Thin-walled beams: the case of the rectangular cross-section

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Abstract

In this paper we present an asymptotic analysis of the three-dimensional problem for a thin linearly elastic cantilever $\Omega_\varepsilon = \omega_\varepsilon \times (0, l)$ with rectangular cross-section ω_ε of sides ε and ε^2, as ε goes to zero. Under suitable assumptions on the given loads, we show that the three-dimensional problem converges in a variational sense to the classical one-dimensional model for extension, flexure and torsion of thin-walled beams.

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1 Introduction

It is common practice in structural engineering to consider structures with extension in one or more directions small compared to the remaining. Such a situation arises, for instance, in the study of flat domains with small thickness (plates) or of cylinders with transversal section having small diameter (beams).

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Approximate mechanical models for thin structures are thousand of years old and go back to the pioneering works in Mechanics of Euler, D. Bernoulli, Navier and Kirchhoff, see [13]. The classical theories are usually based on some a-priori assumptions, motivated by the smallness of certain dimensions with respect to others, on the deformation of the body or on the induced stress field.

In the last two-three decades a considerable amount of work has been done in order to rigorously justify the a-priori assumptions on which classical theories are based. In particular, approaches based on rigorous asymptotic expansion (mainly due to the French school) or inspired by the Γ-convergence of energy functionals (proposed by E. De Giorgi in [8]) have been successfully used, in deriving one or two-dimensional classical mechanical models for thin structures in linear and non-linear elasticity starting from three-dimensional problems.

In this paper we shall be concerned with an asymptotic analysis for a class of linearly elastic thin beams when the thickness of the transversal cross-section goes to zero. Dimension reduction problems from three dimensions to one have received a great deal of attention in recent years and numerous and interesting results have been obtained, see [5] for a comprehensive bibliographical and historical survey. In [1] a general framework based on the Γ-convergence on varying domains has been applied to give a justification of the classical one-dimensional mechanical model for extension, flexure and torsion of slender cylinders having circular cross-section. An extension to slender cylinders under more general boundary conditions and with cross-sections having Lipschitz boundary has been presented in [18]. By adapting and refining the ideas introduced by Ciarlet and Destuynder of rescaling domains and field displacements, see [6], Le Dret showed in [12] the convergence of the displacements and stresses for slender cylinders with Lipschitz cross-section and also discussed how to treat some more general cases, involving beam shapes with spikes and holes.

All of the above results are based on a common assumption, namely the three-dimensional variational problem is formulated on a family of cylinders which are obtained by scaling a reference cylindrical domain $\Omega_\varepsilon \subset \mathbb{R}^3$ by a *single* factor ε, $\varepsilon > 0$, in its cross-section plane, that is $\Omega_\varepsilon = \omega_\varepsilon \times (0, l) \subset \mathbb{R}^3$, where l is the length of the cylinder, $\omega_\varepsilon = \varepsilon\omega$ is its cross-section and $\omega \subset \mathbb{R}^2$ is a (simply connected) open bounded set with Lipschitz boundary.

A variant of these cases has been considered in [4] where the cross-section ω was scaled not simply by ε but by a factor $r_\varepsilon(x_3)$ (not depending on the coordinates x_1 and x_2 of the cross-section plane). This way allows very rapid variations of the thickness of the domain and produces a one-dimensional model for thin notched beams.

In several areas of civil, aeronautic and mechanical engineering, design and technological requirements force the use of the so-called *thin-walled cross-section* *beams*, that is slender cylinders in which the transversal cross-section is the union of several walls, whose thickness is very small compared with the diameter of the cross-section. To give an example, thin-walled tubes are often used in the structures where beams are subjected to high twisting moment or to important transversal forces. Hollow cross-sections are in fact most efficient in resisting torsion and flexure because, as a consequence of the advantageous distribution of stresses, they ensure high rigidity and strength with relatively low weight.

From the mathematical point of view, the main novelty in dealing with thin-walled cross-section beams is the presence of *two scaling factors*: one factor is the ratio between the diameter of the cross-section and the beam length, say ε; the other is the ratio between the wall thickness and the diameter of the cross-section, say ε^α, with $\alpha > 1$. As far as the asymptotic behavior of the three-dimensional energy functional when ε goes to zero, known results for thin-walled cross-section beams are based on the De Saint-Venant classical principle and, therefore, they essentially involve an asymptotic study of certain two-dimensional Neumann problems defined on the transversal cross-section when the thickness of the walls goes to zero. The limit behavior of the torsion problem for thin-walled beams has been recently studied in [19], [20], see also [14], [15] for an alternative development based on Γ-convergence arguments applied on varying domains and [16] for an asymptotic analysis of the flexure problem. The acceptation of the De Saint-Venant's principle has important consequences since, roughly speaking, it implies the loss of one dimension. In fact, all the above investigations involve a reduction from two dimensions to one.

This paper represents a first step of a line of research which aims to a rigorous deduction of the one-dimensional theory for thin-walled beams from the three-dimensional linear elasticity via Γ-convergence techniques. Here we consider a thin-walled cantilever $\Omega_\varepsilon = \omega_\varepsilon \times (0, l)$, made of homogeneous linear isotropic material, with a rectangular cross-section ω_ε of sides ε and ε^2. By merging and refining the different techniques of [1], [2] and [18], we prove that the three-dimensional elasticity problem converges in a variational sense to a one-dimensional problem as ε goes to zero. The limit problem is defined by a functional which includes the extension, the flexure and the torsion energies of the classical thin-walled beam model, see Theorem 5.2 for a precise statement. A further step of the analysis, which takes into account the case where the cross-section ω_ε has a multi-rectangular shape, is developed in a forthcoming paper [9].

The Γ-convergence of the family of energy functionals defined on Ω_ε gives not only the convergence of the energy of the three-dimensional problem to the corresponding energy of the limit problem, but permits also to obtain a remarkable amount of information about the structure and the behavior of the minimizers of the three-dimensional problem as ε goes to zero. In particular, the recovering sequence used in the proof of the limsup inequality of Theorem 5.2, allows to obtain a good approximation of the displacement field solution of the three-dimensional problem. The components of this recovering sequence show scaling with different powers of ε, and this reflect the fact, well known in practical applications, that for beams with thin-walled rectangular section some displacements are bigger than others.

The plan of the paper is as follows. In the next Section 2 we shall introduce the three-dimensional variational problem and some notations. In Section 3 we rewrite the three-dimensional problem as a variational problem for a rescaled energy defined in a fixed domain Ω_1, which is obtained by making a dilatation of Ω_ε in the cross-section plane. Section 4 is devoted to the proof of some compactness results for suitable families of functions defined on Ω_1. These compactness results will be obtained through Korn inequalities, stated in two and three dimensions, with a constant independent of ε. In Section 5 we prove the Γ-convergence of the family of three-dimensional energy functionals to the limit energy as ε goes to zero and the variational consequences are discussed in Section 6 and 7. Finally, the strong convergence of minimizers is proved in Section 8.

Notation. Throughout this article, and unless otherwise specified, we use the Einstein summation convention. Moreover we use the following convention for indexing vector and tensor components: Greek indices α, β and γ take their values in the set $\{1, 2\}$ and Latin indices i, j in the set $\{1, 2, 3\}$. The symbols $L^2(A; B)$ and $H^s(A; B)$ denote the standard Lebesgue and Sobolev spaces of functions defined on the domain A and taking values in B, with the usual norms $\|\cdot\|_{L^2(A;B)}$ and $\|\cdot\|_{H^s(A;B)}$, respectively. When $B = \mathbb{R}$ or when the right set B is clear from the context, we will simply write $L^2(A)$ or $H^s(A)$, sometimes even in the notation used for norms. With a little abuse of notation, and because this is a common practice and does not give rise to any mistake, we use to call "sequences" even those families indicized by a continuous parameter $\varepsilon \in (0, 1]$.

2 The 3-dimensional problem

We consider a three-dimensional body which is at rest in the placement

$$\Omega_\varepsilon := \omega_\varepsilon \times (0, \ell) \subset \mathbb{R}^3,$$

where

$$\omega_\varepsilon := \{(x_1, x_2) : |x_1| < a\varepsilon^2/2, \quad |x_2| < b\varepsilon/2\} \subset \mathbb{R}^2$$

and $\varepsilon \in (0, 1]$. For any $x_3 \in (0, \ell)$ we further set $S_\varepsilon(x_3) := \omega_\varepsilon \times \{x_3\}$.

Henceforth we shall refer to Ω_ε as the reference configuration of the body and denote by

$$\mathbf{Eu}(\mathbf{x}) := \mathrm{sym}(D\mathbf{u}(\mathbf{x})) := \frac{D\mathbf{u}(\mathbf{x}) + D\mathbf{u}^T(\mathbf{x})}{2}, \tag{1}$$

the strain of $\mathbf{u} : \Omega_\varepsilon \to \mathbb{R}^3$.

In what follows we consider the situation in which the body is subject only to *dead body forces* $\mathbf{b}^\varepsilon$, so that the equilibrium equations write as

$$\begin{cases} \mathrm{div}\mathbf{T} + \mathbf{b}^\varepsilon = \mathbf{0} & \text{in } \Omega_\varepsilon, \\ \mathbf{T} = \mathbb{C}\mathbf{Eu} & \text{in } \Omega_\varepsilon, \\ \mathbf{Tn} = \mathbf{0} & \text{on } \partial\Omega_\varepsilon \backslash S_\varepsilon(0), \\ \mathbf{u} = \mathbf{0} & \text{on } S_\varepsilon(0). \end{cases} \tag{2}$$

We consider an homogeneous isotropic material, so that

$$\mathbb{C}\mathbf{A} = 2\mu\mathbf{A} + \lambda(\mathrm{tr}\mathbf{A})\,\mathbf{I}$$

for every symmetric matrix $\mathbf{A}$. $\mathbf{I}$ denotes the identity matrix of order 3. We assume $\mu > 0$ and $\lambda \geq 0$ so to have, for every symmetric tensor $\mathbf{A}$,

$$\mathbb{C}\mathbf{A} \cdot \mathbf{A} \geq \mu|\mathbf{A}|^2, \tag{3}$$

where $\cdot$ denotes the scalar product. Define

$$H^1_\#(\Omega_\varepsilon; \mathbb{R}^3) := \left\{\mathbf{u} \in H^1(\Omega_\varepsilon; \mathbb{R}^3) : \mathbf{u} = \mathbf{0} \text{ on } S_\varepsilon(0)\right\}.$$

Due to the coercivity condition (3) and the strict convexity of the integrand, the energy functionals

$$J_\varepsilon(\mathbf{u}) := \frac{1}{2}\int_{\Omega_\varepsilon} \mathbb{C}\mathbf{Eu} \cdot \mathbf{Eu}\, dx - \int_{\Omega_\varepsilon} \mathbf{b}^\varepsilon \cdot \mathbf{u}\, dx, \tag{4}$$

admit, for every $\varepsilon > 0$, a unique minimizer among all competing displacements $\mathbf{u} \in H^1_\#(\Omega_\varepsilon; \mathbb{R}^3)$. As already explained in the introduction our aim is to study the asymptotic behavior of such minimizers as ε goes to 0, through the theory of Γ-convergence, for an account of it we refer to the books of Braides [3] and Dal Maso [7].

3 The rescaled problem

To state our results it is convenient to stretch the domain Ω_ε along the transverse directions x_1 and x_2 in a way that the transformed domain does not depend on ε. Let us therefore set $\omega := \omega_1$, $\Omega := \Omega_1$, $S(x_3) := S_1(x_3)$ and let

$$p_\varepsilon : \Omega \to \Omega_\varepsilon$$

be defined by

$$p_\varepsilon(\mathbf{y}) = p_\varepsilon(y_1, y_2, y_3) = (\varepsilon^2 y_1, \varepsilon y_2, y_3).$$

Let us consider the following 3×3 matrix

$$\mathbf{H}^\varepsilon \mathbf{v} := \left(\frac{D_1 \mathbf{v}}{\varepsilon^2}, \frac{D_2 \mathbf{v}}{\varepsilon}, D_3 \mathbf{v} \right), \tag{5}$$

where $D_i \mathbf{v}$ denotes the column vector of the partial derivatives of $\mathbf{v}$ with respect to x_i, $i = 1, 2, 3$. We will use moreover the following notation

$$\mathbf{E}^\varepsilon \mathbf{v} := \mathrm{sym}(\mathbf{H}^\varepsilon \mathbf{v}), \quad \mathbf{W}^\varepsilon \mathbf{v} := \mathrm{skw}(\mathbf{H}^\varepsilon \mathbf{v}), \tag{6}$$

and also denote by $\mathbf{W}\mathbf{v} := \mathbf{W}^1 \mathbf{v}$, the skew symmetric part of the gradient.

Let

$$H^1_\#(\Omega; \mathbb{R}^3) := \left\{ \mathbf{v} \in H^1(\Omega; \mathbb{R}^3) : \mathbf{v} = \mathbf{0} \text{ on } S(0) \right\};$$

then we can consider the rescaled energy $I_\varepsilon : H^1_\#(\Omega; \mathbb{R}^3) \to \mathbb{R} \cup \{+\infty\}$ defined by $I_\varepsilon(\mathbf{v}) := \frac{1}{\varepsilon^3} J_\varepsilon(\mathbf{v} \circ p_\varepsilon^{-1})$, i.e.,

$$I_\varepsilon(\mathbf{v}) = \frac{1}{2} \int_\Omega \mathbb{C} \mathbf{E}^\varepsilon \mathbf{v} \cdot \mathbf{E}^\varepsilon \mathbf{v} \, dy - \int_\Omega \mathbf{b}^\varepsilon \circ p_\varepsilon \cdot \mathbf{v} \, dy.$$

We further suppose the loads to have the following form

$$\begin{aligned} b_1^\varepsilon \circ p_\varepsilon(y) &= \varepsilon^4 b_1(y) - \varepsilon^3 \frac{m(y_3)}{I_0} y_2, \\ b_2^\varepsilon \circ p_\varepsilon(y) &= \varepsilon^3 b_2(y) + \varepsilon^2 \frac{m(y_3)}{I_0} y_1, \\ b_3^\varepsilon \circ p_\varepsilon(y) &= \varepsilon^2 b_3(y), \end{aligned} \tag{7}$$

with $\mathbf{b} = (b_1, b_2, b_3) \in L^2(\Omega; \mathbb{R}^3)$, and $m \in L^2(0, \ell)$. Above I_0 denotes the polar moment of inertia of the section ω,

$$I_0 := \int_\omega y_1^2 + y_2^2 \, dy_1 \, dy_2 = \frac{1}{12}(a^3 b + a b^3).$$

We note that while $\mathbf{b}$ has the units of a force per unit of volume, m has the units of a force, or, equivalently, of a moment per unit of length. The scalings of the loads are chosen in a way to keep the displacements bounded as ε goes to zero. With the loads given by (7) the energy $I_\varepsilon(\mathbf{v})$ can be rewritten as

$$\begin{aligned} I_\varepsilon(\mathbf{v}) = \frac{1}{2} \int_\Omega \mathbb{C} \mathbf{E}^\varepsilon \mathbf{v} \cdot \mathbf{E}^\varepsilon \mathbf{v} \, dy \; &- \varepsilon^4 \int_\Omega \mathbf{b} \cdot (v_1, \frac{v_2}{\varepsilon}, \frac{v_3}{\varepsilon^2}) \, dy + \\ &- \varepsilon^4 \int_0^\ell m \, \vartheta^\varepsilon(\mathbf{v}) \, dy_3, \end{aligned} \tag{8}$$

where we have set

$$\vartheta^\varepsilon(\mathbf{v})(y_3) := \frac{1}{I_0}\int_\omega \frac{y_1}{\varepsilon^2} v_2(y_1,y_2,y_3) - \frac{y_2}{\varepsilon} v_1(y_1,y_2,y_3)\,dy_1\,dy_2. \tag{9}$$

We note that if $\mathbf{v} \in L^2(\Omega;\mathbb{R}^3)$ then $\vartheta^\varepsilon(\mathbf{v}) \in L^2(0,\ell)$. A similar statement holds if we replace L^2 with H^1.

4 Compactness lemmata

On the untransformed domain Ω_ε, the following Korn-like inequality holds.

Theorem 4.1 *There exists a constant $C > 0$ such that*

$$\int_{\Omega_\varepsilon} \left(|\mathbf{u}|^2 + |D\mathbf{u}|^2\right)\,dx \le \frac{C}{\varepsilon^4}\int_{\Omega_\varepsilon} |\mathbf{E}\mathbf{u}|^2\,dx \tag{10}$$

for every $\mathbf{u} \in H^1(\Omega_\varepsilon;\mathbb{R}^3)$ with $\mathbf{u} = \mathbf{0}$ on $S_\varepsilon(0)$.

Proof. Divide the section ω_ε in squares of size ε^2 and apply Korn's inequality (the one obtained by Anzellotti, Baldo and Percivale in [1]; see also [18] and Kondrat'ev and Oleinik [10], Theorem 2) to each beam of length ℓ and with section a square with side proportional to ε^2. Then sum over all the obtained inequalities. □

To prove the compactness of the displacements we need the following scaled Korn inequality.

Theorem 4.2 *There exists a constant $C > 0$ such that*

$$\int_\Omega \left(|(u_1, u_2/\varepsilon, u_3/\varepsilon^2)|^2 + |\mathbf{H}^\varepsilon\mathbf{u}|^2\right)\,dy \le \frac{C}{\varepsilon^4}\int_\Omega |\mathbf{E}^\varepsilon\mathbf{u}|^2\,dy \tag{11}$$

for every $\mathbf{u} \in H^1_\#(\Omega;\mathbb{R}^3)$ and every $0 < \varepsilon \le 1$.

Proof. The inequality $\int_\Omega |\mathbf{H}^\varepsilon\mathbf{u}|^2\,dy \le \frac{C}{\varepsilon^4}\int_\Omega |\mathbf{E}^\varepsilon\mathbf{u}|^2\,dy$ is simply obtained by rescaling inequality (10). To show that

$$\int_\Omega |(u_1, u_2/\varepsilon, u_3/\varepsilon^2)|^2\,dy \le \frac{C}{\varepsilon^4}\int_\Omega |\mathbf{E}^\varepsilon\mathbf{u}|^2\,dy,$$

it suffices to set $\mathbf{v} := (u_1, u_2/\varepsilon, u_3/\varepsilon^2)$, notice that $|\mathbf{E}^\varepsilon\mathbf{u}| \ge \varepsilon^2|\mathbf{E}\mathbf{v}|$ and apply the standard Korn inequality to $\mathbf{v}$ on the domain Ω (see for instance [17], Theorem 2.7). □

Let

$$H_{BN}(\Omega;\mathbb{R}^3) := \left\{\mathbf{v} \in H^1_\#(\Omega;\mathbb{R}^3) : (\mathbf{E}\mathbf{v})_{i\alpha} = 0 \text{ for } i = 1,2,3\ \alpha = 1,2\right\}, \tag{12}$$

be the space of Bernoulli-Navier displacements on Ω. This space can be characterized also as follows (see Le Dret [11], Section 4.1)

$$H_{BN}(\Omega;\mathbb{R}^3) = \big\{\mathbf{v} \in H^1_\#(\Omega;\mathbb{R}^3) : \begin{array}{l} \exists\, \xi_\alpha \in H^2_\#(0,\ell), \\ \exists\, \xi_3 \in H^1_\#(0,\ell) \text{ such that} \\ v_\alpha(y) = \xi_\alpha(y_3), \\ v_3(y) = \xi_3(y_3) - y_\alpha \xi'_\alpha(y_3)\big\}. \end{array} \tag{13}$$

The remaining part of this section will be devoted to prove some compactness lemmata which will be stated under the common assumption that $\mathbf{u}^\varepsilon$ be a sequence of functions in $H^1_\#(\Omega;\mathbb{R}^3)$ such that

$$\|\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C\varepsilon^2, \tag{14}$$

for some constant C and every $0 < \varepsilon \leq 1$.

Lemma 4.3 *Let us assume* (14)*. Then, for any sequence of positive numbers* ε_n *converging to* 0 *there exist a subsequence (not relabeled) and a pair of functions* $\mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3)$ *and* $\vartheta \in L^2(\Omega)$ *such that (as* $n \to \infty$*)*

$$\Big(u_1^{\varepsilon_n}, \frac{u_2^{\varepsilon_n}}{\varepsilon_n}, \frac{u_3^{\varepsilon_n}}{\varepsilon_n^2}\Big) \to \mathbf{v} \quad \textit{weakly in } H^1(\Omega;\mathbb{R}^3), \tag{15}$$

$$\mathbf{W}^{\varepsilon_n}\mathbf{u}^{\varepsilon_n} \to \begin{pmatrix} 0 & -\vartheta & D_3 v_1 \\ \vartheta & 0 & 0 \\ -D_3 v_1 & 0 & 0 \end{pmatrix} \quad \textit{weakly in } L^2(\Omega;\mathbb{R}^{3\times 3}). \tag{16}$$

Proof. It is convenient to set $\mathbf{v}^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon/\varepsilon, u_3^\varepsilon/\varepsilon^2)$. It is easily checked that for $\varepsilon \leq 1$, $|\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon| \geq \varepsilon^2|\mathbf{E}\mathbf{v}^\varepsilon|$, hence, by (14), $\mathbf{E}\mathbf{v}^\varepsilon$ is uniformly bounded in $L^2(\Omega;\mathbb{R}^{3\times 3})$, and by Korn's inequality $\mathbf{v}^\varepsilon$ is uniformly bounded in $H^1(\Omega;\mathbb{R}^3)$. It then exists a $\mathbf{v} \in H^1_\#(\Omega;\mathbb{R}^3)$, and a subsequence (not relabeled) of ε_n such that $\mathbf{v}^{\varepsilon_n} \to \mathbf{v}$ weakly in $H^1(\Omega;\mathbb{R}^3)$. Again, it is easy to check that $|(\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon)_{i\alpha}| \geq \varepsilon|(\mathbf{E}\mathbf{v}^\varepsilon)_{i\alpha}|$, thus, using (14), we deduce that $C\varepsilon \geq \|(\mathbf{E}\mathbf{v}^\varepsilon)_{i\alpha}\|_{L^2(\Omega)}$ and consequently $(\mathbf{E}\mathbf{v})_{i\alpha} = 0$ for $i = 1,2,3$ and $\alpha = 1,2$. Hence $\mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3)$.

Using assumption (14) together with Theorem 4.2 we obtain that the sequence $\mathbf{H}^{\varepsilon_n}\mathbf{u}^{\varepsilon_n}$ is bounded in L^2 so that, up to subsequences, it weakly converges in $L^2(\Omega;\mathbb{R}^{3\times 3})$ to some $\mathbf{H} \in L^2(\Omega;\mathbb{R}^{3\times 3})$. Since, from (14), $\mathbf{E}^{\varepsilon_n}\mathbf{u}^{\varepsilon_n} \to \mathbf{0}$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$ we have $\mathbf{W}^{\varepsilon_n}\mathbf{u}^{\varepsilon_n} \to \mathbf{H}$ weakly in $L^2(\Omega;\mathbb{R}^{3\times 3})$. In particular,

$\mathbf{H}$ is, almost everywhere, a skew-symmetric matrix. Since, on the other hand, $(\mathbf{H}^\varepsilon \mathbf{u}^\varepsilon)_{13} = D_3 u_1^\varepsilon = D_3 v_1^\varepsilon$, and $(\mathbf{H}^\varepsilon \mathbf{u}^\varepsilon)_{23} = D_3 u_2^\varepsilon = \varepsilon D_3 v_2^\varepsilon$ we immediately deduce that $(\mathbf{H})_{13} = D_3 v_1$ and $(\mathbf{H})_{23} = 0$. Denoting $(\mathbf{H})_{12} = -\vartheta$ we obtain (16). □

Let $\mathcal{E}_{\alpha\beta}$ denote the Ricci's symbol, thus $\mathcal{E}_{11} = \mathcal{E}_{22} = 0$, $\mathcal{E}_{12} = 1$ and $\mathcal{E}_{21} = -1$. Define (using the summation convention)

$$\Re_2 = \{\mathbf{r} \in L^2(\omega; \mathbb{R}^2) : \exists\, \varphi \in \mathbb{R},\, \mathbf{c} \in \mathbb{R}^2 \text{ s.t. } r_\alpha(y) = \mathcal{E}_{\beta\alpha}\, y_\beta\, \varphi + c_\alpha\}.$$

The elements of $\Re_2$ are the infinitesimal rigid displacements on ω. It is easy to see that $\Re_2 \subset H^1(\omega; \mathbb{R}^2)$, moreover, since $\Re_2$ is a finite-dimensional vector subspace, it is closed in $H^1(\omega; \mathbb{R}^2)$. Let $\Re_2^\perp$ be the Hilbertian orthogonal complement of $\Re_2$ in $L^2(\omega; \mathbb{R}^2)$, i.e.,

$$\Re_2^\perp = \{\mathbf{v} \in L^2(\omega; \mathbb{R}^2) : \int_\omega \mathbf{v} \cdot \mathbf{r}\, dy_1\, dy_2 = 0 \text{ for every } \mathbf{r} \in \Re_2\}. \tag{17}$$

Then $L^2(\omega; \mathbb{R}^2) = \Re_2 \oplus \Re_2^\perp$. Let $\wp$ be the projection of $L^2(\omega; \mathbb{R}^2)$ on $\Re_2$. Then if $\mathbf{w} \in L^2(\omega; \mathbb{R}^2)$ and $\{\mathbf{e}_1, \mathbf{e}_2\}$ denotes the canonical basis of $\mathbb{R}^2$, it is easily seen, taking as test function $\mathbf{r} = \mathbf{e}_\alpha$ and $\mathbf{r} = \mathcal{E}_{\beta\alpha}\, y_\beta \mathbf{e}_\alpha$, that

$$\wp \mathbf{w}_\alpha = \mathcal{E}_{\beta\alpha}\, y_\beta \left(\frac{1}{I_0} \int_\omega \mathcal{E}_{\gamma\delta}\, y_\gamma w_\delta\, dy_1\, dy_2 \right) + \frac{1}{|\omega|} \int_\omega w_\alpha\, dy_1\, dy_2. \tag{18}$$

The two-dimensional Korn's inequality then writes as

$$\|\mathbf{w} - \wp \mathbf{w}\|_{H^1(\omega;\mathbb{R}^3)} \leq C\, \|\mathbf{E}\mathbf{w}\|_{L^2(\omega;\mathbb{R}^{2\times 2})}, \tag{19}$$

for all $\mathbf{w} \in H^1(\omega; \mathbb{R}^2)$.

Lemma 4.4 *Under assumption* (14) *and the notation of Lemma* 4.3 *and of (9) we have*

$$\vartheta^\varepsilon(\mathbf{u}^\varepsilon) \to \vartheta \quad \textit{weakly in } L^2(\Omega).$$

Therefore, ϑ does not depend on y_1 and y_2.

Proof. It is convenient to set $\mathbf{w}^\varepsilon := (u_1^\varepsilon/\varepsilon, u_2^\varepsilon/\varepsilon^2, u_3^\varepsilon/\varepsilon^3)$. Let $\wp$ be the projection of $L^2(\omega; \mathbb{R}^2)$ on $\Re_2$. Then for almost every $y_3 \in (0, \ell)$ we consider the projection of $\mathbf{w}^\varepsilon(\cdot, y_3)$. From equation (18), and recalling (9), we find

$$\wp \mathbf{w}_\alpha^\varepsilon = \mathcal{E}_{\beta\alpha}\, y_\beta\, \vartheta^\varepsilon(\mathbf{u}^\varepsilon) + \frac{1}{|\omega|} \int_\omega w_\alpha^\varepsilon\, dy_1\, dy_2. \tag{20}$$

Since, furthermore, $(\mathbf{E}\mathbf{w}^\varepsilon)_{11} = \varepsilon(\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon)_{11}$, $(\mathbf{E}\mathbf{w}^\varepsilon)_{12} = (\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon)_{12}$, and $(\mathbf{E}\mathbf{w}^\varepsilon)_{22} = (\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon)_{22}/\varepsilon$, we have

$$\|(\mathbf{E}\mathbf{w}^\varepsilon)_{\alpha\beta}\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} \leq \frac{1}{\varepsilon} \|(\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon)_{\alpha\beta}\|_{L^2(\Omega;\mathbb{R}^{2\times 2})}. \tag{21}$$

Hence, integrating (19) on $(0,\ell)$ and taking into account (21) and (14), we deduce that

$$\|D_\alpha(\mathbf{w}^\varepsilon - \wp\mathbf{w}^\varepsilon)\|_{L^2(\Omega;\mathbb{R})} \to 0, \tag{22}$$

for $\alpha = 1,2$. Since $(\mathbf{W}\wp\mathbf{w}^\varepsilon)_{12} = -\vartheta^\varepsilon(\mathbf{u}^\varepsilon)$ and $(\mathbf{W}\mathbf{w}^\varepsilon)_{12} = (\mathbf{W}^\varepsilon\mathbf{u}^\varepsilon)_{12}$ we find from the identity

$$\vartheta^\varepsilon(\mathbf{u}^\varepsilon) = -(\mathbf{W}\wp\mathbf{w}^\varepsilon)_{12} = -(\mathbf{W}^\varepsilon\mathbf{u}^\varepsilon)_{12} + (\mathbf{W}(\mathbf{w}^\varepsilon - \wp\mathbf{w}^\varepsilon))_{12} \tag{23}$$

the first claim of the Lemma, by letting ε to 0 and recalling (16). From the fact that $\vartheta^\varepsilon(\mathbf{u}^\varepsilon)$ does not depend on y_1 and y_2 follows the second claim. □

That ϑ does not depend on y_1 and y_2, can be also easily proved by using (14) and (16). Indeed, it suffices to take $\psi \in C_0^\infty(\Omega)$ and to note that

$$\int_\Omega \frac{D_2 u_1^\varepsilon}{\varepsilon} D_1\psi\, dy = \int_\Omega \frac{D_1 u_1^\varepsilon}{\varepsilon} D_2\psi\, dy = \int_\Omega \varepsilon(\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon)_{11} D_2\psi\, dy.$$

Finally, taking the limit as ε goes to zero, we find

$$\int_\Omega \vartheta D_1\psi\, dy = 0,$$

and hence that ϑ is independent of y_1. A similar argument shows also that ϑ does not depend on y_2.

Remark 4.5 From (19), (21) and (23) follows that

$$\|\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\|_{L^2(\Omega)} \le \|(\mathbf{W}\mathbf{w}^\varepsilon)\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} + C\frac{1}{\varepsilon}\|\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})},$$

and hence from Theorem 4.2 we deduce

$$\|\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\|_{L^2(\Omega)} \le C\frac{1}{\varepsilon^2}\|\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}. \tag{24}$$

We now prove that indeed $\vartheta \in H^1_\#(\Omega)$.

Lemma 4.6 *Under assumption* (14) *and with the notation of Lemma* 4.3 *we have* $\vartheta \in H^1_\#(\Omega)$.

Proof. As before, it is convenient to set $\mathbf{w}^\varepsilon := (u_1^\varepsilon/\varepsilon, u_2^\varepsilon/\varepsilon^2, u_3^\varepsilon/\varepsilon^3)$. Let $\xi \in C_0^\infty(\omega)$ be such that

$$\int_\omega \xi\, dy_1\, dy_2 = -\frac{I_0}{2}.$$

Then, taking into account (20), we have

$$\begin{aligned}
I_0\vartheta^\varepsilon(\mathbf{u}^\varepsilon) &= -2\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\int_\omega \xi\, dy_1\, dy_2 = -\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\int_\omega \xi D_\alpha y_\alpha\, dy_1\, dy_2\\
&= \vartheta^\varepsilon(\mathbf{u}^\varepsilon)\int_\omega D_\alpha\xi\, y_\alpha\, dy_1\, dy_2 = \vartheta^\varepsilon(\mathbf{u}^\varepsilon)\int_\omega \mathcal{E}_{\alpha\gamma}\mathcal{E}_{\beta\gamma}D_\alpha\xi\, y_\beta\, dy_1\, dy_2\\
&= \int_\omega \mathcal{E}_{\alpha\gamma}D_\alpha\xi(\mathcal{E}_{\beta\gamma}y_\beta\vartheta^\varepsilon(\mathbf{u}^\varepsilon))\, dy_1\, dy_2\\
&= \int_\omega \mathcal{E}_{\alpha\gamma}D_\alpha\xi(\wp\mathbf{w}^\varepsilon_\gamma - \frac{1}{|\omega|}\int_\omega w^\varepsilon_\gamma\, dy_1\, dy_2)\, dy_1\, dy_2\\
&= \int_\omega \mathcal{E}_{\alpha\gamma}D_\alpha\xi\, \wp\mathbf{w}^\varepsilon_\gamma\, dy_1\, dy_2\\
&= \int_\omega \mathcal{E}_{\alpha\gamma}D_\alpha\xi\, w^\varepsilon_\gamma\, dy_1\, dy_2 - \int_\omega \mathcal{E}_{\alpha\gamma}D_\alpha\xi(\mathbf{w}^\varepsilon - \wp\mathbf{w}^\varepsilon)_\gamma\, dy_1\, dy_2.
\end{aligned}$$

Hence denoting by

$$\tilde{\vartheta}^\varepsilon = \frac{1}{I_0}\int_\omega \mathcal{E}_{\alpha\gamma}D_\alpha\xi\, w^\varepsilon_\gamma\, dy_1\, dy_2,$$

and recalling (22), we find

$$\vartheta^\varepsilon(\mathbf{u}^\varepsilon) - \tilde{\vartheta}^\varepsilon \to 0 \ \text{ strongly in } L^2(\Omega). \tag{25}$$

We now show that $D_3\tilde{\vartheta}^\varepsilon$ is bounded in L^2. Since $\mathcal{E}_{\alpha\gamma}D_\alpha D_\gamma\xi = 0$ everywhere in ω and $D_\alpha\xi = 0$ on $\partial\omega$, we have

$$\begin{aligned}
I_0 D_3\tilde{\vartheta}^\varepsilon &= \int_\omega \mathcal{E}_{\alpha\gamma}D_\alpha\xi\, D_3 w^\varepsilon_\gamma\, dy_1\, dy_2\\
&= 2\int_\omega \mathcal{E}_{\alpha\gamma}D_\alpha\xi\, (\mathbf{E}\mathbf{w}^\varepsilon)_{\gamma 3}\, dy_1\, dy_2 - \int_\omega \mathcal{E}_{\alpha\gamma}D_\alpha\xi\, D_\gamma w^\varepsilon_3\, dy_1\, dy_2\\
&= 2\int_\omega \mathcal{E}_{\alpha\gamma}D_\alpha\xi\, (\mathbf{E}\mathbf{w}^\varepsilon)_{\gamma 3}\, dy_1\, dy_2 - \int_\omega D_\gamma(\mathcal{E}_{\alpha\gamma}D_\alpha\xi\, w^\varepsilon_3)\, dy_1\, dy_2 +\\
&\qquad + \int_\omega \mathcal{E}_{\alpha\gamma}D_\alpha D_\gamma\xi w^\varepsilon_3\, dy_1\, dy_2\\
&= 2\int_\omega \mathcal{E}_{\alpha\gamma}D_\alpha\xi\, (\mathbf{E}\mathbf{w}^\varepsilon)_{\gamma 3}\, dy_1\, dy_2,
\end{aligned}$$

but $(\mathbf{E}\mathbf{w}^\varepsilon)_{13} = (\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon)_{13}/\varepsilon$ and $(\mathbf{E}\mathbf{w}^\varepsilon)_{23} = (\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon)_{23}/\varepsilon^2$, and therefore $D_3\tilde{\vartheta}^\varepsilon$ is bounded in $L^2(0,\ell)$. Thus, from (25) and Lemma 4.4 we conclude that

$$\tilde{\vartheta}^\varepsilon \to \vartheta \text{ weakly in } H^1(\Omega).$$

Therefore, since $\tilde{\vartheta}^\varepsilon(0) = 0$, we conclude that $\vartheta \in H^1_\#(\Omega)$. □

Lemma 4.7 *Under the same assumptions and with the notation of Lemma 4.3 we have, up to subsequences,*

$$\frac{(\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon)_{33}}{\varepsilon^2} \to D_3 v_3 \quad \textit{weakly in } L^2(\Omega), \tag{26}$$

$$\frac{(\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon)_{23}}{\varepsilon^2} \to y_1 D_3 \vartheta + \eta \quad \textit{weakly in } L^2(\Omega), \tag{27}$$

where $\eta \in L^2(\Omega)$ *is independent of* y_1.

Proof. To prove (26) it suffices to notice that $\frac{(\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon)_{33}}{\varepsilon^2} = D_3 \frac{u_3^\varepsilon}{\varepsilon^2}$ and apply (15).

From (14) we deduce that, up to subsequences, $\frac{(\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon)_{23}}{\varepsilon^2} \to E_{23}$ weakly in $L^2(\Omega)$. To characterize $E_{23} \in L^2(\Omega)$ note that

$$\begin{aligned} 2D_3(\mathbf{W}^\varepsilon \mathbf{u}^\varepsilon)_{12} &= D_3\left(\frac{D_2 u_1^\varepsilon}{\varepsilon} - \frac{D_1 u_2^\varepsilon}{\varepsilon^2}\right) \\ &= D_2\left(\frac{D_3 u_1^\varepsilon}{\varepsilon} + \frac{D_1 u_3^\varepsilon}{\varepsilon^3}\right) - D_1\left(\frac{D_2 u_3^\varepsilon}{\varepsilon^3} + \frac{D_3 u_2^\varepsilon}{\varepsilon^2}\right) \\ &= 2D_2 \frac{(\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon)_{13}}{\varepsilon} - 2D_1 \frac{(\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon)_{23}}{\varepsilon^2}, \end{aligned}$$

in the sense of distributions. Hence for $\psi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega (\mathbf{W}^\varepsilon \mathbf{u}^\varepsilon)_{12} D_3 \psi \, dy = \int_\Omega \frac{(\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon)_{13}}{\varepsilon} D_2 \psi \, dy - \int_\Omega \frac{(\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon)_{23}}{\varepsilon^2} D_1 \psi \, dy.$$

On the other hand, using (14) we have that $\frac{(\mathbf{E}^\varepsilon \mathbf{u}^\varepsilon)_{13}}{\varepsilon} \to 0$ weakly in $L^2(\Omega)$. Hence, passing to the limit in the previous equality we find

$$\int_\Omega -\vartheta D_3 \psi \, dy = -\int_\Omega E_{23} D_1 \psi \, dy.$$

Thus $D_1 E_{23} = D_3 \vartheta$, in the sense of distributions, and therefore, taking into account that ϑ is independent of y_1 we have that $E_{23} = y_1 D_3 \vartheta + \eta$, with η like in the statement of the Lemma. □

5 The limit energy

Define

$$f_0(\alpha, \beta) := \min\{f(\mathbf{A}) \ : \ \mathbf{A} \in \mathrm{Sym},\ A_{23} = \alpha,\ A_{33} = \beta\}$$

where

$$f(\mathbf{A}) = \frac{1}{2}\mathbb{C}\mathbf{A}\cdot\mathbf{A} = \mu|\mathbf{A}|^2 + \frac{\lambda}{2}|\mathrm{tr}\mathbf{A}|^2. \tag{28}$$

A simple computation shows that

$$f_0(\alpha,\beta) := 2\mu\alpha^2 + \frac{1}{2}E\beta^2$$

where E is the Young modulus

$$E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}.$$

Lemma 5.1 *Let $\mathbf{u}^\varepsilon$ be a sequence of functions in $H^1_\#(\Omega;\mathbb{R}^3)$. If*

$$\sup_\varepsilon \frac{1}{\varepsilon^4} I_\varepsilon(\mathbf{u}^\varepsilon) < +\infty, \tag{29}$$

then (14) *holds for some constant C.*

Proof. It is convenient to set $\mathbf{v}^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon/\varepsilon, u_3^\varepsilon/\varepsilon^2)$. With this notation and using (3), (8), (11) and (24) we find

$$\begin{aligned}\frac{1}{\varepsilon^4}I_\varepsilon(\mathbf{u}^\varepsilon) &= \frac{1}{2}\int_\Omega \mathbb{C}\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2}\cdot\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2}\,dy - \int_\Omega \mathbf{b}\cdot\mathbf{v}^\varepsilon\,dy - \int_0^\ell m\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\,dy_3\\
&\geq \frac{\mu}{2}\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2}\|^2_{L^2(\Omega)} - \|\mathbf{b}\|_{L^2(\Omega)}\|\mathbf{v}^\varepsilon\|_{L^2(\Omega)} - \|m\|_{L^2(0,\ell)}\|\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\|_{L^2(0,\ell)}\\
&\geq \frac{\mu}{2}\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2}\|^2_{L^2(\Omega)} - \frac{1}{2C_1}\|\mathbf{b}\|^2_{L^2(\Omega)} - \frac{C_1}{2}\|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)} +\\
&\qquad - \frac{1}{2C_2}\|m\|^2_{L^2(0,\ell)} - \frac{C_2}{2}\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2}\|^2_{L^2(\Omega)},\end{aligned}$$

whenever $\frac{1}{\varepsilon^2}\|\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon\|_{L^2(\Omega)} \geq 1$, and where C_1 and C_2 are arbitrary positive constants. Choosing $C_2 = \mu/2$, we get

$$\begin{aligned}\frac{1}{\varepsilon^4}I_\varepsilon(\mathbf{u}^\varepsilon) \geq \frac{\mu}{4}\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2}\|^2_{L^2(\Omega)} - \frac{1}{2C_1}\|\mathbf{b}\|^2_{L^2(\Omega)} &- \frac{C_1}{2}\|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)} +\\
&- \frac{1}{\mu}\|m\|^2_{L^2(0,\ell)}.\end{aligned} \tag{30}$$

From Theorem 4.2 we have

$$\begin{aligned}\frac{1}{\varepsilon^4}I_\varepsilon(\mathbf{u}^\varepsilon) \geq \frac{\mu}{4C}\|\mathbf{H}^\varepsilon\mathbf{u}^\varepsilon\|^2_{L^2(\Omega)} + \big(\frac{1}{C} - \frac{C_1}{2}\big)\|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)} &- \frac{1}{2C_1}\|\mathbf{b}\|^2_{L^2(\Omega)}\\
&- \frac{1}{\mu}\|m\|^2_{L^2(0,\ell)}\end{aligned}$$

where C is the constant of Theorem 4.2. By choosing for instance $C_1 = 1/C$, and using assumption (29), we obtain that there exists a constant $M > 0$ such that

$$M \geq \frac{\mu}{4C} \|\mathbf{H}^\varepsilon \mathbf{u}^\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2C} \|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)}$$

from which follows that the sequence $\mathbf{v}^\varepsilon$ is bounded in $L^2(\Omega; \mathbb{R}^3)$. Using this fact in (30) we finally get the estimate (14). □

The above Lemma 5.1 and Lemma 4.3 imply that the family of functionals $\frac{1}{\varepsilon^4} I_\varepsilon$ is coercive with respect to the weak convergence of the sequence

$$q_\varepsilon(\mathbf{u}^\varepsilon) := \left(u_1^\varepsilon, \frac{u_2^\varepsilon}{\varepsilon}, \frac{u_3^\varepsilon}{\varepsilon^2}, (\mathbf{W}^\varepsilon \mathbf{u}^\varepsilon)_{12}\right) \tag{31}$$

in the space $H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R})$, uniformly with respect to ε. Hence, for any sequence $\mathbf{u}^\varepsilon$ which is bounded in energy, that is $\frac{1}{\varepsilon^4} I_\varepsilon(\mathbf{u}_\varepsilon) \leq C$ for a suitable constant $C > 0$, and satisfies the boundary conditions, the corresponding sequence $q_\varepsilon(\mathbf{u}^\varepsilon)$ is weakly relatively compact in $H^1(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R})$, and the following convergence result characterizes the weak limits of such sequences.

Theorem 5.2 *Let* $I : H^1_\#(\Omega; \mathbb{R}^3) \times H^1_\#(\Omega; \mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ *be defined by*

$$I(\mathbf{v}, \vartheta) := \int_\Omega f_0(y_1 D_3 \vartheta, D_3 v_3)\, dy - \int_\Omega \mathbf{b} \cdot \mathbf{v}\, dy - \int_0^\ell m\, \vartheta\, dy_3 \tag{32}$$

if $\mathbf{v} \in H_{BN}(\Omega; \mathbb{R}^3)$, *and* $+\infty$ *otherwise.*

As $\varepsilon \to 0^+$, *the sequence of functionals* $\frac{1}{\varepsilon^4} I_\varepsilon$ Γ*-converges to the functional* I, *in the following sense:*

1. *[liminf inequality] for every sequence of positive numbers* ε_k *converging to* 0 *and for every sequence* $\{\mathbf{u}^{\varepsilon_k}\} \subset H^1_\#(\Omega; \mathbb{R}^3)$ *such that* $\left(u_1^{\varepsilon_k}, \frac{u_2^{\varepsilon_k}}{\varepsilon_k}, \frac{u_3^{\varepsilon_k}}{\varepsilon_k^2}\right) \to \mathbf{v}$ *weakly in* $H^1(\Omega; \mathbb{R}^3)$, *and* $(\mathbf{W}^{\varepsilon_k} \mathbf{u}^{\varepsilon_k})_{12} \to -\vartheta$ *weakly in* $L^2(\Omega)$,
$$\liminf_{k \to +\infty} \frac{1}{\varepsilon_k^4} I_{\varepsilon_k}(\mathbf{u}^{\varepsilon_k}) \geq I(\mathbf{v}, \vartheta);$$

2. *[recovering sequence] for every sequence of positive numbers* ε_k *converging to* 0 *and for every* $(\mathbf{v}, \vartheta) \in H^1_\#(\Omega; \mathbb{R}^3) \times H^1_\#(\Omega; \mathbb{R})$ *there exists a subsequence* ε_{k_n} *and a sequence* $\{\mathbf{u}^n\} \subset H^1_\#(\Omega; \mathbb{R}^3)$ *such that* $\left(u_1^n, \frac{u_2^n}{\varepsilon_{k_n}}, \frac{u_3^n}{\varepsilon_{k_n}^2}\right) \to \mathbf{v}$ *weakly in* $H^1(\Omega; \mathbb{R}^3)$, $(\mathbf{W}^{\varepsilon_{k_n}} \mathbf{u}^n)_{12} \to -\vartheta$ *weakly in* $L^2(\Omega)$ *and*
$$\limsup_{n \to +\infty} \frac{1}{\varepsilon_{k_n}^4} I_{\varepsilon_{k_n}}(\mathbf{u}^n) \leq I(\mathbf{v}, \vartheta).$$

Proof. We start by proving the liminf inequality. Without loss of generality we may suppose that

$$\liminf_{k\to+\infty}\frac{1}{\varepsilon_k^4}I_{\varepsilon_k}(\mathbf{u}^{\varepsilon_k})=\lim_{k\to+\infty}\frac{1}{\varepsilon_k^4}I_{\varepsilon_k}(\mathbf{u}^{\varepsilon_k})<+\infty,$$

hence the results of Lemma 4.7 hold. Looking at the expression (8) of the functional I_ε and observing that, from the definitions of f and f_0 given at the beginning of this section,

$$\frac{1}{2}\mathbb{C}\mathbf{A}\cdot\mathbf{A}\geq f_0(A_{23},A_{33}),$$

then we have

$$\begin{aligned}\frac{1}{\varepsilon_k^4}I_{\varepsilon_k}(\mathbf{u}^{\varepsilon_k})\geq\ &\int_\Omega f_0(\frac{(\mathbf{E}^{\varepsilon_k}\mathbf{u}^{\varepsilon_k})_{23}}{\varepsilon_k^2},\frac{(\mathbf{E}^{\varepsilon_k}\mathbf{u}^{\varepsilon_k})_{33}}{\varepsilon_k^2})\,dy+\\ &-\int_\Omega\mathbf{b}\cdot\left(u_1^{\varepsilon_k},\frac{u_2^{\varepsilon_k}}{\varepsilon_k},\frac{u_3^{\varepsilon_k}}{\varepsilon_k^2}\right)\,dy-\int_0^\ell m\,\vartheta^{\varepsilon_k}(\mathbf{u}^{\varepsilon_k})\,dy_3.\end{aligned}$$

Using the convexity of f_0, Lemma 4.4 and Lemma 4.7 we find

$$\begin{aligned}\liminf_{k\to+\infty}\frac{1}{\varepsilon_k^4}I_{\varepsilon_k}(\mathbf{u}^{\varepsilon_k})\ &\geq\int_\Omega f_0(y_1D_3\vartheta+\eta,D_3v_3)\,dy-\int_\Omega\mathbf{b}\cdot\mathbf{v}\,dy+\\ &\qquad-\int_0^\ell m\,\vartheta\,dy_3\\ &=\int_\Omega f_0(y_1D_3\vartheta,D_3v_3)\,dy-\int_\Omega\mathbf{b}\cdot\mathbf{v}\,dy-\int_0^\ell m\,\vartheta\,dy_3+\\ &\qquad+4\int_\Omega\mu y_1D_3\vartheta(y_3)\eta(y_2,y_3)\,dy+2\int_\Omega\mu\eta^2\,dy.\end{aligned}$$

The first integral on the line above is equal to zero, and hence, taking into account that the second integral in the line above is positive we deduce

$$\liminf_{k\to+\infty}\frac{1}{\varepsilon_k^4}I_{\varepsilon_k}(\mathbf{u}^{\varepsilon_k})\geq\int_\Omega f_0(y_1D_3\vartheta,D_3v_3)\,dy-\int_\Omega\mathbf{b}\cdot\mathbf{v}\,dy-\int_0^\ell m\,\vartheta\,dy_3.$$

We now find a recovering sequence. We first note that

$$f_0(\alpha,\beta)=f(\mathbf{\Lambda}(\alpha,\beta)),$$

with $\mathbf{\Lambda}(\alpha,\beta)$ a symmetric matrix with

$$\begin{aligned}&\Lambda_{11}(\alpha,\beta)=\Lambda_{22}(\alpha,\beta)=-\nu\beta,\quad \Lambda_{33}(\alpha,\beta)=\beta,\\ &\Lambda_{12}(\alpha,\beta)=\Lambda_{13}(\alpha,\beta)=0,\qquad\ \ \Lambda_{23}(\alpha,\beta)=\alpha,\end{aligned}\tag{33}$$

where ν denotes the Poisson's coefficient,

$$\nu = \frac{\lambda}{2(\lambda+\mu)}.$$

If $I(\mathbf{v},\vartheta) = +\infty$ there is nothing to prove. Let $I(\mathbf{v},\vartheta) < +\infty$, then $\mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3)$ and $\vartheta \in H^1_{\#}(\Omega;\mathbb{R})$.

We first further assume $\mathbf{v}$ and ϑ smooth and equal to zero near $y_3 = 0$. By (13) there exists $\boldsymbol{\xi}$ smooth and equal to zero near $y_3 = 0$ such that $v_\alpha(y) = \xi_\alpha(y_3)$, and $v_3(y) = \xi_3(y_3) - y_\alpha \xi'_\alpha(y_3)$. Then the function $\mathbf{u}^{0,\varepsilon}$, defined by

$$\begin{aligned}
u_1^{0,\varepsilon} &= \xi_1 - \varepsilon y_2 \vartheta - \nu\varepsilon^4\big(y_1\xi_3' + \frac{1}{2}(-y_1^2 + \frac{1}{\varepsilon^2}y_2^2)\xi_1'' - y_1y_2\xi_2''\big),\\
u_2^{0,\varepsilon} &= \varepsilon\xi_2 + \varepsilon^2 y_1\vartheta - \nu\varepsilon^3\big(y_2\xi_3' - y_1y_2\xi_1'' + \frac{1}{2}(-y_2^2 + \varepsilon^2 y_1^2)\xi_2''\big),\\
u_3^{0,\varepsilon} &= \varepsilon^2\big(\xi_3 - y_1\xi_1' - y_2\xi_2'\big) + \varepsilon^3 y_1y_2\vartheta' + \frac{1}{2}\nu\varepsilon^4 y_1y_2^2\xi_1''',
\end{aligned} \tag{34}$$

is equal to zero in $y_3 = 0$ and satisfies the following estimates

$$\begin{gathered}
\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^{0,\varepsilon}}{\varepsilon^2} - \boldsymbol{\Lambda}(y_1D_3\vartheta, D_3v_3)\|_{L^2(\Omega)} \le \varepsilon C(\mathbf{v},\vartheta),\\
\|(\mathbf{W}^\varepsilon\mathbf{u}^{0,\varepsilon})_{12} + \vartheta\|_{L^2(\Omega)} \le \varepsilon C(\mathbf{v},\vartheta),\\
\|\Big(u_1^{0,\varepsilon}, \frac{u_2^{0,\varepsilon}}{\varepsilon}, \frac{u_3^{0,\varepsilon}}{\varepsilon^2}\Big) - \mathbf{v}\|_{H^1(\Omega)} \le \varepsilon C(\mathbf{v},\vartheta),
\end{gathered} \tag{35}$$

where $C(\mathbf{v},\vartheta)$ depends only on $\mathbf{v}$ and ϑ. Hence in this case $(\mathbf{u}^{0,\varepsilon_k})$ is a recovering sequence.

In the general case, i.e., $\mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3)$ and $\vartheta \in H^1_{\#}(\Omega;\mathbb{R})$, we can find, by convolution, functions $\mathbf{v}^n \in H_{BN}(\Omega;\mathbb{R}^3)$ and $\vartheta^n \in H^1_{\#}(\Omega;\mathbb{R})$ which are smooth, equal to zero near $y_3 = 0$ and such that

$$\begin{gathered}
\|\boldsymbol{\Lambda}(y_1D_3\vartheta^n, D_3v_3^n) - \boldsymbol{\Lambda}(y_1D_3\vartheta, D_3v_3)\|_{L^2(\Omega)} \le \tfrac{1}{n},\\
\|\vartheta^n - \vartheta\|_{L^2(\Omega)} \le \tfrac{1}{n},\\
\|\mathbf{v}^n - \mathbf{v}\|_{H^1(\Omega)} \le \tfrac{1}{n},
\end{gathered}$$

for every n. Denoting by $\mathbf{u}^{n,\varepsilon}$ the sequence defined as $\mathbf{u}^{0,\varepsilon}$ in (34) but with $(\mathbf{v},\vartheta)$ replaced by $(\mathbf{v}^n,\vartheta^n)$, given a sequence ε_k converging to zero, we can find a diagonal $\mathbf{u}^n := \mathbf{u}^{n,\varepsilon_{k_n}}$ such that

$$\begin{gathered}
\|\frac{\mathbf{E}^{\varepsilon_{k_n}}\mathbf{u}^n}{{\varepsilon_{k_n}}^2} - \boldsymbol{\Lambda}(y_1D_3\vartheta^n, D_3v_3^n)\|_{L^2(\Omega)} \le \tfrac{1}{n},\\
\|(\mathbf{W}^{\varepsilon_{k_n}}\mathbf{u}^n)_{12} + \vartheta^n\|_{L^2(\Omega)} \le \tfrac{1}{n},\\
\|\Big(u_1^n, \tfrac{u_2^n}{\varepsilon_{k_n}}, \tfrac{u_3^n}{\varepsilon_{k_n}^2}\Big) - \mathbf{v}^n\|_{H^1(\Omega)} \le \tfrac{1}{n}.
\end{gathered}$$

Therefore, the sequence $\mathbf{u}^n$ satisfies the recovering sequence condition.

□

Remark 5.3 As a consequence of the weak metrizability of compact subsets of $H^1(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R})$ and of the Urysohn property of Γ-convergence (see for instance Dal Maso [7], Chapter 8), conditions *1* and *2* of Theorem 5.2 imply that the sequence of functionals $\frac{1}{\varepsilon^4} I_\varepsilon$ Γ-converges to I with respect to the weak convergence in $H^1(\Omega;\mathbb{R}^3) \times L^2(\Omega;\mathbb{R})$ of the sequence $q_\varepsilon(\mathbf{u}^\varepsilon)$ (see (31)).

6 Convergence of minima and minimizers

For every $\varepsilon \in (0,1]$ let us denote by $\tilde{\mathbf{u}}^\varepsilon$ the solution of the following minimization problem

$$\min\{I_\varepsilon(\mathbf{u}) \ : \ \mathbf{u} \in H^1(\Omega_\varepsilon;\mathbb{R}^3),\, \mathbf{u} = \mathbf{0} \text{ on } S_\varepsilon(0)\}. \tag{36}$$

The existence of the solution can be proved by the direct method of the Calculus of Variations and the uniqueness follows by the strict convexity of the functionals I_ε.

Corollary 6.1 *The following minimization problem for the* Γ*-limit functional* I *defined in* (32)

$$\min\{I(\mathbf{v},\vartheta) : \mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3),\, \vartheta \in H^1(0,\ell),\, \mathbf{v} = \mathbf{0} \, on \, S_\varepsilon(0),\, \vartheta(0) = 0\}$$

admits a unique solution $(\tilde{\mathbf{v}},\tilde{\vartheta})$*. Moreover, as* $\varepsilon \to 0^+$*,*

1. $\left(\tilde{u}_1^\varepsilon, \frac{\tilde{u}_2^\varepsilon}{\varepsilon}, \frac{\tilde{u}_3^\varepsilon}{\varepsilon^2}\right) \to \tilde{\mathbf{v}}$ *weakly in* $H^1(\Omega;\mathbb{R}^3)$*,*
2. $(\mathbf{W}^\varepsilon \tilde{\mathbf{u}}^\varepsilon)_{12} \to -\tilde{\vartheta}$ *weakly in* $L^2(\Omega)$*,*
3. $\frac{1}{\varepsilon^4} I_\varepsilon(\tilde{\mathbf{u}}^\varepsilon)$ *converges to* $I(\tilde{\mathbf{v}},\tilde{\vartheta})$*.*

Proof. Follows from well known properties of Γ-limits and in particular by putting together Propositions 6.8 and 8.16 (lower semicontinuity of sequential Γ-limits), Theorem 7.8 (coercivity of the Γ-limit) and Corollary 7.24 (convergence of minima and minimizers) of Dal Maso [7].

□

7 The equations of equilibrium

The limit energy functional $I(\mathbf{v}, \vartheta)$ can be written in a more explicit form by using (13) and the fact that ϑ depends only on y_3. Indeed, the limit strain energy rewrites as

$$\begin{aligned}\int_\Omega f_0(y_1 D_3\vartheta, D_3 v_3)\, dy &= \int_\Omega f_0(y_1 D_3\vartheta, \xi_3' - y_1\xi_1'' - y_2\xi_2'')\, dy \\ &= \frac{1}{2}E\int_\Omega \left(\xi_3' - y_1\xi_1'' - y_2\xi_2''\right)^2 dy + 2\mu\int_\Omega y_1^2 \vartheta'^2\, dy \\ &= \int_0^\ell \frac{1}{2}EA\xi_3'^2 + \frac{1}{2}EJ_2\xi_1''^2 + \frac{1}{2}EJ_1\xi_2''^2 + \frac{1}{2}\mu J\vartheta'^2\, dy_3\end{aligned}$$

where

$$\begin{aligned} A &:= \int_\omega dy_1 dy_2 = ab, & J_1 &:= \int_\omega y_2^2\, dy_1 dy_2 = \frac{1}{12}ab^3, \\ J_2 &:= \int_\omega y_1^2\, dy_1 dy_2 = \frac{1}{12}a^3 b, & J &:= 4\int_\omega y_1^2\, dy_1 dy_2 = \frac{1}{3}a^3 b,\end{aligned}$$

and the work done by the external forces can be written as

$$\int_\Omega \mathbf{b}\cdot\mathbf{v}\, dy = \int_0^\ell \int_\omega b_\alpha \xi_\alpha + b_3(\xi_3 - y_\alpha \xi_\alpha')\, da\, dy_3 = \int_0^\ell \langle b_i\rangle \xi_i - \langle y_\alpha b_3\rangle \xi_\alpha'\, dy_3,$$

where $\langle\cdot\rangle = \int_\omega \cdot\, da$ denotes integration over the cross-section ω. Thus $\langle b_i\rangle$, with $i = 1, 2, 3$, are forces per unit of length and $\langle y_\alpha b_3\rangle$, for $\alpha = 1, 2$, are moments per unit of length. The energy of the beam $I(\mathbf{v}, \vartheta)$ can therefore be rewritten, with an abuse of notation, as

$$\begin{aligned} I(\boldsymbol{\xi}, \vartheta) = \int_0^\ell & \frac{1}{2}EA\xi_3'^2 + \frac{1}{2}EJ_2\xi_1''^2 + \frac{1}{2}EJ_1\xi_2''^2 + \frac{1}{2}\mu J\vartheta'^2\, dy_3 + \\ & + \int_0^\ell \langle b_i\rangle\xi_i - \langle y_\alpha b_3\rangle \xi_\alpha' - m\,\vartheta\, dy_3,\end{aligned}$$

which has to be minimized over all functions $(\boldsymbol{\xi}, \vartheta)$ with $\xi_\alpha \in H^2_\#(0,\ell)$, $\xi_3 \in H^1_\#(0,\ell)$ and $\vartheta \in H^1_\#(0,\ell)$. Clearly the minimization problem can be split in four independent problems, as well as the Euler-Lagrange equations which write as follows

$$\begin{cases} EJ_2\xi_1^{(4)} + \langle y_1 b_3\rangle' + \langle b_1\rangle = 0 \\ EJ_1\xi_2^{(4)} + \langle y_2 b_3\rangle' + \langle b_2\rangle = 0 \\ EA\xi_3'' - \langle b_3\rangle = 0 \\ \mu J\vartheta'' + m = 0. \end{cases} \tag{37}$$

From Corollary 6.1 we know that if $\tilde{\mathbf{u}}^k$ is a minimizer of I_{ε_k}, then

$$\Big(\tilde{u}_1^{\varepsilon_k}, \frac{\tilde{u}_2^{\varepsilon_k}}{\varepsilon_k}, \frac{\tilde{u}_3^{\varepsilon_k}}{\varepsilon_k^2}\Big) \to \tilde{\mathbf{v}} \text{ weakly in } H^1(\Omega;\mathbb{R}^3),$$

$$(\mathbf{W}^{\varepsilon_k}\tilde{\mathbf{u}}^{\varepsilon_k})_{12} \to -\tilde{\vartheta} \text{ weakly in } L^2(\Omega),$$

where $(\tilde{\mathbf{v}},\tilde{\vartheta})$ is the minimizer of I.

Conversely, if $(\tilde{\mathbf{v}},\tilde{\vartheta})$ is the minimizer of I we can find approximate minimizers of I_{ε_k}. Indeed, in this case we have $\tilde{v}_\alpha(y) = \tilde{\xi}_\alpha(y_3)$ and $\tilde{v}_3(y) = \tilde{\xi}_3(y_3) - y_\alpha\tilde{\xi}'_\alpha(y_3)$ where $\tilde{\xi}_\alpha \in H^3_\#(0,\ell)$, $\tilde{\xi}_3 \in H^2_\#(0,\ell)$ and $\tilde{\vartheta} \in H^2_\#(0,\ell)$ are the solutions of the equilibrium equations (37) (the minimizers of the functionals $I(\boldsymbol{\xi},\vartheta)$ defined above). A consequence of this gain of regularity is that the sequence $\boldsymbol{a}^\varepsilon$ defined as the sequence $\mathbf{u}^{0,\varepsilon}$ in (34) with $(\mathbf{v},\vartheta)$ replaced by $(\tilde{\mathbf{v}},\tilde{\vartheta})$, i.e.,

$$\begin{aligned} a_1^\varepsilon &= \tilde{\xi}_1 - \varepsilon y_2\tilde{\vartheta} - \nu\varepsilon^4\big(y_1\tilde{\xi}'_3 + \frac{1}{2}(-y_1^2 + \frac{1}{\varepsilon^2}y_2^2)\tilde{\xi}''_1 - y_1y_2\tilde{\xi}''_2\big), \\ a_2^\varepsilon &= \varepsilon\tilde{\xi}_2 + \varepsilon^2 y_1\tilde{\vartheta} - \nu\varepsilon^3\big(y_2\tilde{\xi}'_3 - y_1y_2\tilde{\xi}''_1 + \frac{1}{2}(-y_2^2 + \varepsilon^2 y_1^2)\tilde{\xi}''_2\big), \\ a_3^\varepsilon &= \varepsilon^2\big(\tilde{\xi}_3 - y_1\tilde{\xi}'_1 - y_2\tilde{\xi}'_2\big) + \varepsilon^3 y_1y_2\tilde{\vartheta}' + \frac{1}{2}\nu\varepsilon^4 y_1y_2^2\tilde{\xi}'''_1, \end{aligned} \tag{38}$$

is well defined. Even if we cannot say that it is a recovering sequence, for it does not satisfy the boundary conditions, the estimates (35) with $(\mathbf{v},\vartheta)$ replaced by $(\tilde{\mathbf{v}},\tilde{\vartheta})$ holds and therefore

$$\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^4} I_\varepsilon(\boldsymbol{a}^\varepsilon) = I(\tilde{\mathbf{v}},\tilde{\vartheta}).$$

On the other hand, looking at the minimizers $\tilde{\mathbf{u}}^\varepsilon$ of I_ε, by Corollary 6.1 we have $\lim_{\varepsilon\to 0^+} \frac{1}{\varepsilon^4} I_\varepsilon(\tilde{\mathbf{u}}^\varepsilon) = I(\tilde{\mathbf{v}},\tilde{\vartheta})$. Thus for ε small enough we have

$$\Big|\frac{1}{\varepsilon^4} I_\varepsilon(\boldsymbol{a}^\varepsilon) - \frac{1}{\varepsilon^4} I_\varepsilon(\tilde{\mathbf{u}}^\varepsilon)\Big| \le 1,$$

or, said differently,

$$|I_\varepsilon(\boldsymbol{a}^\varepsilon) - \min I_\varepsilon| \le \varepsilon^4, \tag{39}$$

which shows that $\boldsymbol{a}^\varepsilon$ is an approximate minimizer. Of course this sequence can be modified as done in the proof of Theorem 5.2, in order to obtain approximate minimizers which satisfy also the boundary conditions.

We notice that the components of $\boldsymbol{a}^\varepsilon$ scale with different powers of ε; this reflects the fact that some displacements are bigger than others. The form of a_3^ε is also quite interesting: the terms multiplied by ε^2 are the classical displacements found by De Saint-Venant, while the term multiplied by ε^3 takes into account the axial displacements due to the non-uniform warping of the section. The term which multiplies ϑ' is, in classical beam theory, called the warping function and is usually denoted by $\Psi = y_1 y_2$. This, as well as the value of J, is in perfect accordance with the results of the approximate theory of thin-walled cross-section beams.

8 Strong convergence of minimizers

Hereafter it is noticed that in the proof of Theorem 5.2 we have proved more than what is claimed. Indeed, the recovering sequence $\mathbf{u}^n$ is not only weakly but in fact strongly convergent, in the sense precised in part *2* of the statement of the theorem.

The aim of this section is to prove that also the convergence of minimizers stated in Corollary 6.1 is strong.

Theorem 8.1 *With the same notation of Corollary 6.1 we have that* $\left(\tilde{u}_1^\varepsilon, \frac{\tilde{u}_2^\varepsilon}{\varepsilon}, \frac{\tilde{u}_3^\varepsilon}{\varepsilon^2}\right) \to \tilde{\mathbf{v}}$ *strongly in* $H^1(\Omega;\mathbb{R}^3)$ *and* $(\mathbf{W}^\varepsilon \tilde{\mathbf{u}}^\varepsilon)_{12} \to -\tilde{\vartheta}$ *strongly in* $L^2(\Omega)$*.*

Let us start by proving the following lemma.

Lemma 8.2 *Denoted by* $\tilde{\mathbf{u}}^\varepsilon$ *the solution of the minimization problem* (36) *and by* $\boldsymbol{a}^\varepsilon$ *the approximate minimizers defined in* (38)*, we have that*

$$\lim_{\varepsilon\to 0^+} \left\| \frac{\mathbf{E}^\varepsilon(\tilde{\mathbf{u}}^\varepsilon - \boldsymbol{a}^\varepsilon)}{\varepsilon^2} \right\|_{L^2(\Omega)} = 0.$$

Proof. Let us begin by observing that the quadratic form $f(\mathbf{A})$ defined in (28) satisfies the identity

$$f(\tilde{\mathbf{A}}) = f(\mathbf{A}) + \mathbb{C}\mathbf{A}\cdot(\tilde{\mathbf{A}} - \mathbf{A}) + f(\tilde{\mathbf{A}} - \mathbf{A})$$

for every pair of 3×3 matrices $\mathbf{A}$ and $\tilde{\mathbf{A}}$. By (3) we thus obtain the inequality

$$f(\tilde{\mathbf{A}}) \geq f(\mathbf{A}) + \mathbb{C}\mathbf{A}\cdot(\tilde{\mathbf{A}} - \mathbf{A}) + \mu|\tilde{\mathbf{A}} - \mathbf{A}|^2,$$

which can be used in the expression of the integral functional I_ε defined in (8) to obtain that

$$
\begin{aligned}
I_\varepsilon(\tilde{\mathbf{u}}^\varepsilon) \geq I_\varepsilon(\boldsymbol{a}^\varepsilon) \; & + \int_\Omega \mathbb{C}\mathbf{E}^\varepsilon(\boldsymbol{a}^\varepsilon)\cdot\mathbf{E}^\varepsilon(\tilde{\mathbf{u}}^\varepsilon-\boldsymbol{a}^\varepsilon)\,dy+ \\
& + \mu\int_\Omega |\mathbf{E}^\varepsilon(\tilde{\mathbf{u}}^\varepsilon)-\mathbf{E}^\varepsilon(\boldsymbol{a}^\varepsilon)|^2\,dy+ \\
& + \varepsilon^4\int_\Omega \mathbf{b}\cdot\big(\tilde{u}_1^\varepsilon-a_1^\varepsilon,\frac{\tilde{u}_2^\varepsilon-a_2^\varepsilon}{\varepsilon},\frac{\tilde{u}_3^\varepsilon-a_3^\varepsilon}{\varepsilon^2}\big)\,dy+ \\
& + \varepsilon^4\int_0^\ell m\,\vartheta^\varepsilon(\tilde{\mathbf{u}}^\varepsilon-\boldsymbol{a}^\varepsilon)\,dy_3.
\end{aligned}
$$

As, by (39), $I_\varepsilon(\tilde{\mathbf{u}}^\varepsilon)-I_\varepsilon(\boldsymbol{a}^\varepsilon)\leq\varepsilon^4$ for any ε small enough, then for such ε we have

$$
\begin{aligned}
\varepsilon^4 \geq \; & \int_\Omega \frac{\mathbb{C}\mathbf{E}^\varepsilon(\boldsymbol{a}^\varepsilon)\cdot\mathbf{E}^\varepsilon(\tilde{\mathbf{u}}^\varepsilon-\boldsymbol{a}^\varepsilon)}{\varepsilon^4}\,dy+\mu\int_\Omega\frac{|\mathbf{E}^\varepsilon(\tilde{\mathbf{u}}^\varepsilon)-\mathbf{E}^\varepsilon(\boldsymbol{a}^\varepsilon)|^2}{\varepsilon^4}\,dy+ \\
& +\int_\Omega \mathbf{b}\cdot\big(\tilde{u}_1^\varepsilon-a_1^\varepsilon,\frac{\tilde{u}_2^\varepsilon-a_2^\varepsilon}{\varepsilon},\frac{\tilde{u}_3^\varepsilon-a_3^\varepsilon}{\varepsilon^2}\big)\,dy+\int_0^\ell m\,\vartheta^\varepsilon(\tilde{\mathbf{u}}^\varepsilon-\boldsymbol{a}^\varepsilon)\,dy_3.
\end{aligned}
$$

Let us then prove that

$$
\lim_{\varepsilon\to 0^+}\int_\Omega \frac{\mathbb{C}\mathbf{E}^\varepsilon(\boldsymbol{a}^\varepsilon)\cdot\mathbf{E}^\varepsilon(\tilde{\mathbf{u}}^\varepsilon-\boldsymbol{a}^\varepsilon)}{\varepsilon^4}\,dy=0 \tag{40}
$$

and the claim of the lemma is obtained by passing to the upper limit as $\varepsilon\to 0^+$. In order to prove (40) we observe that for every pair of matrices $\mathbf{A}$ and $\mathbf{B}$

$$
\mathbb{C}\mathbf{A}\cdot\mathbf{B}=2\mu\mathbf{A}\cdot\mathbf{B}+\lambda\,\mathrm{tr}(\mathbf{A})\,\mathrm{tr}(\mathbf{B}).
$$

Then

$$
\begin{aligned}
\int_\Omega \frac{\mathbb{C}\mathbf{E}^\varepsilon(\boldsymbol{a}^\varepsilon)\cdot\mathbf{E}^\varepsilon(\tilde{\mathbf{u}}^\varepsilon-\boldsymbol{a}^\varepsilon)}{\varepsilon^4}\;dy = & \; 2\mu\int_\Omega \frac{\mathbf{E}^\varepsilon(\boldsymbol{a}^\varepsilon)\cdot\mathbf{E}^\varepsilon(\tilde{\mathbf{u}}^\varepsilon-\boldsymbol{a}^\varepsilon)}{\varepsilon^4}\,dy+ \\
& +\lambda\int_\Omega\frac{\mathrm{tr}(\mathbf{E}^\varepsilon(\boldsymbol{a}^\varepsilon))\,\mathrm{tr}(\mathbf{E}^\varepsilon(\tilde{\mathbf{u}}^\varepsilon-\boldsymbol{a}^\varepsilon))}{\varepsilon^4}\,dy.
\end{aligned} \tag{41}
$$

In order to perform the computation, it is convenient to shorten some notation by setting

$$
A_{ij}^\varepsilon:=\frac{(\mathbf{E}^\varepsilon(\boldsymbol{a}^\varepsilon))_{ij}}{\varepsilon^2},\quad U_{ij}^\varepsilon:=\frac{(\mathbf{E}^\varepsilon(\tilde{\mathbf{u}}^\varepsilon))_{ij}}{\varepsilon^2},\quad F_{ij}^\varepsilon:=\frac{(\mathbf{E}^\varepsilon(\tilde{\mathbf{u}}^\varepsilon-\boldsymbol{a}^\varepsilon))_{ij}}{\varepsilon^2},
$$

so that the integrand in (41) has the following expression

$$
\sum_{i,j=1}^{3}\big[2\mu(A_{ij}^\varepsilon F_{ij}^\varepsilon)+\lambda A_{ii}^\varepsilon F_{jj}^\varepsilon\big].
$$

By (35) we have

$$\mathbf{A}^\varepsilon \to \mathbf{\Lambda}(y_1 D_3\tilde{\vartheta}, D_3\tilde{v}_3), \text{ strongly in } L^2(\Omega), \tag{42}$$

where $\mathbf{\Lambda}$ is defined in (33), and by the equation above and Lemma 5.1 it follows that $\mathbf{F}^\varepsilon$ is bounded in $L^2(\Omega)$. Thus, from (42) we immediately deduce that

$$\lim_{\varepsilon\to 0^+}\int_\Omega A^\varepsilon_{12}F^\varepsilon_{12}\,dy = \lim_{\varepsilon\to 0^+}\int_\Omega A^\varepsilon_{13}F^\varepsilon_{13}\,dy = 0.$$

From (42) and Corollary 6.1 follows that $F^\varepsilon_{33} \to 0$ weakly in $L^2(\Omega)$, and hence

$$\lim_{\varepsilon\to 0^+}\int_\Omega A^\varepsilon_{ij}F^\varepsilon_{33}\,dy = 0, \quad i,j = 1,2,3.$$

From Corollary 6.1, Lemma 4.3 and Lemma 4.7 it follows that, up to subsequences, U^ε_{23} weakly converges in $L^2(\Omega)$ to $y_1\tilde{\vartheta}'(y_3) + \tilde{\eta}(y_2,y_3)$, for some $\tilde{\eta}$ as specified in Lemma 4.7. By (42) we have that $A^\varepsilon_{23} \to y_1\tilde{\vartheta}'(y_3)$ strongly in $L^2(\Omega)$ and hence, up to subsequences, $F^\varepsilon_{23} \to \tilde{\eta}$ weakly in $L^2(\Omega)$. Thus

$$\lim_{\varepsilon\to 0^+}\int_\Omega A^\varepsilon_{23}F^\varepsilon_{23}\,dy = \int_\Omega y_1\tilde{\vartheta}'(y_3)\tilde{\eta}(y_2,y_3)\,dy = 0.$$

Let F_{11} and F_{22} be, up to subsequences, the weak limits in $L^2(\Omega)$ of F^ε_{11} and F^ε_{22}, respectively. Summarizing and taking the limit as $\varepsilon \to 0^+$ in (41) we obtain (even for the whole sequence)

$$\begin{aligned}\lim_{\varepsilon\to 0^+}\int_\Omega & \frac{\mathbb{C}\mathbf{E}^\varepsilon(\boldsymbol{a}^\varepsilon)\cdot\mathbf{E}^\varepsilon(\tilde{\mathbf{u}}^\varepsilon - \boldsymbol{a}^\varepsilon)}{\varepsilon^4}\,dy = \\ &= \lim_{\varepsilon\to 0^+}\int_\Omega 2\mu(A^\varepsilon_{11}F^\varepsilon_{11} + A^\varepsilon_{22}F^\varepsilon_{22}) + \lambda A^\varepsilon_{ii}F^\varepsilon_{\alpha\alpha}\,dy \\ &= \int_\Omega D_3 v_3(F_{11}+F_{22})[-2\nu(\mu+\lambda)+\lambda]\,dy = 0\end{aligned}$$

because $-2\nu(\mu+\lambda)+\lambda = -\lambda+\lambda = 0$, and the proof is concluded. □

PROOF OF THEOREM 8.1 As already remarked in the proof of Theorem 4.2, setting $\mathbf{v}^\varepsilon = \left(\tilde{u}^\varepsilon_1 - a^\varepsilon_1, \frac{\tilde{u}^\varepsilon_2 - a^\varepsilon_2}{\varepsilon}, \frac{\tilde{u}^\varepsilon_3 - a^\varepsilon_3}{\varepsilon^2}\right)$ we have

$$\|\mathbf{E}\mathbf{v}^\varepsilon\|_{L^2(\Omega)} \le \|\frac{\mathbf{E}^\varepsilon(\tilde{\mathbf{u}}^\varepsilon - \boldsymbol{a}^\varepsilon)}{\varepsilon^2}\|_{L^2(\Omega)}$$

and by the application of the standard Korn inequality to $\mathbf{v}^\varepsilon$, and Lemma 8.2, we obtain that

$$\|\mathbf{v}^\varepsilon\|_{H^1(\Omega)} \to 0.$$

From the third equation of (35) applied to the sequence $\boldsymbol{a}^\varepsilon$ follows that $\left(\tilde{u}_1^\varepsilon, \frac{\tilde{u}_2^\varepsilon}{\varepsilon}, \frac{\tilde{u}_3^\varepsilon}{\varepsilon^2}\right) \to \tilde{\mathbf{v}}$ strongly in $H^1(\Omega;\mathbb{R}^3)$. On the other hand by Theorem 4.2 we have that

$$\lim_{\varepsilon\to 0^+} \int_\Omega |\mathbf{H}^\varepsilon(\tilde{\mathbf{u}}^\varepsilon - \boldsymbol{a}^\varepsilon)|^2\, dy = 0$$

and since the definition of $\mathbf{H}^\varepsilon$ we deduce from here that

$$\frac{1}{\varepsilon^2} D_1(\tilde{u}_2^\varepsilon - a_2^\varepsilon) \to 0, \quad \frac{1}{\varepsilon} D_2(\tilde{u}_1^\varepsilon - a_1^\varepsilon) \to 0, \quad \text{strongly in } L^2(\Omega).$$

Thus, from the second equation of (35) applied to the sequence $\boldsymbol{a}^\varepsilon$ follows that

$$(\mathbf{W}^\varepsilon(\tilde{\mathbf{u}}^\varepsilon))_{12} = \frac{1}{2\varepsilon} D_2(\tilde{u}_1^\varepsilon - a_1^\varepsilon) - \frac{1}{2\varepsilon^2} D_1(\tilde{u}_2^\varepsilon - a_2^\varepsilon) + (\mathbf{W}^\varepsilon(\boldsymbol{a}^\varepsilon))_{12} \to -\vartheta$$

strongly in $L^2(\Omega)$. □

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