{t\to+\infty}\int_{\mathbb{R}^n}|\nabla u(x',x_n)|^2|\nabla\varphi(x')\eta(x_n-t)+\varphi(x')\nabla\eta(x_n-t)|^2\,d(x',x_n) \\
\leq\ & \lim_{t\to+\infty}(1+\mu^{1/4})\int_{\mathbb{R}^n}|\nabla u(x',x_n)|^2|\nabla\varphi(x')|^2|\eta(x_n-t)|^2\,d(x',x_n) \\
&\quad +\frac{2}{\mu^{1/4}}\int_{\mathbb{R}^n}|\nabla u(x',x_n)|^2|\varphi(x')|^2|\eta'(x_n-t)|^2\,d(x',x_n) \\
=\ & \lim_{t\to+\infty}(1+\mu^{1/4})\int_{\mathbb{R}^n}|\nabla u(x',x_n+t)|^2|\nabla\varphi(x')|^2|\eta(x_n)|^2\,d(x',x_n) \\
&\quad +\frac{2\mu^3}{\mu^{1/4}}\int_{\mathbb{R}^n}|\nabla u(x',x_n+t)|^2|\varphi(x')|^2|\tau'(\mu(x_n-s-1))|^2\,d(x',x_n) \\
=\ & (1+\mu^{1/4})\int_{\mathbb{R}^{n-1}}|\nabla u_\infty(x')|^2|\nabla\varphi(x')|^2\,dx'+2\mu^{7/4}\int_{\mathbb{R}^{n-1}}|\nabla u_\infty(x')|^2|\varphi(x')|^2\,dx' \\
&\quad \cdot\int_0^1|\tau'(\theta))|^2\,d\theta
\end{aligned}
$$

The proof of (9.9) then follows by sending $\mu\to 0^+$. □

**Corollary 9.3.** *Let $f$ be locally Lipschitz. Suppose that*

*(C1) either $\Omega=\mathbb{R}^3_+$ and $f(0)\geq 0$,*

*(C2) or that $\Omega\subset\mathbb{R}^3$ is a $C^3$ and uniformly Lipschitz epigraph, and $f(r)\geq 0$ for any $r\geq 0$.*

*Assume that $u\in C^2(\overline{\Omega})\cap L^\infty(\Omega)$ satisfies*

$$\Delta u(x)+f(u(x))=0 \qquad \textit{for any } x\in\Omega. \tag{9.17}$$

*with $u\Big|_{\partial\Omega}=0$ and $\partial_3 u>0$ in $\Omega$.*
*Then, there exists $C>0$ in such a way that*

$$\int_{B_R\cap\Omega}|\nabla u(x)|^2\,dx\leq CR^2 \qquad \textit{for any } R\geq 1. \tag{9.18}$$

*Proof.* By Lemma 9.2, $u_\infty$ is a solution of (9.8) in $\mathbb{R}^2$, satisfying (9.9).
Therefore by [FSV08] (see in particular Corollary 2.6 and Lemma 2.11 there), we get that $u_\infty$ is one-dimensional, that is, there exists $v_\infty:\mathbb{R}\to\mathbb{R}$ and $\omega\in \mathrm{S}^1$ in such a way that

$$u_\infty(x)=v_\infty(\omega\cdot x) \qquad \text{for any } x\in\mathbb{R}^2. \tag{9.19}$$

Thus, from (9.8),

$$v''_\infty(t)+f(v_\infty(t))=0 \qquad \text{for any } t\in\mathbb{R}. \tag{9.20}$$

Cauchy Theorem on ODEs then imply that $v_\infty$ cannot have plateaus (i.e., intervals on which it is constant) unless it is identically constant. Hence, by Lemma 4.10 of [FSV08], we have that, if $v_\infty$ is not constant, then

$$\begin{aligned}&\text{either } v_\infty \text{ has at most one critical point}\\&\text{or there exist } t_-,\, t_+ \in \mathbb{R} \text{ such that}\\&v_\infty(t_-) < v_\infty(t_+) \text{ and } v'_\infty(t_-) = 0 = v'_\infty(t_+).\end{aligned} \tag{9.21}$$

Notice now that, if $F$ is as in (9.2), we have

$$\frac{d}{dt}\left(\frac{|v'_\infty(t)|^2}{2} + F(v_\infty(t))\right) = 0,$$

due to (9.20).

Therefore,

$$\frac{|v'_\infty(t)|^2}{2} + F(v_\infty(t)) = \frac{|v'_\infty(s)|^2}{2} + F(v_\infty(s)) \tag{9.22}$$

for any $s,\, t \in \mathbb{R}$.

Let now $t_n$ be such that

$$\lim_{n\to+\infty} v_\infty(t_n) = \sup_{\mathbb{R}} v_\infty.$$

Let $w_n(t) := v_\infty(t+t_n)$. We have that $w_n$ converges to some $w$ in $C^2_{\rm loc}(\mathbb{R})$. Also,

$$w(0) = \lim_{n\to+\infty} v_\infty(t_n) = \sup_{\mathbb{R}} v_\infty \geq \lim_{n\to+\infty} v_\infty(t+t_n) = w(t)$$

for any $t \in \mathbb{R}$ and so $w'(0) = 0$.

Thence, (9.22) gives that

$$\frac{|v'_\infty(t)|^2}{2} + F(v_\infty(t)) = \lim_{t\to+\infty} \frac{|v'_\infty(t_n)|^2}{2} + F(v_\infty(t_n)) = 0 + c \tag{9.23}$$

for any $t \in \mathbb{R}$, where

$$c := F(\sup_{\mathbb{R}} v_\infty). \tag{9.24}$$

We now claim that

$$\text{if (C2) holds, then } v_\infty \text{ is constant.} \tag{9.25}$$

Indeed, (C2) and (9.20) imply that $v''_\infty \leq 0$ and so (9.25) follows[2] since $v_\infty \in L^\infty(\mathbb{R})$.

We also claim that

$$-F(u(x)) + c \geq 0 \tag{9.26}$$

for any $x \in \Omega$.

Indeed, (9.26) is obvious when (C2) holds, because of (9.24) and (9.25).

On the other hand, when (C1) holds, we argue as follows in order to prove (9.26). We observe that (1.7) holds, thanks to our sign assumptions on $f$. Hence, from (1.9) and (9.24), we conclude that

$$\sup_{r\in[0,\|u\|_{L^\infty(\mathbb{R}^3_+)}]} F(r) = c,$$

[2] We would like to point out the following alternative proof of (9.25) which does not use the results of [FSV08]: when (C2) holds, we have $\Delta u_\infty \leq \Delta u_\infty + f(u_\infty) = 0$ in $\mathbb{R}^2$, so $u_\infty$ is constant by Liouville Theorem (see, for instance, Theorem 3.1 in [Far07]).

which gives (9.26).

We are now in position of strengthening (9.21), namely:

$$v_\infty \text{ has at most one critical point, unless it is constant.} \tag{9.27}$$

Since (9.27) is obvious when (C2) holds, due to (9.25), we now prove it when (C1) holds. For this, we argue by contradiction, supposing that (C1) holds and (9.27) does not hold. Then, by (9.21), there exist $t_-$, $t_+ \in \mathbb{R}$ such that $v_\infty(t_-) < v_\infty(t_+)$ and $v'_\infty(t_-) = 0 = v'_\infty(t_+)$. Then, by (9.23),

$$c = F(v_\infty(t_-)). \tag{9.28}$$

Also, since

$$\lim_{x_n\to+\infty} u(\omega t_+, x_n) = u_\infty(\omega t_+) = v_\infty(t_+) > v_\infty(t_-)$$

and

$$u(\omega t_-, 1) < \lim_{x_n\to+\infty} u(\omega t_-, x_n) = u_\infty(\omega t_-) = v_\infty(t_-),$$

we deduce that there must exist $x_* \in \mathbb{R}^3_+$ for which $u(x_*) = v_\infty(t_-)$.

Consequently, (1.8), (9.26) and (9.28) give that

$$\begin{aligned}\frac{1}{2}|\nabla u(x_*)|^2 &\le \sup_{r\in[0,\|u\|_{L^\infty(\mathbb{R}^3_+)}]} F(r) - F(u(x_*))\\ &= c - F(v_\infty(t_-)) = 0.\end{aligned}$$

This would imply that $\partial_3 u(x_*) = 0$, against our assumptions. Therefore, this contradiction proves (9.27).

From (9.27) we know that there exists $b \in \mathbb{R}$ in such a way that $v_\infty$ is monotone (though, maybe, not strictly monotone) in $(-\infty, b)$ and in $(b, +\infty)$.

Consequently, for any $A > 0$,

$$\begin{aligned}\int_{b-A}^{b} |v'_\infty(t)|^2\,dt &\le C_1 \int_{b-A}^{b} |v'_\infty(t)|\,dt\\ &= C_1 \left| \int_{b-A}^{b} v'_\infty(t)\,dt \right| = C_1 |v_\infty(b-A) - v_\infty(b)| \le C_2,\end{aligned}$$

where $C_1$, $C_2 > 0$ are suitable quantities independent of $A$.

That is,

$$\int_{-\infty}^{b} |v'_\infty(t)|^2\,dt \le C_2.$$

Analogously,

$$\int_{b}^{+\infty} |v'_\infty(t)|^2\,dt \le C_2.$$

Therefore,

$$\int_{-\infty}^{+\infty} |v'_\infty(t)|^2\,dt \le 2C_2. \tag{9.29}$$

Furthermore, (9.23) and (9.29) give that

$$\int_{-\infty}^{+\infty} \frac{|v_\infty'(t)|^2}{2} - F(v_\infty(t)) + c\,dt = \int_{-\infty}^{+\infty} |v_\infty'(t)|^2\,dt \leq 2C_2.$$

As a consequence, (9.19) implies that

$$\int_{B_R} \frac{|\nabla u_\infty(x')|^2}{2} - F(u_\infty(x')) + c\,dx' \leq C_3 R \tag{9.30}$$

for a suitable $C_3 > 0$, with $B_R \subset \mathbb{R}^2$.

Note that the ball $B_R$ in (9.30) is two-dimensional, while the one in (9.3) is three-dimensional, therefore (9.30) implies that

$$\mathcal{E}_{R,c}(u_\infty) \leq C_4 R^2.$$

From this and Lemma 9.1, we see that

$$\mathcal{E}_{R,c}(u) \leq \lim_{t\to+\infty} \mathcal{E}_{R,c}(u^t) + CR^2 \leq \mathcal{E}_{R,c}(u_\infty) + CR^2 \leq (C_4 + C)R^2.$$

This and (9.26) imply (9.18). □

**Corollary 9.4.** *Let $\Omega_o \subset \mathbb{R}^3$ be a continuous epigraph.*

*Suppose that $u \in C^2(\Omega_o) \cap C(\overline{\Omega_o})$ satisfies (3.14), that $u > 0$ in $\Omega_o$, that $u = 0$ on $\partial\Omega_o$ and that*

$$\int_{B_R \cap \Omega_o} |\nabla u(x)|^2\,dx \leq CR^2 \qquad \text{for any } R \geq 1. \tag{9.31}$$

*Assume also that $\{\nabla u = 0\} = \emptyset$.*

*Then, $\Omega_o = \mathbb{R}^3_+$ up to isometry and there exists $u_o : \mathbb{R} \to \mathbb{R}$ in such a way that $u(x) = u_o(x_n)$ for any $x \in \Omega_o$.*

*Proof.* Fubini's Theorem and (9.31) yield that

$$\begin{aligned}
&\int_{(B_R\setminus B_{\sqrt{R}})\cap\Omega_o} \frac{|\nabla u|^2}{|x|^2}\,dx \\
&\quad= \int_{(B_R\setminus B_{\sqrt{R}})\cap\Omega_o} \int_{|x|}^{R} \frac{|\nabla u|^2}{2t^3}\,dt\,dx + \int_{(B_R\setminus B_{\sqrt{R}})\cap\Omega_o} \frac{|\nabla u|^2}{R^2}\,dx \\
&\quad= \int_{\sqrt{R}}^{R} \int_{(B_t\setminus B_{\sqrt{R}})\cap\Omega_o} \frac{|\nabla u|^2}{2t^3}\,dx\,dt + \int_{(B_R\setminus B_{\sqrt{R}})\cap\Omega_o} \frac{|\nabla u|^2}{R^2}\,dx \\
&\quad\leq \int_{\sqrt{R}}^{R} \frac{Ct^2}{2t^3}\,dt + \frac{CR^2}{R^2} \leq C' \ln R,
\end{aligned}$$

as long as $R \geq 2$, for a suitable $C' > 0$.

Therefore, we take $\varphi_R$ as in (5.1) as test function in (3.14). We obtain from the preceding estimate that

$$\begin{aligned}
&\int_{B_{\sqrt{R}}\cap\{\nabla u\neq 0\}} \Big( |\nabla u|^2 \mathcal{K}^2 + |\nabla_T |\nabla u||^2 \Big) \\
&\qquad\leq \frac{C''}{(\ln R)^2} \int_{B_R\setminus B_{\sqrt{R}}} \frac{|\nabla u|^2}{|x|^2} \leq \frac{C'''}{\ln R},
\end{aligned}$$

for appropriate constants $C''$, $C''' > 0$.

Accordingly, by taking $R$ as large as we wish, we deduce that $\mathcal{K}$ and $|\nabla_T|\nabla u||$ vanish identically. The desired result then follows from Lemma 2.3. $\square$

**Corollary 9.5.** *Suppose that*

- *either* $\Omega = \mathbb{R}^3_+$ *and* $f(0) \geq 0$,
- *or that* $\Omega \subset \mathbb{R}^3$ *is a* $C^3$ *and uniformly Lipschitz epigraph, and* $f(r) \geq 0$ *for any* $r \geq 0$.

*Assume that* $u \in C^2(\overline{\Omega}) \cap L^\infty(\Omega)$ *satisfies*

$$\Delta u(x) + f(u(x)) = 0 \qquad \text{for any } x \in \Omega.$$

*with* $u\Big|_{\partial\Omega} = 0$, *and* $\partial_3 u > 0$ *in* $\Omega$.

*Suppose that* (3.14) *holds.*

*Then,* $\Omega = \mathbb{R}^3_+$ *up to isometry and there exists* $u_o : \mathbb{R} \to \mathbb{R}$ *in such a way that*

$$u(x) = u_o(x_n) \qquad \text{for any } x \in \Omega.$$

*Proof.* By Corollary 9.3, we see that (9.31) holds true, thence the claim follows from Corollary 9.4. $\square$

# 10 Proof of Theorems 1.5, 1.6, 1.7 and 1.8

To make the notation uniform, we take $\Omega := \mathbb{R}^3_+$ if we are under the assumptions of Theorems 1.5.

We have that

$$\partial_n u > 0 \text{ in } \Omega. \tag{10.1}$$

Indeed, if we are under the hypotheses of Theorem 1.5, we have that (10.1) is a consequence of Theorem 1.1 in [BCN97b]. If, on the other hand, we are in the assumptions of Theorem 1.6, we argue by contradiction, supposing that we have a coercive epigraph: then (10.1) follows from [EL83] (see also Theorem 1.3 in [BCN97a]).

Finally, when we are in the assumptions of Theorem 1.8, then (10.1) follows from (1.12) and Theorem 1.1 of [BCN97a].

When $n = 2$, Theorems 1.6 and 1.8 are consequence of (10.1) and Theorem 1.2.

Also, thanks to (10.1), we have that conditions (3.2), (3.11) and either (3.12) or (3.13) are satisfied, and so (3.14) holds true, in the light of Corollary 3.4.

The proof of Theorems 1.5, 1.6 for $n = 3$ and 1.8 for $n = 3$ is then finished, thanks to Corollary 9.5 (for this, we recall that, due to Lemma 2.4, we may also assume that $f \geq 0$ under the hypotheses of Theorem 1.8).

Theorem 1.7 is then an obvious consequence of Theorem 1.8. $\square$

## 11 Proof of Theorem 1.9

The proof is by contradiction, assuming that a solution of (1.13) exists.

First of all, we show that then

$$\lambda > 0. \tag{11.1}$$

The proof of (11.1) is by contradiction. If $\lambda \le 0$,

$$\Delta u = -\lambda u \ge 0$$

in $\Omega$, while $u = 0$ on $\partial\Omega$.

So, by Maximum Principle (see Lemma 2.1 in [BCN97a]), we would have that $u \le 0$ in $\Omega$. Since this is against our assumption, we get that (11.1) holds true.

Then, from (11.1), by suitably choosing $f$, the fact that $u$ is bounded and (11.1) give (1.12). Thus, by Theorem 1.8, we obtain that $\Omega$ is a halfspace, say, up to isometries, $\Omega = \mathbb{R}^n_+$.

As a consequence, Theorems 1.3 or 1.5, depending on the dimension $n$, imply that $u(x) = u_o(x_n)$ and (1.13) gives that

$$u_o''(t) = -\lambda u_o(t), \qquad \text{with} \qquad u_o(0) = 0.$$

Hence, (11.1) implies that

$$u_o(t) = A \sin(\sqrt{\lambda}\, t).$$

This contradicts the assumption that $u > 0$ in $\Omega$ and it thus proves Theorem 1.9. $\square$

## 12 Proof of Theorem 1.11

By possibly adding a dummy variable, we may suppose that $n = 5$. By Lemma 2.4, we may also assume that $f \ge 0$.

Furthermore, by Theorem 1.1 in [BCN97b], we have that

$$\partial_5 u > 0.$$

Hence, by Lemma 9.2, we may define

$$u_\infty(x') := \lim_{x_5 \to +\infty} u(x', x_5),$$

for any $x' \in \mathbb{R}^4$ and obtain that

$$\Delta u_\infty(x') + f(u_\infty(x')) = 0 \tag{12.1}$$

for any $x' \in \mathbb{R}^4$, and that

$$\int_{\mathbb{R}^4} |\nabla \varphi|^2 - f'(u_\infty)\varphi^2 \ge 0 \tag{12.2}$$

for any $\varphi \in W_0^{1,\infty}(\mathbb{R}^4)$ (note that here we use the continuity of $f'$).

From this and Theorem 1.1 of [DF08], we conclude that $u_\infty$ is constant.

Therefore, (12.1) boils down to

$$0 = \Delta u_\infty(x') = f(u_\infty(x')) = f\Big(\sup_{\mathbb{R}^4} u_\infty\Big). \tag{12.3}$$

Since

$$\sup_{\mathbb{R}^5_+} u = \sup_{\mathbb{R}^4} u_\infty,$$

we obtain from (12.3) that

$$f\Big(\sup_{\mathbb{R}^5_+} u\Big) = 0.$$

This and Theorem 1.4 of [BCN97b] imply the claim of Theorem 1.11. □

## Acknowlegments

EV has been supported by *MIUR Metodi variazionali ed equazioni differenziali nonlineari* and *FIRB Analysis and Beyond*.

We thank an anonymous referee for her or his careful work.

## References

[AAC01] Giovanni Alberti, Luigi Ambrosio, and Xavier Cabré. On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property. *Acta Appl. Math.*, 65(1-3):9–33, 2001. Special issue dedicated to Antonio Avantaggiati on the occasion of his 70th birthday.

[BCN93] H. Berestycki, L. A. Caffarelli, and L. Nirenberg. Symmetry for elliptic equations in a half space. In *Boundary value problems for partial differential equations and applications*, volume 29 of *RMA Res. Notes Appl. Math.*, pages 27–42. Masson, Paris, 1993.

[BCN97a] H. Berestycki, L. A. Caffarelli, and L. Nirenberg. Monotonicity for elliptic equations in unbounded Lipschitz domains. *Comm. Pure Appl. Math.*, 50(11):1089–1111, 1997.

[BCN97b] Henri Berestycki, Luis Caffarelli, and Louis Nirenberg. Further qualitative properties for elliptic equations in unbounded domains. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 25(1-2):69–94 (1998), 1997. Dedicated to Ennio De Giorgi.

[CGS94] Luis Caffarelli, Nicola Garofalo, and Fausto Segàla. A gradient bound for entire solutions of quasi-linear equations and its consequences. *Comm. Pure Appl. Math.*, 47(11):1457–1473, 1994.

[DF08] Louis Dupaigne and Alberto Farina. Stable solutions of $-\Delta u = f(u)$ in $\mathbb{R}^N$. *Preprint*, 2008.

[Ebe93] Peter Ebenfelt. Some results on the Pompeiu problem. *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 18(2):323–341, 1993.

[EL83] Maria J. Esteban and P.-L. Lions. Existence and nonexistence results for semilinear elliptic problems in unbounded domains. *Proc. Roy. Soc. Edinburgh Sect. A*, 93(1-2):1–14, 1982/83.

[Far07] Alberto Farina. Liouville-type theorems for elliptic problems. In M. Chipot, editor, *Handbook of Differential Equations: Stationary Partial Differential Equations. Vol. IV*, pages 61–116. Elsevier B. V., Amsterdam, 2007.

[FSV08] Alberto Farina, Berardino Sciunzi, and Enrico Valdinoci. Bernstein and De Giorgi type problems: new results via a geometric approach. *To appear in Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, 2008. `http://www.math.utexas.edu/mp_arc/`.

[FV08] Alberto Farina and Enrico Valdinoci. The state of the art for a conjecture of De Giorgi and related problems. *To appear in Ser. Adv. Math. Appl. Sci., World Sci. Publ.*, 2008.

[Giu84] Enrico Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[GS93] Nicola Garofalo and Fausto Segàla. Another step toward the solution of the Pompeiu problem in the plane. *Comm. Partial Differential Equations*, 18(3-4):491–503, 1993.

[GT83] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1983.

[HKM06] Juha Heinonen, Tero Kilpeläinen, and Olli Martio. *Nonlinear potential theory of degenerate elliptic equations*. Dover Publications Inc., Mineola, NY, 2006. Unabridged republication of the 1993 original.

[Mod85] Luciano Modica. A gradient bound and a Liouville theorem for nonlinear Poisson equations. *Comm. Pure Appl. Math.*, 38(5):679–684, 1985.

[PS07] Patrizia Pucci and James Serrin. *The maximum principle*. Progress in Nonlinear Differential Equations and their Applications, 73. Birkhäuser Verlag, Basel, 2007.

[Ser71] James Serrin. A symmetry problem in potential theory. *Arch. Rational Mech. Anal.*, 43:304–318, 1971.

[Ser94] Edoardo Sernesi. *Geometria 2*. Bollati Boringhieri, Torino, 1994.

[SZ98a] Peter Sternberg and Kevin Zumbrun. Connectivity of phase boundaries in strictly convex domains. *Arch. Rational Mech. Anal.*, 141(4):375–400, 1998.

[SZ98b] Peter Sternberg and Kevin Zumbrun. A Poincaré inequality with applications to volume-constrained area-minimizing surfaces. *J. Reine Angew. Math.*, 503:63–85, 1998.

[WCG95] N. B. Willms, Marc Chamberland, and G. M. L. Gladwell. A duality theorem for an overdetermined eigenvalue problem. *Z. Angew. Math. Phys.*, 46(4):623–629, 1995.

[Wei71] H. F. Weinberger. Remark on the preceding paper of Serrin. *Arch. Rational Mech. Anal.*, 43:319–320, 1971.

[Yau82] Shing Tung Yau. Problem section. In *Seminar on Differential Geometry*, volume 102 of *Ann. of Math. Stud.*, pages 669–706. Princeton Univ. Press, Princeton, N.J., 1982.

*Alberto Farina*
LAMFA – CNRS UMR 6140
Université de Picardie Jules Verne
Faculté de Mathématiques et d'Informatique
33, rue Saint-Leu
80039 Amiens CEDEX 1, France
alberto.farina@u-picardie.fr

*Enrico Valdinoci*
Università di Roma Tor Vergata
Dipartimento di Matematica
via della ricerca scientifica, 1
I-00133 Rome, Italy
enrico@mat.uniroma3.it