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Finsler metrics in Optimization Problems and Hamilton-Jacobi equations

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Introduction

A Finsler metric on a differentiable manifold M is a map $\varphi : TM \to [0, +\infty)$ such that $\varphi(x, \cdot)$ is convex and positively 1-homogeneous on T_xM for every $x \in M$. The restriction of φ to each tangent space T_xM gives rise to what is known in literature as *Minkowski norm.* Its definition differs from that of a usual norm by the fact that it is in general non-even, that is $\varphi(x, \xi)$ is in general different from $\varphi(x, -\xi)$. In particular, Finsler metrics generalize the notion of Riemannian ones, which correspond to the case when $\varphi(x, \cdot)$ is the square root of a positive quadratic form, *i.e.* $\varphi(x, \xi) := \left(\sum_{i,j} a_{ij}(x)\, \xi_i \xi_j\right)^{1/2}$.
A non-symmetric distance on M can be associated to φ as follows:

$$d_\varphi\,(x, y) := \inf \left\{ \int_0^1 \varphi(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \;:\; \gamma \in \mathrm{Lip}\,([0,1], M)\,,\ \gamma(0) = x,\ \gamma(1) = y \right\} \qquad (1)$$

for $x, y \in M$, provided the class of admissible paths γ is non-void (for instance, assume M connected). Distances of this kind are usually called *Finsler distances.*

Finsler distances arise naturally in many physical contexts. For instance, consider geometric optics in an anisotropic medium: the speed of light depends on its direction of travel, and the time to move from the location x to $x + \mathrm{d}x$ is given by $\varphi(x, \mathrm{d}x)$. Hence, $d_\varphi(x, y)$ represents, in this setting, the time employed by a particle of light to move from the location x to y. Analogously, $d_\varphi(x, y)$ may represent the time employed to move from x to y on a hillside: the premise here is that one's walking speed depends heavily on the slope of the terrain, and hence on one's direction of travel. More esoteric examples are provided by mathematical ecology: for example, x could stand for the state of a coral reef, and $\varphi(x, \mathrm{d}x)$ for the amount of energy required by the system to evolve from the state x to the neighboring state $x + \mathrm{d}x$. In this setting, then, $d_\varphi(x, y)$ describes the minimal amount of energy needed by the system to evolve from the state x to the state y.

The case of smooth Finsler metrics has been largely investigated in the last century in the framework of differential geometry. The literature on the subject is very wide; an introduction is supplied for instance by [6].

In this thesis we will consider instead the case of Finsler metrics which satisfy weak regularity assumptions; more precisely, we allow φ to be only Borel-measurable. This case is clearly of physical interest, since the irregularities of the metric may actually represent the irregular or non-homogeneous character of the materials where physical phenomena take place: think, for instance, to geometric optic, or heat diffusion in a medium made up of two materials with different thermal conductivity. Moreover, this generality is necessary in view of applications to optimization problems, where one is often interested in minimizing (or maximizing) a cost functional which takes the form $\mathcal{C}(d_\varphi)$, where φ has to be chosen in some suitable class of Finsler metrics. Then, if one restricts himself to consider smooth or continuous metrics only, the problem may not have a solution: indeed, when attacking the problem via the direct methods of the Calculus of Variations, it might happen that minimizing (or maximizing) sequences converge to distances deriving from Finsler metrics that are shown to be Borel-measurable only (see [1, 16, 23]).

Shape optimization problems fall into this framework. In fact, φ describes the geometric properties of the metric space (M, d_φ), and may possibly depend on some quantities one is allowed to vary. For example, if $a(\cdot)$ is the density of a viscous material that fills a region M of the space, the distance between points is expected to be proportional to the resistance opposed by the medium and may be described by a distance d_a defined by (1) with $\varphi(x,\xi) := a(x)|\xi|$. If now we let $a(\cdot)$ vary in a suitable family of functions satisfying integral and pointwise constraints, the corresponding shape optimization problem is that of finding a "best" way of distributing a fixed amount of the given material on M.

The starting point of this thesis is indeed a problem of this kind and is described in Chapter 2. There, M is assumed to be the closure of a bounded and connected open subset Ω of $\mathbb{R}^N$, and $a(\cdot)$ is allowed to vary in the following class:

$$\mathcal{A} := \left\{ a(x) \text{ Borel measurable } \; : \; \alpha \le a(x) \le \beta \; , \; \int_\Omega a(x)\, dx \le m \right\} ,$$

where α, β, m are positive constants satisfying the compatibility conditions

$$\alpha \mathcal{L}^N(\Omega) \le m \le \beta \mathcal{L}^N(\Omega) \; .$$

The cost functional is related to Kantorovich's formulation of optimal mass transportation: given two probability measures f^+ and f^- on $\overline{\Omega}$, the cost functional is given by

$$\mathcal{C}(d) := \min\left\{ \iint_{\overline{\Omega}\times\overline{\Omega}} d(x,y)\, d\nu(x,y) \right\} , \tag{2}$$

where ν varies among all probability measures defined in $\overline{\Omega}\times\overline{\Omega}$ whose marginals are f^+ and f^-, that is

$$\nu(E\times\overline{\Omega}) = f^+(E), \qquad \nu(\overline{\Omega}\times E) = f^-(E)$$

for every Borel subset E of $\overline{\Omega}$. We now let d vary in the class $\mathcal{D}(\mathcal{A}) := \{d_a \ : \ a\in\mathcal{A}\ \}$, and we look for an optimal distance d_a which prevents as much as possible the transfer of f^+ into f^-: higher values of the Riemannian coefficient make the connection more difficult, but the problem is non-trivial due to the presence of the integral constraint. Our main result shows that a solution does exist in the initial class of Riemannian distances. By similar arguments, we are also able to treat more general maximization problems on $\mathcal{D}(\mathcal{A})$, where (2) is replaced by an arbitrary cost functional $\mathcal{C}$ satisfying monotonicity and semicontinuity properties.

The optimization problem outlined above is attacked by using the direct methods of the Calculus of Variations. The main difficulty here is that $\mathcal{D}(\mathcal{A})$ is not closed with respect to the natural convergence which ensures the continuity of the functional $\mathcal{C}$, namely the uniform convergence: consequently, each time we consider a maximizing sequence $(d_n)_n$ in $\mathcal{D}(\mathcal{A})$, by Ascoli-Arzelà Theorem we can always extract a subsequence uniformly converging to some distance d, but in general the latter does not belong to $\mathcal{D}(\mathcal{A})$ any longer. This phenomenon has been first pointed out by Acerbi and Buttazzo in [1], where the following example is provided: in dimension $N=2$, consider a sequence of periodic coefficients $(a_n)_{n\in\mathbb{N}}$ of the form $a_n(x) = a(nx)$, where the function a takes only two different values $\beta>\alpha>0$ respectively on the white and black squares of a chessboard. Then, for fixed points x,y, there holds

$$\lim_{n\to\infty} d_{a_n}(x,y) = \inf\left\{ \int_0^1 \varphi(\gamma')\, dt \ : \ \gamma\in \mathrm{Lip}([0,1];\overline{\Omega}) \ , \ \gamma(0)=x \ , \ \gamma(1)=y \right\} ,$$

where φ is a non-Riemannian Finsler metric independent of the position (for example, when the quotient β/α is sufficiently large, the unit ball $B_\varphi := \{\xi\in\mathbb{R}^2 \ : \ \varphi(\xi)\le 1\}$ is a regular octagon). Thus, in this case, the uniform limit of d_{a_n} cannot be written under the form d_a with $a\in\mathcal{A}$.

In view of these remarks, it is natural to enlarge the class of admissible competitors by considering all Finsler distances arising as limits of sequences belonging to $\mathcal{D}(\mathcal{A})$. To this aim, we embed $\mathcal{A}$ in the following family of Finsler metrics:

$$\mathcal{M} = \mathcal{M}(\alpha, \beta, \Omega) := \left\{\varphi \text{ Finsler metrics on } \overline{\Omega} \;:\; \alpha|\xi| \le \varphi(x,\xi) \le \beta|\xi| \quad \text{in } \overline{\Omega} \times \mathbb{R}^N \right\},$$

and we consider the wider class of Finsler distances

$$\mathcal{D} = \mathcal{D}(\mathcal{M}) := \left\{ d_\varphi \text{ distance on } \overline{\Omega} \text{ given by (1)} \;:\; \varphi \in \mathcal{M} \right\}. \tag{3}$$

The space $\mathcal{D}$ is compact, as proved in [23]; in particular, the existence of a solution d is *a priori* guaranteed only in the class $\overline{\mathcal{D}(\mathcal{A})}$ and is possibly associated to a non-Riemannian Finsler metric. The key point of the proof amounts to clarifying the effect produced by the integral constraint. In Section 2.3 we then show that each element d of $\overline{\mathcal{D}(\mathcal{A})}$ satisfies the following relation:

$$\int_\Omega \sup_{|\xi|=1} \varphi_d(x,\xi)\, \mathrm{d}x \le m,$$

where φ_d is the Finsler metric associated to d by derivation. The arguments we use seem sufficiently general to treat a wider class of integral constraints. A question, however, is left open: is it possible to give a characterization of the class $\overline{\mathcal{D}(\mathcal{A})}$? More generally, we wonder what kind of distances arises as limits of distances associated to (continuous) isotropic Riemannian metrics, and, moreover, when similar integral constraints are imposed on the metrics $a(\cdot)$, how they reflect on the limit distances. This issue is investigated in Chapter 3, where we consider an integral functional of the form

$$\mathcal{F}(d_a) := \int_\Omega F(x, a(x))\, \mathrm{d}x, \tag{4}$$

defined on the family $\mathcal{D}(\mathcal{I})$ of distances d_a induced by isotropic, continuous Riemannian metrics $a : \overline{\Omega} \to [\alpha, \beta]$ through formula (1) with $\varphi(x,\xi) := a(x)|\xi|$. The hypothesis that Ω is bounded is now replaced by the assumption that $F(x,\beta)$ is summable. Clearly, $\mathcal{D}(\mathcal{I})$ is included in the metric space of Finsler distances $\mathcal{D}$ defined by (3), endowed with the topology of the uniform convergence on compact subset of $\overline{\Omega} \times \overline{\Omega}$. In Section 3.3 we prove that $\mathcal{D}(\mathcal{I})$ is dense in the space of symmetric distances belonging to $\mathcal{D}$, namely in the set defined as follows:

$$\mathcal{D}_S := \{d \in \mathcal{D} \;:\; d(x,y) = d(y,x) \text{ for all } x, y \in \overline{\Omega}\},$$

and, under suitable monotonicity and convexity assumptions on F (which includes, in particular, the case $F(x,s)=s$ considered in Chapter 2), that the relaxed functional of (4) has the following integral representation:

$$\overline{\mathcal{F}}(d)=\int_\Omega F\big(x,\sup_{|\xi|=1}\varphi_d(x,\xi)\big)\,\mathrm{d}x \qquad \text{for every } d\in\mathcal{D}_S.$$

In the specific situation considered in Chapter 2, such results yield in particular the following characterization:

$$\overline{\mathcal{D}(\mathcal{A})}=\Big\{d\in\mathcal{D}_S\ :\ \int_\Omega \sup_{|\xi|=1}\varphi_d(x,\xi)\,\mathrm{d}x\le m\ \Big\}.$$

Another interesting result proved in Section 3.3 is given by Theorem 3.11, which amounts to saying that any Finsler symmetric distance, locally equivalent to the Euclidean one, can always be obtained from a suitable Borel measurable, isotropic Riemannian metric $a:\overline{\Omega}\to[\alpha,\beta]$, according to definition (1) with $\varphi(x,\xi):=a(x)|\xi|$.

Our proofs rely basically on techniques related to Γ-convergence, a notion that was introduced by De Giorgi and Franzoni in [46], and that is nowadays a tool widely used in the description of the asymptotic behavior of families of minimum problems, also outside the field of the Calculus of Variations and of Partial Differential Equations. The link with the topics we consider is immediately clear by observing that Finsler distances are defined through (1) in terms of minima of variational problems. In particular, we point out that the (uniform) convergence of a sequence of Finsler distances of the form $(d_{\varphi_n})_n$ is strictly related to Γ-convergence of the length functionals

$$\mathbb{L}_{\varphi_n}(\gamma):=\int_0^1\varphi_n(\gamma(t),\dot\gamma(t))\,\mathrm{d}t \qquad \gamma\in\mathrm{Lip}\left([0,1],\overline{\Omega}\right), \tag{5}$$

as shown by Buttazzo, De Pascale and Fragalà in [23]. The example given by Acerbi and Buttazzo in [1] and previously mentioned reveals that, in contrast with the case of quadratic forms corresponding to elliptic operators (*cf.* [36, Chapters 13 and 22]), Riemannian metrics are not closed with respect to the Γ-convergence of the corresponding length functionals. This case, which is in some sense "classical", emphasizes the flexibility of the tool of Γ-convergence, which is linked to no *a priori* ansatz on the form of minimizers, hence not bounded to any specific setting. It is quite natural to wonder if it is possible to give a characterization of metrics arising as Γ-limits of Riemannian ones. This problem and related issues of Γ-convergence have been considered by several authors. Besides [1], we quote, among others, the works by Amar and Vitali [4], Amar, Bellettini and Venturini [3], Braides [14], Braides, Buttazzo and Fragalà [16], and the books by Braides [15], Braides and Defranceschi [17], Dal Maso [36]. Similar problems, related to the structure of stable norms arising by homogenization of Riemannian metrics, have been considered by Burago [19] and by Burago, Ivanov and Kleiner [20] in a more geometric setting. It is worth mention that homogenization of Riemannian and Finsler metrics has been also considered in the framework of PDE. This topic is in fact related to the homogenization of Hamilton-Jacobi equations. Besides the celebrated (and unpublished) paper by Lions, Papanicolaou and Varadhan [64], we recall, among others, the works by Concordel [33, 34], E [48], Evans [49], Horie and Ishii [57], Alvarez [2], Souganidis [70].

The density result proven in Chapter 3 solves in particular a conjecture raised in [23], and partially answered in [16], where Finsler metrics were additionally assumed to be lower semicontinuous. Our proof makes use of similar techniques, but the underlying idea is quite different; in particular, no extra regularity assumptions on the metrics are needed. Consequently, the class of distances $\mathcal{D}$ here considered is, *a priori*, larger; actually, it coincides with the family of geodesic symmetric distances satisfying suitable equivalence relations with the Euclidean one, as we will see.

The arguments used in Chapter 3 are generalized to obtain an analogous density result for non-symmetric Finsler distances. In Chapter 4 we show that any element of $\mathcal{D}$ is the uniform limit of a sequence of distances derived through (1) from smooth Finsler metrics. This result covers also the case when the constant α, introduced in the definition of $\mathcal{M}$, is equal to zero. In this case, the associated family $\mathcal{D}(\mathcal{M})$ includes distances for which the local equivalence with the Euclidean one fails to hold somewhere and, in particular, it might happen that distinct points have reciprocal null distance with respect to some element of this family. The interest for this class of degenerate distances is motivated by the study of Hamilton-Jacobi equations of eikonal type in the critical case (see [29, 54]). The main difference with respect to the non-degenerate case relies on the fact that $\mathcal{D}$ is no longer closed when $\alpha = 0$. This fact is investigated in more detail in Section 4.4.

In Section 4.5 we compare definition (1) with another way of deriving a distance from a Finsler metric. The latter was introduced by De Cecco and Palmieri [41, 42, 43, 44, 45] to suitably generalize the notions of Riemannian and Finsler metrics for Lipschitz manifolds, namely topological manifolds with a countable basis, whose changes of coordinates are Lipschitz maps. Lipschitz manifolds are a generalization of polyhedra, and were introduced to treat the case of manifolds with singularities, such as vertices, edges, conical points, even not isolated. Their definition, specialized to the cases considered in this thesis, amounts to "smoothing" the metric, providing a definition of distance which is not affected if the metric is bad-behaved on negligible sets. We prove that the family of distances thus obtained gives rise to a dense and proper subset of $\mathcal{D}$.

The general results obtained in the framework of Finsler distances are used in Chapter 5 to study Hamilton-Jacobi equations of eikonal type with measurable ingredients. The metric character of equations of this kind has been recognized and explored by several authors in the case of continuous Hamiltonians [30, 31, 54, 62, 63, 68]. A central role is played, in fact, by a Finsler distance associated to the equation, the so called *optical length function,* through which a class of fundamental (viscosity) subsolutions can be defined. The study of Finsler metrics in the measurable setting acquires therefore further interest in view of generalizations of the theory of viscosity solutions for Hamilton-Jacobi equations with discontinuous Hamiltonians, a topic that is the object of growing attention [27, 28, 29, 37, 38, 69]

The measurable case, however, remains largely unsettled. Important results are provided by Caffarelli, Crandall, Kocan and Swiech in [25] for fully nonlinear equations of second order. Yet, their techniques are based on the strong maximum principle, so they do not apply to the first order case.

First order equations have been less studied; we quote in particular [9, 28, 59, 66, 69]. Our approach is analogous to that chosen in [66]. Indeed, we work in the framework of Monge solutions, introduced by Newcomb and Su, and we extend their results to a wider class of Hamiltonians by using the metric devices developed in the previous chapters. In particular, we establish the comparison principle for Monge sub and supersolutions, and, consequently, existence and uniqueness of a Monge solution for the Dirichlet problem obtained by coupling the Hamilton-Jacobi equation with a boundary condition. Strong stability results are also provided. The relation between Monge and Lipschitz subsolutions is discussed in Section 5.5, while Section 5.6 contains some examples. Example 5.23 shows in particular that Monge solutions are variational, *i.e.* they can be always obtained as limits of classical viscosity solutions of Hamilton-Jacobi equations with continuous Hamiltonians.

Properties and results about Finsler metrics and associated distances that will be needed in this thesis are collected and proved in Chapter 1. Almost all results there stated are known in literature; the main references used are the lecture notes by Ambrosio and Tilli [5], the works by De Cecco and Palmieri [41, 42, 43, 44, 45] and an unpublished preprint by Venturini [72]. Though, some proofs have been simplified; moreover, we have underlined the relation with the classical theory of metrics spaces due to Busemann and his school, by extending definitions and results to cover the case of (possibly degenerate) non-symmetric distances.

We wish to point out that our results also hold if $\mathbb{R}^N$ is replaced by a N-dimensional, differential manifold without boundary and of class C^1. Since all arguments exploit local properties, proofs can actually be rephrased by using local coordinates. We have preferred, however, to consider this more special case to not add further technicalities.

The results of Chapter 2 are obtained in collaboration with Giuseppe Buttazzo, Ilaria Fragalà and Fabricio Macià and will appear on [22]. The contents of Chapter 3 and Chapter 4 correspond to papers [39] and [40] respectively. The results of Chapter 5 are obtained in collaboration with Ariela Briani in [18]

Chapter 1

Finsler metrics

The aim of the current chapter is to present the background material needed in the remainder of this thesis.

The main notation used are collected in Section 1.1. In Section 1.2 we extend some definitions and results usually given in literature for classical metric spaces to cover the case of (possibly degenerate) non-symmetric distances. The treatment of metric spaces in the classical framework goes back to Busemann and his school. A presentation of these topics is provided, for instance, by [5].

Then we begin to introduce the objects of main interest for our future analysis, that is Finsler metrics and associated distances. Their main properties are presented and proved in the subsequent sections. These results are essentially known in literature; the main references used are the works by De Cecco and Palmieri [41, 42, 43, 44, 45] and an unpublished preprint by Venturini [72]. All this material has been reorganized and presented in a convenient form for later use. In particular, we have underlined the relation with the (by know) classical analysis in metric spaces and with the objects there introduced (such as, for instance, the *metric derivative*), a thing that was basically around in the quoted papers but never made explicit. Some proofs have been also simplified.

1.1 Notation

We write here a list of symbols used throughout this thesis.

N	a positive integer number
$\mathbb{S}^{N-1}$	$(N-1)$-dimensional unitary sphere of $\mathbb{R}^N$
$B_r(x), B_r$	open ball in $\mathbb{R}^N$ of radius r, centred in x and 0 respectively
I	closed interval $[0,1]$
$\mathcal{L}^k$	k-dimensional Lebesgue measure
$\mathcal{H}^k$	k-dimensional Hausdorff measure
$\lvert x\rvert$	Euclidean norm of the vector $x \in \mathbb{R}^N$
$\mathbb{R}_+$	non-negative real numbers
χ_E	characteristic function of the set E
$\mathrm{dist}(x, C)$	distance of x from the set C, *i.e.* the value $\inf\limits_{y\in C} \lvert x-y\rvert$

Given a set of points X, a function d defined in $X \times X$ will be said a *distance* on X if it satisfies the following properties:

(i) $d(x,x) = 0$ for every $x \in X$;

(ii) $d(x,y) \le d(x,z) + d(z,y)$ for every $x, y, z \in X$.

With respect to the classical definition, two conditions are not required: first, d may be *non-symmetric*, *i.e.* the identity $d(x,y) = d(y,x)$ may fail to hold in $X \times X$; second, d may possibly be *degenerate*, namely it might happen that $d(x,y) = 0$ for some $x \neq y$. This generality is needed in view of later use. Let us point out, however, that in the subsequent chapters we will usually deal with non-degenerate distances, restraining the treatment of degenerate ones to Section 4.4.

The set X, endowed with the topology induced by d, will be called a *metric space*, and will be denoted by the couple (X,d). Given an interval J of $\mathbb{R}$, we say that a curve $\gamma : J \to (X,d)$ is Lipschitz if there exists a finite constant C such that $d(\gamma(t),\gamma(s)) \le C|t-s|$ for all $t, s \in J$. The family of Lipschitz curves $\gamma : J \to (X,d)$ will be denoted by $\mathrm{Lip}(J,(X,d))$. We will say that $(\gamma_n)_n$ *(uniformly) converges* to γ in $\mathrm{Lip}(J,(X,d))$ if $\sup_{t\in J} d(\gamma(t),\gamma_n(t)) \vee d(\gamma_n(t),\gamma(t))$ tends to 0 as n goes to $+\infty$. For every fixed couple of points x, y in X, $\mathrm{Lip}_{x,y}(J,(X,d))$ stands for the family of Lipschitz curves connecting x to y, *i.e.* such that $\gamma(0) = x$ and $\gamma(1) = y$.

When X is a subset of $\mathbb{R}^N$ and d is the Euclidean distance, the foregoing spaces of curves will be more briefly denoted by $\mathrm{Lip}(J, X)$, $\mathrm{Lip}_{x,y}(J, X)$, and by $\mathrm{Lip}_{x,y}$ when $J = I := [0, 1]$. Unless otherwise stated, in this case all curves are also assumed to be parametrized by constant speed, *i.e.* in such a way that $|\dot\gamma(t)|$ is constant for $\mathcal{L}^1$-a.e. $t \in J$. A map $f : J \to \mathbb{R}^N$ will be said to be *transversal* to a set $E \subset \mathbb{R}^N$ if $\mathcal{L}^1\big(\{t \in J\ :\ f(t) \in E\}\big) = 0$. A subset E of $\mathbb{R}^N$ will be said to be *negligible* if its N-dimensional Lebesgue measure is null. Last, given two distances d, d' on X, we will say that $d(x, y) \le d'(x, y)$ *locally in* X when this holds for every x, y belonging to an open neighborhood of x_0, for any choice of x_0 in X.

1.2 An overview on metric spaces

We begin by extending to the case of possibly degenerate, non-symmetric distances some well known definitions and results usually given in the framework of classical metric spaces [5]. Let (X, d) be a metric space. Let us define the metric d-length of a curve $\gamma \in \mathrm{Lip}([a, b], X)$, obtained as the supremum of the d-lengths of inscribed polygonal curves:

$$\mathrm{L}_d(\gamma) := \sup\left\{\sum_{i=0}^{m-1} d\big(\gamma(t_i), \gamma(t_{i+1})\big)\ :\ a = t_0 < t_1 < \ldots < t_m = b,\ m \in \mathbb{N}\right\}. \tag{1.1}$$

Next result is a trivial consequence of definition (1.1).

Proposition 1.1. *The length functional* L_d *is lower semicontinuous on* $\mathrm{Lip}([a, b],(X, d))$ *with respect to the uniform convergence of paths, namely if* $(\gamma_n)_n$ *converges to* γ *in* $\mathrm{Lip}([a, b],(X, d))$ *then*

$$\mathrm{L}_d(\gamma) \le \liminf_{n\to+\infty} \mathrm{L}_d(\gamma_n).$$

We can give the definition of metric derivative of a curve, by slightly modifying the one given in the classical case in order to take into account the non-symmetric character of the distance d.

Definition 1.2 (Metric derivative). *Given a curve* $\gamma \in \mathrm{Lip}([a,b],(X,d))$*, we define the metric derivative* $|\dot\gamma|_d(t)$ *of* γ *at the point* $t \in (a,b)$ *as*

$$|\dot\gamma|_d(t) := \limsup_{h\to 0^+} \frac{d(\gamma(t),\gamma(t+h))}{h}. \tag{1.2}$$

The metric length of a curve can be expressed in terms of its metric derivative, as stated in the following

Theorem 1.3. *For every curve* γ *the* $\limsup$ *at the right-hand side of (1.2) is actually a limit for* $\mathcal{L}^1$*-a.e.* $t \in (a,b)$*. Moreover we have*

$$\mathrm{L}_d(\gamma) = \int_a^b |\dot\gamma|_d(t)\,\mathrm{d}t.$$

Proof. The proof is taken from [5]. Let us set $J := [a,b]$. Since γ is continuous, its range $\Gamma := \gamma(J)$ is a compact metric space, hence it is separable. Let $(x_n)_n$ be a dense sequence in Γ. For each $n \in \mathbb{N}$, we define the function $\varphi_n(t) := d(x_n,\gamma(t))$. By the triangular inequality we get

$$d(x_n,\gamma(t)) - d(x_n,\gamma(s)) \le d(\gamma(s),\gamma(t)), \tag{1.3}$$

therefore, interchanging the roles of t and s, we get

$$|\varphi_n(t) - \varphi_n(s)| \le \mathrm{Lip}(\gamma)|t-s|. \tag{1.4}$$

In particular, $\varphi_n \in \mathrm{Lip}(J,\mathbb{R})$, hence, by Rademacher Theorem, its derivative $\dot\varphi_n(t)$ exists at $\mathcal{L}^1$-a.e. point $t \in J$. Let us define

$$m(t) := \sup_n |\dot\varphi_n(t)|.$$

We will prove that

$$|\dot\gamma|_d(t) = m(t) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in J. \tag{1.5}$$

First, by using (1.3) we get

$$\liminf_{h\to 0^+} \frac{d(\gamma(t),\gamma(t+h)}{h} \ge \liminf_{h\to 0^+} \frac{\varphi_n(t+h)-\varphi_n(t)}{h} = \dot\varphi_n(t)$$

for $\mathcal{L}^1$-a.e. $t \in J$, hence, taking the sup over $n \in \mathbb{N}$, we obtain:

$$\liminf_{h\to 0^+} \frac{d(\gamma(t),\gamma(t+h))}{h} \ge m(t) \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in J. \tag{1.6}$$

On the other hand we have, for every $t \geq s$

$$d(\gamma(s), \gamma(t)) = \sup_n \Big(d(x_n, \gamma(t)) - d(x_n, \gamma(s)) \Big) = \sup_n \int_s^t \dot{\varphi}_n(\tau)\, \mathrm{d}\tau \leq \int_s^t m(\tau)\, \mathrm{d}\tau. \tag{1.7}$$

By (1.4) we have that $\mathrm{Lip}(\varphi_n) \leq \mathrm{Lip}(\gamma)$, hence $m(t) \leq \mathrm{Lip}(\gamma)$ and m is integrable over J. If t is a Lebesgue point for m, we obtain

$$\limsup_{h \to 0^+} \frac{d(\gamma(t), \gamma(t+h))}{h} \leq \limsup_{h \to 0^+} \frac{1}{h} \int_t^{t+h} m(\tau)\, \mathrm{d}\tau = m(t). \tag{1.8}$$

Since almost every $t \in J$ is a Lebesgue point for m, the last inequality, combined with (1.6) gives (1.5).

We now prove the claim. From (1.7) and (1.5) it follows

$$\sum_{i=0}^{m-1} d(\gamma(t_i), \gamma(t_{i+1})) \leq \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} |\dot{\gamma}|_d(\tau)\, \mathrm{d}\tau = \int_a^b |\dot{\gamma}|_d(t)\, \mathrm{d}t$$

for every choiche $a = t_0 < t_1 < ... < t_m = b,\ m \in \mathbb{N}$. Taking the sup over all such partitions we obtain $\mathrm{L}_d(\gamma) \leq \int_a^b |\dot{\gamma}|_d(t)\, \mathrm{d}t$.

In order to prove the opposite inequality, choose $\varepsilon > 0$ and let $h = (b-a)/m$ and $t_i = a + ih$, with $m \geq 2$ such that $h \leq \varepsilon$. We then observe that

$$\begin{aligned} \frac{1}{h} \int_a^{b-\varepsilon} d(\gamma(t), \gamma(t+h))\, \mathrm{d}t &\leq \frac{1}{h} \int_0^h \sum_{i=0}^{m-2} d(\gamma(t_i + \tau), \gamma(t_{i+1} + \tau))\, \mathrm{d}\tau \\ &\leq \frac{1}{h} \int_0^h \mathrm{L}_d(\gamma)\, \mathrm{d}\tau = \mathrm{L}_d(\gamma). \end{aligned}$$

From the definition of $|\dot{\gamma}|_d$ and Fatou's Lemma we get

$$\begin{aligned} \int_a^{b-\varepsilon} |\dot{\gamma}|_d(t)\, \mathrm{d}t &= \int_a^{b-\varepsilon} \liminf_{h \to 0^+} \frac{d(\gamma(t), \gamma(t+h))}{h}\, \mathrm{d}t \\ &\leq \liminf_{h \to 0^+} \frac{1}{h} \int_a^{b-\varepsilon} d(\gamma(t), \gamma(t+h))\, \mathrm{d}t \leq \mathrm{L}_d(\gamma), \end{aligned}$$

hence the claim by the arbitrariness of ε. □

Remark 1.4. Let us observe that, by definition, $\mathrm{L}_d(\gamma)$ does not depend on the way the Lipschitz curve γ is parametrized, namely if $\sigma = \gamma \circ \rho$ where $\rho : \mathbb{R} \to \mathbb{R}$ is an order-preserving, Lipschitz continuous diffeomorphism, then $\mathrm{L}_d(\gamma) = \mathrm{L}_d(\sigma)$ (if d is symmetric, this holds for an order-reversing diffeomorphism too). In particular, up to a reparametrization, any curve γ can be always assumed to be defined on $I := [0, 1]$.

Definition 1.5 (Geodesic distance). *We will say that d is a* geodesic distance *if it satisfies the following identity:*

$$d(x,y) = \inf\left\{\mathrm{L}_d(\gamma)\ :\ \gamma \in \mathrm{Lip}_{x,y}(I,(X,d))\right\} \qquad \textit{for every } (x,y) \in X \times X.$$

A metric space (X,d) such that d is geodesic is called a length space*.*

Arguing as in [5], we may extend to the case of non-symmetric distances the following theorem due to Busemann.

Theorem 1.6 (Busemann). *Assume that d is a non-degenerate, non-symmetric distance, and that every closed ball in (X,d) is compact. Let $x, y \in X$. Then the minimum problem*

$$\min\left\{\mathrm{L}_d(\gamma)\ :\ \gamma \in \mathrm{Lip}_{x,y}(I,(X,d))\right\} \tag{1.9}$$

admits a solution, provided the family $\mathrm{Lip}_{x,y}(I,(X,d))$ is not empty. In particular, if d is a geodesic distance, there exists a curve $\gamma \in \mathrm{Lip}_{x,y}(I,(X,d))$ which is of minimal d-length, i.e. *such that $\mathrm{L}_d(\gamma) = d(x,y)$.*

1.3 Finsler metrics and induced distances

Let us now consider an open and connected subset Ω of $\mathbb{R}^N$. The definition of (weak) Finsler metric is given as follows.

Definition 1.7. A Borel function $\varphi : \overline{\Omega} \times \mathbb{R}^N \to [0,+\infty)$ is said to be a *Finsler metric* on $\overline{\Omega}$ if

(i) $\varphi(x,\cdot)$ is positively 1-homogeneous for every $x \in \overline{\Omega}$;

(ii) $\varphi(x,\cdot)$ is convex on $\mathbb{R}^N$ for $\mathcal{L}^N$-a.e. $x \in \overline{\Omega}$;

(iii) for every compact set $K \subset \overline{\Omega}$ there exist two non-negative real constants α_K, β_K such that
$$0 \le \alpha_K|\xi| \le \varphi(x,\xi) \le \beta_K|\xi| \quad \text{for all } (x,\xi) \in K \times \mathbb{R}^N.$$

We will say that the metric φ is convex if (ii) holds for all $x \in \overline{\Omega}$.

Given a Finsler metric φ, we can define the Finslerian length functional $\mathbb{L}_\varphi$ through the formula

$$\mathbb{L}_\varphi(\gamma) := \int_a^b \varphi(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t \qquad \gamma \in \mathrm{Lip}([a,b], \overline{\Omega}). \tag{1.10}$$

Notice that $\mathbb{L}_\varphi$ is well defined. Indeed, the map $t \mapsto \big(\gamma(t), \dot{\gamma}(t)\big)$ is Lebesgue measurable on I and φ is Borel-measurable on $\overline{\Omega} \times \mathbb{R}^N$, hence their composition $\varphi(\gamma(t), \dot{\gamma}(t))$ is Lebesgue measurable. Moreover, by assumption (i) of Definition 1.7, $\mathbb{L}_\varphi(\gamma)$ does not depend on the chosen parametrization for γ, that is, if $\rho : I \to I$ is an order preserving, Lipschitz continuous diffeomorphism, then $\mathbb{L}_\varphi(\gamma \circ \rho) = \mathbb{L}_\varphi(\gamma)$. In particular, it is not restrictive to assume γ to be defined on the closed interval $I := [0,1]$.
The Finsler length functional (1.10) induces a distance d_φ on $\overline{\Omega}$ as follows:

$$d_\varphi(x,y) := \inf \left\{ \mathbb{L}_\varphi(\gamma) : \ \gamma \in \mathrm{Lip}_{x,y} \right\}. \tag{1.11}$$

A distance deriving from a Finsler metric through (1.11) is said to be of *Finsler type.* Note that, as $\varphi(x,\xi)$ is in general not even in ξ, the distance d_φ may be non-symmetric. When $\inf\limits_{K \subset \overline{\Omega}} \alpha_K = 0$, the distance d_φ may be degenerate as well.

Proposition 1.8. *Let $d := d_\varphi$ for some Finsler metric φ. Then $\mathrm{L}_d(\gamma) \leq \mathbb{L}_\varphi(\gamma)$ for every curve γ. In particular, d is a distance of geodesic type according to Definition 1.5.*

Proof. Let $\gamma \in \mathrm{Lip}_{x,y}$ and let $0 = t_0 < t_1 < .. < t_m = 1$ be a partition of I. By the definition of d and the 1-homogeneity of φ, for each i we have

$$d(\gamma(t_i), \gamma(t_{i+1})) \leq \int_{t_i}^{t_{i+1}} \varphi(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t,$$

and by summing up for all i we obtain

$$\sum_{i=0}^{m-1} d\big(\gamma(t_i), \gamma(t_{i+1})\big) \leq \int_0^1 \varphi(\gamma(t), \dot{\gamma}(t))\,\mathrm{d}t = \mathbb{L}_\varphi(\gamma). \tag{1.12}$$

By taking the supremum in (1.12) over all possible partitions of I we get that $\mathrm{L}_d(\gamma) \leq \mathbb{L}_\varphi(\gamma)$.
To prove that d is of geodesic type, we remark that, by the triangular inequality and the first part of the proof, there holds

$$d(x,y) \leq \mathrm{L}_d(\gamma) \leq \mathbb{L}_\varphi(\gamma). \tag{1.13}$$

The claim then follows by taking the infimum in (1.13) over all possible $\gamma \in \mathrm{Lip}_{x,y}$ and by the definition of d. $\square$

Remark 1.9. The inequality in the previous proposition may be strict. For example, take $\Omega := (-1,1) \times (-1,1)$, $\Gamma := \{0\} \times [-1,1]$ and $a(x) := \chi_{\overline{\Omega}}(x) + \chi_\Gamma(x)$ for all $x \in \overline{\Omega}$. Then $d_a(y,z) = |y-z|$ for all $y, z \in \overline{\Omega}$. If now we take $\gamma(t) := (0,-1/2)(1-t) + (0,1/2)t$, it is easily seen that $\mathrm{L}_{d_a}(\gamma) = 1 < 2 = \mathbb{L}_a(\gamma)$.

1.3.1 Finsler metrics on $\mathbb{R}^N$

Let us now focus our attention to the case $\Omega := \mathbb{R}^N$.

Definition 1.10. Let d be a distance on $\mathbb{R}^N$. We define the function φ_d associated to d by derivation as

$$\varphi_d(x,\xi) := \limsup_{t \to 0^+} \frac{d(x, x+t\xi)}{t} \qquad (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{1.14}$$

Assume that for every compact set $K \subset \mathbb{R}^N$, there exists a positive constant β_K such that $d(x,y) \le \beta_K |x-y|$ for every $x, y \in K$. Then it is not difficult to show that for any curve $\gamma \in \mathrm{Lip}(I, \mathbb{R}^N)$ there holds $|\dot\gamma|_d(t) = \varphi_d(\gamma(t), \dot\gamma(t))$ for $\mathcal{L}^1$-a.e. $t \in I$ (*cf.* [45, Theorem 2.5]). Comparing this remark with Theorem 1.3, we have in particular that

$$\mathrm{L}_d(\gamma) = \int_0^1 \varphi_d(\gamma(t), \dot\gamma(t))\,\mathrm{d}t, \tag{1.15}$$

that is $\mathrm{L}_d = \mathbb{L}_{\varphi_d}$ on $\mathrm{Lip}(I, \mathbb{R}^N)$. Next proposition shows that φ_d is actually a Finsler metric.

Theorem 1.11. *Let d be a Finsler distance on $\mathbb{R}^N$. Then the function $\varphi_d : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ is a Borel-measurable Finsler metric. In particular we have:*

(i) $\varphi_d(x,\cdot)$ *is convex for* $\mathcal{L}^N$*-a.e.* $x \in \mathbb{R}^N$*;*

(ii) $|\varphi_d(x,\xi) - \varphi_d(x,\nu)| \le \beta_r |\xi - \nu|$ *for every* $x \in B_r$ *and every* $\xi, \nu \in \mathbb{R}^N$*, where* $\beta_r := \beta_{\overline{B}_r}$ *according to Definition 1.7 (iii).*

If moreover d is symmetric, $\varphi_d(x,\cdot)$ is an even function (in particular, a norm) for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$.

The proof of the previous result is based on the auxiliary Lemma 1.12 below. In what follows, we denote by d_z the function $x \mapsto d(z,x)$ defined on $\mathbb{R}^N$, where $z \in \mathbb{R}^N$.

Lemma 1.12. *Let d be a distance on $\mathbb{R}^N$, and let φ be a Borel function defined on $\mathbb{R}^N \times \mathbb{R}^N$ which satisfies the following conditions:*

(i) $\varphi(x,\cdot)$ is continuous for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$;

(ii) for every compact set $K \subset \mathbb{R}^N$, there exists a positive constant β_K such that

$$|\varphi(x,\xi)| \leq \beta_K |\xi| \quad \text{for all } (x,\xi) \in K \times \mathbb{R}^N.$$

Let F be a negligible subset of $\mathbb{R}^N$ and assume that every curve γ belonging to $\mathrm{Lip}([0,t],\mathbb{R}^N)$, $t > 0$, and transversal to F, satisfies the following property:

$$d(\gamma(0),\gamma(t)) \leq \int_0^t \varphi(\gamma(s),\dot{\gamma}(s))\,\mathrm{d}s.$$

Then for any fixed $z \in \mathbb{R}^N$, for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$ and for all $\xi \in \mathbb{R}^N$:

$$\langle Dd_z(x),\xi\rangle \leq \liminf_{t\to 0^+} \frac{d(x,x+t\xi)}{t} \leq \limsup_{t\to 0^+} \frac{d(x,x+t\xi)}{t} \leq \varphi(x,\xi).$$

Proof. Fix $\xi \in \mathbb{R}^N$. Let us first notice that, by Fubini's Theorem and Lebesgue-Besicovitch Differentiation Theorem, the following properties hold for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$:

(i) the curve $\gamma_x(t) := x + t\xi$ is transversal to F;

(ii) $\varphi(x,\xi) = \lim_{t\to 0^+} \frac{1}{t}\int_0^t \varphi(x+s\xi,\xi)\,\mathrm{d}s.$

Let $z \in \mathbb{R}^N$ and denote by $E(z,\xi)$ be the set of points x satisfying conditions (i)-(ii) above and such that d_z is differentiable at x. By Rademacher's Theorem and what previously remarked, it follows that $\mathbb{R}^N \setminus E(z,\xi)$ has zero Lebesgue measure. Moreover, if $x \in E(z,\xi)$ we have:

$$\begin{aligned} \langle Dd_z(x),\xi\rangle &= \lim_{t\to 0^+} \frac{d(z,x+t\xi)-d(z,x)}{t} \leq \liminf_{t\to 0^+} \frac{d(x,x+t\xi)}{t} \\ &\leq \limsup_{t\to 0^+} \frac{d(x,x+t\xi)}{t} \leq \lim_{t\to 0^+} \frac{1}{t}\int_0^t \varphi(x+s\xi,\xi)\,\mathrm{d}s = \varphi(x,\xi). \end{aligned} \tag{1.16}$$

Now take a dense sequence $(\xi_n)_n$ in $\mathbb{R}^N$ and set $E(z) := \bigcap_{n\in\mathbb{N}} E(z,\xi_n) \cap \{x \in \mathbb{R}^N : \varphi(x,\cdot) \text{ is continuous}\}$. Clearly $\mathbb{R}^N \setminus E(z)$ has zero Lebesgue measure. Moreover, for every $x \in E(z)$ the previous inequality holds with $\xi := \xi_n$ for each $n \in \mathbb{N}$, hence for all $\xi \in \mathbb{R}^N$ since all functions appearing in (1.16) are continuous in ξ. $\square$

Proof of Theorem 1.11. The Borel-measurable character of φ_d as well as property (i) of Definition 1.7 trivially comes from the definition.

Let us prove claim (i). Take a countable dense subset G of $\mathbb{R}^N$. For each $z \in G$, let $d_z(x) := d(z,x)$ for all $x \in \mathbb{R}^N$ and Σ_z a negligible Borel subset of $\mathbb{R}^N$ which contains all the points where d_z is not differentiable. For every $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$ we define

$$\psi(x,\xi) := \sup_{z\in G} \begin{cases} \langle Dd_z(x), \xi\rangle & \text{if } x \in \mathbb{R}^N \setminus \Sigma_z \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\psi(x,\xi)$ is Borel-measurable and convex in ξ. Now, set $\Sigma := \cup_{z\in G}\Sigma_z$ and observe that Σ is negligible, as G is countable. For every curve $\gamma \in \mathrm{Lip}([0,t],\mathbb{R}^N)$, $t > 0$, transversal to Σ we have:

$$\begin{aligned} d(\gamma(0),\gamma(t)) &= \sup_{z\in G}\Big(d(z,\gamma(t)) - d(z,\gamma(0))\Big) = \sup_{z\in G}\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\, d_z(\gamma(s))\,\mathrm{d}s \\ &= \sup_{z\in G}\int_0^t \langle Dd_z(\gamma(s)), \dot\gamma(s)\rangle\,\mathrm{d}s \le \int_0^t \psi(\gamma(s),\dot\gamma(s))\,\mathrm{d}s. \end{aligned}$$

We can therefore apply Lemma 1.12 to obtain

$$\langle Dd_z(x),\xi\rangle \le \liminf_{t\to 0^+}\frac{d(x,x+t\xi)}{t} \le \limsup_{t\to 0^+}\frac{d(x,x+t\xi)}{t} \le \psi(x,\xi)$$

for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$ and for all $\xi \in \mathbb{R}^N$. The claim then easily follows by taking the supremum over $z \in G$ of the left-hand side term.

To prove claim (ii), fix $r > 0$, $x \in B_r$ and $\xi,\eta \in \mathbb{R}^N$ and observe that, by the triangular inequality, we have

$$d(x,x+t\xi) \le d(x,x+t\eta) + d(x+t\eta,x+t\xi) \le d(x,x+t\eta) + \beta_r t|\xi-\eta|,$$

where t is a suitably small positive number. By dividing the above inequality by t and by taking the limsup as t decreases to zero we get

$$\varphi_d(x,\xi) \le \varphi_d(x,\eta) + \beta_r|\xi-\eta|,$$

and the claim follows by interchanging the roles of ξ and η.

Assume now d symmetric and let us prove that $\varphi_d(x,\cdot)$ is even for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$. We argue by contradiction: we then assume that the set $F := \{x \in \mathbb{R}^N \,:\, \varphi(x,\xi) \neq \varphi(x,-\xi)$ for some $\xi \in \mathbb{R}^N\}$ has positive Lebesgue measure. Fubini's Theorem then implies that there exist a Lipschitz curve $\gamma : [a,b] \to \mathbb{R}^N$ and a real number $\varepsilon > 0$ such that the set $I_0 := \{t \in \mathbb{R} \,:\, \varphi_d(\gamma(t),\dot\gamma(t)) > \varphi_d(\gamma(t),-\dot\gamma(t)) + \varepsilon\}$ has positive $\mathcal{L}^1$-measure. Take a point $t_0 \in (a,b)$ of density 1 for I_0. Set $\sigma := \gamma_{|[t_0-\delta,t_0+\delta]}$ and $\breve\sigma(t) := \sigma(2t_0 - t)$ for $t \in [t_0-\delta, t_0+\delta]$, with $\delta > 0$ suitably chosen. If δ is small enough, we have

$$\mathrm{L}_d(\sigma) = \int_{t_0-\delta}^{t_0+\delta} \varphi_d(\gamma(s),\dot\gamma(s))\,\mathrm{d}s > \int_{t_0-\delta}^{t_0+\delta} \varphi_d(\gamma(s),-\dot\gamma(s))\,\mathrm{d}s = \mathrm{L}_d(\breve\sigma),$$

a contradiction since $\mathrm{L}_d(\sigma) = \mathrm{L}_d(\breve\sigma)$ in view of Remark 1.4. $\square$

As a consequence, we derive the following result.

Theorem 1.13. *The space of Finsler distances on $\mathbb{R}^N$ coincide with the family of geodesic distances on $\mathbb{R}^N$ satisfying the following property:*

$$\begin{gathered}\text{for every compact set } K \subset \mathbb{R}^N\text{, there exist } \alpha_K,\, \beta_K \in \mathbb{R}_+ \text{ such that}\\ \alpha_K|x-y| \le d(x,y) \le \beta_K|x-y| \qquad \forall\, x,\, y \in K.\end{gathered} \tag{1.17}$$

Proof. One inclusion is obvious in view of Proposition 1.8. Conversely, assume d is a geodesic distance on $\mathbb{R}^N$ satisfying (1.17). Then we can define the function φ_d through (1.14). Arguing as in the proof of Theorem 1.11, we may prove that φ_d is a Finsler metric on $\mathbb{R}^N$. Then $d = d_{\varphi_d}$ in view of (1.15) and since d is a geodesic distance. $\square$

To sum up, any Finsler metric φ on $\mathbb{R}^N$ gives rise to a distance $d := d_\varphi$ through (1.11). To such a distance, one can associate by derivation the Finsler metric φ_d given by (1.14). The next example shows that φ_d is in general different from φ.

Example 1.14. It is possible to construct a Finsler distance $d := d_\varphi$, where φ is a metric of the form $a\,(x)\,|\xi|$, such that the corresponding φ_d is non-Riemannian. This is due to the possible lack of regularity of Finsler metrics. An example of this singular behavior is the following: let $E := \big\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{Q} \text{ or } x_2 \in \mathbb{Q}\big\}$ and define $\varphi(x,\xi) := a(x)|\xi|$, being the coefficient $a(x)$ given by

$$a\,(x) = \chi_E\,(x) + \beta\chi_{\mathbb{R}^2\setminus E}\,(x)\ .$$

If $\beta > 0$ is sufficiently large (*i.e.* , such that $\beta\sqrt{|x_1|^2+|x_2|^2} \geq |x_1|+|x_2|$ for every $x \in \mathbb{R}^2$), the induced distance $d := d_\varphi$ is precisely $d\,(x,y) = |x_1-y_1|+|x_2-y_2|$. Consequently, we have $\varphi_d\,(x,\xi) = |\xi_1|+|\xi_2|$, so that φ_d is everywhere different from φ.

Even if φ_d need not be equal to φ, some relations between them can be deduced.

Proposition 1.15. *Let φ be a Finsler metric on $\mathbb{R}^N$ and $d := d_\varphi$. Then there exists a negligible set $N \subset \mathbb{R}^N$ such that*

$$\varphi_d(x,\xi) \leq \varphi(x,\xi) \quad \textit{for every } (x,\xi) \in \left(\mathbb{R}^N \setminus N\right) \times \mathbb{R}^N.$$

Moreover the following holds:

(i) if $\varphi(\cdot,\xi)$ is upper semicontinuous for every $\xi \in \mathbb{R}^N$, then $\varphi_d(x,\xi) \leq \varphi(x,\xi)$ for every $(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$;

(ii) if $\varphi(x,\cdot)$ is convex on $\mathbb{R}^N$ for every $x \in \mathbb{R}^N$, $\varphi(\cdot,\xi)$ is lower semicontinuous for every $\xi \in \mathbb{R}^N$ and $\varphi(x,\xi) \geq \alpha|\xi|$ on $\mathbb{R}^N \times \mathbb{R}^N$ for some $\alpha > 0$, then

$$\varphi_d(x,\xi) \geq \liminf_{t\to 0^+} \frac{d(x,x+t\xi)}{t} \geq \varphi(x,\xi) \quad \textit{for every } (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N.$$

In particular, $\varphi_d(x,\xi) = \varphi(x,\xi)$ on $\left(\mathbb{R}^N \setminus N\right) \times \mathbb{R}^N$.

Proof. Let us fix a vector $\xi \in \mathbb{R}^N$ and, for every $x_0 \in \mathbb{R}^N$, let us define the curve $\gamma_{x_0}(s) := x_0 + s\xi$. Let t be a Lebesgue point for the map $s \mapsto \varphi(\gamma_{x_0}(s),\xi)$. For $h > 0$ we have

$$\frac{1}{h}\int_t^{t+h} \varphi(\gamma_{x_0}(s),\xi)\,\mathrm{d}s = \frac{1}{h}\int_0^1 \varphi(\gamma_{x_0}(t+h\tau),h\xi)\,\mathrm{d}\tau \geq \frac{d(\gamma_{x_0}(t),\gamma_{x_0}(t)+h\xi)}{h},$$

so, by taking the limsup as $h \to 0^+$, we get $\varphi_d(\gamma_{x_0}(t),\xi) \leq \varphi(\gamma_{x_0}(t),\xi)$. Since $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ is a Lebegue point for $\varphi(\gamma_{x_0}(\cdot),\xi)$ and x_0 was arbitrarily chosen in $\mathbb{R}^N$, Fubini's Theorem implies that $\varphi_d(x,\xi) \leq \varphi(x,\xi)$ for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$. Then we can take a dense sequence $(\xi_n)_n$ in $\mathbb{R}^N$ and repeat the above argument for each ξ_n. Recalling that the functions $\varphi_d(x,\cdot)$ and $\varphi(x,\cdot)$ are continuous for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$, we get the claim by the density of $(\xi_n)_n$.

(i) Fix $(x,\xi) \in \mathbb{R}^N \times \mathbb{S}^{N-1}$. By the upper-semicontinuity assumption, there exists an

$r > 0$ such that $B_r(x) \subset \mathbb{R}^N$ and $\varphi(y, \xi) < \varphi(x, \xi) + \varepsilon$ for every $y \in B_r(x)$. For t small enough the curve $\gamma_t(s) := x + s(t\xi)$ lies within $B_r(x)$, so we have

$$d(x, x + t\xi) \leq \int_0^1 \varphi(x + s(t\xi), t\xi)\,\mathrm{d}s \leq \int_0^1 (\varphi(x, t\xi) + \varepsilon t)\,\mathrm{d}s = t(\varphi(x, \xi) + \varepsilon),$$

and hence

$$\frac{d(x, x + t\xi)}{t} \leq \varphi(x, \xi) + \varepsilon. \tag{1.18}$$

By taking the limsup in (1.18) as $t \to 0^+$ and since $\varepsilon > 0$, $x \in \mathbb{R}^N$ and $\xi \in \mathbb{S}^{N-1}$ were arbitrary, we obtain the claim.

(ii) Let us assume φ lower semicontinuous in x and convex in ξ and fix $(x, \xi) \in \mathbb{R}^N \times \mathbb{S}^{N-1}$. By the lower semicontinuity, for every $\varepsilon > 0$ there exists $r = r(\varepsilon, x) > 0$ such that $B_r(x) \subset \mathbb{R}^N$ and $\varphi(y, \xi) > \varphi(x, \xi) - \varepsilon$ for every $y \in B_r(x)$. Moreover, by the Lipschitz continuity of φ in ξ and by possibly choosing a smaller r, the previous inequality holds in $B_r(x) \times B_r(\xi)$. A standard compactness argument then guarantees the existence of a suitable $r > 0$ such that

$$\varphi(y, \eta) \geq \varphi(x, \eta) - \varepsilon \quad \text{for every } (y, \eta) \in B_r(x) \times \mathbb{S}^{N-1}.$$

Choose a d-minimizing sequence of paths $(\gamma_n)_n \subset \mathrm{Lip}_{x,x+t\xi}$. For t small enough, the curves γ_n lie within $B_r(x)$. Then, for n big enough, we have

$$\mathbb{L}_\varphi(\gamma_n) = \int_0^1 \varphi(\gamma_n(s), \dot{\gamma_n}(s))\,\mathrm{d}s \geq \int_0^1 (\varphi(x, \dot{\gamma_n}(s)) - \varepsilon|\dot{\gamma_n}(s)|)\,\mathrm{d}s \geq t\left(\varphi(x, \xi) - 2\frac{\beta_r}{\alpha}\varepsilon\right),$$

where for the last estimate we have used Jensen's inequality applied to the convex function $\varphi(x, \cdot)$ and the fact that $\alpha \int_0^1 |\dot{\gamma_n}|\,\mathrm{d}s \leq \mathbb{L}_\varphi(\gamma_n) \leq 2d(x, x + t\xi) \leq 2\beta_R t$ if n is large enough, where $R > r + |x|$. Letting n go to $+\infty$ in the above inequality we obtain

$$\frac{d(x, x + t\xi)}{t} \geq \varphi(x, \xi) - 2\frac{\beta_R}{\alpha}\varepsilon. \tag{1.19}$$

By taking the liminf of (1.19) as $t \to 0^+$ and since $\varepsilon > 0$, $x \in \mathbb{R}^N$ and $\xi \in \mathbb{S}^{N-1}$ were arbitrary we obtain

$$\varphi_d(x, \xi) \geq \liminf_{t \to 0^+} \frac{d(x, x + t\xi)}{t} \geq \varphi(x, \xi) \qquad \text{for every } (x, \xi) \in \mathbb{R}^N \times \mathbb{S}^{N-1},$$

and the claim follows by 1-homogeneity in ξ. □

1.3.2 Finsler metrics on $\overline{\Omega}$

Let Ω be an open and connected set in $\mathbb{R}^N$ and denote by $d_\Omega(x,y)$ the Euclidean geodesic distance in $\overline{\Omega}$, that is $d_\Omega := d_\varphi$ according to (1.11) with $\varphi(x,\xi) := |\xi|$ (note that d_Ω locally coincides in Ω with the Euclidean distance). Some topological conditions on Ω have to be assumed to prevent Finsler distances on $\overline{\Omega}$ to degenerate: in fact, if $\partial\Omega$ is sufficiently bad-behaved, it might happen that $d_\Omega(x,y) = +\infty$ for some couple of points $x, y \in \overline{\Omega}$. Therefore, in the sequel we will always assume that

$$\forall r > 0 \;\; \exists C_r \geq 1 \quad \text{such that} \quad d_\Omega(x,y) \leq C_r|x-y| \quad \forall\, x, y \in \overline{\Omega} \cap B_r. \tag{Ω}$$

Condition (Ω) is related to the regularity of $\partial\Omega$. When the latter can be (locally) expressed as a graph of a function h, (Ω) is equivalent to require that h is Lipschitz continuous.

Proposition 1.16. *Let us suppose that condition* (Ω) *holds. Let* φ *be a Finsler metric on* $\overline{\Omega}$ *and and let* $d := d_\varphi$ *be the distance on* $\overline{\Omega}$ *defined through (1.11). Then* d *can be extended to a Finsler distance on* $\mathbb{R}^N$*.*

Proof. Let us define

$$\overline{\varphi}(x,\xi) := \begin{cases} \varphi(x,\xi) & \text{if } x \in \overline{\Omega} \text{ and } \xi \in \mathbb{R}^N \\ \beta_n C_n |\xi| & \text{if } x \in B_n \setminus (\overline{B}_{n-1} \cup \overline{\Omega}) \text{ and } \xi \in \mathbb{R}^N,\ n \in \mathbb{N} \end{cases}$$

where C_n are positive constants chosen according to condition (Ω), and $\beta_n := \beta_{\overline{B_n}}$ according to Definition 1.7 (iii). The Finsler metric $\overline{\varphi}$ defines a distance $\overline{d} := d_{\overline{\varphi}}$ on $\mathbb{R}^N$ through (1.11). We claim that $\overline{d}$ is the required extension of d. In fact, when connecting two points of $\overline{\Omega}$ in $\mathbb{R}^N$, if one is interested in minimizing the Finslerian length $L_{\overline{\varphi}}$ there is no advantage to choosing a path which gets out of $\overline{\Omega}$, as $\overline{\varphi}$ is "high" outside Ω and one would pay too much. This means that

$$\overline{d}(x,y) = \inf\left\{\mathbb{L}_\varphi(\gamma) :\ \gamma \in \mathrm{Lip}_{x,y}(I, \overline{\Omega})\right\} \quad \text{for all } x, y \in \overline{\Omega}.$$

Since $\overline{\varphi} = \varphi$ on $\overline{\Omega} \times \mathbb{R}^N$, this immediately gives that $\overline{d} = d$ on $\overline{\Omega} \times \overline{\Omega}$. Therefore $\overline{d}$ provides the required extension of d. □

Remark 1.17. As we will always work with sets Ω which satisfy condition (Ω), in the sequel we will identify, if needed, a Finsler distance d with its extension $\overline{d}$ to $\mathbb{R}^N$.

Let now d be a Finsler distance on $\overline{\Omega}$. We may define the function φ_d associated to d by derivation as follows:

$$\varphi_d\,(x,\xi) := \limsup_{t\to 0^+} \frac{d\,(x, x+t\xi)}{t} \qquad (x,\xi) \in \overline{\Omega}\times\mathbb{R}^N, \tag{1.20}$$

where we have taken Remark 1.17 into account to give a meaning to the above expression for those points x which belong to $\partial\Omega$. By Theorem 1.11 there follows that φ_d is in fact a Finsler metric on $\overline{\Omega}$. Moreover, the analogous of Proposition 1.15 holds, with $\mathbb{R}^N$ replaced by $\overline{\Omega}$.

1.4 A family of Finsler metrics

In this section we will focus our attention on a specific family of Finsler metrics. Let Ω be a connected open set in $\mathbb{R}^N$ satisfying assumption (Ω). Given two positive constants α, β (with $\beta \geq \alpha > 0$), we set

$$\mathcal{M} = \mathcal{M}(\Omega,\alpha,\beta) := \left\{\, \varphi \text{ Finsler metric on } \overline{\Omega} \,:\, \alpha|\xi| \leq \varphi(x,\xi) \leq \beta|\xi| \text{ on } \overline{\Omega}\times\mathbb{R}^N \,\right\}, \tag{1.21}$$

and we consider the family of Finsler distances generated by elements of $\mathcal{M}$ through (1.11), namely

$$\mathcal{D} = \mathcal{D}(\mathcal{M}) := \left\{\, d_\varphi \text{ distance on } \overline{\Omega} \text{ given by (1.11)} \,:\, \varphi \in \mathcal{M} \,\right\}. \tag{1.22}$$

The fact that α and β are now fixed allows us to get further results with respect to those obtained in the previous section. The following is an improvement of Proposition 1.8.

Theorem 1.18. *Let $d := d_\varphi$ for some Finsler metric $\varphi \in \mathcal{M}$. Then for any $\gamma \in \mathrm{Lip}(I,\overline{\Omega})$ we have:*

$$\mathrm{L}_d(\gamma) = \inf\left\{ \liminf_{n\to+\infty} \mathbb{L}_\varphi(\gamma_n) \,:\, (\gamma_n)_n \text{ converges to } \gamma \text{ in } \mathrm{Lip}(I,\overline{\Omega}) \right\}, \tag{1.23}$$

namely L_d is the relaxed functional of $\mathbb{L}_\varphi$ on $\mathrm{Lip}(I,\overline{\Omega})$.

Proof. Let us fix a curve γ in $\mathrm{Lip}(I,\overline{\Omega})$. From Proposition 1.1 and Proposition 1.8, we already now that $\mathrm{L}_d(\gamma)$ is less or equal than the right-hand side of (1.23). To show the equality, we have to find a sequence $(\gamma_n)_n$ in $\mathrm{Lip}(I,\overline{\Omega})$ converging to γ and such that $\mathrm{L}_d(\gamma) \geq \limsup_n \mathbb{L}_\varphi(\gamma_n)$. We claim that for every $\varepsilon > 0$ there exists a curve γ_ε

such that $\sup_{t\in I}|\gamma(t)-\gamma_\varepsilon(t)|<\varepsilon$ and $\mathrm{L}_d(\gamma)+\varepsilon>\mathbb{L}_\varphi(\gamma_\varepsilon)$. Obviously, that is enough to conclude. Let us choose a partition $0=t_0<t_1<\ldots<t_m=1$ and, for each i, a curve $\sigma_i\in\mathrm{Lip}_{\gamma(t_i),\gamma(t_{i+1})}(I,\overline{\Omega})$ such that $\mathbb{L}_\varphi(\sigma_i)\le d(\gamma(t_i),\gamma(t_{i+1}))+\varepsilon/m$. Since $\alpha\int_0^1|\dot\sigma_i(t)|\,\mathrm{d}t\le\mathbb{L}_\varphi(\sigma_i)$, the curves σ_i lie in an ε-neighborhood of γ if $\sup_i|t_i-t_{i+1}|$ is sufficiently small (and up to choosing a finer partition, this can be always assumed). Let γ_ε be the curve otained by gluing up all these curves σ_i. We have

$$\mathbb{L}_\varphi(\gamma_\varepsilon)=\sum_{i=0}^{m-1}\mathbb{L}_\varphi(\sigma_i)\le\varepsilon+\sum_{i=0}^{m-1}d(\gamma(t_i),\gamma(t_{i+1}))\le\varepsilon+\mathrm{L}_d(\gamma),$$

which is the claim. □

Remark 1.19. By Theorem 1.18, $\mathbb{L}_\varphi$ will coincide with L_d whenever $\mathbb{L}_\varphi$ is lower semicontinuous on $\mathrm{Lip}(I,\overline{\Omega})$. This happens, for instance, when φ is lower semicontinuous on $\overline{\Omega}\times\mathbb{R}^N$ and $\varphi(x,\cdot)$ is convex on $\mathbb{R}^N$ for every $x\in\overline{\Omega}$ (*cf.* [21, Theorem 4.1.1]).

Any distance $d\in\mathcal{D}$ is such that $\alpha d_\Omega(x,y)\le d(x,y)\le\beta d_\Omega(x,y)$ for all $x,y\in\overline{\Omega}$, which means that d induces a topology on $\overline{\Omega}$ which is equivalent to the Euclidean one. We can therefore apply the results recalled in section 1.2, specialized for $X:=\overline{\Omega}$. In particular, Busemann Theorem 1.6 immediately gives the following

Proposition 1.20. *For any couple of points x,y in $\overline{\Omega}$, there exists a curve $\gamma\in\mathrm{Lip}_{x,y}$ of minimal d-length,* i.e. *such that $\mathrm{L}_d(\gamma)=d(x,y)$.*

We endow $\mathcal{D}$ with the topology given by the uniform convergence on compact subset of $\overline{\Omega}\times\overline{\Omega}$. We will write $d_n\xrightarrow{\mathcal{D}}d$ to mean that the sequence $(d_n)_n\subset\mathcal{D}$ converges to $d\in\mathcal{D}$ with respect to this topology. The next result can be easily obtained by adapting the proof of [23, Theorem 3.1] to our setting.

Theorem 1.21. *Let Ω be a domain in $\mathbb{R}^N$ satisfying (Ω), and let d,d_n belong to $\mathcal{D}$ for each $n\in\mathbb{N}$. Then $d_n\xrightarrow{\mathcal{D}}d$ if and only if L_{d_n} Γ-convergence to L_d on $\mathrm{Lip}(I,\overline{\Omega})$. Moreover, $\mathcal{D}$ is a metrizable compact space.*

Let us stress that the interesting part of the result provided by Theorem 1.21 corresponds to the closed character of the space $\mathcal{D}$, since the compactness trivially follows from Ascoli-Arzelà Theorem. A trivial consequence of Ascoli-Arzelà Theorem is next lemma too.

Lemma 1.22. *Let $(d_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}$ which converges pointwise to some $d \in \mathcal{D}$ on a dense subset of $\overline{\Omega} \times \overline{\Omega}$. Then $d_n \xrightarrow{\mathcal{D}} d$.*

Lemma 1.22 is immediately applied to show the next proposition.

Proposition 1.23. *Let φ, $\varphi_n \in \mathcal{M}$ and d and d_n be the distances associated respectively to φ and φ_n through (1.11). If $\mathbb{L}_{\varphi_n}$ Γ-converges to $\mathbb{L}_\varphi$ on $\mathrm{Lip}_{x,y}(I,\overline{\Omega})$ for all x, y belonging to a dense subset of $\overline{\Omega} \times \overline{\Omega}$, then $d_n \xrightarrow{\mathcal{D}} d$.*

Proof. Choose $x, y \in \overline{\Omega}$ as in the statement and set $X := \mathrm{Lip}_{x,y}(I,\overline{\Omega})$. By hypothesis, $\mathbb{L}_{\varphi_n}$ Γ-converge to $\mathbb{L}_\varphi$ on X. Moreover the length functionals $\mathbb{L}_{\varphi_n}$,$n \in \mathbb{N}$ are equicoercive on X (since $\mathbb{L}_{\varphi_n}(\gamma) \geq \alpha \int_0^1 |\dot\gamma|(t)\,\mathrm{d}t$). Hence, by the crucial result of Γ-convergence (see [36, Theorem 7.4]), we have that the sequence $(\inf_X \mathbb{L}_{\varphi_n})_n$ converges to $\inf_X \mathbb{L}_\varphi$, namely $\lim_n d_n(x,y) = d(x,y)$ by definition of d_n and d. The claim follows by Lemma 1.22. $\square$

Proposition 1.24. *Let φ, $\varphi_n \in \mathcal{M}$ and d and d_n be the distances associated respectively to φ and φ_n through (1.11). Then $d_n \xrightarrow{\mathcal{D}} d$ in the following cases:*

(i) φ_n are lower semicontinuous, convex in ξ and converge increasingly to φ pointwise on $\overline{\Omega} \times \mathbb{R}^N$;

(ii) $(\varphi_n)_n$ converges uniformly to φ on compact subset of $\overline{\Omega} \times \mathbb{R}^N$;

(iii) $(\varphi_n)_n$ converges decreasingly to φ pointwise on $\overline{\Omega} \times \mathbb{R}^N$.

Proof. Assume condition (i) holds. Then, for each $x, y \in \overline{\Omega}$, $(\mathbb{L}_{\varphi_n})_n$ is an increasing sequence of lower semicontinuous functionals on $\mathrm{Lip}_{x,y}(I,\overline{\Omega})$ (*cf.* Remark 1.19), which converges pointwise to $\mathbb{L}_\varphi$ (*i.e.* $\lim_n \mathbb{L}_{\varphi_n}(\gamma) = \mathbb{L}_\varphi(\gamma)$ for each $\gamma \in \mathrm{Lip}_{x,y}(I,\overline{\Omega})$). By [36, Remark 5.5] the functionals $\mathbb{L}_{\varphi_n}$ Γ-converge to $\mathbb{L}_\varphi$, and the claim follows by Proposition 1.23.

To establish the result under the hypotheses (ii) and (iii) respectively, it will be enough to prove that $(d_n)_n$ converges pointwise to d in view of Lemma 1.22. For any fixed $(x, y) \in \overline{\Omega}\times\overline{\Omega}$ let us then prove that $\lim_n d_n(x,y) = d(x,y)$.

(ii) For $\gamma \in \mathrm{Lip}_{x,y}(I,\overline{\Omega})$ and the uniform convergence of φ_n we have

$$\mathbb{L}_\varphi(\gamma) = \lim_{n\to+\infty} \mathbb{L}_{\varphi_n}(\gamma) \geq \limsup_{n\to+\infty} d_n(x,y),$$

which entails $d(x,y) \geq \limsup_n d_n(x,y)$ by taking the infimum over all possible curves $\gamma \in \mathrm{Lip}_{x,y}(I,\overline{\Omega})$. Let us now choose a sequence of curves $(\gamma_n)_n \subset \mathrm{Lip}_{x,y}(I,\overline{\Omega})$ such that $\mathbb{L}_{\varphi_n}(\gamma_n) \leq d_n(x,y) + 1/n$. Since the curves γ_n are equi-Lipschitz continuous (as $\mathbb{L}_{\varphi_n}(\gamma_n) \geq \alpha \int_0^1 |\dot{\gamma}(t)|\, dt$ and all curves are parametrized by constant speed), the uniform convergence of the metrics implies that $\limsup_n |\mathbb{L}_{\varphi_n}(\gamma_n) - \mathbb{L}_{\varphi}(\gamma_n)| = 0$, therefore

$$d(x,y) \leq \limsup_{n\to+\infty} \mathbb{L}_{\varphi}(\gamma_n) = \liminf_{n\to+\infty} \mathbb{L}_{\varphi_n}(\gamma_n) = \liminf_{n\to+\infty} d_n(x,y),$$

hence the claim.

(iii) By monotonicity we get $d(x,y) \leq \inf_n d_n(x,y)$. To show the reverse inequality, take a curve $\gamma \in \mathrm{Lip}_{x,y}(I,\overline{\Omega})$. By the monotone convergence theorem and by the definition of $d_n(x,y)$ we have

$$\mathbb{L}_{\varphi}(\gamma) = \inf_n \mathbb{L}_{\varphi_n}(\gamma) \geq \inf_n d_n(x,y),$$

and the claim easily follows by taking the infimum over all curves in $\mathrm{Lip}_{x,y}(I,\overline{\Omega})$. □

Remark 1.25. The proof of claim (iii) in Proposition 1.24 does not rely upon the fact that the constant α, introduced in the definition of $\mathcal{M}$, is strictly positive. Actually, the result still holds even in the case $\alpha := 0$. This fact will be used in the proof of Theorem 4.7.

Remark 1.26. Let $d := d_\varphi$ for some $\varphi \in \mathcal{M}$, and let φ_d be obtained through (1.20). As previously seen, φ_d is a Finsler metric on $\overline{\Omega}$. Nevertheless, we can not say that φ_d belongs to $\mathcal{M}$, as the inequality $\varphi_d(x,\xi) \leq \beta|\xi|$ may fail when $x \in \partial\Omega$ (*cf.* Proposition 1.16 and Remark 1.17). For later use, it is convenient to fix a "canonical" way to associate to d a Finsler metric $\psi \in \mathcal{M}$ such that $d = d_\psi$. This can be performed by slightly modifying the definition of φ_d. Indeed, we may replace (1.20) with the following:

$$\varphi_d\,(x,\xi) := \limsup_{t\to 0^+} \left(\frac{d\,(x, x+t\xi)}{t} \wedge \beta|\xi| \right) \qquad (x,\xi) \in \overline{\Omega} \times \mathbb{R}^N. \tag{1.24}$$

Clearly, formula (1.24) defines a Finsler metric belonging to $\mathcal{M}$. Moreover, $d = d_{\varphi_d}$. To see this, take a curve $\gamma \in \mathrm{Lip}(I,\overline{\Omega})$ and pick up a differentiability point $t \in (0,1)$ for γ. By arguing as in [45, Theorem 2.5], we get:

$$\varphi_d(\gamma(t),\dot{\gamma}(t)) = \limsup_{h\to 0^+} \frac{d(\gamma(t),\gamma(t+h))}{h},$$

where Remark 1.17 has been taken into account to ensure that the above expression makes always sense. If t is also a Lebesgue point for $|\dot{\gamma}(s)|$, we have moreover:

$$\varphi_d(\gamma(t),\dot{\gamma}(t)) \leq \limsup_{h\to 0^+} \frac{1}{h}\int_t^{t+h} \varphi(\gamma(s),\dot{\gamma}(s))\,\mathrm{d}s \leq \limsup_{h\to 0^+} \frac{\beta}{h}\int_t^{t+h} |\dot{\gamma}(s)|\,\mathrm{d}s = \beta|\dot{\gamma}(t)|.$$

In particular, we conclude that the following holds:

$$\alpha \leq \varphi_d\left(\gamma(t), \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}\right) \leq \beta \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \in I. \tag{1.25}$$

To sum up, with respect to definition (1.20) formula (1.24) amounts to modifying the metric φ_d only on a subset E of $\partial\Omega \times \mathbb{R}^N$ which, in view of (1.25), plays no role in this setting: indeed, the map $t \mapsto (\gamma(t),\dot{\gamma}(t))$ is transversal to E for any choice of γ in $\mathrm{Lip}(I,\overline{\Omega})$.

Chapter 2

Optimal Riemannian distances preventing mass transfer

2.1 Introduction

The classical mass transport problem, introduced by Monge in [65], and reformulated by Kantorovich in [60, 61], has been widely investigated in recent years with a renewed interest (see, for instance, references [10, 12, 13, 26, 47, 50, 51, 55, 56, 67, 71]). It can be roughly described as follows: given two mass distributions, find the most efficient way to move one on the other. By efficiency it is intended that the mass transportation plan must minimize some average cost. In the original problem suggested by Monge, a pile of soil (which can be represented as a Borel probability measure f^+ on $\mathbb{R}^N$) was to be transported to some final configuration (given through a probability measure f^-). Monge wondered about the existence of a transportation map $T : \mathbb{R}^N \to \mathbb{R}^N$ minimizing the average work performed

$$\int_{\mathbb{R}^N} |x - T(x)| \, df^+(x),$$

among all the admissible transport maps T which send f^+ into f^-, i.e. $T_\# f^+ = f^-$, where $T_\#$ denotes the push-forward operator between measures.

Kantorovich's reformulation of the mass transportation problem consists in the following relaxation procedure: the minimum is now sought in the larger class of *admissible transport plans* (also known as stochastic transport maps). These are Borel probability measures ν

defined on the product $\mathbb{R}^N \times \mathbb{R}^N$ whose marginals are precisely (f^+, f^-), that is,

$$f^+(E) = \nu(E \times \mathbb{R}^N), \qquad f^-(E) = \nu(\mathbb{R}^N \times E)$$

for every Borel subset E of $\mathbb{R}^N$. One then tries to minimize

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} |x - y| \, d\nu(x, y),$$

among such admissible plans. An admissible transport map T corresponds to a transport plan ν concentrated on the graph of T. Since the constraint appearing now in this relaxed version is linear, an optimal transport plan can always be shown to exist.
This problem finds a natural setting in a metric space (X, d): for a given pair (f^+, f^-) of Borel probability measures on X, the Kantorovich formulation of the mass transport problem reads as

$$\min \left\{ \iint_{X \times X} c(x, y) \, d\nu(x, y) \;:\; \nu \text{ admissible plan} \right\}, \tag{2.1}$$

where $c(x, y)$ is a given nonnegative continuous function on $X \times X$, which represents the cost of transporting a point mass from x into y. The most studied situation is when the cost density $c(x, y)$ is a function of the distance d:

$$c(x, y) = \Phi(d(x, y)) \,, \qquad (x, y) \in X \times X \,,$$

where $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing and continuous. It is by now well known that the minimum (2.1) is realized by an optimal admissible plan. With the choice $\Phi(t) = t^p$, the quantity (2.1) (to the power $1/p$) is known as the p-Wasserstein distance between the measures f^+ and f^-. The case $p = 1$, the classical one considered by Monge, is related to several results in shape optimization theory (see [12, 13]); the case $p = 2$ is also widely studied for its implications in fluid mechanics (see [10]); the case $p < 1$, or more generally the case when $\Phi(t)$ is a concave function, seems to be the most realistic for several applications, and has been studied in [56].

In the present chapter, we want to investigate an optimization problem which occurs when we are allowed to vary the distance d in a suitable admissible class. More precisely, we consider as X the closure $\overline{\Omega}$ of an open bounded subset Ω of the Euclidean space $\mathbb{R}^N$

with Lipschitz boundary. We let d vary among the distances generated by a conformally flat Riemannian metric in the following sense:

$$d_a(x,y) := \inf\left\{ \int_0^1 a(\gamma)|\gamma'|\,dt \ :\ \gamma \in \mathrm{Lip}(]0,1[;\Omega)\ ,\ \gamma(0^+) = x\ ,\ \gamma(1^-) = y \right\} . \tag{2.2}$$

The problem we are interested in is the following: for fixed marginals f^+ and f^-, we consider the cost functional

$$F(a) := \min\left\{ \iint_{\overline{\Omega}\times\overline{\Omega}} \Phi(d_a(x,y))\,d\nu(x,y) \ :\ \nu \text{ admissible plan } \right\} , \tag{2.3}$$

defined for every nonnegative Borel coefficient $a(x)$. We want to prevent as much as possible the transportation of f^+ into f^-, by maximizing the cost $F(a)$ among all a belonging to the class

$$\mathcal{A} := \left\{ a(x) \text{ Borel-measurable } \ :\ \alpha \leq a(x) \leq \beta\ ,\ \int_\Omega a(x)\,dx \leq m \right\} , \tag{2.4}$$

the constants α, β, m being positive numbers, satisfying the compatibility conditions

$$\alpha \mathcal{L}^N(\Omega) \leq m \leq \beta \mathcal{L}^N(\Omega) \ .$$

For instance, when $\Phi(t) = t$ and $f^+ = \delta_x$, $f^- := \delta_y$ are Dirac masses concentrated in two fixed points $x, y \in \Omega$, the problem of maximizing F is nothing else than that of proving the existence of a conformally flat Euclidean metric whose length-minimizing geodesics joining x and y are as long as possible.

This problem seems to be unexplored in the literature on Calculus of Variations, though its study can be supported by natural motivations. Indeed, in many concrete examples, one can be interested in making as difficult as possible the communication between some masses f^+ and f^-. For instance, it is easy to imagine that this situation may arise in economics, or in medicine, or simply in traffic planning, each time the connection between two "enemies" is undesired. Of course, the problem is made non trivial by the integral constraint in (2.4), which has a physical meaning: it prescribes the quantity of material at one's disposal to solve the problem; in particular, it expresses that such quantity is finite. (On the other hand, the pointwise constraint in (2.4) is somehow of technical nature, as it is used to get compactness).

We would also like to point out that the similar problem of minimizing the cost functional $F(a)$ over the class $\mathcal{A}$, which corresponds to favor the transportation of f^+ into f^-, is trivial, since

$$\inf\Big\{F(a)\ :\ a\in\mathcal{A}\Big\} = F(\alpha)\ .$$

In fact, it is enough to approximate f^+ and f^- by finite sums of weighted Dirac masses $f_n^+ = \sum_{i=1}^n p_i\delta_{x_i}$ and $f_n^- = \sum_{i=1}^n q_i\delta_{y_i}$, and to put $a(x)=\alpha$ in all Euclidean geodesic lines connecting every x_i to every y_j, with $a(x) = m/\mathcal{L}^N(\Omega)$ elsewhere.
On the other hand, the existence of a solution for the maximization problem

$$\sup\Big\{F(a)\ :\ a\in\mathcal{A}\Big\} \tag{2.5}$$

is a delicate matter. Indeed, maximizing sequences $\{a_n\}\subset\mathcal{A}$ could develop an oscillatory behavior producing only a relaxed solution. This phenomenon has been first pointed out in [1], and later investigated in more detail in [16, 23]. These works reveal that the traditional approach to attack the maximization problem (2.5), namely the direct methods of the Calculus of Variations, cannot be used to obtain the existence of a solution. Basically, the reason is that the class $\mathcal{A}$ is *not closed* with respect to the natural convergence which ensures the continuity of the functional F. Indeed, given a maximizing sequence $(a_n)_{n\in\mathbb{N}}\subset\mathcal{A}$, it is not difficult to prove (see for instance [23]) that d_{a_n} converge uniformly on $\overline{\Omega}\times\overline{\Omega}$ to some distance d, and there holds

$$\lim_{n\to\infty} F(a_n) = \min\left\{\iint_{\overline{\Omega}\times\overline{\Omega}} \Phi(d(x,y))\,d\nu(x,y)\ :\ \nu \text{ admissible plan}\right\}.$$

Thus, if we could write $d = d_a$ for some $a\in\mathcal{A}$, we would have $\lim_{n\to\infty} F(a_n) = F(a)$, and a would be a solution to problem (2.5). The point is that the limit distance d in general cannot be associated with a Riemannian coefficient in the class $\mathcal{A}$. For instance, consider in dimension $N=2$ a sequence of periodic coefficients $(a_n)_{n\in\mathbb{N}}$ of the form $a_n(x) = a(nx)$, where the function a takes only two different values $\beta>\alpha>0$ respectively on the white and black squares of a chessboard. It has been shown in [1] that, for fixed points x, y, there holds

$$\lim_{n\to\infty} d_{a_n}(x,y) = \inf\left\{\int_0^1 \varphi(\gamma')\,dt\ :\ \gamma\in\mathrm{Lip}([0,1];\overline{\Omega})\ ,\ \gamma(0)=x\ ,\ \gamma(1)=y\right\},$$

where φ is a Finsler metric independent of the position. Moreover, when the quotient β/α is sufficiently large, the unit ball $B_\varphi := \{\xi\in\mathbb{R}^2\ :\ \varphi(\xi)\le 1\}$ is a polytope (precisely, a regular octagon). Thus φ is non-Riemannian, and in this case the uniform limit of d_{a_n} cannot be written under the form d_a with $a \in \mathcal{A}$.

In view of these considerations, it is natural to relax problem (2.5), enlarging the class of admissible competitors to all Finsler metrics arising as limits of sequences $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}$. The existence of a solution in such a relaxed class may be easily deduced. Then, in order to understand whether a solution exists for the original problem, the effect produced by the integral constraint $\int_\Omega a_n(x)dx \leq m$ on the Finsler limit of a sequence $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ must be clarified. To this aim, we embed the class $\mathcal{A}$ into a family $\mathcal{M}$ of Finsler metrics, where the functional F admits a natural extension $\overline{F}$ (see Section 2.2). We also endow $\mathcal{M}$ with a suitable topology τ, that guarantees both the compactness of $\mathcal{M}$ and the continuity of $\overline{F}$ on $\mathcal{M}$ (cf. respectively Propositions 2.1 and 2.2). Then in Section 2.3 we show that the crucial condition satisfied by the Finsler metrics belonging to the τ-closure of the class $\mathcal{A}$ in the wider class $\mathcal{M}$ is an integral inequality for their largest eigenvalue Λ_φ:

$$\Lambda_\varphi(x) := \max\Big\{\varphi(x,\xi) \ : \ \xi \in \mathbb{R}^N \ , \ |\xi| \leq 1\Big\} \tag{2.6}$$

(see Theorem 2.4). As a consequence of this fact, we can prove that the optimization problem (2.5) of preventing the mass transfer of f^+ into f^- admits at least a solution in the original class $\mathcal{A}$ (see Theorem 2.3). By similar arguments, we also are able to treat more general maximization problems of the form (2.5), when F is replaced by an arbitrary cost functional satisfying suitable monotonicity and semicontinuity properties (see Theorem 2.5).

In some sense, our results may be read as regularity theorems, as they ensure the existence of a solution to the relaxed problem within the smaller class $\mathcal{A}$ of Riemannian coefficients, which is considerably more manageable than $\mathcal{M}$. (In particular, in the concrete frameworks mentioned above, the optimal metric turns out to be easier to manufacture.) However, let us stress that the uniqueness of solution for the relaxed problem when the cost function Φ is strictly increasing is, at present, an open question which, in our opinion, deserves further investigation.

2.2 Notation and preliminaries

In this section we precise the notation adopted throughout the chapter and we prove some preliminary results. We will denote by Ω an open, bounded and connected subset of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$. We set as usual

$$\mathcal{M} := \left\{\varphi \text{ Finsler metrics on } \overline{\Omega} \ : \ \alpha\,|\xi| \le \varphi\,(x,\xi) \le \beta\,|\xi|\right\},$$

and we denote by $\mathcal{D} = \mathcal{D}(\mathcal{M})$ the space of Finsler distances on $\overline{\Omega}$ generated by the metrics $\mathcal{M}$, namely

$$\mathcal{D} = \mathcal{D}(\mathcal{M}) := \{d_\varphi \text{ distance on } \overline{\Omega} \text{ given by (1.11) } : \ \varphi \in \mathcal{M}\,\}.$$

The class $\mathcal{A}$ defined in (2.4) is trivially included in $\mathcal{M}$ identifying a coefficient $a(x)$ with the metric $a(x)|\xi|$. According to Remark 1.26, any distance in $\mathcal{D}$ gives rise to a Finsler metric $\varphi_d \in \mathcal{M}$ as follows:

$$\varphi_d\,(x,\xi) := \limsup_{t\to 0^+} \left(\frac{d\,(x, x+t\xi)}{t} \wedge \beta|\xi|\right) \qquad (x,\xi) \in \overline{\Omega} \times \mathbb{R}^N. \tag{2.7}$$

By Theorem 1.11, φ_d satisfies the following property:

$$|\varphi_d(x,\xi) - \varphi_d(x,\nu)| \le \beta|\xi - \nu| \text{ for all } x \in \Omega \text{ and all } \xi, \nu \in \mathbb{R}^N \tag{2.8}$$

We recall that $\mathcal{D}$ is endowed with the metrizable topology of uniform convergence on $\overline{\Omega}\times\overline{\Omega}$ (as Ω is bounded), and that $\mathcal{D}$ is a metrizable compact space by the Lipschitz assumption on $\partial\Omega$ (*cf.* Theorem 1.21).

We introduce the following definition of τ-convergence for sequences of metrics in $\mathcal{M}$:

$$\varphi_n \xrightarrow{\tau} \varphi \quad \Longleftrightarrow \quad d_{\varphi_n} \text{converge uniformly to } d \text{ on } \overline{\Omega}\times\overline{\Omega} \text{ and } \varphi = \varphi_d\ .$$

Notice that, if $d_{\varphi_n} \to d$ uniformly on $\overline{\Omega}\times\overline{\Omega}$ and $d = d_\varphi$ for some $\varphi \in \mathcal{M}$, the τ-limit of φ_n is φ_d, which, in view of Example 1.14, is in general different from φ. We have:

Proposition 2.1. *The class $\mathcal{M}$ is sequentially τ-compact, namely every sequence $(\varphi_n)_n \subset \mathcal{M}$ admits a subsequence that converges to some metric $\varphi \in \mathcal{M}$.*

Proof. Let $(\varphi_n)_n$ be a sequence in the class $\mathcal{M}$. Then the associated distances $d_n := d_{\varphi_n}$ lie in the class $\mathcal{D}$. By compactness, we can find a subsequence $(d_{n_i})_i$ and a distance $d \in \mathcal{D}$ such that $d_{n_i} \to d$ uniformly on $\overline{\Omega}\times\overline{\Omega}$. Then, by definition, $\varphi_{n_i} \xrightarrow{\tau} \varphi_d$. □

The functional F defined by (2.3) may be extended in the natural way to the class $\mathcal{M}$ by setting, for φ in $\mathcal{M}$,

$$\overline{F}(\varphi) := \min\Big\{ \iint_{\overline{\Omega}\times\overline{\Omega}} \Phi(d_\varphi(x,y))\, \mathrm{d}\nu(x,y) \ : \ \nu \text{ admissible plan } \Big\}. \tag{2.9}$$

Proposition 2.2. *The functional $\overline{F}$ is sequentially τ-continuous on the class $\mathcal{M}$, namely if $\varphi_n \xrightarrow{\tau} \varphi$ then $\lim_n \overline{F}(\varphi_n) = \overline{F}(\varphi)$.*

Proof. Assume that $\varphi_n \xrightarrow{\tau} \varphi$. Then, by definition, the distances d_{φ_n} converge uniformly on $\overline{\Omega}\times\overline{\Omega}$ to $d \in \mathcal{D}$ and $\varphi = \varphi_d$. Next we observe that, for any sequence $(\nu_n)_{n\in\mathbb{N}}$ of nonnegative measures defined on $\overline{\Omega}\times\overline{\Omega}$ and weakly converging to some measure ν, there holds:

$$\lim_{n\to\infty} \int_{\overline{\Omega}} \Phi\left(d_{\varphi_n}(x,y)\right) d\nu_n(x,y) = \int_{\overline{\Omega}} \Phi(d(x,y)) d\nu(x,y)\,. \tag{2.10}$$

Now, for every n, let σ_n be a plan that realizes the minimum $\overline{F}(\varphi_n)$ according to definition (2.9); then there exists a subsequence $(\sigma_{n_i})_{i\in\mathbb{N}}$ weakly converging to some admissible plan σ and such that $\lim_i \overline{F}(\varphi_{n_i}) = \liminf_n \overline{F}(\varphi_n)$. Then, using (2.10) and the identity $d = d_\varphi$, we obtain

$$\begin{aligned} \liminf_{n\to\infty} \overline{F}(\varphi_n) &= \lim_{i\to\infty} \iint_{\overline{\Omega}\times\overline{\Omega}} \Phi\left(d_{\varphi_{n_i}}(x,y)\right) d\sigma_{n_i}(x,y) = \iint_{\overline{\Omega}\times\overline{\Omega}} \Phi(d(x,y)) d\sigma(x,y) \\ &= \iint_{\overline{\Omega}\times\overline{\Omega}} \Phi(d_\varphi(x,y)) d\sigma(x,y) \geq \overline{F}(\varphi)\,. \end{aligned}$$

To show that $\overline{F}(\varphi) \geq \limsup_n \overline{F}(\varphi_n)$, we may argue in a similar way: we apply (2.10) taking as $(\nu_n)_{n\in\mathbb{N}}$ a constant sequence equal to a measure σ that realizes the minimum $\overline{F}(\varphi)$ in (2.9). □

2.3 The existence results

Our main existence results are stated as follows.

Theorem 2.3. *Let $\mathcal{A}$ be the class of Borel coefficients given by (2.4), and let F be the functional defined by (2.3). Under the assumption that the cost density Φ is non-decreasing on $\mathbb{R}_+$, there exists at least an element $\overline{a} \in \mathcal{A}$ such that*

$$F(\overline{a}) = \sup\{F(a) : a \in \mathcal{A}\}\,.$$

The main tool for the proof of the above existence result is the next theorem. It states that the largest eigenvalue of Finsler metrics belonging to the τ-adherence of the class $\mathcal{A}$ must satisfy the same integral constraint as the elements of $\mathcal{A}$.

Theorem 2.4. *Let $(a_n)_n \subset \mathcal{A}$, and set $\varphi_n(x,\xi) := a_n(x)|\xi|$. If $\varphi_n \xrightarrow{\tau} \varphi$, then we have*

$$\int_\Omega \Lambda_\varphi(x)\,\mathrm{d}x \le m \ , \tag{2.11}$$

where $\Lambda_\varphi(x)$ is the largest eigenvalue of $\varphi(x,\cdot)$ defined by (2.6).

We now prove Theorem 2.3 using Theorem 2.4, whose proof is postponed.

Proof of Theorem 2.3. Let $(a_n)_{n\in\mathbb{N}} \subset \mathcal{A}$ be a maximizing sequence for the functional F, and set $\varphi_n(x,\xi) := a_n(x)|\xi|$. By Proposition 2.1, up to subsequences we have $\varphi_n \xrightarrow{\tau} \varphi$, and, by Proposition 2.2, we have

$$\overline{F}(\varphi) = \lim_{n\to\infty} \overline{F}(\varphi_n) = \lim_{n\to\infty} F(a_n) = \sup\{F(a) \ : \ a \in \mathcal{A}\} \ .$$

We are thus reduced to show that there exists at least an element $\overline{a} \in \mathcal{A}$ such that $F(\overline{a}) \ge \overline{F}(\varphi)$. We set

$$\overline{a}(x) := \Lambda_\varphi(x) \qquad \text{for} \ \ x \in \overline{\Omega} \ .$$

We first remark that the coefficient $\overline{a}$ is Borel-measurable. Indeed $\varphi = \varphi_d$ for some distance $d \in \mathcal{D}$ by definition (since φ is the τ-limit of a sequence of metrics $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$), so φ satisfies (2.8). In particular, if $(\xi_k)_{k\in\mathbb{N}}$ is a dense sequence in $\mathbb{S}^{N-1}$, we have that $\Lambda_\varphi(x) = \sup_k \varphi(x,\xi_k)$, which implies that $\overline{a}$ is Borel-measurable and satisfies the bounds $\alpha \le \overline{a}(x) \le \beta$. By Theorem 2.4, it satisfies also the integral constraint $\int_\Omega \overline{a}(x)dx \le m$. Hence $\overline{a} \in \mathcal{A}$. Moreover, since

$$\overline{a}\,(x)\,|\xi| \ge \varphi\,(x,\xi) \qquad (x,\xi) \in \overline{\Omega} \times \mathbb{R}^N \ ,$$

we have $d_{\overline{a}} \ge d_\varphi$ and then, by the monotonicity of Φ, $F\,(\overline{a}) \ge \overline{F}\,(\varphi)$.

Arguing as in the proof of Theorem 2.3, we may obtain the following formulation of the existence result for functionals defined on distances, where we denote by $\mathcal{D}(\mathcal{A})$ the class of distances on $\overline{\Omega}$ of the form d_a with $a \in \mathcal{A}$.

Theorem 2.5. *Let $\mathcal{C}$ be a functional defined on $\mathcal{D}$. We assume that* *(i) $\mathcal{C}$ is upper semicontinuous for the uniform convergence;*

(ii) $\mathcal{C}$ is non-decreasing for the usual order on distances.

Then the maximization problem

$$\max\left\{\mathcal{C}(d) \ : \ d \in \mathcal{D}(\mathcal{A})\right\}$$

admits at least a solution.

The remaining part of this section is devoted to the proof of Theorem 2.4. It is based on the auxiliary Propositions 2.6 and 2.9 below.

Proposition 2.6. *Let $(\varphi_n)_{n\in\mathbb{N}} \subset \mathcal{M}$, with $\varphi_n \xrightarrow{\tau} \varphi$. Then, for every Borel set $\omega \subset \Omega$ and every $\xi \in \mathbb{R}^N$, we have*

$$\int_\omega \varphi(x,\xi)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_\omega \varphi_n(x,\xi)\,\mathrm{d}x\ .$$

Proof. By the homogeneity property of φ, it is not restrictive to assume that $|\xi| = 1$. Thus, let us fix an element $\xi \in \mathbb{S}^{N-1}$. We claim that it is possible to find a subsequence of $(\varphi_n)_{n\in\mathbb{N}}$ and a sequence of positive numbers $t_n \to 0$ such that, for a.e. $x \in \Omega$,

$$\varphi(x,\xi) = \lim_{n\to\infty} \chi_{\Omega_n}(x)\,\frac{d_{\varphi_n}(x, x+t_n\xi)}{t_n}, \tag{2.12}$$

where $\Omega_n := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > t_n\}$. Indeed, we first remark that, almost everywhere in x, the limsup appearing in the right hand side of (2.7) is actually a limit (see [45, Corollary 2.7]). Thus, denoting by d the uniform limit of d_{φ_n}, we have

$$\varphi(x,\xi) = \lim_{t\to 0^+} \frac{d(x, x+t\xi)}{t} \qquad \text{for a.e. } x \in \Omega\ . \tag{2.13}$$

Next we observe that, by uniform convergence, there exists a sequence $(\varepsilon_n)_n$ tending to zero such that,

$$|d_{\varphi_n}(x, x+t\xi) - d(x, x+t\xi)| \le \varepsilon_n\ .$$

for every $x \in \Omega$ and every $t > 0$ with $x + t\xi \in \Omega$. Therefore, for a.e. $x \in \Omega$ and any $t_n \to 0$, we have

$$\left|\varphi(x,\xi) - \chi_{\Omega_n}(x)\,\frac{d_{\varphi_n}(x, x+t_n\xi)}{t_n}\right| \le \frac{\varepsilon_n}{t_n} + \left|\varphi(x,\xi) - \chi_{\Omega_n}(x)\,\frac{d(x, x+t_n\xi)}{t_n}\right|\ .$$

Then (2.12) follows choosing $t_n := \sqrt{\varepsilon_n}$, taking into account (2.13).
Now, integrating (2.12) over ω and using Fatou's lemma we get:

$$\int_\omega \varphi(x,\xi)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_\omega \chi_{\Omega_n}(x)\,\frac{d_{\varphi_n}(x, x+t_n\xi)}{t_n}\,\mathrm{d}x. \tag{2.14}$$

Since $d_{\varphi_n}(x, x+t_n\xi)$ is less than or equal to the (Finslerian) length of the straight line segment joining x and $x+t_n\xi$, we have

$$d_{\varphi_n}(x, x+t_n\xi) \le \int_0^1 \varphi_n(x+st_n\xi, t_n\xi)\,\mathrm{d}s. \tag{2.15}$$

Combining (2.14) and (2.15), we obtain

$$\begin{aligned}
\int_\omega \varphi(x,\xi)\,\mathrm{d}x &\le \liminf_{n\to\infty} \int_\Omega \chi_{\Omega_n\cap\omega}(x) \int_0^1 \varphi_n(x+st_n\xi, \xi)\,\mathrm{d}s\,\mathrm{d}x \\
&= \liminf_{n\to\infty} \int_0^1 \int_\Omega \chi_{\Omega_n\cap\omega}(x-st_n\xi)\,\varphi_n(x,\xi)\,\mathrm{d}x\,\mathrm{d}s \\
&\le \liminf_{n\to\infty} \int_\omega \varphi_n(x,\xi)\,\mathrm{d}x\ ,
\end{aligned}$$

the last inequality being a consequence of

$$\int_\Omega \left|\chi_{\Omega_n\cap\omega}(x-st_n\xi) - \chi_\omega(x)\right|\mathrm{d}x \to 0 \qquad \text{as } n\to\infty \qquad \text{for every } s\in(0,1)\,.$$

$\square$

We next state and prove two lemmas which will be used in the proof of Proposition 2.9. For every $i\in\mathbb{Z}^N$ and every $\delta>0$ we set $D_i^\delta := \Omega\cap\delta\big(2i+[-1,1)^N\big)$.

Lemma 2.7. *Let $\varphi\in\mathcal{M}$ be a continuous Finsler metric. Then, for every $\varepsilon>0$ there exists $\delta>0$ such that*

$$\int_{D_i^\delta} \Lambda_\varphi(x)\mathrm{d}x \le \sup_{|\xi|=1} \int_{D_i^\delta} [\varphi(x,\xi)+\varepsilon]\,\mathrm{d}x \qquad \textit{for all } i\in\mathbb{Z}^N.$$

Proof. Since φ is uniformly continuous on $\overline{\Omega}\times\mathbb{S}^{N-1}$, given $\varepsilon>0$ it is possible to find $\delta>0$ in such a way that

$$|\varphi(x,\xi)-\varphi(y,\xi)| < \varepsilon \qquad \text{for every } \xi\in\mathbb{S}^{N-1}\,,\ x,y\in D_i^\delta\,,\ i\in\mathbb{Z}^N\ .$$

As the function Λ_φ is continuous, there exist points $x_i^\delta\in D_i^\delta$ such that

$$\int_{D_i^\delta} \Lambda_\varphi(x)\mathrm{d}x = \int_{D_i^\delta} \sup_{|\xi|=1} \varphi(x_i^\delta,\xi)\mathrm{d}x\ .$$

Therefore,

$$\int_{D_i^\delta} \Lambda_\varphi(x)\mathrm{d}x = \sup_{|\xi|=1} \int_{D_i^\delta} \varphi(x_i^\delta, \xi)\mathrm{d}x \le \sup_{|\xi|=1} \int_{D_i^\delta} \varphi(x,\xi)\,\mathrm{d}x + \sup_{|\xi|=1} \int_{D_i^\delta} \left[\varphi(x_i^\delta,\xi) - \varphi(x,\xi)\right]\mathrm{d}x$$

and the statement of the lemma follows. □

Lemma 2.8. *Let $\varphi \in \mathcal{M}$ such that $\varphi(x,\cdot)$ is convex for every x. Then for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \overline{\Omega}$ such that $\mathcal{L}^N\left(\overline{\Omega} \setminus K_\varepsilon\right) < \varepsilon$ and φ is continuous on $K_\varepsilon \times \mathbb{R}^N$.*

Proof. Let us take a sequence of vectors $(\xi_k)_{k\in\mathbb{N}}$ dense in $\mathbb{S}^{N-1}$. For every fixed k, Lusin's Theorem ensures the existence of a compact set $C_k \subset \overline{\Omega}$ such that $\varphi(\cdot,\xi_k)$ is continuous on C_k, and $\mathcal{L}^N\left(\overline{\Omega} \setminus C_k\right) < \varepsilon 2^{-k}$. Define

$$K_\varepsilon := \bigcap_{k\in\mathbb{N}} C_k.$$

Obviously, $\mathcal{L}^N\left(\overline{\Omega} \setminus K_\varepsilon\right) < \varepsilon$ and $\varphi(\cdot,\xi_k)$ is continuous on K_ε for all k. We claim that φ is actually continuous on $K_\varepsilon \times \mathbb{S}^{N-1}$ (and hence on $K_\varepsilon \times \mathbb{R}^N$ by the homogeneity property of φ). In fact, since by convexity φ enjoys (2.8), for $\xi \in \mathbb{S}^{N-1}$ and $x, y \in K_\varepsilon$ we get

$$|\varphi(x,\xi) - \varphi(y,\xi)| \le 2\beta\,|\xi - \xi_k| + |\varphi(x,\xi_k) - \varphi(y,\xi_k)|\,,$$

and we conclude by the density of $(\xi_k)_{k\in\mathbb{N}}$ and the continuity of $\varphi(\cdot,\xi_k)$ on K_ε. □

Proposition 2.9. *Let $\varphi \in \mathcal{M}$. Assume that, for a sequence $(\mu_n)_{n\in\mathbb{N}}$ of nonnegative Borel measures on Ω, the following property holds:*

$$\sup_{|\xi|=1} \int_\omega \varphi(x,\xi)\,\mathrm{d}x \le \liminf_{n\to\infty} \mu_n(\omega) \qquad \textit{for every Borel set } \omega \subset \Omega\ . \tag{2.16}$$

Then

$$\int_\Omega \Lambda_\varphi(x)\mathrm{d}x \le \liminf_{n\to\infty} \mu_n(\Omega)\,. \tag{2.17}$$

Proof. We proceed in three steps.

Step 1. We prove the result for φ continuous. Fix $\varepsilon > 0$ and take $\delta > 0$ given by Lemma 2.7. We have:

$$\int_\Omega \Lambda_\varphi(x)\mathrm{d}x = \sum_{i\in\mathbb{Z}^N} \int_{D_i^\delta} \Lambda_\varphi(x)\mathrm{d}x \le \sum_{i\in\mathbb{Z}^N} \sup_{|\xi|=1} \int_{D_i^\delta} \left[\varphi(x,\xi) + \varepsilon\right]\mathrm{d}x\ .$$

By assumption

$$\sum_{i\in\mathbb{Z}^N}\sup_{|\xi|=1}\int_{D_i^\delta}\left[\varphi\left(x,\xi\right)+\varepsilon\right]\mathrm{d}x\leq\sum_{i\in\mathbb{Z}^N}\left[\liminf_{n\to\infty}\mu_n(D_i^\delta)+\int_{D_i^\delta}\varepsilon\mathrm{d}x\right]\leq\liminf_{n\to\infty}\mu_n\left(\Omega\right)+\varepsilon\mathcal{L}^N\left(\Omega\right)$$

and since ε is arbitrary (2.17) follows.

Step 2. We show that (2.17) holds when $\varphi(x,\cdot)$ is convex and $\varphi\left(\cdot,\xi\right)$ is lower semicontinuous for every fixed x and ξ. Indeed in this case, thanks to Lemma 2.2.3 of [21], and since $\varphi(x,\cdot)$ is positively one-homogeneous, there exists a sequence of continuous functions $a_j:\mathbb{R}^N\to\mathbb{R}^N$ such that, for $(x,\xi)\in\Omega\times\mathbb{R}^N$,

$$\varphi\left(x,\xi\right)=\sup_{j\in\mathbb{N}}\left\{a_j\left(x\right)\cdot\xi\right\}$$

Then, defining

$$\varphi_k\left(x,\xi\right):=\sup_{j\leq k}\left\{a_j\left(x\right)\cdot\xi\right\}\vee\alpha\left|\xi\right| \ ,$$

we obtain a sequence of continuous elements of $\mathcal{M}$ which converges increasingly to φ. Each of the metrics φ_k satisfies the property (2.16) because, for every Borel set $\omega\subset\Omega$ and every $\xi\in\mathbb{S}^{N-1}$, we have

$$\int_\omega\varphi_k\left(x,\xi\right)\mathrm{d}x\leq\int_\omega\varphi\left(x,\xi\right)\mathrm{d}x\leq\liminf_{n\to\infty}\mu_n\left(\omega\right).$$

Therefore, by Step 1, we get

$$\sup_{k\in\mathbb{N}}\int_\Omega\Lambda_{\varphi_k}(x)\mathrm{d}x\leq\liminf_{n\to\infty}\mu_n\left(\Omega\right).\tag{2.18}$$

Now, let us take a dense set $(\xi_h)_{h\in\mathbb{N}}$ in $\mathbb{S}^{N-1}$. We have:

$$\int_\Omega\Lambda_\varphi(x)\mathrm{d}x=\int_\Omega\sup_{h\in\mathbb{N}}\varphi\left(x,\xi_h\right)\mathrm{d}x=\int_\Omega\sup_{h\in\mathbb{N}}\sup_{k\in\mathbb{N}}\varphi_k\left(x,\xi_h\right)\mathrm{d}x=\int_\Omega\sup_{k\in\mathbb{N}}\sup_{h\in\mathbb{N}}\varphi_k\left(x,\xi_h\right)\mathrm{d}x.$$

By the Monotone Convergence Theorem and (2.18), we finally obtain

$$\int_\Omega\sup_{k\in\mathbb{N}}\sup_{h\in\mathbb{N}}\varphi_k\left(x,\xi_h\right)\mathrm{d}x=\sup_{k\in\mathbb{N}}\int_\Omega\sup_{h\in\mathbb{N}}\varphi_k\left(x,\xi_h\right)\mathrm{d}x=\sup_{k\in\mathbb{N}}\int_\Omega\Lambda_{\varphi_k}(x)\mathrm{d}x\leq\liminf_{n\to\infty}\mu_n\left(\Omega\right) \ .$$

Step 3: We now make no additional regularity assumptions on φ. First observe that we may assume that $\varphi(x,\cdot)$ is convex for all $x\in\overline{\Omega}$. Indeed, if this is not the case, take a negligible Borel set $E \subset \overline{\Omega}$ which contains the points x where $\varphi(x,\cdot)$ is not convex. Then we can replace $\varphi(x,\xi)$ with $\varphi(x,\xi)\chi_{\overline{\Omega}\setminus E}(x) + C\beta\,\chi_E(x)|\xi|$ without affecting the validity of (2.16).

Hence φ suits the assumptions of Lemma 2.8: we deduce that, for every $\varepsilon > 0$, there exists a compact set $K_\varepsilon \subset \overline{\Omega}$ such that $\mathcal{L}^N\left(\overline{\Omega} \setminus K_\varepsilon\right) < \varepsilon$ and $\varphi|_{K_\varepsilon\times\mathbb{R}^N}$ is continuous. We define

$$\varphi^\varepsilon(x,\xi) := \begin{cases} \varphi(x,\xi) & \text{if } x \in K_\varepsilon, \\ \beta\,|\xi| & \text{otherwise.} \end{cases}$$

Notice that, as K_ε is closed, φ^ε is lower semicontinuous and

$$\varphi^\varepsilon(x,\xi) \geq \varphi(x,\xi) \qquad \text{for every } x \in \overline{\Omega} \text{ and every } \xi \in \mathbb{R}^N.$$

Moreover, for every Borel set $\omega \subset \Omega$,

$$\begin{aligned} \sup_{|\xi|=1} \int_\omega \varphi^\varepsilon(x,\xi)\,\mathrm{d}x &\leq \sup_{|\xi|=1} \int_\omega \varphi(x,\xi)\,\mathrm{d}x + \beta\mathcal{L}^N(\omega \setminus K_\varepsilon) \\ &\leq \liminf_{n\to\infty} \mu_n(\omega) + \beta\mathcal{L}^N(\omega \setminus K_\varepsilon) \end{aligned}$$

Applying Step 2 with $\tilde{\mu}_n := \mu_n + \beta\chi_{\Omega\setminus K_\varepsilon}\mathrm{d}x$, we get

$$\int_\Omega \Lambda_{\varphi^\varepsilon}(x)\mathrm{d}x \leq \liminf_{n\to\infty} \tilde{\mu}_n(\Omega) \leq \liminf_{n\to\infty} \mu_n(\Omega) + \beta\varepsilon.$$

Since

$$\int_\Omega \Lambda_\varphi(x)\mathrm{d}x \leq \int_\Omega \Lambda_{\varphi^\varepsilon}(x)\mathrm{d}x$$

and ε is arbitrary, we have

$$\int_\Omega \Lambda_\varphi(x)\mathrm{d}x \leq \liminf_{n\to\infty} \mu_n(\Omega),$$

as claimed. $\square$

We are finally in position to give the

Proof of Theorem 2.4. Let a_n, φ_n and φ be as in the statement. Then, by Proposition 2.6, the limit metric φ satisfies condition (2.16) if we take as μ_n the Lebesgue measure on Ω with density a_n, for each $n \in \mathbb{N}$, namely

$$\mu_n(\omega) = \int_\omega a_n(x)\mathrm{d}x \qquad \text{for every Borel set } \omega \subseteq \Omega\,.$$

We can therefore apply Proposition 2.9 to infer

$$\int_\Omega \Lambda_\varphi(x)\mathrm{d}x \leq \liminf_{n\to\infty} \mu_n(\Omega) = \liminf_{n\to\infty} \int_\Omega a_n(x)\mathrm{d}x \leq m \ .$$

Chapter 3

Relaxation of integral constraints on Riemannian metrics

3.1 Introduction

The existence of a solution to the optimization problems studied in the previous chapter (*i.e.* Theorems 2.3 and 2.5) relies on the fact that all distances arising as limits of sequences belonging to $\mathcal{D}(\mathcal{A})$ must satisfy the relaxed integral constraint (2.11). The employed arguments seem sufficiently general to extend to a wider class of integral constraints.

In the current chapter we will be concerned with the study of an integral functional of the form

$$\mathcal{F}(d_a) := \int_\Omega F(x, a(x))\,\mathrm{d}x, \tag{3.1}$$

defined on the family $\mathcal{D}(\mathcal{I})$ of distances d_a induced by isotropic, continuous Riemannian metrics through the formula

$$d_a(x,y) := \inf\Big\{\mathbb{L}_a(\gamma) \ :\ \gamma \in \mathrm{Lip}([0,1];\overline{\Omega}),\ \gamma(0)=x,\ \gamma(1)=y\ \Big\} \tag{3.2}$$

for every $(x,y) \in \overline{\Omega}\times\overline{\Omega}$, where the length functional $\mathbb{L}_a$ is defined as follows

$$\mathbb{L}_a(\gamma) := \int_0^1 a(\gamma(t))|\dot{\gamma}(t)|\,\mathrm{d}t. \tag{3.3}$$

Here a ranges in the family $\mathcal{I}$ of positive continuous functions from $\overline{\Omega}$ to the interval $[\alpha,\beta]$, where α and β are fixed positive constants. We point out that the map that associates

to each metric of $\mathcal{I}$ an element in $\mathcal{D}(\mathcal{I})$ through (3.2) is injective, that is two continuous, isotropic Riemannian metrics which induce the same distance through (3.2) actually coincide (*cf.* Remark 3.2). In particular, that shows that the functional (3.1) is well defined.
The set $\mathcal{D}(\mathcal{I})$ can be seen as a subspace of the space of Finsler distances $\mathcal{D}$ defined by (1.22), endowed with the metrizable topology given by the uniform convergence on compact subset of $\overline{\Omega} \times \overline{\Omega}$. We recall that this is equivalent to the Γ-convergence of the associated metric length functionals in view of Theorem 1.21.

If $\mathcal{C}$ is a lower semicontinuous cost functional defined on $\mathcal{D}$, an optimization problem analogous to the ones considered in Chapter 2 may be defined as follows:

$$\min \{\mathcal{C}(d_a) \,:\, a \in \mathcal{I},\ \mathcal{F}(d_a) \leq m\} \tag{3.4}$$

where m is a suitable constant. For reasons already discussed in Chapter 2, the main problem arising in the study of problem (3.4) is that $\mathcal{D}(\mathcal{I})$ is a non-closed subspace of $\mathcal{D}$. In particular, the existence of a solution is guaranteed only in the the following class of distances

$$\left\{ d \in \overline{\mathcal{D}(\mathcal{I})} \,:\, \overline{\mathcal{F}}(d) \leq m \right\},$$

where $\overline{\mathcal{F}}$ is the *relaxed functional* of (3.1), namely

$$\overline{\mathcal{F}}(d) := \inf\{\liminf_n \mathcal{F}(d_n) \,:\, d_n \xrightarrow{\mathcal{D}} d,\ (d_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathcal{I})\}, \tag{3.5}$$

defined for every d belonging to the closure of $\mathcal{D}(\mathcal{I})$.

In Section 3.3 we will prove that $\mathcal{D}(\mathcal{I})$ is dense in $\mathcal{D}_S$, where the latter denotes the family of symmetric distances belonging to $\mathcal{D}$. Moreover, under suitable monotonicity and convexity assumptions on the integrand F (which include, in particular, the case $F(x,s) = s$ considered in Chapter 2), we will show that the relaxed functional (3.5), which is consequently defined on all $\mathcal{D}_S$, has the following integral representation:

$$\overline{\mathcal{F}}(d) = \int_\Omega F(x, \Lambda_d(x))\,\mathrm{d}x, \tag{3.6}$$

where $\Lambda_d(x) := \sup_{|\xi|=1} \varphi_d(x,\xi)$ and φ_d is the Finslerian metric associated to d by derivation (*cf.* Definition 1.24). In view of Proposition 3.1, that also implies that the functional $\overline{\mathcal{F}}$ coincides with $\mathcal{F}$ on $\mathcal{D}(\mathcal{I})$. In the specific situation considered in Chapter 2, such results yield in particular the following characterization:

$$\overline{\mathcal{D}(\mathcal{A})} = \left\{ d \in \mathcal{D}_S \,:\, \int_\Omega \sup_{|\xi|=1} \varphi_d(x,\xi)\,\mathrm{d}x \leq m \right\}.$$

We conclude this introduction with some considerations. Definition (3.5) clearly implies that $\overline{\mathcal{F}}$ is lower semicontinuous. Moreover, it can be shown that it is the greatest among all lower semicontinuous ones which are bounded from above by $\mathcal{F}$ on $\mathcal{D}(\mathcal{I})$ (see [21] for various results on this topic). Therefore, in order to prove our relaxation result, we have to show first that the functional (3.6) is lower semicontinuous. The proof of this issue is just a technical adaptation of the arguments described in Chapter 2. To prove its maximality, we will approximate each $d \in \mathcal{D}$ by means of a sequence of suitably chosen distances $d_n \in \mathcal{D}(\mathcal{I})$, namely such that

$$\limsup_n \int_\Omega F(x, \Lambda_{d_n}(x))\,\mathrm{d}x \leq \int_\Omega F(x, \Lambda_d(x))\,\mathrm{d}x.$$

Then, by a standard argument (see Section 3.3), the maximality of (3.6) follows.

Indeed, finding such an approximating sequence is a delicate matter. In fact, one should define the Riemannian metrics a_n in such a way to have Γ-convergence of the relative length functionals $\mathbb{L}_{a_n}$ to $\mathbb{L}_{\varphi_d}$ (*cf.* Theorem 1.21 and Remark 1.19) and this problem is non-trivial even in the simplified situation of an isotropic Riemannian metric φ_d, *i.e.* such that $\varphi_d = b(x)|\xi|$ where b is a Borel function from $\overline{\Omega}$ to $[\alpha, \beta]$. It is clear, in fact, that this convergence strongly relies upon the convergence of the approximating metrics on curves, which is much finer than convergence almost everywhere in $\overline{\Omega}$. Moreover, we do not have many informations on the properties of the metric φ_d; we only know it is Borel-measurable and such that the associated length functional $\mathbb{L}_{\varphi_d}$ is lower semicontinuous on $\mathrm{Lip}(I, \overline{\Omega})$ with respect to the uniform convergence of curves (see Chapter 1). In the general case of a non-isotropic metric the situation is obviously more delicate.

The key idea of our proof is that it is sufficient to control the convergence of the approximating distances only on a fixed countable and dense subset of $\overline{\Omega} \times \overline{\Omega}$ (in view of Lemma 1.22). Therefore, when we define the Riemannian metrics, we have only to control the value of the associated distance d_n on the first n points of this set. This will be done by approximating the Finsler metric φ_d along geodesics (*cf.* Theorem 3.11). With regard to that, let us notice that Theorem 3.11 is not just a technical result in order to prove our main theorems, but has an interesting consequence it is worth underline: every Finsler distance $d \in \mathcal{D}_S$ can indeed be seen as generated by a suitable Borel-measurable, isotropic Riemannian metric $a : \overline{\Omega} \to [\alpha, \beta]$ according to definition (3.2). In other words, by allowing the isotropic metric a to vary in a somehow "uncontrolled" way, one can recover all the possible anisotropies of φ_d.

The problem of the density of (smooth) isotropic, Riemannian metrics in Finsler ones has already been studied. The question was raised in [23], and partially answered in [16] in the case $\Omega := \mathbb{R}^N$, where Finsler metrics were additionally assumed to be lower semicontinuous. Our proof makes use of Γ-convergence techniques as well, but the underlying idea is quite different; in particular, we remark that no extra regularity assumptions on the metrics are needed. Consequently, the class of distances $\mathcal{D}_S$ here considered is, *a priori*, larger; actually, it coincides with the family of geodesic distances d such that $\alpha d_\Omega \leq d \leq \beta d_\Omega$, in view of Theorem 1.13. The results proved here, then, completely settle the question mentioned above. Indeed, as pointed out in [16], once the density result for continuous and isotropic Riemannian metrics is established, the analogous result for smooth ones is easily recovered via a standard mollification argument (*cf.* Remark 3.8).

3.2 Notation and preliminary results

Throughout this chapter α and β denote two fixed positive constants with $\beta \geq \alpha$, and Ω an open connected subset of $\mathbb{R}^N$ which enjoys condition (Ω) in Section 1.3.2.
In the sequel, we will write argmin$(\mathcal{P})$ to denote the set of minimizers of the problem $(\mathcal{P})$. We will denote by $\mathcal{M}$ the class (1.21) of Finsler metrics on $\overline{\Omega}$, by $\mathcal{D}$ the family (1.22) of Finsler distances on $\overline{\Omega} \times \overline{\Omega}$ and by $\mathcal{D}_S$ the family of symmetric distances belonging to $\mathcal{D}$, namely

$$\mathcal{D}_S := \{d \in \mathcal{D} \,:\, d(x,y) = d(y,x) \text{ for all } x, y \in \overline{\Omega}\,\}.$$

Obviously, $\mathcal{D}_S$ is a closed subspace of $\mathcal{D}$. The function $F : \Omega \times [\alpha, \beta] \to \mathbb{R}_+$ appearing in the integrand of (3.1) is assumed to be continuous and to fulfill the following conditions:

$$\begin{aligned} &\text{(i) the function } F(x,\cdot) \text{ is convex and nondecreasing for every } x \in \Omega; \\ &\text{(ii) } \textstyle\int_\Omega F(x,\beta)\,\mathrm{d}x < +\infty. \end{aligned} \tag{3.7}$$

Given a distance $d \in \mathcal{D}$, we define for every $x \in \Omega$

$$\Lambda_d(x) := \sup_{|\xi|=1} \varphi_d(x,\xi),$$

which represents, with analogy to the Riemannian case $\varphi_d(x,\xi) = (B(x)\xi\cdot\xi)^{\frac{1}{2}}$ with $B(x)$ a symmetric and positive definite matrix, the largest "eigenvalue" of $\varphi_d(x,\cdot)$ at the point x. We notice that $\Lambda_d(x)$ is a Borel-measurable function. Indeed, if $(\xi_n)_{n\in\mathbb{N}}$ is a dense sequence in $\mathbb{S}^{N-1}$, by Theorem 1.11-(ii) we have that $\Lambda_d(x)$ coincides with the function $\sup_n \varphi_d(x,\xi_n)$, which is Borel-measurable since it is the supremum of Borel-measurable functions.

In the remainder of this section we state and prove some results which will be needed in the sequel. The first one is a trivial consequence of Proposition 1.15.

Proposition 3.1. *Let $\varphi \in \mathcal{M}$ and $d := d_\varphi$. If $\varphi(x,\xi) := a(x)|\xi|$ with $a : \overline{\Omega} \to [\alpha,\beta]$ lower semicontinuous, then $\Lambda_d(x) = a(x)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$.*

Remark 3.2. If a and b are two continuous isotropic metrics which give rise to the same distance function d through (3.2), then $a(x) = b(x)$ for every $x \in \overline{\Omega}$. Indeed, Proposition 3.1 implies the equality to hold almost everywhere, hence everywhere by the continuity of the metrics. In particular, this shows that the functional (3.1) is well defined.

Let us now prove two lemmas.

Lemma 3.3. *Let $\{(x_i,y_i) : i \in \mathbb{N}\}$ be a countable collection of points in $\overline{\Omega}\times\overline{\Omega}$. Then it is possible to find a family of curves $\{\gamma_i : \gamma_i \in \mathrm{Lip}_{x_i,y_i},\, i \in \mathbb{N}\}$ such that*

(i) $\mathrm{L}_d(\gamma_i) = d(x_i,y_i)$ and γ_i is injective for every $i \in \mathbb{N}$;

(ii) $\gamma_i(I)\cap\gamma_j(I)$ is a (possibly void) disjoint finite union of closed arcs for every $i,j \in \mathbb{N}$.

Proof. First we remark that for every $i \in \mathbb{N}$ the set

$$\mathcal{R}_i := \mathrm{argmin}\{\mathrm{L}_d(\gamma) : \gamma \in \mathrm{Lip}_{x_i,y_i}\}.$$

is non-void by Proposition 1.20. Moreover, any curve in $\mathcal{R}_i$ is injective by minimality, hence it satisfies point (i) of the claim. In order to prove the Lemma, it will be enough to show that the following holds for every $n \in \mathbb{N}$:

Claim: Let $\{\gamma_i : \gamma_i \in \mathrm{Lip}_{x_i,y_i},\, i \le n-1\}$ be a collection of curves satisfying claim (i)-(ii) above. Then it is possible to find $\gamma_n \in \mathrm{Lip}_{x_n,y_n}$ such that the curves $\{\gamma_i : i \le n\}$ still satisfy claim (i)-(ii).

For $n = 1$ the claim is satisfied by choosing a γ_1 which belongs to $\mathcal{R}_1$. Let then $n > 1$ and choose a curve σ in $\mathcal{R}_n$. For every $j \leq n-1$, let us set $t_j := \min\{t \in I \,:\, \sigma(t) \in \gamma_j(I)\}$, $T_j := \max\{t \in I \,:\, \sigma(t) \in \gamma_j(I)\}$ (we agree that $t_j = T_j = +\infty$ if such minima do not exist), and $J := \{j \leq n-1 \,:\, t_j < T_j < +\infty\}$. If J is void, the claim is proved by setting $\gamma_n := \sigma$. Otherwise, we can suppose, up to reordering the curves γ_j, that $t_1 = \min\{t_j \,:\, j \in J\}$. Then we define $\tau_1 \in \mathrm{Lip}_{x_n,y_n}$ to be the curve obtained by moving from $\sigma(0)$ to $\sigma(t_1)$ along σ, from $\sigma(t_1)$ to $\sigma(T_1)$ along γ_1 and from $\sigma(T_1)$ to $\sigma(1)$ along σ again. Remark that, by minimality, γ_1 is a path which connects $\sigma(t_1)$ to $\sigma(T_1)$ in the shortest way and so we have not increased the length, *i.e.* $\mathrm{L}_d(\tau_1) \leq \mathrm{L}_d(\sigma)$, hence $\tau_1 \in \mathcal{R}_n$. Moreover $\tau_1([0,T_1]) \cap \gamma_i(I)$ is a disjoint finite union of closed arcs for every $1 \leq i \leq n-1$. Then we set $\sigma := \tau_1 \mid_{[T_1,1]}$ and we repeat the above argument to obtain a curve $\tau_2 : [T_1, 1] \to \overline{\Omega}$. By iterating this procedure, we eventually find a finite number of curves $\{\tau_h \,:\, 1 \leq h \leq M\}$ for some $M < n$. Then we define

$$\gamma_n(t) := \begin{cases} \tau_1(t) & \text{if } t \in [0, T_1] \\ \tau_h(t) & \text{if } t \in [T_{h-1}, T_h] \ \text{ and } 1 < h < M \\ \tau_M(t) & \text{if } t \in [T_{M-1}, 1]. \end{cases}$$

By what previously observed, we have that γ_n still belongs to $\mathcal{R}_n$ and is therefore injective by minimality. Moreover, it is such that $\gamma_n(I) \cap \gamma_i(I)$ is a disjoint finite union of closed arcs for every $i \leq n-1$ by construction. The claim is thus proved. $\square$

Lemma 3.4. *Let γ be an injective Lipschitz curve, $\Gamma := \gamma((0,1)) \subset \overline{\Omega}$ and $a : \overline{\Omega} \to [\alpha, \beta]$ a Borel function. Then there exists a sequence of continuous functions $\sigma_k : \Gamma \to [\alpha, \beta]$ such that $\sigma_k(x)$ converge to $a(x)$ for $\mathcal{H}^1$-a.e. $x \in \Gamma$. Moreover, for every $\varepsilon > 0$ there exists a Borel subset $B_\varepsilon \subset \Gamma$ such that $\mathcal{H}^1(\Gamma \setminus B_\varepsilon) < \varepsilon$ and σ_k converge uniformly to a on B_ε.*

Proof. The function $a \circ \gamma : (0,1) \to [\alpha, \beta]$ is Borel-measurable, therefore there exists a sequence $(f_k)_{k \in \mathbb{N}}$ of continuous functions $f_k : (0,1) \to [\alpha, \beta]$ such that $f_k(t)$ converges to $a \circ \gamma(t)$ for a.e. $t \in (0,1)$. Moreover, by Severini-Egoroff's theorem [52, Section 1.2, Theorem 3], for every $\widetilde{\varepsilon} > 0$ there exist an infinitesimal sequence $(\delta_k)_{k \in \mathbb{N}}$ and a Borel set $E_{\widetilde{\varepsilon}}$ such that $\mathcal{H}^1((0,1) \setminus E_{\widetilde{\varepsilon}}) < \widetilde{\varepsilon}$ and $|f_k(t) - a \circ \gamma(t)| < \delta_k$ for every $t \in E_{\widetilde{\varepsilon}}$. The claim then follows by choosing $\widetilde{\varepsilon} := \varepsilon / \mathrm{Lip}(\gamma)$ and setting $\sigma_k(x) := f_k(\gamma^{-1}(x))$, $B_{\widetilde{\varepsilon}} := \gamma(E_{\widetilde{\varepsilon}})$. $\square$

3.3 Main results

Our main result is stated as follows.

Theorem 3.5. *Let $\mathcal{F}$ be the functional defined on $\mathcal{D}(\mathcal{I})$ by (3.1), where $F : \Omega \times [\alpha, \beta] \to \mathbb{R}_+$ is continuous and satisfies conditions (3.7). Then its relaxed functional (3.5) has the following integral representation:*

$$\overline{\mathcal{F}}(d) = \int_\Omega F(x, \Lambda_d(x))\, dx \tag{3.8}$$

for all $d \in \mathcal{D}_S$. In particular, $\overline{\mathcal{F}}(d) = \mathcal{F}(d)$ for all $d \in \mathcal{D}(\mathcal{I})$.

The proof of the previous theorem is based on the following two results which we state separately.

Theorem 3.6. *If $d_n \xrightarrow{\mathcal{D}} d$, then $\liminf\limits_{n\to+\infty} \int_\Omega F(x, \Lambda_{d_n}(x))\, dx \geq \int_\Omega F(x, \Lambda_d(x))\, dx$.*

Theorem 3.7. *The family $\mathcal{D}(\mathcal{I})$ of distances induced by continuous and isotropic Riemannian metrics is dense in $\mathcal{D}_S$. Moreover, for every $d \in \mathcal{D}_S$ we can choose a sequence $(d_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathcal{I})$ such that $d_n \xrightarrow{\mathcal{D}} d$ and*

$$\limsup_{n\to+\infty} \int_\Omega F(x, \Lambda_{d_n}(x))\, dx \leq \int_\Omega F(x, \Lambda_d(x))\, dx. \tag{3.9}$$

Remark 3.8. The class of distances induced by smooth isotropic Riemannian metrics is dense in $\mathcal{D}(\mathcal{I})$, hence in $\mathcal{D}_S$ by Theorem 3.7. In fact, let us take a distance d in $\mathcal{D}(\mathcal{I})$. Then $d = d_a$ for some continuous metric $a : \overline{\Omega} \to [\alpha, \beta]$. By Tietze's Lemma, we may extend a continuously to the whole $\mathbb{R}^N$ in such a way that $\alpha \leq a(x) \leq \beta$ for all $x \in \mathbb{R}^N$. Then, by taking a sequence of convolution kernels ρ_n, we define the sequence of smooth isotropic metrics $a_n : \overline{\Omega} \to [\alpha, \beta]$ by mollification, *i.e.* $a_n(x) := \rho_n * a(x)$, and we call d_n the induced distances. Since the functions a_n converge to a uniformly on compact subset of $\overline{\Omega} \times \overline{\Omega}$, it can be easily shown that the length functionals $\mathbb{L}_{a_n}$ Γ-converge to $\mathbb{L}_a$ on $\mathrm{Lip}(I, \overline{\Omega})$ with respect to the uniform convergence of curves. Then, by Remark 1.19 and Theorem 1.21, we have that $d_n \xrightarrow{\mathcal{D}} d$, as claimed.

Once Theorem 3.6 and Theorem 3.7 will be proven, Theorem 3.5 will trivially follow. In fact, Theorem 3.6 gives that the functional (3.8) is lower semicontinuous with respect to the uniform convergence of distances, and Theorem 3.7 implies it is the greatest lower semicontinuous functional defined on $\mathcal{D}_S$ which is bounded from above by $\mathcal{F}$ on $\mathcal{D}(\mathcal{I})$. Indeed, let $\mathcal{G}$ be another candidate and let $d \in \mathcal{D}_S$. Choose a sequence $(d_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathcal{I})$ as in the statement of Theorem 3.7. We have

$$\mathcal{G}(d) \leq \liminf_{n\to+\infty} \mathcal{G}(d_n) \leq \liminf_{n\to+\infty} \mathcal{F}(d_n) \leq \limsup_{n\to+\infty} \mathcal{F}(d_n) \leq \int_\Omega F(x, \Lambda_d(x))\,\mathrm{d}x,$$

hence the claim. The last statement in the claim of Theorem 3.5 is an immediate consequence of Proposition 3.1.

Let us then start by proving Theorem 3.6.

Proof of Theorem 3.6. The proof will be just sketched, since it is essentially an adaptation of the arguments described in Chapter 2, where the case $F(x, s) := s$ was considered.

Let d_n, d be as in the statement of the theorem. We recall that the function $F : \Omega \times [\alpha, \beta] \to \mathbb{R}_+$ is continuous and fulfills conditions (3.7). The first result we state is the following:

- *Claim:* for every bounded Borel set $\omega \subset\subset \Omega$ and every $\xi \in \mathbb{S}^{N-1}$, we have

$$\int_\omega F\left(x, \varphi_d(x,\xi)\right) \mathrm{d}x \leq \liminf_{n\to\infty} \int_\omega F\left(x, \varphi_{d_n}(x,\xi)\right) \mathrm{d}x\ . \tag{3.10}$$

The previous statement is analogous to Proposition 2.6 and can be proved similarly. Inequality (3.10) immediately gives the following:

$$\sup_{|\xi|=1} \int_\omega F\left(x, \varphi_d(x,\xi)\right) \mathrm{d}x \leq \liminf_{n\to\infty} \int_\omega F\left(x, \Lambda_{d_n}(x)\right) \mathrm{d}x, \tag{3.11}$$

where we have also used the monotonicity assumption (3.7)-(i) made on F. In order to conclude, it will be therefore enough to prove the following:

- *Claim:* Let $\varphi \in \mathcal{M}$. Assume that, for a sequence $(\mu_n)_{n\in\mathbb{N}}$ of non-negative Borel measures on Ω, the following property holds:

$$\sup_{|\xi|=1} \int_\omega F\left(x, \varphi(x,\xi)\right) \mathrm{d}x \leq \liminf_{n\to\infty} \mu_n(\omega) \qquad \text{for every Borel set } \omega \subset\subset \Omega\ .$$

  Then

$$\int_\Omega F\big(x, \sup_{|\xi|=1} \varphi(x,\xi)\big)\mathrm{d}x \leq \liminf_{n\to\infty} \mu_n(\Omega)\,. \tag{3.12}$$

Indeed, in view of (3.11), the claim of the theorem would follow by applying the previous statement with $\varphi := \varphi_d$ and $\mu_n(\omega) := \int_\omega F\left(x, \Lambda_{d_n}(x)\right) \mathrm{d}x$.

Let us prove (3.12). First, we reduce to consider the case of a bounded domain Ω. Indeed, if this is not the case, we take a sequence $(\Omega_l)_{l\in\mathbb{N}}$ of bounded and connected open sets well contained in Ω such that $\overline{\Omega}_l \subset \Omega_{l+1}$ and $\Omega = \bigcup_{l\in\mathbb{N}} \Omega_l$, and we notice that it is enough to prove that (3.12) holds for $\Omega := \Omega_l$ for each $l \in \mathbb{N}$.

Let us then assume that Ω is bounded and set $\Lambda_\varphi(x) := \sup_{|\xi|=1} \varphi(x,\xi)$ for all $x \in \Omega$. Following the proof of Proposition 2.9, we consider three cases:

(1) let φ be continuous. Then (3.12) easily comes by arguing as in the proof of Proposition 2.9 and by using, in place of Lemma 2.7, the following

Lemma 3.9. *Let $\varphi \in \mathcal{M}$ be a continuous Finsler metric. Then, for every bounded open set $A \subset\subset \Omega$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\int_{D_i^\delta \cap A} F\left(x, \Lambda_\varphi(x)\right) \mathrm{d}x \le \sup_{|\xi|=1} \int_{D_i^\delta \cap A} \left[F\left(x, \varphi(x,\xi)\right) + \varepsilon\right] dx \qquad \text{for all } i \in \mathbb{Z}^N,$$

where we have set $D_i^\delta := \Omega \cap \delta\big(2i + [-1,1)^N\big)$.

(2) Let φ be lower semicontinuous and $\varphi(x,\cdot)$ convex for every $x \in \Omega$. By arguing as in the proof of Proposition 2.9, we can find an increasing sequence of continuous Finsler metrics $(\varphi_k)_{k\in\mathbb{N}} \subset \mathcal{M}$ such that $\varphi(x,\xi) = \sup_{k\in\mathbb{N}} \varphi_k(x,\xi)$ for all $(x,\xi) \in \Omega \times \mathbb{R}^N$, and we conclude analogously in view of the previous step. Hence, (3.12) holds in this case too.

(3) We now make no additional regularity hypotheses on φ: we only assume it belongs to $\mathcal{M}$. First, notice that it is not restrictive to assume that $\varphi(x,\cdot)$ is convex for every $x \in \overline{\Omega}$: it is actually sufficient to redefine the metric φ by setting $\varphi(x,\xi) := \beta|\xi|$ on a negligible Borel subset of $\overline{\Omega}$ which contains all the points where the metric is not convex. We can then apply Lemma 2.8: for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \overline{\Omega}$ such that $\mathcal{L}^N(\overline{\Omega} \setminus K_\varepsilon) < \varepsilon$ and $\varphi|_{K_\varepsilon \times \mathbb{R}^N}$ is continuous. We define

$$\varphi^\varepsilon(x,\xi) := \begin{cases} \varphi(x,\xi) & \text{if } x \in K_\varepsilon, \\ \beta\,|\xi| & \text{otherwise.} \end{cases}$$

Notice that φ^ε is lower semicontinuous, so we can apply the previous step with μ_n replaced by $\widetilde{\mu}_n := \mu_n + F(x,\beta)\chi_{\Omega\setminus K_\varepsilon}(x)\, d\mathcal{L}^N$ to get

$$\begin{aligned}\int_\Omega F\left(x, \Lambda_\varphi(x)\right) \mathrm{d}x &\leq \int_\Omega F\left(x, \Lambda_{\varphi^\varepsilon}(x)\right) \mathrm{d}x \leq \liminf_{n\to\infty} \widetilde{\mu}_n\left(\Omega\right) \\ &= \liminf_{n\to\infty} \mu_n\left(\Omega\right) + \int_{\Omega\setminus K_\varepsilon} F(x,\beta)\, \mathrm{d}x.\end{aligned}$$

As $|F(x,\beta)|$ is summable over Ω (condition (3.7)-(ii)), the integral appearing in the most right-hand side of the above inequality goes to 0 as $\varepsilon \to 0^+$. The claim hence follows letting ε go to 0. $\square$

Remark 3.10. The above proof still works for slightly more general functionals. Indeed, it is sufficient that there exists a sequence of continuous functions $F_k : \Omega \times [\alpha, \beta] \to \mathbb{R}_+$ which satisfy conditions (3.7) and such that $F(x,\xi) = \sup_k F_k(x,\xi)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$. In fact, one can apply the above argument to each F_k to get

$$\int_\Omega F_k\left(x, \Lambda_d(x)\right) \mathrm{d}x \leq \liminf_{n\to\infty} \int_\Omega F\left(x, \Lambda_{d_n}(x)\right) \mathrm{d}x,$$

and the claim immediately follows by taking the supremum over k of the left-hand side term and by the monotone convergence theorem.

We now come to the proof of Theorem 3.7: for any fixed $d \in \mathcal{D}_S$, we want to find a sequence $(d_n)_{n\in\mathbb{N}} \subset \mathcal{D}(\mathcal{I})$ which converges to d and enjoys (3.9). In view of Lemma 1.22, in the approximating procedure one only needs to control convergence of distances on a dense subset of $\overline{\Omega} \times \overline{\Omega}$. To this aim, we set $S := \mathbb{Q}^N \cap \overline{\Omega}$. Obviously $S \times S$ is countable and dense in $\overline{\Omega} \times \overline{\Omega}$, so we write $S \times S := \{(x_i, y_i) \,:\, i \in \mathbb{N}\}$.

As a preliminary step, we first approximate $d \in \mathcal{D}_S$ with distances induced by a sequence of Borel-measurable and isotropic Riemannian metrics.

Theorem 3.11. *Let $d \in \mathcal{D}_S$. Then there exists a decreasing sequence of Borel-measurable isotropic metrics $a_n : \overline{\Omega} \to [\alpha, \beta]$ such that*

(i) $d_{a_n}(x_i, y_i) = d(x_i, y_i)$ for each $i \leq n$;

(ii) $a_n(x) = \Lambda_d(x)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$.

In particular $d_{a_n} \xrightarrow{\mathcal{D}} d$. *Moreover, if we set* $a(x) := \inf_{n\in\mathbb{N}} a_n(x)$, *we have that* $d_a = d$ *on* $\overline{\Omega}\times\overline{\Omega}$, *that is every Finsler distance is induced by a Borel-measurable, isotropic Riemannian metric.*

Proof. For each $(x_i, y_i) \in S \times S$ let $\gamma_i \in \mathrm{Lip}_{x_i,y_i}$ be a path of minimal d-length, *i.e.* $\mathrm{L}_d(\gamma_i) = d(x_i, y_i)$. Such family of curves $\{\gamma_i \,:\, i \in \mathbb{N}\}$ can be chosen in such a way to satisfy conditions (i) and (ii) of Lemma 3.3 (this assumption is not really needed here, but will be important in the proof of Theorem 3.7). By condition (ii), each non-empty set $\gamma_i(I) \cap \gamma_j(I)$ is a disjoint finite union of closed arcs. Let us fix $n \in \mathbb{N}$ and denote by T_n the finite set given by the extreme points of such arcs for every $1 \le i \le j \le n$. Set $N_n := \cup_{i\le n}\gamma_i(I)$ and let Σ_n be a Borel $\mathcal{H}^1$-negligible subset of N_n which contains the points where the 1-rectifiable set N_n is not differentiable (this is possible by the Borel regularity of the measure $\mathcal{H}^1$ and by the differentiability properties of rectifiable sets [53, Theorem 1.6, Theorems 3.8 and 3.14]). Then we define the function $a_n : \overline{\Omega} \to [\alpha,\, \beta]$ as

$$a_n(x) := \begin{cases} \beta & \text{if } x \in \partial\Omega \setminus N_n \\ \Lambda_d(x) & \text{if } x \in \Omega \setminus N_n \\ \alpha & \text{if } x \in T_n \cup \Sigma_n \\ \varphi_d\left(\gamma_i(t), \dfrac{\dot\gamma_i(t)}{|\dot\gamma_i(t)|}\right) & \text{if } x = \gamma_i(t) \in N_n \setminus (T_n \cup \Sigma_n) \end{cases} \tag{3.13}$$

The function a_n is well defined and Borel-measurable, provided the set Σ_n is suitably chosen. Moreover it is clear that a_n satisfies claim (ii). Let d_{a_n} be the distance generated by such metric a_n for each $n \in \mathbb{N}$. We want to prove claim (i). Let us fix an $i \le n$. Then we have

$$d_{a_n}(x_i, y_i) \le \int_0^1 a_n(\gamma_i)|\dot\gamma_i|\,\mathrm{d}t = \int_0^1 \varphi_d(\gamma_i, \dot\gamma_i)\,\mathrm{d}t = d(x_i, y_i).$$

To prove the reverse inequality, choose a curve $\sigma \in \mathrm{Lip}_{x_i,y_i}$ and, for every $1 \le j \le n$, set $I_{j+1} := \{t \in I \setminus \cup_{h\le j} I_h \;:\; \sigma(t) \in \gamma_{j+1}(I)\}$, $I_1 := \{t \in I \;:\; \sigma(t) \in \gamma_1(I)\}$, and $I_0 := I \setminus \cup_{j\le n} I_j$. We remark that, for each $1 \le j \le n$, the vector $\dot\sigma(t)$ is parallel to $\dot\gamma_j(t)$ $\mathcal{L}^1$-a.e. on I_j and so $a_n(\sigma)|\dot\sigma| = \varphi_d(\sigma, \dot\sigma)$ $\mathcal{L}^1$-a.e. on I_j. Therefore we have

$$\begin{aligned} \mathbb{L}_{a_n}(\sigma) &= \int_0^1 a_n(\sigma)|\dot\sigma|\,\mathrm{d}t = \sum_{j=1}^n \int_{I_j} a_n(\sigma)|\dot\sigma|\,\mathrm{d}t + \int_{I_0} a_n(\sigma)|\dot\sigma|\,\mathrm{d}t \\ &\ge \sum_{j=1}^n \int_{I_j} \varphi_d(\sigma, \dot\sigma)\,\mathrm{d}t + \int_{I_0} \varphi_d(\sigma, \dot\sigma)\,\mathrm{d}t \ge d(x_i, y_i), \end{aligned}$$

where we have used the fact that $a_n(\sigma)|\dot\sigma| \geq \varphi_d(\sigma, \dot\sigma)$ on I_0. By passing to the infimum over all possible curves $\sigma \in \mathrm{Lip}_{x_i,y_i}$ we get the claim.
Notice that $N_n \subset N_{n+1}$, and we may as well suppose that $\Sigma_n \subset \Sigma_{n+1}$ (otherwise, replace Σ_{n+1} with $\Sigma_n \cup \Sigma_{n+1}$), therefore $(a_n)_{n\in\mathbb{N}}$ is a decreasing sequence of metrics. If we set $a(x) := \inf_{n\in\mathbb{N}} a_n(x)$, Proposition 1.24 gives $d_{a_n} \xrightarrow{\mathcal{D}} d_a$. In particular we have

$$d_a(x_i, y_i) = \lim_{n\to+\infty} d_{a_n}(x_i, y_i) = d(x_i, y_i)$$

for every $i \in \mathbb{N}$. Therefore $d_a = d$ on a dense subset of $\overline{\Omega} \times \overline{\Omega}$ and hence d_a coincides with d by continuity. That concludes the proof of the claim. □

The metrics $(a_n)_{n\in\mathbb{N}}$ above defined will be now used to construct the required approximating sequence of distances.

Proof of Theorem 3.7. The proof is organized in two steps.

Step 1. We first remark that the closure of $\mathcal{D}(\mathcal{I})$ contains the family of distances generated by lower semicontinuous isotropic Riemannian metrics. In fact, let $b : \Omega \to [\alpha, \beta]$ be a lower semicontinuous metric. It is well known that $b(x) = \sup_{n\in\mathbb{N}} \tilde a_n(x)$ for suitable continuous functions $\tilde a_n$ (and we may as well suppose that $\alpha \leq \tilde a_n \leq \beta$ by possibly replacing the function $\tilde a_n$ with $\tilde a_n \vee \alpha$). Setting $a_n(x) := \sup_{i\leq n} \tilde a_i(x)$, we have that $d_{a_n} \xrightarrow{\mathcal{D}} d_b$ by Proposition 1.24. Moreover, by Proposition 3.1 we have that $\Lambda_{d_b}(x) = b(x)$ and $\Lambda_{d_{a_n}}(x) = a_n(x)$ almost everywhere on Ω and therefore, by the monotonicity assumption (3.7)-(i) made on F, we obviously have

$$\limsup_{n\to+\infty} \int_\Omega F(x, \Lambda_{d_{a_n}}(x))\,\mathrm{d}x \leq \int_\Omega F(x, \Lambda_b(x))\,\mathrm{d}x.$$

To prove the theorem, it is then sufficient to find a sequence of lower semicontinuous metrics $b_n : \Omega \to [\alpha, \beta]$ such that the generated distances d_{b_n} satisfy the claim of the theorem. Indeed, by combining the idea just described with a diagonal argument, the conclusion would follow at once.

Step 2. To get the desired approximation of the distance $d \in \mathcal{D}_S$ via lower semicontinuous isotropic metrics, it is enough to prove that, for every fixed $n \in \mathbb{N}$, there exists a sequence of lower semicontinuous isotropic metrics $b_k : \Omega \to [\alpha, \beta]$ such that

(i) $\lim_{k\to+\infty} d_{b_k}(x_i, y_i) = d(x_i, y_i)$ for every $i \leq n$;

(ii) $\limsup_{k\to+\infty} \int_\Omega F(x, b_k(x))\,\mathrm{d}x \leq \int_\Omega F(x, a_n(x))\,\mathrm{d}x$

where a_n are the Borel isotropic metrics defined in the proof of Theorem 3.11.

In fact the desired sequence of lower semicontinuous metrics is then obtained via a diagonal argument and taking into account that $a_n(x) = \Lambda_d(x)$ almost everywhere on Ω by Theorem 3.11.

Let us then fix $n \in \mathbb{N}$ and let a_n be the Borel metric defined by (3.13). Keeping the notations used in the proof of Theorem 3.11, we observe that the set $N_n \setminus T_n$ is a finite, disjoint union of open arcs. Therefore, by applying Lemma 3.4 to each arc, we can find a sequence of continuous functions $\sigma_k : N_n \setminus T_n \to [\alpha, \beta]$ which converge to a_n $\mathcal{H}^1$-a.e. on $N_n \setminus T_n$. Let us set $A_k := \{x \in \Omega : \mathrm{dist}(x, N_n) < 1/k\}$. Let $(\Omega_k)_{k\in\mathbb{N}}$ be a sequence of bounded open sets well contained in Ω such that $\overline{\Omega}_k \subset \Omega_{k+1}$ and $\Omega = \bigcup_{k\in\mathbb{N}} \Omega_k$. By Lusin's theorem we may find a sequence of closed set $K_k \subset \overline{\Omega}_k \setminus A_k$ such that $a_n|_{K_k}$ is continuous and $\mathcal{L}^N((\overline{\Omega}_k \setminus A_k) \setminus K_k) < 1/k$. Then we define $b_k : \overline{\Omega} \to [\alpha, \beta]$ as

$$b_k(x) := \begin{cases} \sigma_k(x) & \text{if } x \in N_n \setminus T_n \\ \alpha & \text{if } x \in T_n \\ a_n(x) & \text{if } x \in K_k \\ \beta & \text{elsewhere.} \end{cases}$$

Notice that b_k is lower semicontinuous. Moreover we have

$$\limsup_{k\to+\infty} \int_\Omega F(x, b_k(x))\,\mathrm{d}x = \limsup_{k\to+\infty} \left(\int_{K_k} F(x, a_n(x))\,\mathrm{d}x + \int_{\Omega\setminus K_k} F(x, \beta)\,\mathrm{d}x \right). \tag{3.14}$$

Recalling that $|F(x, \beta)|$ is summable over Ω (condition (3.7)-(ii)), we have that the second integral in the right-hand side of (3.14) goes to zero. In fact

$$\int_{\Omega\setminus K_k} F(x, \beta)\,\mathrm{d}x = \int_{\Omega\setminus\Omega_k} F(x, \beta)\,\mathrm{d}x + \int_{\Omega_k\setminus K_k} F(x, \beta)\,\mathrm{d}x, \tag{3.15}$$

and the first and second term of the right-hand side of (3.15) go to zero, respectively by the dominated convergence theorem and the absolute continuity of the integral. Therefore

$$\limsup_{k\to+\infty} \int_\Omega F(x, b_k(x))\,\mathrm{d}x \le \int_\Omega F(x, a_n(x))\,\mathrm{d}x,$$

so point (ii) of the claim is satisfied.

Let us show now that (i) holds. For $i \le n$ we have by definition

$$d_{b_k}(x_i, y_i) \le \mathbb{L}_{b_k}(\gamma_i) = \int_0^1 \sigma_k(\gamma_i)|\dot\gamma_i|\,\mathrm{d}t,$$

therefore by the dominated convergence theorem we get

$$\begin{aligned}\limsup_{k\to+\infty} d_{b_k}(x_i, y_i) &\le \limsup_{k\to+\infty} \int_0^1 \sigma_k(\gamma_i)|\dot\gamma_i|\,\mathrm{d}t = \int_0^1 a_n(\gamma_i)|\dot\gamma_i|\,\mathrm{d}t \\ &= \int_0^1 \varphi_d(\gamma_i, \dot\gamma_i)\,\mathrm{d}t = d(x_i, y_i). \end{aligned} \tag{3.16}$$

Now, let us take for every $k \in \mathbb{N}$ a curve $\tilde\gamma_k \in \mathrm{Lip}_{x_i,y_i}$ such that

$$\mathbb{L}_{b_k}(\tilde\gamma_k) = d_{b_k}(x_i, y_i). \tag{3.17}$$

Notice that such a curve exists in view of Remark 1.19 and Theorem 1.21. Once again, we remark that, by Lemma 3.3, it is not restrictive to suppose that such curves are injective. Since $\alpha \int_I |\dot{\tilde\gamma}_k|\,\mathrm{d}t \le \mathbb{L}_{b_k}(\tilde\gamma_k)$, by (3.17) and (3.16) we get that $\limsup_k \int_I |\dot{\tilde\gamma}_k|\,\mathrm{d}t < +\infty$. Let us choose an $\varepsilon > 0$. By applying Lemma 3.4 to each open arc of $N_n \setminus T_n$, we can find a Borel set $B_\varepsilon \subset N_n \setminus T_n$ and an infinitesimal sequence of positive numbers $(\delta_k)_{k\in\mathbb{N}}$ such that $\mathcal{H}^1(N_n \setminus B_\varepsilon) < \varepsilon$ and $|\sigma_k(x) - a_n(x)| < \delta_k$ for every $x \in B_\varepsilon$. Let us set $I_k := \{t \in I : \tilde\gamma_k(t) \in N_n \setminus B_\varepsilon\}$. Then $b_k(\tilde\gamma_k) \ge a_n(\tilde\gamma_k) - \delta_k$ $\mathcal{L}^1$-a.e. on $I \setminus I_k$. Let us write

$$\mathbb{L}_{b_k}(\tilde\gamma_k) = \int_{I_k} b_k(\tilde\gamma_k)|\dot{\tilde\gamma}_k|\mathrm{d}t + \int_{I\setminus I_k} b_k(\tilde\gamma_k)|\dot{\tilde\gamma}_k|\mathrm{d}t.$$

We remark that, as $\tilde\gamma_k(I_k) \subset N_n \setminus B_\varepsilon$ for every $k \in \mathbb{N}$, by the Area-formula we have

$$\int_{I_k} |\dot{\tilde\gamma}_k|\,\mathrm{d}t = \mathcal{H}^1(\tilde\gamma_k(I_k)) \le \mathcal{H}^1(N_n \setminus B_\varepsilon) < \varepsilon.$$

Taking this remark into account we get

$$\begin{aligned}\int_{I_k} b_k(\tilde\gamma_k)|\dot{\tilde\gamma}_k|\mathrm{d}t &= \int_{I_k} a_n(\tilde\gamma_k)|\dot{\tilde\gamma}_k|\mathrm{d}t + \int_{I_k} (b_k(\tilde\gamma_k) - a_n(\tilde\gamma_k))|\dot{\tilde\gamma}_k|\mathrm{d}t \\ &\ge \int_{I_k} a_n(\tilde\gamma_k)|\dot{\tilde\gamma}_k|\mathrm{d}t - (\beta - \alpha)\varepsilon.\end{aligned}$$

Then we have

$$\begin{aligned} \mathbb{L}_{b_k}(\tilde{\gamma}_k) &\geq \int_0^1 a_n(\tilde{\gamma}_k)|\dot{\tilde{\gamma}}_k|\,\mathrm{d}t - \delta_k \int_{I\setminus I_k} |\dot{\tilde{\gamma}}_k|\,\mathrm{d}t - (\beta-\alpha)\varepsilon \\ &\geq d_{a_n}(x_i,y_i) - \delta_k \int_0^1 |\dot{\tilde{\gamma}}_k|\,\mathrm{d}t - (\beta-\alpha)\varepsilon \end{aligned}$$

and therefore, as $\delta_k \int_0^1 |\dot{\tilde{\gamma}}_k|\,\mathrm{d}t$ goes to zero when $k \to +\infty$, we obtain

$$\liminf_{k\to+\infty} d_{b_k}(x_i,y_i) \geq \liminf_{k\to+\infty} \mathbb{L}_{b_k}(\tilde{\gamma}_k) \geq d_{a_n}(x_i,y_i) - (\beta-\alpha)\varepsilon.$$

Since ε was arbitrary, the above inequality coupled with (3.16) gives the claim. □

Remark 3.12. It should be noticed that the proof of Theorem 3.7 holds under very general assumptions on the function F, namely it is sufficient to take an F which is Borel-measurable and satisfies assumption (ii) of (3.7), and such that the function $F(x,\cdot)$ is non-decreasing for $\mathcal{L}^N$-a.e. $x \in \Omega$. This consideration, together with Remark 3.10, enables us to conclude that our relaxation result, namely Theorem 3.5, holds under the following milder conditions on $F : \Omega \times [\alpha,\beta] \to \mathbb{R}_+$:

(i) there exists a sequence of continuous functions $F_k : \Omega \times [\alpha,\beta] \to \mathbb{R}_+$ satisfying conditions (3.7) and such that $F(x,\xi) = \sup_{k\in\mathbb{N}} F_k(x,\xi)$ for $\mathcal{L}^N$-a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^N$;

(ii) $\int_\Omega F(x,\beta)\,\mathrm{d}x < +\infty$.

Chapter 4

Smooth approximation of Finsler metrics

4.1 Introduction

In this chapter we consider the space $\mathcal{D}_\alpha$ of non-symmetric distances defined through (1.11), where Ω is a connected open subset of $\mathbb{R}^N$, and φ varies in the family $\mathcal{M}_\alpha$ of Borel-measurable Finsler metrics that satisfy the following bounds for two fixed positive constants α and β:

$$\alpha|\xi| \leq \varphi(x,\xi) \leq \beta|\xi| \quad \text{on } \overline{\Omega} \times \mathbb{R}^N.$$

The aim is to show that continuous (smooth) Finsler metrics are dense in Borel ones. More precisely, we will show that any element of $\mathcal{D}_\alpha$ is the uniform limit of a suitable sequence of distances derived through (1.11) from continuous (smooth) metrics belonging to $\mathcal{M}_\alpha$ (Theorems 4.2 and 4.6). The heavy part of the job corresponds to Theorem 4.2: indeed, once the density result is proved for continuous metrics, the analogous result for smooth ones is obtained through a standard mollification argument.

These results can be read as the counterpart of those obtained in Chapter 3, where the case of symmetric distances was considered, and analogous density theorems for continuous and smooth Riemannian metrics were obtained (*cf.* Theorem 3.7 and Remark 3.8). As a matter of fact, the proofs exploit similar ideas; in particular, the key observation still corresponds to Lemma 1.22, which allows to replace the uniform convergence of distances with a pointwise convergence on a fixed, countable subset of $\overline{\Omega}\times\overline{\Omega}$. On the other hand, new arguments have to be introduced to overcome the difficulties produced by the non-symmetric character of distances here considered.

We also wish to underline the content of Theorem 4.3: it amounts to saying that any geodesic distance d, locally equivalent to the Euclidean one, can be obtained from a convex Finsler metric through (1.11), a fact which is not trivial at all: notice indeed that the Finsler metric $\varphi_d(x,\xi)$, obtained from d by derivation (*cf.* formula (1.20)), is proved to be convex in ξ for almost every x only (*cf.* Theorem 1.8).

The density results obtained for $\alpha > 0$ are then extended to the case $\alpha = 0$. The main difference between the two cases relies on the fact that, while $\mathcal{D}_\alpha$ is a closed metric space when $\alpha > 0$, this is no longer true when $\alpha = 0$. This fact is investigated in more detail in Section 4.4 through suitable, explicit examples. We remark that the family $\mathcal{D}_0$ includes distances for which the local equivalence with the Euclidean one fails to hold somewhere. The interest for this class of degenerate distances is motivated by the study of Hamilton-Jacobi equations of eikonal type in the critical case (see [29, 54]).

Last, in Section 4.5 we compare definition (1.11) with a different way of deriving a distance from a Finsler metric, introduced by De Cecco and Palmieri in [41, 42, 43, 44, 45]. The results there provided will be used in Chapter 5.

4.2 Notation and preliminary results

Throughout this chapter α and β denote two fixed constants with $\beta > \alpha \geq 0$, and Ω an open connected subset of $\mathbb{R}^N$ enjoying condition (Ω) of Section 1.3.2. All curves considered in the sequel will be always assumed to be Lipschitz continuous and parametrized by constant speed. We denote by $\mathrm{Lip}_{x,y}$ the family of curves $\gamma : I \to \overline{\Omega}$ connecting x to y, namely such that $\gamma(0) = x$ and $\gamma(1) = y$, where $I := [0,1]$.

Given a measurable function $f : I \to \mathbb{R}^N$, $\|f\|_\infty$ stands for $\sqrt{\sum_{i=0}^N \|f_i\|^2_{L^\infty(I)}}$, where f_i and $\|f_i\|_{L^\infty(I)}$ denote the i-th component of f and the L^∞-norm of f_i respectively.

The closed, convex hull of a subset E of $\mathbb{R}^N$ will be denoted by $\overline{\mathrm{co}}(E)$. If C is a closed and convex subset of $\mathbb{R}^N$, we will denote by σ_C the support function of C, namely

$$\sigma_C(\xi) := \sup\{\langle \xi, p\rangle \,|\, p \in C\}.$$

We set

$$\mathcal{M}_\alpha := \left\{\varphi \text{ Finsler metric on } \overline{\Omega} \ : \ \alpha\,|\xi| \leq \varphi\,(x,\xi) \leq \beta\,|\xi|\right\},$$

and we denote by $\mathcal{D}_\alpha = \mathcal{D}(\mathcal{M}_\alpha)$ the space of Finsler distances on $\overline{\Omega}$ generated by the metrics $\mathcal{M}_\alpha$ through (1.11), namely

$$\mathcal{D}_\alpha = \mathcal{D}(\mathcal{M}_\alpha) := \{d_\varphi \text{ distance on } \overline{\Omega} \text{ given by (1.11)} \ : \ \varphi \in \mathcal{M}_\alpha\}.$$

In the sequel, we will need the following C^1-approximation result, which is just a restatement of Theorem 1 in [52, Section 6.6].

Theorem 4.1. *Suppose $\gamma : I \to \mathbb{R}^N$ is a (Lispchitz) curve. Then for each $\varepsilon > 0$, there exists a C^1 curve $\overline{\gamma} : I \to \mathbb{R}^N$ such that:*

$$\mathcal{L}^1\left(\{t \in I \ : \ \overline{\gamma}(t) \neq \gamma(t) \ \text{or} \ \dot{\overline{\gamma}}(t) \neq \dot{\gamma}(t)\}\right) \leq \varepsilon.$$

In addition

$$\left\|\dot{\overline{\gamma}}\right\|_\infty \leq c\,\|\dot{\gamma}\|_\infty$$

for some constant c depending only on N.

4.3 Approximation results for non-degenerate distances

Throughout this section, α is always assumed to be strictly positive. Our first density result is then stated as follows:

Theorem 4.2. *Let $d \in \mathcal{D}_\alpha$. Then there exists a sequence $(\varphi_n)_n$ of continuous and convex Finlser metrics in $\mathcal{M}_\alpha$ such that $d_{\varphi_n} \xrightarrow{\mathcal{D}_\alpha} d$, where d_{φ_n} is the distance associated to φ_n through (1.11).*

The strategy to prove the density result is the same used in Chapter 3: when defining the approximating sequence φ_n, we only need to control convergence of the induced distances d_{φ_n} to d on a dense subset of $\overline{\Omega}\times\overline{\Omega}$ (in view of Lemma 1.22). To this aim, we set $S := \mathbb{Q}^N \cap \overline{\Omega}$. Obviously $S \times S$ is countable and dense in $\overline{\Omega} \times \overline{\Omega}$, so we write $S \times S := \{(x_i, y_i) \ : \ i \in \mathbb{N}\}$. This notation will be adopted in the remainder of this section.

Instead of providing a direct proof to Theorem 4.2, we prefer to break it into intermediate propositions, which will be proved separately. The first one is an interesting result, which can be read as a "dual" formulation of Theorem 3.11.

Theorem 4.3. *Let $d \in \mathcal{D}_\alpha$. Then there exists a decreasing sequence of convex metrics $\psi_n \in \mathcal{M}_\alpha$ such that*

$$d_{\psi_n}(x_i, y_i) = d(x_i, y_i) \quad \text{for each } i \leq n. \tag{4.1}$$

Moreover, if we set $\psi(x,\xi) := \inf_n \psi_n(x,\xi)$ for every $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^N$, then $d = d_\psi$, that is every Finsler distance is induced by a convex Finsler metric.

Proof. For each $(x_i, y_i) \in S \times S$, let $\gamma_i \in \mathrm{Lip}_{x_i,y_i}$ be a path of minimal d-length, *i.e.* $\mathrm{L}_d(\gamma_i) = d(x_i, y_i)$, and set $\Gamma_i := \gamma_i\left((0,1)\right)$. For every $x \in \Gamma_i$, let $\{\xi_1^i(x), \xi_2^i(x), \ldots, \xi_N^i(x)\}$ be an ortonormal basis of $\mathbb{R}^N$ such that $\xi_1^i(\gamma_i(t)) = \dot\gamma_i(t)/\|\dot\gamma_i\|_\infty$ for $\mathcal{L}^1$-a.e $t \in I$ (recall that all curves are parametrized by constant speed), and set $a^i(x) := \varphi_d(x, \xi_1^i(x))$. Such vectors can be chosen in such a way that the map $x \to \xi_j^i(x)$ is Borel-measurable on Γ_i for each $1 \leq j \leq N$. We define the set-valued map $C^i(\cdot)$ on $\overline{\Omega}$ as follows:

$$C^i(x) := \begin{cases} \overline{\mathrm{co}}\left\{a^i(x)\,\xi_1^i(x), -\beta\,\xi_1^i(x), \pm\beta\xi_2(x), \ldots, \pm\beta\,\xi_N(x)\right\} & \text{if } x \in \Gamma_i \\ B_\alpha & \text{if } x \in \{x_i, y_i\} \\ B_\beta & \text{otherwise.} \end{cases}$$

Let $\sigma^i(x,\xi) := \sigma_{C^i(x)}(\xi)$ for every $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^N$. The map σ^i is easily seen to be Borel-measurable, and in particular to belong to $\mathcal{M}_\alpha$. The required metrics ψ_n are now defined as follows:

$$\psi_n(x,\xi) := \inf_{1 \leq i \leq n} \sigma^i(x,\xi) \quad \text{for every } (x,\xi) \in \overline{\Omega} \times \mathbb{R}^N, \tag{4.2}$$

for each $n \in \mathbb{N}$. Obvioulsy, ψ_n belongs $\mathcal{M}_\alpha$. Let us show now that (4.1) holds. For a fixed index $1 \leq i \leq n$ we have

$$d_{\psi_n}(x_i, y_i) \leq \int_0^1 \psi_n(\gamma_i, \dot\gamma_i)\,\mathrm{d}t = \int_0^1 \varphi_d(\gamma_i, \dot\gamma_i)\,\mathrm{d}t = d(x_i, y_i).$$

To prove the reverse inequality, choose a curve $\gamma \in \mathrm{Lip}_{x_i,y_i}$ and, for every $1 \leq j \leq n-1$, set $I_{j+1} := \{t \in I \setminus \cup_{h \leq j} I_h \ :\ \gamma(t) \in \gamma_{j+1}(I)\}$, $I_1 := \{t \in I \ :\ \gamma(t) \in \gamma_1(I)\}$ and $I_0 := I \setminus \cup_{j \leq n} I_j$. We remark that, for each $1 \leq j \leq n$, the vector $\dot\gamma(t)$ is parallel to $\dot\gamma_j(t)$ $\mathcal{L}^1$-a.e. on I_j and so $\psi_n(\gamma, \dot\gamma) \geq \varphi_d(\gamma, \dot\gamma)$ $\mathcal{L}^1$-a.e. on I_j (depending on whether the two vectors are equally oriented or not). Note that this trivially holds for $j = 0$ too. Therefore we have

$$\mathbb{L}_{\psi_n}(\gamma) = \int_0^1 \psi_n(\gamma, \dot\gamma)\,\mathrm{d}t = \sum_{j=0}^n \int_{I_j} \psi_n(\gamma, \dot\gamma)\,\mathrm{d}t \geq \sum_{j=0}^n \int_{I_j} \varphi_d(\gamma, \dot\gamma)\,\mathrm{d}t \geq d(x_i, y_i),$$

By passing to the infimum over all possible curves $\gamma \in \mathrm{Lip}_{x_i,y_i}$ we get the claim.
Last, set $\psi(x,\xi) := \inf_n \psi_n(x,\xi)$ for every $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^N$. By applying Proposition 1.24 (iii), we obtain

$$d_\psi(x_i,y_i) = \lim_{n\to+\infty} d_{\psi_n}(x_i,y_i) = d(x_i,y_i) \quad \text{for every } i \in \mathbb{N},$$

namely d_ψ coincide with d on $S \times S$, hence everywhere in $\overline{\Omega} \times \overline{\Omega}$ by density. □

Next, we prove the following

Proposition 4.4. *Let $d \in \mathcal{D}_\alpha$. Then, for every $n \in \mathbb{N}$ and every $\varepsilon > 0$, there exists a lower semicontinuous, convex metric $\phi_\varepsilon \in \mathcal{M}_\alpha$ such that*

$$|d_{\phi_\varepsilon}(x_i,y_i) - d(x_i,y_i)| < \varepsilon \quad \textit{for all } i \le n. \tag{4.3}$$

Proof. Note first that, if the functions σ^i defined in Proof of Theorem 4.3 were continuous on Γ_i, a possible choice for ϕ_ε would be ψ_n. Hence, the idea is that of modifying the definition of σ^i in order to get a continuous function on Γ_i for each i. We adopt the notation used in the proof of Theorem 4.3.
Let $n \in \mathbb{N}$ be fixed, let γ_i, Γ_i and $a^i(x)$ be defined as above for each $i \in \mathbb{N}$, and set $N_n = \cup_{i=1}^n \Gamma_i$. Let us choose $\lambda > 0$ and fix an index $1 \le i \le n$. Note that γ_i is injective by minimality, hence we can apply Lemma 3.4 to obtain a sequence of continuous functions $a^i_k : \Gamma_i \to [\alpha,\beta]$, $k \in \mathbb{N}$, converging pointwise to a^i $\mathcal{H}^1$-a.e. on Γ_i as k goes to infinity. Let $\overline{\gamma}^i$ be the C^1-continuous curve obtained by applying Theorem 4.1 with $\gamma := \gamma^i$ and $\varepsilon := \lambda$. For every $x \in \Gamma_i$, let $\{\overline{\xi}^i_1(x), \overline{\xi}^i_2(x), \dots, \overline{\xi}^i_N(x)\}$ be an orthogonal basis of $\mathbb{R}^N$ such that:

(i) $\overline{\xi}^i_1(\gamma_i(t)) = \dot{\overline{\gamma}}_i(t)/\|\dot{\gamma}_i\|_\infty$ for $\mathcal{L}^1$-a.e $t \in I$;

(ii) $|\overline{\xi}^i_j(x)| = 1$ for every $2 \le j \le N$.

Such vectors can be chosen in such a way that the map $x \to \overline{\xi}^i_j(x)$ is continuous on Γ_i for each $1 \le j \le N$. For each $k \in \mathbb{N}$, the set-valued map $\overline{C}^i_k(\cdot)$ is defined on $\overline{\Omega}$ as follows:

$$\overline{C}^i_k(x) := \begin{cases} \overline{\mathrm{co}}\left\{a^i_k(x)\,\overline{\xi}^i_1(x), -\beta\,\overline{\xi}^i_1(x), \pm\beta\overline{\xi}_2(x), \dots, \pm\beta\,\overline{\xi}_N(x)\right\} & \text{if } x \in \Gamma_i \\ B_\alpha & \text{if } x \in \{x_i, y_i\} \\ B_\beta & \text{otherwise.} \end{cases}$$

Let $\overline{\sigma}^i_k(x,\xi) := \left(\sigma_{C^i_k(x)}(\xi) \wedge \beta|\xi|\right) \vee \alpha|\xi|$ for every $(x,\xi) \in \overline{\Omega} \times \mathbb{R}^N$. The map $\overline{\sigma}^i_k$ is lower semicontinuous by definition, in particular it belongs to $\mathcal{M}_\alpha$. For each $k \in \mathbb{N}$, we define the metric ϕ_k as follows:

$$\phi_k(x,\xi) := \inf_{1\le i\le n} \overline{\sigma}^i_k(x,\xi) \quad \text{for every } (x,\xi) \in \overline{\Omega} \times \mathbb{R}^N. \tag{4.4}$$

Obviously, ϕ_k is lower semicontinuous and belongs $\mathcal{M}_\alpha$. We claim that the following holds:

$$\lim_{k\to+\infty} |d_{\phi_k}(x_i,y_i) - d(x_i,y_i)| \le (n\beta L)\,\lambda \quad \text{for all } 1 \le i \le n, \tag{4.5}$$

where $L := \max_{1\le i\le n} \|\dot\gamma_i\|_\infty$ This will be enough to conclude: indeed, it is sufficient to take $\lambda < \varepsilon/(n\beta L)$ and $\phi_\varepsilon := \phi_k$ with k suitably large.
Let us then show (4.5). For $1 \le i \le n$ we have by definition

$$d_{\phi_k}(x_i,y_i) \le \mathbb{L}_{\phi_k}(\gamma_i) = \int_0^1 \phi_k(\gamma_i,\dot\gamma_i)\,\mathrm{d}t.$$

Let $J_i := \{\, t \in I \,:\, \dot{\overline{\gamma}}_i(t) = \dot\gamma_i(t) \,\}$ and recall that, by Theorem 4.1, $\mathcal{L}^1(I \setminus J_i) \le \lambda$. By the dominated convergence theorem we have

$$\begin{aligned} \limsup_{k\to+\infty} d_{\phi_k}(x_i,y_i) &\le \limsup_{k\to+\infty} \left(\int_{J_i} a^i_k(\gamma_i)|\dot\gamma_i|\,\mathrm{d}t + \int_{I\setminus J_i} \beta\,|\dot\gamma_i(t)|\,\mathrm{d}t \right) \\ &\le \int_0^1 a^i(\gamma_i)|\dot\gamma_i|\,\mathrm{d}t + \beta L\lambda = d(x_i,y_i) + \beta L\lambda. \end{aligned} \tag{4.6}$$

Now, set $\Gamma_0 := \cup_{i=1}^n \gamma_i(I \setminus J_i)$ and remark that $\mathcal{H}^1(\Gamma_0) \le nL\lambda$. Fix an index $1 \le i \le n$. For each $k \in \mathbb{N}$, pick up a curve $\tilde\gamma_k \in \mathrm{Lip}_{x_i,y_i}$ such that

$$\mathbb{L}_{\phi_k}(\tilde\gamma_k) = d_{\phi_k}(x_i,y_i). \tag{4.7}$$

Note that such curves are injective by minimality. Since $\alpha \int_I |\dot{\tilde\gamma}_k|\,\mathrm{d}t \le \mathbb{L}_{\phi_k}(\tilde\gamma_k)$, by (4.7) and (4.6) we get that $\limsup_k \int_I |\dot{\tilde\gamma}_k|\,\mathrm{d}t < +\infty$. Let us choose an $\tilde\varepsilon > 0$. By applying Lemma 3.4 to each open arc Γ_i, we can find a Borel set $B_{\tilde\varepsilon} \subset N_n$ and an infinitesimal sequence of positive numbers $(\delta_k)_{k\in\mathbb{N}}$ such that $\mathcal{H}^1(N_n \setminus B_{\tilde\varepsilon}) < \tilde\varepsilon$ and $|a^i_k(x) - a^i(x)| < \delta_k$ for every $x \in B_{\tilde\varepsilon}$, $1 \le i \le n$ and $k \in \mathbb{N}$. Let us set $I_k := \{t \in I \,:\, \tilde\gamma_k(t) \in \Gamma_0 \cup (N_n \setminus B_{\tilde\varepsilon})\}$. Then $\phi_k(\tilde\gamma_k,\dot{\tilde\gamma}_k) \ge \psi_n(\tilde\gamma_k,\dot{\tilde\gamma}_k) - \delta_k|\dot{\tilde\gamma}_k|$ $\mathcal{L}^1$-a.e. on $I \setminus I_k$. Let us write

$$\mathbb{L}_{\phi_k}(\tilde\gamma_k) = \int_{I_k} \phi_k(\tilde\gamma_k,\dot{\tilde\gamma}_k)\,\mathrm{d}t + \int_{I\setminus I_k} \phi_k(\tilde\gamma_k,\dot{\tilde\gamma}_k)\,\mathrm{d}t.$$

We remark that, as $\tilde{\gamma}_k(I_k) \subset \Gamma_0 \cup (N_n \setminus B_{\tilde{\varepsilon}})$ for every $k \in \mathbb{N}$, by the Area-formula we have

$$\int_{I_k} |\dot{\tilde{\gamma}}_k| \,\mathrm{d}t = \mathcal{H}^1(\tilde{\gamma}_k(I_k)) \leq \mathcal{H}^1(N_n \setminus B_{\tilde{\varepsilon}}) + \mathcal{H}^1(\Gamma_0) < \tilde{\varepsilon} + nL\lambda.$$

Taking this remark into account we get

$$\begin{aligned} \int_{I_k} \phi_k(\tilde{\gamma}_k, \dot{\tilde{\gamma}}_k)\mathrm{d}t &= \int_{I_k} \psi_n(\tilde{\gamma}_k, \dot{\tilde{\gamma}}_k)\,\mathrm{d}t + \int_{I_k} \big(\phi_k(\tilde{\gamma}_k, \dot{\tilde{\gamma}}_k) - \psi_n(\tilde{\gamma}_k, \dot{\tilde{\gamma}}_k)\big)\,\mathrm{d}t \\ &\geq \int_{I_k} \psi_n(\tilde{\gamma}_k, \dot{\tilde{\gamma}}_k)\,\mathrm{d}t - \beta\,(\tilde{\varepsilon} + nL\lambda)\,. \end{aligned}$$

Then we have

$$\begin{aligned} \mathbb{L}_{\phi_k}(\tilde{\gamma}_k) &\geq \int_0^1 \psi_n(\tilde{\gamma}_k, \dot{\tilde{\gamma}}_k)\,\mathrm{d}t - \delta_k \int_{I\setminus I_k} |\dot{\tilde{\gamma}}_k|\,\mathrm{d}t - \beta\,(\tilde{\varepsilon} + nL\lambda) \\ &\geq d_{\psi_n}(x_i, y_i) - \delta_k \int_0^1 |\dot{\tilde{\gamma}}_k|\,\mathrm{d}t - \beta\,(\tilde{\varepsilon} + nL\lambda) \end{aligned}$$

and therefore, as $\delta_k \int_0^1 |\dot{\tilde{\gamma}}_k|\,\mathrm{d}t$ goes to zero when $k \to +\infty$, we obtain

$$\liminf_{k\to+\infty} d_{\phi_k}(x_i, y_i) \geq \liminf_{k\to+\infty} \mathbb{L}_{\phi_k}(\tilde{\gamma}_k) \geq d_{\psi_n}(x_i, y_i) - \beta\,(\tilde{\varepsilon} + nL\lambda)\,.$$

Since $\tilde{\varepsilon}$ was arbitrary and $d_{\psi_n}(x_i, y_i) = d(x_i, y_i)$, the above inequality coupled with (4.6) gives the claim. $\square$

We are now ready to give the

Proof of Theorem 4.2. First, we claim that, for every $\varepsilon > 0$, there exists a continuous, convex metric $\varphi_\varepsilon \in \mathcal{M}_\alpha$ such that

$$|d_{\varphi_\varepsilon}(x_i, y_i) - d(x_i, y_i)| < \varepsilon \quad \text{for all } i \leq n. \tag{4.8}$$

Indeed, Proposition 4.4 provides a convex, lower semicontinuous metric ϕ_ε satisfying (4.8). By setting $\phi_\varepsilon(x, \xi) := \beta|\xi|$ when $x \in \mathbb{R}^N \setminus \overline{\Omega}$, ϕ_ε can be extended to $\mathbb{R}^N \times \mathbb{R}^N$. Then Lemma 2.2.3 of [21] easily implies the existence of an increasing sequence $(\Phi_n)_n$ of continuous, convex Finsler metrics on $\mathbb{R}^N$ such that $\Phi_n|_{\overline{\Omega}\times\mathbb{R}^N} \in \mathcal{M}_\alpha$ for each $n \in \mathbb{N}$ and $\phi_\varepsilon(x, \xi) = \sup_{n\in\mathbb{N}} \Phi_n(x, \xi)$ in $\overline{\Omega} \times \mathbb{R}^N$ (*cf.* Proof of [16, Theorem 3.1]). By Proposition 1.24 (i), the sequence of distances $(d_{\Phi_n})_n$ converges to d_{ϕ_ε} in $\mathcal{D}_\alpha$, hence (4.8) is proved by setting $\varphi_\varepsilon := \Phi_n$ for n sufficiently large.

It is now clear how to conclude: for each $n \in \mathbb{N}$, define $\varphi_n := \varphi_{\varepsilon_n}$ with $\varepsilon_n := 1/n$. The distances d_{φ_n} converge to d pointwise on $S \times S$, which is dense in $\overline{\Omega} \times \overline{\Omega}$, so the claim follows in view of Lemma 1.22. $\square$

Remark 4.5. Note that, by construction, the metrics ϕ_n obtained in Proposition 4.4 are actually defined and continuous in all $\mathbb{R}^N \times \mathbb{R}^N$.

The result established in Theorem 4.2 can now be easily improved by requiring the approximating metrics to be smooth.

Theorem 4.6. *Let $d \in \mathcal{D}_\alpha$. Then there exists a sequence $(\tilde{\varphi}_n)_n$ of smooth and convex Finlser metrics in $\mathcal{M}_\alpha$ such that $d_{\tilde{\varphi}_n} \xrightarrow{\mathcal{D}_\alpha} d$.*

Proof. Theorem 4.2 provides a sequence $(\varphi_n)_n$ of continuous, convex metrics in $\mathcal{M}_\alpha$ satisfying the claim. Such metrics are actually defined and continuous in $\mathbb{R}^N \times \mathbb{R}^N$ (*cf.* Remark 4.5). Now, take a sequence $(\rho_k)_k$ of convolution kernels and, for each $n \in \mathbb{N}$, let $(\rho_k * \varphi_n)(x,\xi) := \int_{\mathbb{R}^N} \rho_k(x-y)\varphi_n(y,\xi)\,\mathrm{d}y$. Clearly $(\rho_k * \varphi_n)_k$ is a sequence of convex, smooth metrics in $\mathcal{M}_\alpha$, uniformly converging to φ_n on compact subset of $\overline{\Omega} \times \mathbb{R}^N$. In view of Proposition 1.24 (ii), the claim follows by setting $\tilde{\varphi}_n := \rho_k * \varphi_n$ for each $n \in \mathbb{N}$, with $k := k(n)$ suitably large. $\square$

4.4 The degenerate case

We want to extend the density results obtained in the previous section to the degenerate case, namely when $\alpha = 0$. In fact, the analogous of Theorem 4.6 holds.

Theorem 4.7. *Let $d \in \mathcal{D}_0$. Then there exists a sequence $(\tilde{\varphi}_n)_n$ of smooth and convex Finsler metrics in $\mathcal{M}_0$ such that $d_{\tilde{\varphi}_n} \xrightarrow{\mathcal{D}_0} d$.*

Proof. Let $\varphi \in \mathcal{M}_0$ such that $d = d_\varphi$. For each $k \in \mathbb{N}$, set $\varphi_k(x,\xi) := \varphi(x,\xi) \vee \frac{1}{k}|\xi|$ in $\overline{\Omega} \times \mathbb{R}^N$. As $\varphi_k \in \mathcal{M}_{1/k}$, we can apply Theorem 4.6 to d_{φ_k} for each $k \in \mathbb{N}$, to obtain a smooth Finsler metric $\tilde{\varphi}_k \in \mathcal{M}_{1/k}$ such that

$$|d_{\varphi_k}(x_i, y_i) - d_{\tilde{\varphi}_k}(x_i, y_i)| \leq \frac{1}{k} \quad \text{for every } i \leq k.$$

Now the claim easily follows, since the sequence $(d_{\varphi_k})_k$ converges to d in $\mathcal{D}_0$ in view of Remark 1.25. □

As already remarked in Section 1.4, the space $\mathcal{D}_\alpha$ is closed when $\alpha > 0$, that is, if $(d_n)_n$ is a sequence in $\mathcal{D}_\alpha$ converging to d, then d belongs to $\mathcal{D}_\alpha$. Does the same property hold for $\alpha = 0$ too?

The point is to show that d is still of geodesic type. When $\alpha > 0$, this is basically due to the fact that the corresponding length functionals are equi-coercive, namely $\mathrm{L}_{d_n}(\gamma) \geq \alpha \int_0^1 |\dot\gamma(t)|\,\mathrm{d}t$ for any curve γ and for each $n \in \mathbb{N}$, and this means that any sequence of curves $(\gamma_n)_n \subset \mathrm{Lip}_{x,y}$ such that $\limsup_n \mathrm{L}_{d_n}(\gamma_n) < +\infty$ admits (at least) a cluster point $\gamma \in \mathrm{Lip}_{x,y}$. In particular, for a suitable choice of $(\gamma_n)_n$ and of γ, that yields

$$\mathrm{L}_d(\gamma) \leq \liminf_{n\to+\infty} \mathrm{L}_{d_n}(\gamma_n) = \lim_{n\to+\infty} d_n(x,y) = d(x,y),$$

which obviously means that d is a geodesic distance. When $\alpha = 0$, instead, it may happen that the Euclidean lengths of the curves γ_n diverge, and two critical phenomena may basically occur in this case: either the curves go to infinity and disappear in the limit, or they stay bounded and converge to a non-rectifiable curve (hence, no longer Lipschitz continuous). In both cases, the limit distance might be not of geodesic type; hence, the answer to the question previously raised is no: $\mathcal{D}_0$ is not closed. Explicit examples of these possible situations are provided below.

Example. Let $\Omega := \mathbb{R}^2$, and let us set $\Gamma_n := \{(x,y) \,:\, x \in \{0,1\},\, y \in [0,n]\,\} \cup [0,1] \times \{n\}$ for each $n \in \mathbb{N}$, $\Gamma_\infty := \{(x,y) \,:\, x \in \{0,1\},\, y \in [0,+\infty]\,\}$. For each $n \in \mathbb{N}$, we define the metric φ_n on $\mathbb{R}^2 \times \mathbb{R}^2$ as follows:

$$\varphi_n(x,\xi) := \begin{cases} \frac{1}{4n^2}|\xi| & \text{if } x \in \Gamma_n \text{ and } \xi \in \mathbb{R}^2 \\ \\ 2|\xi| & \text{if } x \in \mathbb{R}^2 \setminus \Gamma_n \text{ and } \xi \in \mathbb{R}^2 \end{cases}$$

Let $x_0 := (0,0)$, $y_0 := (0,1)$, and let $d_n := d_{\varphi_n}$ be the distance defined on $\mathbb{R}^2 \times \mathbb{R}^2$ via (1.11) for each $n \in \mathbb{N}$. Let us notice that $d_n(x_0,y_0) = (2n+1)/4n^2$: in fact, the d_n-minimizing path connecting x_0 to y_0 is given by the polygonal arc Γ_n. Up to subsequences, $(d_n)_n$ converges (uniformly on compact subset of $\mathbb{R}^2 \times \mathbb{R}^2$) to some distance d, by Ascoli-Arzelà Theorem. Now we have that $d(x_0,y_0) = 0$, while the d-metric length of any curve connecting x_0 to y_0

is at least 2. To see this, simply notice that $\varphi_d(x,\xi) = 2|\xi|$ for every $x \in \mathbb{R}^2 \setminus \Gamma_\infty$. Hence d is not of geodesic type, though uniform limit of geodesic distances.

Example. Let $\Omega := (0,1) \times (0,1)$ and let $q = (1/2, 0), q' = (1, 1/2)$. Our example relies upon the construction provided by Whitney in [73]. In this remarkable paper, Whitney recursively defines an arc A joining q to q' of infinite length, and a function f, defined on $\overline{\Omega}$ and of class C^1, whose gradient is null on A, but $f(q) = 0$ and $f(q') = 1$ (in particular, f is not constant on A).

Such an arc can be represented as the image of a continuous function $\gamma : I \to \overline{\Omega}$ with $\gamma(0) = q$, $\gamma(1) = q'$, as explained by the author. Our goal is to define a sequence of Lipschitz curves uniformly converging to γ. This can be easily done by exploiting Whitney's construction. Let us keep the same notation of [73] and assume we are at the n-th step of the iterative procedure leading to the definition of A, *i.e.* we have already defined squares $Q_{i_1\cdots i_t}$, points $q_{i_1\cdots i_t}$, $q'_{i_1\cdots i_t}$ and lines $A_{j_1\cdots j_t}$ (each $i_k = 0,1,2,3$, each $j_k = 0,1,2,3,4$) for $t \le n$. Then, an arc A_n of finite length, joining q to q', can be obtained by connecting each point $q'_{i_1\cdots i_s}$ to $q_{i_1\cdots i_s+1}$, if $i_s \le 2$, and $q'_{i_1\cdots i_{s-1},3}$ to $q_{i_1\cdots i'_{s-1}}$ by means of a segment, and by gluing all these segments with all the lines $A_{j_1\cdots j_t}$, $t \le n$. Each arc A_n may be represented as the image of a Lipschitz curve γ_n. Up to a reparametrization (not by constant speed, in particular), the curves γ_n uniformly converge to γ, as easily seen.

Let us now denote by l_n the Euclidean length of the curve γ_n, and define the following sequence $(a_n)_n$ of Riemannian metrics:

$$a_n(x) := \begin{cases} \frac{1}{n l_n} & \text{if } x \in A_n \\ \\ 1 & \text{if } x \in \overline{\Omega} \setminus A_n. \end{cases}$$

Let d_n the distances on $\overline{\Omega} \times \overline{\Omega}$ associated to $a_n(x)|\xi|$ through (1.11). Obviously, $d_n(q,q') \le 1/n$. Up to subsequences, $(d_n)_n$ converges to some distance d, by Ascoli-Arzelà Theorem, and obviously $d(q,q') = 0$. Notice also that $\varphi_d(x,\xi) = |Df(x)||\xi|$ for every $x \in \overline{\Omega} \setminus A$. Now, let us take a curve $\xi \in \mathrm{Lip}_{q,q'}$. We have

$$\mathrm{L}_d(\xi) = \int_0^1 \varphi_d(\xi(t), \dot{\xi}(t))\,\mathrm{d}t \ge \int_0^1 |Df(\xi(t))|\,|\dot{\xi}(t)|\,\mathrm{d}t \ge f(q') - f(q) = 1,$$

so d is not of geodesic type.

4.5 Comparison with a different definition of distance

In this section we present a different way to derive a distance from an element of $\mathcal{M}_\alpha$, where α is now assumed to be a fixed non-negative constant. We recall that a curve γ is said to be *transversal* to the set E if $\mathcal{H}^1(\gamma(I) \cap E) = 0$. Then, for each $\varphi \in \mathcal{M}_\alpha$, we define a function $\widetilde{d}_\varphi$ on $\overline{\Omega} \times \overline{\Omega}$ through the following formula:

$$\widetilde{d}_\varphi(x, y) := \sup_{\mathcal{L}^N(E)=0} \left\{ \inf \left\{ \mathbb{L}_\varphi(\gamma) : \ \gamma \in \mathrm{Lip}_{x,y}, \ \gamma \text{ transversal to } E \right\} \right\}. \tag{4.9}$$

Definition (4.9) was introduced by De Cecco and Palmieri in [41, 42, 43, 44, 45] to generalize the notions of Riemannian and Finsler metric to a Lipschitz manifold, namely a topological manifold (with countable basis) whose changes of charts are Lipschitz functions. In such a framework, there was the need of giving an intrinsic definition of length of a curve, *i.e.* compatible with the changes of coordinates. Since the latter are Lipschitz continuous, hence differentiable outside a N-dimensional negligible set, a right notion must be independent of sets of null measure. Lipschitz manifolds are a generalization of polyhedra, and were introduced to treat the case of manifolds with singularities, such as vertices, edges, conical points, even not isolated.

On the other hand, our point of view is different: we are concerned with the singularities carried by the metric, rather than dealing with those of the manifold. In this setting, definition (4.9) amounts to "smoothing" the metric, providing a definition of distance which is not affected if the metric is bad-behaved on negligible sets.

The purpose of this section is to compare definition (4.9) with definition (1.11) and to understand the relations between them. The results proved here will be used in Chapter 5. Throughout the present section, we make the additional assumption that $\partial\Omega$ is locally Lipschitz, *i.e.* locally coincides with the graph of a Lipschitz continuous function.

Let us denote by $\widetilde{\mathcal{D}}_\alpha := \{\widetilde{d}_\varphi : \varphi \in \mathcal{M}_\alpha\}$ the space of distances generated by the elements of $\mathcal{M}_\alpha$ through (4.9). We have the following result.

Theorem 4.8. *Let $\varphi \in \mathcal{M}_\alpha$ and let $\widetilde{d}_\varphi$ be the distance defined by (4.9). Then there exists a negligible set $F \subset \overline{\Omega}$ such that*

$$\widetilde{d}_\varphi(x, y) = \inf \left\{ \mathbb{L}_\varphi(\gamma) : \ \gamma \in \mathrm{Lip}_{x,y}, \ \gamma \textit{ transversal to } F \right\}. \tag{4.10}$$

Moreover, if we set $\widetilde{\varphi}(x,\xi) := \varphi(x,\xi)\chi_{\overline{\Omega}\setminus F}(x) + \beta|\xi|\chi_F(x)$*, we have that* $\widetilde{d}_\varphi = d_{\widetilde{\varphi}}$*, where* $d_{\widetilde{\varphi}}$ *is the distance associated to* $\widetilde{\varphi}$ *through (1.11). In particular, we have that* $\widetilde{\mathcal{D}}_\alpha \subset \mathcal{D}_\alpha$*.*

In order to prove Theorem 4.8, we need a preliminary lemma.

Lemma 4.9. *Let* $\gamma \in \mathrm{Lip}_{x,y}$ *with* $x, y \in \overline{\Omega}$ *and let* E *be a negligible subset of* $\overline{\Omega}$*. Then for every* $\varepsilon > 0$ *there exists a curve* $\gamma_\varepsilon \in \mathrm{Lip}_{x,y}$ *transversal to* E *and such that* $\|\gamma_\varepsilon - \gamma\|_{W^{1,\infty}} := \|\gamma_\varepsilon - \gamma\|_\infty + \|\dot{\gamma}_\varepsilon - \dot{\gamma}\|_\infty < \varepsilon$*.*

Proof. Let $\gamma \in \mathrm{Lip}_{x,y}$ and let $g(t) \in C^1(I)$ be a non negative function such that $g(t) = 0$ for $t = 0$ and $t = 1$ only (take for example $g(t) := \sin(\pi t)$). First, let us prove that for $\mathcal{L}^N$-a.e. $v \in \mathbb{R}^N$ the curve $\gamma_v(t) := \gamma(t) + vg(t)$ is transversal to the set E. Set $F(t,v) := \gamma(t) + vg(t)$ and let A be the set of points $(t,v) \in I \times \mathbb{R}^N$ such that $F(t,v)$ belongs to E. For every fixed $t \in (0,1)$, the section $A_t := \{v \in \mathbb{R}^N \,:\, (t,v) \in A\}$ has zero Lebesgue measure in $\mathbb{R}^N$, therefore A has zero Lebesgue measure in $I \times \mathbb{R}^N$. This implies that for every $v \in \mathbb{R}^N \setminus N_0$ the section $A_v := \{t \in I \,:\, (t,v) \in A\}$ is $\mathcal{L}^1$-negligible in I, where N_0 is a negligible set in $\mathbb{R}^N$. Therefore, since $\gamma_v(t)$ is Lipschitz, for every $v \in \mathbb{R}^N \setminus N_0$ the set $\gamma_v(A_v)$ is $\mathcal{H}^1$-negligible in $\mathbb{R}^N$, hence the curve γ_v is transversal to E, as $\gamma_v(A_v) = \gamma_v(I) \cap E$. Remark that $\|\gamma_v - \gamma\|_{W^{1,\infty}} = |v|\|g\|_{W^{1,\infty}}$.

If γ lies inside Ω, then for $|v|$ small enough the curves γ_v lie inside Ω. The claim follows by setting $\gamma_\varepsilon := \gamma_v$ with $v \in \mathbb{R}^N \setminus N_0$ and $|v| < \varepsilon/\|g\|_{W^{1,\infty}}$.

Otherwise, let us assume that the curve γ touches the boundary in a point x_0. By possibly subdividing $\gamma(I)$ into small subarcs, we may suppose that the curve γ lies in $\overline{\Omega} \cap B$, where B is a ball centred in x_0. This ball can be chosen small enough in such a way that there exists a cone $C := \{v \in B_\delta(0) \,:\, \langle v, \xi\rangle > \delta|v|\}$, with $\delta > 0$ and $\xi \in \mathbb{S}^{N-1}$ suitably chosen, such that $z + C \subset \Omega$ for every $z \in \partial\Omega \cap B$. Remark that, if $v \in C$, the curve γ_v lies inside Ω. Therefore, by arguing as above, the claim is achieved by setting $\gamma_\varepsilon := \gamma_v$ with $v \in C \setminus N_0$ and $|v| < \varepsilon/\|g\|_{W^{1,\infty}}$. □

Proof of Theorem 4.8: The existence of a negligible set F which satisfies the first assertion of the claim follows by Proposition 3.5 of [28]. Up to enlarging this set if necessary, we may as well suppose that F is Borel-measurable.

Set $\widetilde{\varphi}(x,\xi) := \varphi(x,\xi)\chi_{\overline{\Omega}\setminus F}(x) + \beta|\xi|\chi_F(x)$ and let $d_{\widetilde{\varphi}}$ be the associated distance defined according to (1.11). Since $\mathbb{L}_{\widetilde{\varphi}}(\gamma) = \mathbb{L}_\varphi(\gamma)$ if γ is transversal to F, we obviously have that

$d_{\widetilde{\varphi}} \leq \widetilde{d}_{\varphi}$. We want to prove the reverse inequality. It will be enough to show that for every $\gamma \in \mathrm{Lip}_{x,y}$ and every $\varepsilon > 0$ there exists a curve $\gamma_\varepsilon \in \mathrm{Lip}_{x,y}$ transversal to F such that $\varepsilon + \mathbb{L}_{\widetilde{\varphi}}(\gamma) > \mathbb{L}_{\varphi}(\gamma_\varepsilon)$, with x and y arbitrarily chosen in $\overline{\Omega}$. Then, let $\gamma \in \mathrm{Lip}_{x,y}$ and let $A := \{t \in (0,1) : \gamma(t) \in F\}$. Fix $\varepsilon > 0$ and assume $0 < \mathcal{L}^1(A) < 1$, being the other cases trivial. Choose an open set $J \supset A$ in $(0,1)$ such that $\mathcal{L}^1(J \setminus A) < \varepsilon$. The open set J is a countable disjoint union of intervals of the form $J_k := (a_k, b_k)$ with $k \in \mathbb{N}$. Applying Lemma 4.9, we choose, for each $k \in \mathbb{N}$, a curve $\sigma_k : [a_k, b_k] \to \overline{\Omega}$ transversal to F such that $\sigma_k(a_k) = \gamma(a_k)$, $\sigma_k(b_k) = \gamma(b_k)$ and $\|\sigma_k - \gamma\|_{W^{1,\infty}(J_k, \overline{\Omega})} < \varepsilon/2^k$. For each $n \in \mathbb{N}$ let us set:

$$\gamma^n(t) := \begin{cases} \sigma_k(t) & \text{if } t \in \ [a_k, b_k] \text{ for each } k \leq n \\ \gamma(t) & \text{otherwise.} \end{cases} \tag{4.11}$$

Let γ_ε be the curve defined by (4.11) with $n = +\infty$. It is easily seen that $(\gamma^n)_n$ is a Cauchy sequence in $W^{1,\infty}(I, \overline{\Omega})$ and converges to γ_ε, which is therefore Lipschitz too. We claim that γ_ε is the desired curve. Indeed, it connects x and y in $\overline{\Omega}$ and is transversal to F by construction. Moreover we have:

$$\begin{aligned} \int_{J_k} \Big(\varphi(\sigma_k, \dot\sigma_k) - \widetilde{\varphi}(\gamma, \dot\gamma)\Big)\, \mathrm{d}t \ &\leq\ \beta \|\dot\sigma_k\|_\infty \mathcal{L}^1(J_k \setminus A) + \int_{J_k \cap A} \beta\Big(|\dot\sigma_k(t)| - |\dot\gamma(t)|\Big)\, \mathrm{d}t \\ &<\ C\mathcal{L}^1(J_k \setminus A) + \beta \frac{\varepsilon}{2^k}, \end{aligned}$$

where C is a constant depending only on β and $\|\dot\gamma\|_\infty$. Therefore

$$\mathbb{L}_{\varphi}(\gamma_\varepsilon) - \mathbb{L}_{\widetilde{\varphi}}(\gamma) = \sum_{k=1}^{+\infty} \int_{J_k} \Big(\varphi(\sigma_k, \dot\sigma_k) - \widetilde{\varphi}(\gamma, \dot\gamma)\Big)\, \mathrm{d}t < C\mathcal{L}^1(J \setminus A) + \beta\varepsilon < (C + \beta)\varepsilon,$$

and the claim follows. □

Remark 4.10. Let us remark that formula (4.9) is invariant with respect to modifications of the function φ on negligible subsets of $\overline{\Omega}$. Therefore, since $\widetilde{\varphi}(x,\xi) = \varphi(x,\xi)$ for $\mathcal{L}^N$-a.e. $x \in \overline{\Omega}$ and every $\xi \in \mathbb{R}^N$, we also have that $\widetilde{d}_{\widetilde{\varphi}} = \widetilde{d}_{\varphi} = d_{\widetilde{\varphi}}$.

Corollary 4.11. *$\widetilde{\mathcal{D}}_\alpha$ is a proper subset of $\mathcal{D}_\alpha$.*

Proof. Let $\varphi(x,\xi)$ be equal to $\alpha|\xi|$ on a segment Γ contained in Ω and $\beta|\xi|$ elsewhere, and let $d := d_\varphi$ be the distance associated to φ through (1.11). If d belonged to $\widetilde{\mathcal{D}}_\alpha$, by taking into account Theorem 4.8 and Remark 4.10, we would have $d = d_\psi = \widetilde{d}_\psi$ for a function

$\psi \in \mathcal{M}_\alpha$. Proposition 1.15 and the definition of φ_d would imply $\psi(x,\xi) \geq \varphi_d(x,\xi) = \beta|\xi|$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^N$, hence $\psi(x,\xi) = \beta|\xi|$ $\mathcal{L}^N$-a.e. on $\overline{\Omega}$. Then, by Remark 4.10, we would have $d = \widetilde{d}_\psi = \beta d_\Omega$, which is obviously impossible since $d(x,y) = \alpha|x-y|$ if x and y belong to the segment Γ. □

Definitions (1.11) and (4.9) individuate two different ways to derive a distance from a given $\varphi \in \mathcal{M}_\alpha$. In general, we have that $d_\varphi \leq \widetilde{d}_\varphi$, and the inequality may be strict, as shown by the function φ defined in the proof of Corollary 4.11. It seems a difficult task to characterize the functions φ for which equality holds. We therefore restrict to look for sufficient conditions which entail equivalence between the two definitions. The next proposition shows that the upper semicontinuity property of the length functional $\mathbb{L}_\varphi$ plays a role in this issue.

Proposition 4.12. *Let $\varphi \in \mathcal{M}_\alpha$ be such that the length functional $\mathbb{L}_\varphi$ is upper semicontinuous on $W^{1,\infty}(I,\overline{\Omega})$ with respect to the strong topology. Then $d_\varphi = \widetilde{d}_\varphi$.*

Proof. Let F be a Borel negligible subset of $\overline{\Omega}$ satisfying (4.10), according to Theorem 4.8. Fix x and y in $\overline{\Omega}$ and let $\gamma \in \mathrm{Lip}_{x,y}$. By applying Lemma 4.9, we find a sequence of curves $(\gamma_n)_n \subset \mathrm{Lip}_{x,y}$ transversal to F which converges to γ in $W^{1,\infty}(I,\overline{\Omega})$. By the upper semicontinuity of $\mathbb{L}_\varphi$ we get

$$\mathbb{L}_\varphi(\gamma) \geq \limsup_{n\to+\infty} \mathbb{L}_\varphi(\gamma_n) \geq \widetilde{d}_\varphi(x,y).$$

By taking the infimum over all possible curves in $\mathrm{Lip}_{x,y}$ we obtain $d_\varphi(x,y) \geq \widetilde{d}_\varphi(x,y)$, hence the claim. □

We want to understand under which conditions on the function φ the hypothesis of the previous proposition is satisfied.

Proposition 4.13. *Let $\varphi \in \mathcal{M}_\alpha$ be upper semicontinuous in $\overline{\Omega} \times \mathbb{R}^N$. Then the length functional $\mathbb{L}_\varphi$ is upper semicontinuous on $W^{1,\infty}(I,\overline{\Omega})$ with respect to the strong topology. In particular, $d_\varphi = \widetilde{d}_\varphi$.*

Proof. Let $(\gamma_n)_n$ be a sequence in $W^{1,\infty}(I,\overline{\Omega})$ which strongly converges to γ. Using Fatou's Lemma and the upper semicontinuity of φ we get

$$\int_0^1 \varphi(\gamma,\dot\gamma)\,\mathrm{d}t \geq \int_0^1 \limsup_{n\to+\infty} \varphi(\gamma_n,\dot\gamma_n)\,\mathrm{d}t \geq \limsup_{n\to+\infty} \int_0^1 \varphi(\gamma_n,\dot\gamma_n)\,\mathrm{d}t$$

and so the claim. $\square$

Remark 4.14. Propositions 4.12 and 4.13 are an adaptation to our setting of similar results proved in [44].

As a consequence of what seen so far, we obtain the following

Proposition 4.15. *$\widetilde{\mathcal{D}}_\alpha$ is a proper and dense subset of $\mathcal{D}_\alpha$. In particular, it is not closed.*

Proof. Proposition 4.13 implies that $\widetilde{\mathcal{D}}_\alpha$ contains the distances d_φ with $\varphi \in \mathcal{M}_\alpha$ continuous, so the density follows by Theorem 4.2. $\square$

In conclusion, the upper semicontinuity of φ is a sufficient condition to entail equivalence of (1.11) and (4.9) (for instance, the function φ defined in the proof of Corollary 4.11 was lower semicontinuous). On the other hand, it is clear that this condition is far from being optimal: if the set where φ fails to be upper semicontinuous is not too bad, equivalence between (1.11) and (4.9) still holds. A naive example of this situation is given by a function $\varphi(x, \xi)$ of the form $a(x)|\xi|$ with a equal to 2 on $\mathbb{R} \times (0, +\infty)$ and to 1 on $\mathbb{R} \times (-\infty, 0]$. The proposition that follows generalizes this idea.

Proposition 4.16. *Assume that $\overline{\Omega} := \cup_{i=1}^m \overline{\Omega}_i$, where the sets Ω_i are domains with Lipschitz boundaries such that $\overline{\Omega}_i \cap \overline{\Omega}_j = \partial\Omega_i \cap \partial\Omega_j$ if $i \neq j$, and every $x \in \Omega$ belongs to at most two subdomains $\overline{\Omega}_i$. Let $\varphi \in \mathcal{M}_\alpha$ and suppose that φ is upper semicontinuous in each Ω_i. Moreover, let us assume that for every $x \in \cup_{i=1}^m \partial\Omega_i$ there exist an index i_0 and a real number $\rho(x) > 0$ such that $x \in \partial\Omega_{i_0}$ and φ is upper semicontinuous in $\overline{\Omega}_{i_0} \cap B_\rho(x)$. Then $d_\varphi(x, y) = \widetilde{d}_\varphi(x, y)$ on $\overline{\Omega} \times \overline{\Omega}$.*

Proof. Let F be a Borel negligible subset of $\overline{\Omega}$ satisfying (4.10) in Theorem 4.8. It will be enough to show that for every $\gamma \in \mathrm{Lip}_{x,y}$ and every $\varepsilon > 0$ there exists a curve $\gamma_\varepsilon \in \mathrm{Lip}_{x,y}$ transversal to F such that $\mathbb{L}_\varphi(\gamma) + \varepsilon > \mathbb{L}_\varphi(\gamma_\varepsilon)$, with $x, y \in \overline{\Omega}$.

Let us then take a curve $\gamma \in \mathrm{Lip}_{x,y}$ and fix $\varepsilon > 0$. If $\gamma(I)$ is contained in Ω_i for some index i, one can apply Lemma 4.9 with $\Omega := \Omega_i$ and conclude by remarking that $\mathbb{L}_\varphi$ is upper semicontinuous in $W^{1,\infty}(I, \Omega_i)$.

Otherwise, there exists a point $x \in \gamma(I) \cap \bigcup_{i=1}^m \partial\Omega_i$. Up to subdividing $\gamma(I)$ into a finite number of small subarcs, we can assume that γ lies in $B_r(x) \cap \overline{\Omega}$, where $r < \rho(x)$ is a sufficiently small radius. The case of x belonging to $\partial\Omega_i$ for just one index i is easy to deal: for r small enough $B_r(x) \cap \overline{\Omega} = B_r(x) \cap \overline{\Omega}_i$ for some index i and φ is upper semicontinuous in $B_r(x) \cap \overline{\Omega}_i$ by hypothesis, so $\mathbb{L}_\varphi$ is upper semicontinuous in $W^{1,\infty}(I, B_r(x) \cap \overline{\Omega})$ and the claim follows by applying Lemma 4.9 again.

Let us then suppose that x belongs to $\gamma(I) \cap \partial\Omega_i$ for two distinct i. Up to reordering the indexes and to choosing a smaller r, we may suppose $x \in \partial\Omega_1 \cap \partial\Omega_2$, $B_r(x) \subset \Omega$, $B_r(x) \cap \Omega_i = \emptyset$ for each $i \geq 3$ and φ upper semicontinuous in $\overline{\Omega}_1 \cap B_r(x)$. Assume also that r has been chosen so small that there exists a cone $C := \{v \in B_\delta(0) : \langle v, \xi\rangle > \delta|v|\}$ (for suitable $\delta > 0$ and $\xi \in \mathbb{S}^{N-1}$) such that $z + C \subset \Omega_1$ for every $z \in \partial\Omega_1 \cap B_r(x)$. Arguing as in the proof of Lemma 4.9, we can take a sequence $(v_n)_n \subset C$ converging to 0 such that the curves $\gamma_n(t) := \gamma(t) + v_n \sin(\pi t)$ are transversal to F and $\|\gamma - \gamma_n\|_{W^{1,\infty}(I,\overline{\Omega})} \leq 2|v_n|$. Let us set $I_1 := \{t \in I : \gamma(t) \in \overline{\Omega}_1\}$ and $I_2 := \{t \in I : \gamma(t) \in \Omega_2\}$. Notice that, if $\gamma(t) \in \overline{\Omega}_1$, then $\gamma_n(t) := \gamma(t) + v_n \sin(\pi t) \in \overline{\Omega}_1$ for every $n \in \mathbb{N}$, since the translation by the vector $\sin(\pi t)v_n$ has the effect of moving points on $\partial\Omega_1$ inside Ω_1. On the other hand, it is clear that if $\gamma(t) \in \Omega_2$ then $\gamma_n(t) \in \Omega_2$ for n big enough. Therefore, by Fatou's Lemma and taking into account the upper semicontinuity properties enjoyed by φ, we get

$$\begin{aligned}\int_0^1 \varphi(\gamma, \dot{\gamma})\,\mathrm{d}t &= \int_{I_1} \varphi(\gamma, \dot{\gamma})\,\mathrm{d}t + \int_{I_2} \varphi(\gamma, \dot{\gamma})\,\mathrm{d}t \geq \int_{I_1} \limsup_{n\to+\infty} \varphi(\gamma_n, \dot{\gamma}_n)\,\mathrm{d}t \\ &+ \int_{I_2} \limsup_{n\to+\infty} \varphi(\gamma_n, \dot{\gamma}_n)\,\mathrm{d}t \geq \limsup_{n\to+\infty} \int_0^1 \varphi(\gamma_n, \dot{\gamma}_n)\,\mathrm{d}t.\end{aligned}$$

The claim follows by setting $\gamma_\varepsilon := \gamma_n$ for n big enough. $\square$

Chapter 5

Monge solutions for discontinuous Hamiltonians

5.1 Introduction

We consider the Hamilton-Jacobi equation

$$H(x, Du) = 0 \qquad x \in \Omega \subset \mathbb{R}^N, \tag{5.1}$$

where Du is the gradient of the unknown function $u : \Omega \to \mathbb{R}$ and $H : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ is the Hamiltonian. We are concerned with the study of equation (5.1) in the framework of discontinuous Hamiltonians: indeed, H will be assumed to be only Borel-measurable, and quasi-convex in the p-variable for every $x \in \overline{\Omega}$. The interest of this issue is easily motivated by the applications: Hamilton-Jacobi equations with discontinuous ingredients arise naturally in several models, as, for example, propagation of fronts in non-homogeneous media, geometric optics in presence of layers, shape-from-shading problems.

One of the main theory concerning Hamilton-Jacobi equations is that of viscosity solutions, developed in the last twenty years. The literature on this subject is wide, as main reference we recall the books [7], [8] and [63], and the references therein.

With regard to the discontinuous case, measurable fully nonlinear equations of second order have been studied in [25], however the techniques exploited there are based on the strong maximum principle so they do not apply to first order equations.

The first order case has been less studied; we recall, among others (see e.g. [9] and [59]), [28] and [69]. In the first one Camilli and Siconolfi study equation (5.1) and give a notion of viscosity solution making use of suitable measure-theoretic devices. They prove a comparison result, and consequently, when equation (5.1) is coupled with a boundary datum, they get unicity of the solution and an integral representation formula, generalizing the one valid for the continuous case. Moreover, such a solution is proven to be the maximal among Lipschitz subsolutions, in analogy with the classical setting.

In [69], Soravia studies the following Hamilton-Jacobi equation related to optimal control problems

$$\lambda u(x) + \sup_{a \in A}\{-f(x,a)Du(x) - h(x,a)\} = g(x)$$

where g is only Borel-measurable. The viscosity solutions are defined by taking the lower and upper semicontinuous envelopes of g following [58]. Uniqueness and stability results are given.

Both the recalled works start by comparing their definitions with a slightly different one, given by Newcomb and Su in [66]. The authors studied the equation of eikonal type

$$H(Du) = n(x) \tag{5.2}$$

where the discontinuity is in n only, which is assumed to be lower semicontinuous. They introduce the definition of *Monge solution*, which is shown to be consistent with the viscosity notion when n is continuous. In this framework they establish the comparison principle for sub and supersolutions, existence and uniqueness results for (5.2) with Dirichlet boundary conditions, and a stability result.

In this chapter we want to extend this definition to equations of the more general form (5.1) and to generalize to this case the above-mentioned results. In order to be more precise about the type of discontinuities we admit, let us specify that we will deal with Borel-measurable Hamiltonians H such that $Z(x) := \{p \in \mathbb{R}^N \,:\, H(x,p) \le 0\}$ is closed and convex and $\partial Z(x) = \{p \in \mathbb{R}^N \,:\, H(x,p) = 0\}$ for every $x \in \overline{\Omega}$. Moreover, we assume that there exist two positive constants α and β such that $B_\alpha(0) \subset Z(x) \subset B_\beta(0)$ for every $x \in \overline{\Omega}$.

In analogy with [66], we need to recall that the *optical length function* relative to the

Hamiltonian H is the map $S : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ defined as follows:

$$S(x,y) := \inf \Big\{ \int_0^1 \sigma(\gamma(t),\dot\gamma(t))\,\mathrm{d}t \,:\, \gamma \in \mathrm{Lip}\left([0,1],\overline{\Omega}\right),\ \gamma(0)=x,\ \gamma(1)=y \Big\} \tag{5.3}$$

for every $x, y \in \overline{\Omega}$, where σ is the support function of the section $Z(x)$, namely $\sigma(x,\xi) := \sup\{\langle -\xi, p\rangle \,:\, p \in Z(x)\}$. Given $u \in C(\Omega)$, we say that u is a *Monge solution* (resp. *subsolution, supersolution*) of (5.1) in Ω if for each $x_0 \in \Omega$ there holds

$$\liminf_{x \to x_0} \frac{u(x) - u(x_0) + S(x_0,x)}{|x - x_0|} = 0 \quad (\text{resp. } \geq, \leq).$$

As it should be clear by the above definition, the properties of Monge sub and supersolutions strictly depend on those enjoyed by the optical length function S. Note that the function S is nothing else than the geodesic, non-symmetric distance d_σ defined by (1.11). The results of Chapter 1 (and in particular of Section 1.4) are then specialized to S to carry on the study of Monge solutions. With this regard, we underline that the lower semicontinuity of the function n in (5.2) is mainly used in [66] to obtain lower semicontinuity of the length functional $\mathbb{L}_\sigma$ (*cf.* Remark 1.19), and therefore the existence of an optimal path for $S(x,y)$, *i.e.* a path of minimal $\mathbb{L}_\sigma$-length. This technical difficulty is overcome here by introducing the metric length of a curve with respect to the non-symmetric distance S (see (1.1)), which is the relaxed functional of $\mathbb{L}_\sigma$ (in view of Theorem 1.18). The existence of a minimal path (with respect to the metric S-length) for $S(x,y)$ for all $x, y \in \overline{\Omega}$ is then assured by the results of Chapter 1 (*cf.* Proposition 1.20). Consequently, under the above-stated conditions for the Hamiltonian, we obtain a comparison result among Monge sub and supersolutions of equation (5.1) (Theorem 5.8). This implies moreover that, under certain compatibility conditions for the boundary data, the Dirichlet problem

$$\begin{cases} H(x,Du) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{5.4}$$

has a unique Monge solution u, given by Lax formula

$$u(x) := \inf_{y \in \partial\Omega} \{S(x,y) + g(y)\} \quad \text{for all } x \in \overline{\Omega}, \tag{5.5}$$

thus recovering a well known result in the case of a continuous Hamiltonian.

In the continuous case, moreover, the function defined by (5.5) is also the maximal element in the class of Lipschitz subsolutions of (5.4). As already remarked in [66, 69], this is no longer true in general when dealing with Monge solutions of discontinuous Hamilton-Jacobi equations. However, under mild discontinuous assumptions, the previous maximality property still holds. This issue will be investigated in a more detailed way in Section 5.5. As a matter of fact, this will be done by comparing the definition of Monge solution adopted here with that of viscosity solution introduced by Camilli and Siconolfi in [28]. The main difference between the two approaches relies upon the definition of optical length function: while here S is defined by (5.3) through an infimum, the corresponding function L^Ω in [28] is defined through a sup-inf process (*cf.* Section 5.5 for the definition). The latter has the effect of rendering the function L^Ω independent of modifications of the Hamiltonian H (and consequently of the support function σ) on negligible subset of $\overline{\Omega}$ with respect to the x-variable, a property which is necessary if one is interested in keeping the equivalence (holding in the continuous setting, see [7]) between Lipschitz and viscosity subsolutions of (5.1). This in particular gives the maximality of the viscosity solution of (5.4) among Lipschitz subsolutions (*cf.* [28, Proposition 3.6]). Some problems arise instead when one deals with sequences of solutions: in [28, Example 7.2], the authors consider a sequence of continuous Hamilton-Jacobi equations converging to a limit equation for which it is easy to exhibit a corresponding sequence of viscosity solutions (in the classical sense) uniformly converging to a function which is not the viscosity solution, in the sense there considered, of the limit equation (actually, it turns out to be a Monge solution, see Example 5.17). The main reason of this behavior is that the family of distances that can be obtained through such a sup-inf process is not closed for the uniform convergence.

On the other hand, the definition of optical length function given here strictly depends on the pointwise behavior of the Hamiltonian and changing it in the x-variable over negligible sets does count. Moreover, the class of distances obtained through (5.3) is closed for the uniform convergence (in fact, it is compact, *cf.* Theorem 1.21). In particular, with this approach one can treat optimization problems such as

$$\min\left\{\int_\Omega |u_a - f|^2\,\mathrm{d}x \ : \ a:\overline{\Omega}\to[\alpha,\beta] \text{ Borel-measurable}, \ \int_\Omega a(x)\,\mathrm{d}x \le m\right\},$$

where α,β and m are suitable positive constants, $f:\Omega\to\mathbb{R}$ is a given function and u_a is the Monge solution of the following equation, depending on the control a:

$$\begin{cases} |Du| = a(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.6}$$

Indeed, the problem can be attacked using the direct method of the Calculus of Variations: chosen a minimizing sequence $(a_n)_n$, it is easy to see, using the representation formula (5.5) and the recalled compactness result, that the corresponding solutions u_{a_n} converge uniformly to a function u. To show that u is the Monge solution of problem (5.6) for an admissible control a one can refer to the results proved in Chapter 3 (specifically, Theorem 3.6 and Theorem 3.11, *cf.* also Example 5.24).

Throughout all this chapter, Ω will denote a bounded and connected open set of $\mathbb{R}^N$ with Lipschitz boundary.

5.2 Monge solutions: definitions and main properties

In this section we study the main properties of Monge sub and supersolutions for the equation

$$H(x, Du) = 0 \qquad x \in \Omega \subset \mathbb{R}^N. \tag{5.7}$$

We will deal with Hamiltonians H satisfying the following set of assumptions (H):

(H1) $H : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ is Borel-measurable;

(H2) For every $x \in \overline{\Omega}$ the 0-sublevel set

$$Z(x) := \{p \in \mathbb{R}^N \,:\, H(x, p) \le 0\} \tag{5.8}$$

is closed and convex. Moreover $\partial Z(x) = \{p \in \mathbb{R}^N \,:\, H(x, p) = 0\,\}$ for all $x \in \overline{\Omega}$;

(H3) there exist $\alpha, \beta > 0$ such that $B_\alpha(0) \subset Z(x) \subset B_\beta(0)$ for every $x \in \overline{\Omega}$.

We recall the definition of optical length function relative to the the Hamiltonian H, that is the map $S : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$ defined by:

$$S(x, y) := \inf \left\{ \int_0^1 \sigma(\gamma(t), \dot\gamma(t))\, \mathrm{d}t \,:\, \gamma \in \mathrm{Lip}_{x,y} \right\} \tag{5.9}$$

for every $x, y \in \overline{\Omega}$, where σ is the support function of the 0-sublevel set $Z(x)$, namely

$$\sigma(x, \xi) := \sup\left\{\langle -\xi, p\rangle \,:\, p \in Z(x)\,\right\}. \tag{5.10}$$

Note that, when it will be needed, given an Hamiltonian H, we will respectively denote by $Z_H(x), S_H(x, y), \sigma_H(x, \xi)$ the corresponding 0-sublevel set (5.8), optical length function (5.9) and support function (5.10). The definition of Monge solution is given as follows.

Definition 5.1. Let $u \in C(\Omega)$. We say that u is a *Monge solution* (resp. *subsolution, supersolution*) of (5.7) in Ω, if for each $x_0 \in \Omega$ there holds

$$\liminf_{x \to x_0} \frac{u(x) - u(x_0) + S(x_0, x)}{|x - x_0|} = 0 \quad (\text{resp. } \geq, \leq). \tag{5.11}$$

The general results obtained in Chapter 1 are now applied to derive the main properties of the optical length function S. Note that S is indeed the non-symmetric distance d_φ defined in (1.11) with $\varphi := \sigma$. We start by studying the regularity of σ in the following lemma.

Lemma 5.2. *If H is an Hamiltonian satisfying (H), then the function $\sigma : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}_+$ belongs to $\mathcal{M}$ and $\sigma(x, \cdot)$ is convex on $\mathbb{R}^N$ for every $x \in \overline{\Omega}$.*
Moreover

(i) if $H(\cdot, p)$ is upper semicontinuous on $\overline{\Omega}$ for every $p \in \mathbb{R}^N$, then $\sigma(\cdot, \xi)$ is lower semicontinuous on $\overline{\Omega}$ for every $\xi \in \mathbb{R}^N$;

(ii) if $H(\cdot, p)$ is lower semicontinuous on $\overline{\Omega}$ for every $p \in \mathbb{R}^N$, then $\sigma(\cdot, \xi)$ is upper semicontinuous on $\overline{\Omega}$, for every $\xi \in \mathbb{R}^N$.

Proof. In order to prove that $\sigma \in \mathcal{M}$, it will be enough to show σ is Borel-measurable, since all the other properties immediately follow from the definition of σ and assumptions (H). Let $(p_i)_i$ be a countable dense subset of $\mathbb{R}^N$. By (H2) and (H3), it is easily seen that

$$\sigma(x, \xi) = \sup_{i \in \mathbb{N}} \{\langle -\xi, p_i \rangle \ : \ p_i \in Z(x)\} = \sup_{i \in \mathbb{N}} \{\langle -\xi, p_i \rangle \, \chi_{E_i}(x)\} \tag{5.12}$$

where $E_i := \{x \in \overline{\Omega} \ : \ H(x, p_i) < 0 \}$. Notice that, by assumption (H1), E_i is a Borel set, hence each function $(x, \xi) \mapsto \langle -\xi, p_i \rangle \, \chi_{E_i}(x)$ is Borel-measurable and the claim follows. In order to prove (i), we remark that, by assumption (H3), one can replace the functions $\langle -\xi, p_i \rangle \, \chi_{E_i}(x)$ with $(\langle -\xi, p_i \rangle \vee \alpha |\xi|) \, \chi_{E_i}(x)$ in (5.12) without affecting the equality. Then, as E_i is open for every $i \in \mathbb{N}$, each function $x \mapsto (\langle -\xi, p_i \rangle \vee \alpha |\xi|) \, \chi_{E_i}(x)$ is lower semicontinuous for every fixed $\xi \in \mathbb{R}^N$, and so is $\sigma(\cdot, \xi)$. The remainder of the claim easily follows by assumptions (H) and the definition of support function σ. $\square$

Remark 5.3. Comparing Lemma 5.2 with Proposition 1.8, we have that the function S is well-defined. Moreover, it is a non-symmetric geodesic distance such that:

(i) $\alpha|x-y| \leq S(x,y) \leq \beta|x-y|$ locally in Ω;

(ii) S is Lipschitz on $\overline{\Omega} \times \overline{\Omega}$, with Lipschitz constant equal to $2\beta C$, where $C \geq 1$ is a Lipschitz constant for $\partial\Omega$.

In particular, by Proposition 1.20, for every $x, y \in \overline{\Omega}$, there exists a curve $\gamma \in \mathrm{Lip}_{x,y}$ such that $S(x,y) = \mathrm{L}_S(\gamma)$, where $\mathrm{L}_S(\gamma)$ is the length of the curve γ defined according to (1.1) for the non-symmetric distance S.

We want to show now that the definitions of Monge sub and supersolution are consistent with those given in the viscosity sense in the classical setting of a continuous Hamiltonian.

Definition 5.4. A function $u \in C(\Omega)$ is a *viscosity subsolution* of (5.7) in Ω if

$$H(x_0, q) \leq 0 \quad \text{for every } x_0 \in \Omega \text{ and every } q \in D^+u(x_0).$$

Similarly, $u \in C(\Omega)$ is a *viscosity supersolution* of (5.7) in Ω if

$$H(x_0, q) \geq 0 \quad \text{for every } x_0 \in \Omega \text{ and every } q \in D^-u(x_0).$$

We say that $u \in C(\Omega)$ is a *viscosity solution* of (5.7) in Ω if it is both a subsolution and a supersolution in the viscosity sense. Here we have denoted by $D^+u(x_0)$ and $D^-u(x_0)$ the classical superdifferential and subdifferential of u at x_0.

Proposition 5.5. *Let H be a continuous Hamiltonian satisfying (H). Then $v \in C(\Omega)$ is a Monge supersolution (resp. subsolution) of (5.7) if and only if it is a viscosity supersolution (resp. subsolution) of (5.7).*

Proof. To prove that any viscosity supersolution (resp. subsolution) in $C(\Omega)$ is a Monge supersolution (resp. subsolution), one can argue as in [66].

Conversely, let $v \in C(\Omega)$ be a Monge supersolution. Let $x_0 \in \Omega$ and $q \in D^-v(x_0)$. By definition we have

$$0 \geq \liminf_{x \to x_0} \frac{v(x) - v(x_0) + S(x_0, x)}{|x - x_0|} \geq \liminf_{x \to x_0} \left(\langle q, \frac{x - x_0}{|x - x_0|} \rangle + \frac{S(x_0, x)}{|x - x_0|} \right). \tag{5.13}$$

Let $(x_n)_n$ be a minimizing sequence for the most right-hand side of (5.13). We set

$$\xi_n := \frac{x_n - x_0}{|x_n - x_0|}, \qquad t_n := |x_n - x_0|.$$

Up to subsequences, we have that $\xi_n \to \xi \in \mathbb{S}^{N-1}$. Moreover

$$\liminf_{n\to+\infty} \frac{S(x_0, x_0 + t_n\xi_n)}{t_n} = \liminf_{n\to+\infty} \frac{S(x_0, x_0 + t_n\xi)}{t_n} \geq \sigma(x_0, \xi).$$

Indeed, the first equality comes from

$$\left| \frac{S(x_0, x_0 + t_n\xi_n) - S(x_0, x_0 + t_n\xi)}{t_n} \right| \leq \beta|\xi_n - \xi|,$$

while the second follows by the continuity of H (and therefore of σ by Lemma 5.2) and Proposition 1.15 (ii). Therefore by (5.13) we obtain

$$0 \geq \lim_{n\to+\infty} \left(\langle q, \xi_n \rangle + \frac{S(x_0, x_0 + t_n\xi_n)}{t_n} \right) \geq \langle q, \xi \rangle + \sigma(x_0, \xi), \tag{5.14}$$

that is $\langle -\xi, q \rangle \geq \sigma(x_0, \xi) = \sup \{ \langle -\xi, p \rangle : p \in Z(x_0) \}$. By Hahn-Banach theorem we get that $H(x_0, q) \geq 0$.

Let $v \in C(\Omega)$ be a Monge subsolution. Let $x_0 \in \Omega$ and $q \in D^+v(x_0)$. We have

$$0 \leq \liminf_{x\to x_0} \frac{v(x) - v(x_0) + S(x_0, x)}{|x - x_0|} \leq \limsup_{x\to x_0} \left(\langle q, \frac{x - x_0}{|x - x_0|} \rangle + \frac{S(x_0, x)}{|x - x_0|} \right). \tag{5.15}$$

If it were $H(x_0, q) > 0$, by Hahn-Banach theorem there would exist a vector $\xi \in \mathbb{S}^{N-1}$ such that $\langle -\xi, q \rangle > \sup \{ \langle -\xi, p \rangle : p \in Z(x_0) \} = \sigma(x_0, \xi)$. But that is impossible, since, by taking the sequence $x_n = x_0 + t_n\xi$ with $t_n = 1/n$, from inequality (5.15) and Proposition 1.15 (i) we get

$$0 \leq \langle q, \xi \rangle + \limsup_{n\to+\infty} \frac{S(x_0, x_0 + t_n\xi)}{t_n} \leq \langle q, \xi \rangle + \sigma(x_0, \xi). \tag{5.16}$$

$\square$

In the measurable setting, the following pointwise description of the behavior of Monge sub and supersolutions holds.

Proposition 5.6. *Let v be a Lipschitz function in Ω and H satisfy (H).*

(i) If v is a Monge subsolution of (5.7), then it is a Lipschitz subsolution*, i.e.*

$$H(x, Dv(x)) \leq 0 \quad \textit{for } \mathcal{L}^N\textit{-a.e. } x \in \Omega.$$

(ii) If $\sigma(\cdot,\xi)$ is lower semicontinuous for every $\xi \in \mathbb{R}^N$ and v is a Monge supersolution of (5.7), then it is a Lipschitz supersolution, i.e.

$$H(x, Dv(x)) \geq 0 \quad \textit{for } \mathcal{L}^N\textit{-a.e. } x \in \Omega.$$

In particular, a Monge solution is a Lipschitz solution, i.e. *it solves (5.7) almost everywhere in Ω.*

For the proof, the reader may follow word by word that of Proposition 5.5, using Proposition 1.15 instead of the continuity of the support function σ.

The next proposition says that any Monge subsolution is locally 1-Lipschitz continuous with respect to the non-symmetric distance S (*cf.* [66, Lemma 3.1]).

Proposition 5.7. *Let H be an Hamiltonian satisfying (H) and $u \in C(\Omega)$ be a Monge subsolution of (5.7). Then u is locally Lipschitz in Ω, with* $\mathrm{Lip}(u,\Omega) \leq \beta$*. Moreover, for every $x_0 \in \Omega$ there exists an $r > 0$, depending only on* $\mathrm{dist}(x_0, \partial\Omega)$*, α, β, such that*

$$u(x) - u(y) \leq S(x,y) \quad \textit{for every } x, y \in B_r(x_0). \tag{5.17}$$

Proof. First remark that the function u is Lipschitz continuous on Ω. Indeed, by the fact that u is a Monge subsolution and Remark 5.3, we have that u is a Monge subsolution of $|Dv| = \beta$, hence a (classical) viscosity subsolution. That gives $\mathrm{Lip}(u,\Omega) \leq \beta$. Now, fix a point $x_0 \in \Omega$. Then we can choose an $r > 0$ small enough so that every optimal path for $S(x,y)$ with $x, y \in B_r(x_0)$ lies inside Ω. Observe that r is only dependent on $\mathrm{dist}(x_0, \partial\Omega)$, α, β (*cf.* Remark 5.3). Fix $x, y \in B_r(x_0)$ and take an optimal path $\gamma \in \mathrm{Lip}_{x,y}$ for $S(x,y)$. By Remark 5.3 the function $f(t) := S(x, \gamma(t))$ is Lipschitz continuous. Therefore the function $u\circ\gamma(t) + f(t)$ is Lipschitz continuous and we can compute its derivative for $\mathcal{L}^1$-a.e. $t \in I$. We have then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\big(u\circ\gamma + f\big)(t) &= \lim_{s\to t^+} \frac{u(\gamma(s)) - u(\gamma(t)) + S(x,\gamma(s)) - S(x,\gamma(t))}{s-t} \\ &= |\dot\gamma(t)| \lim_{s\to t^+} \frac{u(\gamma(s)) - u(\gamma(t)) + S(\gamma(t),\gamma(s))}{|\gamma(s)-\gamma(t)|} \geq 0 \end{aligned}$$

for $\mathcal{L}^1$-a.e. $t \in I$, where we have used the optimality of γ and the definition of Monge subsolution. By integrating the above inequality we get (5.17), that is the claim. $\square$

5.3 The comparison result and solvability of the Dirichlet problem

Our comparison result is stated as follows.

Theorem 5.8 (Comparison Theorem). *Let H be an Hamiltonian satisfying (H) and let $u, v \in C(\overline{\Omega})$ be, respectively, a Monge subsolution and a Monge supersolution of (5.7) in Ω. If $u \leq v$ on $\partial\Omega$ then $u \leq v$ in Ω.*

The proof is based on the following Lemma (*cf.* [66, Lemma 3.2]).

Lemma 5.9. *Let H and K be two Hamiltonians satisfying (H) and suppose that there exists a $\delta \in (0, 1)$ such that*

$$Z_K(x) \subseteq \delta Z_H(x) \qquad \forall x \in \overline{\Omega}. \tag{5.18}$$

Let $u \in C(\overline{\Omega})$ be a Monge subsolution of $K(x, Du) = 0$ and $v \in C(\overline{\Omega})$ be a Monge supersolution of $H(x, Dv) = 0$. Then $u \leq v$ on $\partial\Omega$ implies $u \leq v$ in Ω.

Proof. Consider the function

$$f(x, y) := u(x) - v(y) - \frac{S_H(x, y)^2}{\varepsilon}.$$

By the continuity of S_H we have that $f \in C(\overline{\Omega} \times \overline{\Omega})$. Let us argue by contradiction and suppose that the claim is false. Hence that implies, for an $\varepsilon > 0$ sufficiently small, the existence of a point $(x_\varepsilon, y_\varepsilon) \in \Omega \times \Omega$ where f reaches its maximum. Choose an optimal path $\gamma \in \mathrm{Lip}_{x_\varepsilon, y_\varepsilon}$ and set $h(t) := \big(S_H(x_\varepsilon, y_\varepsilon) + S_H(\gamma(t), y_\varepsilon)\big)/\varepsilon$. We claim that $h(0) \leq \delta < 1$. We may as well suppose that $x_\varepsilon \neq y_\varepsilon$, otherwise the statement is trivial. The function $f(\cdot, y_\varepsilon)$ has a local maximum in x_ε, therefore for t small enough we have $f(x_\varepsilon, y_\varepsilon) \geq f(\gamma(t), y_\varepsilon)$, *i.e.*

$$u(x_\varepsilon) - u(\gamma(t)) \geq \frac{1}{\varepsilon}\left(S_H(x_\varepsilon, y_\varepsilon)^2 - S_H(\gamma(t), y_\varepsilon)^2\right) = h(t) S_H(x_\varepsilon, \gamma(t)).$$

Since u is a subsolution of $K(x, Du) = 0$ we can apply (5.17) of Proposition 5.7 and get $S_K(x_\varepsilon, \gamma(t)) \geq h(t) S_H(x_\varepsilon, \gamma(t))$. Assumption (5.18) clearly gives us $S_K(x, y) \leq \delta S_H(x, y)$ for every $x, y \in \overline{\Omega}$, so we are lead to $\delta S_H(x_\varepsilon, \gamma(t)) \geq h(t) S_H(x_\varepsilon, \gamma(t))$ which clearly gives the claim.

Now $f(x_\varepsilon, \cdot)$ has a local maximum in y_ε, so, for y close enough to y_ε, we have

$$\begin{aligned} v(y_\varepsilon) - v(y) &\le \frac{S_H(x_\varepsilon, y) + S_H(x_\varepsilon, y_\varepsilon)}{\varepsilon}\Big(S_H(x_\varepsilon, y) - S_H(x_\varepsilon, y_\varepsilon)\Big) \\ &\le \frac{S_H(x_\varepsilon, y) + S_H(x_\varepsilon, y_\varepsilon)}{\varepsilon} S_H(y_\varepsilon, y) < \frac{1+\delta}{2} S_H(y_\varepsilon, y), \end{aligned}$$

that is

$$\frac{v(y) - v(y_\varepsilon) + S_H(y_\varepsilon, y)}{|y - y_\varepsilon|} \ge \left(\frac{1-\delta}{2}\right)\frac{S_H(y_\varepsilon, y)}{|y - y_\varepsilon|} \ge \alpha\,\frac{1-\delta}{2} > 0,$$

which is clearly in contradiction with the fact that v is a Monge supersolution. $\square$

Proof of Theorem 5.8: Up to replacing u with $u + c$ and v with $v + c$ for a positive constant c large enough, we may as well suppose that u is positive on $\overline{\Omega}$. Let $\delta \in (0, 1)$ and $H_\delta(x, p) := H(x, p/\delta)$. Notice that $Z_{H_\delta}(x) = \delta Z_H(x)$ for every $x \in \overline{\Omega}$, therefore $S_{H_\delta} = \delta S_H$. In particular, this implies that δu is a Monge subsolution of $H_\delta(x, Dw) = 0$. Moreover, $\delta u \le u \le v$ on $\partial\Omega$, so we can apply Lemma 5.9 with $K := H_\delta$ to obtain that $\delta u \le v$ in Ω. The claim then follows by letting δ increase to 1. $\square$

We address now our attention to the Dirichlet problem

$$\begin{cases} H(x, Du) = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{5.19}$$

More precisely, we will prove that the function u given by the Lax formula

$$u(x) := \inf_{y \in \partial\Omega} \{S(x, y) + g(y)\} \quad \text{for } x \in \overline{\Omega}, \tag{5.20}$$

is a Monge solution of the Dirichlet problem (5.19) according to the following definition.

Definition 5.10. We will say that a function $u \in C(\overline{\Omega})$ is a *Monge solution of the Dirichlet problem* (5.19) if it is a Monge solution of equation $H(x, Du) = 0$ in Ω and $u(x) = g(x)$ for each $x \in \partial\Omega$.

Our result is the following.

Theorem 5.11 (Solvability of the Dirichlet Problem). *Let H be an Hamiltonian satisfying (H) and assume that the boundary datum $g : \partial\Omega \to \mathbb{R}$ satisfies the compatibility condition*

$$g(x) - g(y) \le S(x, y) \qquad \text{for every } x, y \in \partial\Omega. \tag{5.21}$$

The function u *given by the Lax formula (5.20) is the unique Monge solution of the Dirichlet problem (5.19). Moreover,* u *is the maximal element of the set*

$$\mathcal{S}_M := \{v \in C(\overline{\Omega}) \, : \, v \textit{ Monge subsolution of (5.7) in } \Omega,\ v \leq g \quad \textit{on } \partial\Omega\,\}. \tag{5.22}$$

The result that follows is preliminary to the proof of the theorem and underlines that most of the properties enjoyed by the function u defined by (5.20) do not depend in fact on the compatibility condition (5.21).

Proposition 5.12. *Let* H *be an Hamiltonian satisfying (H) and* $g : \partial\Omega \to \mathbb{R}$ *be a function bounded from below. The function* u *defined by (5.20) is Lipschitz continuous on* $\overline{\Omega}$*. Moreover,* u *is a Monge solution of (5.7) in* Ω*.*

Proof. As g is bounded from below, u is well defined on $\overline{\Omega}$ by formula (5.20). One can verify, by definition, that $|u(x) - u(y)| \leq \max\{S(x,y), S(y,x)\}$ on $\overline{\Omega} \times \overline{\Omega}$, therefore u is Lipschitz continuous on $\overline{\Omega}$ (*cf.* Remark 5.3), in particular it is of class $C(\overline{\Omega})$.
To show that u is a Monge subsolution, fix $x_0 \in \Omega$ and an arbitrary sequence $(x_n)_n$ in Ω which converges to x_0. For every $n \in \mathbb{N}$ choose a point $y_n \in \partial\Omega$ such that $u(x_n) \geq S(x_n, y_n) + g(y_n) - o(|x_0 - x_n|)$. Then

$$u(x_n) + S(x_0, x_n) \geq S(x_0, y_n) + g(y_n) - o(|x - x_n|) \geq u(x_0) - o(|x - x_n|)$$

and, by taking the liminf as n goes to $+\infty$ in the above expression, we conclude that u is a Monge subsolution of (5.7) by the arbitrariness of $(x_n)_n$.
Let us prove that u is a Monge supersolution. Fix $x_0 \in \Omega$ and, for $n \in \mathbb{N}$ big enough, consider the ball $B_{1/n}(x_0) \subset \Omega$. Choose an $y_n \in \partial\Omega$ such that $u(x_0) \geq S(x_0, y_n) + g(y_n) - 1/n^2$. Let $\gamma_n \in \mathrm{Lip}_{x_0, y_n}$ be an optimal path for $S(x_0, y_n)$ and take a point $z_n \in \gamma_n(I) \cap \partial B_{1/n}(x_0)$. By definition we have that $u(z_n) \leq S(z_n, y_n) + g(y_n)$. Hence, using also the optimality of γ_n, we have

$$u(z_n) - u(x_0) \leq S(z_n, y_n) - S(x_0, y_n) + 1/n^2 = -S(x_0, z_n) + 1/n^2.$$

This implies

$$\liminf_{n \to +\infty} \frac{u(z_n) - u(x_0) + S(x_0, z_n)}{|z_n - x_0|} \leq \liminf_{n \to +\infty} \frac{1}{n} = 0,$$

which obviously implies that u is a Monge supersolution. □

Proof of Theorem 5.11. Uniqueness in the class $C(\overline{\Omega})$ is a consequence of the Comparison Theorem. By Proposition 5.12 we have that the function u defined by (5.20) is Lipschitz continuous on $\overline{\Omega}$, in particular of class $C(\overline{\Omega})$, and is a Monge solution of (5.7) in Ω. We have, by definition, that $u(x) \leq g(x)$ for every $x \in \partial\Omega$ (just choose $y = x$ in formula (5.20)), while the opposite inequality holds by the compatibility condition (5.21). Hence $u = g$ on $\partial\Omega$, therefore u is the unique solution of class $C(\overline{\Omega})$ of the Dirichlet problem (5.19). Last, the maximality of u in the set $\mathcal{S}_M$ easily follows from Theorem 5.8. $\square$

5.4 The stability result

We start this section by introducing a suitable convergence on Hamiltonians under which we will prove a stability result for Monge solutions.

Definition 5.13. Let $(H_n)_n$, H be Hamiltonians satisfying assumptions (H) and $(S_n)_n$ and S be the relative optical length functions defined according to (5.9). We say that H_n *τ-converges to H* and write $H_n \xrightarrow{\tau} H$ if $(S_n)_n$ converges uniformly to S on $\overline{\Omega} \times \overline{\Omega}$.

Remark 5.14. Note that the convergence of the Hamiltonians above defined is equivalent, by Theorem 1.21, to the Γ-convergence of the length functions $(\mathrm{L}_{S_n})_n$ to the length functions L_S with respect to the uniform convergence of paths. This, in fact, mainly motivate our definition.

Since our definition does not give a condition one can check on the sequence $(H_n)_n$, we will see, in the next proposition, which conditions on the Hamiltonians imply $H_n \xrightarrow{\tau} H$.

Proposition 5.15. *Let the Hamiltonians $H, (H_n)_n$ satisfy (H). Then $H_n \xrightarrow{\tau} H$ if one of the following conditions holds:*

(i) For each $n \in \mathbb{N}$ and $p \in B_\beta(0)$ the function $H_n(\cdot, p)$ is upper semicontinuous on $\overline{\Omega}$ and $(H_n)_n$ converge decreasingly to H on $\overline{\Omega} \times B_\beta(0)$.

(ii) $(H_n)_n$ converges uniformly to H on $\overline{\Omega} \times B_\beta(0)$.

(iii) $(H_n)_n$ converges increasingly to H on $\overline{\Omega} \times B_\beta(0)$.

Proof. By Definition 5.13 the claim will be proved if we show that $(S_n)_n$ converge uniformly to S in $\overline{\Omega} \times \overline{\Omega}$. This easily follows by applying Proposition 1.24 with $\varphi := \sigma$ and $\varphi_n := \sigma_n$ for each $n \in \mathbb{N}$. Indeed hypothesis (i), (ii), and (iii) respectively imply (i), (ii), and (iii) in Proposition 1.24 (to obtain (i) we also used Lemma 5.2), and then we can conclude that the distances associated to σ_n, *i.e.* S_n, converge uniformly to the distance associated to σ, *i.e.* S. □

We are now ready to show our stability result.

Theorem 5.16 (Stability Theorem). *Fix $\alpha, \beta > 0$ and let us consider an Hamiltonian H and a sequence of Hamiltonians $(H_n)_n$ satisfying assumption (H) for every $n \in \mathbb{N}$. Suppose that:*

1. $H_n \xrightarrow{\tau} H$ *as* $n \to \infty$,
2. $u_n \in C(\Omega)$ *is a Monge solution of* $H_n(x, Du_n) = 0$ *in* Ω *for each* $n \in \mathbb{N}$;
3. *the sequence* $(u_n)_n$ *converges uniformly to* $u \in C(\Omega)$ *on compact subsets of* Ω.

Then u is a Monge solution of $H(x, Du) = 0$ in Ω.

Proof. Fix a point $x_0 \in \Omega$. By Proposition 5.7, there exists an $r > 0$ independent of n such that (5.17) holds for each S_n. Therefore we have

$$u_n(x) = \inf_{y \in \partial B_r(x_0)} \{S_n(x, y) + u_n(y)\} \quad \text{for every } x \in B_r(x_0). \tag{5.23}$$

By Definition 5.13 $(S_n)_n$ converge uniformly to S on $\overline{\Omega} \times \overline{\Omega}$ and, by hypothesis 3, u_n converge uniformly to u in $B_r(x_0)$, thus, letting $n \to \infty$ in (5.23) we obtain

$$u(x) = \inf_{y \in \partial B_r(x_0)} \{S(x, y) + u(y)\} \quad \text{for every } x \in B_r(x_0).$$

So, by Theorem 5.11, u is a Monge solution of $H(x, Du) = 0$ in $B_r(x_0)$. The claim then follows since (5.11) is a local property and $x_0 \in \Omega$ was arbitrary. □

We end this section describing an example already studied in [28, Example 7.2]. We observe that, with our definitions, a stability result holds, while this is not obtained in [28], as stressed by the authors. Note that the difference is in the definition of the optical length function: indeed, we both consider the same discontinuous Hamiltonian H which is the pointwise limit of a sequence of continuous ones $(H_n)_n$, but while, using our definition, the corresponding optical length functions S_n converge uniformly to the optical length function S corresponding to H, with their definition (*cf.* also Section 5.5) the sequence $(L_n^\Omega)_n$ does not converge to L^Ω (note that $S_n = L_n^\Omega$ for each $n \in \mathbb{N}$ as H_n are continuous, *cf.* Theorem 5.20).

Example 5.17. Let $\Omega := (0,1) \times (-2,2)$ and consider a sequence of continuous functions $a_n : \overline{\Omega} \to \mathbb{R}$ defined by

$$a_n(x_1, x_2) := \begin{cases} 1 & \text{if } |x_2| \geq 1/n \\ 1/2 + |x_2|n/2 & \text{otherwise.} \end{cases}$$

The functions a_n converge increasingly to the function $a(x) := \chi_{\overline{\Omega}}(x) - 1/2\chi_\Gamma(x)$ pointwise on $\overline{\Omega} \times \mathbb{R}^N$, where Γ is the x_1-axis $\mathbb{R} \times \{0\}$. Let us define the Hamiltonians $H_n(x,p) := |p| - a_n(x)$ and $H(x,p) := |p| - a(x)$. Obviously, $(H_n)_n$ and H satisfy assumptions (H) with, for instance, $\alpha := 1/2$ and $\beta := 1$. By Proposition 5.15 (i), we immediately have that $H_n \xrightarrow{\tau} H$, therefore the Stability Theorem holds. In particular, if g is a continuous function on $\partial\Omega$ satisfying the compatibility condition (5.21) for H and H_n for each $n \in \mathbb{N}$ (take, for instance $g(x) := 2|x|$ for $x \in \partial\Omega$), then the Monge solutions u_n of the Dirichlet problems

$$\begin{cases} |Dv| = a_n(x) & \text{in } \Omega \\ v = g & \text{on } \partial\Omega \end{cases}$$

are classical viscosity solutions (as the Hamiltonians H_n are continuous) and converge uniformly on $\overline{\Omega} \times \overline{\Omega}$ to a function u which is the unique Monge solution of

$$\begin{cases} |Dv| = a(x) & \text{in } \Omega \\ v = g & \text{on } \partial\Omega. \end{cases}$$

5.5 Pointwise behavior of Monge subsolutions

In this section we will study the pointwise properties enjoyed by the Monge subsolutions of problem (5.19) and the relation between Monge and Lipschitz subsolutions, in particular we are interested in investigating maximality properties of the function u defined by the Lax formula (5.20).

We recall that a function $v : \overline{\Omega} \to \mathbb{R}$ is said to be a *Lipschitz subsolution* of the Dirichlet problem (5.19) if $v \in W^{1,\infty}(\Omega)$, $H(x, Dv(x)) \leq 0$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ and $v \leq g$ on $\partial\Omega$. It is well known that in the classical context of a continuous Hamiltonian H the function u defined in (5.20) is the maximum element of the set

$$\mathcal{S}_P := \{\, v \in W^{1,\infty}(\Omega) \;:\; H(x, Dv(x)) \leq 0 \;\mathcal{L}^N\text{-a.e. in } \Omega,\; v \leq g \text{ on } \partial\Omega \,\}$$

of Lipschitz subsolutions of (5.19). We wonder if this maximality property is maintained when the Hamiltonian H satisfies the more general hypotheses (H). Indeed, by Proposition 5.12, the function u is a Lipschitz continuous Monge solution of (5.7), therefore is a Lipschitz subsolution of (5.19), by Proposition 5.6. But in general it is not the maximum element of $\mathcal{S}_P$, not even in the case of a boundary datum g satisfying the compatibility condition (5.21), as the following example shows.

Example 5.18. Let $\Omega := (0,1) \times (-1,1)$ and let $H(x,p) := |p| - a(x)$, where $a(x) := 2\chi_{\overline{\Omega}}(x) - \chi_\Gamma(x)$ and Γ denotes the x_1-axis $\mathbb{R} \times \{0\}$. Let $v(x_1, x_2) := 1/2|x_2| + 3/2|x_1|$. Then the inequality $H(x, Dv) < 0$ holds true for every differentiability point of v in Ω. Let u be the function given by formula (5.20) with $g := v_{|\partial\Omega}$. Observe that g satisfy the compatibility condition (5.21). Nevertheless, we have $u(x_1, 0) = S\big((x_1,0),(0,0)\big) = |x_1| < 3/2|x_1| = v(x_1, 0)$. Hence, u is not the maximum element of $\mathcal{S}_P$.

Therefore we are led to seek for sufficient conditions which guarantee the maximality of the function u among all Lipschitz subsolution of (5.19).
Let H be an Hamiltonian fulfilling assumptions (H). Following the approach of Camilli and Siconolfi in [28], we define a slightly different optical length function:

$$L^\Omega(x,y) := \sup_{\mathcal{L}^N(E)=0} \left\{ \inf \left\{ \int_0^1 \sigma(\gamma(t), \dot\gamma(t))\,\mathrm{d}t \;:\; \gamma \in \mathrm{Lip}_{x,y},\; \gamma \text{ transversal to } E \right\} \right\}$$

for every $x, y \in \overline{\Omega}$. We remark that L^Ω is nothing else that the distance $\widetilde{d_\sigma}$ defined according to (4.9). The following result holds (*cf.* [28]).

Theorem 5.19. *Let H be an Hamiltonian satisfying (H). Assume that $g : \partial\Omega \to \mathbb{R}$ is a function bounded from below and that $S(x, y) = L^{\Omega}(x, y)$ for every $x, y \in \overline{\Omega}$. Then any Lipschitz subsolution of (5.19) is a Monge subsolution. Moreover, the function u defined by Lax formula (5.20) is maximal in $\mathcal{S}_P$.*

The previous theorem gives a first answer to the question raised before. Unfortunately, the above condition, stated in terms of equality of the optical length functions S and L^{Ω}, is quite indirect. In order to derive conditions on the Hamiltonian, we now use the results obtained in Section 4.5. The next theorem will indeed follow quite easily from Proposition 4.16. We remark that our result is more general than those obtained by Newcomb and Su [66, Theorem 5.4] and by Soravia [69, Theorem 4.7]: indeed, the Hamiltonian H is not assumed to be piecewise constant in the x-variable near the interface of two contiguous subdomains.

Theorem 5.20. *Assume that $\overline{\Omega} := \cup_{i=1}^{m} \overline{\Omega}_i$, where the sets Ω_i are bounded domains with Lipschitz boundaries such that $\overline{\Omega}_i \cap \overline{\Omega}_j = \partial\Omega_i \cap \partial\Omega_j$ if $i \neq j$, and every $x \in \Omega$ belongs to at most two subdomains $\overline{\Omega}_i$.*
Let H be an Hamiltonian satisfying (H) and lower semicontinuous in $\Omega_i \times \mathbb{R}^N$ for each i. Moreover, assume that for every $x \in \cup_{i=1}^{m} \partial\Omega_i$ there exist an index i_0 and a real number $\rho > 0$ such that $x \in \partial\Omega_{i_0}$ and H is lower semicontinuous in $\overline{\Omega}_{i_0} \cap B_\rho(x)$.
Then $S(x, y) = L^{\Omega}(x, y)$ for every $x, y \in \overline{\Omega}$. In particular, the claim of Theorem 5.19 holds.

Proof. The claim directly follows by applying Proposition 4.16 with $\varphi := \sigma$ (as $S = d_\sigma$ and $L^{\Omega} = \widetilde{d}_\sigma$). Since the hypotheses on Ω are satisfied, we only have to check those on σ. Since $\sigma(x, \cdot)$ is convex on $\mathbb{R}^N$ for every $x \in \overline{\Omega}$, when checking the upper semicontinuity properties of σ, we can reduce to consider the function $\sigma(\cdot, \xi)$ for every fixed $\xi \in \mathbb{R}^N$. Now, it is easy to prove that $\sigma(\cdot, \xi)$ is upper semicontinuous on X if H is lower semicontinuous on $X \times \mathbb{R}^N$, being X a subspace of $\mathbb{R}^N$ and ξ a fixed vector in $\mathbb{R}^N$. This argument, applied with $X := \Omega_i$ and $X := \overline{\Omega}_{i_0} \cap B_\rho(x)$ with x, i_o and ρ as in the statement of the theorem, shows that the assumptions of Proposition 4.16 are fulfilled. $\square$

Another question that could be raised is whether the last part of the claim of Theorem 5.11 is still true even when the g does not satisfy the compatibility condition (5.21), that is we wonder if the function u defined by (5.20) is the maximum element of the set $\mathcal{S}_M$ for a generic boundary datum. The following example shows that such a maximality property can not be expected in general.

Example 5.21. Let $\Omega := (0,1) \times (0,1)$ and let $H(x,p) := |p| - a(x)$, $K(x,p) := |p| - b(x)$, where $a(x) := \chi_{\overline{\Omega}}(x) + \chi_{\Omega}(x)$ and $b(x) := 2\chi_{\overline{\Omega}}(x)$. Notice that $S_K(x,y) = 2|x-y|$ and that $S_H = S_K$ in a suitable neighborhood of every point of Ω. Let $g(x) := 2|x|$ and set, for every $x \in \overline{\Omega}$,

$$u(x) := \inf_{y \in \partial\Omega} \{S_H(x,y) + g(y)\}, \qquad v(x) := \inf_{y \in \partial\Omega} \{S_K(x,y) + g(y)\}.$$

Notice that g satisfies the compatibility condition (5.21) with respect to the Hamiltonian K (but not with respect to H). In particular, that implies $v = g$ on $\partial\Omega$. By Proposition 5.12, u and v are a Monge solutions (in particular, Monge subsolutions) of equation (5.7) with Hamiltonian H and K respectively. Moreover, since $S_H = S_K$ locally in Ω and (5.11) is a local property, we have that v is a Monge subsolution with respect to H too. Let us show now that u is not greater than v, *i.e.* that there exists a point $x_0 \in \overline{\Omega}$ such that $u(x_0) < v(x_0)$. To this aim, take $x_0 := (1/2, 0)$. Indeed, $v(x_0) = g(x_0) = 1$, while $u(x_0) \leq S_H(x_0, 0) + g(0) = 1/2$.

We look for conditions sufficient to guarantee the maximality in $\mathcal{S}_M$ of the function u defined in (5.20). A sufficient condition we found is that the optical length function S defined in (5.9) can be obtained by taking the infimum only over those curves in $\mathrm{Lip}_{x,y}$ which lie in the interior of Ω, possibly except for their endpoints. Note the this condition is not true in general, as can be easily seen by considering S_H in Example 5.21.

Theorem 5.22. *Let H be an Hamiltonian satisfying (H). If, for every $x, y \in \overline{\Omega}$,*

$$S(x,y) = \inf \Big\{ \int_0^1 \sigma(\gamma(t), \dot\gamma(t))\, \mathrm{d}t \ : \gamma \in \mathrm{Lip}_{x,y},\ \gamma(t) \in \Omega \textit{ for all } t \in (0,1) \Big\}, \tag{5.24}$$

then u defined by (5.20) is maximal in $\mathcal{S}_M$.

Proof. Let γ be a curve in $\mathrm{Lip}_{x,y}$ such that $\gamma(t) \in \Omega$ for all $t \in (0,1)$ and let $v \in \mathcal{S}_M$. For a fixed positive $\delta < 1/2$, let $\Gamma_\delta := \gamma([\delta, 1-\delta])$. The set Γ_δ is compact and contained in Ω, therefore, by Proposition 5.7, we may find a finite partition $\delta = t_0 < t_1 < .. < t_m = 1-\delta$ such that $v(\gamma(t_i)) - v(\gamma(t_{i+1})) \le S(\gamma(t_i), \gamma(t_{i+1}))$ for each i. Therefore

$$v(\gamma(\delta)) - v(\gamma(1-\delta)) \le \sum_{i=0}^{m-1} S(\gamma(t_i), \gamma(t_{i+1})) \le \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \sigma(\gamma, \dot\gamma)\, \mathrm{d}t. \tag{5.25}$$

By letting δ go to 0 and by taking the infimum of (5.25) over all curves $\gamma \in \mathrm{Lip}_{x,y}$ with $\gamma(t) \in \Omega$ for all $t \in (0,1)$, we obtain, in view of assumption (5.24) and the continuity of v, that

$$v(x) - v(y) \le S(x,y).$$

In particular the above inequality is true for every $y \in \partial\Omega$, therefore, recalling also that $v \le g$ on $\partial\Omega$, we have

$$v(x) \le \inf_{y \in \partial\Omega} \{S(x,y) + g(y)\},$$

which gives the claim. □

5.6 Examples

We conclude by discussing some examples. Before going on, we introduce some preliminary notation. Given a closed subset C of $\mathbb{R}^N$, we will denote by $\mathrm{dist}^{\#}(x, C)$ the signed distance from the set C, namely the function defined as follows

$$\mathrm{dist}^{\#}(x, C) := \mathrm{dist}(x, C) - \mathrm{dist}(x, \mathbb{R}^N \setminus C) \quad \text{for every } x \in \mathbb{R}^N.$$

The dual metric of a Finsler metric $\varphi \in \mathcal{M}$ is the function φ^* defined by

$$\varphi^*(x,p) := \sup\{\langle p, \xi\rangle \,:\, \varphi(x,\xi) \le 1\} \quad \text{for every } (x,p) \in \overline{\Omega} \times \mathbb{R}^N.$$

When the metric φ is convex, *i.e.* $\varphi(x,\cdot)$ is convex for every $x \in \overline{\Omega}$, the following holds (see [32]):

$$\sup\{\langle \xi, p\rangle \,:\, \varphi^*(x,p) \le 1\} = \varphi(x,\xi) \quad \text{for every } (x,\xi) \in \overline{\Omega} \times \mathbb{R}^N. \tag{5.26}$$

Example 5.23. Let us consider the Hamilton-Jacobi equation

$$H(x, Du) = 0 \quad \text{in } \Omega, \tag{5.27}$$

where H satisfies assumptions (H), and let S be the associated length function. As S is a Finsler distance, it is actually the uniform limit of a sequence of distances $(d_{\varphi_n})_n$, where φ_n is a continuous Finsler metric belonging $\mathcal{M}$ for each $n \in \mathbb{N}$ (by Theorem 4.2). For each $n \in \mathbb{N}$, let us set

$$Z_n(x) := \{p \in \mathbb{R}^N \,:\, \varphi_n^*(x, -p) \leq 1\} \quad \text{for every } x \in \overline{\Omega},$$

and $H_n(x,p) := \operatorname{dist}^{\#}(p, Z_n(x))$ for every $(x,p) \in \overline{\Omega} \times \mathbb{R}^N$. For each $n \in \mathbb{N}$, H_n is continuous, and it is convex since $Z_n(x)$ is a convex set for every x. Moreover, if S_n is the associated optical length function for each $n \in \mathbb{N}$, then $S_n = d_{\varphi_n}$ in view of (5.26) and by definition of optical length function. Therefore, if g is a boundary datum satisfying the compatibility condition (5.21) with respect to the length function S, the Monge solution u of

$$\begin{cases} H(x, Dv) = 0 & \text{in } \Omega \\ v = g & \text{on } \partial\Omega, \end{cases}$$

is the uniform limit of the unique maximal viscosity solutions u_n of the problems

$$\begin{cases} H_n(x, Dv) = 0 & \text{in } \Omega \\ v \leq g & \text{on } \partial\Omega. \end{cases}$$

Indeed, by the standard theory of viscosity solutions for continuous Hamiltonians, we know that $u_n(x) = \inf_{y \in \partial\Omega}\{S_n(x,y) + g(y)\}$ in $\overline{\Omega}$, so the claim easily follows in view of Theorem 5.11 and by the uniform convergence of S_n to S.

Example 5.24. In equation (5.27), assume in addition that the Hamiltonina H is such that the associated optical length function S is symmetric, *i.e.* $S(x,y) = S(y,x)$ for all $x, y \in \overline{\Omega}$ (this happens, for instance, when $H(x,p)$ is even in p). Then, by Theorem 3.11, there exists a Borel function $a : \overline{\Omega} \to [\alpha, \beta]$ such that

$$S(x,y) = \inf\left\{ \int_0^1 a(\gamma(t))|\dot{\gamma}(t)|\,\mathrm{d}t \,:\, \gamma \in \operatorname{Lip}_{x,y} \right\} \quad \text{for all } x, y \in \overline{\Omega}.$$

Therefore, with regard to Monge sub and supersolutions, equation (5.27) is equivalent to the eikonal equation

$$|Du| = a(x) \quad \text{in } \Omega, \tag{5.28}$$

that is, equations (5.27) and (5.28) have the same Monge subsolutions and the same Monge supersolutions, since they have the same optical length functions. Moreover, by the density result proven in Section 3.3 (*cf.* Theorem 3.7), the continuous Hamiltonians H_n of Example 5.23 can be chosen in such a way that $H_n(x,p) := |p| - a_n(x)$, for a suitable sequence of Borel-measurable functions $a_n : \overline{\Omega} \to [\alpha, \beta]$.

Inspired by Example 5.17, we use the same idea to construct an evolutive Hamilton-Jacobi equation with continuous coefficients, for which standard results of the theory of Hamilton-Jacobi equations apply. The Cauchy problem obtained by coupling this equation with a null boundary datum has therefore a unique viscosity solution, which is shown to tend asymptotically to the Monge solution of a stationary Hamilton-Jacobi equation.

Example 5.25. Let $\Omega := (0,1) \times (-2,2)$ and, for each $t > 0$, consider the continuous function $a_t : \overline{\Omega} \to \mathbb{R}$ defined by

$$a_t(x_1, x_2) := \begin{cases} 1 & \text{if } |x_2| \geq 1/t \\ 1/2 + |x_2|t/2 & \text{otherwise.} \end{cases}$$

Let us define on $\overline{\Omega} \times (0, +\infty)$ a function a by setting $a(x,t) := a_t(x)$ for each $t > 0$ and $x \in \overline{\Omega}$. We consider the following evolutive Cauchy problem:

$$\begin{cases} \partial_t v(x,t) + |Dv|(x,t) = a(x,t) & \text{in } Q := \Omega \times (0, +\infty) \\ v(x,t) = 0 & \text{on } \partial Q. \end{cases} \tag{5.29}$$

Since $a(x,t)$ is continuous, we know, by the standard theory of Hamilton-Jacobi equations [63], that the above Cauchy problem admits a unique viscosity solution, given by the following formula:

$$u(x,t) := \inf_{(y,s) \in \partial Q} S((x,t),(y,s)) \quad \text{for all } (x,t) \in Q, \tag{5.30}$$

where S is the function defined on $\overline{Q} \times \overline{Q}$ as follows:

$$S((x,t),(y,s)) := \inf \left\{ \int_s^t a(\gamma(\tau), \tau) + H^*(\dot{\gamma}(\tau))\, \mathrm{d}\tau \ : \ \gamma \in \mathrm{Lip}_{y,x}([s,t], \overline{\Omega}) \right\}, \tag{5.31}$$

where $\mathrm{Lip}_{y,x}([s,t], \overline{\Omega})$ denotes the space of curves $\gamma \in \mathrm{Lip}([s,t], \overline{\Omega})$ such that $\gamma(s) = y$, $\gamma(t) = x$. When $s > t$ or $s = t$ and $x \neq y$ this family is empty: in that case we agree that

$S((x,t),(y,s)) = +\infty$. In the above formula we have denoted by H^* the Fenchel transform of $H(p) := |p|$, namely $H^*(\xi) := \sup_{p\in\mathbb{R}^N}\langle \xi, p\rangle - H(p)$. Notice that, in this case, H^* coincides with the indicator function of the closed ball $\overline{B}_1(0)$, *i.e.* $H^*(\xi)$ is equal to 0 if $|\xi| \le 1$ and to $+\infty$ otherwise. In particular, S degenerates outside a cone of vertex (x,t), *i.e.* $S((x,t),(y,s)) = +\infty$ if $t - s < |x-y|$.

We want to study the asymptotic behavior of the solution $u(x,t)$ of (5.29). Since the functions a_t converge pointwise and increasingly on $\overline{\Omega}$, as t tends to $+\infty$, to the discontinuous function $a_\infty(x) := \chi_{\overline{\Omega}}(x) - 1/2\chi_\Gamma(x)$ (where we have denoted by Γ the x_1-axis $\mathbb{R}\times\{0\}$), we expect the asymptotic limit of $u(x,t)$ to solve the stationary Hamilton-Jacobi equation

$$|Dv| = a_\infty(x) \quad \text{in } \Omega.$$

In fact, we will show that $u(x,t)$ tends asymptotically, uniformly in t, to the Monge solution of the following Dirichlet problem:

$$\begin{cases} |Dv| = a_\infty(x) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.32}$$

To this goal, we first recall (see for instance [63, Theorem 5.2]) that, if in (5.31) the function a is replaced by a function $b : \overline{\Omega} \to [\alpha, \beta]$, $0 < \alpha < \beta$ that does not depend on t, then, for fixed (x,t) and (y,s) in $\overline{Q}$, we have:

$$S((x,t),(y,s)) \ge \inf\Big\{ \int_0^T b(\gamma(t))|\dot\gamma(t)|\,\mathrm{d}t \,:\, \gamma \in \mathrm{Lip}_{y,x}([0,T],\overline{\Omega}),\, T>0\Big\} = d_b(y,x),$$

with equality holding if $t-s \ge |y-x|\,\beta/\alpha$. In particular, by taking into account this remark and using in (5.31) the fact that $a(x,t) \le a_\infty(x)$ for all $(x,t)\in\overline{Q}$, one easily obtains that $S((x,t),(y,s)) \le 2\,\mathrm{diam}(\Omega) \vee d_{a_\infty}(x,y) \le 2\,\mathrm{diam}(\Omega)$ for all (x,t) and (y,s) in $\overline{Q}$ such that $S(x,t),(y,s)) < +\infty$ (we have denoted by $\mathrm{diam}(\Omega)$ the diameter of the set Ω). Let us now fix $(x,t)\in\overline{Q}$ and let $\gamma \in \mathrm{Lip}([s,t],\overline{\Omega})$, $0 \le s < t$, be a minimizing path of (5.30). Then we have

$$\frac{1}{2}(t-s) \le \int_s^t a(\gamma(\tau),\tau) + H^*(\dot\gamma(\tau))\,\mathrm{d}\tau = u(x,t) \le 2\,\mathrm{diam}(\Omega),$$

that is $0 \le t - s \le r := 4\,\mathrm{diam}(\Omega)$. Then, for $t > r$, any path $\gamma \in \mathrm{Lip}([s,t],\overline{\Omega})$, which is minimal for (5.30), is such that $s \ge t - r > 0$, in particular $\gamma(s) \in \partial\Omega$. Therefore, for $t > r$, it is not restrictive to assume that the infimum in (5.30) is taken letting (y,s) vary over the set $\partial\Omega \times [t-r, t]$ only. In particular, as $a_{t-r}(z) \le a(\tau, z) \le a_\infty(z)$ for every $z \in \overline{\Omega}$ and $s \le \tau \le t$, we obtain that

$$\int_s^t a_{t-r}(\gamma) + H^*(\dot\gamma)\,\mathrm{d}\tau \le \int_s^t a(\gamma,\tau) + H^*(\dot\gamma)\,\mathrm{d}\tau \le \int_s^t a_\infty(\gamma) + H^*(\dot\gamma)\,\mathrm{d}\tau. \tag{5.33}$$

Taking the infimum over all possible curves γ joining $(y,s) \in \partial\Omega \times [t-r,t]$ to (x,t) and letting (y,s) vary in $\partial\Omega \times [t-r,t]$, by what previously remarked we eventually get

$$\inf_{y\in\partial\Omega} d_{a_{t-r}}(x,y) \le u(x,t) \le \inf_{y\in\partial\Omega} d_{a_\infty}(x,y).$$

The claim now follows as a_t is an increasing sequence of isotropic Riemannian metrics converging pointwise to a_∞ on $\overline{\Omega}$ and therefore, by Proposition 1.24, the distance d_{a_t} uniformly converges to d_{a_∞} on $\overline{\Omega} \times \overline{\Omega}$ as t goes to $+\infty$. In particular, this easily implies that $u(x,t)$ asymptotically converges, uniformly in t, to $\inf_{y\in\partial\Omega} d_{a_\infty}(x,y)$, which is the Monge solution of (5.32) (remark that d_{a_∞} is nothing else than the optical length function associated to the Hamiltonian $H(x,p) = |p| - a_\infty(x)$).

The result of the previous example was obtained in a very special case. Nevertheless, with the same idea, one can obtain an analogous result for Monge solutions of eikonal equations of the following form:

$$\begin{cases} |Dv| = a_\infty(x) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.34}$$

where $a_\infty : \overline{\Omega} \to [\alpha, \beta]$ is lower or upper semicontinuous and α and β are, as usual, fixed positive constants. Indeed, let us assume, for instance, a_∞ lower semicontinuous, being the other case analogous. As well known, it is possible to find an increasing sequence of continuous functions $a_n : \overline{\Omega} \to [\alpha, \beta]$, $n \in \mathbb{N}$, such that $a_\infty(x) = \sup_n a_n(x)$ for all $x \in \overline{\Omega}$. Let us define on $\overline{\Omega} \times (0, +\infty)$ a continuous function a by setting $a(x,t) := (n+1-t)a_n(x) + (t-n)a_{n+1}(x)$ for all $x \in \overline{\Omega}$, $t \in (n, n+1]$ and $n \in \mathbb{N}$. Arguing as above, one immediately gets that the viscosity solution of

$$\begin{cases} \partial_t v(x,t) + |Dv|(x,t) = a(x,t) & \text{in } Q := \Omega \times (0, +\infty) \\ v(x,t) = 0 & \text{on } \partial Q. \end{cases}$$

tends asymptotically to the Monge solution of (5.34).

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