ON SOME VARIATIONAL PROBLEMS INVOLVING VOLUME AND SURFACE ENERGIES

L. Granieri[†] and F. Maddalena[‡]
Dipartimento di Matematica, Politecnico di Bari,
via Re David 200, 70125 Bari, Italy

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[†]Corresponding author. Email: l.granieri@poliba.it
[‡]Email: f.maddalena@poliba.it

Abstract. We show how some problems coming from different fields of applied sciences such as physics, engineering, biology, admit a common variational formulation characterized by the competition of two energetic terms. We discuss related problems and techniques studied by the authors and collaborators in the recent past as well open problems and further possible research directions in these topics.

Keywords. Calculus of variations, shape optimization, applied variational methods.

1. Introduction

The main purpose of this paper relies in showing how some problems coming from different fields of applied sciences admit a common variational formulation characterized by the competition of two energetic terms. More precisely, the mathematical problem modeling such phenomena can be put in the following general form

$$\min\left\{\mathcal{F}(\mathcal{B})=\int_{\mathcal{B}} g(x)\ dv+\int_{\partial\mathcal{B}}\Gamma(y,\nu)\ ds,\ \mathcal{B}\subset\mathbb{R}^N,\ |\mathcal{B}|=1\right\}, \tag{1}$$

where ν denotes the outward unit normal to the boundary of the set $\mathcal{B}\subset\mathbb{R}^N$, $|\mathcal{B}|$ is the volume of $\mathcal{B}$, $\Gamma(y,\nu)$ is the surface energy density and $g(x)$ is the volume energy density. The main features of the problem stay in the interplay between the two energetic contributions and, in general, its mathematical treatment is difficult since the minimization is carried among the subsets of $\mathbb{R}^N$ with fixed volume. Note that, when $g\equiv$ const, problem (1) reduces to some classical problems such as *Wulff problem* (crystal shape) or *Newton problem* (least resistance shape for a body moving in a fluid), under suitable choices of Γ. If $\Gamma\equiv$ const, then (1) corresponds to the classical problem of *prescribed mean curvature*. We refer to $[1-4]$ for an introduction to these problems. Actually, many problems arising in material science and structural

optimization are related to the determination of material regions minimizing certain energy functionals. In this area, a classical problem is the equilibrium shape of a crystal in the presence of an external field and it consists in minimizing a functional involving bulk and interfacial energies. The literature on the subject is very extensive from both the mathematical and physical sides. Here, we want to show that some particular choices of the energy densities in problem (1) and/or some generalizations of its formulation, can be suited to model some problems from very different fields of applied sciences, thus producing different variational problems, which, in our opinion, deserve a deep study in connection with important mathematical problems in physics, engineering, biology.

2. Forces and shapes

Forces manifest themselves through their action on the matter, that means that the presence of a force field in a material region is detectable just by the interaction of the field with the matter. Indeed, the *visualization* of the force field is made by introducing subtle materials in the influence region of the field. In such a way, it is possible to highlight the force lines of the field and then recover its structure (see for

instance [5]). These arguments suggest a kind of *duality* occurring in the interaction between forces and matter. We try to clarify this idea and some of its consequences.

Let $E \subset \mathbb{R}^3$ be any homogeneous material region on which the force field $F(x)$ is supposed to act; we can model an elementary displacement or velocity of the matter at a point $x \in E$ by taking a vector field $\varphi(x)$. Therefore, the total work done or power expended by the force field F in the region E is given by

$$T(\varphi) := \int_E \langle F, \varphi \rangle \, dx. \tag{2}$$

Let us note that, in geometric measure theory, the relation (2), after identifying the components of φ as the coefficients of a 1-form ω, gives the evaluation on E of a 1-current $\omega \mapsto T(\omega)$ of $\mathbb{R}^3$ (see [6]). By Helmholtz decomposition, we know that any smooth vector field can be split into the sum of a curl-free and a divergence-free components, i.e. $\varphi = \nabla\psi + \operatorname{curl} h$, which induces the corresponding splitting in (2). Then the action of the force field F on the material region E can be viewed as the superposition of two contributions and so, by assigning an energy content, a natural variational question is the determination of material regions minimizing such energies related to the two contributions. Partially, this strategy is realized in the paper [7],

where only the curl-free contribution of the displacement field is considered. This choice corresponds to conservative (or irrotational on simply-connected regions), displacements φ.

In the language of currents, in this case (2) corresponds to consider the *boundary current*

$$\partial T(f) = T(df) = \int_E \langle F, df \rangle \; dx,$$

where df is the 1-form corresponding to $\nabla\psi$. Moreover, to deal with general not homogeneous distributions of materials, it is natural to relax the formulation in (2) by considering probability measures σ in place of domains E. To this aim, let us introduce the space $\mathcal{P}(\Omega)$ of the probability measures defined on $\Omega \subset \mathbb{R}^3$. The probability constraint is needed to deal with a fixed amount of material. The energy considered in [7] is just the mass of the boundary current ∂T given by the functional

$$E(\sigma) = \sup \left\{ \int_{\overline{\Omega}} \langle F, df \rangle d\sigma \;\; : \;\; \|f\|_\infty \leq 1 \right\}.$$

Therefore, if $\Omega \subset \mathbb{R}^3$ plays the role of the ambient space, we are interested in investigating the material shapes inside Ω (represented by probability measures)

solving the following variational problem

$$\min\left\{E(\sigma)\ :\ \sigma\in\mathcal{P}(\overline{\Omega})\right\}. \tag{3}$$

Let us observe that problem (3) is strictly related to problem (1). Indeed, suppose that F is of class $\mathcal{C}^1$ and assume that $E\subset\overline{\Omega}$ is a given subset having prescribed volume and finite perimeter, denoting by ν the outward normal unit vector. By the Gauss-Green theorem, we have:

$$\int_E\langle F,df\rangle dx=-\int_E f\mathrm{div}F dx+\int_{\partial E} fF\cdot\nu\ d\mathcal{H}^{N-1}.$$

Taking the supremum with respect to f such that $\|f\|_\infty\leq 1$, we obtain

$$\sup_{\|f\|_\infty\leq 1}\int_E\langle df,F\rangle dx=\left(\int_E|\mathrm{div}F|dx+\int_{\partial E}|F\cdot\nu|\ d\mathcal{H}^{N-1}\right), \tag{4}$$

which is a particular form of the functional $\mathcal{F}$ in (1) corresponding to take energy densities both related to the same vector field. Actually, this last fact was the basic motivation to introduce the formulation (3) in [7]. Observe that the variational problem (3) involves a min-max procedure. In [7], the existence of solutions and some regularity properties of problem (3) are investigated. Naturally, the optimal shapes (minimizers of (3)) are strictly influenced by the structure of the force field

F. For instance, if $F(\bar{x}) = 0$, then an optimal shape is the atomic measure $\delta_{\bar{x}}$, i.e. all the matter is concentrated in just one point. Moreover, in general, one cannot expect the uniqueness of minimizers.

Example 2.1. Let $\overline{\Omega} = [0,1]^2$. We consider the constant vector field $F(x) = e_1 = (1,0)$. Let us fix $\bar{x} \in \overline{\Omega}$, $\bar{x} = (0, \bar{x}_2)$. It is possible to show that both the measures $\tilde{\sigma} = \mathcal{H}^1 \llcorner [0,1] \otimes \delta_{\bar{x}_2}$ and $\bar{\sigma} = \mathcal{H}^2 \llcorner [0,1]^2$ are optimal shapes. Therefore, we see that, in general, the optimal shapes can exhibit very different regularity properties.

Figure 1.

In [7], conditions on F ensuring the existence of absolutely continuous optimal shapes are discussed. It is interesting to notice that the optimal shapes are also influenced by the ambient space.

Example 2.2. Let $v = (1/\sqrt{2}, 1/\sqrt{2})$, $F(x) = v$ and let $\overline{\Omega} \subset \mathbb{R}^2$ be a unit square oriented with one edge direction equal to v. It results that the measure $\bar{\sigma} = \mathcal{H}^2 \llcorner \Omega$ is an optimal shape.

Figure 2.

Now, we address the question of changing the ambient space. Let $\overline{\Omega} = [0,1]^2$. In this case, the optimal shapes are absolutely continuous measures σ with density $a \in L^1(\Omega)$. We inscribe $\overline{\Omega}$ in a square Σ with edge of length $\sqrt{2}$ and parallel to v (see Figure 2). However, this time it is possible to prove that the optimal shapes are not reached by sets of finite perimeter, while on Σ we have the optimal measures as in Example 2.1.

If the force field has a radial structure, i.e. $F = \varphi(|x|)x$, then it is possible to find radially symmetric shapes. Force fields taking the form $F(x) = p|x|^{p-2}x$ deserve particular attention. In the solenoidal case, i.e. for $p = -1$, it turns out that the unit ball is an optimal shape. However this is not always the case. For instance, for $p > 2$ it results that there not exist absolutely continuous optimal shapes.

3. Mass reduction for elastic bodies

In this section, we study a classical problem in structural optimization (see [8, 9]) consisting in placing a hole of fixed volume inside an elastic body in equilibrium under given external loads. In addition, we impose a *failure constraint* by bounding

the maximum stress. For a deeper discussion we refer the reader to [10]. Let us consider an elastic body occupying a regular region $\Omega \subset \mathbb{R}^3$, in equilibrium under the given traction f on the boundary $\partial\Omega$. Its elastic state is completely determined by the triplet (u, E, T), representing the displacement vector, the strain tensor and the stress tensor fields, respectively, and satisfying the kinematical condition $E = \frac{1}{2}(\nabla u + \nabla u^T)$, the constitutive equation $T = \mathbb{C}[E]$ and the equilibrium conditions

$$\operatorname{div} T = 0, \quad \text{in } \Omega, \qquad T\nu = f, \quad \text{on } \partial\Omega, \tag{5}$$

where ν is the outward unit normal to the boundary $\partial\Omega$. According to the well known principle of minimum of complementary elastic energy (see [11]), it results that, at the equilibrium, the material body Ω is characterized by the energy

$$\mathcal{E}(\Omega) = \min\left\{ \frac{1}{2}\int_\Omega T \cdot \mathbb{C}^{-1}[T]\ dx \mid T \text{ satisfies (5)} \right\}.$$

Here we are interested in analyzing the consequence of making a *hole* inside Ω and in particular, in producing this mass reduction under a failure constraint and by optimizing a quantity which is particularly relevant for the elastic state of the body.

Let $K \subset\subset \Omega$ be a compact set representing the hole. By subtracting to the body the material inside K we get a new material body $\Omega \setminus K$ whose equilibrium

configuration is characterized by the elastic state (u_K, E_K, T_K). The new stress field T_K satisfies the equilibrium conditions

$$\text{div } T_K = 0, \quad \text{in } \Omega \setminus K, \quad T_K \nu = f, \quad \text{on } \partial\Omega, \quad T_K \nu = 0, \quad \text{on } \partial K. \tag{6}$$

Let us notice that we cannot subtract too much material from the body without exceeding its strength. Therefore, we need to impose a *failure* constraint that, usually, takes the form of an inequality such as $F(T_K) \leq C$, where the choice of the mapping F relies on constitutive assumptions. We shall assume as failure constraint

$$\| T_K \|_\infty \leq C. \tag{7}$$

We remark that, assuming the volume of the hole as the unique descriptor of the problem is not enough. Indeed, the hole K could be placed everywhere inside Ω and this, in general, seems not an optimal choice due to the inhomogeneity of the stress state. Indeed, the geometry of K should be related to the stress distribution T_K and the dependence on K makes this problem difficult, since it requires the study of the equations (6) for varying domains. To avoid this difficulty, a reasonable choice could be to place the hole where the stress tensor T of the reference state Ω takes lower values, in order to get a nearly uniform engagement of the material.

By arguing as in shape optimization ([11 − 14]) we are led to consider the elastic stored energy. The following two simple assertions proved in [10] constitute the main reason to reduce the present problem to a variational problem involving a surface like functional.

Proposition 3.1. *Let $K \subset\subset \Omega$ be any compact subset. Then,*

$$\mathcal{E}(\Omega_K) - \mathcal{E}(\Omega) \geq 0.$$

Moreover, we also have that the energy change $\mathcal{E}(\Omega_K) - \mathcal{E}(\Omega)$ is concentrated on the boundary of the hole K (see also [9]).

Theorem 3.2. *Let $K \subset\subset \Omega$ be any compact subset and let (u, E, T), (u_K, E_K, T_K) be the solutions of the elastic problem in the configurations Ω and $\Omega \setminus K$ respectively. Then,*

$$\mathcal{E}(\Omega_K) - \mathcal{E}(\Omega) = \int_{\partial K} T\nu \cdot u_K \; d\mathcal{H}^{N-1}. \tag{8}$$

By following a direct approach, one could consider the term u_K in (8) as a state variable of an optimal control problem (see [14]) but this approach seems to be

difficult because of the dependence of u_K from the variable domain $\Omega \setminus K$. Therefore, we observe that, according to (8), the energy change occurring in the body after the creation of the hole K can be reduced to a surface like energy term localized on ∂K. Indeed, by using the Poincaré inequality and by considering that the elastic state u_K is determined up to a constant, we have

$$\begin{aligned} \mathcal{E}(\Omega_K) - \mathcal{E}(\Omega) &= \int_{\partial K} T\nu \cdot u_K \, d\mathcal{H}^{N-1} \leq \|u_K\|_\infty \int_{\partial K} |T\nu| \, d\mathcal{H}^{N-1} \\ &\leq C\|\nabla u_K\|_\infty \int_{\partial K} |T\nu| \, d\mathcal{H}^{N-1}. \end{aligned} \tag{9}$$

Now, by virtue of the failure constraint (7) and the constitutive equation, assuming that the elasticity tensor $\mathbb{C}$ is invertible, we have $\|\nabla u_K\|_\infty \leq M$, for some positive constant M. Therefore, according to (9), we can estimate the lower bound of the energy increment due to the creation of a hole by minimizing the energy, whose density is the norm of the stress vector acting on the boundary of the hole. Then, we are lead to study the following minimization problem.

$$\min \left\{ \int_{\partial K} |T\nu| \, d\mathcal{H}^{N-1} \mid K \subset \Omega, \ |K| = V \right\} \tag{10}$$

Let us notice that problem (10) is a Wulff like problem ([1]) for the density $\Gamma(x, \nu(x)) = |T\nu|$, where $|\cdot|$ denotes the Euclidean norm. The Wulff problem is

known as describing the equilibrium shape of a perfect crystal of one material in contact with a single surrounding medium for which the dependence of the surface energy on the normal ν relates the surface tension with the bulk crystalline lattice. If the property

$$\Gamma(x, \nu(x)) > \alpha > 0 \tag{11}$$

holds, the existence of minimizers for problem (10) can be proved by applying the direct methods of the calculus of variations. Indeed, we have compactness among the sets of finite perimeter and by the lower semicontinuity of the surface energy we get minimizers. If (11) fails, the existence problem is more involved and in many cases it is an open problem. However, for particular stress tensor T, problem (10) could be reduced to problem (3) for a vector field having constant divergence. For instance, as one expects, for stress tensor T having eigenvalues given by $\lambda_i = \lambda x_i$, $i = 1, 2, 3$, the optimal hole is a ball, while it is an ellipsoid for constant non-vanishing eigenvalues. For more details and other examples we refer to [10]. To develop more general tools to treat problem (10) is an interesting open question.

4. Morphological stability of crystals

In material science, a typical experimental observation reveals that a non-hydrostatically strained solid that is in contact with a more compliant phase, as its own melt or vapor, can partially release its elastic energy by a sudden morphological transition at the interface. This occurs for instance when the strained solid has a surface at which the material can be redistributed and thus it may reduce its elastic energy via surface undulations. Roughly speaking, these transitions lead to a decrease of the energy associated to a new stress distribution by decreasing the stress inhomogeneities.

This kind of instabilities is usually drawn back to the studies by Asaro and Tiller ([15]) of corrosion phenomena in metals, but the recognition of the general nature of these kind of instabilities can instead be referred to the studies of Grinfeld ([16−19]), hence they are often referred to as *Grinfeld or Asaro-Tiller-Grinfeld instability.*

The shape of a crystal Ω can be taken as the actual configuration, under a deformation mapping $y : \Omega_0 \mapsto \Omega$ of a reference state $\Omega_0 \subset \mathbb{R}^3$. Then the rearrangement of material occurring at the free boundary $\partial\Omega$ turns out to be the result of the counteraction between bulk and surface energies.

In Section 4.1 we show, through a simple case, that in general one cannot expect morphological stability by considering the only contribution of the bulk energy. Indeed, by considering a plane region with a density energy $g(x, y)$, taking a rectangular configuration, we find a family of perturbations of the horizontal edge which preserves the volume and on which the energy decreases. In fact, under some conditions on the density energy $g(x, y)$, the rectangle is a local maximum of the energy with respect to such perturbations. In Section 4.2 we show that, by adding a surface like energy, as in problem (1), the rectangle becomes a local minimum for the total energy. Thus, in the subsequent Section 4.3, we use these results to state an example of Grinfeld instability in linear elasticity.

4.1. **Instability for the Bulk Energy.** Let $g : \mathbb{R}^2 \to \mathbb{R}$ be a regular bulk energy density and let $\Omega = [0, 1] \times [0, h]$ be the reference configuration of an elastic crystal. Let us denote by $\bar{\gamma}(x) = h$ the upper edge of Ω. The bulk energy associated to the configuration Ω is

$$J(\bar{\gamma}) = \int_0^1 \int_0^h g(x, y) dx dy.$$

More generally, if $\gamma : [0,1] \to \mathbb{R}$ is a function such that $\gamma(0) = h = \gamma(1)$, we consider the bulk energy

$$J(\gamma) = \int_0^1 \left(\int_0^{\gamma(x)} g(x,y)dy \right) dx. \tag{12}$$

We are going to study the behavior of the above energy under area preserving perturbations of γ. Therefore, let $\eta : [0,1] \to \mathbb{R}$ such that $\eta(0) = \eta(1) = 0$ and $\int_0^1 \eta(x)dx = 0$. To fix the ideas, we consider functions such as $\eta(x) = \sin(2k\pi x)$. For $\varepsilon > 0$, we consider the perturbations $\gamma_\varepsilon(x) = \gamma(x) + \varepsilon\eta(x)$. Let us evaluate the first and second variations of the functional J. By the definition of variation it results

$$\delta J(\gamma, \eta) := \lim_{\varepsilon \to 0^+} \frac{J(\gamma_\varepsilon) - J(\gamma)}{\varepsilon}.$$

Observe that

$$\frac{J(\gamma_\varepsilon) - J(\gamma)}{\varepsilon} = \frac{1}{\varepsilon} \left(\int_0^1 \left(\int_0^{\gamma_\varepsilon(x)} g(x,y)dy - \int_0^{\gamma(x)} g(x,y)dy \right) dx \right) =$$

$$= \int_0^1 \left(\frac{1}{\varepsilon} \int_{\gamma(x)}^{\gamma_\varepsilon(x)} g(x,y)dy \right) dx.$$

After applying the fundamental theorem of calculus, we get

$$\delta J(\gamma, \eta) = \int_0^1 g(x, \gamma(x))\eta(x)dx. \tag{13}$$

By requiring the condition

$$g(x, \gamma(x)) = \text{const}, \tag{14}$$

we get

$$\delta J(\gamma, \eta) = C \int_0^1 \eta(x)dx = 0, \tag{15}$$

since the perturbations η are area preserving. Therefore, in this case, the configuration γ is a critical point for J. Whereas, as it is standard in calculus of variations, $\delta J = 0$ for every η implies $g(x, \gamma(x)) = \text{const}$.

The nature of this critical point is revealed by studying the second variation

$$\delta^2 J(\gamma, \eta) := \lim_{\varepsilon \to 0^+} \frac{\delta J(\gamma_\varepsilon, \eta) - \delta J(\gamma, \eta)}{\varepsilon}.$$

By (13), we obtain

$$\frac{\delta J(\gamma_\varepsilon, \eta) - \delta J(\gamma, \eta)}{\varepsilon} = \int_0^1 \frac{g(x, \gamma_\varepsilon(x)) - g(x, \gamma(x))}{\varepsilon}\eta(x)dx.$$

Passing to the limit as $\varepsilon \to 0$ under the integral sign, we get

$$\delta^2 J(\gamma, \eta) = \int_0^1 \frac{\partial}{\partial y} g(x, \gamma(x))\eta(x)^2 dx. \tag{16}$$

Therefore, by requiring the condition

$$\frac{\partial}{\partial y} g(x, \gamma(x)) < \alpha < 0, \tag{17}$$

we obtain $\delta^2 J(\gamma, \eta) < 0$. Thus, in such a case, the configuration γ is a local maximum for J. If we consider a flat configuration $\bar{\gamma} = h$, in order to have an instability property it is sufficient for instance that the volume density does not depend on the first variable, that is $g = g(y)$, and that $g'(h) < \alpha < 0$. In particular this means that the bulk energy decreases by passing from the flat configuration to corrugated ones.

4.2. **Stabilizing Effect of Surface Energies.** We add a surface contribution. The simplest choice is the length of the boundary configuration, which leads to the following total energy

$$E(\gamma) = \int_0^1 \left(\int_0^{\gamma(x)} g(x, y) dy \right) dx + l(\gamma), \tag{18}$$

where $l(\gamma) = \int_0^1 \sqrt{1 + \gamma'(x)^2} dx$ is the length of the curve γ. We claim that, this time, the flat configuration $\bar{\gamma} = h$ is a local minimum for E whenever it satisfies the

necessary condition (14). By standard computations we have

$$\delta l(\gamma,\eta)=\int_0^1 \frac{\gamma'(x)\eta'(x)}{\sqrt{1+\gamma'(x)^2}}dx,$$

while

$$\delta^2 l(\gamma,\eta)=-\int_0^1 \frac{\gamma'(x)^2\eta'(x)^2}{(1+\gamma'(x)^2)^{\frac{3}{2}}}dx+\int_0^1 \frac{\eta'(x)^2}{\sqrt{1+\gamma'(x)^2}}dx.$$

Then, we obtain

$$\delta^2 E(\bar\gamma,\eta)=\int_0^1 \frac{\partial}{\partial y}g(x,h)\eta(x)^2dx+\int_0^1 \eta'(x)^2dx.$$

If $\eta(x)=\sin(2k\pi x)$, we get

$$\delta^2 E(\bar\gamma,\eta)=\int_0^1 \frac{\partial}{\partial y}g(x,h)\sin^2(2k\pi x)dx+4k^2\pi^2\int_0^1\cos^2(2k\pi x)dx=$$

$$=\int_0^1 \frac{\partial}{\partial y}g(x,h)\sin^2(2k\pi x)dx+2k^2\pi^2\geq\frac{m}{2}+2k^2\pi^2,$$

where $m=\min\{\,\partial_y g(x,h)\,|x\in[0,1]\}$. For $k\in\mathbb{N}$ large enough, it results $\delta^2E(\bar\gamma,\eta)>0$ and then the flat configuration is a local minimum for E. Thus, we see that, in some sense, the flat configuration is stable for the total energy at least for thick corrugations.

4.3. **Instability in Linear Elasticity.** Let $\Omega=[0,1]\times[0,h]$ be the reference configuration for an elastic crystal. We denote by $\partial_2\Omega$ the upper edge of Ω and we

require $\partial_2\Omega$ to be a free boundary. On $\partial_1\Omega = \partial\Omega \setminus \partial_2\Omega$, we assign a displacement u_0. By assuming a linear isotropic elastic behavior, the constitutive equation takes the form

$$T = 2\mu E + \lambda\, tr(E) I, \tag{19}$$

where $\lambda, \mu > 0$ are the Lamé coefficients, tr is the trace operator and I is the identity matrix. Therefore, the elastic energy corresponding to u is given by

$$I(u) = \mu \int_\Omega tr(E^2)\; dxdy + \frac{\lambda}{2} \int_\Omega (tr(E))^2\; dxdy. \tag{20}$$

Any minimizer of (20) satisfies the equilibrium equations

$$\text{div}\, T = 0, \quad \text{in } \Omega, \qquad T\nu = 0, \quad \text{on } \partial_2\Omega, \tag{21}$$

where ν denotes the outer unit normal to $\partial_2\Omega$. We set

$$\mathcal{E}(\Omega) = \min\{I(u) \mid u \in C^1(\bar{\Omega}, \mathbb{R}^2),\ u = u_0 \text{ on } \partial_1\Omega\}. \tag{22}$$

In [16] it is shown that the elastic energy (22) is unstable for perturbations of the free boundary $\partial_2\Omega$, whence an affine displacement u_0 is prescribed. Here, we apply the results of the previous section to state an analogous instability occurrence for a general nonlinear displacement u_0.

Let $u_0(x,y) = (axy + bx, cy^2 + dx^2 + ey)$, where the coefficients will be chosen in order to satisfy the equilibrium equations (21). Therefore we evaluate

$$E = \begin{pmatrix} ay+b & \frac{1}{2}(a+2d)x \\ \frac{1}{2}(a+2d)x & 2cy+e \end{pmatrix},$$

$$T = 2\mu \begin{pmatrix} ay+b & \frac{1}{2}(a+2d)x \\ \frac{1}{2}(a+2d)x & 2cy+e \end{pmatrix} + \lambda \begin{pmatrix} (a+2c)y+b+e & 0 \\ 0 & (a+2c)y+b+e \end{pmatrix}.$$

Then, we get

$$\text{div } T = \begin{pmatrix} 0 \\ \mu(a+2d) + 4c\mu + \lambda(a+2c) \end{pmatrix},$$

$$T\nu = T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mu(a+2d)x \\ 4\mu cy + 2\mu e + \lambda(a+2c)y + \lambda b + \lambda e \end{pmatrix}.$$

Hence, u_0 is an equilibrium configuration after choosing the coefficients according to the following:

$$a + 2d = 0,\ 2(2\mu+\lambda)c + \lambda a = 0,\ 2h(2\mu+\lambda)c + \lambda h a + 2\mu e + \lambda b + \lambda e = 0. \quad (23)$$

In particular, (23) is satisfied by the following choice of the coefficients:

$$d = 1, \quad a = -2, \quad c = \frac{\lambda}{2\mu+\lambda}, \quad (2\mu+\lambda)e + \lambda b = 0. \quad (24)$$

By (20) and (24), we compute

$$I(u_0) = \mu \int_\Omega (ay+b)^2 dxdy + \mu \int_\Omega (2cy+e)^2 dxdy + \frac{\lambda}{2} \int_\Omega [(a+2c)y + b + e]^2 dxdy.$$

Setting $g(y) = \mu(ay+b)^2 + \mu(2cy+e)^2 + \frac{\lambda}{2}[(a+2c)y+b+e]^2$, we have $I(u_0) = \int_\Omega g(y)\, dxdy$. Observe that g is defined on the whole $\mathbb{R}^2$. Therefore, the results of the previous section hold true. Hence, by (17), if the following instability condition holds true:

$$g'(h) = 2\mu a(ah+b) + 4\mu c(2ch+e) + \lambda(a+2c)[(a+2c)h+b+e] < 0, \qquad (25)$$

it follows that the energy $I_\Omega(u_0) := \int_\Omega g(y)\, dxdy = I(u_0)$ is unstable for area-preserving perturbations of the domain Ω. According to (24), setting $a = -2$ in (25), we get

$$g'(h) = 8\mu h - 4\mu b + 8\mu c^2 h + 4\mu ce + 4\lambda(c-1)^2 h + 2\lambda(c-1)(b+e).$$

Since, by using again (24), $c = \frac{\lambda}{2\mu+\lambda}$, we obtain

$$g'(h) = (8\mu + 8\mu c^2 + 4\lambda(c-1)^2)h + 2(2\mu+\lambda)ce + 2\lambda cb - 4b\mu - 2\lambda b - 2\lambda e =$$

$$= (8\mu + 8\mu c^2 + 4\lambda(c-1)^2)h + 2\lambda cb - 4b\mu - 2\lambda b =$$

$$= (8\mu + 8\mu c^2 + 4\lambda(c-1)^2)h - 8\mu\left(\frac{\mu+\lambda}{2\mu+\lambda}\right) b.$$

Hence, in order to satisfy (25), it suffices to take large enough values of $b > 0$ or to take $b > 0$ and h small enough. Let Ω_ε be the ε-corrugated configuration given by

$$\Omega_\varepsilon = \{(x, y) \in \mathbb{R}^2 \mid x \in [0, 1],\ 0 \leq y \leq h + \varepsilon\eta(x)\}$$

with $\int_0^1 \eta \ dx = 0$ and $\eta(0) = \eta(1) = 0$. Let $I_{\Omega_\varepsilon}(u)$ be the energy associated to the displacement u for the configuration Ω_ε. Since g and u_0 are well defined on the whole $\mathbb{R}^2$, applying the results of the previous section we obtain

$$\mathcal{E}(\Omega_\varepsilon) \leq I_{\Omega_\varepsilon}(u_0) \leq I_\Omega(u_0) = \mathcal{E}(\Omega).$$

Therefore the elastic energy decreases by passing from the flat initial configuration to the corrugated one. The previous arguments suggest the possibility to construct an initial displacement for which the energy is not decreasing by passing to corrugated configurations of the reference domain. Thus, at least from the mathematical point of view, this seems to suggest that the instability phenomenon, as focused by Grinfeld in [16], is strictly related to the affine displacements. Indeed, in such a case, the stored energy identically satisfies the necessary condition (14). We think that these instability phenomena deserve a deep understanding and in [20] we will propose a

more general variational framework in order to treat some stability or instability properties of mechanical interest.

5. Manufacturing flat surfaces and fish population dynamic

Here we address two problems related to the industrial production of flat surfaces and the stock-recruitment relationship in fish population dynamic. These problems involve complex physical and biological phenomenologies, thus, by arguing as in Section 4, we try to capture the essential of the phenomena by means of a simple variational argument.

5.1. **Flatness Manufacturing Operations.**

The assessment of the output of manufacturing operations is, for a large part, a problem of evaluating geometric shapes (e.g. straight line, planes, circles, etc.) and feature forms tolerances (e.g. flatness, straightness, cylindricity, etc.) of the components. Here, we focus on the industrial process producing flat surfaces by proposing an energy approach to control the quality of the production. As a starting point, we assume that, whatever the industrial process actually works, the machine production, in his perfect configuration,

proceeds by minimizing an energy functional or at least by selecting critical points, among all other possible geometric product configurations with the same amount of material. To simplify the description, we discuss a 2-dimensional model. Let $\Omega = [0,1] \times [0,h]$ be the reference of an ideal material product (the density is assumed to be uniform). In general, the processing produces a corrugated output Ω_p. Let us assume that the upper edge of Ω_p is the graph of the function $y(x)$. If $W(x,y)$ denotes the cost to leave material (W could also depend on the initial configuration before the processing) in the position (x,y), the cost to obtain the configuration Ω_p is given by

$$C_W(\Omega_p) = \int_0^1 \int_0^{y(x)} W(x,y)\ dxdy.$$

If the material is homogeneous, the condition $\int_0^1 y(x)\ dx = K$, for a fixed constant K, corresponds to fixing the amount of material employed in the production. We also require that the cost to leave the material, or to take away material, depends on the geometry of the assumed configuration, namely on the normal vector, and then on the first derivative $y'(x)$ of the upper edge of Ω_p. If $\sigma(x,y)$ denotes the unitary

cost to produce the derivative $y'(x)$ at the position (x, y), the total cost is given by

$$C_\sigma(\Omega_p) = \int_0^1 \int_0^{y'(x)} \sigma(x, y)\ dxdy.$$

Therefore, the total cost to realize the configuration Ω_p is given by the functional

$$J(y) := C(\Omega_p) = C_W(\Omega_p) + C_\sigma(\Omega_p). \tag{26}$$

The problem is then to study the behavior of the above energy under area preserving perturbations of y. Therefore, consider a function $\eta : [0, 1] \to \mathbb{R}$ such that $\eta(0) = \eta(1) = 0$ and $\int_0^1 \eta(x)dx = 0$. For $\varepsilon > 0$, we consider the usual perturbations $y_\varepsilon(x) = y(x) + \varepsilon\eta(x)$. The first variation of J is given by

$$\delta J(y, \eta) = \int_0^1 \left(W(x, y(x)) - \frac{d}{dx}\sigma(x, y'(x)) \right) \eta(x)dx.$$

The process selects a critical point if this first variation vanishes, namely if and only if the following condition holds:

$$W(x, y(x)) - \frac{d}{dx}\sigma(x, y'(x)) = \text{const.} \tag{27}$$

Observe that, in general, the equation (27) can be solved with respect to y just for special forms of W and σ. The condition (27) could be taken as a characterization of the process once the energy density functions are given, by means of physical

inspection or by experimental derivation. We remark that, if the energy densities W and σ depend only on the height y, then the flat configuration $y(x) = h$ satisfies the condition (27). Therefore, the flatness condition can be regarded as a critical point for an energy functional. It could be useful to determine the expression of the density energies W and σ through precise measurements at the beginning of the production. Thus, condition (27) can be used as a test of production; i.e. whenever a measurement on a product piece does not satisfy condition (27), one can suspect that a modification has occurred in the production process, such as worn-out machine tools, modified energy level, and so on.

The above variational paradigm makes sense only if an optimal configuration y actually exists. Applying the direct methods of the calculus of variations, it is not difficult to find conditions ensuring the existence of the minimum. To this aim we set

$$L(x, y, y') = \int_0^y W(x, t)\ dt + \int_0^{y'} \sigma(x, t)\ dt.$$

Then, the functional J can be written in the standard form

$$J(y) = \int_0^1 L(x, y(x), y'(x))\ dx. \tag{28}$$

Typically, to get existence results, we have to impose conditions on the density $L(x, y, y')$ and/or on the space of admissible functions y. Let $L(x, y, y')$ be continuous and satisfy the following conditions:

(i) Growth Condition. There exists constants $a > 0, b, c \in \mathbb{R}$ and exponents $p > q \geq 1$ such that $L(x, y, y') \geq a|y'|^p + b|y|^q + c$;

(ii) Convexity condition. For fixed (x, y), the function $L(x, y, z)$ is convex with respect to z.

Under (i) and (ii), a classical existence result (see for instance [21]) states that the functional $J(y)$ admits minimizers in the Sobolev space $W^{1,p}([0, 1])$. Moreover, every solution of (27) is in fact a minimizer of J. Finally, if $L(x, y, \cdot)$ is strictly convex, then we have a unique minimizer of J. Observe that the previous assumptions are conditions on the integrand $L(x, y, y')$ and so they impose some restrictions on the expression of the cost functions W and σ. In this way, we are making assumptions on the machine process; indeed, to produce perfectly flat surfaces, the machine has

to work with a homogeneous energy density, (i.e. independent on the $x-$variable), otherwise the necessary condition (27) cannot be satisfied. However, since the cost densities W and σ are unknown in the model, it could be preferable to make some more assumptions on the space of admissible functions. Indeed, to get compactness, we need some more conditions on the admissible functions, in particular the equi-boundedness of y'' allows us to use the Ascoli-Arzelà theorem. This corresponds to introducing restrictions on the geometry of the surfaces produced, but it does not yield restrictions on the densities W and σ. Moreover, existence of minimizers follows since the functional J is lower semicontinuous. It is in fact continuous with respect to the pointwise convergence. However, although in such a case we have almost no restrictions on W and σ, and we find a regular minimizer of the functional J, in general in this case we cannot expect the validity of the condition (27). Indeed, in such a case, the variations $h_\varepsilon = h + \varepsilon\eta$ could not be admissible. The question of finding explicit forms of W and σ in order to make this program work for appropriate production processes will be investigated in a future work [22].

5.2. **A Problem in Fish Population Dynamic.** The cycle of regeneration of a population is crucial to maintain the population at a stable level. For fish populations, a crucial step in this cycle is the so-called recruitment phase, i.e. the age at which the born fishes become vulnerable to the fishing activities. In fishery sciences, the stock-recruitment problem (SR problem) concerns the relationship between the amount of spawning stock biomass (SSB) and the number of fish at the subsequent recruitment (R). Despite the fundamental importance of the SR process in fish dynamic, the mathematical modeling of the subject is yet not really satisfactory. We want to apply a variational model to describe some aspects of this phenomenon. A basic observation is that many fish species appear to maintain a constant mean R level, at least in some SSB range. This is the reason for the assumption of constant recruitment R in many fish population models. We assume this fact as a first principle which could be mathematically stated by asking that any SR relationship, say $R = h(S)$, has a fixed integral mean in a reference SSB range $[S_1, S_2]$, i.e. it satisfies the following requirement.

Principle 1. *The function $h(S)$ satisfies the following integral mean condition*

$$\int_{S_1}^{S_2} h(S)\ dS = R_m(S_1 - S_2).$$

At this point we need some criteria in order to select the function h among all the possible ones. To this aim, we suppose that the biological phenomenon underlying the recruitment, being submitted to natural selection, is related to a SR relationship obeying some optimality condition. Then, the function h is selected as the best possible to guarantee the survival of the stock. Of course, we need some more precise statement in order to translate these hypothesis into a useful mathematical principle. We assume that, whatever the biological processes actually work, the SR relationship, in his equilibrium configuration, operates by minimizing an *energy-like functional* or at least by selecting critical points. Therefore, we ask that the SR relationship realizes the minimum or at least a critical point, among all other possible configurations. To define a suitable functional we denote by $W(S, R)$ the positive cost for the amount S of SSB which yields the recruitment level R. It is reasonable to suppose that the total cost in $[S_1, S_2]$ depends on the area spanned by the function h. In other words, very expensive R levels are possible but only in a

very small range of SSB. Therefore, the total cost associated to the function h could be defined as

$$J(h) = \int_{S_1}^{S_2} \left(\int_0^{h(S)} W(S,R)\, dR \right)\, dS. \tag{29}$$

The above functional is exactly the functional defined in (26) in the case $\sigma \equiv 0$. We can now state a second variational principle.

Principle 2. *The function $R = h(S)$ for a SR relationship realizes the minimum or at least a critical point, of the functional (29).*

In this formulation, the total cost $J(h)$ depends only on the amount R of the recruitment. Of course the SR phenomenon could also depend on the geometric shape of h, for instance from the rate of change at which variations on the R level happen. It is possible to take into account these requirements by adding in (29) another term depending on the first derivative h' analogously to the approach pursued for flat surfaces. In [23], a preliminary study of this model is performed and we refer the reader to that paper for more details and applications.

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List of Figures.

(1) Figure 1. Two optimal measures.
(2) Figure 2. Changing the ambient space.

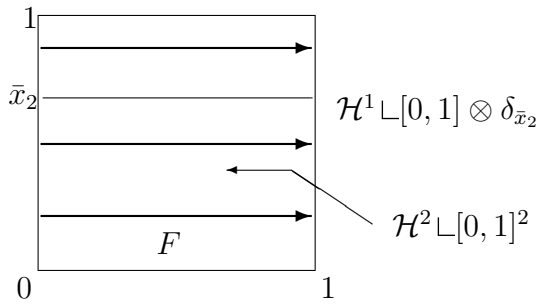

FIGURE 1. Two optimal measures

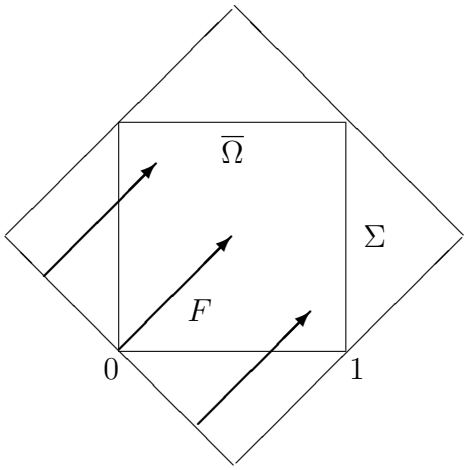

FIGURE 2. Changing the ambient space