

# Quasilinear elliptic inequalities on complete Riemannian manifolds <sup>☆</sup>

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## Abstract

We prove maximum and comparison principles for weak distributional solutions of quasilinear, possibly singular or degenerate, elliptic differential inequalities in divergence form on complete Riemannian manifolds. A new definition of ellipticity for nonlinear operators on Riemannian manifolds is introduced, covering the standard important examples. As an application, uniqueness results for some related boundary value problems are presented.

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## Résumé

Nous démontrons des principes de maximum et de comparaison pour les solutions distributionnelles faibles d’inégalités différentielles quasi-linéaires elliptiques, éventuellement singulières ou dégénérées, sous forme divergence sur les variétés riemanniennes complètes. On présente une nouvelle définition d’ellipticité pour les opérateurs non-linéaires sur des variétés riemanniennes, en couvrant les exemples importants standards. Comme application, nous présentons quelques résultats d’unicité pour des problèmes aux limites.

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## 1. Introduction

Our main interest is in developing maximum principles and comparison results for weak distributional solutions of quasilinear, possibly singular or degenerate, elliptic inequalities in divergence form on complete Riemannian manifolds as

$$\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) \geq 0$$

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or

$$\operatorname{div} A(x, v, \nabla v) + B(x, v, \nabla v) \leq 0,$$

where  $A$  and  $B$  are very general nonlinear continuous functions, but no differentiability assumptions are required either on  $A$  when the gradient variable is  $\mathbf{0}$ , or on  $B$ . For this purpose in Section 5 we give a new, but natural, concept of ellipticity for nonlinear operators  $A$  on Riemannian manifolds, which covers the standard definition in the Euclidean setting. Furthermore, the well known operators as the Laplace–Beltrami  $\Delta$ , the more general  $p$ -Laplace–Beltrami,  $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ ,  $p > 1$ , and the mean curvature operators are still elliptic in the sense of Definition 5.2 below. Moreover, the approach used here lets us treat nonlinear operators having general growth of power  $p$ , even including the case  $p = 1$ .

Maximum principles for differential inequalities on complete Riemannian manifolds were already given in [28] and [31], but our approach and the spirit of the results are quite different. Indeed, in [28] the main subject is the Omori–Yau maximum principle (and various generalizations with geometrical applications), while we are interested in classical pointwise versions of comparison and maximum principles for solutions of differential inequalities, in the spirit of Calabi [9].

The results of the paper are obtained adapting a technique introduced by Pucci and Serrin in [30] and [32] for the proof of the validity of the comparison principle for elliptic inequalities in Euclidean domains, also developed in [29].

In the last section, as a consequence and application of the comparison theorems of Sections 4 and 5, we establish some uniqueness results for singular or degenerate elliptic problems on complete Riemannian manifolds. As a corollary of Theorem 6.3 we present the following standard prototype of uniqueness results in smooth domains  $\Omega$  of  $\mathcal{M}$  and for  $B$  regular, in the sense that for every compact set  $K \subset T\Omega \times_{\Omega} \mathbb{R}$  there exists a constant  $L > 0$  such that  $|B(x, z, \xi) - B(x, z, \eta)| \leq L|\xi - \eta|$  for every  $(x, z, \xi), (x, z, \eta) \in K$ . For the precise definitions we refer to Sections 2 and 5.

**Theorem 1.1.** *Let  $u_0 \in C(\partial\Omega)$ , and let  $u, v$  be weak solutions of class  $H_{\text{loc}}^{1,\infty}(\Omega)$  of the Dirichlet problem:*

$$\begin{cases} \Delta_p u + B(x, u, \nabla u) = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where  $B = B(x, z, \xi)$  is regular and nonincreasing in  $z$ . Moreover, assume either:

- (i)  $1 < p \leq 2$ , or
- (ii)  $p > 2$  and  $\operatorname{ess\,inf}_{\Omega} \{|\nabla u| + |\nabla v|\} > 0$ .

Then  $u = v$ .

As already noted in [30,32], assumption (ii) of Theorem 1.1 is somewhat sharp. Indeed, the problem

$$\begin{cases} \Delta_4 u + |\nabla u|^2 = 0 & \text{in } B_R \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial B_R, \end{cases}$$

where  $\Delta_4$  is an analytic elliptic operator, admits the smooth solutions  $u(x) = 0$  and  $v(x) = \frac{1}{8}(R^2 - |x|^2)$ , but

$$\inf_{B_R} \{|\nabla u| + |\nabla v|\} = 0.$$

A special case of (1.1), and a natural extension of the classical Yamabe problem discussed below, is the *generalized scalar curvature equation*

$$\Delta_p u - a(x)u^{p-1} + b(x)u^{p^*-1} = 0, \quad u \geq 0, \tag{1.2}$$

where  $p \in (1, n)$ ,  $n \geq 2$ ,  $p^* = pn/(n - p)$ , while  $a$  and  $b$  are smooth functions. When  $p = 2$  Trudinger has shown in [39] that actually any solution of (1.2) is of class  $C^\infty(\mathcal{M})$ , but in the general case any solution of (1.2) is only of class  $C_{\text{loc}}^{1,\alpha}(\mathcal{M})$  for some  $\alpha \in (0, 1)$  by the well-known regularity results of [14]. In the degenerate case  $p > 2$ , when the uniform ellipticity is lost, this regularity is what we can expect at most. Indeed, even in the Euclidean case the function,

$$u(x) = |x|^{p'}, \quad 1/p + 1/p' = 1, \quad p > 2,$$

solves  $\Delta_p u = np'$  in  $\mathbb{R}^n$ , but it is not of class  $C^2$ . For this reason, it is natural to deal with *weak solutions*, which will be introduced in Section 3 for more general elliptic differential inequalities. We remark that our *distributional* approach is different from the one of Calabi (see [9]), who considered weak solutions in the sense of viscosity.

In [5] and [14] the authors have proved that under suitable assumptions on  $\mathcal{M}$ ,  $a$  and  $b$ , there exists at least one positive solution for (1.2); while Theorem 1.1 shows that the boundary value problem associated to (1.2) admits a unique solution when  $a \geq 0$  and  $b \leq 0$ .

The famous and widely studied special example of (1.2), and so of (1.1), is the Yamabe problem on a Riemannian manifold  $\mathcal{M}$  of dimension  $n \geq 3$ ; that is, given a metric  $g$  the problem consists in finding another metric  $\tilde{g}$  with prescribed scalar curvature  $R_{\tilde{g}}$  on  $(\mathcal{M}, \tilde{g})$  which is conformally equivalent to  $g$ . Setting  $\tilde{g} = u^{4/(n-2)}g$ , then  $R_{\tilde{g}}$  is related to scalar curvature  $R_g$  of  $(\mathcal{M}, g)$  by the formula:

$$4\frac{n-1}{n-2}\Delta u - R_g u + R_{\tilde{g}} u^{(n+2)/(n-2)} = 0, \quad u > 0. \tag{1.3}$$

Hence the original question reduces to finding a smooth solution  $u$  of (1.3). In the classical Yamabe problem the conformal metric  $\tilde{g}$  is assumed to have *constant scalar curvature*  $R_{\tilde{g}} = R_0 \in \mathbb{R}$ . The exponent  $(n + 2)/(n - 2)$  appearing in (1.3) is critical in the sense of the Sobolev embedding, so that a usual variational approach exhibits a lack of compactness. Nevertheless, existence theorems have been given starting from Trudinger in [39], essentially when  $R_g \leq 0$ , and by Aubin in [2] and Schoen in [34] when  $\mathcal{M} = S^n$ , the unit sphere in  $\mathbb{R}^n$ , and  $\tilde{g}$  is the standard metric  $g_0$  of  $S^n$  (so that  $R_0$  is positive), basically for  $R_g \geq 0$ . In the latter case some multiplicity results for a perturbed problem are established in [1] provided the dimension  $n$  is sufficiently large. Among more recent results, we quote the one in [21], where conditions on the Weyl tensor and the Ricci curvature are required, in [6], where  $R_g$  is 0 and  $\mathcal{M}$  is compact, and in [7], where  $R_g$  is a negative constant. See also [23] and [38, case  $n = 2$ ] for a detailed description of existence or nonexistence results according to the sign or behaviour of  $R_g$  and  $R_{\tilde{g}}$ . Recently some existence results for boundary value problems associated to (1.3) have been established in [17] and [26] under mixed boundary conditions.

The homogeneous Dirichlet problem associated to (1.3) is considered in [11] when  $\Omega$  is a smooth bounded domain,  $R_g$  and  $R_{\tilde{g}}$  are smooth functions, as well as in [12], where a more general selfadjoint, negative definite second-order differential operator is considered in a Lipschitz domain  $\Omega$  of  $\mathcal{M}$  in presence of a nonlinear  $C^1$  function  $B(x, u)$ . In the special case that  $\mathcal{M}$  is an annulus of  $\mathbb{R}^n$ ,  $n \geq 4$ ,  $R_g, R_{\tilde{g}}$  are positive numbers, the existence of infinitely many solutions is proved in [11], while in [12] a uniqueness result is established when  $\partial_u B \leq 0$  and the asymptotic behavior of the solution is studied in [4]. On the other hand, as an immediate consequence of Theorem 1.1, it is easily seen that when  $R_g \geq 0$  and  $R_{\tilde{g}} \leq 0$ , the homogeneous Dirichlet problem associated to (1.3) has a unique solution of class  $H_{loc}^{1,\infty}(\Omega)$ . The example in [12, p. 1388] illustrates the uniqueness result for,

$$\Delta u - |u|^q u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = g \in L^p(\partial\Omega),$$

whose solution is unique, provided that  $0 \leq q < 2p/(n - 1)$ , a result which we can easily cover when  $g \in C(\partial\Omega)$ .

Finally, we remark that versions of the comparison and uniqueness results in presence of the mean curvature operator seem to be fairly new. In the case of  $B$  independent of  $u$  and  $\nabla u$ , Spruck shows that if  $\Omega$  is a bounded  $C^2$  domain,  $H > 0$  is the mean curvature of a nonparametric surface  $u$  and  $u_0$  is a continuous datum on  $\partial\Omega$ , then, under additional assumptions on the geometry of  $\Omega$ , the Dirichlet problem:

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = nH & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

is uniquely solvable [36, Theorem 1.4]; such a result is immediately reobtained in Section 6. In the same way, we extend the uniqueness results proved for several other special cases, both for the manifold and for the equation, treated, i.e. in [10,16,27]. We recall that this problem arises when considering the isoperimetric problem of the least surface area bounding a given volume.

Let us note that the equation appearing in (1.4), which can be easily handled in the comparison and uniqueness theorems provided in the sections below, is equivalent to

$$(1 + |\nabla u|^2)\Delta u - \langle D^2 u \nabla u, \nabla u \rangle = nH(x)(1 + |\nabla u|^2)^{3/2}, \tag{1.5}$$

and that similar equations appear in several other applications of physical and geometric interest, in particular when  $H$  is a function of  $(x, u, \nabla u)$ . In the following we recall some of the main examples, and we refer to [32] for a detailed description.

1. The surface of a fluid under the combined action of gravity and surface tension, *capillary surface*, verifies:

$$(1 + |\nabla u|^2)\Delta u - (D^2 u \nabla u, \nabla u) - \kappa(1 + |\nabla u|^2)^{3/2}u = 0,$$

where  $\kappa$  is a physical constant.

2. The  $p$ -Dirichlet norm on  $S^n$ ,  $p > 1$ , is minimized by functions  $u$  on  $S^n$  which satisfy

$$\operatorname{div}_{S^n}(|\nabla u|^{p-2}\nabla u) = 0.$$

Since  $S^n$  can be mapped conformally onto the Euclidean tangent space  $\mathbb{R}^n$  at the South Pole by means of the *stereographic projection* from the North Pole, in the stereographic variables  $x$  we have:

$$\rho^{-n} \operatorname{div}(\rho^{n-p}|\nabla u|^{p-2}\nabla u) = 0, \quad \rho(x) = 1/(1 + |x|^2).$$

This is a particular example where the vector  $A$  depends on both  $x$  and  $\nabla u$  on the manifold  $\mathbb{R}^n$ . Of course, general variational integrals on  $S^n$  can be treated in the same way.

3. In *subsonic gas dynamics* the velocity potential  $\varphi$  satisfies,

$$\operatorname{div}(\rho \nabla \varphi) = 0,$$

where the velocity  $\nabla \varphi$  and the density  $\rho$  are related through Bernoulli's law.

4. The general equation of *radiative cooling* has the form

$$\operatorname{div}(\kappa |\nabla u|^{p-2}\nabla u) = \sigma u^4, \quad p > 1,$$

where  $\kappa$  is the coefficient of heat conduction, depending on  $x$  and possibly also on  $u$ , while  $\sigma$  is the radiation, assumed to be constant. Replacing the right-hand side by various functions  $f = f(x, u)$  yields further examples of physical interest.

## 2. Preliminaries

In this section we introduce the main notation. From now on  $\mathcal{M}$  will denote a smooth complete Riemannian  $n$ -manifold with metric tensor  $g \in C^\infty(\mathcal{M}, T^*\mathcal{M} \otimes T^*\mathcal{M})$ .

**Definition 2.1.** The *fibred product bundle* of two bundles  $(E, \pi_1, \mathcal{M})$  and  $(F, \pi_2, \mathcal{M})$  is the manifold,

$$E \times_{\mathcal{M}} F = \{(e, f) \in E \times F: \pi_1(e) = \pi_2(f)\},$$

with the induced vector bundle structure.

In the sequel  $\Omega$  will denote a regular domain of  $\mathcal{M}$  and, for shortness, we shall write  $T\Omega \times_{\Omega} \mathbb{R}$  in place of  $T\Omega \times_{\Omega} (\Omega \times \mathbb{R})$  to denote the fibred product bundle  $T\Omega \times_{\Omega} (\Omega \times \mathbb{R})$ . Of course  $T\Omega \times_{\Omega} (\Omega \times \mathbb{R}) \cong T\Omega \times \mathbb{R}$ , and in turn the notation is not ambiguous. In analogy with the Euclidean case, points of  $T\Omega \times_{\Omega} \mathbb{R}$  will be denoted with  $(x, z, \xi)$ , where  $(x, \xi) \in T\Omega$  and  $(x, z) \in \Omega \times \mathbb{R}$ .

Integrals will be taken with respect to the natural Riemannian measure. For example, if  $(U, \Phi)$  is a coordinate chart and  $u$  is a continuous function compactly supported in  $U$ , we define:

$$\int_U u \, d\mathcal{M} = \int_{\Phi(U)} (\sqrt{G}u) \circ \Phi^{-1} \, dx,$$

where  $dx$  stands for the Lebesgue measure on  $\mathbb{R}^n$  and  $G$  is the absolute value of the determinant of the metric tensor in the coordinate chart  $(U, \Phi)$ . With the help of smooth partitions of unity, the construction above defines a canonical positive Radon measure on  $\mathcal{M}$ , which is the natural Lebesgue measure on  $\mathcal{M}$ , denoted simply by  $|\cdot|$ . In particular

we can speak of measurable vector fields, that is measurable sections of the tangent bundle. For a section  $V$  defined on  $\Omega$  and  $p \geq 1$ , introduce the Lebesgue  $p$ -norm,

$$\|V\|_{p,\Omega} = \left( \int_{\Omega} |V|^p \, d\mathcal{M} \right)^{1/p}. \tag{2.1}$$

The integrand at the right-hand side of (2.1) is  $|V|(x) = \sqrt{g(V(x), V(x))}$ . In fact, we shall write  $|\cdot|$  to denote, according to the cases, the real modulus, the Riemannian norm of tangent vectors and the measure of measurable subsets of  $\mathcal{M}$ .

Let  $H^{1,p}(\Omega)$  be the closure of  $C^\infty(\Omega)$  in the Sobolev norm,

$$\|u\| = \|u\|_{p,\Omega} + \|\nabla u\|_{p,\Omega}, \tag{2.2}$$

where  $\|u\|_{p,\Omega} = \|u\|_{L^p(\Omega)}$  and let

$$H_{\text{loc}}^{1,p}(\Omega) = \{u : \Omega \rightarrow \mathbb{R} : u|_{\Omega'} \in H^{1,p}(\Omega') \text{ for all open sets } \Omega' \Subset \Omega\}.$$

Finally, denote by  $H_0^{1,p}(\Omega)$  the closure of  $C_c^\infty(\Omega)$  with respect to the Sobolev norm (2.2).

We recall that, if  $u \in L^1_{\text{loc}}(\Omega)$ , a locally integrable vector field  $\mathbf{H} \in L^1_{\text{loc}}(\Omega, T\Omega)$  is a *weak gradient* for  $u$  if

$$\int_{\Omega} \langle \mathbf{H}, V \rangle \, d\mathcal{M} = - \int_{\Omega} u \operatorname{div} V \, d\mathcal{M}$$

for every vector field  $V \in C_c^\infty(\Omega, T\Omega)$  (see [22]). Since  $\mathbf{H}$  is unique, we put  $\mathbf{H} = \nabla u$ . Of course, if  $u$  is a smooth function its usual Riemannian gradient is a weak gradient.

We shall use the following result, which extends [15, Theorem 7.8] to a general domain  $\Omega$  of  $\mathcal{M}$ . Such an extension is well known when  $\Omega = \mathcal{M}$  (see for instance [19, Proposition 2.5]).

**Lemma 2.2.** *Let  $u \in H^{1,p}(\Omega)$ ,  $p \geq 1$ , and let  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  be a piecewise smooth function, with  $\psi' \in L^\infty(\mathbb{R})$ . When  $|\Omega| = \infty$ , also assume that  $\psi(0) = 0$ . Then  $\psi \circ u \in H^{1,p}(\Omega)$ . Moreover, if  $S$  denotes the set where  $\psi$  is not differentiable, then*

$$\nabla(\psi \circ u) = \begin{cases} \psi'(u)\nabla u, & \text{if } u \notin S, \\ 0, & \text{if } u \in S. \end{cases}$$

**Proof.** Let us first suppose  $\psi \in C^1(\mathbb{R})$ , with  $|\psi'| \leq M$ . Set  $v = \psi \circ u$  and take  $u_k \in C^\infty(\Omega)$  such that  $u_k \rightarrow u$  in  $H^{1,p}(\Omega)$  and a.e. in  $\Omega$ . Then  $v_k := \psi \circ u_k \in C^1(\Omega)$  and  $\nabla v_k = \psi'(u_k)\nabla u_k$  for any  $k \in \mathbb{N}$ . Moreover,  $|v_k| \leq M|u_k|$  and  $|\nabla v_k| \leq M|\nabla u_k|$ , so that they are both  $p$ -integrable on  $\Omega$ .

It remains to prove that  $v_k \rightarrow v$  in  $L^p(\Omega)$  and  $\nabla v_k \rightarrow \nabla v = \psi'(u)\nabla u$  in  $L^p(\Omega, T\Omega)$ , so that  $\nabla v$  is the weak gradient of  $v$ , since for every vector field  $V \in C_c^\infty(\Omega, T\Omega)$ ,

$$\int_{\Omega} \langle \nabla v, V \rangle \, d\mathcal{M} = \lim_{k \rightarrow \infty} \int_{\Omega} \langle \nabla v_k, V \rangle \, d\mathcal{M} = - \lim_{k \rightarrow \infty} \int_{\Omega} v_k \operatorname{div} V \, d\mathcal{M} = - \int_{\Omega} v \operatorname{div} V \, d\mathcal{M}.$$

Of course, by the mean value theorem,

$$\int_{\Omega} |v_k - v|^p \, d\mathcal{M} \leq M^p \int_{\Omega} |u_k - u|^p \, d\mathcal{M} \rightarrow 0.$$

Moreover,

$$\begin{aligned} |\nabla v_k - \nabla v| &= |\psi'(u_k)\nabla u_k - \psi'(u)\nabla u| \\ &\leq |\psi'(u_k)\nabla u_k - \psi'(u_k)\nabla u| + |\psi'(u_k)\nabla u - \psi'(u)\nabla u| \\ &\leq M|\nabla u_k - \nabla u| + |\psi'(u_k) - \psi'(u)| \cdot |\nabla u|. \end{aligned}$$

Consequently, by the Lebesgue theorem, as  $k \rightarrow \infty$ ,

$$\int_{\Omega} |\nabla v_k - \nabla v|^p \, d\mathcal{M} \rightarrow 0,$$

since  $|\nabla u_k - \nabla u| \rightarrow 0$  in  $L^p(\Omega)$ , while  $|\psi'(u_k) - \psi'(u)| \cdot |\nabla u| \rightarrow 0$  a.e. in  $\Omega$  and  $|\psi'(u_k) - \psi'(u)| \cdot |\nabla u| \leq 2M|\nabla u| \in L^p(\Omega)$ . The first case is complete.

Now let  $\psi$  be a piecewise smooth function. By iterating the following argument, we can assume that  $\psi$  is not differentiable at only one point, say  $u_0$ . Without loss of generality, we can assume that  $u_0 = 0$  and, moreover, that  $\psi(0) = 0$  also when  $\Omega$  is bounded. Now take  $\psi_1, \psi_2 \in C^1(\mathbb{R})$  with bounded derivatives such that  $\psi_1(u) = \psi(u)$  for  $u \geq 0$  and  $\psi_2(u) = \psi(u)$  for  $u \leq 0$ . Then  $\psi(u) = \psi_1(u^+) + \psi_2(u^-)$ . For the first step, it is enough to show that  $u^\pm \in H^{1,p}(\Omega)$ .

Take  $\varepsilon > 0$  and define:

$$\psi_\varepsilon(t) = \begin{cases} \sqrt{t^2 + \varepsilon^2} - \varepsilon, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Of course  $\psi_\varepsilon \in C^1(\mathbb{R})$  and  $|\psi'_\varepsilon| \leq 1$ . For the first step  $\psi_\varepsilon \circ u \in H^{1,p}(\Omega)$  and

$$\int_{\Omega} \psi_\varepsilon(u) \operatorname{div} V \, d\mathcal{M} = - \int_{\Omega} \langle \nabla \psi_\varepsilon(u), V \rangle \, d\mathcal{M} = - \int_{\Omega^+} \frac{u}{\sqrt{u^2 + \varepsilon^2}} \langle \nabla u, V \rangle \, d\mathcal{M} \tag{2.3}$$

for any vector field  $V \in C_c^\infty(\Omega, T\Omega)$ , where  $\Omega^+ = \{x \in \Omega : u(x) > 0\}$ . Moreover, a.e. in  $\Omega$  we have  $|\psi'_\varepsilon(u)| = \psi'_\varepsilon(u) \leq u^+$  and  $\psi_\varepsilon(u) \rightarrow u^+$  as  $\varepsilon \rightarrow 0$ . Furthermore  $u/\sqrt{u^2 + \varepsilon^2} \rightarrow 1$  a.e. in  $\Omega^+$  and of course

$$\left| \frac{u}{\sqrt{u^2 + \varepsilon^2}} \langle \nabla u, V \rangle \right| \leq |\nabla u| \cdot |V|.$$

Therefore, by the Lebesgue theorem, we can pass to the limit in (2.3) proving that

$$\nabla(u^+) = (\nabla u)\chi_{\Omega^+}.$$

The rest of the proof is straightforward.  $\square$

We shall also use this fact:

**Lemma 2.3.** [19, Proposition 2.4] *If  $u : \Omega \rightarrow \mathbb{R}$  is a Lipschitz function with compact support, then  $u \in H^{1,p}(\Omega)$  for every  $p \geq 1$ .*

### 3. Maximum principles for homogeneous inequalities

In this section we shall follow the work of Pucci and Serrin [32] and extend to a Riemannian setting their results about  $p$ -homogeneous inequalities in  $\mathbb{R}^n$ , including in particular inequalities involving the  $p$ -Laplacian operator  $\Delta_p$ . Recall that  $\Omega$  is a bounded and regular domain of  $\mathcal{M}$ , so that  $\overline{\Omega}$  is a smooth manifold with boundary.

We shall treat inequalities of the form

$$\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) \geq 0 \quad \text{in } \Omega, \tag{3.1}$$

where divergence and gradient are taken with respect to the Riemannian structure.

Assume that  $A : T\Omega \times_{\Omega} \mathbb{R} \rightarrow T\Omega$  is continuous, and  $A(x, z, \xi) \in T_x\mathcal{M}$  for all  $x \in \Omega, z \in \mathbb{R}$  and  $\xi \in T_x\mathcal{M}$ . Let  $B$  be a real function defined in  $T\Omega \times_{\Omega} \mathbb{R}$ . We also suppose that there exist  $p \geq 1, a_1 > 0$  and  $a_2, b_1 \geq 0$  such that for all  $(x, z, \xi) \in T\Omega \times_{\Omega} \mathbb{R}$  there holds

$$\langle A(x, z, \xi), \xi \rangle \geq a_1 |\xi|^p - a_2 |z|^p, \quad B(x, z, \xi) \leq b_1 |\xi|^{p-1}. \tag{3.2}$$

**Definition 3.1.** A (weak) solution of (3.1) is a function  $u \in L^1_{\text{loc}}(\Omega)$  such that it is weakly differentiable in  $\Omega$ ,

$$A(\cdot, u, \nabla u) \in L^1_{\text{loc}}(\Omega, T\Omega), \quad B(\cdot, u, \nabla u) \in L^{p'}_{\text{loc}}(\Omega),$$

where  $p' = p/(p - 1)$  if  $p > 1$  and  $p' = \infty$  if  $p = 1$ , and finally such that

$$\int_{\Omega} \langle \mathbf{A}(x, u, \nabla u), \nabla \phi \rangle d\mathcal{M} \leq \int_{\Omega} B(x, u, \nabla u) \phi d\mathcal{M} \tag{3.3}$$

for all nonnegative  $\phi \in H^{1,p}(\Omega)$  such that  $\phi = 0$  a.e. in some neighborhood of  $\partial\Omega$ . We say that  $u$  is a  $p$ -regular solution, if in addition,

$$\mathbf{A}(\cdot, u, \nabla u) \in L^{p'}_{loc}(\Omega). \tag{3.4}$$

By  $u \leq M$  on  $\partial\Omega$  we mean that for every  $\delta > 0$  there exists a neighborhood of  $\partial\Omega$  in which  $u \leq M + \delta$ . We need the following result, which corresponds to [32, Lemma 3.1.2].

**Lemma 3.2.** *Let  $\psi : \mathbb{R} \rightarrow \mathbb{R}_0^+$  be a nondecreasing continuous function such that  $\psi(t) = 0$  for  $t \in (-\infty, \ell]$ ,  $\ell > 0$ , and  $\psi$  is of class  $C^1$  in  $[\ell, \infty)$  with  $\psi'$  bounded. If  $u \in H^{1,p}_{loc}(\Omega)$ ,  $p \geq 1$ , is a  $p$ -regular solution of (3.1) with  $u \leq 0$  on  $\partial\Omega$ , then (3.3) is valid for  $\phi = \psi \circ u$ , in the sense that*

$$\int_{\Omega} \langle \mathbf{A}(x, u, \nabla u), \nabla \phi \rangle d\mathcal{M} \leq \int_{\Omega} [B(x, u, \nabla u)]^+ \phi d\mathcal{M}, \tag{3.5}$$

where  $\nabla \phi = \psi'(u) \nabla u$  when  $u \neq \ell$ .

**Proof.** Let  $\phi_k = \psi_k \circ u$  be the truncation of  $\psi \circ u$  at the level  $k$ , that is,  $\phi_k(u) = \psi(u)$  when  $u < k$  and  $\phi_k(u) = \psi(k)$  when  $u \geq k$ . By the properties of  $\psi$  and by Lemma 2.2,  $\phi_k \in H^{1,p}(\Omega)$ . Clearly  $\phi_k$  has compact support in  $\Omega$  when  $k > 0$ , so that in this case  $\phi_k$  can be used as a test function in (3.3). Hence

$$\int_{\Omega} \langle \mathbf{A}(x, u, \nabla u), \nabla \phi_k \rangle d\mathcal{M} \leq \int_{\Omega} [B(x, u, \nabla u)]^+ \phi_k d\mathcal{M}.$$

Of course

$$\|\nabla \phi_k - \nabla \phi\|_{p,\Omega} = \|\nabla \phi\|_{p,\{u \geq k\}} \rightarrow 0$$

as  $k \rightarrow \infty$ . Finally, by the Beppo Levi theorem ( $\phi_k \uparrow \phi$ ), the result follows at once.  $\square$

The integral  $\int B^+ \phi$  in (3.5) can possibly be infinite, though in the sequel it will be proved to be actually finite in our applications. The next result corresponds to Theorem 3.2.1 in [32].

**Theorem 3.3 (Maximum principle).** *Assume that  $\mathbf{A}$  and  $B$  satisfy (3.2) with  $a_2 = 0$ . Let  $u$  be a  $p$ -regular solution of (3.1) of class  $H^{1,p}_{loc}(\Omega)$ ,  $p \geq 1$ . If  $u \leq M$  on  $\partial\Omega$  for some constant  $M \geq 0$ , then  $u \leq M$  in  $\Omega$ .*

**Proof.** Since  $a_2 = 0$ , it is enough to consider only the case  $M = 0$ . Set  $V = \text{ess sup}_{\Omega} u$  and suppose by contradiction that  $V > 0$ .

First assume  $0 < V < \infty$ . For  $\ell \in (V/2, V)$  we define  $\psi(t) = (t - \ell)^+$ . By Lemma 3.2 we can take  $\psi(u)$  as nonnegative test function for (3.1), so that

$$\int_{\Gamma} \langle \mathbf{A}(x, u, \nabla u), \nabla \psi \rangle d\mathcal{M} \leq \int_{\Gamma} [B(x, u, \nabla \psi)]^+ (u - \ell) d\mathcal{M},$$

where  $\Gamma = \{x \in \Omega : \ell < u(x) \leq V\}$ . By Lemma 2.2

$$\nabla(\psi \circ u) = \begin{cases} \nabla u & \text{in } \Gamma, \\ 0 & \text{in } \Omega \setminus \Gamma. \end{cases}$$

Then applying (3.2) we have:

$$a_1 \int_{\Gamma} |\nabla u|^p d\mathcal{M} \leq b_1 \int_{\Gamma} |\nabla u|^{p-1} w d\mathcal{M}, \tag{3.6}$$

where  $w = (u - \ell)^+$ . Now take  $s = p^* = np/(n - p)$  if  $p < n$ , and  $s = 2p$  (or any exponent  $s > p$ ) if  $p \geq n$ . Applying the Hölder inequality to the right-hand side of (3.6) yields

$$a_1 \|\nabla u\|_{p,\Gamma}^p \leq b_1 |\Gamma|^{1/n} \|w\|_{s,\Omega} \|\nabla u\|_{p,\Gamma}^{p-1}. \tag{3.7}$$

If  $p > 1$ , divide by  $\|\nabla u\|_{p,\Gamma}^{p-1}$  (which is strictly positive, since  $V > 0$ ), so that

$$a_1 \|\nabla u\|_{p,\Gamma} \leq b_1 |\Gamma|^{1/p-1/s} \|w\|_{s,\Omega}. \tag{3.8}$$

From the Sobolev embedding and from the Poincaré inequality (see Appendix A), there exists a positive constant  $C$  such that

$$\|w\|_{s,\Omega} \leq C \|\nabla w\|_{p,\Omega} = C \|\nabla u\|_{p,\Gamma}. \tag{3.9}$$

Combining (3.8) and (3.9), dividing by  $\|w\|_{s,\Omega} (> 0)$  we get the inequality,

$$1 \leq C(b_1/a_1)^p |\Gamma|^{1-p/s},$$

which is also true when  $p = 1$  by (3.7) and (3.8). But such an inequality is impossible, since  $|\Gamma| \rightarrow 0$  when  $k \rightarrow V$ .

If now  $\text{ess sup}_\Omega u = \infty$ , take  $\ell \in (1, \infty)$  and set  $\Gamma = \{x \in \Omega : u(x) > \ell\}$ . The rest of the proof can be repeated word-by-word until the final step, where we let  $\ell \rightarrow \infty$ . Again  $|\Gamma| \rightarrow 0$  and  $\|w\|_{s,\Omega}$  is finite thanks to the Sobolev inequality.  $\square$

The next result covers the case  $b_1 = 0$  in (3.2) and corresponds to [32, Theorem 3.2.2].

**Theorem 3.4 (Maximum principle).** *Assume that  $A$  and  $B$  satisfy (3.2) with  $b_1 = 0$  and let  $u \in H_{\text{loc}}^{1,p}(\Omega)$ ,  $p > 1$ , be a  $p$ -regular solution of (3.1) in  $\Omega$ . If  $u \leq 0$  on  $\partial\Omega$ , then  $u \leq 0$  in  $\Omega$ .*

**Proof.** Assume by contradiction that  $V = \text{ess sup}_\Omega u > 0$  (possibly  $V = \infty$ ), and for any  $\varepsilon > 0$  define:

$$\psi(t) = \begin{cases} 0, & t \leq \varepsilon, \\ 1 - (\varepsilon/t)^{p-1}, & t > \varepsilon. \end{cases}$$

By Lemma 3.2 we can take  $\phi = \psi(u)$  as an admissible test function for (3.1). Since  $b_1 = 0$ , (3.5) implies

$$\int_\Omega \langle A(x, u, \nabla u), \nabla \phi \rangle \, d\mathcal{M} \leq 0. \tag{3.10}$$

Set  $\Gamma = \{x \in \Omega : u(x) > \varepsilon\}$ . Then  $\phi = 0$  in  $\Omega \setminus \Gamma$ , and by Lemma 2.2

$$\nabla \phi = \frac{(p-1)\varepsilon^{(p-1)}}{u^p} \nabla u \quad \text{in } \Gamma.$$

By (3.2) and (3.10) we get

$$0 \geq \int_\Gamma \frac{\langle A(x, u, \nabla u), \nabla u \rangle}{u^p} \, d\mathcal{M} \geq \int_\Gamma \frac{a_1 |\nabla u|^p - a_2 u^p}{u^p} \, d\mathcal{M}. \tag{3.11}$$

Hence

$$a_1 \int_\Gamma |\nabla \log u|^p \, d\mathcal{M} \leq a_2 |\Gamma|. \tag{3.12}$$

Now, put  $w = (u - \varepsilon)^+$  and take  $s = p^*$  if  $p < n$ , and simply  $s > p$  if  $p \geq n$ . Reasoning as in Theorem 3.3, by the Sobolev and Poincaré inequalities, there exists a positive number  $C$  such that

$$\left\| \log \frac{w + \varepsilon}{\varepsilon} \right\|_{s,\Omega} \leq C \left\| \nabla \log \frac{w + \varepsilon}{\varepsilon} \right\|_{p,\Omega} = C \left\| \nabla \log \frac{u}{\varepsilon} \right\|_{p,\Gamma} = C \|\nabla \log u\|_{p,\Gamma}, \tag{3.13}$$



since  $\log[(w + \varepsilon)/\varepsilon] = 0$  on  $\Omega \setminus \Gamma$ . Now take  $\varepsilon \leq \min\{1, V/2\}$  and set:

$$\Sigma = \begin{cases} \{x \in \Omega: V/2 \leq u(x) \leq V\} & \text{if } V < \infty, \\ \{x \in \Omega: u(x) \geq 1\} & \text{if } V = \infty. \end{cases}$$

If  $V < \infty$ , note that  $w \geq V/2 - \varepsilon$  in  $\Sigma$ . Then (3.12) and (3.13) imply:

$$|\Sigma|^{1/s} \log \frac{V}{2\varepsilon} \leq C \left( \frac{a_2}{a_1} |\Gamma| \right)^{1/p}.$$

This is a contradiction, since  $\varepsilon \rightarrow 0$ , while  $\Sigma$  is independent of  $\varepsilon$ .

If  $V = \infty$ , we get in a similar way

$$|\Sigma|^{1/s} \log \frac{1}{2\varepsilon} \leq C \left( \frac{a_2}{a_1} |\Gamma| \right)^{1/p},$$

which is a contradiction.  $\square$

**Remarks.** 1. An alternative formulation of the boundary condition requires that  $(u - M)^+ \in H_0^{1,p}(\Omega)$ . In this case, (3.4) must be strengthened to

$$A(\cdot, u, \nabla u) \in L^{p'}(\Omega, T\Omega), \quad B^+(\cdot, u, \nabla u) \in L^{p'}(\Omega),$$

and the corresponding changes are needed for the proofs.

2. It is obvious that in the previous theorems condition (3.2) needs to hold only for the range of values of  $u(x)$  and  $\nabla u(x)$ ,  $x \in \Omega$ . We shall take advantage of this remark in the following, if  $u$  is assumed to be of class  $H_{loc}^{1,\infty}(\Omega)$  rather than  $u \in H_{loc}^{1,p}(\Omega)$ .

3. The conclusions of Theorems 3.3 and 3.4 still hold even if  $\Omega$  is unbounded, provided the boundary condition is understood to include the condition

$$\limsup_{x \in \Omega, r(x) \rightarrow \infty} u(x) \leq M,$$

where  $r(x) = \text{dist}(x, O)$  denotes the Riemannian distance of  $x$  from a fixed origin  $O \in \Omega$ .

**Theorem 3.5.** *In Theorem 3.3 the coefficient  $b_1$  can be taken as a function in the following Lebesgue spaces:*

$$b_1 \in \begin{cases} L_{loc}^{n/(1-\theta)}(\Omega), & \text{when } 1 < p \leq n, \\ L_{loc}^p(\Omega), & \text{when } p > n, \end{cases}$$

for some  $\theta \in (0, 1]$ .

The same result holds for Theorem 3.4, provided that  $a_2 \in L^1(\Omega)$ .

**Proof.** When  $1 < p \leq n$  the proof of Theorem 3.3 is valid exactly as before, with (3.7) replaced by:

$$a_1 \|\nabla u\|_{p,\Gamma}^p \leq |\Gamma|^{\theta/n} \|b_1\|_{n/(1-\theta),\Gamma} \|w\|_{s,\Omega} \|\nabla u\|_{p,\Gamma}^{p-1}.$$

For the case  $p > n$  it is enough to use the Morrey theorem, see Theorem A.1 in Appendix A.

The second result is obvious from the proof itself.  $\square$

**Theorem 3.6.** *The conclusions of Theorems 3.3 and 3.4 remain valid when the second inequality in (3.2) is replaced by:*

$$B(x, z, \xi) \leq b_1 (|\xi|^{p-1} + |\xi|^{q-1}), \tag{3.14}$$

with  $1 < q < p$ .

**Proof.** The proofs are essentially the same as before, except for the estimate of the right side of (3.6). Indeed, (3.7) becomes

$$a_1 \int_{\Gamma} |\nabla u|^p \leq b_1 \left( \int_{\Gamma} \{ |\nabla u|^{p-1} + |\nabla u|^{q-1} \} \cdot |u| \right).$$

One then applies the Hölder inequality to separate terms on the right side, as before.  $\square$

#### 4. A first comparison results for singular or degenerate inequalities

In this section and in the following we consider the pair of differential inequalities:

$$\operatorname{div} \mathbf{A}(x, u, \nabla u) + B(x, u, \nabla u) \geq 0 \quad \text{in } \Omega, \tag{4.1}$$

$$\operatorname{div} \mathbf{A}(x, v, \nabla v) + B(x, v, \nabla v) \leq 0 \quad \text{in } \Omega, \tag{4.2}$$

where  $\Omega$ , as in the previous section, is a regular bounded domain of  $\mathcal{M}$ ,  $\mathbf{A} : T\Omega \times_{\Omega} \mathbb{R} \rightarrow T\Omega$  is continuous and  $B : T\Omega \times_{\Omega} \mathbb{R} \rightarrow \mathbb{R}$ , and  $u$  and  $v$  satisfy the inequality according to Definition 3.1.

Our first comparison result is concerned with an operator  $\mathbf{A}$  which has no further regularity properties, but which is independent of  $z$  and monotone in the variable  $\xi$ , that is for all  $x \in \Omega$  and  $\xi, \eta \in T_x(\mathcal{M})$ , with  $\xi \neq \eta$ , it results

$$\langle \mathbf{A}(x, \xi) - \mathbf{A}(x, \eta), \xi - \eta \rangle > 0. \tag{4.3}$$

**Theorem 4.1.** *Let  $u$  and  $v$  be  $p$ -regular solutions in  $H_{\text{loc}}^{1,p}(\Omega)$  of (4.1) and (4.2), respectively, with  $p \geq 1$ . Suppose that  $\mathbf{A} = \mathbf{A}(x, \xi)$  is independent of  $z$  and satisfies (4.3). Moreover, assume that  $B = B(x, z)$  is independent of  $\xi$  and nonincreasing in  $z$ .*

*If  $u \leq v$  on  $\partial\Omega$ , then  $u \leq v$  in  $\Omega$ .*

**Proof.** By definition of solution and by subtraction, we get that for any  $\phi \geq 0$  in  $H^{1,p}(\Omega)$  which is 0 near  $\partial\Omega$ ,

$$\int_{\Omega} \langle \mathbf{A}(x, \nabla u) - \mathbf{A}(x, \nabla v), \nabla \phi \rangle d\mathcal{M} \leq \int_{\Omega} [B(x, u) - B(x, v)] \phi d\mathcal{M}.$$

Take  $\varepsilon > 0$  and  $\phi = (u - v - \varepsilon)^+$ . Of course this  $\phi \in H_0^{1,p}(\Omega)$  can be used as test function and Lemmas 3.2 and 2.2 imply:

$$\int_{\Gamma} \langle \mathbf{A}(x, \nabla u) - \mathbf{A}(x, \nabla v), \nabla(u - v) \rangle d\mathcal{M} \leq \int_{\Gamma} [B(x, u) - B(x, v)]^+ (u - v - \varepsilon) d\mathcal{M},$$

where  $\Gamma := \{x \in \Omega : u(x) - v(x) > \varepsilon\}$ . Since  $B$  is nonincreasing in  $z$  and  $\mathbf{A}$  is monotone in  $\xi$ , both sides of the previous inequality equal 0. By (4.3),  $\nabla u = \nabla v$  a.e. in  $\Gamma$ . Moreover,  $(u - v - \varepsilon)^+ = 0$  a.e. in  $\Omega \setminus \Gamma$ , so that

$$\nabla(u - v - \varepsilon)^+ = 0 \quad \text{a.e. in } \Omega.$$

Since  $\Omega$  is connected,  $(u - v - \varepsilon)^+ = c$  for some  $c \in \mathbb{R}$ . In turn  $c = 0$  since  $\phi \in H_0^{1,p}(\Omega)$  and so  $u \leq v + \varepsilon$  in  $\Omega$ . Letting  $\varepsilon \rightarrow 0$  completes the proof.  $\square$

**Remark 4.2.** It is clear that the previous result applies to the  $p$ -Laplace operator  $\Delta_p$ , when  $\mathbf{A}(\xi) = |\xi|^{p-2}\xi$ ,  $p > 1$ . But, even more interesting, it also applies to the mean curvature operator, when  $\mathbf{A}(\xi) = \xi/\sqrt{1 + |\xi|^2}$ .

Indeed,  $p$ -regularity and 1-regularity of solutions belonging to  $H_{\text{loc}}^{1,p}(\Omega)$  and  $H_{\text{loc}}^{1,1}(\Omega)$  in presence of the  $p$ -Laplace operator or the mean curvature operator respectively, are automatic, as well as (4.3).

#### 5. Comparison results for singular or degenerate inequalities

In this section we prove a comparison result for a large class of inequalities which include, as special case, the  $p$ -Laplace operator. To this aim, we first settle down the general framework in which the comparison holds, giving a new definition of ellipticity in terms of the canonical lift (see Section 5.1 below), and then we establish the general comparison result in Section 5.2.

In addition to the continuity of  $\mathbf{A}$ , from now on we also assume that

(A1)  $\mathbf{A}$  is continuously differentiable in  $(T\Omega \setminus \mathbf{0}) \times_{\Omega} \mathbb{R}$ , where  $\mathbf{0}$  stands for the zero section;

(A2)  $B$  is locally bounded in  $T\Omega \times_{\Omega} \mathbb{R}$ .

The requirement of smoothness of  $A$  only away from the zero section is essential in order to cover the case of singular as well as degenerate elliptic inequalities. Indeed, the ellipticity condition fails e.g. for the standard  $p$ -Laplace operator  $\Delta_p$ ,  $p > 1$ ,  $A(\xi) = |\xi|^{p-2}\xi$ , as  $\xi \rightarrow \mathbf{0}$  (see below).

5.1. Ellipticity through the canonical lift

Here we introduce a definition of ellipticity for nonlinear operators seen as fiber-preserving mappings  $A : T\Omega \rightarrow T\Omega$ . For the ease of notation, we write  $\xi_x$  for any tangent vector of  $T_x\mathcal{M}$ ,  $x \in \Omega$ , or simply  $\xi$  if there is no ambiguity.

Roughly speaking,  $A$  is said to be *elliptic at some vector*  $\xi \in T_x\mathcal{M}$  if the tangent mapping of  $A(x, \cdot) : T_x\mathcal{M} \rightarrow T_x\mathcal{M}$  at  $\xi$  is *positive definite*, after the identification of  $T_\xi(T_x\mathcal{M})$  with  $T_x\mathcal{M}$ . More precisely, consider the second tangent bundle of  $\Omega$ ,  $T^2\Omega = T(T\Omega)$ , and let  $\pi_*$  be the tangent mapping of the projection  $\pi : T\Omega \rightarrow \Omega$ . The space  $VB(T\Omega) = \{v \in T^2\Omega : \pi_*(v) = \mathbf{0}\}$  has a natural structure of vector bundle on  $\Omega$ . It is called the *vertical bundle* over  $\Omega$ , since it is the space of vectors which are tangent to the fibers  $T_x\mathcal{M}$  of  $T\Omega$ .

The mapping  $vl : T\Omega \times_\Omega T\Omega \rightarrow VB$  defined as

$$vl(\xi_x, \eta_x) = \frac{d}{dt}(\xi_x + t\eta_x)|_{t=0}, \tag{5.1}$$

is called the *vertical lift* and plays a crucial role in the theory of connections (see [25]). Clearly  $vl$  induces an inner product along the fibers  $(VB)_x$  for every  $x \in \Omega$ .

**Definition 5.1.** We say that  $A$  is *elliptic at*  $\xi \in T_x\mathcal{M}$  if for every  $\eta \in T_x\mathcal{M}$

$$\langle pr_2 \circ vl^{-1} \circ A_* \circ vl(\xi, \eta), \eta \rangle \geq 0, \tag{5.2}$$

where  $pr_2$  is the second factor projection from  $T\Omega \times_\Omega T\Omega$  to  $T\Omega$ .

Note that (5.2) is well defined since  $A$  is fiber-preserving, so its differential  $A_*T^2(\Omega) \rightarrow T^2(\Omega)$  induces a continuous mapping from  $VB(\Omega)$  to  $VB(\Omega)$ .

Now, for any  $b > 0$  let  $P_{b,x} = \overline{B}(\mathbf{0}_x, b) \setminus \{\mathbf{0}_x\}$  be the punctured ball at  $\mathbf{0}_x$  of radius  $b$  in  $T_x\mathcal{M}$ . If  $K \subset \Omega$  set,

$$P_{b,K} = \bigcup_{x \in K} P_{b,x} \subset T\Omega.$$

**Definition 5.2.** We say that  $A$  is *quasi-uniformly elliptic* in  $P_{b,K}$  if there exists  $c = c(b, K) > 0$  such that

$$\langle pr_2 \circ vl^{-1} \circ A_* \circ vl(\xi, \eta), \eta \rangle \geq c|\eta|^2 \tag{5.3}$$

for all  $x \in K$  and  $\xi, \eta \in P_{b,x}$ .

We also say that  $A$  is *elliptic* in  $P_{b,K}$  if (5.3) holds with  $c \geq 0$ .

In other words,  $A$  is quasi-uniformly elliptic in  $P_{b,K}$  if the tangent mapping  $A_*$  of  $A$  is uniformly positive definite in  $P_{b,K}$  by means of the canonical lift.

Concerning  $B$ , as already stated in the Introduction, we say that  $B$  is *regular* in the set  $P_{b,V,K} := \bigcup_{x \in K} P_{b,x} \times [-V, V]$  if it is locally uniformly Lipschitz continuous with respect to  $\xi \in P_{b,K}$ .

Again, also the requirement of Lipschitz continuity of  $B$  only away from the zero section is essential for degenerate equations.

**Remarks 1.** In the special case of linear operators, it is natural to compare our definition to the one given in [8,24,33, 37], where linear uniformly elliptic operators are defined. In particular, in [33] a linear uniformly elliptic operator in divergent form is considered, so that it is possible to adopt an approach which is simpler than ours, but which does not cover the nonlinear cases that we can handle. On the other hand, in [8,24,37] the authors consider operators of the nondivergent form

$$Lu = \text{tr}(D^2u \circ A_x),$$

where  $x \in \mathcal{M}$ ,  $A_x$  is a positive definite symmetric endomorphism of  $T_x(\mathcal{M})$  such that for all  $x \in \mathcal{M}$  and  $\xi \in T_x(\mathcal{M})$

$$c_0|\xi|^2 \leq g(A_x \xi, \xi) \leq C_0|\xi|^2.$$

Therefore, though they can treat nondivergent equations, as in [33], they require an upper bound which we avoid.

2. The definition of ellipticity and of quasi-uniform ellipticity is also valid for operators  $A$  that depend explicitly on  $z$ , since we can write (5.2) and (5.3), respectively, for the mapping  $A(\cdot, z, \cdot)$  with fixed  $z$ . By quasi-uniform ellipticity in  $P_{b,V,K}$  we shall mean that the analogue of (5.3) holds for any  $z \in [-V, V]$ .

3. The concept of ellipticity is well illustrated by the  $p$ -Laplace operator  $\Delta_p$ . Indeed, if  $p > 2$ , then  $\Delta_p$  is elliptic for  $\xi \neq 0$ ; while  $\Delta_p$  is quasi-uniformly elliptic in  $P_{b,\Omega}$  for all  $b > 0$  when  $1 < p \leq 2$  (see Corollary 5.4).

On the other hand, the mean curvature operator is quasi-uniformly elliptic in  $P_{b,\Omega}$  for all  $b > 0$ .

### 5.2. Comparison principle

**Theorem 5.3** (Comparison principle). *Suppose that  $A = A(x, \xi)$  is independent of  $z$  and quasi-uniformly elliptic in  $P_{b,\Omega}$  for all  $b > 0$ . Assume additionally that  $B$  is nonincreasing in  $z$  and regular in  $P_{b,V,\Omega}$  for all  $b > 0$  and  $V > 0$ . Let  $u$  and  $v$  be solutions of (4.1) and (4.2), respectively, of class  $H_{loc}^{1,\infty}(\Omega)$ .*

*If  $u \leq v + M$  on  $\partial\Omega$  for some constant  $M \geq 0$ , then  $u \leq v + M$  in  $\Omega$ .*

**Proof.** It is enough to treat the case  $M = 0$ , since the case of arbitrary values of  $M$  reduces to  $M = 0$  by setting  $\bar{v} = v + M$ , since  $A$  is independent of  $z$ . Fix  $\xi_x$  and  $\eta_x \in P_{b,\Omega}$  and suppose that the line segment  $\zeta_x(t) = t\xi_x + (1-t)\eta_x$ ,  $t \in [0, 1]$ , is contained in  $P_{b,\Omega}$  (hence is never zero). Dropping  $x$  for the ease of notation, by the integral mean value theorem and the regularity assumption on  $A$ , since  $\zeta$  is a vector field along the constant curve  $\gamma(t) = x$ , we have

$$\langle A(\xi) - A(\eta), \xi - \eta \rangle = \int_0^1 \frac{d}{dt} \langle A(\zeta(t)), \xi - \eta \rangle dt = \langle \nabla(A \circ \zeta)(s_0), \xi - \eta \rangle \tag{5.4}$$

for a suitable  $s_0 \in [0, 1]$ . Now, observe that

$$\langle \nabla(A \circ \zeta)(s_0), \xi - \eta \rangle = \langle pr_2 \circ vl^{-1} \circ A_* \circ vl(\zeta(s_0), \xi - \eta), \xi - \eta \rangle. \tag{5.5}$$

Indeed, let  $U$  be a coordinate neighborhood containing  $x$ . Let  $\{e_i\}_{i=1}^n$  be a frame of  $T_x M$ , so that  $\xi = \sum_{i=1}^n \xi_i e_i$  and  $\eta = \sum_{i=1}^n \eta_i e_i$  for suitable components  $\xi_i$  and  $\eta_i$ ,  $i = 1, \dots, n$ . Finally denote by  $A_i(x, y_1, \dots, y_n)$ ,  $i = 1, \dots, n$ , the components of a locale representation of  $A$  on  $U$ . Then in local coordinates

$$vl^{-1} \circ A_* \circ vl(\zeta(s_0), \xi - \eta) = \left( A(\zeta(s_0)), \sum \partial_{y_i} A_i(x, \zeta(s_0))(\xi_i - \eta_i) e_i \right),$$

so that

$$pr_2 \circ vl^{-1} \circ A_* \circ vl(\zeta(s_0), \xi - \eta) = \sum \partial_{y_i} A_i(x, \zeta(s_0))(\xi_i - \eta_i) e_i.$$

On the other hand it is well known (for example, see [13, Eq. (1), p. 51]) that

$$\nabla(A \circ \zeta)(s_0) = \sum_{i=1}^n \frac{d}{dt} A_i(\zeta(t)) \Big|_{s_0} e_i = \sum_{i,j=1}^n \partial_{y_j} A_i(x, \zeta(s_0))(\xi_j - \eta_j) e_i,$$

since the remainder term in the general formula for covariant derivatives of a vector field along a curve  $\gamma$  is  $0$ , being  $\gamma$  a constant curve. Then (5.5) follows.

Therefore (5.3), (5.4) give

$$\langle A(\xi) - A(\eta), \xi - \eta \rangle \geq C|\xi - \eta|^2, \tag{5.6}$$

where  $C = C(b, \Omega)$ .

Let us show, by a continuity argument, that (5.4) remains valid even if  $\zeta(t)$  passes through the origin in the tangent space. Indeed, if necessary, we replace  $\xi$  by a suitable nearby vector  $\xi'$ , so that the new line segment joining  $\xi'$  and  $\eta$

is contained in  $P_{b,\Omega}$ . Therefore (5.6) holds true with  $\xi$  replaced by  $\xi'$ . Now let  $\xi' \rightarrow \xi$ ; since  $A$  is continuous, then (5.6) holds without the previous restriction  $\zeta(t) \neq 0$ , as claimed.

If  $V$  is a fixed positive number, an argument similar to that for obtaining (5.4) together with the regularity of  $B$ , yields

$$B(x, u, \xi) - B(x, v, \eta) \leq b_2|\xi - \eta| + B(x, u, \eta) - B(x, v, \eta),$$

in  $P_{b,\Omega}$  and  $|u|, |v| \leq V$ , where  $b_2$  is a constant depending on the Lipschitz regularity of  $B$ . In particular, since  $B$  is monotone in  $z$ ,

$$B(x, u, \xi) - B(x, v, \eta) \leq b_2|\xi - \eta| \quad \text{when } u \geq v. \tag{5.7}$$

Let  $y = u - v$  and  $Y = \sup_{\Omega} y$ , which is finite, since  $y \leq 0$  on  $\partial\Omega$  by assumption. Let us also note that  $Y$  is actually a “sup” and not an “ess sup” by the Morrey theorem (see Theorem A.1). Suppose by contradiction that  $Y > 0$ . For  $k \in [Y/2, Y)$  we define  $w = (y - k)^+$ . Obviously  $w = 0$  in  $\Omega \setminus [\Gamma \cup y^{-1}(Y)]$ , where

$$\Gamma = \{x \in \Omega : k < y(x) < Y\}.$$

Since  $u, v \in H^{1,\infty}(\Omega)$ , there exist  $V, W > 0$  such that

$$\|u\|_{\infty,\Gamma}, \|v\|_{\infty,\Gamma} \leq V, \quad \|\nabla u\|_{\infty,\Gamma}, \|\nabla v\|_{\infty,\Gamma} \leq W.$$

By subtracting (4.1) from (4.2) we obtain the principal relation

$$\operatorname{div}\{A(x, \nabla u) - A(x, \nabla v)\} + B(x, u, \nabla u) - B(x, v, \nabla v) \geq 0. \tag{5.8}$$

Now  $w$  is a compactly-supported Lipschitz function, with  $\nabla w = \nabla y$  on  $\Gamma$  and  $\nabla w = 0$  on  $\Omega \setminus \Gamma$  by Lemma 2.2. Therefore  $w$  can be taken as a nonnegative test function for inequality (5.8), that is

$$\int_{\Gamma} \{A(x, \nabla u) - A(x, \nabla v), \nabla w\} d\mathcal{M} \leq \int_{\Omega} \{B(x, u, \nabla u) - B(x, v, \nabla v)\} w d\mathcal{M}. \tag{5.9}$$

From (5.6) and (5.7), with  $\xi = \nabla u$  and  $\eta = \nabla v$ , we get (note that  $y > w > 0$  and  $y = u - v > 0$  in  $\Gamma$ )

$$C \int_{\Gamma} |\nabla y|^2 d\mathcal{M} \leq b_2 \int_{\Gamma} |\nabla y| y d\mathcal{M}.$$

Now proceed as in the proof of Theorem 3.3, since  $C = C(b, \Omega)$ .  $\square$

The case of the  $p$ -Laplace operator when  $1 < p \leq 2$  is particularly important.

**Corollary 5.4.** *Theorem 5.3 is valid when  $A$  corresponds to the  $p$ -Laplace operator for  $1 < p \leq 2$ , namely*

$$A(x, \nabla u) = \Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

**Proof.** We have only to show that when  $1 < p \leq 2$ , the  $p$ -Laplace operator is quasi-uniformly elliptic, that is (5.3) is satisfied. Here  $A : T\Omega \rightarrow T\Omega$  is the mapping  $\xi \mapsto |\xi|^{p-2} \xi$  for  $\xi \neq 0$  (and  $A(0) = 0$ ). Obviously,

$$\frac{d}{dt} A(\xi + t\eta)|_{t=0} = \nu(|\xi|^{p-2} \xi, (p-2)|\xi|^{p-4} \langle \xi, \eta \rangle \xi + |\xi|^{p-2} \eta).$$

So that, when  $|\xi| \leq V$

$$\begin{aligned} \langle pr_2 \circ \nu|^{-1} \circ A_* \circ \nu(\xi, \eta), \eta \rangle &= (p-2)|\xi|^{p-4} \langle \xi, \eta \rangle^2 + |\xi|^{p-2} |\eta|^2 \\ &\geq (p-1)|\xi|^{p-2} |\eta|^2 \geq (p-1)V^{p-2} |\eta|^2, \end{aligned}$$

since  $1 < p \leq 2$ .  $\square$

However, we shall show later that a comparison result for the  $p$ -Laplace operator holds true also when  $p > 2$ , under different assumptions (see Corollary 5.8).

In [32] there is another comparison result (Theorem 3.5.3) where  $A$  is permitted to depend on  $z$ , but  $B$  does not depend on the vector variable  $\xi$ . We present here the case in which the concept of boundedness of  $A_*(\partial_z)$  is meant in terms of the canonical lift.

**Theorem 5.5.** *Let  $A$  be quasi-uniformly elliptic in  $P_{b,\Omega}$  for all  $b > 0$  and  $A_*(\partial_z)$  is locally bounded in  $T\Omega \times_{\Omega} \mathbb{R}$ . Assume additionally that  $B = B(x, z)$  does not depend on  $\xi$  and is nonincreasing in the variable  $z$ . Let  $u$  and  $v$  be solutions of (4.1) and (4.2), respectively, of class  $H_{\text{loc}}^{1,\infty}(\Omega)$ .*

*If  $u \leq v$  on  $\partial\Omega$  then  $u \leq v$  in  $\Omega$ .*

**Proof.** The proof is similar to the proof of Theorem 5.3, with the only novelty that (5.6) has to be treated taking into account the derivative of  $A$  with respect to  $z$ , which is assumed to be bounded. More precisely

$$\begin{aligned} \langle A(x, u, \xi) - A(x, v, \eta), \xi - \eta \rangle &= \langle A(x, u, \xi) - A(x, u, \eta), \xi - \eta \rangle \\ &\quad + \langle A(x, u, \eta) - A(x, v, \eta), \xi - \eta \rangle. \end{aligned}$$

As in (5.6), we can find  $c_1 > 0$  such that

$$\langle A(x, u, \xi) - A(x, u, \eta), \xi - \eta \rangle \geq c_1 |\xi - \eta|^2.$$

Moreover, by the mean value theorem there exists  $t$  in the bounded interval between  $u$  and  $v$  such that

$$\langle A(x, u, \eta) - A(x, v, \eta), \xi - \eta \rangle = \langle A_*(\partial_z)(x, t, \eta), \xi - \eta \rangle (u - v),$$

and since  $A_*(\partial_z)$  is locally bounded, there exists  $c_2 > 0$  such that

$$\langle A(x, u, \eta) - A(x, v, \eta), \xi - \eta \rangle \geq -c_2 |\xi - \eta| \cdot |u - v|.$$

By the Cauchy inequality

$$\langle A(x, u, \eta) - A(x, v, \eta), \xi - \eta \rangle \geq -\frac{c_1}{2} |\xi - \eta|^2 - c_3 |u - v|^2,$$

where  $c_3 = 2c_2^2/c_1$ . Therefore,

$$\langle A(x, u, \xi) - A(x, v, \eta), \xi - \eta \rangle \geq \frac{c_1}{2} |\xi - \eta|^2 - c_3 |u - v|^2. \tag{5.10}$$

Next by the monotonicity of  $B$  in  $z$ ,

$$B(x, u) - B(x, v) \leq 0 \quad \text{when } u > v. \tag{5.11}$$

Following the proof of Theorem 5.3, subtract (4.2) from (4.1) to get

$$\text{div} \{ A(x, u, \nabla u) - A(x, v, \nabla v) \} + B(x, u) - B(x, v) \geq 0. \tag{5.12}$$

As before, set  $w = u - v$  and let  $Y = \sup_{\Omega} w$ . Again  $Y$  is finite, since  $w \leq 0$  on  $\partial\Omega$ . Assume by contradiction that  $Y > 0$  and for any  $\varepsilon \in (0, Y/2)$  define  $\psi(t) = (1 - \varepsilon/t)^+$  and  $\varphi = \psi(w)$ . Of course  $w = 0$  in  $\Omega \setminus \Gamma$  and  $w > 0$  in  $\Gamma$ , where

$$\Gamma = \{x \in \Omega: \varepsilon < w(x)\}.$$

By Lemma 3.2 we can take  $\varphi$  as a nonnegative test function for (5.12), so that

$$\int_{\Gamma} \langle A(x, u, \nabla u) - A(x, v, \nabla v), \nabla \varphi \rangle d\mathcal{M} \leq \int_{\Omega} [B(x, u) - B(x, v)]^+ \varphi d\mathcal{M}. \tag{5.13}$$

By (5.11) and the fact that  $w > 0$  in  $\Gamma$  we immediately get that the right-hand side of (5.13) is zero. Moreover,  $\nabla \varphi = \nabla w/w^2$ , so that (5.10), with  $\nabla w = \xi - \eta$ , and (5.13) imply

$$\int_{\Gamma} \left( \frac{c_1}{2} \cdot \frac{|\nabla w|^2}{w^2} - c_3 \right) \leq 0.$$

Now the proof proceeds as in Theorem 3.4 from (3.11) onward.  $\square$

In the next result we remove the condition of quasi-uniformly ellipticity by adding a further hypothesis, which allow us to treat the  $p$ -Laplace operator also when  $p > 2$ . This theorem corresponds to Theorem 3.6.1 in [32].

**Theorem 5.6.** Assume that  $A = A(x, \xi)$  is independent of  $z$  and elliptic in  $T\Omega \setminus \mathbf{0}$ . Assume additionally that  $B$  is regular in  $P_{b,V,\Omega}$  for all  $b, V > 0$  and nonincreasing in  $z$ . Let  $u$  and  $v$  be solutions of (4.1) and (4.2), respectively, of class  $H_{loc}^{1,\infty}(\Omega)$ . Suppose that

$$\operatorname{ess\,inf}_{\Omega} \{ |\nabla u| + |\nabla v| \} > 0.$$

If  $u \leq v + M$  on  $\partial\Omega$  for some constant  $M \geq 0$ , then  $u \leq v + M$  in  $\Omega$ .

The proof relies on the following

**Lemma 5.7.** Let  $\hat{\Omega}$  be a compact subset of  $\Omega$  and let  $x \mapsto \xi_x, x \mapsto \eta_x$  be continuous vector fields in  $\Omega$  satisfying,

$$|\xi_x| \leq W, \quad |\eta_x| \leq W \quad \text{and} \quad |\xi_x| + |\eta_x| \geq 4d,$$

for all  $x \in \hat{\Omega}$  and for some positive constants  $W$  and  $d$ . Then, under the assumptions of Theorem 5.6, there exists a constant  $a_3 > 0$  such that for all  $x \in \hat{\Omega}$ ,

$$\langle A(x, \xi_x) - A(x, \eta_x), \xi_x - \eta_x \rangle \geq a_3 |\xi_x - \eta_x|^2,$$

and

$$|B(x, z, \xi) - B(x, z, \eta)| \leq b_3 |\xi_x - \eta_x|, \tag{5.14}$$

where

$$b_3 = L + 2d^{-1} \sup_{P_{W,V,\Omega}} B$$

and  $L$  is the Lipschitz coefficient of  $B$  over the set

$$S = \{ (x, z, \xi) : x \in \hat{\Omega}, z \in [0, V], d \leq |\xi| \leq W \}.$$

**Proof.** Fix a nonzero tangent vector  $w \in T_x\mathcal{M} \setminus \mathbf{0}$  at some point  $x \in \hat{\Omega}$ , and define the bilinear form  $Q_w : T_x\mathcal{M} \times T_x\mathcal{M} \rightarrow \mathbb{R}$ , as

$$Q_w(\xi, \eta) = \langle pr_2 \circ vl^{-1} \circ A_* \circ vl(w, \xi), \eta \rangle. \tag{5.15}$$

It is clear that  $Q$  is positive definite by ellipticity. Hence we can adapt the Euclidean argument of Lemma 10.2 in [30] with the formalism of the first part of the proof of Theorem 5.3 (here the zero section plays the role of the singular set  $Q$  in [32]). In particular we can take

$$a_3 = \frac{1}{2} \inf_S \{ \min \text{eigenvalue } Q_{w_x} \},$$

where  $S = \{ w_x \in T\Omega, d \leq |w_x| \leq W, x \in \hat{\Omega} \}$ .

Inequality (5.14) is very simple to prove, since the first term in the sum for  $b_3$  applies when  $d \leq |\xi_x|, |\eta_x| \leq W$ , and the second term when  $|\xi_x|, |\eta_x| \geq 2d$ .  $\square$

**Proof of Theorem 5.6.** Of course, since  $A$  is independent of  $z$ , it is enough to treat the case  $M = 0$ .

We proceed as in the proof of Theorem 5.3. Let  $y = u - v$  and  $Y = \sup_{\Omega} y$ , which is finite. Suppose by contradiction that  $Y > 0$ . For  $k \in [Y/2, Y)$  we define  $w = (y - k)^+$ . Of course  $w = 0$  in  $\Omega \setminus [\Gamma \cup y^{-1}(Y)]$ , where

$$\Gamma = \{ x \in \Omega : k < y(x) < Y \}.$$

Moreover

$$\Gamma \subset \bar{\Sigma} \subset \Omega,$$

where  $\Sigma = \{ x \in \Omega : Y/2 < y(x) < Y \}$ . Since  $\Sigma$  is pre-compact in  $\Omega$ , and  $\operatorname{ess\,inf}_{\Omega} \{ |\nabla u| + |\nabla v| \} > 0$ , there exists a number  $d > 0$  such that

$$\operatorname{ess\,inf}_{\Sigma} \{ |\nabla u| + |\nabla v| \} \geq 4d.$$

By assumption there exist  $V, W > 0$  such that

$$\|u\|_{\infty, \Sigma}, \|v\|_{\infty, \Sigma} \leq V, \quad \|\nabla u\|_{\infty, \Sigma}, \|\nabla v\|_{\infty, \Sigma} \leq W.$$

As before,  $w$  can be taken as a nonnegative test function for (4.1) and (4.2), that is (5.9) holds. From Lemma 5.7, applied with  $\nabla u = \xi_x$  and  $\nabla v = \eta_x$ , we have that (5.6) and (5.7) are valid (with  $C = C(b, \Omega)$  replaced by  $a_3$  and  $b_2$  replaced by  $b_3$ ). Now we can proceed exactly as in the proof of Theorem 5.3.  $\square$

**Remark.** With the same notation of Lemma 5.7 above, from the proof of Corollary 5.4 we see that the bilinear form  $Q$  corresponding to the  $p$ -Laplace operator is:

$$Q_w(\xi, \eta) = (p/2 - 1)|w|^{p-4} \langle w, \eta \rangle \cdot \langle w, \xi \rangle + |w|^{p-2} \langle \xi, \eta \rangle$$

(foot points are omitted for the ease of notation). When  $p > 2$  the smaller eigenvalue of  $Q$  is easily shown to be  $|w|^{p-2}$ , that is we can take  $a_3 = d^{p-2}/2$  in Lemma 5.7. For the case  $1 < p \leq 2$  under more general conditions see Corollary 5.4.

**Corollary 5.8.** *Assume that  $B$  is regular in  $P_{b,V,\Omega}$  for all  $b, V > 0$  and nonincreasing in  $z$ . Let  $u$  and  $v$  be solutions of class  $H_{loc}^{1,\infty}(\Omega)$ ,  $p > 1$ , of the inequalities,*

$$\Delta_p u + B(x, u, \nabla u) \geq 0 \quad \text{in } \Omega,$$

and

$$\Delta_p v + B(x, v, \nabla v) \leq 0 \quad \text{in } \Omega,$$

respectively. Suppose that

$$\text{ess inf}_{\Omega} \{ |\nabla u| + |\nabla v| \} > 0.$$

If  $u \leq v + M$  on  $\partial\Omega$  for some constant  $M \geq 0$ , then  $u \leq v + M$  in  $\Omega$ .

The following final comparison result is a general comparison result similar to Theorem 5.5, but  $A$  needs not to be quasi-uniformly elliptic and  $A_*(\partial_z)$  needs not to be bounded in  $P_{b,\Omega}$  for all  $b > 0$ . For the proof it is enough to combine the ideas used in the proofs of Theorems 5.5 and 5.6.

**Theorem 5.9.** *Assume that  $A$  is elliptic in  $P_{b,V,\Omega}$  for all  $b, V > 0$ . Moreover, suppose that  $B = B(x, z)$  does not depend on  $\xi$  and is nonincreasing in  $z$ . Let  $u$  and  $v$  be solutions of (4.1) and (4.2), respectively, of class  $H_{loc}^{1,\infty}(\Omega)$  such that*

$$\text{ess inf}_{\Omega} \{ |\nabla u| + |\nabla v| \} > 0.$$

If  $u \leq v$  on  $\partial\Omega$ , then  $u \leq v$  in  $\Omega$ .

All the results of this section continue to hold if  $\Omega$  is unbounded, provided that the boundary condition is understood to include the limit relation

$$\limsup_{x \in \Omega, r(x) \rightarrow \infty} \{ u(x) - v(x) \} \leq M.$$

Indeed, in the proofs above we use the Sobolev inequalities to functions which are compactly supported on  $\mathcal{M}$ , so that Theorem A.1 can be still applied.

### 6. Uniqueness results

In this section the results of Sections 4 and 5.2 are employed to prove uniqueness of solutions for the Dirichlet problem

$$\begin{cases} \operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$



where  $u_0 \in C(\partial\Omega)$  is a given boundary datum. Every comparison result gives immediately rise to a uniqueness result for the corresponding Dirichlet problem, so that here we report the general cases and the important subcases of the mean curvature operator and of the  $p$ -Laplace operator. Let us also note that existence results for quasilinear problems of the form (6.1) are well known in Euclidean domains since the pioneering paper of Serrin [35].

First, Theorem 4.1 gives a uniqueness result whenever  $A = A(x, \xi)$  is independent of  $z$  and satisfies (4.3), while  $B = B(x, z)$  is independent of  $\xi$  and nonincreasing in  $z$ . As an application, consider the following result.

**Theorem 6.1.** *Assume that  $B = B(x, z)$  is independent of  $\xi$  and nonincreasing in  $z$ . Then the Dirichlet problem:*

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + B(x, u) = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \end{cases}$$

$u_0 \in C(\partial\Omega)$ , has at most one solution in  $H_{\text{loc}}^{1,1}(\Omega)$ .

**Proof.** It is a consequence of Theorem 4.1.  $\square$

**Theorem 6.2.** *Suppose that  $A = A(x, \xi)$  is independent of  $z$  and quasi-uniformly elliptic in  $P_{b,\Omega}$  for all  $b > 0$ . Assume additionally that  $B$  is regular in  $P_{b,V,\Omega}$  for all  $b, V > 0$  and nonincreasing in  $z$ . Then problem (6.1) can have at most one solution in  $H_{\text{loc}}^{1,\infty}(\Omega)$ .*

**Proof.** It is an immediate application of the comparison principle given in Theorem 5.3.  $\square$

**Theorem 6.3.** *Assume that  $A = A(x, \xi)$  is independent of  $z$  and elliptic in  $T\Omega \setminus \mathbf{0}$ . Suppose additionally that  $B$  is regular in  $P_{b,V,\Omega}$  for all  $b, V > 0$  and nonincreasing in  $z$ . Let  $u$  and  $v$  be solutions of (6.1) of class  $H_{\text{loc}}^{1,\infty}(\Omega)$  such that*

$$\operatorname{ess\,inf}_{\Omega}\{|\nabla u| + |\nabla v|\} > 0.$$

Then  $u = v$  in  $\Omega$ .

**Proof.** It is a straightforward consequence of Theorem 5.6.  $\square$

As a general and final application of Theorem 6.3, let us consider in particular the  $p$ -Laplace operator, which is for sure one of the most relevant cases.

**Corollary 6.4.** *Let  $u_0 \in C(\partial\Omega)$ , and let  $u, v$  be weak solutions of class  $H_{\text{loc}}^{1,\infty}(\Omega)$  of the Dirichlet problem:*

$$\begin{cases} \Delta_p u + B(x, u, \nabla u) = 0 & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \end{cases} \quad (6.2)$$

where  $B = B(x, z, \xi)$  is regular in  $P_{b,V,\Omega}$  for all  $b, V > 0$  and nonincreasing in  $z$ . Moreover, assume either,

- (i)  $1 < p \leq 2$ , or
- (ii)  $p > 2$  and  $\operatorname{ess\,inf}_{\Omega}\{|\nabla u| + |\nabla v|\} > 0$ .

Then  $u = v$ .

**Proof.** In the first case it is a consequence of Corollary 5.4, in the second case of Corollary 5.8.  $\square$

**Remark.** Consider problem (6.2) in  $\mathbb{R}^n$ , with  $B(x, z, \xi) = |\xi|^2 - 1$  and  $u_0(x) = x_1$ , for  $x = (x_1, \dots, x_n)$ . Then the unique solution of this problem is  $u(x) = x_1$  whatsoever  $\Omega$  is, since  $|\nabla u| = 1$  in  $\mathbb{R}^n$ .

### Appendix A

We recall here the Sobolev embedding theorem in the framework of Riemannian manifolds with boundary, together with the proof of the validity of an Euclidean-type Poincaré’s inequality. For an extensive treatment of this subject see [3] and [19]. Let  $\Omega$  be a bounded regular domain of  $\mathcal{M}$  so that  $\overline{\Omega}$  is an  $n$ -manifold with  $C^1$  boundary. Of course we do not write the most general version of these results, for which we refer to Theorem 2.30 of [3], Chapters 2 and 10 of [18] and to [20].

**Theorem A.1.** *For the compact manifold  $\overline{\Omega}$  with  $C^1$  boundary, the following Sobolev embeddings hold.*

First part (Sobolev). *If  $1 \leq p < n$ , the embedding  $H^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$  is continuous, where  $p^* = np/(n - p)$ . Moreover, there exists  $\kappa > 0$  such that for any  $u \in H_0^{1,p}(\Omega)$ ,*

$$\|u\|_{p^*,\Omega} \leq \kappa \|\nabla u\|_{p,\Omega}.$$

*If  $p = n$ , the embedding  $H^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$  is continuous for any  $q \in [1, \infty)$ .*

Second part (Morrey). *If  $p > n$ , the embedding  $H^{1,p}(\Omega) \hookrightarrow C^\alpha(\overline{\Omega})$  is continuous for  $0 \leq \alpha \leq 1 - n/p$ .*

Here  $C^\alpha(\Omega)$ ,  $0 < \alpha < 1$ , is the space of Hölder continuous functions of exponent  $\alpha$  with norm

$$\|u\|_{C^\alpha} = \sup_{x \in \Omega} |u(x)| + \sup_{x, y \in \Omega, x \neq y} \{|u(x) - u(y)| \cdot d(x, y)^{-\alpha}\},$$

where  $d(x, y)$  is the Riemannian distance on  $\mathcal{M}$ .

The next result is the analogue of the Kondrachov theorem.

**Theorem A.2.** [3, Theorem 2.34] *If  $\Omega$  is a bounded submanifold with  $C^1$  boundary, the following embeddings are compact.*

- (i) *If  $1 \leq p < n$ , then  $H^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ , with  $q \in [1, p^*)$ . If  $p = n$  the result continues to hold for all  $q \in [1, \infty)$ .*
- (ii) *If  $p > n$ , then  $H^{1,p}(\Omega) \hookrightarrow C^\alpha(\overline{\Omega})$ , with  $0 \leq \alpha \leq 1 - n/p$ .*

Finally, we give a simple version of the Poincaré inequality.

**Lemma A.3.** *Let  $\Omega$  be a bounded submanifold of  $\mathcal{M}$ . If  $p \geq 1$ , then there exists  $C > 0$  such that for all  $u \in H_0^{1,p}(\Omega)$  there holds*

$$\int_{\Omega} |u|^p \leq C \int_{\Omega} |\nabla u|^p.$$

**Proof.** It is the same of the proof of the Poincaré inequality, Theorem 2.10 of [18], established in  $H^{1,p}(\Omega)$ , that is with an extra mean term, which, however, we give for completeness, when  $p > 1$  and without using Theorem A.1.

We show first that the functional  $L(u) = \int_{\Omega} |\nabla u|^p$  has a positive minimum  $\alpha$  on

$$\mathcal{H} = \left\{ u \in H_0^{1,p}(\Omega) : \int_{\Omega} |u|^p = 1 \right\}.$$

Let  $\alpha = \inf_{u \in \mathcal{H}} L(u)$  and let  $k \mapsto v_k$  be a minimizing sequence. Since  $H_0^{1,p}(\Omega)$  is reflexive for  $p > 1$  (see Proposition 2.2 in [18]), by the Kondrachov Theorem A.2, the sequence  $(v_k)_k$  has a subsequence which converges weakly in  $H_0^{1,p}(\Omega)$  and strongly in  $L^p(\Omega)$ , since  $H_0^{1,p}(\Omega)$  is compactly embedded in  $L^p(\Omega)$ . Let  $v$  be its limit. From the strong convergence we see that  $v \in \mathcal{H}$ , while from the weak convergence we have

$$\int_{\Omega} |\nabla v|^p \leq \liminf_{k \rightarrow \infty} \int_{\Omega} |\nabla v_k|^p = \alpha.$$

In turn  $\alpha = L(v) > 0$ , because  $v$  cannot be zero and the proof is completed.  $\square$

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