Thin walled beams with residual stress

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Abstract

A one-dimensional variational problem for an anisotropic, partially inhomogeneous, residually stressed, rectangular thin-walled beam is derived, by Γ-convergence, from the three dimensional theory of linear elasticity with residual stress.

Keywords: thin-walled cross-section beams, residual stress, linearized elasticity, Γ-convergence, dimension reduction, anisotropic beams.

1 Introduction

The history of linear elasticity with residual stress is quite controversial. Cauchy in 1829 derived the equations of a pre-stressed elastic material but his results were misunderstood and partially forgotten. Recently, motivated by the necessity to obtain more accurate models, the theory of linear elasticity with residual stress has found a renewed interest.

Inspired by works of Freddi, Morassi and Paroni (see [5] and [6]) and of Trabucho and Viano (see [14]), we derive, by means of Γ-convergence, a variational model for a thin-walled beam with residual stress, which, to our knowledge, is new.

We start by considering a sequence of regions

$$\Omega_\varepsilon = \left(-\frac{a\varepsilon^2}{2}, \frac{a\varepsilon^2}{2}\right) \times \left(-\frac{b\varepsilon}{2}, \frac{b\varepsilon}{2}\right) \times (0, \ell) \subset \mathbb{R}^3,$$

where $\varepsilon \in (0, 1]$ is a smallness parameter and ℓ is the length of the beam. The different scalings in the first and second variable are introduced to model the thin-walled beam. We denote by $S_\varepsilon(x_3)$ a generic cross-section relative to the abscissa x_3. The body is clamped on $S_\varepsilon(0)$ and subject to dead-body forces of density $\mathbf{b}^\varepsilon$ and null contact-forces over their lateral surfaces. Moreover, the material is assumed to be completely anisotropic and non homogeneous along the x_3-axis, so to model beams made of several different thin layers jointed along the longitudinal axis. Following Truesdell [15], the constitutive equation in terms of the Piola-Kirchhoff stress tensor $\mathbf{S}$ is

$$\mathbf{S} = \mathring{\mathbf{T}}^\varepsilon + \mathbf{Du}\mathring{\mathbf{T}}^\varepsilon + \mathbb{L}^\varepsilon\mathbf{Eu},$$

where $\mathbf{Du}$ denotes the gradient of the displacement $\mathbf{u}$, $\mathbf{Eu}$ the symmetric part of $\mathbf{Du}$, $\mathring{\mathbf{T}}^\varepsilon$ the residual stress in the reference configuration and $\mathbb{L}^\varepsilon$ the incremental elasticity tensor. The $\mathbf{Du}$-term, which comes into play because of material frame indifference, leads to some differences from the classical theory of linear elasticity; in particular, the elastic energy is

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not convex, see Paroni [13]. After rescaling the problem to a fixed domain, we find the right scalings of the displacement components and of the residual stress tensor. By letting ε go to zero we then find, using Γ-convergence, the one-dimensional limit problem.

The paper is organized as follows. In Section 2 we briefly describe the 3d-problem. We first state the strong formulation of the 3d-equilibrium problem, and then its weak form. The problem of existence of solutions is discussed in Section 3, where we make a wide use of results obtained by Paroni [13]. In Section 4, following a standard procedure, we rewrite the three dimensional variational problem on a fixed domain Ω. Section 5 is devoted to the Γ-convergence theoretical result. In our proof we use some compactness theorems obtained by Freddi, Morassi and Paroni [5]. The difficulty in passing to the limit is due to the lack of convexity of the rescaled energy. Subsequently, we study the convergence of minima and minimizers, in Section 6, and we deduce the equations of equilibrium. In particular, it is interesting to note that the equations of longitudinal extension and bending in the (x_2, x_3) plane are uncoupled, while the equations involving the twist and the displacement along the x_1-axis are coupled, see (23). Moreover, the residual stress appears only in the coupled equations. For a material with monoclinic symmetry several coefficients of the incremental elasticity tensor and components of the residual stress are equal to zero. For this symmetry all equations decouple and the residual stress does not enter into the limit problem, see (27).

Notation: repeated Latin indices are summed from 1 to 3 while repeated Greek indices are summed from 1 to 2. The gradient is denoted by $\mathbf{D}$. The notation used for Lebesgue and Sobolev spaces is standard (see Adams [1]) while the notation used to describe the operations on tensorial quantities is similar to that used by Gurtin [9]. Convergence in the norm will be denoted by $\rightarrow$ while weak convergence is denoted by $\rightharpoonup$.

2 The 3-dimensional problem

We consider the following sequence of open subsets of $\mathbb{R}^3$

$$\Omega_\varepsilon := \omega_\varepsilon \times (0, \ell) \subset \mathbb{R}^3,$$

where

$$\omega_\varepsilon := \Big\{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < \frac{a\varepsilon^2}{2}, |x_2| < \frac{b\varepsilon}{2}\Big\},$$

$\varepsilon \in (0, 1]$ and $\ell > 0$. For any $x_3 \in (0, \ell)$ we further set $S_\varepsilon(x_3) := \omega_\varepsilon \times \{x_3\}$. Henceforth we shall refer to Ω_ε as the reference configuration of an elastic body.

We assume that the body responds elastically to deformations from the reference configuration. If $\mathbf{u}$ is a smooth displacement field defined on Ω_ε and $\mathbf{Du}$ denotes the gradient of $\mathbf{u}$, the first Piola-Kirchhoff stress field, $\mathbf{S}$, can be expressed as (see Truesdell [15])

$$\mathbf{S}(\mathbf{x}) = \mathring{\mathbf{T}}^\varepsilon(\mathbf{x}) + \mathbf{Du}(\mathbf{x})\mathring{\mathbf{T}}^\varepsilon(\mathbf{x}) + \mathbb{L}^\varepsilon(\mathbf{x})\mathbf{Eu}(\mathbf{x}), \tag{1}$$

where $\mathbb{L}^\varepsilon(\mathbf{x})$ is the incremental elasticity tensor, $\mathring{\mathbf{T}}^\varepsilon$ is the Cauchy stress field present in the reference configuration and

$$\mathbf{Eu}(\mathbf{x}) := \mathrm{sym}(\mathbf{Du}(\mathbf{x})) := \frac{\mathbf{Du}(\mathbf{x}) + \mathbf{Du}^T(\mathbf{x})}{2}$$

denotes the strain of $\mathbf{u}$. By using the property $\mathbb{L}^\varepsilon\mathbf{W} = \mathbf{0}$ for every skew-symmetric tensor $\mathbf{W}$ (see Paroni [13]) and the balance of angular momentum, we have $\mathbb{L}^\varepsilon_{ijkl} = \mathbb{L}^\varepsilon_{ijlk} = \mathbb{L}^\varepsilon_{jikl}$. We further assume that $\mathbb{L}^{\varepsilon^T} = \mathbb{L}^\varepsilon$, i.e. $\mathbb{L}^\varepsilon_{ijkl} = \mathbb{L}^\varepsilon_{klij}$.

In what follows we consider the situation in which the body is clamped on $S_\varepsilon(0)$ and is subject only to dead body forces $\mathbf{b}^\varepsilon$, so that the equilibrium equations can be written as

$$\begin{cases} \operatorname{div}\mathbf{S} + \mathbf{b}^\varepsilon = \mathbf{0} & \text{in } \Omega_\varepsilon, \\ \mathbf{S} = \mathring{\mathbf{T}}^\varepsilon + \mathbf{Du}\mathring{\mathbf{T}}^\varepsilon + \mathbb{L}^\varepsilon\mathbf{Eu} & \text{in } \Omega_\varepsilon, \\ \mathbf{Sn} = \mathbf{0} & \text{on } \partial\Omega_\varepsilon \setminus S_\varepsilon(0), \\ \mathbf{u} = \mathbf{0} & \text{on } S_\varepsilon(0), \end{cases} \tag{2}$$

where $\mathbf{n}$ denotes the outward unit normal to the boundary of Ω_ε.

We assume that $\mathring{\mathbf{T}}^\varepsilon$ is a *residual stress*, so that it satisfies the following equations:

$$\begin{cases} \operatorname{div}\mathring{\mathbf{T}}^\varepsilon = \mathbf{0} & \text{in } \Omega_\varepsilon, \\ \mathring{\mathbf{T}}^\varepsilon = \left(\mathring{\mathbf{T}}^\varepsilon\right)^T & \text{in } \Omega_\varepsilon, \\ \mathring{\mathbf{T}}^\varepsilon\mathbf{n} = \mathbf{0} & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{3}$$

Hereafter, we suppose that $\mathbb{L}^\varepsilon \in L^\infty(\Omega_\varepsilon; \mathbb{R}^{3\times3\times3\times3})$, $\mathring{\mathbf{T}}^\varepsilon \in L^\infty(\Omega_\varepsilon; \mathbb{R}^{3\times3})$ and $\mathbf{b}^\varepsilon \in L^2(\Omega_\varepsilon; \mathbb{R}^3)$. The weak form of the problem defined by equations (2) and (3) can be written as: find $\mathbf{u} \in H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3)$ such that

$$\int_{\Omega_\varepsilon} \left(\mathbf{Du}\mathring{\mathbf{T}}^\varepsilon \cdot \mathbf{Dv} + \mathbb{L}^\varepsilon\mathbf{Eu}\cdot\mathbf{Ev}\right) dx = \int_{\Omega_\varepsilon} \mathbf{b}^\varepsilon \cdot \mathbf{v}\, dx, \tag{4}$$

for every $\mathbf{v} \in H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3)$, where

$$H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3) := \{\mathbf{u} \in H^1(\Omega_\varepsilon; \mathbb{R}^3) : \mathbf{u} = \mathbf{0} \text{ on } S_\varepsilon(0)\}.$$

3 Existence of the solution

In this section we study the existence of a solution to the problem defined by equation (4), following the same approach of Paroni [13]. A key ingredient in proving the existence of a solution is Korn's inequality. The following version was proven by Freddi, Morassi and Paroni [5], Theorem 4.1.

Theorem 3.1. *There exists a constant $C > 0$, independent of ε, such that*

$$\int_{\Omega_\varepsilon} \left(|\boldsymbol{u}|^2 + |\boldsymbol{Du}|^2\right) dx \le \frac{C}{\varepsilon^4} \int_{\Omega_\varepsilon} |\boldsymbol{Eu}|^2 dx, \tag{5}$$

for every $\boldsymbol{u} \in H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3)$ and for every $\varepsilon \in (0, 1]$.

Let C_K denote the smallest of all constants C for which inequality (5) holds. We assume that there exists a constant $C > 0$, independent of ε, such that

$$\mathbb{L}^\varepsilon(\mathbf{x})\mathbf{E} \cdot \mathbf{E} \ge C|\mathbf{E}|^2, \tag{6}$$

for every $\mathbf{E} \in \mathbb{R}^{3\times3}_{\mathrm{sym}}$ and for a.e. $\mathbf{x} \in \Omega_\varepsilon$ and we denote by C_L the largest of all such constants C.

We shall prove the existence of a solution provided that the absolute value of the smallest eigenvalue of $\mathring{\mathbf{T}}^{\varepsilon}$,

$$|\mathring{\tau}^{\varepsilon}_m| := \Big|\operatorname*{essinf}_{\mathbf{x}\in\Omega_\varepsilon} \min_{\mathbf{a}\in\mathbb{R}^3} \Big\{\mathring{\mathbf{T}}^{\varepsilon}(\mathbf{x})\mathbf{a}\cdot\mathbf{a} \;:\; |\mathbf{a}|=1\Big\}\Big|,$$

is not too large.

It can be proved that the smallest eigenvalue of $\mathring{\mathbf{T}}^{\varepsilon}$, is either identically equal to 0 or that it also takes negative value (see Lemma 3.2 of Paroni [13]).

By using the inequality

$$\mathbf{HT}\cdot\mathbf{H} \geq \tau_m|\mathbf{H}|^2,$$

for every $\mathbf{T}\in\mathbb{R}^{3\times3}_{\mathrm{sym}}$, $\mathbf{H}\in\mathbb{R}^{3\times3}$, where τ_m is the smallest eigenvalue of $\mathbf{T}$, and the Lax-Milgram lemma, we deduce the following existence theorem.

Theorem 3.2. *Assume that*

$$C_L > C_K\frac{|\mathring{\tau}^{\varepsilon}_m|}{\varepsilon^4}. \tag{7}$$

Then there exists a unique solution $\boldsymbol{u}^\varepsilon\in H^1_\flat(\Omega_\varepsilon;\mathbb{R}^3)$ *of problem* (4).

Proof. From equations (3), inequality (6) and Theorem 3.1 we have

$$\begin{aligned}\int_{\Omega_\varepsilon}\Big(\mathbb{L}^\varepsilon\mathbf{Ev}\cdot\mathbf{Ev}+\mathbf{Dv}\mathring{\mathbf{T}}^{\varepsilon}\cdot\mathbf{Dv}\Big)\,dx &\geq C_L\|\mathbf{Ev}\|^2_{L^2(\Omega_\varepsilon)} - (|\mathring{\tau}^{\varepsilon}_m|)\|\mathbf{Dv}\|^2_{L^2(\Omega_\varepsilon)}\\ &\geq \Big(C_L - C_K\frac{|\mathring{\tau}^{\varepsilon}_m|}{\varepsilon^4}\Big)\|\mathbf{Ev}\|^2_{L^2(\Omega_\varepsilon)},\end{aligned}$$

where in the last inequality we used Theorem 3.1. Existence and uniqueness of the solution of problem (4) follow from an application of the Lax-Milgram lemma. □

Hereafter, we will always assume inequality (7) to hold. Since we have supposed $\mathbb{L}^\varepsilon = \mathbb{L}^{\varepsilon^T}$, the energy functionals

$$J_\varepsilon(\mathbf{u}) := \frac{1}{2}\int_{\Omega_\varepsilon}\Big(\mathbf{Du}\mathring{\mathbf{T}}^{\varepsilon}\cdot\mathbf{Du}+\mathbb{L}^\varepsilon\mathbf{Eu}\cdot\mathbf{Eu}\Big)\,dx - \int_{\Omega_\varepsilon}\mathbf{b}^\varepsilon\cdot\mathbf{u}\,dx,$$

admit, for $\varepsilon>0$, a unique minimizer among all functions $\mathbf{u}\in H^1_\flat(\Omega_\varepsilon;\mathbb{R}^3)$.

4 The rescaled problem

To state our results it is convenient to stretch the domain Ω_ε along the transverse directions x_1 and x_2 in a way that the transformed domain does not depend on ε. Let us therefore set $\Omega:=\Omega_1$, $\omega:=\omega_1$, $S(x_3):=S_1(x_3)$ and let $p_\varepsilon:\Omega\to\Omega_\varepsilon$ be defined by $p_\varepsilon(\mathbf{y}) = p_\varepsilon(y_1,y_2,y_3) = (\varepsilon^2y_1,\varepsilon y_2,y_3)$. Let us consider the following 3×3 matrix

$$\mathbf{H}^\varepsilon\mathbf{v} := (\frac{D_1\mathbf{v}}{\varepsilon^2},\frac{D_2\mathbf{v}}{\varepsilon},D_3\mathbf{v}),$$

where $D_i\mathbf{v}$ denotes the column vector of the partial derivatives of $\mathbf{v}$ with respect to y_i. We will use moreover the following notation

$$\mathbf{E}^\varepsilon\mathbf{v} := \mathrm{sym}(\mathbf{H}^\varepsilon\mathbf{v}),\quad \mathbf{W}^\varepsilon\mathbf{v} := \mathrm{skw}(\mathbf{H}^\varepsilon\mathbf{v})$$

and also denote by $\mathbf{Wv}:=\mathbf{W}^1\mathbf{v}$ the skew symmetric part of the gradient of $\mathbf{v}$. We define

$$\mathbb{L}^\varepsilon = \mathbb{L}\circ p_\varepsilon^{-1} \quad\text{and}\quad \mathring{\mathbf{T}}^{\varepsilon} = \varepsilon^4\mathring{\mathbf{T}}\circ p_\varepsilon^{-1},$$

where $\mathbb{L} \in L^\infty(\Omega;\mathbb{R}^{3\times3\times3\times3})$ and $\mathring{\mathbf{T}} \in L^\infty(\Omega;\mathbb{R}^{3\times3})$. In what follows we shall always assume that inequality (7) holds and we shall denote by $|\mathring{\tau}_m|$ the absolute value of the smallest eigenvalue of $\mathring{\mathbf{T}}$, so that

$$C_L > C_K|\mathring{\tau}_m|, \tag{8}$$

where C_K is Korn's constant for Ω and $\mathbb{L}(\mathbf{y})\mathbf{E}\cdot\mathbf{E} \geq C_L|\mathbf{E}|^2$, for every $\mathbf{E} \in \mathbb{R}^{3\times3}_{\rm sym}$ and a.e. $\mathbf{y} \in \Omega$.

The rescaled energy $F_\varepsilon : H^1_\flat(\Omega;\mathbb{R}^3) \to \mathbb{R}$ is defined by

$$F_\varepsilon(\mathbf{v}) := \frac{1}{\varepsilon^3} J_\varepsilon(\mathbf{v}\circ p_\varepsilon^{-1}) = I_\varepsilon(\mathbf{v}) - \int_\Omega \mathbf{b}^\varepsilon \circ p_\varepsilon \cdot \mathbf{v}\, dy,$$

where

$$I_\varepsilon(\mathbf{v}) := \frac{1}{2}\int_\Omega \Big(\mathbb{L}\mathbf{E}^\varepsilon\mathbf{v}\cdot\mathbf{E}^\varepsilon\mathbf{v} + \varepsilon^4\mathbf{H}^\varepsilon\mathbf{v}\mathring{\mathbf{T}}\cdot\mathbf{H}^\varepsilon\mathbf{v}\Big)\, dy.$$

We suppose the loads to have the following form

$$\begin{aligned}
\mathbf{b}_1^\varepsilon \circ p_\varepsilon(\mathbf{y}) &= \varepsilon^4 b_1(\mathbf{y}) + \varepsilon^3\frac{m(y_3)}{I_3}y_2,\\
\mathbf{b}_2^\varepsilon \circ p_\varepsilon(\mathbf{y}) &= \varepsilon^3 b_2(\mathbf{y}) + \varepsilon^2\frac{m(y_3)}{I_3}y_1,\\
\mathbf{b}_3^\varepsilon \circ p_\varepsilon(\mathbf{y}) &= \varepsilon^2 b_3(\mathbf{y}),
\end{aligned} \tag{9}$$

with $\mathbf{b} = (b_1, b_2, b_3) \in L^2(\Omega;\mathbb{R}^3)$, $m \in L^2(0,\ell)$, while $I_3 := \int_\omega (y_1^2 + y_2^2)\, dy_1\, dy_2$ denotes the polar moment of inertia of the section ω. With the loads given by (9), the energy $F_\varepsilon(\mathbf{v})$ can be rewritten as

$$F_\varepsilon(\mathbf{v}) = I_\varepsilon(\mathbf{v}) - \varepsilon^4\int_\Omega \mathbf{b}\cdot\big(v_1, \frac{v_2}{\varepsilon}, \frac{v_3}{\varepsilon^2}\big)\, dy - \varepsilon^4\int_0^\ell m\,\vartheta^\varepsilon(\mathbf{v})\, dy_3, \tag{10}$$

where we have set

$$\vartheta^\varepsilon(\mathbf{v})(y_3) := \frac{1}{I_3}\int_\omega \Big(\frac{y_1}{\varepsilon^2}v_2(y_1,y_2,y_3) - \frac{y_2}{\varepsilon}v_1(y_1,y_2,y_3)\Big)\, dy_1 dy_2. \tag{11}$$

We note that if $\mathbf{v} \in L^2(\Omega;\mathbb{R}^3)$ then $\vartheta^\varepsilon(\mathbf{v}) \in L^2(0,\ell)$. A similar statement holds if we replace L^2 with H^1.

5 The limit energy

Hereafter, we assume the material to be anisotropic and non homogeneous along the y_3-axis only, i.e. the incremental elasticity tensor does not depend on y_1 and y_2. This assumption allows to consider beams built with thin layers of different mechanical properties. Moreover we assume that $\mathbb{L} \in W^{2,\infty}((0,\ell);\mathbb{R}^{3\times3\times3\times3})$.

Define

$$f_0(\alpha,\beta) := \min\{f(\mathbf{A}) : \mathbf{A} \in \mathrm{Sym},\ A_{23} = \alpha,\ A_{33} = \beta\}, \tag{12}$$

where

$$f(\mathbf{A}) := \mathbb{L}(y_3)\mathbf{A}\cdot\mathbf{A}\ .$$

With a simple but long computation, we can check that

$$f_0(\alpha,\beta) = f(\mathbf{\Lambda}(\alpha,\beta)), \tag{13}$$

where $\boldsymbol{\Lambda}(\alpha,\beta)$ is a symmetric matrix whose components are given by

$$\begin{pmatrix} \mathbb{L}_{1111} & \mathbb{L}_{1122} & 2\mathbb{L}_{1131} & 2\mathbb{L}_{1112} & 0 & 0 \\ \mathbb{L}_{1112} & \mathbb{L}_{2212} & 2\mathbb{L}_{3112} & 2\mathbb{L}_{1212} & 0 & 0 \\ \mathbb{L}_{1131} & \mathbb{L}_{2213} & 2\mathbb{L}_{3131} & 2\mathbb{L}_{3112} & 0 & 0 \\ \mathbb{L}_{1122} & \mathbb{L}_{2222} & 2\mathbb{L}_{2231} & 2\mathbb{L}_{2212} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \Lambda_{11} \\ \Lambda_{22} \\ \Lambda_{31} \\ \Lambda_{21} \\ \Lambda_{33} \\ \Lambda_{23} \end{pmatrix} = \begin{pmatrix} -2\mathbb{L}_{1123}\alpha - \mathbb{L}_{1133}\beta \\ -2\mathbb{L}_{1223}\alpha - \mathbb{L}_{1233}\beta \\ -2\mathbb{L}_{3123}\alpha - \mathbb{L}_{3133}\beta \\ -2\mathbb{L}_{2223}\alpha - \mathbb{L}_{2233}\beta \\ \beta \\ \alpha \end{pmatrix}.$$

We note that the components of $\boldsymbol{\Lambda}$ can be written as

$$\begin{aligned} \Lambda(\alpha,\beta)_{11} &= C_1(y_3)\alpha + C_2(y_3)\beta, & \Lambda(\alpha,\beta)_{12} &= C_3(y_3)\alpha + C_4(y_3)\beta, \\ \Lambda(\alpha,\beta)_{22} &= C_5(y_3)\alpha + C_6(y_3)\beta, & \Lambda(\alpha,\beta)_{13} &= C_7(y_3)\alpha + C_8(y_3)\beta, \\ \Lambda(\alpha,\beta)_{33} &= \beta, & \Lambda(\alpha,\beta)_{23} &= \alpha, \end{aligned} \tag{14}$$

where, for $i = 1, \dots, 8$, $C_i(y_3)$ are a combination of the components of $\mathbb{L}(y_3)$ only. By using (13) and (14), we find that f_0 can be expressed as

$$f_0\,(\alpha,\beta) = 4\widetilde{\mu}(y_3)\alpha^2 + \widetilde{E}(y_3)\beta^2 + 2\widetilde{\gamma}(y_3)\alpha\beta\;, \tag{15}$$

where the coefficients $\widetilde{\mu}(y_3), \widetilde{E}(y_3)$ and $\widetilde{\gamma}(y_3)$ depend on the components of the incremental elasticity tensor only.

By inequality (8), it follows that $\widetilde{\mu}(y_3), \widetilde{E}(y_3) > 0$ and $\widetilde{\gamma}(y_3)^2 < 4\widetilde{\mu}(y_3)\widetilde{E}(y_3)$ almost everywhere in $(0,\ell)$.

We introduce the space of Bernoulli-Navier displacements on Ω,

$$H_{BN}(\Omega;\mathbb{R}^3) := \{\mathbf{v} \in H^1_\flat(\Omega;\mathbb{R}^3)\ :\ (\mathbf{E}\mathbf{v})_{i\alpha} = 0 \text{ for } i = 1,2,3,\ \alpha = 1,2\},$$

which can be characterized also as follows (see Le Dret [11], Section 4.1)

$$\begin{aligned} H_{BN}(\Omega;\mathbb{R}^3) = \{\ \ &\mathbf{v} \in H^1_\flat(\Omega;\mathbb{R}^3) : \exists\,\xi_\alpha \in H^2_\flat(0,\ell), \exists\,\xi_3 \in H^1_\flat(0,\ell) \\ &\text{s.\,t. } v_\alpha(\mathbf{y}) = \xi_\alpha(y_3),\ v_3(\mathbf{y}) = \xi_3(y_3) - y_\alpha\xi'_\alpha(y_3)\}. \end{aligned} \tag{16}$$

Theorem 5.1. *Let $F : H^1_\flat(\Omega;\mathbb{R}^3) \times H^1_\flat(\Omega;\mathbb{R}) \to \mathbb{R}\cup\{+\infty\}$ be defined by*

$$\begin{aligned} F(\boldsymbol{v},\vartheta) :=\ & \frac{1}{2}\int_\Omega f_0\left(y_1 D_3\vartheta - \frac{\widetilde{\gamma}}{4\widetilde{\mu}}\,(D_3v_3 + y_1D_{33}v_1)\,, D_3v_3\right) + \boldsymbol{Hv}\mathring{\boldsymbol{T}}\cdot\boldsymbol{Hv}\,dy \\ & - \int_\Omega \boldsymbol{b}\cdot\boldsymbol{v}\,dy - \int_0^\ell m\vartheta\,dy_3 \end{aligned} \tag{17}$$

if $\boldsymbol{v} \in H_{BN}(\Omega;\mathbb{R}^3)$, and $+\infty$ otherwise, where

$$\boldsymbol{Hv} := \begin{pmatrix} 0 & -\vartheta & D_3v_1 \\ \vartheta & 0 & 0 \\ -D_3v_1 & 0 & 0 \end{pmatrix}. \tag{18}$$

As $\varepsilon \to 0$, the sequence of functionals $(1/\varepsilon^4)F_\varepsilon$ defined in (10) and (11) Γ-converges to the functional F, in the following sense:

1. *(liminf inequality) for every sequence of positive numbers ε_k converging to* 0 *and for every sequence* $\{\boldsymbol{u}^k\} \subset H^1_\flat(\Omega;\mathbb{R}^3)$ *such that*

$$(u_1^k, \frac{u_2^k}{\varepsilon_k}, \frac{u_3^k}{\varepsilon_k^2}) \rightharpoonup \boldsymbol{v} \text{ in } H^1(\Omega;\mathbb{R}^3), \qquad (W^{\varepsilon_k} u^k)_{12} \to -\vartheta \text{ in } L^2(\Omega),$$

   we have

$$\liminf_{k\to+\infty} \frac{F_{\varepsilon_k}(\boldsymbol{u}^k)}{\varepsilon_k^4} \geq F(\boldsymbol{v},\vartheta);$$

2. *(recovery sequence) for every sequence of positive numbers ε_k converging to* 0 *and for every* $(\boldsymbol{v},\vartheta) \in H^1_\flat(\Omega;\mathbb{R}^3) \times H^1_\flat(\Omega;\mathbb{R})$, *there exists a sequence* $\{\boldsymbol{u}^k\} \subset H^1_\flat(\Omega;\mathbb{R}^3)$ *such that*

$$(u_1^k, \frac{u_2^k}{\varepsilon_k}, \frac{u_3^k}{\varepsilon_k^2}) \rightharpoonup \boldsymbol{v} \text{ in } H^1(\Omega;\mathbb{R}^3), \qquad (W^{\varepsilon_k} u^k)_{12} \to -\vartheta \text{ in } L^2(\Omega),$$

   and

$$\limsup_{k\to+\infty} \frac{F_{\varepsilon_k}(\boldsymbol{u}^k)}{\varepsilon_k^4} \leq F(\boldsymbol{v},\vartheta).$$

To prove Theorem 5.1 above, we need some auxiliary results. We briefly recall a rescaled Korn's inequality obtained by Freddi, Morassi and Paroni [5].

Theorem 5.2. *There exists a positive constant K, independent of ε, such that*

$$\int_\Omega \Big(\big|(u_1, \frac{u_2}{\varepsilon}, \frac{u_3}{\varepsilon^2})\big|^2 + |\boldsymbol{H}^\varepsilon \boldsymbol{u}|^2 \Big) dy \leq \frac{K}{\varepsilon^4} \int_\Omega |\boldsymbol{E}^\varepsilon \boldsymbol{u}|^2 \, dy,$$

for every $\boldsymbol{u} \in H^1_\flat(\Omega;\mathbb{R}^3)$ *and every* $\varepsilon \in (0,1]$.

The following lemma was proven in [5].

Lemma 5.3. *Let $\boldsymbol{u}^\varepsilon$ be a sequence of functions in $H^1_\flat(\Omega;\mathbb{R}^3)$ such that*

$$\|\boldsymbol{E}^\varepsilon \boldsymbol{u}^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \leq C\varepsilon^2,$$

for some constant C and for every $\varepsilon \in (0,1]$. Then

1. *for any sequence of positive numbers ε_n converging to 0, there exist a subsequence (not relabelled) and a couple of functions $\boldsymbol{v} \in H_{BN}(\Omega;\mathbb{R}^3)$ and $\vartheta \in L^2(\Omega)$ such that (as $n \to +\infty$)*

$$(u_1^{\varepsilon_n}, \frac{u_2^{\varepsilon_n}}{\varepsilon_n}, \frac{u_3^{\varepsilon_n}}{\varepsilon_n^2}) \rightharpoonup \boldsymbol{v} \text{ in } H^1(\Omega;\mathbb{R}^3), \tag{19}$$

   and

$$\boldsymbol{W}^{\varepsilon_n} \boldsymbol{u}^{\varepsilon_n} \rightharpoonup \boldsymbol{H}\boldsymbol{v} = \begin{pmatrix} 0 & -\vartheta & D_3 v_1 \\ \vartheta & 0 & 0 \\ -D_3 v_1 & 0 & 0 \end{pmatrix} \text{ in } L^2(\Omega;\mathbb{R}^{3\times 3});$$

2. $\vartheta^\varepsilon(\boldsymbol{u}^\varepsilon) \to \vartheta$ *in $L^2(\Omega)$; therefore ϑ does not depend on y_1 and y_2;*

3. $\|\vartheta^\varepsilon(\boldsymbol{u}^\varepsilon)\|_{L^2(\Omega)} \leq (K/\varepsilon^2)\|\boldsymbol{E}^\varepsilon \boldsymbol{u}^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}$ *holds for some constant $K > 0$;*

4. $\vartheta \in H^1_\flat(\Omega)$;

5. *the following identities hold in* $L^2(\Omega)$

$$E_{33} = D_3 v_3, \qquad \text{and} \qquad E_{23} = y_1 D_3\vartheta + \eta,$$

where $\eta \in L^2(\Omega)$ *is independent on* y_1 *and, up to subsequences,* E_{33} *and* E_{23} *are the limits of* $(E^\varepsilon u^\varepsilon)_{33}/\varepsilon^2$ *and* $(E^\varepsilon u^\varepsilon)_{23}/\varepsilon^2$ *in the weak convergence of* $L^2(\Omega)$ *respectively.*

Remark 5.4. The strong convergence in 2. was shown in [5], Lemma 4.6, even if not explicity stated. From Lemma 4.3 and Lemma 4.4 by Freddi, Morassi and Paroni [5], it follows that

$$(H^\varepsilon u^\varepsilon)_{12} \to -\vartheta \ \text{ in } L^2(\Omega), \qquad \text{and} \qquad (H^\varepsilon u^\varepsilon)_{21} \to \vartheta \ \text{ in } L^2(\Omega). \tag{20}$$

Lemma 5.5. *Let* $\boldsymbol{u}^\varepsilon$ *be a sequence of functions in the space* $H^1_\flat(\Omega;\mathbb{R}^3)$*. If*

$$\sup_\varepsilon \big(F_\varepsilon(\boldsymbol{u}^\varepsilon)/\varepsilon^4\big) < +\infty,$$

then

$$\|\boldsymbol{E}^\varepsilon \boldsymbol{u}^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times3})} \le C\varepsilon^2, \tag{21}$$

holds for some constant $C > 0$ *and for every* $\varepsilon \in (0,1]$.

Proof. It is convenient to set $\mathbf{v}^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon/\varepsilon, u_3^\varepsilon/\varepsilon^2)$ and $R := C_L - C_K|\mathring{\tau}_m|$. By inequality (8), we have $R > 0$. With this notation and by using (6), (10), Theorem 5.2, and 3. of Lemma 5.3 we obtain

$$\begin{aligned}
\frac{1}{\varepsilon^4}F_\varepsilon(\mathbf{u}^\varepsilon) &= \frac12\int_\Omega \mathbb{L}\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2}\cdot\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2} + \mathbf{H}^\varepsilon\mathbf{u}^\varepsilon\mathring{\mathbf{T}}\cdot\mathbf{H}^\varepsilon\mathbf{u}^\varepsilon\,dy + \\
&\quad - \int_\Omega \mathbf{b}\cdot\mathbf{v}^\varepsilon\,dy - \int_0^\ell m\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\,dy_3 \\
&\ge \frac{R}{2}\Big\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2}\Big\|^2_{L^2(\Omega)} - \|\mathbf{b}\|_{L^2(\Omega)}\|\mathbf{v}^\varepsilon\|_{L^2(\Omega)} - \|m\|_{L^2(0,\ell)}\|\vartheta^\varepsilon(\mathbf{u}^\varepsilon)\|_{L^2(0,\ell)} \\
&\ge \frac{R}{2}\Big\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2}\Big\|^2_{L^2(\Omega)} - \frac{1}{2C_1}\|\mathbf{b}\|^2_{L^2(\Omega)} - \frac{C_1}{2}\|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)} + \\
&\quad - \frac{1}{2C_2}\|m\|^2_{L^2(0,\ell)} - \frac{C_2}{2}\Big\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2}\Big\|^2_{L^2(\Omega)},
\end{aligned}$$

where C_1 and C_2 are arbitrary positive constants. Choosing $C_2 = R/2$ we have

$$\frac{1}{\varepsilon^4}F_\varepsilon(\mathbf{u}^\varepsilon) \ge \frac{R}{4}\Big\|\frac{\mathbf{E}^\varepsilon\mathbf{u}^\varepsilon}{\varepsilon^2}\Big\|^2_{L^2(\Omega)} - \frac{1}{2C_1}\|\mathbf{b}\|^2_{L^2(\Omega)} - \frac{C_1}{2}\|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)} - \frac{1}{R}\|m\|^2_{L^2(0,\ell)}. \tag{22}$$

By Theorem 5.2, we deduce that

$$\begin{aligned}
\frac{1}{\varepsilon^4}F_\varepsilon(\mathbf{u}^\varepsilon) \ge{}& \frac{R}{4K}\|\mathbf{H}^\varepsilon\mathbf{u}^\varepsilon\|^2_{L^2(\Omega)} + \Big(\frac1K - \frac{C_1}{2}\Big)\|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)} + \\
& - \frac{1}{2C_1}\|\mathbf{b}\|^2_{L^2(\Omega)} - \frac1R\|\ m\|^2_{L^2(0,\ell)}.
\end{aligned}$$

By choosing, for instance, $C_1 = 1/K$, we find that there exists a constant $M > 0$ such that

$$M \ge \frac{R}{4K}\|\mathbf{H}^\varepsilon\mathbf{u}^\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2K}\|\mathbf{v}^\varepsilon\|^2_{L^2(\Omega)}$$

from which follows that the sequence $\mathbf{v}^\varepsilon$ is bounded in $L^2(\Omega;\mathbb{R}^3)$. Using this fact in (22) we get the estimate (21). □

Lemma 5.5, and 1. of Lemma 5.3 imply that the family of functionals $(1/\varepsilon^4)F_\varepsilon$ is coercive in the space $H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{R})$ with respect to the weak convergence of the sequence $q_\varepsilon(\mathbf{u}^\varepsilon) := \Big(u_1^\varepsilon, u_2^\varepsilon/\varepsilon, u_3^\varepsilon/\varepsilon^2, (W^\varepsilon u^\varepsilon)_{12}\Big)$, uniformly with respect to ε. Hence, for any sequence $\mathbf{u}^\varepsilon$ which is bounded in energy, that is $(1/\varepsilon^4)F_\varepsilon \le C$ for a suitable constant $C>0$, and satisfies the boundary conditions $\mathbf{u}^\varepsilon = 0$ on $S(0)$, the corresponding sequence $q_\varepsilon(\mathbf{u}^\varepsilon)$ is weakly relatively compact in $H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{R})$.

We briefly recall a lemma of Paroni [13].

Lemma 5.6. *Let $V\subset\mathbb{R}^N$ be open and bounded and $\mathbb{G}\in L^\infty(V;\mathbb{R}^{d\times N\times d\times N})$. For $\boldsymbol{\varphi}\in H^1(V;\mathbb{R}^d)$ let*

$$I(\boldsymbol{\varphi}) := \int_V \mathbb{G}\boldsymbol{D}\boldsymbol{\varphi}\cdot\boldsymbol{D}\boldsymbol{\varphi}\,dx,$$

be such that $I(\boldsymbol{\varphi})\ge 0$ for every $\boldsymbol{\varphi}\in H^1(V;\mathbb{R}^d)$. If $\boldsymbol{\varphi}_k \rightharpoonup \boldsymbol{\varphi}$ in $H^1(V;\mathbb{R}^d)$ then

$$\liminf_{k\to\infty} I(\boldsymbol{\varphi}_k) \ge I(\boldsymbol{\varphi}),$$

i.e. I is lower semicontinuous with respect to the weak topology of $H^1(V;\mathbb{R}^d)$.

of Theorem 5.1. Let us prove the liminf inequality. Without loss of generality we may suppose that

$$\liminf_{k\to+\infty}\frac{F_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^4} = \lim_{k\to+\infty}\frac{F_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^4} < +\infty.$$

Then Lemma 5.5 applies to the sequence $(1/\varepsilon_k^4)F_{\varepsilon_k}(\mathbf{u}^k)$. Hence (21) is fulfilled and the results of Lemma 5.3 hold true.

From (10) and (11), and with $L_\varepsilon := I_\varepsilon - F_\varepsilon$ the work done by the loads, we see that, by using Lemma 5.3,

$$\frac{L_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^4} = \int_\Omega \mathbf{b}\cdot(u_1^k, \frac{u_2^k}{\varepsilon_k}, \frac{u_3^k}{\varepsilon_k^2})\,dy + \int_0^\ell m\vartheta^{\varepsilon_k}(\mathbf{u}^k)\,dy_3 \to \int_\Omega \mathbf{b}\cdot\mathbf{v}\,dy + \int_0^\ell m\vartheta\,dy_3.$$

Thus we have only to prove that

$$\liminf_{k\to+\infty}\frac{I_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^4} \ge \frac{1}{2}\int_\Omega f_0\left(y_1D_3\vartheta - \frac{\widetilde{\gamma}}{4\widetilde{\mu}}\left(D_3v_3 + y_1D_{33}v_1\right), D_3v_3\right) + \mathbf{H}\mathbf{v}\mathring{\mathbf{T}}\cdot\mathbf{H}\mathbf{v}\,dy.$$

From the definition (12) of f_0, we find

$$\frac{I_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^4} \ge \frac{1}{2}\int_\Omega f_0\left(\frac{(E^{\varepsilon_k}u^k)_{23}}{\varepsilon_k^2}, \frac{(E^{\varepsilon_k}u^k)_{33}}{\varepsilon_k^2}\right) + \mathbf{H}^{\varepsilon_k}\mathbf{u}^k\mathring{\mathbf{T}}\cdot\mathbf{H}^{\varepsilon_k}\mathbf{u}^k\,dy.$$

If we define $\mathbf{z}^k := (u_1^k, u_2^k/\varepsilon_k, u_3^k/\varepsilon_k^2)$, we obtain

$$
\begin{aligned}
&\frac{I_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^4} \geq \\
&\geq \frac{1}{2}\int_\Omega f_0\left(\frac{(Ez^k)_{23}}{\varepsilon_k}, (Ez^k)_{33}\right) + \mathring{T}_{11}(z_{3,1}^k)^2 + \mathring{T}_{33}(z_{1,3}^k)^2 + 2C_L\sum_{i=1}^{3}\sum_{\alpha=1}^{2}|(Ez^k)_{i\alpha}|^2 \\
&\quad -2C_L\sum_{i=1}^{3}\sum_{\alpha=1}^{2}|(Ez^k)_{i\alpha}|^2 + \mathring{T}_{11}\left(\frac{(z_{1,1}^k)^2}{\varepsilon_k^4} + \frac{(z_{2,1}^k)^2}{\varepsilon_k^2}\right) + \mathring{T}_{33}\left(\varepsilon_k^2(z_{2,3}^k)^2 + \varepsilon_k^4(z_{3,3}^k)^2\right) \\
&\quad +\mathring{T}_{22}\left(\frac{(z_{1,2}^k)^2}{\varepsilon_k^2} + (z_{2,2}^k)^2 + \varepsilon_k^2(z_{3,2}^k)^2\right) + 2\mathring{T}_{12}\left(\frac{z_{1,1}^k}{\varepsilon_k^2}\frac{z_{1,2}^k}{\varepsilon_k} + \frac{z_{2,1}^k}{\varepsilon_k}z_{2,2}^k + \varepsilon_k z_{3,1}^k z_{3,2}^k\right) \\
&\quad +2\mathring{T}_{13}\left(\frac{z_{1,1}^k}{\varepsilon_k^2}z_{1,3}^k + z_{2,1}^k z_{2,3}^k + \varepsilon_k^2 z_{3,1}^k z_{3,3}^k\right) \\
&\quad +2\mathring{T}_{23}\left(\frac{z_{1,2}^k}{\varepsilon_k}z_{1,3}^k + \varepsilon_k z_{2,2}^k z_{2,3}^k + \varepsilon_k^3 z_{3,2}^k z_{3,3}^k\right)\, dy.
\end{aligned}
$$

By using (19), (20) and (21), the last four lines in the inequality above converge to $(\mathring{T}_{11} + \mathring{T}_{22})\vartheta^2 - 2\mathring{T}_{23}\vartheta D_3 v_1$, while by using 5. of Lemma 5.3, the inequality

$$
\int_\Omega f_0\left(\frac{(Ez^k)_{23}}{\varepsilon_k}, (Ez^k)_{33}\right) + 2C_L\sum_{i=1}^{3}\sum_{\alpha=1}^{2}|(Ez^k)_{i\alpha}|^2\, dy \geq C_L\|\mathbf{E}\mathbf{z}^k\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})},
$$

and Lemma 5.6 on the second line, we obtain

$$
\begin{aligned}
\liminf_{k\to+\infty}\frac{I_{\varepsilon_k}(\mathbf{u}^k)}{\varepsilon_k^4} &\geq \frac{1}{2}\int_\Omega f_0(y_1 D_3\vartheta + \eta,\, D_3 v_3) + \mathbf{Hv}\mathring{\mathbf{T}}\cdot\mathbf{Hv}\, dy \\
&\geq \frac{1}{2}\min_\eta\int_\Omega f_0(y_1 D_3\vartheta + \eta,\, D_3 v_3) + \mathbf{Hv}\mathring{\mathbf{T}}\cdot\mathbf{Hv}\, dy\ ,
\end{aligned}
$$

where the minimun is taken over all functions η in $L^2(\Omega)$ independent of y_1. It is easy to see that the minimum is achieved for $\bar{\eta} := -(\widetilde{\gamma}/4\widetilde{\mu})\,(D_3 v_3 + y_1 D_{33} v_1)$, which depends only on y_2 and y_3 since $\mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3)$. Hence we have the liminf inequality.

Let us now find a recovery sequence. Let $F(\mathbf{v},\vartheta) < +\infty$, otherwise there is nothing to prove. Then $\mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3)$ and $\vartheta \in H^1_\flat(\Omega;\mathbb{R})$.

We first assume further that $\mathbf{v}$ and ϑ are smooth and equal to zero near by $y_3 = 0$. By (16), there exists $\boldsymbol{\xi}$ smooth and equal to zero near by $y_3 = 0$ such that $v_\alpha(\mathbf{y}) = \xi_\alpha(y_3)$, and $v_3(\mathbf{y}) = \xi_3(y_3) - y_\alpha\xi'_\alpha(y_3)$. Let $\mathbf{u}^{0,\varepsilon}$ be the sequence defined by

$$
\mathbf{u}^{0,\varepsilon} := \mathbf{u}^{f,\varepsilon} + \mathbf{u}^{t,\varepsilon},
$$

with

$$
\begin{aligned}
u_1^{f,\varepsilon} &:= \xi_1 + \varepsilon^2 C_6 \frac{y_2^2}{2}\xi_1'' + \varepsilon^4 C_2 \left(y_1\xi_3' - \frac{y_1^2}{2}\xi_1'' - y_1y_2\xi_2'' \right) + \\
&\quad -\varepsilon^4 C_1 \frac{\widetilde{\gamma}}{4\widetilde{\mu}} y_1 \left(\xi_3' - y_2\xi_2''\right), \\
u_2^{f,\varepsilon} &:= \varepsilon\xi_2 + \varepsilon^3 \left(C_6 \left(y_2\xi_3' - \frac{y_2^2}{2}\xi_2'' - y_1y_2\xi_1'' \right) - C_5 \frac{\widetilde{\gamma}}{4\widetilde{\mu}} y_2 \left(\xi_3' - \frac{y_2}{2}\xi_2'' \right) \right) + \\
&\quad +\varepsilon^4 \left(C_4 \left(2y_1\xi_3' - y_1^2\xi_1'' - 2y_1y_2\xi_2''\right) - C_3 \frac{\widetilde{\gamma}}{2\widetilde{\mu}} y_1 \left(\xi_3' - y_2\xi_2''\right) \right), \\
u_3^{f,\varepsilon} &:= \varepsilon^2(\xi_3 - y_1\xi_1' - y_2\xi_2') - \varepsilon^3 \frac{\widetilde{\gamma}}{4\widetilde{\mu}} y_2 \left(2\xi_3' - y_2\xi_2''\right) + \\
&\quad +\varepsilon^4 \left(C_8 \left(2y_1\xi_3' - y_1^2\xi_1'' - 2y_1y_2\xi_2''\right) - C_7 \frac{\widetilde{\gamma}}{2\widetilde{\mu}} y_1 \left(\xi_3' - y_2\xi_2''\right) \right) + \\
&\quad +\varepsilon^4 \left(-C_6 \frac{y_2^2}{2} y_1\xi_1''' - D_3(C_6)\frac{y_2^2}{2} y_1\xi_1'' \right),
\end{aligned}
$$

and

$$
\begin{aligned}
u_1^{t,\varepsilon} &:= -\varepsilon y_2\vartheta - \varepsilon^2 C_5 \frac{y_2^2}{2} D_3\vartheta + \varepsilon^4 C_1 \frac{y_1^2}{2} D_3\vartheta, \\
u_2^{t,\varepsilon} &:= \varepsilon^2 y_1\vartheta + \varepsilon^3 C_5 y_1y_2 D_3\vartheta + \varepsilon^4 C_3 y_1^2 D_3\vartheta, \\
u_3^{t,\varepsilon} &:= \varepsilon^3 y_1y_2 D_3\vartheta + \varepsilon^4 C_7 y_1^2 D_3\vartheta + \varepsilon^4 \left(C_5 \frac{y_2^2}{2} y_1 D_{33}\vartheta + D_3(C_5)\frac{y_2^2}{2} y_1 D_3\vartheta \right),
\end{aligned}
$$

where, for $i = 1, \ldots, 8$, $C_i = C_i(y_3)$ are defined by equation (14). We have that $\mathbf{u}^{0,\varepsilon}$ is equal to zero in $y_3 = 0$ and satisfies the following estimates

$$
\begin{aligned}
&\left\| \frac{\mathbf{E}^\varepsilon \mathbf{u}^{0,\varepsilon}}{\varepsilon^2} - \mathbf{\Lambda}\left(y_1 D_3\vartheta - \frac{\widetilde{\gamma}}{4\widetilde{\mu}}(D_3v_3 + y_1D_{33}v_1), D_3v_3 \right) \right\|_{L^2(\Omega)} \le \varepsilon C(\mathbf{v},\vartheta), \\
&\|\mathbf{H}^\varepsilon \mathbf{u}^{0,\varepsilon} - \mathbf{H}\mathbf{v}\|_{L^2(\Omega)} \le \varepsilon C(\mathbf{v},\vartheta), \\
&\|(W^\varepsilon u^{0,\varepsilon})_{12} + \vartheta\|_{L^2(\Omega)} \le \varepsilon C(\mathbf{v},\vartheta), \\
&\left\| \left(u_1^{0,\varepsilon}, \frac{u_2^{0,\varepsilon}}{\varepsilon}, \frac{u_3^{0,\varepsilon}}{\varepsilon^2} \right) - \mathbf{v} \right\|_{H^1(\Omega)} \le \varepsilon C(\mathbf{v},\vartheta),
\end{aligned}
$$

where $C(\mathbf{v},\vartheta)$ depends only on $\mathbf{v}$ and ϑ and $\mathbf{\Lambda}$ is defined in equation (13). Hence, in this case, $(\mathbf{u}^{0,\varepsilon_k})$ is a recovery sequence.

In the general case, i.e. $\mathbf{v} \in H_{BN}(\Omega;\mathbb{R}^3)$ and $\vartheta \in H^1_\flat(\Omega;\mathbb{R})$, a standard diagonal argument concludes the proof. □

6 Convergence of minima and minimizers

For every $\varepsilon \in (0,1]$ let us denote by $\tilde{\mathbf{u}}^\varepsilon$ the solution of the following minimization problem

$$
\min\left\{F_\varepsilon(\mathbf{u}) : \mathbf{u} \in H^1(\Omega;\mathbb{R}^3),\ \mathbf{u} = 0 \text{ on } S(0)\right\}.
$$

Corollary 6.1. *The following minimization problem for the* Γ*-limit functional* F *defined in* (17)

$$\min\left\{F(\boldsymbol{v},\vartheta):\boldsymbol{v}\in H_{BN}(\Omega;\mathbb{R}^3),\ \vartheta\in H^1(0,\ell),\ \boldsymbol{v}=\boldsymbol{0}\ on\ S(0),\ \vartheta(0)=0\right\}$$

admits a unique solution $(\tilde{\boldsymbol{v}},\tilde{\vartheta})$*. Moreover, as* $\varepsilon\to 0$*,*

1. $(\tilde{u}_1^\varepsilon,\tilde{u}_2^\varepsilon/\varepsilon,\tilde{u}_3^\varepsilon/\varepsilon^2)\rightharpoonup\tilde{\boldsymbol{v}}$ *in* $H^1(\Omega;\mathbb{R}^3)$*;*
2. $(W^\varepsilon\tilde{u}^\varepsilon)_{12}\to-\tilde{\vartheta}$ *in* $L^2(\Omega)$*;*
3. $(1/\varepsilon^4)F_\varepsilon(\tilde{\boldsymbol{u}}^\varepsilon)$ *converges to* $F(\tilde{\boldsymbol{v}},\tilde{\vartheta})$*.*

Proof. Property *3.* and the weak convergence in *1.* and *2.* follow from the Γ-convergence Theorem 5.1, the uniform coercivity of the sequence $(1/\varepsilon^4)F_\varepsilon$ and the variational property of Γ-convergence (see for instance Dal Maso [3] or Freddi and Paroni [7], Proposition 3.4). The strong convergence in *2.* follows from Remark 5.4. □

7 The equation of equilibrium

The limit energy functional $F(\mathbf{v},\vartheta)$ defined in (17) can be written in a more explicit form by using (16), the fact that ϑ depends only on y_3 and the expressions (15) of f_0 and (18) of $\mathbf{Hv}$. By using

$$\int_\Omega y_\alpha\xi_3'\xi_\alpha''\,dy=0,\qquad\int_\Omega y_\alpha\xi_3'\,dy=0,$$

for $\alpha=1,2$ and

$$\int_\Omega y_1y_2\xi_1''\xi_2''\,dy=0,\qquad\int_\Omega y_1y_2\xi_2''\,dy=0\ ,$$

the limit strain energy can be rewritten as

$$\begin{aligned}
I(\mathbf{v},\vartheta)\ &=\frac{1}{2}\int_\Omega f_0\left(y_1D_3\vartheta-\frac{\widetilde{\gamma}}{4\widetilde{\mu}}\left(D_3v_3+y_1D_{33}v_1\right),\ D_3v_3\right)+\mathbf{Hv}\mathring{\mathbf{T}}\cdot\mathbf{Hv}\,dy\\
&=\frac{1}{2}\int_\Omega f_0\left(y_1D_3\vartheta-\frac{\widetilde{\gamma}}{4\widetilde{\mu}}\left(\xi_3'-y_2\xi_2''\right),\ \xi_3'-y_1\xi_1''-y_2\xi_2''\right)dy+\\
&\quad+\frac{1}{2}\int_\Omega\left(\mathring{T}_{11}+\mathring{T}_{33}\right)\xi_1'^2+\mathring{T}_{\alpha\alpha}\vartheta^2-2\mathring{T}_{23}\vartheta\xi_1'\,dy\\
&=\int_0^\ell\frac{1}{2}\left(\widetilde{E}-\frac{\widetilde{\gamma}^2}{4\widetilde{\mu}}\right)A\xi_3'^2+\frac{1}{2}\left(\widetilde{E}-\frac{\widetilde{\gamma}^2}{4\widetilde{\mu}}\right)J_1\xi_2''^2+\frac{1}{2}\widetilde{E}J_2\xi_1''^2\,dy_3+\\
&\quad+\int_0^\ell\frac{1}{2}\langle\mathring{T}_{11}+\mathring{T}_{33}\rangle\xi_1'^2-\widetilde{\gamma}J_2\vartheta'\xi_1''-\langle\mathring{T}_{23}\rangle\vartheta\xi_1'\,dy_3+\\
&\quad+\int_0^\ell\frac{1}{2}\widetilde{\mu}J\vartheta'^2+\frac{1}{2}\langle\mathring{T}_{\alpha\alpha}\rangle\vartheta^2\,dy_3,
\end{aligned}$$

where

$$\begin{aligned}
A&:=\int_\omega dy_1dy_2=ab, & J_1&:=\int_\omega y_2^2\,dy_1dy_2=\frac{1}{12}ab^3,\\
J_2&:=\int_\omega y_1^2\,dy_1dy_2=\frac{1}{12}a^3b, & J&:=4\int_\omega y_1^2\,dy_1dy_2=\frac{1}{3}a^3b,
\end{aligned}$$

and $\langle\cdot\rangle = \int_\omega \cdot\, dy_1 dy_2$ denotes integration over the cross section ω. The work done by the external forces rewrites as

$$\int_\Omega \mathbf{b}\cdot\mathbf{v}\, dy = \int_0^\ell \langle b_i\rangle\xi_i - \langle y_\alpha b_3\rangle\xi_\alpha'\, dy_3.$$

From (3), we obtain

$$\langle \mathring{T}_{1i,1} + \mathring{T}_{2i,2} + \mathring{T}_{3i,3}\rangle = 0,$$

for $i = 1, 2, 3$. Using the Divergence theorem on the first two terms of the equation above and recalling that $\mathring{\mathbf{T}}\mathbf{n} = \mathbf{0}$ on $\partial\Omega$, see (3), we find $\langle\mathring{T}_{3i}\rangle = 0$, for $i = 1, 2, 3$.

The energy of the beam $F(\mathbf{v},\ \vartheta)$ can be rewritten, with a small abuse of notation, as

$$\begin{aligned} F(\boldsymbol{\xi},\vartheta) &= \int_0^\ell \frac{1}{2}\left(\widetilde{E} - \frac{\widetilde{\gamma}^2}{4\widetilde{\mu}}\right) A\xi_3'^2 + \frac{1}{2}\left(\widetilde{E} - \frac{\widetilde{\gamma}^2}{4\widetilde{\mu}}\right) J_1\xi_2''^2 + \frac{1}{2}\widetilde{E}J_2\xi_1''^2\, dy_3 + \\ &+ \int_0^\ell \frac{1}{2}\langle\mathring{T}_{11}\rangle\xi_1'^2 - \widetilde{\gamma}J_2\vartheta'\xi_1'' + \frac{1}{2}\widetilde{\mu}J\vartheta'^2 + \frac{1}{2}\langle\mathring{T}_{\alpha\alpha}\rangle\vartheta^2\, dy_3 + \\ &- \int_0^\ell \langle b_i\rangle\xi_i - \langle y_\alpha b_3\rangle\xi_\alpha' + m\vartheta\, dy_3, \end{aligned}$$

which has to be minimized over all functions $(\boldsymbol{\xi},\vartheta)$ with $\xi_\alpha \in H^2_\flat(0,\ell)$, $\xi_3 \in H^1_\flat(0,\ell)$ and $\vartheta \in H^1_\flat(0,\ell)$. The Euler-Lagrange equations can be written as

$$\begin{cases} J_2\left(\widetilde{E}\xi_1''\right)'' - \left(\langle\mathring{T}_{11}\rangle\xi_1'\right)' - J_2\left(\widetilde{\gamma}\vartheta'\right)'' - \langle b_1\rangle - \langle y_1 b_3\rangle' = 0, \\ J_1\left(\left(\widetilde{E} - \dfrac{\widetilde{\gamma}^2}{4\widetilde{\mu}}\right)\xi_2''\right)'' - \langle b_2\rangle - \langle y_2 b_3\rangle' = 0, \\ A\left(\left(\widetilde{E} - \dfrac{\widetilde{\gamma}^2}{4\widetilde{\mu}}\right)\xi_3'\right)' + \langle b_3\rangle = 0, \\ J\left(\widetilde{\mu}\vartheta'\right)' - \langle\mathring{T}_{11} + \mathring{T}_{22}\rangle\vartheta - J_2\left(\widetilde{\gamma}\xi_1''\right)' + m = 0. \end{cases} \tag{23}$$

Remark 7.1. If we suppose that each (y_1, y_2) plane is a plane of symmetry, i.e. the material is monoclinic with uniform axis of symmetry identified with the y_3-axis, the term $\widetilde{\gamma}$ vanishes and, hence, the set of equilibrium equations (23) becomes uncoupled. Moreover, see Hoger [8],

$$\mathbf{Q}\left(\mathbb{L}^\varepsilon(\cdot)\,\mathbf{E}\right)\mathbf{Q}^T = \mathbb{L}^\varepsilon(\cdot)\left(\mathbf{QEQ}^T\right), \tag{24}$$

and

$$\mathring{\mathbf{T}}^\varepsilon = \mathbf{Q}\mathring{\mathbf{T}}^\varepsilon\mathbf{Q}^T, \tag{25}$$

for all symmetric tensor $\mathbf{E}$ and for all $\mathbf{Q}$ in the symmetric group. From equation (24), for a monoclinic material as specified above, it follows that (see Gurtin [10])

$$\mathbb{L}^\varepsilon_{\alpha\beta\gamma3} = 0, \qquad \mathbb{L}^\varepsilon_{\alpha333} = 0,$$

for all $\alpha,\beta,\gamma = 1, 2$, while from equation (25) we deduce

$$\mathring{T}^\varepsilon_{\alpha3} = 0, \tag{26}$$

for all $\alpha = 1, 2$. Then, the geometry of the domain and the equations of equilibrium (3) imply that (see Hoger [8])

$$\mathring{T}^\varepsilon_{33} = 0.$$

Hence, for this symmetry, we only have $\mathring{T}_{\alpha\beta} \neq 0$. But, using equations (3) and (26), we find

$$\langle \mathring{T}^{\varepsilon}_{\alpha\alpha} \rangle = \langle x_\alpha (\mathring{T}^{\varepsilon}_{\alpha 3,3}) \rangle = 0$$

for $\alpha = 1, 2$. Hence, recalling equations (23), we find that the equilibrium equations of the limit problem do not depend on the components of the residual stress. Moreover it can be easily shown that, substituting the condition on $\mathbb{L}$ and $\mathring{\mathbf{T}}$ above in (23), the Euler-Lagrange equations become

$$\begin{cases} J_2 \left(\widetilde{E}_m \xi_1'' \right)'' - \langle b_1 \rangle - \langle y_1 b_3 \rangle' = 0, \\ J_1 \left(\widetilde{E}_m \xi_2'' \right)'' - \langle b_2 \rangle - \langle y_2 b_3 \rangle' = 0, \\ A \left(\widetilde{E}_m \xi_3' \right)' + \langle b_3 \rangle = 0, \\ J \left(\widetilde{\mu}_m \vartheta' \right)' + m = 0, \end{cases} \tag{27}$$

where $\widetilde{E}_m(y_3)$ and $\widetilde{\mu}_m(y_3)$ are obtained by the definition (12) of f_0 and by the assumptions of monoclinic material on $\mathbb{L}$.

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References

[1] R. A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics, Vol. 65, Academic Press, New York, 1975.

[2] G. Anzellotti, S. Baldo and D. Percivale, *Dimension reduction in variational problems, asymptotic development in* Γ*-convergence and thin structures in elasticity*, Asymptot. Anal. **9** (1994), 61–100.

[3] G. Dal Maso, *An Introduction to* Γ*-Convergence*, Birkhäuser, 1993.

[4] E. De Giorgi and T. Franzoni, *Su un tipo di convergenza variazionale*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **58** (1975), 842–849.

[5] L. Freddi, A. Morassi and R. Paroni, *Thin-walled beams: the case of the rectangular cross-section*, J. Elasticity **76** (2005), 45–66.

[6] L. Freddi, A. Morassi and R. Paroni, *Thin-walled beams: a derivation of Vlassov theory via* Γ*-convergence*, J. Elasticity **86** (2007), 263–296.

[7] L. Freddi and R. Paroni, *The energy density of martensitic thin films via dimension reduction*, Interfaces Free Bound. **6** (2004), 439–459.

[8] A. Hoger, *On the residual stress possible in an elastic body with material symmetry*, Archive for Rational Mechanics and Analysis **88** (1985), 271–290.

[9] M. Gurtin, *An Introduction to Continuum Mechanics*, Mathematics in Science and Engineering, Vol. 158, Academic Press, New York, 1981.

[10] M. Gurtin, *The linear theory of elasticity*, Handbuch der Physic, Band VIa/2, Springer (1965), 1–295.

[11] H. Le Dret, *Problemes Variationnels dans les Multi-domaines. Modélisation des Jonctions et Applications*, Masson, 1991.

[12] O. A. Oleinik, A. S. Shamaev and G. A. Yosifian, *Mathematical Problems in Elasticity and Homogenization*, North-Holland, 1992.

[13] R. Paroni, *Theory of linearly elastic residually stressed plates*, Mathematics and Mechanics of Solids **11** (2006), 137–159.

[14] L. Trabucho, J. M. Viano, *Mathematical modelling of rods*, Handbook of Numerical Analysis **4** North-Holland, Amsterdam (1996), 487–974.

[15] C. Truesdell, *The Elements of Continuum Mechanics*, Springer-Verlag, New York, 1966.