CORRIGENDUM TO "CHARACTERIZATION OF SOBOLEV AND BV SPACES'

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ABSTRACT. We correct a mistake in [3].

1. INTRODUCTION

In the paper [3] we study the asymptotics of the functional

$$J^{p,q}_{\epsilon,\lambda}(f) := \int_{\Omega_\lambda} \left(\int_{\Omega_\lambda} \left(\frac{|f(x)-f(y)|^p}{|x-y|^p} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx, \quad f \in L^1_{\mathrm{loc}}(\Omega), \tag{1.1}$$

as $\epsilon \to 0$ and $\lambda \to 0$, where $\Omega \subset \mathbb{R}^N$ is an arbitrary open set and $1 \leq p, q < \infty$. The main result of the paper is a characterization of Sobolev and BV spaces for arbitrary open sets, which in turn provides answers to conjectures of Brézis and Ponce in the case $q = 1$.

While all of the results in [3] are true for $q = 1$, in the case $q > 1$ there is a fundamental mistake in Lemma 3.3 (which pertains to Sobolev functions), repeated in Lemma 3.5 (which pertains to BV functions). Precisely, on page 2941 the change of variables $y = x + th$ is unjustified, since when $q > 1$ we are not free to use Fubini's or Tonelli's theorem to interchange the integrals and make such a substitution. This error is then propagated through the paper in Theorems 4.4, 4.6 and in the sufficiency part of Theorems 1.5 and 1.9. Theorems 4.1 and 4.2 hold for all $q \geq 1$ and so does the necessary part of Theorems 1.5 and 1.9.

The following counterexample establishes that Theorem 4.6 is false when $q > 1$.

Counterexample 1.1. Let $\Omega = (-1, 1)$, $q > 1$, and $\rho_\epsilon(x) = c\epsilon \frac{1}{|x|^{1-\epsilon}}$. Define

$$f(x) = \begin{cases} 1 & \text{if } 0 < x \leq 1, \\ 0 & \text{if } -1 \leq x \leq 0. \end{cases}$$

Then we claim

$$\lim_{\epsilon \to 0} \int_{-1+\lambda}^{1-\lambda} \left(\int_{-1+\lambda}^{1-\lambda} \left(\frac{|f(x)-f(y)|}{|x-y|} \right)^q \rho_\epsilon(x-y)\, dy \right)^{\frac{1}{q}} dx = +\infty \tag{1.2}$$

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for every λ sufficiently small. But we have that

$$\begin{aligned}
&\int_{-1+\lambda}^{1-\lambda}\left(\int_{-1+\lambda}^{1-\lambda}\left(\frac{|f(x)-f(y)|}{|x-y|}\right)^q\rho_\epsilon(x-y)\,dy\right)^{\frac{1}{q}}dx\\
&=\tilde{c}\epsilon^{\frac{1}{q}}\int_{-1+\lambda}^{0}\left(\int_0^{1-\lambda}\left(\frac{1}{|x-y|}\right)^{q+1-\epsilon}dy\right)^{\frac{1}{q}}dx\\
&=\tilde{c}\epsilon^{\frac{1}{q}}\int_{-1+\lambda}^{0}\left(\left[\frac{|x-y|^{-q+\epsilon}}{-q+\epsilon}\right]_0^{1-\lambda}\right)^{\frac{1}{q}}dx\\
&\geq \overline{c}\epsilon^{\frac{1}{q}}\int_{-1+\lambda}^{0}|x|^{-1+\frac{\epsilon}{q}}\,dx=\overline{c}\epsilon^{\frac{1}{q}}\frac{q}{\epsilon}|-1+\lambda|^{\frac{\epsilon}{q}}.
\end{aligned}$$

Thus, taking the limit as $\epsilon\to 0$ we obtain (1.2) as claimed.

The validity of Theorem 4.4 for $p>1$ and $1\leq q<\infty$ will be addressed in future work. In the remaining of this erratum we show that when $q>1$, Theorems 1.5 and 1.9 continue to hold for radial mollifiers when pointwise convergence of the functional $J^{p,q}_{\epsilon,\lambda}$ is replaced by Γ-convergence. From the standpoint of approximation of a local functional this is actually a more natural convergence, since it implies that any limit point of a sequence of minimizers of the nonlocal problems is a minimizer of the local problem.

For simplicity, we replace the sets

$$\Omega_\lambda:=\left\{x\in\Omega:\ \operatorname{dist}(x,\partial\Omega)>\lambda,\ |x|<\frac{1}{\lambda}\right\}\tag{1.3}$$

in the definition of $J^{p,q}_{\epsilon,\lambda}$ with Lipschitz domains. Precisely, for every $\lambda>0$ let U_λ be an open set with Lipschitz boundary such that

$$\Omega_{2\lambda}\subseteq U_\lambda\subseteq\Omega_\lambda\tag{1.4}$$

and define

$$F^{p,q}_{\epsilon,\lambda}(f):=\int_{U_\lambda}\left(\int_{U_\lambda}\left(\frac{|f(x)-f(y)|^p}{|x-y|^p}\right)^q\rho_\epsilon(x-y)\,dy\right)^{\frac{1}{q}}dx,\quad f\in L^1_{\mathrm{loc}}(\Omega).$$

Note that the specific form of the invading sequence $\{\Omega_\lambda\}$ played no special role in [3].

We now proceed to state and prove the Γ-convergence result asserted previously, valid for all $1\leq p<\infty$ and $1\leq q<\infty$ (notice here we can use all values of q as opposed to the restrictions on q that were required in Theorems 1.5 and 1.9). For $1\leq p<\infty$ and $1\leq q<\infty$ define the functional

$$F^{p,q}(f):=\begin{cases}K_{p,q,N}\int_\Omega|\nabla f|^p\,dx & \text{if } f\in W^{1,p}_{\mathrm{loc}}(\Omega) \text{ and } \nabla f\in L^p\left(\Omega;\mathbb{R}^N\right),\\ \infty & \text{otherwise in } L^1_{\mathrm{loc}}(\Omega),\end{cases}\tag{1.5}$$

when $p>1$ and

$$F^{1,q}(f):=\begin{cases}K_{1,q,N}|Df|(\Omega) & \text{if } f\in BV_{\mathrm{loc}}(\Omega) \text{ and } Df\in M_b\left(\Omega;\mathbb{R}^N\right),\\ \infty & \text{otherwise in } L^1_{\mathrm{loc}}(\Omega).\end{cases}\tag{1.6}$$

Theorem 1.2. *Let $\Omega\subset\mathbb{R}^N$ be open, let ρ_ϵ satisfy* (1.3), (1.4), *and* (1.5) *in* [3], *let $1\leq p<\infty$ and $1\leq q<\infty$. Then*

$$\Gamma\text{-}\lim_{\lambda\to 0}\left(\Gamma\text{-}\lim_{\epsilon\to 0}F^{p,q}_{\epsilon,\lambda}\right)=F^{p,q},$$

where the Γ limit is taken with respect to the $L^1_{\mathrm{loc}}(\Omega)$-strong topology.

Proof. We begin by showing that for $\lambda > 0$ fixed

$$\Gamma\text{-}\lim_{\epsilon\to 0} F^{p,q}_{\epsilon,\lambda} = F^{p,q}_{\lambda},$$

where

$$F^{p,q}_{\lambda}(f) := \begin{cases} K_{p,q,N}\int_{U_\lambda}|\nabla f|^p\,dx & \text{if } f\in W^{1,p}_{\mathrm{loc}}(\Omega) \text{ and } \nabla f\in L^p\left(\Omega;\mathbb{R}^N\right), \\ \infty & \text{otherwise in } L^1_{\mathrm{loc}}(\Omega), \end{cases}$$

when $p > 1$ and

$$F^{1,q}_{\lambda}(f) := \begin{cases} K_{1,q,N}\,|Df|(U_\lambda) & \text{if } f\in BV_{\mathrm{loc}}(\Omega) \text{ and } Df\in M_b\left(\Omega;\mathbb{R}^N\right), \\ \infty & \text{otherwise in } L^1_{\mathrm{loc}}(\Omega) \end{cases}$$

for $p = 1$.

Step 1: Let $\epsilon_n \to 0^+$. We claim that for every $f \in L^1_{\mathrm{loc}}(\Omega)$ there exists a sequence $\{f_n\} \subset L^1_{\mathrm{loc}}(\Omega)$ converging to f in $L^1_{\mathrm{loc}}(\Omega)$ such that

$$\limsup_{n\to\infty} F^{p,q}_{\epsilon_n,\lambda}(f_n) \le F^{p,q}_{\lambda}(f).$$

If $F^{p,q}_{\lambda}(f) = \infty$, we can take $f_n :\equiv f$ and there is nothing to prove. Thus, we can assume that $F^{p,q}_{\lambda}(f) < \infty$. Moreover, in the case $p = 1$, we also assume that

$$|Df|(\partial U_\lambda) = 0. \tag{1.7}$$

Extend f to be zero outside Ω and consider the sequence of functions $f_n := f * \psi^{\delta_n}$, where ψ^{δ_n} are smooth mollifiers and $\delta_n > 0$ will be chosen later. By standard properties of mollifiers, we have that $f_n \to f$ in $L^1_{\mathrm{loc}}(\Omega)$.

For $0 < \delta_n < \frac{1}{2}\lambda$, we are in a position to apply Lemma 3.1 in [3] to f_n, to obtain that

$$|f_n(x) - f_n(y) - \nabla f_n(x)\cdot x - y| \le C^{f_n}|x-y|^2$$

for all $x \in U_\lambda$ and $y \in U_{\lambda/2}$, where

$$C^{f_n} := C(N)||\nabla^2 f_n||_{L^\infty(\overline{U_{\lambda/2}})} + \frac{8}{\lambda^2}||f_n||_{L^\infty(\overline{U_{\lambda/2}})} + \frac{2}{\lambda}||\nabla f_n||_{L^\infty(\overline{U_{\lambda/2}})}. \tag{1.8}$$

In turn, reasoning exactly as in the proof of Step 1 of Lemma 3.2 in [3], we have

$$\begin{aligned} \int_{U_\lambda}\left|\left(\frac{|f_n(x)-f_n(y)|^p}{|x-y|^p}\right)^q - \left|\nabla f_n(x)\cdot\frac{x-y}{|x-y|}\right|^{pq}\right|\rho_{\epsilon_n}(x-y)\,dy \\ \le C^{f_n}_{p,q}\int_{U_\lambda}|x-y|\rho_{\epsilon_n}(x-y)\,dy, \end{aligned} \tag{1.9}$$

and so

$$\begin{aligned} \int_{U_\lambda}\left(\frac{|f_n(x)-f_n(y)|^p}{|x-y|^p}\right)^q\rho_{\epsilon_n}(x-y)\,dy &\le \int_{U_\lambda}\left|\nabla f_n(x)\cdot\frac{x-y}{|x-y|}\right|^{pq}\rho_{\epsilon_n}(x-y)\,dy \\ &+ C^{f_n}_{p,q}\int_{U_\lambda}|x-y|\rho_{\epsilon_n}(x-y)\,dy =: I + II, \end{aligned}$$

where $C^{f_n}_{p,q} := pqM^{pq-1}_{f_n}C^{f_n}$ and $M_{f_n} := ||\nabla f_n||_{L^\infty(\overline{U_{\lambda/2}})}$. Since $\rho_\epsilon(z) = \hat{\rho}_\epsilon(|z|)$, we may use polar coordinates centered at x to write

$$\begin{aligned} I &\le \int_{S^{N-1}}|\nabla f_n(x)\cdot\sigma|^{pq}\,d\mathcal{H}^{N-1}(\sigma)\int_0^\infty\hat{\rho}_{\epsilon_n}(t)t^{N-1}\,dt \\ &= |\nabla f_n(x)|^{pq}\frac{1}{|S^{N-1}|}\int_{S^{N-1}}|e_1\cdot\sigma|^{pq}\,d\mathcal{H}^{N-1}(\sigma), \end{aligned}$$

where we have used the fact that $\left|S^{N-1}\right|\int_0^\infty\hat{\rho}_{\epsilon_n}(t)t^{N-1}\,dt = \int_{\mathbb{R}^N}\rho_\epsilon(x)\,dx = 1$. Similarly, using the fact that $|x| < \frac{1}{\lambda}$ and $|y| < \frac{1}{\lambda}$ (see (1.3) and (1.4)), we have

$$II \le C^{f_n}_{p,q}\left|S^{N-1}\right|\int_0^{\frac{2}{\lambda}}\hat{\rho}_{\epsilon_n}(t)t^N\,dt.$$

Using the inequality $(a+b)^{1/q} \leq a^{1/q} + b^{1/q}$, $a, b \geq 0$, gives

$$F_{\epsilon_n,\lambda}^{p,q}(f_n) \leq K_{p,q,N} \int_{U_\lambda} |\nabla f_n(x)|^p \, dx + \left(C_{p,q}^{f_n}\right)^{\frac{1}{q}} |U_\lambda| \left(\int_{B(0,2/\lambda)} |h| \rho_{\epsilon_n}(h) \, dh \right)^{\frac{1}{q}},$$

where[1]

$$K_{p,q,N} := \left(\frac{1}{|S^{N-1}|} \int_{S^{N-1}} |e_1 \cdot \sigma|^{pq} \, d\mathcal{H}^{N-1}(\sigma) \right)^{\frac{1}{q}}.$$

By (2.4) in [3],

$$\lim_{n\to\infty} \int_{B(0,2/\lambda)} |h| \rho_{\epsilon_n}(h) \, dh = 0. \tag{1.10}$$

On the other hand, by the properties of mollifiers,

$$||f_n||_{L^\infty(\overline{U_{\lambda/2}})} \leq \frac{C(N,p)}{\delta_n^{N/p}} ||f||_{L^p(\overline{U_{\lambda/2}})}, \quad ||\nabla f_n||_{L^\infty(\overline{U_{\lambda/2}})} \leq \frac{C(N,p)}{\delta_n^{N/p}} ||\nabla f||_{L^p(\overline{U_{\lambda/2}})},$$
$$||\nabla^2 f_n||_{L^\infty(\overline{U_{\lambda/2}})} \leq \frac{C(N,p)}{\delta_n^{1+N/p}} ||\nabla f||_{L^p(\overline{U_{\lambda/2}})},$$

for $p > 1$, while

$$||f_n||_{L^\infty(\overline{U_{\lambda/2}})} \leq \frac{C(N)}{\delta_n^{N}} ||f||_{L^1(\overline{U_{\lambda/2}})}, \quad ||\nabla f_n||_{L^\infty(\overline{U_{\lambda/2}})} \leq \frac{C(N)}{\delta_n^{N}} |Df| \left(\overline{U_{\lambda/2}}\right),$$
$$||\nabla^2 f_n||_{L^\infty(\overline{U_{\lambda/2}})} \leq \frac{C(N)}{\delta_n^{1+N}} |Df| \left(\overline{U_{\lambda/2}}\right)$$

for $p = 1$. In turn,

$$\left(C_{p,q}^{f_n}\right)^{\frac{1}{q}} \leq \frac{C_f}{\delta_n^{\ell}}$$

for some constant C_f depending on f and for some $\ell > 0$. In view of (1.10), by choosing δ_n which tends to zero slowly enough we can ensure that

$$\left(C_{p,q}^{f_n}\right)^{\frac{1}{q}} |U_\lambda| \left(\int_{B(0,2/\lambda)} |h| \rho_{\epsilon_n}(h) \, dh \right)^{\frac{1}{q}} \to 0$$

as $n \to \infty$. It follows that

$$\limsup_{n\to\infty} F_{\epsilon_n,\lambda}^{p,q}(f_n) \leq K_{p,q,N} \limsup_{n\to\infty} \int_{U_\lambda} |\nabla f_n(x)|^p \, dx = F_\lambda^{p,q}(f),$$

where we have used the fact that $\nabla f_n \to \nabla f$ in $L^p\left(U_\lambda; \mathbb{R}^N\right)$ for $p > 1$, while $\int_{U_\lambda} |\nabla f_n| \, dx \to |Df|(U_\lambda)$ for $p = 1$ in view of (1.7).

To remove the additional condition (1.7), consider an arbitrary $f \in BV_{\text{loc}}(\Omega)$ with $Df \in M_b\left(\Omega; \mathbb{R}^N\right)$ and denote by $f|_{U_\lambda}$ the restriction of f to U_λ. Since U_λ is Lipschitz, we can extend $f|_{U_\lambda}$ to a function $g \in BV(\Omega)$ with $|Dg|(\partial U_\lambda) = 0$ (see Proposition 3.21 in [1]). Construct a sequence of cut-off functions $\varphi_k \in C_c^\infty(\Omega; [0,1])$ such that $\varphi_k = 1$ in U_λ and $\varphi_k = 0$ outside $(U_\lambda)^{\frac{1}{k}}$ and define

$$g_k := \varphi_k g + (1 - \varphi_k) f.$$

[1]There is a misprint in the definition of $K_{p,q,N}$ in Theorem 1.5 in [3]. The factor $\frac{1}{|S^{N-1}|}$ is missing.

Then on any compact set $K \subset \Omega$,

$$\int_K |g_k - f| \, dx \le \int_{(U_\lambda)^{\frac{1}{k}} \setminus U_\lambda} |g - f| \, dx \to 0$$

as $k \to \infty$, where we used the fact that $g_k = f$ in U_λ. Thus, $g_k \to f$ in $L^1_{\mathrm{loc}}(\Omega)$. Moreover, $|Dg_k|(\partial U_\lambda) = |Dg|(\partial U_\lambda) = 0$. By the lower semicontinuity of the Γ-upper limit (see, e.g., Proposition 6.8 in [2]), the first part of the proof applied to g_k, and the fact that $g_k = f$ in U_λ, we have

$$\begin{aligned} |Df|(U_\lambda) &= \liminf_{k\to\infty} |Dg_k|(U_\lambda) = \liminf_{k\to\infty} \left(\Gamma\text{-}\limsup_{n\to\infty} F^{p,q}_{\epsilon_n,\lambda}(g_k) \right) \\ &\ge \Gamma\text{-}\limsup_{n\to\infty} F^{p,q}_{\epsilon_n,\lambda}(f). \end{aligned}$$

Given the arbitrariness of ϵ_n, we have shown that

$$\Gamma\text{-}\limsup_{\epsilon\to 0} F^{p,q}_{\epsilon,\lambda} \le F^{p,q}_\lambda.$$

Step 2: We prove that

$$\Gamma\text{-}\liminf_{\epsilon\to 0} F^{p,q}_{\epsilon,\lambda} \ge F^{p,q}_\lambda. \tag{1.11}$$

Let $\epsilon_n \to 0^+$. We claim that for every $f \in L^1_{\mathrm{loc}}(\Omega)$ and for every sequence $\{f_n\} \subset L^1_{\mathrm{loc}}(\Omega)$ converging to f in $L^1_{\mathrm{loc}}(\Omega)$ we have

$$\liminf_{n\to\infty} F^{p,q}_{\epsilon_n,\lambda}(f_n) \ge F^{p,q}_\lambda(f).$$

If the left-hand size is infinite, then there is nothing to prove, thus we assume that

$$\liminf_{n\to\infty} F^{p,q}_{\epsilon_n,\lambda}(f_n) < \infty. \tag{1.12}$$

For $0 < \delta < \lambda$, let $f_n^\delta := f_n * \psi^\delta$, where ψ^δ is a smooth mollifier. Since $f_n \to f$ in $L^1_{\mathrm{loc}}(\Omega)$, by standard properties of mollifiers, we have that $f_n^\delta \to f^\delta$ in $C^2_{\mathrm{loc}}(\Omega)$. By applying Lemma 3.6 to f_n (and using $((U_\lambda)_\eta)^\eta \subset U_\lambda$ along with non-negativity of the integrand), we obtain

$$\begin{aligned} &\int_{(U_\lambda)_\eta} \left(\int_{(U_\lambda)_\eta} \frac{|f_n^\delta(x) - f_n^\delta(y)|^{pq}}{|x-y|^{pq}} \rho^\eta_{\epsilon_n}(x-y) \, dy \right)^{\frac{1}{q}} dx \\ &\le \int_{U_\lambda} \left(\int_{U_\lambda} \frac{|f_n(x) - f_n(y)|^{pq}}{|x-y|^{pq}} \rho^\eta_{\epsilon_n}(x-y) \, dy \right)^{\frac{1}{q}} dx \end{aligned}$$

for $0 < \delta < \eta < \lambda$. Further, Lemma 3.2 may be modified to demonstrate the convergence

$$\lim_{n\to\infty} \int_{(U_\lambda)_\eta} \frac{|f_n^\delta(x) - f_n^\delta(y)|^{pq}}{|x-y|^{pq}} \rho^\eta_{\epsilon_n}(x-y) \, dy = |\nabla f_n(x)|^{pq} \frac{1}{|S^{N-1}|} \int_{S^{N-1}} |e_1 \cdot \sigma|^{pq} \, d\mathcal{H}^{N-1}(\sigma), \tag{1.13}$$

Indeed, reasoning as in Step 1 of the proof of Lemma 3.2 (see also (1.9) above), we have

$$\begin{aligned} &\int_{(U_\lambda)_\eta} \left| \frac{|f_n^\delta(x) - f_n^\delta(y)|^{pq}}{|x-y|^{pq}} - \left| \nabla f_n^\delta(x) \cdot \frac{x-y}{|x-y|} \right|^{pq} \right| \rho^\eta_{\epsilon_n}(x-y) \, dy \\ &\le C^{f_n^\delta}_{p,q} \int_{(U_\lambda)_\eta} |x-y| \rho^\eta_{\epsilon_n}(x-y) \, dy, \end{aligned} \tag{1.14}$$

where $C^{f_n^\delta}_{p,q} := pq M^{pq-1}_{f_n^\delta} C^{f_n^\delta}$, with $M_{f_n^\delta} := ||\nabla f_n^\delta||_{L^\infty(\overline{(U_\lambda)_\eta})}$ and

$$C^{f_n^\delta} := C(N) ||\nabla^2 f_n^\delta||_{L^\infty(\overline{(U_\lambda)_\eta})} + \frac{2}{\eta^2} ||f_n^\delta||_{L^\infty(\overline{(U_\lambda)_\eta})} + \frac{1}{\eta} ||\nabla f_n^\delta||_{L^\infty(\overline{(U_\lambda)_\eta})}.$$

Since $f_n^\delta \to f^\delta$ in $C^2_{\mathrm{loc}}(\Omega)$, we have that $C^{f_n^\delta}_{p,q} \to C^{f^\delta}_{p,q} < \infty$ as $n \to \infty$, where $f^\delta := f * \psi^\delta$. Hence, by (2.4) in [3], the right-hand side of (1.14) converges to 0 as $n \to \infty$. We can now proceed as in the second step of Lemma 3.2 to conclude (1.13). In turn, Fatou's lemma, and the previous inequality implies that

$$\begin{aligned}
\liminf_{n\to\infty} F^{p,q}_{\epsilon_n,\lambda}(f_n) &\geq \liminf_{n\to\infty} \int_{(U_\lambda)_\eta} \left(\int_{(U_\lambda)_\eta} \frac{|f_n^\delta(x)-f_n^\delta(y)|^{pq}}{|x-y|^{pq}} \rho^\eta_{\epsilon_n}(x-y)\,dy \right)^{\frac{1}{q}} dx \\
&\geq \int_{(U_\lambda)_\eta} \liminf_{n\to\infty} \left(\int_{(U_\lambda)_\eta} \frac{|f_n^\delta(x)-f_n^\delta(y)|^{pq}}{|x-y|^{pq}} \rho^\eta_{\epsilon_n}(x-y)\,dy \right)^{\frac{1}{q}} dx \\
&= K_{p,q,N} \int_{(U_\lambda)_\eta} |\nabla f^\delta(x)|^p\,dx.
\end{aligned}$$

In view of (1.12), this implies that $f \in W^{1,p}((U_\lambda)_\eta)$ for $p>1$ and $f \in BV((U_\lambda)_\eta)$ for $p=1$. Thus we may send $\delta \to 0$ along a subsequence and use lower semicontinuity with respect to the convergence $\nabla f^\delta \rightharpoonup \nabla f$ weakly in $L^p((U_\lambda)_\eta;\mathbb{R}^{N\times N})$ if $p>1$ and $\nabla f^\delta \overset{*}{\rightharpoonup} Df$ weakly-star in $\left(C_0((U_\lambda)_\eta;\mathbb{R}^{N\times N})\right)'$ if $p=1$ to conclude that

$$\liminf_{n\to\infty} F^{p,q}_{\epsilon_n,\lambda}(f_n) \geq K_{p,q,N} \int_{(U_\lambda)_\eta} |\nabla f(x)|^p\,dx,$$

if $p>1$ while

$$\liminf_{n\to\infty} F^{1,q}_{\epsilon_n,\lambda}(f_n) \geq K_{1,q,N}|Df|((U_\lambda)_\eta),$$

if $p=1$. But then sending $\eta \to 0$ and applying Lebesgue's monotone convergence theorem we conclude that the inequality (1.11) holds.

Step 3: The fact that

$$\Gamma\text{-}\lim_{\lambda\to 0} F^{p,q}_\lambda = F^{p,q}$$

is straightforward. We omit the details. □

Finally, let us mention two small misprints. At the end of page 2927 the mollifiers ρ_ϵ should be

$$\rho_\epsilon(x) = \frac{\chi_{B(0,R)}(|x|)}{|S^{N-1}|}\frac{\epsilon p c_\epsilon}{|x|^{N-\epsilon p}},$$

which is the correct normalization to ensure that they satisfy

$$\int_{\mathbb{R}^N} \rho_\epsilon(x)\,dx = 1.$$

Also in Remark 2.2 we should have $\mu_\epsilon = \frac{\mathcal{H}^{N-1}}{|S^{N-1}|}$.

References

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