EVOLUTION BY CURVATURE OF NETWORKS OF CURVES IN THE PLANE

CARLO MANTEGAZZA – JOINT PROJECT WITH MATTEO NOVAGA AND VINCENZO MARIA TORTORELLI

ABSTRACT. This survey describes our project to study the motion by curvature of a network of smooth curves with multiple junctions in the plane, that is, the geometric gradient flow associated to the *Length* functional.
Such a flow can represent the evolution of a two–dimensional multiphase system where the energy is simply the sum of the lengths of the interfaces, in particular it is a possible model for the growth of grain boundaries.
Moreover, the motion of these networks of curves is the simplest example of curvature flow for sets which are "essentially" non regular.
In this paper, we introduce the problem and we present some results and open problems about existence, uniqueness and, in particular, the global regularity of the flow.

1. INTRODUCTION

In this survey we describe the first steps of our project to address the problem of the motion by curvature of a network of curves in the plane, where by network of curves we mean a connected planar graph without self–intersections.
The evolution by curvature of such a network is the geometric gradient flow with respect to the energy given by the *Length functional* which is simply the sum of the lengths of all the curves of the network (see [7]).

We point out two motivations to study this evolution. The first is the analysis of models of two–dimensional multiphase systems, where the problem of the structure and regularity of the interfaces between different phases arises naturally. As an example, the model where the energy of a configuration is simply the total length has proven useful in the analysis of the growth of grain boundaries, see [7, 8, 15, 22], the papers by Herring and Mullins in [6] and *http://mimp.mems.cmu.edu*.
The second motivation is more theoretical: the evolution of such a network of curves in the plane is the simplest example of motion by mean curvature of a set which is *essentially* singular.
In the literature there are various generalized definitions of flow by mean curvature for non regular sets (see [1, 7, 11, 13, 21, 25], for instance). All of them are fairly general, but usually lack uniqueness and a satisfactory regularity theory, even in simple situations.

We consider a connected network $\mathbb{S} = \cup_{i=1}^{n} \sigma^i$ in a smooth domain $\Omega \subset \mathbb{R}^2$ to be a finite family of regular curves $\sigma^i(x) : [0, 1] \to \overline{\Omega}$ which can intersect each other or self–intersect only at their end points. We call "multi–points" the vertices of such a smooth graph $\mathbb{S}$ whose order is greater than one. Moreover, we assume that all the other ends of the curves (if there are any) are some fixed points P^l on the boundary of Ω.
The problem is to analyse the existence, uniqueness, regularity and asymptotic behavior of the evolution by curvature of this network, under the constrain that the end points $P^l \in \partial\Omega$ stay fixed.

Inspired by Grayson's Theorem in [14], stating that any smooth closed curve embedded in $\mathbb{R}^2$ evolves by curvature without singularities before vanishing, and by the new approach to such result by Huisken in [20], one can reasonably expect that an "embedded" network of smooth curves does not develop singularities during the flow if its "topological structure" does not change (we

Date: June 10, 2003.
1991 *Mathematics Subject Classification.* Primary 53C44; Secondary 53A04, 35K55.
Key words and phrases. Curvature, evolution, triple junction.

will be more precise about this point in the sequel) and asymptotically converges to a critical configuration for the *Length* functional.

In this survey we present our first results, discussing some missing key points and open problems, and we suggest possible future research directions.

In many places this paper will be somehow roughly, we preferred to discuss mainly the general problems than to mention all the technical points.
We refer the interested reader to the paper [23] for proofs and all the details.

2. Basic Definitions and Small Time Existence of the Flow

Since the sets we consider are singular because of the presence of the multi–points, first of all we need to decide what definition of flow by mean curvature we adopt.
As previously underlined, the existing weak definitions of curvature motion do not give uniqueness of the flow or allow "fattening" phenomena (see [13], for instance), which we would like to avoid. Among the existing notions, the most suitable to our point of view seems to be Brakke's one (see Definition 2.2), which also lacks uniqueness, but maintains at least the Hausdorff dimension of the sets (thus, preventing the event of fattening).

At the moment we are able to show a satisfactory *small time* existence result (Theorem 2.4) of a smooth motion for a special class of networks, that is, the ones having only multi–points with at most three concurrent curves (triple–points) forming angles of 120 degrees between them (this last property is called *Herring condition*).

We set in a precise analytical way the curvature evolution problem for an embedded special network with only triple–points.

Definition 2.1. We say that the family of networks of curves $\mathbb{S}_t = \cup_{i=1}^n \gamma^i(\cdot, t)$ in $\Omega \subset \mathbb{R}^2$, with end points $P^j \in \partial\Omega$ and only triple–points, evolves by curvature (remaining embedded) in the time interval $[0, T)$ if the functions $\gamma^i : [0, 1] \times [0, T) \to \overline{\Omega}$ are of class C^2 in space and C^1 in time, at least, and satisfy the following quasilinear parabolic system

$$\begin{cases} \gamma^i_x(x,t) \neq 0 & \text{regularity} \\ \gamma^i(x,0) = \sigma^i(x) & \text{initial data} \\ \gamma^i_t(x,t) = \frac{\gamma^i_{xx}(x,t)}{|\gamma^i_x(x,t)|^2} & \text{motion by curvature} \end{cases} \tag{2.1}$$

for every $x \in [0, 1]$, $t \in [0, T)$ and $i \in \{1, \ldots, n\}$, moreover, the following conditions hold,

(1) at every time, the curves can intersect each other or self–intersect only at their ends (embeddedness of the network);
(2) every end of a curve, either is a 3–point or it coincides with one of the fixed end points of the network P^j on the boundary of Ω (there are no "free" end points of the curves). In this latter case, for example, if $\sigma^i(0) = P^j$, for some index j, then $\gamma^i(0, t) = P^j$ for every $t \in [0, T)$;
(3) every three curves meeting at a 3–point of the network form three angles of 120 degrees;
(4) the only curves which "touch" the boundary of Ω are the ones with fixed end points coinciding with the points P^j.

Notice that the evolution equation

$$\gamma^i_t = \frac{\gamma^i_{xx}}{|\gamma^i_x|^2}\,, \tag{2.2}$$

is not the usual way to describe the motion by curvature, that is,

$$\gamma^i_t = \frac{\langle \gamma^i_{xx} \,|\, \nu^i \rangle}{|\gamma^i_x|^2}\,\nu^i = k^i \nu^i$$

where we denoted with ν^i the unit normal to the curve γ^i and k^i its curvature.
The two velocities differ by a tangential component which actually affects the motions of the

single points (Lagrangian point of view), but it does not affect the local motion of a curve as a whole subset of $\mathbb{R}^2$ (Eulerian point of view).
In our situation this extra component becomes necessary in order to allow the motion of the 3–points. Indeed, since we look for a C^2 solution of problem (2.1), if the velocity would be in normal direction at every point of three concurrent curves, the 3–point should move in a direction which is normal to all three, thus the only possibility would be that it does not move at all (see the discussions and examples in [7, 8, 22]).

Definition 2.2. We will speak of *Brakke flow with equality* of an initial network $\mathbb{S}_0$ in $[0,T)$, for a family of C^2 networks $\mathbb{S}_t$ in Ω, all with the same end points as $\mathbb{S}_0$ and satisfying the equation

$$\frac{d}{dt}\int_{\mathbb{S}_t}\varphi(\gamma,t)\,ds = -\int_{\mathbb{S}_t}\varphi(\gamma,t)k^2\,ds + \int_{\mathbb{S}_t}\langle\nabla\varphi(\gamma,t)\,|\,\underline{k}\rangle\,ds + \int_{\mathbb{S}_t}\varphi_t(\gamma,t)\,ds\,, \tag{2.3}$$

for every smooth function with compact support $\varphi : \Omega\times[0,T)\to\mathbb{R}$ and $t\in[0,T)$.
This means also that the time derivative at the hand side has to exist.
Notice that the right hand side does not give any problem since the curves are at least C^2.

It is straightforward to check that a solution of problem (2.1) is also a smooth Brakke flow with equality.

Remark 2.3. The original definition of Brakke flow stated in [7, Section 3.3] allows equality (2.3) to be an inequality (and networks $\mathbb{S}_t$ to be one–dimensional countably rectifiable subsets of $\mathbb{R}^2$ with a distributional notion of curvature, called *varifolds*, see [24]), precisely,

$$\begin{aligned}\overline{\frac{d}{dt}}\int_{\mathbb{S}_t}\varphi(x,t)\,d\mathcal{H}^1(x) \leq -\int_{\mathbb{S}_t}\varphi(x,t)k^2\,d\mathcal{H}^1(x) &+ \int_{\mathbb{S}_t}\langle\nabla\varphi(x,t)\,|\,\underline{k}\rangle\,d\mathcal{H}^1(x)\\ &+\int_{\mathbb{S}_t}\varphi_t(x,t)\,d\mathcal{H}^1(x)\,,\end{aligned}$$

must hold for every *positive* smooth function with compact support $\varphi : \Omega\times[0,T)\to\mathbb{R}$ and $t\in[0,T)$, where $\overline{\frac{d}{dt}}$ is the *upper* derivative (the $\overline{\lim}$ of the incremental ratios) and $\mathcal{H}^1$ is the Hausdorff one–dimensional measure in $\mathbb{R}^2$.
This weaker condition was introduced by Brakke in order to prove an existence result [7, Section 4.13] for a family of initial sets much wider than the networks of curves, but, on the other hand, it let open the possibility of instantaneous vanishing of some parts of the set.
Since for our networks we are able to show, via a different method, the existence of a Brakke flow composed of smooth curves and satisfying the equality, for sake of simplicity, we included such extra properties in the definition.

Theorem 2.4. *If $\mathbb{S}_0$ is a C^2 initial embedded network, then there exists a Brakke flow with equality $\mathbb{S}_t$ for some positive time interval $[0,T)$.*
Moreover, the networks $\mathbb{S}_t$ it is described by a smooth family of curves solving problem (2.1) *in every time interval $[\varepsilon,T)$, for any $\varepsilon>0$. Hence, all the curvatures k^i belong to $C^\infty([0,1]\times[\varepsilon,T))$, hence the flow is a* smooth *Brakke flow with equality for every positive time.*
Finally, the unit tangents τ^i are continuous in space and time and the function $\int_{\mathbb{S}_t}k^2\,ds$ is continuous on $[0,T)$, where ds is the arclength measure on the curves of the network.

Remark 2.5. The fact that we have a solution of problem (2.1) (which is stronger than a Brakke flow) only after some positive time ε, is due to the eventuality that the *parabolic compatibility conditions* at time $t=0$ are not satisfied by the initial network. If such conditions hold, we have a *unique* solution of system (2.1) on all $[0,T)$, for every parametrization of the initial network $\mathbb{S}_0$ (see [23] for details).

The proof of the theorem is based on a result of Bronsard and Reitich in their paper [8], where they show that if the parabolic compatibility conditions are satisfied by the initial network, then the system (2.1) has a *unique* solution on some positive time interval $[0,T)$. Then, by means of a priori estimates and approximation, it is possible to get the existence of a Brakke flow for any C^2 network like above.

If the compatibility conditions are not satisfied, the uniqueness of a smooth Brakke flow with equality for these special networks, is an open problem, even restricting the admissible class of Brakke flow to the ones with continuous unit tangents.

We discuss now a little the difficulties in extending our analysis to a *general* network.
In the case of the presence in the initial network of a "bad" 3–point, not satisfying the Herring condition, that is, the three concurrent curves do not form angles of 120 degrees, we are not able at the moment to show the existence of a flow, smooth for every positive time, satisfying a "robust" definition (at least as Definition 2.2).
Actually, one would expect that the desired good definition should give uniqueness of the motion and force, by an instantaneous regularization, the three angles to become immediately of 120 degrees and to remain so. This is sustained by the fact that, by an energy argument ([7]), any smooth Brakke flow has to share such a property (which is also suggested by numerical and physical experiments, see at *http://mimp.mems.cmu.edu* and also the discussions in [6, 7, 8, 15, 22]). Notice that, by the variational nature of the problem it is appealing to guess that some sort of parabolic regularization could play a role here.

The second "big trouble" is when one tries to consider networks with multi–points of order greater than three.
We remark that if a multi–point has only two concurrent curves, it can be shown, by the regularizing effect of the evolution by curvature (see [2, 4, 5, 14]), that the two curves together become instantaneously a single smooth curve moving by curvature. Hence, the 2–point has vanished but this particular event is so "soft" (and topologically null) that we can avoid to consider it as a real structural change.
In the case of a 4–point instead (and clearly also of a higher order multi–point), for instance considering the network described by two curves crossing each other, there are really several possible candidates for the flow, even excluding a priori "fattening" phenomena. One cannot easily decide how the angles must behave, like in the 3–point case above, moreover, one can allow the four concurrent curves to separate in two pairs of curves moving independently each other and it could even be taken into account the "creation" of new multi–points from such a single one (these events are actually possible in Brakke's definition).
In these latter cases, the topology of the network changes dramatically, forcing us to change the "structure" of the system (2.1) governing the evolution or the family of curves composing the network.

Finally, it should be noticed that, the previous situation can appear even for our special networks with only 3–points after some time, since two (or many) of them could possibly "collapse" together creating a 4–point (or a multi–point), modifying the topological structure of the network. This clearly happens when the length of at least one curve of the network goes to zero (and there is no reason to exclude such an event). In this case, like at the previous point, one possibly has to "restart" the evolution with a different set of curves.
Again, a "collapse" could also produce a "bad" 3–points (think of three 3–points collapsing together along three curves connecting them).

Anyway, it seems actually reasonable that the configurations with multi–points of order greater than three or 3–points with angles different to 120 degrees should be unstable (they are actually unstable for the *Length* functional), with the meaning that they can appear at some discrete set of times (and probably in some cases are unavoidable), but they must vanish immediately after.

3. A Priori Estimates

Once the flow starts, by Theorem 2.4, we can suppose that it is described by a family of smooth curves $\gamma^i : [0,1] \times [0,T) \to \overline{\Omega}$, solving problem (2.1) in a *maximal time interval* $[0,T)$, hence, moving according to the law

$$\gamma^i_t(x,t) = \frac{\gamma^i_{xx}(x,t)}{|\gamma^i_x(x,t)|^2}\,.$$

After some time, it could happen that the network "hits" the boundary of Ω, so the flow would stop, by the definition above, without developing a *real* analytical singularity in the maps γ^i. Excluding for the moment this *geometric* event which will be discussed in the next section, there can be only two reasons why the time T is maximal: either the length of one curve γ^i has gone to zero (putting us in one of the situations discussed in the previous section) which implies that $\gamma^i_x = 0$, or some derivative of a curve γ^i is not bounded.
Hence, in order to analyse the long term behavior, a priori estimates are needed. They are strongly based on several relations holding at the 3–points of the network $\mathbb{S}_t$ between the curvatures of the three concurrent curves and their space derivatives. For instance, the sum of the three curvatures at every 3–point is zero at every time.
Moreover, even if these 3–points (by the Herring condition) are *interior points* of the network, from the *distributional* point of view, every argument based on the maximum principle has to consider them as "genuine" *boundary points*. This fact forces us to change approach, with respect to the standard smooth case, and to resort to integral estimates obtained by means of interpolation inequalities on the L^2 norms of the curvature and its derivatives. Pointwise estimates then will follow using Sobolev embeddings.

The first, immediate, estimate is on the lengths of the curves: since we are moving by the gradient of the *Length* functional, the length of every curve of the network is uniformly bounded above by a constant independent of time.

After some heavy computation one is able to get the following proposition which control these lengths also away from zero.

Proposition 3.1. *For every $M > 0$ there exists a time $T_M \in (0, T)$ such that for every $\mu > 0$, the inverses of the lengths of the curves of $\mathbb{S}_0$, the curvature of every $\mathbb{S}_t$ and all its space derivatives are uniformly bounded in the time interval $[\mu, T_M]$ by some constants depending only on μ, the L^2 norm of the curvature of $\mathbb{S}_0$ and the inverses of the lengths of the curves of the network at time zero.*

By means of these a priori estimates, which are similar to the ones in [3, 4, 5, 18] for the mean curvature evolution of a smooth curve or hypersurface, we can work out some results about the flow and, in particular, what happens at the maximal time of existence.

Theorem 3.2. *If $[0, T)$ is the maximal time interval of existence of a smooth solution $\mathbb{S}_t$ with $T < +\infty$ of problem* (2.1)*, then*

(1) *either the inferior limit of the length of at least one curve of $\mathbb{S}_t$ goes to zero as $t \to T$,*
(2) *or $\overline{\lim}_{t\to T} \max_{\mathbb{S}_t} k^2\, ds = +\infty$.*

Moreover, if the lengths of the curves are uniformly bounded away from zero, then the superior limit in (2) *is a limit and there exists a positive constant C such that*

$$\max_{\mathbb{S}_t} k^2 \geq \frac{C}{\sqrt{T-t}} \to +\infty\,, \tag{3.1}$$

for every $t \in [0, T)$.

Remark 3.3. In the case of the evolution γ_t of a single closed curve in the plane there exist a constant $C > 0$ such that if at time $T > 0$ a singularity develops, then

$$\max_{\gamma_t} k^2 \geq \frac{C}{T-t}$$

for every $t \in [0, T)$ (see [19]). If this lower bound on the rate of blowing up of the curvature (which is clearly stronger than the one in inequality (3.1)) holds also in the case of the evolution of a network is an open problem.

If we now suppose that no curve of the network collapses, the analysis proceed like in the standard case of mean curvature evolution of an hypersurface in $\mathbb{R}^n$.
What is needed is the understanding of the structure of the possible *blow up* around the singularities, in order to actually exclude these latter by means of geometric arguments. Some key references for this line are [3, 16, 19, 20].

The most relevant difference between our case and the standard one is that, because of the presence of the multi–points, it is difficult to apply the *maximum principle*, which would be the natural tool to get estimates on the geometric quantities during the flow.

We postpone the study of the possible singularities to Section 5, as we want now to discuss some geometric properties of the flow.

4. Geometric Properties of the Flow

As we said in the previous section, there are two "catastrophic" geometric events that do not allow the solution of system (2.1) to be defined for all times, one is the collapse of a curve that, at the moment, even requires a definition to be dealt with, the other is if the network during the flow "hits" the boundary of the domain $\Omega \in \mathbb{R}^2$.

Assuming the hypothesis that all the curves have lengths uniformly bounded away from zero, a simple condition to avoid the second situation is the convexity of Ω.

Proposition 4.1. *All the networks $\mathbb{S}_t$ intersect the boundary of Ω only at their end points. Moreover, for every positive time such intersections are transversal.*

Another important geometric property of the flow, in particular if one is thinking to modelize a physical multiphase system in $\mathbb{R}^2$, for instance a situation with several non–mixing liquid, is the *embeddedness* of the network.
We required that all networks are embedded (condition (1) in problem (2.1)) but actually, one can get a solution for small time also starting from a self–intersecting network.
Clearly, if the initial network is embedded, by continuity, it remains so for some time, what is more interesting is that actually it remains embedded *for every time*, till the solution γ^i is smooth. Moreover, this geometric property plays a key role also in the analysis of the singularities.

We assume a couple of technical hypotheses, that we would like to drop in the future.

- The domain Ω is *strictly* convex.
- The network $\mathbb{S}_0$ is a *tree*, that is, it does not contain *loops*.

Borrowing ideas from [17], [20] (see also [10]), we associate to the flow $\mathbb{S}_t$ a quantity $E(t)$ as follows: considering two points p, q belonging to $\mathbb{S}_t$, we define $\Gamma_{p,q}$ to be the *geodesic* curve contained in $\mathbb{S}_t$ connecting p and q (such *geodesic* is uniquely defined because there are no loops in $\mathbb{S}_t$), then we let $A_{p,q}$ to be the area of the open region $\mathcal{A}_{p,q}$ in $\mathbb{R}^2$ enclosed by the segment $[p,q]$ and the curve $\Gamma_{p,q}$. When the region $\mathcal{A}_{p,q}$ is not connected, we let $A_{p,q}$ to be the sum of the areas of its connected components.
We consider the function $\Phi_t : \mathbb{S}_t \times \mathbb{S}_t \to \mathbb{R} \cup \{+\infty\}$ as

$$\Phi_t(p,q) = \begin{cases} \frac{|p-q|^2}{A_{p,q}} & \text{if } p \neq q, \\ 4\sqrt{3} & \text{if } p \text{ and } q \text{ coincide with a 3–point of } \mathbb{S}_t, \\ +\infty & \text{otherwise.} \end{cases}$$

Since $\mathbb{S}_t$ is smooth and the 120 degrees condition holds, it is easy to check that Φ_t is a lower semicontinuous function. Hence, by the compactness of $\mathbb{S}_t$, the following infimum is actually a minimum

$$E(t) = \inf_{p,q \in \mathbb{S}_t} \Phi_t(p,q) \tag{4.1}$$

for every $t \in [0, T)$.

If the network $\mathbb{S}_t$ has no self–intersections we have $E(t) > 0$, the converse is clearly also true. Moreover, $E(t) \leq 4\sqrt{3}$ always holds, since at least one 3–point is present in the network, thus, when $E(t) > 0$ the two points (p,q) of a minimizing pair can coincide if and only if they coincide with a 3–point.
Finally, since the evolution is smooth it is easy to see that the function $E : [0, T) \to \mathbb{R}$ is continuous.

The nice property of this function E is stated in the following proposition.

Proposition 4.2. *The function $E(t)$ is monotone increasing in every time interval where $0 < E(t) < 4\sqrt{3}$ and for at least one minimizing pair (p, q) of Φ_t neither p nor q coincides with one of the end points P^i.*

Since the positivity of $E(t)$ is equivalent to the embeddedness of $\mathbb{S}_t$, our claim above follows. We resume then these geometric properties.

Theorem 4.3. *If Ω is bounded and* strictly *convex, and $\mathbb{S}_0$ is a* tree*, there exists a constant $C > 0$ depending only on $\mathbb{S}_0$ such that $E(t) > C > 0$ for every $t \in [0, T)$.*
Hence, if the initial network was embedded, all the networks $\mathbb{S}_t$ remain embedded in all the maximal interval of existence of the flow.

We remark that the two hypotheses above seems to be really unnecessary and very likely can be avoided, maybe with a smart choice of a slightly different function E, since the property to be a tree is necessary only to have a *single* geodesic in order to well determine the region $\mathcal{A}_{p,q}$. Moreover, we remark that if one is able to extend this theorem to networks with loops, the strict convexity of Ω is no more necessary.

5. Analysis of the Singularities

We recall that we continue to assume that the lengths of the curves of the networks are uniformly bounded away from zero.

Since at the maximal time T the curvature has to explode, like in the standard smooth case we separate the singularities according to its rate of blow up.
We say that a singularity is of *Type I* if for some constant C we have $\max_{\mathbb{S}_t} k^2 \leq C/(T-t)$ as $t \to T$ and it is of *Type II* otherwise.

Following Huisken [19], rescaling properly the flow around a "hypothetical" *Type I* singularity, one gets an evolution of embedded networks (unbounded and without end points) shrinking homothetically during the motion by curvature. Classifying all such particular evolutions, we will show that none of them can arise as a blow up of the flow $\mathbb{S}_t$, this clearly implies that *Type I* singularities cannot develop.

With the same idea, rescaling the flow around a *Type II* singularity, one gets an *eternal* motion by curvature, that is, an evolution of networks defined for every time $t \in \mathbb{R}$.
What is missing at the moment is that this eternal flow is actually simply given by a *translating* network (unbounded and without end points), like it happens in the case of a single smooth curve. Here also, the main difficulty resides in replacing some maximum principle arguments.
If this blow up would be translating, after classifying these latters, we could exclude also this case by means of an argument based on the monotonicity of the quantity E of previous section. Hence, no singularity at all could appear during the flow if the lengths of the curves of the network stay away from zero.

Since the technical details related to the blow up procedures are quite heavy, in this section we will speak a little bit roughly, referring to [23] for more precise statements.

Proposition 5.1. *Blowing up "in a proper way" the flow of the networks $\mathbb{S}_t$, in space and time, around a* Type I *singularity, we can obtain a non empty "limit" flow by curvature of embedded smooth networks of curves in all $\mathbb{R}^2$, with at most one end point or 3–point (still satisfying the* 120 *degrees condition), which move simply by homothety with respect to the origin. Moreover, the curvature of these networks is not identically zero.*

At this point we want to know what are the possible candidate flows satisfying all the properties stated in this proposition.

Lemma 5.2. *The only smooth flows by curvature of embedded networks with at most one end point or 3–point, moving by homothety with respect to the origin of $\mathbb{R}^2$ are:*

(1) *a straight line from the origin;*
(2) *a halfline from the origin;*

(3) *the network composed by three halfline from the origin, forming three angles of* 120 *degrees between them.*

Since in all these three situations the curvature of the network is identically zero, we get a contradiction.

Theorem 5.3. Type I *singularities cannot develop during the smooth flow* $\mathbb{S}_t$ *of an embedded network in a bounded and strictly convex domain* $\Omega \subset \mathbb{R}^2$ *if the lengths of the curves are uniformly bounded away from zero.*

We turn now our attention to the *Type II* singularities, that is,

$$\overline{\lim_{t\to T}}(T-t)\max_{\mathbb{S}_t} k^2 = +\infty\,.$$

Employing a trick by Hamilton, blowing up the flow in a way different to the previous one, we get an *eternal* limit flow by curvature of embedded smooth networks $\mathbb{S}^\infty_t$, all with at most one end point or 3–point (as before), where *eternal* means that the flow is defined for every time $t \in \mathbb{R}$. Moreover, the curvature of the networks achieves a non zero maximum at a point in space and time.

In the case of the evolution of a single closed curve in the plane, it is possible to show that the "limit" flow by curvature obtained in this way has to move by translation. This conclusion is reached in two steps: first, one shows that at every time the limit curve is convex, then, by means of the Harnack estimate proved by Hamilton in [16], it follows that it is simply translating during the motion.
Then, one is able to classify these translating flows and it turns out that all of them are, up to translations and dilations, the so called *grim reaper* (see [3, 16]), which is the curve given by the graph of the function $x = -\log(\cos y)$ in $\mathbb{R}^2$ when y varies in the interval $(-\pi/2, \pi/2)$. Notice

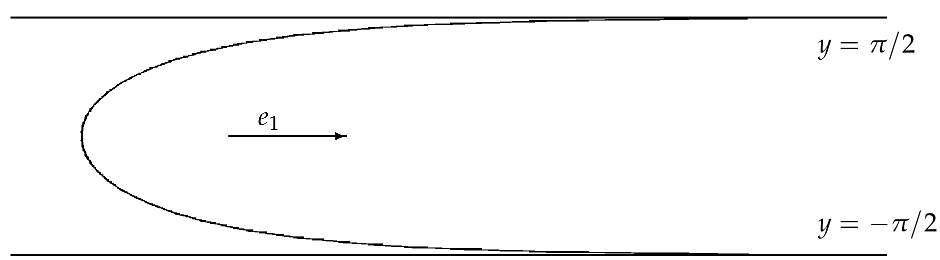

FIGURE 1. The *grim reaper*.

that this curve is contained in a strip or $\mathbb{R}^2$ of finite width.

In our situation the convexity condition above can be translated to "the curvature of the limit flow is never zero".
At the moment we can only state the following two conjectures.

Conjecture 5.4. If the curvature is zero at some point of the limit flow of networks (or curve) $\mathbb{S}^\infty_t$, then it is zero along all the curve containing such a point (everywhere).

Conjecture 5.5. The networks (or curves) $\mathbb{S}^\infty_t$ move by translation.

If this last conjecture is true, so $\mathbb{S}^\infty_t$ are translating networks, we can apply the following classification lemma to describe them.

Lemma 5.6. *The only smooth flows by curvature of embedded networks with at most one end point or 3–point, moving by translation in* $\mathbb{R}^2$ *are:*

(1) *a straight line (not moving at all);*
(2) *a halfline;*

(3) *a* triod *composed by three halflines (like the set at point (3) of Lemma 5.2;*
(4) *the* grim reaper*;*
(5) *a network composed of three unbounded curves which are either halflines or translated copies of pieces of the* grim reaper*, meeting at a single 3-points with angles of* 120 *degrees (it clearly follows that at most one curve can be a halfline).*

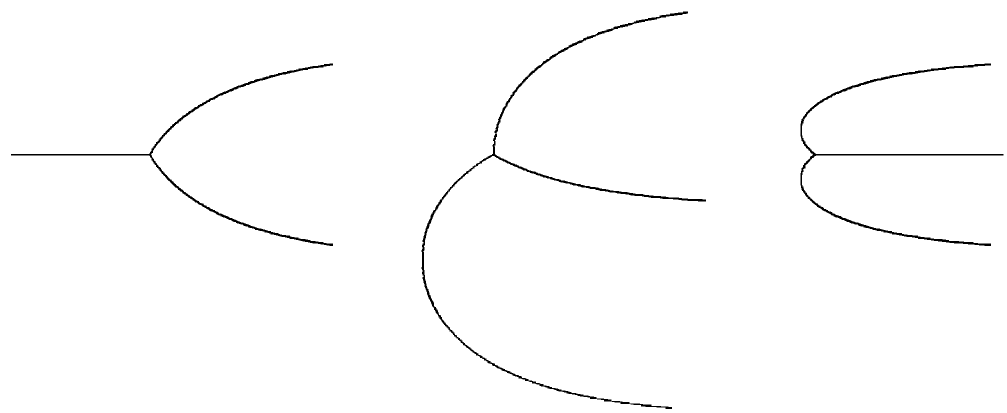

FIGURE 2. Some examples of translating networks.

Since our limit flow has non zero curvature, the first three cases would be excluded and we could conclude that either we are dealing with the *grim reaper* or with a network composed by three pieces of it, or two pieces and a halfline.

If we can get to this conclusion, we consider the analogous of the function E of Section 4 for the limit flow $\mathbb{S}_t^\infty$ (defined on all $\mathbb{R}$ in this case), we call it E^∞ to distinguish it from the one of the original flow $\mathbb{S}_t$. Since, by definition, E is dilation invariant and we got these limit flows by means of a blow up procedure, as we know that $E(t)$ was greater than an universal constant C (by the embeddedness of the initial network), the same estimate holds also for E^∞, that is, $E^\infty(\mathfrak{t}) \geq C$ for every $\mathfrak{t} \in \mathbb{R}$.

Fixing $t \in \mathbb{R}$, if we now consider a pair of points p, q on two curves of the network with opposite convexity (or on the two ends, if $\mathbb{S}_\mathfrak{t}^\infty$ is the *grim reaper*), we see that sending them both to infinity in a way that the length of the segment $[p, q]$ is uniformly bounded (this can be done because the *grim reaper* is contained in a strip of $\mathbb{R}^2$ with finite width), it follows that $E^\infty(\mathfrak{t})$ must be zero, which is in contradiction with $E^\infty(\mathfrak{t}) \geq C$.

This clearly exclude *Type II* singularities.

Remark 5.7. The same argument, but with a different geometric quantity, is used by Huisken in [20] to exclude *Type II* singularities during the motion of a single curve.

Proposition 5.8. *If Conjecture 5.5 is true then* Type II *singularities cannot develop during the smooth flow $\mathbb{S}_t$ of an embedded network in a bounded and strictly convex domain $\Omega \subset \mathbb{R}^2$, if the lengths of the curves are uniformly bounded away from zero.*

Hamilton's proof that a convex blow up of a *Type II* singularity (in the standard smooth case) is translating, is heavily based on the maximum principle which, as we said, is difficult to apply in our situation. So, it could happen that only Conjecture 5.4 can be proved and one could possibly exclude *Type II* singularities without actually show that the limit flow $\mathbb{S}_\mathfrak{t}^\infty$ is translating. For instance, if the curvature is always or never zero on each of the three curves, then they have asymptotic tangents (indeed, they are all convex but some of them in the opposite way, by the fact that the sum of the curvatures at the 3–point is zero), hence in order to apply the previous argument based on the function E, it would be enough to show that two of these limit tangents, belonging to curves with opposite convexity, coincide.

Conjecture 5.9. The flow $\mathbb{S}_t$ of an embedded network in a bounded and strictly convex domain $\Omega \subset \mathbb{R}^2$, such that the lengths of its curves are uniformly bounded away from zero, does not develop singularities at all.

If this conjecture is true, it can also be shown that the networks converge, as $t \to +\infty$ to a configuration which is critical for the *Length* functional among all the networks Ω with the same end points.

6. Open Problems

The main problem left open in the paper is Conjecture 5.5, whose validity would imply Conjectures 5.9.

Other questions in the paper that we would like to set are concerned with the extension of the results to all the networks, possibly with loops, with only 3–points (no 4–points or higher order points), in particular, proving an analogous of Theorem 4.3. This would also make superfluous the requirement that the domain $\Omega \subset \mathbb{R}^2$ is *strictly* convex.

Finally, we conclude by listing some, naturally arising, research directions.

(1) The problem of the uniqueness of Brakke flows (smooth/with equality) of networks, discussed after Theorem 2.4.
(2) The problem of the existence/uniqueness of flow for an initial network not satisfying the 120 degrees condition at the 3–points.
(3) The study of the singularities such that the curvature blows up but the lengths are not bounded away from zero. Such analysis requires new estimates and a general classification of all homothetically moving and translating networks, with only 3–points, flowing by curvature.
(4) The "definitory" problem of the motion of networks with multi–points (of order greater than three) and the analysis of the collapsing situations with change of topology (see the introduction and the papers by De Giorgi [12] and Caraballo [9]).

References

1. F. J. Almgren, J. E. Taylor, and L. Wang, *Curvature driven flows: a variational approach*, SIAM J. Cont. and Opt. **31** (1993), 387–438.
2. S. Altschuler and M. Grayson, *Shortening space curves and flow through singularities*, J. Diff. Geom. **35** (1992), 283–298.
3. S. J. Altschuler, *Singularities of the curve shrinking flow for space curves*, J. Diff. Geom. **34** (1991), no. 2, 491–514.
4. S. Angenent, *Parabolic equations for curves on surfaces: curves with p–integrable curvature*, Ann. of Math. (2) **132** (1990), 451–483.
5. ______, *Parabolic equations for curves on surfaces: intersections, blow up and generalized solutions*, Ann. of Math. (2) **133** (1991), 171–215.
6. J. M. Ball, D. Kinderlehrer, P. Podio-Guidugli, and M. Slemrod (eds.), *Fundamental Contributions to the Continuum Theory of Evolving Phase Interfaces in Solids*, Springer, Berlin, 1999.
7. K. A. Brakke, *The Motion of a Surface by its Mean Curvature*, Princeton University Press, Princeton, N.J., 1978.
8. L. Bronsard and F. Reitich, *On three-phase boundary motion and the singular limit of a vector–valued Ginzburg–Landau equation*, Arch. Rat. Mech. Anal. **124** (1993), no. 4, 355–379.
9. D. G. Caraballo, *A variational scheme for the evolution of polycrystals by curvature*, Ph.D. thesis, Princeton University, 1996.
10. K.-S. Chou and X.-P. Zhu, *Shortening complete plane curves*, J. Diff. Geom. **50** (1998), no. 3, 471–504.
11. E. De Giorgi, *Barriers, boundaries and motion of manifolds*, Scuola Normale Superiore di Pisa, 1995.
12. ______, *Motions of partitions*, Variational methods for discontinuous structures (Como, 1994), Progr. Nonlinear Differential Equations Appl., vol. 25, Birkhäuser, Basel, 1996, pp. 1–5.
13. L. C. Evans and J. Spruck, *Motion of level sets by mean curvature I*, J. Diff. Geom. **33** (1991), 635–681.
14. M. A. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Diff. Geom. **26** (1987), 285–314.
15. M. E. Gurtin, *Thermomechanics of Evolving Phase Boundaries in the Plane*, Oxford Science Publication, New York, 1993.
16. R. S. Hamilton, *Harnack estimate for the mean curvature flow*, J. Diff. Geom. **41** (1995), no. 1, 215–226.

17. ______, *Isoperimetric estimates for the curve shrinking flow in the plane*, Modern methods in complex analysis (Princeton, NJ, 1992), Princeton University Press, Princeton, NJ, 1995, pp. 201–222.
18. G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Diff. Geom. **20** (1984), 237–266.
19. ______, *Asymptotic behavior for singularities of the mean curvature flow*, J. Diff. Geom. **31** (1990), 285–299.
20. ______, *A distance comparison principle for evolving curves*, Asian J. Math. **2** (1998), 127–133.
21. T. Ilmanen, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc., vol. 108(520), AMS, 1994.
22. D. Kinderlehrer and C. Liu, *Evolution of grain boundaries*, Math. Models Methods Appl. Sci. **11** (2001), no. 4, 713–729.
23. C. Mantegazza, M. Novaga and V. M. Tortorelli, *Motion by curvature of planar networks*, in preparation.
24. L. Simon, *Lectures on Geometric Measure Theory*, Proc. Center Math. Anal., vol. 3, Australian National University, Canberra, 1983.
25. H. M. Soner, *Motion of a set by the curvature of its boundary*, J. Diff. Eqs. **101** (1993), no. 2, 313–372.

(Carlo Mantegazza) SCUOLA NORMALE SUPERIORE DI PISA, P.ZA CAVALIERI 7, 56126, PISA, ITALY
E-mail address, C. Mantegazza: `mantegazza@sns.it`

(Matteo Novaga) MATH. DEPT. UNIV. PISA, PISA, 56127, ITALY
E-mail address, M. Novaga: `novaga@dm.unipi.it`

(Vincenzo Maria Tortorelli) MATH. DEPT. UNIV. PISA, PISA, 56127, ITALY
E-mail address, V. M. Tortorelli: `tortorel@dm.unipi.it`