A NEGATIVE ANSWER TO THE BERNSTEIN PROBLEM FOR INTRINSIC GRAPHS IN THE HEISENBERG GROUP

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Dedicated to the memory of Guido Stampacchia

Abstract. A negative answer to the Bernstein problem for entire $\mathbb{H}$-perimeter minimizing intrinsic graphs is given in the setting of the first Heisenberg group $\mathbb{H}^1$ endowed with its Carnot-Carathéodory metric structure. Moreover, in all Heisenberg groups $\mathbb{H}^n$ an area formula for intrinsic graphs with Sobolev regularity is provided, together with the associated first and second variation formulae.

1. Introduction

The subgraph of a function $\phi : \omega \to \mathbb{R}$ defined in an open set $\omega \subset \mathbb{R}^{n-1}$ is the set

$$E = \{(x', x_n) = (x_1, \dots, x_{n-1}, x_n) \in \omega \times \mathbb{R} : x_n < \phi(x')\}. \tag{1.1}$$

If ϕ is of class $\mathbf{C}^1$, the perimeter of E in $\Omega = \omega \times \mathbb{R}$, i.e. the area of the graph of ϕ, is given by

$$\|\partial E\|(\Omega) = \int_\omega \sqrt{1 + |\nabla \phi|^2} d\mathcal{L}^{n-1}. \tag{1.2}$$

If E locally minimizes perimeter in $\omega \times \mathbb{R}$ (i.e. E minimizes perimeter under compact perturbations) and $\phi \in \mathbf{C}^2(\omega)$, then ϕ satisfies the classical *minimal surface equation*

$$\mathrm{div}\left(\frac{\nabla\phi}{\sqrt{1+|\nabla\phi^2|}}\right) = 0 \quad \text{in} \quad \omega. \tag{1.3}$$

G. Stampacchia and E. De Giorgi studied in [18] the problem of removable singularities: they proved that any classical analytic solution ϕ of (1.3) in a set $\omega = \mathcal{A} \setminus K$ with $\mathcal{A} \subset \mathbb{R}^{n-1}$ open and $K \subset \mathcal{A}$ compact set such that $\mathcal{H}^{n-2}(K) = 0$, can be extended to an analytic solution in $\mathcal{A}$. This result was improved in [32] assuming K to be a closed subset of $\mathcal{A}$ with $\mathcal{H}^{n-2}(K) = 0$.

The classical Bernstein problem asks to find functions $\phi \in \mathbf{C}^2(\mathbb{R}^{n-1})$ solving (1.3) which are not affine functions. The same question can be asked assuming that the subgraph E locally minimizes perimeter in $\mathbb{R}^n$. Both problems were completely solved thanks to many contributions (see [26], chapter 17, for an interesting account on this problem). In particular, S. Bernstein solved the problem for entire minimal graphs in $\mathbb{R}^n$ with $n = 3$.

Theorem 1.1. *(i) If $n \leqslant 8$ every $\mathbf{C}^2$ solution ϕ of* (1.3) *in $\omega = \mathbb{R}^{n-1}$ is an affine function. If $n \geqslant 9$ there are analytic functions $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ solving* (1.3) *which are not affine.*

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(ii) Suppose the set E in (1.1) with $\omega = \mathbb{R}^{n-1}$ locally minimizes perimeter in $\mathbb{R}^n$. Then either $n \geq 9$ or ∂E is a hyperplane.

In this paper we study the *Bernstein problem* in the Heisenberg group $\mathbb{H}^n$. We investigate whether a local minimizer in $\mathbb{H}^n$ of the Heisenberg perimeter whose boundary is an entire "graph" is necessarily a "half-space". In particular, we give a negative answer to the Bernstein problem for entire intrinsic minimal graphs in $\mathbb{H}^1$. The notions of graph and half-space are defined in a suitable way by means of the algebraic structure of $\mathbb{H}^n$. They have been introduced in [21] and [23] in the setting of Carnot groups and studied in [2] in the case of hypersurfaces.

The Bernstein problem in $\mathbb{H}^1$ was attacked in [25], [9], [16], [43], [3] and [17]. In [39], [25] and [9] the problem was studied for $\mathbf{C}^2$ regular sets which are *t-subgraphs*, i.e. sets of the form

$$E_u^t = \{(x, y, t) \in \mathbb{R}^3 : t < u(x, y)\} \tag{1.4}$$

with $u \in \mathbf{C}^2(\mathbb{R}^2)$. A suitable *minimal surface equation* for u has been obtained and its solutions have been called *H-minimal*. In particular, it turned out that there exist H-minimal functions $u : \mathbb{R}^2 \to \mathbb{R}$ whose t-graph is not an affine plane. On the other hand, $\mathbf{C}^2$ regular entire H-minimal solutions u for which E_u^t is a minimizer have been characterized in [43]. In [40], [10] and [42], t-subgraphs of functions which are less than $\mathbf{C}^2$ regular have been considered, too. For instance, in [10] and [42] there are interesting examples of minimizers E_u^t with $u \in Lip(\mathbb{R}^2)$ (see also Remark 2.2). Very recently J.H. Cheng, M. Ritoré and P. Yang informed us about the possibility to construct minimizers with less than Lipschitz regularity.

The Bernstein problem has been recently studied also in higher dimensional Heisenberg groups $\mathbb{H}^n$ under suitable assumptions (see [3], [17]).

Many other classical problems of Geometric Measure Theory have been considered in Heisenberg groups and in related Carnot-Carathédory structures (see, for instance, [6], [20], [24], [21], [1], [22] and [31] where an interesting survey on this argument can be found). In particular, isoperimetric type inequalities have been studied in [36], [19], [6], [20], [24], [14], [13], [30], [43], [34], [41] and [35], and regularity properties of minimal surfaces in $\mathbb{H}^n$ have been investigated in [39], [9], [8], [40], [11], [4] and [5]. Finally, a counterpart of De Giorgi's result on the structure of finite perimeter sets has been obtained in [22] in the setting of step two Carnot groups (see also [1], [38], [31]).

In this paper, we consider the Bernstein problem for *intrinsic graphs* in $\mathbb{H}^n$. Before stating the problem, we need to recall some preliminary facts (see [7] for a more complete introduction to the Heisenberg group).

We denote the points of $\mathbb{H}^n \equiv \mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$ by

$$P = [z, t] = [x + iy, t] = (x, y, t), \qquad z \in \mathbb{C}^n,\ x, y \in \mathbb{R}^n,\ t \in \mathbb{R}.$$

If $P = [z, t]$, $Q = [\zeta, \tau] \in \mathbb{H}^n$ and $r > 0$, the group operation reads as

$$P \cdot Q := [z + \zeta, t + \tau + 2\Im m(\langle z, \bar{\zeta}\rangle)]. \tag{1.5}$$

The group identity is the origin 0 and one has $[z, t]^{-1} = [-z, -t]$. In $\mathbb{H}^n$ there is a natural one parameter group of non isotropic dilations $\delta_r(P) := [rz, r^2 t]$, $r > 0$.

The group $\mathbb{H}^n$ can be endowed with the homogeneous norm

$$\|P\|_\infty := \max\{|z|, |t|^{1/2}\} \tag{1.6}$$

and with the left-invariant and homogeneous distance

$$d_\infty(P, Q) := \|P^{-1} \cdot Q\|_\infty. \tag{1.7}$$

The metric d_∞ is equivalent to the standard Carnot-Carathéodory distance. It follows that the Hausdorff dimension of $(\mathbb{H}^n, d_\infty)$ is $2n+2$, whereas its topological dimension is $2n+1$.

The Lie algebra $\mathfrak{h}_n$ of left invariant vector fields is (linearly) generated by

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j = 1, \dots, n, \quad T = \frac{\partial}{\partial t} \tag{1.8}$$

and the only nonvanishing commutators are

$$[X_j, Y_j] = -4T, \qquad j = 1, \dots n. \tag{1.9}$$

We also use the notation $X_j := Y_{j-n}$ for $j = n+1, \dots, 2n$.

Let $\Omega \subset \mathbb{H}^n$ be an open set and $\varphi = (\varphi_1, \dots, \varphi_{2n}) \in \mathbf{C}^1_c(\Omega; \mathbb{R}^{2n})$. The Heisenberg divergence of φ is

$$\mathrm{div}_{\mathbb{H}} \varphi := \sum_{j=1}^{m} X_j \varphi_j. \tag{1.10}$$

Following De Giorgi, the $\mathbb{H}$-perimeter in Ω of a measurable set $E \subset \mathbb{H}^n$ was introduced in [6] as

$$\|\partial E\|_{\mathbb{H}}(\Omega) := \sup \left\{ \int_E \mathrm{div}_{\mathbb{H}} \varphi \, d\mathcal{L}^{2n+1} : \varphi \in \mathbf{C}^1_c(\Omega, \mathbb{R}^{2n}), |\varphi| \leqslant 1 \right\}. \tag{1.11}$$

Alternatively, $\|\partial E\|_{\mathbb{H}}(\Omega)$ is the total variation in Ω of the vector valued measure $X\chi_E$.

By the Riesz' representation Theorem, $\|\partial E\|_{\mathbb{H}}$ is a Radon measure on Ω for which there exists a unique $\|\partial E\|_{\mathbb{H}}$-measurable function $\nu_E : \Omega \to \mathbb{R}^{2n}$ such that

$$\begin{aligned} &|\nu_E| = 1 \quad \|\partial E\|_{\mathbb{H}}\text{-a.e. in } \Omega \\ &\int_E \mathrm{div}_{\mathbb{H}} \varphi \, d\mathcal{L}^{2n+1} = - \int_\Omega \langle \varphi, \nu_E \rangle \, d\|\partial E\|_{\mathbb{H}} \quad \text{for all } \varphi \in \mathbf{C}^1_c(\Omega, \mathbb{R}^{2n}). \end{aligned} \tag{1.12}$$

We call ν_E the *horizontal inward normal to* E (see [20]).

A real measurable function f defined on an open set $\Omega \subset \mathbb{H}^n$ is said to be of class $\mathbf{C}^1_{\mathbb{H}}(\Omega)$ if $f \in \mathbf{C}^0(\Omega)$ and the distribution

$$\nabla_{\mathbb{H}} f := (X_1 f, \dots, X_{2n} f)$$

is represented by a continuous function. We say that $S \subset \mathbb{H}^n$ is an $\mathbb{H}$-*regular surface* if for every $P \in S$ there exist a neighbourhood U of P and a function $f \in \mathbf{C}^1_{\mathbb{H}}(U)$ such that $\nabla_{\mathbb{H}} f \neq 0$ and $S \cap U = \{Q \in U : f(Q) = 0\}$, see [23] and [2]. The *horizontal normal* to S at P is

$$\nu_S(P) := -\frac{\nabla_{\mathbb{H}} f(P)}{|\nabla_{\mathbb{H}} f(P)|}.$$

The importance of $\mathbb{H}$-regular surfaces is clear in the theory of rectifiability in $\mathbb{H}^n$ [21]. An $\mathbb{H}$-regular surface can be highly irregular from the Euclidean viewpoint, in fact it can be a fractal set [29]. This not being restrictive, we deal only with surfaces S which are level sets of functions $f \in \mathbf{C}^1_{\mathbb{H}}$ with $X_1 f \neq 0$.

If $n \geqslant 2$, we identify the maximal subgroup $\mathbb{W} = \{(x, y, t) \in \mathbb{H}^n : x_1 = 0\}$ with $\mathbb{R}^{2n}$ by writing $(x_2, \dots, x_n, y_1, \dots, y_n, t)$ instead of $(0, x_2, \dots, x_n, y_1, \dots, y_n, t)$; similarly $\mathbb{W} \equiv \mathbb{R}^2_{y,t}$ if $n = 1$. Moreover, for $s \in \mathbb{R}$ we denote by se_1 the point $\exp(sX_1) = (s, 0, \dots, 0) \in \mathbb{H}^n$.

Now we come to the definition of "intrinsic graph". As in [3], a set $S \subset \mathbb{H}^n$ is called X_1-*graph* of a function $\phi : \omega \subset \mathbb{W} \to \mathbb{R}$ if

$$S = \{A \cdot \phi(A) \, e_1 : A \in \omega\}. \tag{1.13}$$

The notion of X_1-graph is not a pointless generalization: for a more complete introduction see [23]. Analogously, a set $E \subset \mathbb{H}^n$ is called X_1-*subgraph* of ϕ if

$$E = E_\phi := \{A \cdot se_1 \ : \ A \in \omega,\, s < \phi(A)\}\,. \tag{1.14}$$

Given ϕ, we denote by $\Phi : \omega \to \mathbb{H}^n$ the corresponding parametric map which is defined as

$$\Phi(A) = \ A \cdot \phi(A)\, e_1 = \exp(\phi(A)X_1)(A), \quad A \in \omega. \tag{1.15}$$

Figure 1 shows the construction of Φ for $n = 1$.

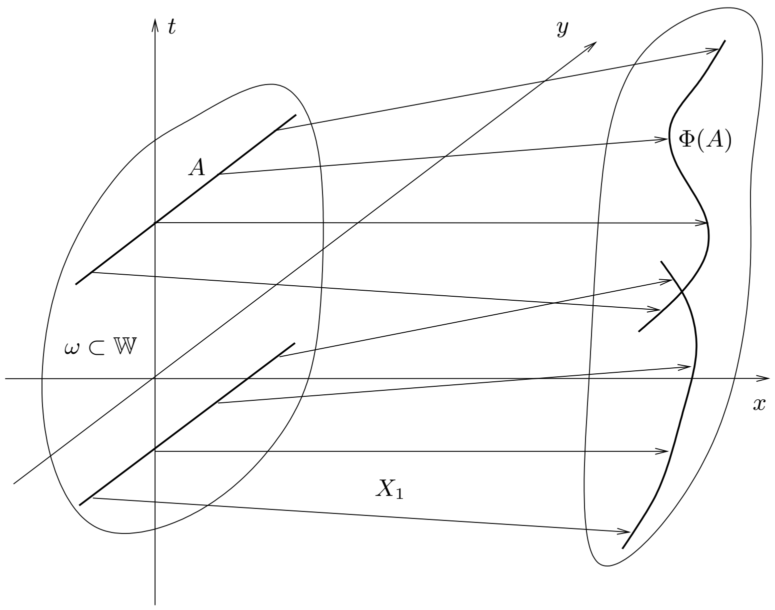

FIGURE 1. Intrinsic graphs.

We are now in a position to state the Implicit Function Theorem (see [21] and [12] for a generalization).

Theorem 1.2 (Implicit Function Theorem)**.** *Let Ω be an open set in $\mathbb{H}^n$ with $0 \in \Omega$ and let $f \in \mathbf{C}^1_{\mathbb{H}}(\Omega)$ be such that $f(0) = 0$ and $X_1 f(0) > 0$. Let*

$$E := \{P \in \Omega : f(P) < 0\} \quad \text{and} \quad S := \{P \in \Omega : f(P) = 0\}.$$

Then:
A) There exist open sets $I \subset \mathbb{W}$ with $0 \in I$ and $J \subset \mathbb{R}$ with $0 \in J$ such that:
- (i) *E has finite $\mathbb{H}$-perimeter in $\mathcal{U} := I \cdot Je_1 = \{A \cdot se_1 : A \in I, s \in J\}$;*
- (ii) *$\partial E \cap \mathcal{U} = S \cap \mathcal{U}$;*
- (iii) *$\nu_E(P) = \nu_S(P)$ for all $P \in S \cap \mathcal{U}$. Here, ν_E is the measure theoretic horizontal inward normal to E defined in (1.12).*

B) There exists a unique continuous function $\phi : I \to \mathbb{R}$ such that $S \cap \mathcal{U} = \Phi(I)$. Moreover, $\mathbb{H}$-perimeter has the integral representation

$$\|\partial E\|_{\mathbb{H}}(\mathcal{U}) = \int_I \frac{|\nabla_{\mathbb{H}} f|}{X_1 f}(\Phi(A))\, d\mathcal{L}^{2n}(A). \tag{1.16}$$

C) There exists a geometric constant $c(n) > 0$ such that $\|\partial E\|_{\mathbb{H}} = c(n)\, \mathcal{S}^{2n+1}_\infty \llcorner S$, where $\mathcal{S}^{2n+1}_\infty$ is the $2n+1$ dimensional spherical Hausdorff measure associated with d_∞.

A characterization of the functions ϕ such that $\Phi(\omega)$ is an $\mathbb{H}$-regular surface is given in [2] (see also [12]). Since $\mathbb{W} = \exp(\mathrm{span}\{X_2, \dots, X_n, Y_1, \dots, Y_n, T\})$ it is possible to define the differential operators given in distributional sense by

$$\begin{aligned} &W^\phi\phi := Y_1\phi - 2T(\phi^2), \\ &\nabla^\phi\phi := \begin{cases} (X_2\phi, \dots, X_n\phi, W^\phi\phi, Y_2\phi, \dots, Y_n\phi) & \text{if } n \geqslant 2 \\ W^\phi\phi & \text{if } n = 1. \end{cases} \end{aligned} \tag{1.17}$$

Theorem 1.3. *Let $\omega \subset \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \to \mathbb{R}$ be a continuous function. The following conditions are equivalent:*

(i) *The set $S := \Phi(\omega)$ is an $\mathbb{H}$-regular surface and $\nu_S^1(P) < 0$ for all $P \in S$, where $\nu_S(P) = (\nu_S^1(P), \dots, \nu_S^{2n}(P))$ is the horizontal normal to S at P.*

(ii) *The distribution $\nabla^\phi\phi$ is represented by a continuous function and there exists a family $(\phi_\epsilon)_{\epsilon>0} \subset \mathbf{C}^1(\omega)$ such that as $\epsilon \to 0^+$*

$$\phi_\epsilon \to \phi \quad \text{and} \quad \nabla^{\phi_\epsilon}\phi_\epsilon \to \nabla^\phi\phi \quad \text{in } L^\infty_{loc}(\omega). \tag{1.18}$$

Moreover, for all $P \in S$ we have

$$\nu_S(P) = \left(-\frac{1}{\sqrt{1 + |\nabla^\phi\phi|^2}}, \frac{\nabla^\phi\phi}{\sqrt{1 + |\nabla^\phi\phi|^2}} \right)(\Phi^{-1}(P)), \tag{1.19}$$

and, with $\Omega = \omega \cdot \mathbb{R}e_1$, we have the area formula

$$\|\partial E_\phi\|_{\mathbb{H}}(\Omega) = c(n)\,\mathcal{S}^{2n+1}_\infty(S) = \int_\omega \sqrt{1 + |\nabla^\phi\phi|^2}\, d\mathcal{L}^{2n}, \tag{1.20}$$

where $c(n)$ is as in Theorem 1.2.

The area formula (1.20) for intrinsic graphs is the exact counterpart of (1.2) for Euclidean graphs. In Section 3, we extend this formula from $\mathbb{H}$-regular X_1-graphs to a class of Sobolev X_1-graphs that we denote by $W^{1,1}_{\mathbb{W}}(\omega)$ (see Definition 3.1 and Theorem 3.4). Then we prove a first and second variation formula for this functional on a suitable subset of $W^{1,1}_{\mathbb{W}}(\omega)$ (see Theorem 3.5).

The notion of intrinsic plane in $\mathbb{H}^n$ arises in a natural way on taking into account Pansu's differentiability theorem in Carnot groups [37]. A function $f : \mathbb{H}^n \to \mathbb{R}$ which is Lipschitz w.r.t. the metric d_∞ can be approximated a.e. by an intrinsic differential, i.e. by a homogeneous linear function $L : \mathbb{H}^n \to \mathbb{R}$. This function is of the form

$$L(x, y, t) = \langle a, x \rangle + \langle b, y \rangle$$

for some $a, b \in \mathbb{R}^n$. It is then natural to define a *vertical plane* V in $\mathbb{H}^n$ as a level set of L, $V = \{(x, y, t) \in \mathbb{H}^n : \langle a, x \rangle + \langle b, y \rangle = c\}$ for some $c \in \mathbb{R}$. It is $V = P_0 \cdot V_0$ for some $P_0 \in V$, where $V_0 := \{(x, y, t) \in \mathbb{H}^n : \langle a, x \rangle + \langle b, y \rangle = 0\}$ is a maximal subgroup of $\mathbb{H}^n$. In [21] it is proved that $\mathbb{H}$-regular surfaces can be approximated at a given point by vertical planes.

We can now give the intrinsic formulation of the Bernstein problem. In [16] and [3] the problem in $\mathbb{H}^n$ has been rephrased replacing the notion of Euclidean t-graph with the notion of intrinsic graph, and the notion of plane with the one of vertical plane. If ϕ is of class $\mathbf{C}^2$, by performing a simple first variation of the functional in (1.20) we can get the following *minimal surface equation for X_1-graphs* (see Section 3)

$$\nabla^\phi \cdot \left(\frac{\nabla^\phi\phi}{\sqrt{1 + |\nabla^\phi\phi|^2}} \right) = 0 \quad \text{in } \omega, \tag{1.21}$$

where, consistently with the distributional definition (1.17), ∇^ϕ is the family of operators

$$\nabla^\phi = (X_2, \dots, X_n, Y_1 - 4\phi T, Y_2, \dots, Y_n) \text{ if } n \geqslant 2, \quad \nabla^\phi = Y_1 - 4\phi T \text{ if } n = 1\,.$$

Vertical planes are parameterized by "affine" functions of the form

$$\phi(x_2, \dots, x_n, y_2, \dots, y_n) = c + \langle (x_2, \dots, x_n, y_2, \dots, y_n), w \rangle, \tag{1.22}$$

with $c \in \mathbb{R}, w \in \mathbb{R}^{2n-1}$ (the previous formula reads $\phi(y,t) = c + wy$ when $n = 1$). These functions are trivial solutions of (1.21) and vertical planes are therefore stationary points of the area functional. In fact, they are minimizers since they have constant horizontal normal (see [3], Example 2.2).

These considerations and Theorem 1.1 suggest the following formulations of the Bernstein problem in the Heisenberg group:

Bernstein problem for X_1-graphs in $\mathbb{H}^n$:

(B_1) Are there entire solutions $\phi \in \mathbf{C}^2(\mathbb{R}^{2n})$ of the minimal surface equation (1.21) which do not parametrize vertical planes?

(B_2) Let $\phi : \mathbb{R}^{2n} \to \mathbb{R}$ be such that E_ϕ is a minimizer for $\mathbb{H}$-perimeter in $\mathbb{H}^n$. Is it true that ∂E_ϕ is a vertical plane?

To our knowledge, Problem (B_1) is answered in the affirmative if $n = 1$ [**?**, 16] and if $n \geq 5$ [3, 17]. In [16] it has been provided a remarkable example in $\mathbb{H}^1$ of a $\mathbf{C}^2$ entire solution ϕ of (1.21) whose subgraph E_ϕ is not a minimizer and ∂E_ϕ is not a vertical plane.This is in contrast with the classical case : Euclidean subgraphs of $\mathbf{C}^2$ solutions to the minimal surface equation (1.3) are also minimizers. Problem (B_2) is answered in the affirmative for $n = 1$ with the additional assumption $\phi \in \mathbf{C}^2(\mathbb{R}^{2n})$ [3]. For $n \geqslant 5$, Problem (B_2) is answered in the negative. Precisely, we have:

Theorem 1.4. *(i) Let $\phi \in \mathbf{C}^2(\mathbb{R}^2)$ and assume that E_ϕ is a minimizer in $\mathbb{H}^1$. Then ∂E_ϕ is a vertical plane, i.e. $\phi(y,t) = wy + c$ for some constants $w, c \in \mathbb{R}$.*

(ii) If $n \geq 5$ there exist functions $\phi \in \mathbf{C}^2(\mathbb{R}^{2n})$ for which E_ϕ is a minimizer in $\mathbb{H}^n$ but ∂E_ϕ is not a vertical plane.

Theorem 1.4 (ii) also yields a negative answer to Problem (B_1) when $n \geq 5$ (see [3]). An extension of Theorem 1.4 (i) to more general $\mathbf{C}^2$ entire graphs without characteristic points has been obtained in [17].

In Section 2, we give a negative answer to Problem (B_2) for $n = 1$ when the regularity assumption $\phi \in \mathbf{C}^2(\mathbb{R}^2)$ of Theorem 1.4 is dropped.

Theorem 1.5. *Let $\vartheta : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$\vartheta(y,t) := -\operatorname{sgn}(t)\sqrt{|t|}. \tag{1.23}$$

Then the subgraph E_ϑ is $\mathbb{H}$-perimeter minimizing in $\mathbb{H}^1$ and

$$\partial E_\vartheta = \{(x, y, 2xy - x|x|) \in \mathbb{H}^1 : x, y \in \mathbb{R}\}$$

is not a vertical plane. (See also Figure 2).

Although the intrinsic graph ∂E_ϑ is of class $C^{1,1}$ from the Euclidean viewpoint, the function ϑ only belongs to $\mathbf{C}^{0,\frac{1}{2}}(\mathbb{R}^2) \setminus Lip_{loc}(\mathbb{R}^2)$. We also show that ϑ satisfies equation (1.21) in a weak sense and that it satisfies a second variation formula for the area functional for X_1-graphs (see Remark 3.9). Regularity results for Lipschitz vanishing viscosity solutions to the intrinsic minimal surface equation (1.21) have been recently announced in [4, 5].

To our knowledge, Problems (B_1) and (B_2) are still open for the cases $n = 2, 3$ and 4. Problem (B_2) still remains open also for $n = 1$ if the boundary ∂E_ϕ of a minimizer E_ϕ is required to be an $\mathbb{H}$-regular surface.

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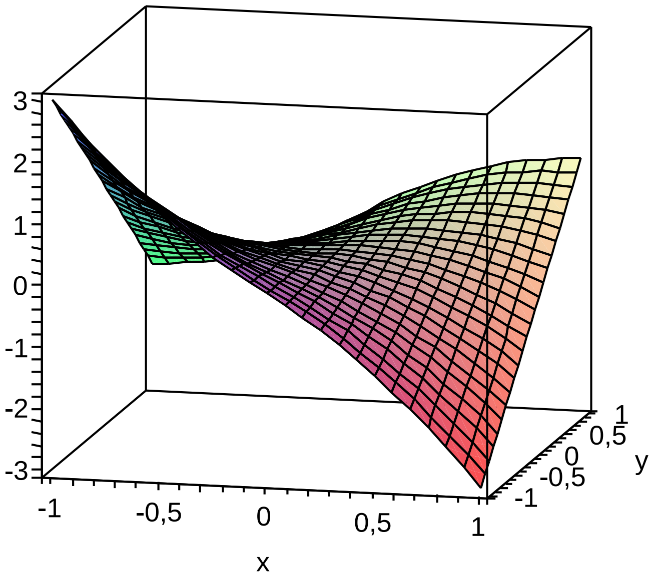

FIGURE 2. The intrinsic graph of ϑ.

2. THE COUNTEREXAMPLE

A set $E \subset \mathbb{H}^n$ with locally finite $\mathbb{H}$-perimeter is said to be *locally minimizing* in a fixed open set $\Omega \subset \mathbb{H}^n$ if for any open subset $\Omega' \Subset \Omega$ one has

$$\|\partial E\|_{\mathbb{H}}(\Omega') \leq \|\partial F\|_{\mathbb{H}}(\Omega') \tag{2.1}$$

for any measurable $F \subset \mathbb{H}^n$ such that $E\Delta F \Subset \Omega'$. We call such a set *minimizer*.

The following calibration result is proved in [3] in the general setting of Carnot groups.

Theorem 2.1. *Let $E \subset \mathbb{H}^n$ be a measurable set, $\Omega \subset \mathbb{H}^n$ be an open set and $\nu : \Omega \to \mathbb{R}^m$ be a Borel map. Assume that:*

(i) *E has locally finite $\mathbb{H}$-perimeter in Ω;*
(ii) *$\nu = \nu_E \quad \|\partial E\|_{\mathbb{H}}$-a.e. in Ω;*
(iii) *there exists an open set $\tilde{\Omega} \subset \Omega$ such that $\|\partial E\|_{\mathbb{H}}(\Omega \setminus \tilde{\Omega}) = 0$ and $\nu \in \mathbf{C}^0(\tilde{\Omega})$;*
(iv) *$\mathrm{div}_{\mathbb{H}}\, \nu = 0$ in distributional sense in Ω.*

Then E is a minimizer of $\mathbb{H}$-perimeter in Ω.

Proof of Theorem 1.5. By (1.14) the (intrinsic) subgraph of ϑ in (1.23) is

$$\begin{aligned} E = E_\vartheta &= \big\{(x,y,t) \in \mathbb{H}^1 : x < \vartheta(y, t-2xy)\big\} \\ &= \big\{(x,y,t) \in \mathbb{R}^3 : f(x,y,t) < 0\big\} \end{aligned} \tag{2.2}$$

where

$$f(x,y,t) := t - 2xy + x|x|, \qquad (x,y,t) \in \mathbb{H}^1.$$

Indeed, the function $g : \mathbb{R} \to \mathbb{R}$, $g(\tau) = -\mathrm{sgn}(\tau)\sqrt{|\tau|}$ and $g(0) = 0$, is strictly decreasing with inverse function $g^{-1}(x) = -x|x|$. The set E can be represented also as an entire t-subgraph of class $\mathbf{C}^{1,1}$, namely it is $E = E^t_u$ with

$$u(x,y) = 2xy - x|x|. \tag{2.3}$$

Thus the boundary $S = \partial E = \{(x,y,t) \in \mathbb{H}^1 : f(x,y,t) = 0\}$ is (Euclidean) $\mathbf{C}^{1,1}$ regular and therefore E has locally finite Euclidean and $\mathbb{H}$-perimeters (see [21], Remark 2.13).

Let $S_0 = S \setminus \{(0,y,t) : y,t \in \mathbb{R}\}$. The inward horizontal normal to E at points in S_0 is

$$\nu_E(x,y,t) = -\left(\frac{1}{\sqrt{5}}, -\frac{x}{|x|}\frac{2}{\sqrt{5}}\right), \qquad (x,y,t) \in S_0. \tag{2.4}$$

In fact, it is $f \in \mathbf{C}^{1,1}(\mathbb{H}^1)$ and

$$X_1 f(x,y,t) = 2|x|, \qquad Y_1 f(x,y,t) = -4x.$$

By Theorem 1.2, the surface S_0 is $\mathbb{H}$-regular with horizontal normal

$$\nu_E = \nu_{S_0} = -\left(\frac{1}{\sqrt{5}}, -\frac{x}{|x|}\frac{2}{\sqrt{5}}\right), \quad (x,y,t) \in S_0,$$

which is (2.4).

The surface S is not $\mathbb{H}$-regular in a neighborhood of any point $(0,y,0) \in S \setminus S_0$, because the normal ν_{S_0} cannot be extended with continuity at such points.

We prove that E is a minimizer in $\mathbb{H}^1$ for $\mathbb{H}$-perimeter by means of Theorem 2.1. Let $\nu = (\nu_1, \nu_2) : \mathbb{H}^1 \setminus \mathbb{W} \to \mathbb{R}^2$ be the map

$$\nu(x,y,t) := -\left(\frac{1}{\sqrt{5}}, -\frac{x}{|x|}\frac{2}{\sqrt{5}}\right).$$

Condition (i) of Theorem 2.1 is trivially satisfied. Condition (iii) is also satisfied with $\tilde{\Omega} = \mathbb{H}^1 \setminus \mathbb{W}$. Indeed, letting $K = \{(0,y,0) : y \in \mathbb{R}\}$, we have

$$\|\partial E\|_{\mathbb{H}}(\Omega \setminus \tilde{\Omega}) = \|\partial E\|_{\mathbb{H}}(K) \le \mathcal{S}^3_\infty(K) \le \mathcal{H}^2(K) = 0,$$

because $\|\partial E\|_{\mathbb{H}} \ll \mathcal{S}^3_\infty$ (see [21], Theorem 7.1) and $\mathcal{S}^3_\infty \ll \mathcal{H}^2$ (see [22], Proposition 4.4), where $\mathcal{H}^2$ denotes the 2-dimensional Euclidean Hausdorff measure in $\mathbb{R}^3$. The same argument shows that also (ii) in Theorem 2.1 is satisfied.

It remains to check (iv), i.e. that $\mathrm{div}_{\mathbb{H}}\, \nu = 0$ in $\mathbb{H}^1$ in distributional sense. In fact, for any $\varphi \in \mathbf{C}^1_c(\mathbb{H}^1)$ we have

$$\begin{aligned}\int_{\mathbb{R}^3} (\nu_1 X_1 \varphi + \nu_2 Y_1 \varphi)\, d\mathcal{L}^3 &= -\frac{1}{\sqrt{5}} \int_{\mathbb{R}^3} (\varphi_x + 2y\varphi_t)\, d\mathcal{L}^3 + \frac{2}{\sqrt{5}} \int_{\mathbb{R}^3} \frac{x}{|x|}(\varphi_y - 2x\varphi_t)\, d\mathcal{L}^3 \\ &= 0,\end{aligned}$$

because both integrals vanish. □

Remark 2.2. An example of a $C^{1,1}$ entire t-graph given by a function similar to the u in (2.3), which locally minimizes the area functional for t-graphs

$$W^{1,1}(\omega) \ni u \mapsto \int_\omega \sqrt{(u_x - 2\,y)^2 + (u_y + 2\,x)^2}\, dxdy, \quad \omega \subset \mathbb{R}^2_{x,y},$$

was already given in [10]. Examples of entire t-graphs of class $C^{1,1}$ which are locally minimizers among C^1 regular surfaces are also given in [41].

3. First and second variation of the area

We extend the area formula (1.20) to intrinsic graphs with Sobolev regularity.

Definition 3.1. Let $\omega \subset \mathbb{R}^{2n}$ be an open set.

(i) We say that a function $\phi \in L^1(\omega) \cap L^2(\omega)$ belongs to the class $W^{1,1}_{\mathbb{W}}(\omega)$ if there exist a sequence $(\phi_j)_{j\in\mathbb{N}} \subset \mathbf{C}^1(\omega)$ and a vector valued map $w \in L^1(\omega, \mathbb{R}^{2n-1})$ such that as $j \to +\infty$

$$\phi_j \to \phi, \quad \phi_j^2 \to \phi^2 \quad \text{and} \quad \nabla^{\phi_j}\phi_j \to w \quad \text{in } L^1(\omega). \tag{3.1}$$

(ii) We say that a function $\phi \in L^1(\omega) \cap L^2(\omega)$ belongs to the class $W^{1,1}_{\mathbb{W},T}(\omega)$ if there exist a sequence $(\phi_j)_{j\in\mathbb{N}} \subset \mathbf{C}^1(\omega)$, a vector valued map $w \in L^1(\omega, \mathbb{R}^{2n-1})$ and a function $v \in L^1(\omega)$ such that (3.1) holds together with

$$T\phi_j \to v \quad \text{in } L^1(\omega). \tag{3.2}$$

(iii) We say that a function $\phi \in \mathbf{C}^0(\omega)$ belongs to the class $\mathbf{C}^1_{\mathbb{W}}(\omega)$ if $\Phi(\omega)$ is an $\mathbb{H}$-regular surface, where $\Phi : \omega \to \mathbb{H}^n$ is the map in (1.15).

Moreover, we say that $\phi \in L^2_{loc}(\omega)$ belongs to the class $W^{1,1}_{\mathbb{W},loc}(\omega)$ (respectively $W^{1,1}_{\mathbb{W},T,loc}(\omega)$) if there exist $(\phi_j)_{j\in\mathbb{N}} \subset \mathbf{C}^1(\omega)$ and $w \in L^1_{loc}(\omega, \mathbb{R}^{2n-1})$ such that all convergences in (3.1) (respectively in (3.1) and (3.2)) hold in $L^1_{loc}(\omega)$.

For a function $\phi \in W^{1,1}_{\mathbb{W},loc}(\omega)$, the distribution $\nabla^\phi\phi$ is represented by an L^1_{loc} function w and namely the function in (3.1). Analogously, if $\phi \in W^{1,1}_{\mathbb{W},T,loc}(\omega)$, $T\phi$ is represented by a function, and precisely the function v in (3.2).

Remark 3.2. By definition we have the trivial inclusions

$$W^{1,1}_{\mathbb{W},T}(\omega) \subset W^{1,1}_{\mathbb{W}}(\omega) \subset L^1(\omega) \cap L^2(\omega), \tag{3.3}$$

as well as the inclusions of the corresponding local classes. Moreover, by Theorem 1.3 we also have

$$\mathbf{C}^1_{\mathbb{W}}(\omega) \subset W^{1,1}_{\mathbb{W},loc}(\omega). \tag{3.4}$$

In general, the inclusion $\mathbf{C}^1_{\mathbb{W}}(\omega) \subset W^{1,1}_{\mathbb{W},T,loc}(\omega)$ does not hold. Indeed, we are able to show that

$$\mathbf{C}^1_{\mathbb{W}}(\omega) \cap W^{1,1}_{\mathbb{W},T,loc} \subset W^{1,1}_{loc}(\omega), \tag{3.5}$$

while an example of a function in $\mathbf{C}^1_{\mathbb{W}}(\omega) \setminus W^{1,1}_{loc}(\omega)$ is provided by [29]. To prove (3.5), let $\phi \in \mathbf{C}^1_{\mathbb{W}}(\omega) \cap W^{1,1}_{\mathbb{W},T,loc}$ be fixed and consider a sequence of mollifications $\phi_j := \phi * \varrho_j$. Here, $\varrho_j(x) := j^{-2n}\varrho(x/j)$ and ϱ is a fixed standard mollifier. Up to subsequences one has

$$\phi_j \to \phi \text{ in } L^\infty_{loc}(\omega) \quad \text{and} \quad T\phi_j \to T\phi \text{ in } L^1_{loc}(\omega),$$

whence $\lim_{j\to\infty}(\phi_j T\phi_j) = \phi T\phi$ in $L^1_{loc}(\omega)$. Therefore, the distribution $T(\phi^2) = 2\phi T\phi$ belongs to $L^1_{loc}(\omega)$. The continuity of $\nabla^\phi\phi$ ensures that $\phi \in W^{1,1}_{loc}(\omega)$.

The class $W^{1,1}_{\mathbb{W},T,loc}(\omega)$ is closed under additive smooth perturbations and it can therefore be used to compute the first variation of the area of intrinsic graphs (see Theorem 3.5 and Remark 3.8).

Lemma 3.3. *If $\phi \in W^{1,1}_{\mathbb{W},T,loc}(\omega)$ and $\psi \in \mathbf{C}^\infty_c(\omega)$ then $\phi + \psi \in W^{1,1}_{\mathbb{W},T,loc}(\omega)$ and*

$$X_i(\phi+\psi) = X_i\phi + X_i\psi \quad i = 2, ..., 2n,\, i \neq n+1,$$

$$W^{\phi+\psi}(\phi+\psi) = W^\phi\phi + W^\psi\psi - 4\,T(\phi\,\psi), \quad T(\phi+\psi) = T\phi + T\psi.$$

Proof. We prove that $\phi+\psi$ fulfills Definition 3.1. Let $(\phi_j)_{j\in\mathbb{N}}$ be as in Definition 3.1 relatively to ϕ. It is sufficient to consider the sequence $(\phi_j+\psi)_{j\in\mathbb{N}}$: indeed, it is trivial to see that $\phi_j\to\phi$, $(\phi_j+\psi)^2\to(\phi+\psi)^2$, $X_i(\phi_j+\psi)\to X_i\phi+X_i\psi$ (if $n\geq 2$) and $T(\phi_j+\psi)\to T\phi+T\psi$ in $L^1(\omega)$. For the last requirement, notice that

$$\begin{aligned} W^{\phi_j+\psi}(\phi_j+\psi) &= W^{\phi_j}\phi_j + W^{\psi}\psi - 4\phi_j T\psi - 4\psi T\phi_j \\ &\to W^{\phi}\phi + W^{\psi}\psi - 4\phi T\psi - 4\psi T\phi \quad \text{in } L^1_{loc}(\omega). \end{aligned}$$

□

An area formula for X_1-graphs of functions in $W^{1,1}_{\mathbb{W}}(\omega)$ is now available.

Theorem 3.4. *Let $\omega\subset\mathbb{R}^{2n}$ be a bounded open set and let E_ϕ be the subgraph of a function $\phi\in W^{1,1}_{\mathbb{W}}(\omega)$. Then we have*

$$\|\partial E_\phi\|_{\mathbb{H}}(\Omega)=\int_\omega\sqrt{1+|\nabla^\phi\phi|^2}\,d\mathcal{L}^{2n}, \tag{3.6}$$

where $\Omega=\omega\cdot\mathbb{R}e_1$.

Proof. Let $(\phi_j)_{j\in\mathbb{N}}$ and w be as in Definition 3.1. Without loss of generality we can also assume that $\phi_j\to\phi$ $\mathcal{L}^{2n}$- a.e. in ω. Let $E:=E_\phi$ and $E_j:=E_{\phi_j}$, and, as in (1.15), set $\Phi_j(A)=A\cdot\phi_j(A)e_1$, $\Phi(A)=A\cdot\phi(A)e_1$. For any $\varphi\in\mathbf{C}^1_c(\Omega,\mathbb{R}^{2n})$ we have

$$\int_{E_j\cap\Omega}\mathrm{div}_{\mathbb{H}}\varphi\,d\mathcal{L}^{2n+1}=-\int_\Omega\langle\nu_{E_j},\varphi\rangle\,d\|\partial E_j\|_{\mathbb{H}}\,. \tag{3.7}$$

Moreover, it is (see [2], Remark 2.23)

$$\|\partial E_j\|_{\mathbb{H}}\llcorner\Omega={\Phi_j}_\#\left(\sqrt{1+|\nabla^{\phi_j}\phi_j|^2}\,\mathcal{L}^{2n}\llcorner\omega\right),$$

and

$$\nu_{E_j}\circ\Phi_j=\left(-\frac{1}{\sqrt{1+|\nabla^{\phi_j}\phi_j|^2}},\frac{\nabla^{\phi_j}\phi_j}{\sqrt{1+|\nabla^{\phi_j}\phi_j|^2}}\right)\in\mathbb{R}\times\mathbb{R}^{2n-1}. \tag{3.8}$$

From (3.7) and (3.8) we obtain

$$\int_{E_j\cap\Omega}\mathrm{div}_{\mathbb{H}}\varphi\,d\mathcal{L}^{2n+1}=\int_\omega\left[\varphi_1\circ\Phi_j-\langle\nabla^{\phi_j}\phi_j,\hat\varphi\circ\Phi_j\rangle\right]d\mathcal{L}^{2n}, \tag{3.9}$$

where $\hat\varphi=(\varphi_2\ldots,\varphi_{2n})$. Now notice that

$$\begin{aligned} &\chi_{E_j}\to\chi_E \quad \mathcal{L}^{2n+1}-\ \text{a.e. in } \Omega \\ &\varphi_i\circ\Phi_j\to\varphi_i\circ\Phi \quad \mathcal{L}^{2n+1}\ \text{a.e. in } \Omega,\ i=1,\ldots,2n. \end{aligned}$$

By the lower semicontinuity of perimeter we have $\|\partial E\|_{\mathbb{H}}(\Omega)<+\infty$. Moreover, letting $j\to+\infty$ in (3.9), by the dominated convergence theorem we get

$$-\int_\Omega\langle\nu_E,\varphi\rangle\,d\|\partial E\|_{\mathbb{H}}=\int_E\mathrm{div}_{\mathbb{H}}\varphi\,d\mathcal{L}^{2n+1}=\int_\omega\left[\varphi_1\circ\Phi-\langle\nabla^\phi\phi,\hat\varphi\circ\Phi\rangle\right]d\mathcal{L}^{2n}, \tag{3.10}$$

where $\nu_E=(\nu_E^{(1)},\ldots,\nu_E^{(2n)})$ denotes the inward generalized horizontal normal to E defined in (1.12).

Identity (3.10) can be also read as

$$\nu_E\,\|\partial E\|_{\mathbb{H}}\llcorner\Omega=\Phi_\#\left(\nu\sqrt{1+|\nabla^\phi\phi|^2}\,\mathcal{L}^{2n}\llcorner\omega\right), \tag{3.11}$$

where

$$\nu=\left(-\frac{1}{\sqrt{1+|\nabla^\phi\phi|^2}},\frac{\nabla^\phi\phi}{\sqrt{1+|\nabla^\phi\phi|^2}}\right)\quad\text{in } \omega.$$

Moreover, we have

$$\nu_E \circ \Phi = \nu \quad \mathcal{L}^{2n} - \text{a.e. in } \omega. \tag{3.12}$$

By (3.11) we get $-\nu_E^{(1)} \|\partial E\|_{\mathbb{H}} \llcorner \Omega = \Phi_{\#}\left(\mathcal{L}^{2n} \llcorner \omega\right)$ and in particular

$$\Phi_{\#}\left(\mathcal{L}^{2n} \llcorner \omega\right) \ll \|\partial E\|_{\mathbb{H}} \llcorner \Omega\,. \tag{3.13}$$

Being $\|\partial E\|_{\mathbb{H}}$ a Radon measure, classical results ensure the existence of a sequence $(\varphi_j)_{j\in\mathbb{N}} \subset \mathbf{C}_c^1(\Omega, \mathbb{R}^{2n})$ with $|\varphi_j| \leqslant 1$ such that

$$\varphi_j \to \nu_E \quad \|\partial E\|_{\mathbb{H}} - \text{a.e. in}\,\Omega. \tag{3.14}$$

By (3.12), (3.13) and (3.14) we get

$$\varphi_j \circ \Phi \to \nu \quad \mathcal{L}^{2n} - \text{a.e. in } \omega. \tag{3.15}$$

Let $\varphi \equiv \varphi_j$ in (3.10), taking the limit as $j \to \infty$ by (3.14) and (3.15) we achieve the thesis. $\square$

If $X : \mathbf{C}_c^\infty(\omega) \to L^1(\omega)$ is an operator we denote by $X^* : L^\infty(\omega) \to \mathcal{D}'(\omega)$ the adjoint operator of X. We have $X_j^*\psi = -X_j\psi$ for all $j = 2, ..., 2n$, $j \neq n+1$, and $T^*\psi = -T\psi$, $\forall \psi \in \mathbf{C}_c^\infty(\omega)$. It is not difficult to check that if $\phi \in L^1(\omega)$ and $T\phi \in L^1(\omega)$ in distributional sense then

$$(W^\phi)^*\psi = -W^\phi\psi + 4\psi T\phi \qquad \text{for } \psi \in \mathbf{C}_c^\infty(\omega),$$

We also set

$$\begin{aligned} \nabla^{\phi^*} &:= \left(X_2^*, \dots, X_n^*, W^{\phi^*}, X_{n+2}^*, \dots, X_{2n}^*\right) && \text{if } n \geqslant 2 \\ \nabla^{\phi^*} &:= W^{\phi^*} && \text{if } n = 1. \end{aligned}$$

We give a weak formulation of the first and second variation of the area functional (3.6) in the Sobolev class $W^{1,1}_{\mathbb{W},T}(\omega)$.

Theorem 3.5. *Let $\phi \in W^{1,1}_{\mathbb{W},T,loc}(\omega)$ be such that the subgraph $E \equiv E_\phi$ is $\mathbb{H}$-perimeter minimizing in $\Omega = \omega \cdot \mathbb{R}e_1$. Then, for any $\psi \in \mathbf{C}_c^\infty(\omega)$*

$$\int_\omega \frac{\langle \nabla^\phi \phi, \nabla^{\phi^*}\psi\rangle}{\sqrt{1 + |\nabla^\phi\phi|^2}}\, d\mathcal{L}^{2n} = 0\,, \tag{3.16}$$

and

$$\int_\omega \frac{(1 + |\nabla^\phi\phi|^2)\left[|\nabla^{\phi^*}\psi|^2 - 8\psi T\psi W^\phi\phi\right] - \langle \nabla^\phi\phi, \nabla^{\phi^*}\psi\rangle^2}{[1 + |\nabla^\phi\phi|^2]^{3/2}}\, d\mathcal{L}^{2n} \geqslant 0 \tag{3.17}$$

where

$$\langle \nabla^\phi\phi, \nabla^{\phi^*}\psi\rangle := \sum_{j\neq n+1} X_j\phi\, X_j^*\psi + W^\phi\phi\, W^{\phi^*}\psi.$$

Here and in the following the sums range over $j = 2, \dots, 2n$ with $j \neq n+1$. When $n = 1$ there is no sum.

Proof. Let $\psi \in \mathbf{C}_c^\infty(\omega)$ and set $\phi_s := \phi + s\psi$ if $s \in \mathbb{R}$. By Lemma 3.3 $\phi_s \in W^{1,1}_{\mathbb{W},T,loc}(\omega)$ and

$$\begin{aligned} W^{\phi_s}\phi_s &= W^\phi\phi + W^{s\psi}(s\psi) - 4sT(\phi\,\psi) \\ &= W^\phi\phi - s\,W^{\phi^*}\psi - 4s^2\psi T\psi. \end{aligned}$$

By Theorem 3.4 we can define the function $g : \mathbb{R} \to [0, +\infty)$

$$g(s) := \|\partial E_{\phi_s}\|_{\mathbb{H}}(\Omega) = \int_\omega \sqrt{1 + |\nabla^{\phi_s}\phi_s|^2} d\mathcal{L}^{2n}$$
$$= \int_\omega \Big[1 + \sum_{j \neq n+1} (X_j\phi + sX_j\psi)^2 + \big(W^\phi\phi - sW^{\phi^*}\psi - 4s^2\psi T\psi\big)^2\Big]^{1/2} d\mathcal{L}^{2n}.$$

It easy to see that g is twice differentiable and it is not difficult to compute

$$g'(s) = \int_\omega \frac{\sum_{j\neq n+1} X_j\phi_s\, X_j\psi + W^{\phi_s}\phi_s\,(-W^{\phi^*}\psi - 8s\psi T\psi)}{\sqrt{1 + |\nabla^{\phi_s}\phi_s|^2}} d\mathcal{L}^{2n}, \tag{3.18}$$

$$\begin{aligned} g''(s) = \int_\omega \frac{1}{1 + |\nabla^{\phi_s}\phi_s|^2} \Bigg\{ & \sqrt{1 + |\nabla^{\phi_s}\phi_s|^2} \times \\ & \times \Bigg[\sum_{j\neq n+1} (X_j\psi)^2 + \big(W^{\phi^*}\psi + 8s\psi T\psi\big)^2 - 8\psi T\psi W^{\phi_s}\phi_s \Bigg] + \\ & - \Bigg[\frac{\Big[\sum_{j\neq n+1} X_j\phi_s\, X_j\psi + W^{\phi_s}\phi_s\big(-W^{\phi^*}\psi - 8s\psi T\psi\big)\Big]^2}{\sqrt{1 + |\nabla^{\phi_s}\phi_s|^2}} \Bigg] \Bigg\} d\mathcal{L}^{2n}. \end{aligned} \tag{3.19}$$

Moreover it is $E\Delta E_{\phi_s} \Subset \Omega$, and since E is a minimizer we have $g'(0) = 0$ and $g''(0) \geq 0$. By (3.18) and (3.19) we get

$$g'(0) = \int_\omega \frac{\sum_{j\neq n+1} X_j\phi\, X_j\psi - W^\phi\phi\, W^{\phi^*}\psi}{\sqrt{1 + |\nabla^\phi\phi|^2}}\, d\mathcal{L}^{2n} = -\int_\omega \frac{\langle \nabla^\phi\phi, \nabla^{\phi^*}\psi\rangle}{\sqrt{1 + |\nabla^\phi\phi|^2}}\, d\mathcal{L}^{2n},$$

$$g''(0) = \int_\omega \frac{(1 + |\nabla^\phi\phi|^2)\Big[|\nabla^{\phi^*}\psi|^2 - 8\psi T\psi W^\phi\phi\Big] - \langle \nabla^\phi\phi, \nabla^{\phi^*}\psi\rangle^2}{[1 + |\nabla^\phi\phi|^2]^{3/2}} d\mathcal{L}^{2n},$$

and then thesis follows. □

Theorem 3.5 can be applied when $\phi \in Lip_{loc}(\omega)$.

Proposition 3.6. *The inclusion* $W^{1,1}_{loc}(\omega) \cap \mathbf{C}^0(\omega) \subset W^{1,1}_{\mathbb{W},T,loc}(\omega)$ *holds.*

Proof. We prove that a function $\phi \in W^{1,1}_{loc}(\omega) \cap \mathbf{C}^0(\omega)$ fulfills the requirements of Definition 3.1 by considering a (sub-)sequence of standard mollifications $\phi_j := \phi * \varrho_j$. Again, $\varrho_j(x) := j^{-2n}\varrho(x/j)$ and ϱ is a fixed standard mollifier.

Well known arguments ensure that $\phi_j \to \phi$ in $L^\infty_{loc}(\omega)$ and, in particular, $\phi_j \to \phi$ and $\phi_j^2 \to \phi^2$ in $L^1_{loc}(\omega)$. Also the convergences

$$T\phi_j \to T\phi, \qquad X_i\phi_j \to X_i\phi, \qquad Y_i\phi_j \to Y_i\phi \quad \text{in } L^1_{loc}(\omega)$$

are immediate for $i = 2, \ldots, n$. Moreover, for any $\omega' \Subset \omega$ we have

$$\int_{\omega'} |2T(\phi_j^2) - 4\phi T\phi|\, d\mathcal{L}^{2n} \leqslant \int_{\omega'} 4|\phi_j - \phi||T\phi_j|\, d\mathcal{L}^{2n} + \int_{\omega'} 4|\phi||T\phi_j - T\phi|\, d\mathcal{L}^{2n} \to 0,$$

because $\phi_j \to \phi$ in $L^\infty(\omega')$, $\|T\phi_j\|_{L^1(\omega')}$ are uniformly bounded, ϕ is bounded on ω' and $T\phi_j \to T\phi$ in $L^1(\omega')$. This implies that, as $j \to +\infty$, we have

$$W^{\phi_j}\phi_j = Y_1\phi_j - 2T(\phi_j^2) \to Y_1\phi - 4\phi T\phi \quad \text{in } L^1_{loc}(\omega).$$

The proof is accomplished. □

Corollary 3.7. *Let* $\phi \in Lip_{loc}(\omega)$ *be such that* E_ϕ *is* $\mathbb{H}$*-perimeter minimizing in* $\Omega = \omega \cdot \mathbb{R}e_1$*. Then* (3.16) *and* (3.17) *hold.*

Proof. Since $Lip_{loc}(\omega) \subset W^{1,1}_{loc}(\omega) \cap \mathbf{C}^0(\omega)$, Theorem 3.5 applies by Proposition 3.6. □

Variation formulae are obtained in [2] under the assumption $\phi \in \mathbf{C}^2(\omega)$ (see also [16] and [15]). A first variation formula for the area functional in $\mathbb{H}^n$ defined on t-graphs of Sobolev type is provided in [10]. Variation formulae for general $\mathbf{C}^2$ surfaces in the setting of CC structures are obtained also in [14], [25], [9], [43], [28], [15] and [33].

Remark 3.8. It is not clear how to compute the first variation for the functional

$$W^{1,1}_{\mathbb{W}}(\omega) \ni \phi \mapsto \int_\omega \sqrt{1+|\nabla^\phi \phi|^2}\, d\mathcal{L}^{2n}.$$

In fact, variations similar to the ones of Lemma 3.3 do not seem to be possible in the class $W^{1,1}_{\mathbb{W}}(\omega)$.

Remark 3.9. By Proposition 3.6, the function ϑ in (1.23) satisfies $\vartheta \in W^{1,1}_{\mathbb{W},T,loc}(\mathbb{R}^2)$. Therefore

$$\int_{\mathbb{R}^2} \frac{W^\vartheta \vartheta\, W^{\vartheta *}\psi}{\sqrt{1+|W^\vartheta\vartheta|^2}}\, d\mathcal{L}^2 = 0 \tag{3.20}$$

and

$$\int_{\mathbb{R}^2} \frac{(1+|W^\vartheta\vartheta|^2)\left[|W^{\vartheta *}\psi|^2 - 8\psi T\psi W^\vartheta\vartheta\right] - (W^\vartheta\vartheta\, W^{\vartheta *}\psi)^2}{\left[1+|W^\vartheta\vartheta|^2\right]^{3/2}}\, d\mathcal{L}^2 \geqslant 0 \tag{3.21}$$

hold for any $\psi \in \mathbf{C}^\infty_c(\mathbb{R}^2)$.

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