The role of monotonicity in some shape optimization problems

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1 Introduction

A general optimization problem is a minimization problem of the form

$$\min\{F(A) : A \in \mathcal{A}\} \tag{1}$$

where $\mathcal{A}$ is the set of *admissible choices* and F is a *cost functional* which has to be minimized over $\mathcal{A}$. We want to stress the fact that in several interesting situations the set $\mathcal{A}$ is not a linear function space and there is no topology a priory given on $\mathcal{A}$. This is the case for instance of shape optimization problems, where $\mathcal{A}$ is a class of *admissible domains*.

It is well known that often problem (1) does not admit any solution, and so, in order to study the behavior of minimizing sequences, a relaxation procedure is necessary (see for instance Belloni, Buttazzo and Freddi [2]). In several applications, to any admissible choice $A \in \mathcal{A}$ we can associate a *state function* u_A which usually comes out by solving a differential equation or a variational problem associated to A. For instance, if $\mathcal{A}$ is a class of domains, and the state function is the temperature of a conducting homogeneous isotropic medium described by the domain $A \in \mathcal{A}$, we have the partial differential equation (stationary heat equation)

$$\begin{cases} -\Delta u_A = f & \text{in } A \\ u_A = 0 & \text{on } \partial A \end{cases}$$

where f is the heat sources density, and where we assumed the Dirichlet boundary conditions on ∂A. Note that in this case the solution u_A also comes out as the solution of the variational problem

$$\min\left\{\frac{1}{2}\int_A |Du|^2 dx - \int_A f u \, dx \; : \; u \in H_0^1(A)\right\},$$

so that, in general, we may assume the *state equation* which links u to A is written in the form

$$u_A \in \operatorname{argmin} G(\cdot, A),$$

where the functional G can be called *state functional*. Therefore, the optimization problem (1) becomes the optimal control problem

$$\min\{J(u,A)\,:\,u\in\mathcal{U},\ A\in\mathcal{A},\ u\in\operatorname{argmin}G(\cdot,A)\}. \tag{2}$$

Here we denoted by $\mathcal{U}$ the *space of states* and by J the *cost functional*, which may depend on both u and A.

Another case which falls into the previous framework is the optimal obstacle problem, where $\mathcal{A}$ is a class of obstacles, the state functional G describes the obstacle variational problem

$$G(u,g)=\begin{cases}\frac{1}{2}\int_\Omega|Du|^2dx-\int_\Omega fu\,dx & \text{if } u\geq g\\ +\infty & \text{otherwise}\end{cases}$$

for $g\in\mathcal{A}$, being Ω a fixed open subset of $I\!R^N$, $N\geq 2$, and $J(u,g)$ a suitable cost functional.

In the literature many problems of this form have been studied (see Attouch and Picard [1], Dal Maso and Longo [12], Longo [15]) from the point of view of the relaxation; in fact the relaxed problem associated to (2) can be identified, and it is a minimization problem of the form

$$\min\{\widehat{J}(u,\widehat{A})\,:u\in\mathcal{U},\ A\in\widehat{\mathcal{A}},\ u\in\operatorname{argmin}\widehat{G}(\cdot,\widehat{A})\} \tag{3}$$

where the relaxed functionals $\widehat{J}$ and $\widehat{G}$, and the set of the relaxed controls $\widehat{\mathcal{A}}$ can be suitably defined. As it usually occurs in relaxation theory, the minimization problem (3) always admits a solution, but in general the optimal solution $\widehat{A}$ only belongs to the set of relaxed controls $\widehat{\mathcal{A}}$ which can be considerably larger than the original admissible class $\mathcal{A}$.

In this paper we want to show how in some particular situations an existence result for the original problem (1) can be obtained. A first possibility is to impose some restrictions to the admissible class $\mathcal{A}$, strong enough to obtain the existence of a solution; this is the case, in shape optimization problem with Dirichlet conditions on the free boundary, of geometrical constraints like a Lipschitz condition, with the Lipschitz constant a priori fixed on the admissible domains, or imposing conditions of a similar type (for instance the exterior cone condition and the capacitary Wiener condition).

We do not want to pursue this direction, and we refer the interested reader to the many papers where this method has been applied to obtain the existence of classical non-relaxed minimizers (see Chenais [11], Bucur and Trebeschi [6], Bucur and Zolésio [7]). On the contrary, we do not impose any geometrical constraint on the admissible controls, but we consider simply more particular cases where the cost function fulfills some additional monotonicity properties. This will enable us to still obtain the existence of classical minimizers.

The paper is organized as follows: in Section 2 we present a general abstract theory of minimum problem for real functions defined on an ordered set which can be suitably topologized; in Sections 3, 4, 5 we show how this abstract theory can be applied to some interesting situations as:

- shape optimization problem;
- optimal partitions problems;
- problems of optimal obstacles.

2 General setting

In this section we present a general abstract framework for optimum problems with monotone cost functionals, large enough to include some of the cases listed in the introduction.

Consider an ordered space $(X, \leq)$ and two functionals

$$F, \Phi : X \to \overline{I\!R}.$$

Suppose that X is endowed with two convergences denoted by γ and $w\gamma$, where the second one is sequentially compact and weaker than the first one. Suppose, moreover, that the functional F is monotone increasing and sequentially γ lower semicontinuous, while Φ is sequentially $w\gamma$ lower semicontinuous.

The problem we want to solve is the following one

$$\min\{F(x) : x \in X, \Phi(x) \leq 0\}. \tag{4}$$

Let us observe that, being X a $w\gamma$ sequentially compact space, if F were sequentially $w\gamma$ lower semicontinuous, then the existence of the minimizers for the problem (4) would follow straightforward from the direct methods of the calculus of variations. Therefore the idea is to find a "relation" between γ and $w\gamma$ such that, under the monotonicity assumption on F, the sequentially γ lower semicontinuity implies the sequentially $w\gamma$ lower semicontinuity for F.

The relation between the two convergences on X is strictly related to the order relation on X, and is the following one:

Assumption (A) *For every $x_n \xrightarrow{w\gamma} x$ there exist a sequence of integers (n_k) and a sequence (y_{n_k}) in X such that $y_{n_k} \leq x_{n_k}$ and $y_{n_k} \xrightarrow{\gamma} x$.*

We emphasize that, in our contest, the monotonicity of F plays an essential role because of the following result.

Proposition 2.1 *Let us suppose that assumption (A) is fulfilled. If $F : X \to \overline{I\!R}$ is monotone increasing and sequentially γ lower semicontinuous, then F is sequentially $w\gamma$ lower semicontinuous.*

Proof Let us consider $x_n \stackrel{w\gamma}{\to} x$, and let (x_{n_k}) be a subsequence such that

$$\lim_{k\to\infty} F(x_{n_k}) = \liminf_{n\to\infty} F(x_n).$$

Using assumption (A) above there exists a subsequence (which we still denote by (x_{n_k})) and $y_{n_k} \le x_{n_k}$ such that $y_{n_k} \stackrel{\gamma}{\to} x$. The γ lower semicontinuity of F gives

$$F(x) \le \liminf_{k\to\infty} F(y_{n_k})$$

and the monotonicity of F gives $F(y_{n_k}) \le F(x_{n_k})$. Therefore

$$F(x) \le \liminf_{k\to\infty} F(y_{n_k}) \le \liminf_{k\to\infty} F(x_{n_k}) = \liminf_{n\to\infty} F(x_n)$$

which concludes the proof. □

We can now formulate the main result of this section which is a straightforward consequence of the direct methods of the calculus of variations and of Proposition 2.1.

Theorem 2.2 *Let us suppose that assumption (A) is fulfilled. Let F be an increasing sequentially γ lower semicontinuous functional and let Φ be sequentially $w\gamma$ lower semicontinuous. Then the minimum problem (4) admits a solution.*

This general framework is very simple but it is very useful since it applies very well in many cases (see sections below). The main difficulty in the concrete problems will be to "identify" the order in X, the two convergences γ and $w\gamma$, and especially to prove that assumption (A) is fulfilled.

3 Shape optimization problems

In this section we consider shape optimization problems of the form

$$\min\{F(A) \,:\, A \in \mathcal{A},\, \Phi(A) \le 0\}$$

where $\mathcal{A}$ is a class of admissible domains, Φ is a constraint functional, and F is a cost functional defined on $\mathcal{A}$. More precisely, given a bounded open subset Ω of $I\!R^N$ and a number $c \in [0, \mathrm{meas}(\Omega)]$ we take as $\mathcal{A}$ the class

$$\mathcal{A} = \{A \text{ quasi-open subset of } \Omega\}$$

and as Φ the functional

$$\Phi(A) = \mathrm{meas}(A) - c,$$

so that the optimization problem above becomes

$$\min\{F(A) \,:\, A \in \mathcal{A},\, \mathrm{meas}(A) \le c\}.$$

We recall that a subset A of Ω is said to be quasi-open if for every $\epsilon > 0$ there exists an open subset A_ϵ of Ω such that $\mathrm{cap}(A \triangle A_\epsilon) < \epsilon$, where $\triangle$ denotes the symmetric difference of sets, or equivalently if $A = \{\tilde{u} > 0\}$ for a suitable function $u \in H^1(\Omega)$, where we denoted by $\tilde{u}$ the quasi-continuous representative of u.

In order to include the shape optimization problem above in the general setting introduced in Section 2 we endow the class $\mathcal{A}$ with the order given by the set inclusion

$$A \leq B \Longleftrightarrow A \supseteq B$$

so that F turns out to be increasing on $\mathcal{A}$ if and only if it is decreasing with respect to the set inclusion.

In order to introduce the γ and $w\gamma$ convergences we define, for every domain $A \in \mathcal{A}$, the solution $w_A \in H_0^1(\Omega)$ of the elliptic boundary value problem formally written as

$$\begin{cases} -\Delta w_A = 1 & \text{in } A \\ w_A = 0 & \text{on } \overline{\Omega} \setminus A. \end{cases} \tag{5}$$

The precise definition of the solution w_A can be given in weak form

$$w_A \in H_0^1(\Omega), \ \int_\Omega Dw_A \, Dv \, dx = \int_\Omega v \, dx \quad \forall v \in H_0^1(A),$$

or as a minimizer of the functional

$$\frac{1}{2} \int_\Omega |Du|^2 dx - \int_\Omega u \, dx \quad (u \in H_0^1(A)).$$

We can now define

$$A_n \xrightarrow{\gamma} A \Longleftrightarrow w_{A_n} \to w_A \text{ weakly in } H_0^1(\Omega)$$

which turns out to be equivalent (see Sverak [17]) to

$$A_n \xrightarrow{\gamma} A \Longleftrightarrow G(\cdot, A_n) \to G(\cdot, A) \text{ in the } \Gamma(L^2(\Omega)) - \text{convergence},$$

where the functional G is given by

$$G(u, A) = \begin{cases} \frac{1}{2} \int_\Omega |Du|^2 dx - \int_\Omega u \, dx & \text{if } u \in H_0^1(A) \\ +\infty & \text{otherwise.} \end{cases}$$

Many cost functionals are known to be γ-lower semicontinuous; for instance (see Buttazzo and Dal Maso [8]) this occurs in the following cases.

Example 3.1 Integral functionals like

$$F(A) = \int_\Omega j(x, u_A) \, dx$$

where u_A is the solution of a second order elliptic state equation

$$\begin{cases} Lu_A = f & \text{in } A \\ u_A \in H_0^1(A), \end{cases}$$

with $Lu = -\mathrm{div}\,(a(x)Du)$ for a given uniformly positive definite, symmetric, bounded and measurable matrix $a(x)$, and $j(x,s)$ is a nonnegative Carathéodory function. The mapping F actually turns out to be γ-continuous as soon as $j(x,s)$ satisfies some growth conditions like

$$0 \leq j(x,s) \leq \alpha(x) + \beta|s|^2$$

with $\alpha \in L^1(\Omega)$ and $\beta \in I\!R^+$.

Example 3.2 Functions of eigenvalues like

$$F(A) = \varphi\left(\lambda_1(A), \cdots, \lambda_k(A)\right)$$

where k is a given integer, φ is lower semicontinuous on $I\!R^k$, and $\lambda_j(A)$ are the eigenvalues on $H_0^1(A)$ of a given second order elliptic operator L of the form seen in Example 3.1. The mapping F actually turns out to be γ-continuous as soon as φ is a continuous function. Similarly, we can consider cost functions which depend on the whole spectrum

$$F(A) = \varphi(\lambda(A))$$

where $\lambda(A)$ denotes the sequence $\left(\lambda_j(A)\right)_{j \in I\!N}$ and φ is a function defined on the set of all nonnegative sequences. We have in this case that F is γ-lower semicontinuous (respectively γ-continuous) whenever φ is lower semicontinuous (respectively γ-continuous) with respect to the pointwise convergence

$$\lambda^n \to \lambda \iff \lambda_j^n \to \lambda_j \text{ for every } j \in I\!N.$$

Example 3.3 Capacity functions like

$$F(A) = \mathrm{cap}_L(\Omega \setminus A)$$

where cap_L is the capacity associated to a second order elliptic operator L of the form seen in Example 3.1

$$\mathrm{cap}_L(E) = \min\left\{ < Lu, u >: \; u \in H_0^1(\Omega),\; u \geq 1 \text{ q.e. on } E \right\}.$$

The mapping F is γ-lower semicontinuous on $\mathcal{A}$.

It is well known that the γ-convergence is not compact on $\mathcal{A}$, and therefore in general a γ-lower semicontinuous functional on $\mathcal{A}$ has not always a minimizer, so that relaxed solutions are needed.

We introduce now the weaker notion of $w\gamma$-convergence, which turns out to be compact on $\mathcal{A}$.

Definition 3.4 *We say that a sequence (A_n) in $\mathcal{A}$ $w\gamma$-converges to a domain $A \in \mathcal{A}$ if the solutions w_{A_n} of problem (5) converge weakly in $H_0^1(\Omega)$ to a function w such that $A = \{w > 0\}$.*

Remark 3.5 In general the function w in Definition 3.4 does not coincide with the solution w_A; this happens if and only if the sequence (A_n) γ-converges to A. Moreover, it is easy to see that the $w\gamma$-convergence defined above is compact on $\mathcal{A}$ and weaker than the γ-convergence (see Bucur, Buttazzo and Henrot [4]).

Proposition 3.6 *The functional $\Phi(A) = \mathrm{meas}(A) - c$ is lower semicontinuous on $\mathcal{A}$ with respect to the $w\gamma$-convergence.*

Proof Let (A_n) be a sequence in $\mathcal{A}$ which $w\gamma$-converges to a domain $A \in \mathcal{A}$; then by Definition 3.4 the solutions w_{A_n} of (5) converge weakly in $H_0^1(\Omega)$ to a function w such that $A = \{w > 0\}$. Possibly passing to subsequences we may assume that w_{A_n} converge to w pointwise almost everywhere and so, if $x \in A$ is such that $w_{A_n}(x) \to w(x)$ we have $w_{A_n}(x) > 0$ for all n large enough. Hence $x \in A_n$ for all n large enough, and this implies

$$1_A(x) \leq \liminf_{n\to+\infty} 1_{A_n}(x) \quad \text{for a.e. } x \in \Omega.$$

The conclusion now follows by Fatou's lemma. □

In order to apply the general setting of Section 2 it remains to prove that the convergences γ and $w\gamma$ fulfill assumption (A). The proof of this fact can be found in Bucur, Buttazzo and Henrot [4] Lemma 2.6, where it is obtained by using some results previously proved in Buttazzo and Dal Maso [8].

Finally, collecting the results above we have obtained the following existence theorem (see Buttazzo and Dal Maso [8]).

Theorem 3.7 *The minimization problem*

$$\min\{F(A) \,:\, A \in \mathcal{A},\ \mathrm{meas}(A) \leq c\}$$

admits at least a solution, provided the functional F verifies the following assumptions:

(i) F is sequentially γ-lower semicontinuous;

(ii) F is monotone decreasing with respect to the set inclusion.

Remark 3.8 If a cost functional F satisfies assumptions (i) and (ii) of Theorem 3.7, it is easy to see that also the problem

$$\min\{F(A) \,:\, A \in \mathcal{A},\ \mathrm{meas}(A) = c\}$$

has a solution. Indeed, once the minimization problem with the constraint $\{\mathrm{meas}(A) \leq c\}$ is solved, by the monotonicity property of F it is enough to enlarge a solution up to obtain its measure be equal to c.

Note that the results above apply to Example 3.1 when the right-hand side f is nonnegative and the integrand $j(x,s)$ in the cost is monotone decreasing in s. Indeed the maximum principle and the monotonicity of $j(x,\cdot)$ imply that the cost $F(A)=\int_\Omega j(x,u_A)\,dx$ is decreasing with respect to the set inclusion.

The case of minimum problems for functions of eigenvalues, considered in Example 3.2, also falls into the framework of the existence Theorem 3.7 provided the function φ is monotone increasing in each of its variables. Indeed, the monotonicity of the cost F follows in this case by the well known fact that all eigenvalues of a second order elliptic operator of the form considered in Examples 3.2, with Dirichlet boundary conditions, are monotone decreasing with respect to the set inclusion.

The case seen in Example 3.3 of the problem of minimal capacity also admits a classical non relaxed solution, thanks again to the existence Theorem 3.7.

Example 3.9 Let $F:\mathcal{A}\to[0,+\infty]$ be a sequentially γ-lower semicontinuous cost functional which is monotone decreasing with respect to the set inclusion, for instance one of the cases seen in Examples 3.1–3.3. By Proposition 2.1 the functional F is sequentially $w\gamma$-lower semicontinuous. Since by Proposition 3.6 the mapping $A\mapsto \mathrm{meas}(A)$ is sequentially $w\gamma$-lower semicontinuous too, we obtain the existence of a minimizer for the problem

$$\min\{F(A)+\mathrm{meas}(A)\ :\ A\in\mathcal{A}\}$$

ore more generally for problems like

$$\min\{F(A,\mathrm{meas}(A))\ :\ A\in\mathcal{A}\}$$

where $F:\mathcal{A}\times I\!\!R^+\to[0,+\infty]$ is sequentially semicontinuous for the ($w\gamma\times$euclidean)-convergence and monotone increasing in the second variable.

Remark 3.10 The monotonicity assumption on the cost functional F is not necessary for obtaining the existence of a classical optimal domain in shape optimization problem (see for instance Bucur, Buttazzo and Figueiredo [3]); a weaker condition than monotonicity which still implies the existence of optimal design is not yet known.

4 Optimal partitions problems

In this section we consider minimization problems of the form

$$\min\{F(A_1,\cdots,A_k)\ :\ A_j\in\mathcal{A},\ \mathrm{cap}(A_i\cap A_j)=0 \text{ if } i\neq i\} \tag{6}$$

where k is a fixed integer and $\mathcal{A}$ is, as in the previous section, the class of all quasi-open subsets of Ω. Then the cost function F is defined on $\mathcal{A}^k$ which can be endowed with the order

$$(A_1,\cdots,A_k)\le(B_1,\cdots,B_k)\Longleftrightarrow A_j\supseteq B_j\quad\forall j=1,\cdots,k$$

and with the convergences γ and $w\gamma$ defined as the γ and $w\gamma$ convergences of the previous section on each component. Therefore assumption (A) is fulfilled, as well as the compactness of $w\gamma$ and the fact that $w\gamma$ is weaker than γ. In order to include problem (6) into the general framework of Section 2 it remains only to prove the following result.

Proposition 4.1 *The constraint*

$$\{(A_1, \cdots, A_k) \ : \ A_j \in \mathcal{A}, \ \operatorname{cap}(A_i \cap A_j) = 0 \ \text{if} \ i \neq j\}$$

is sequentially $w\gamma$-closed.

Proof Let (A_j^n) be a sequences in $\mathcal{A}$ $(j = 1, \cdots, k)$ such that $A_j^n \xrightarrow{w\gamma} A_j$ for all $j = 1, \cdots, k$ and $\operatorname{cap}(A_i^n \cap A_j^n) = 0$ if $i \neq j$; we have to prove that also $\operatorname{cap}(A_i \cap A_j) = 0$ if $i \neq j$. By definition of $w\gamma$-convergence we have

$$w_{A_j^n} \to w_j \quad \text{weakly in } H_0^1(\Omega) \text{ with } \ A_j = \{w_j > 0\}.$$

Since $w_{A_i^n} \cdot w_{A_j^n} = 0$ a.e. in Ω if $i \neq j$ and $w_{A_j^n} \to w_j$ in $L^2(\Omega)$, it follows that $w_i \cdot w_j = 0$ a.e. in Ω if $i \neq j$. Moreover, w_i and w_j are quasi-continuous functions, so that their product $w_i \cdot w_j$ is quasi continuous too, and since a quasi-continuous function which vanishes almost everywhere also vanishes quasi-everywhere (see for instance Heinonen, Kilpelainen and Martio [14]), we obtain $w_i \cdot w_j = 0$ q.e. in Ω, which yields that $\operatorname{cap}(A_i \cap A_j) = 0$. $\square$

We are now in a position to apply the abstract scheme of Section 2 and we obtain the following existence result (see Bucur, Buttazzo and Henrot [4]).

Theorem 4.2 *Let $F : \mathcal{A}^k \to [0, +\infty]$ be a cost functional such that*

(i) F is sequentially γ-lower semicontinuous, that is

$$A_j^n \xrightarrow{\gamma} A_j \ \forall j = 1, \cdots, k \quad \Rightarrow \quad F(A_1, \cdots, A_k) \leq \liminf_{n \to +\infty} F(A_1^n, \cdots, A_k^n);$$

(ii) F is monotone decreasing in each of its variables with respect to the set inclusion, that is

$$A_j \subseteq B_j \ \forall j = 1, \cdots, k \quad \Rightarrow \quad F(A_1, \cdots, A_k) \geq F(B_1, \cdots, B_k).$$

Then the minimization problem (6) admits at least a solution.

Example 4.3 Take $k = 2$ and consider the problem

$$\min \{F(A, B) \ : \ A, B \in \mathcal{A}, \ \operatorname{cap}(A \cap B) = 0\}$$

which, by Theorem 4.2, has a solution if F fulfills assumptions (i) and (ii). Replacing B with $\Omega \setminus A$ we obtain the existence of a solution for the problem

$$\min\{F(A, \Omega \setminus A) \ : \ A \in \mathcal{A}\}.$$

For instance, with the notation of Example 3.2, for fixed integers i and j the optimal partition problem

$$\min\{\varphi(\lambda_i(A), \lambda_j(\Omega \setminus A)) \ : \ A \in \mathcal{A}\}$$

admits a solution in $\mathcal{A}$ whenever $\varphi : I\!R^2 \to I\!R$ is continuous and increasing in each of its variables.

5 Problems of optimal obstacles

In this section we study the optimal obstacle problem with monotone cost functional as a particular case of the abstract framework introduced in Section 2.

Optimal obstacle problems are control problems in which the controls are obstacles, and the state functional depends on the control through a variational inequality.

Let us consider, for example, the following problem: fix an open subset Ω of $I\!R^N$, $N \geq 2$, and a function $\psi \in H_0^{1,p}(\Omega)$, $1 < p < +\infty$. Let us consider the family of "*admissible obstacles*"

$$X_\psi(\Omega) = \{g : \Omega \to \overline{I\!R},\ g \leq \psi,\ \ g \text{ quasi-upper semicontinuous}\}$$

endowed with the natural order on functions. We recall that a function g is said to be *quasi-upper semicontinuous* (shortly quasi u.s.c) if it is upper semicontinuous up to a set of p-capacity zero (see Heinonen, Kilpelainen and Martio [14]). Fix moreover a non negative Charathéodory function $j(x, u)$ and a constant $c \in I\!R$. Then the problem

$$\min\left\{\int_\Omega j(x, u_g)\, dx \ : \ g \in X_\psi(\Omega),\ \int_\Omega g\, dx = c\right\} \tag{7}$$

where u_g is the minimum of the "*energy*"

$$G(u, g) = \begin{cases} \frac{1}{p}\int_\Omega |Du|^p dx & \text{if } u \geq g \\ +\infty & \text{otherwise} \end{cases} \tag{8}$$

is an optimal obstacle problem.

In this section we study more general optimization problem for obstacles, of the form

$$\min\left\{F(g) \ : \ g \in X_\psi(\Omega),\ \int_\Omega g\, dx = c\right\} \tag{9}$$

where F does not need to be of integral type.

In order to include the obstacle problem (9) in the general setting introduced in Section 2, having just fix the order on the set $X_\psi(\Omega)$, we have to choose the two convergences on the family of admissible obstacles. As in the case of shape optimization problem the γ-convergence on $X_\psi(\Omega)$ is the one related to the Γ-convergence of the energy functionals (8) associated to the obstacles.

Definition 5.1 *We say that a sequence (g_n) γ-converges to g (and we write $g_n \xrightarrow{\gamma} g$) if the sequence of functionals $G(\cdot, g_n)$ Γ-converges to $G(\cdot, g)$ in $L^p(\Omega)$.*

Remark 5.2 It is well known that, without any other further assumptions on the class of admissible obstacles, the γ-convergence is not compact; hence the Γ-limit of obstacles functional might not belong to the same class (see for instance Dal Maso and Longo [12], Attouch and Picard [1]) and the existence of the solution of the problem (9) might fail. In this section we do not consider additional constraints on the admissible obstacles, and we will find an optimal solution of problem (9) only by requiring the monotonicity of the cost functional.

In order to use the abstract framework of Section 2, we have now to choose the second suitable $w\gamma$-convergence on $X_\psi(\Omega)$ which has to be sequentially compact and such that the assumption (A) is fulfilled, and then we have to show that the constraint

$$\int_\Omega g\,dx = c \tag{10}$$

can be written in the form

$$\Phi(g) \leq 0,$$

where Φ is a sequentially $w\gamma$-lower semicontinuous functional.

Let us begin with the definition of $w\gamma$. The choice of the $w\gamma$-convergence comes from the study of level sets of the obstacles (see Bucur, Buttazzo and Trebeschi [5]) which are quasi-open sets, in particular it is suggested by the characterization of the γ-convergence of a sequence of obstacles in terms of the γ-convergence (see Section 3) of a "suitable class of level sets".

Theorem 5.3 *Let $g_n, g \in X_\psi(\Omega)$. Then $g_n \xrightarrow{\gamma} g$ if and only if there exists a dense subset $D \subseteq I\!R$ such that*

$$\{g_n < t\} \xrightarrow{\gamma} \{g < t\} \quad \forall t \in D.$$

The choice of the $w\gamma$-convergence comes as a direct consequence of the relations between the γ and $w\gamma$-convergences of sequences of domains (see Section 3, Definition 3.4).

Definition 5.4 *Let $g_n, g \in X_\psi(\Omega)$. We say that (g_n) $w\gamma$-converges to g (and we write $g_n \xrightarrow{w\gamma} g$) if there exists a dense subset $D \subseteq I\!R$ such that*

$$\{g_n < t\} \xrightarrow{w\gamma} \{g < t\} \quad \forall t \in D.$$

Definition 5.4 is well posed and, from the properties of the $w\gamma$-convergence of the domains (see Remark 3.5), we find that the $w\gamma$-convergence on obstacles is sequentially compact. Moreover, from Definition 5.4 and Theorem 5.3, we have that γ-convergence is stronger than $w\gamma$-convergence.

We have now to verify that assumption (A) is fulfilled on the class of obstacles endowed with the γ and $w\gamma$- convergences already defined. This is a very delicate and technical point, and it is proved by the following result (see Bucur, Buttazzo and Trebeschi [5]).

Theorem 5.5 *Let $g_n, g \in X_\psi(\Omega)$ with $g_n \overset{w\gamma}{\rightharpoonup} g$. Then there exist a subsequence (g_{n_k}) and a sequence (f_{n_k}) with $f_{n_k} \le g_{n_k}$ such that $f_{n_k} \xrightarrow{\gamma} g$.*

In order to conclude we have to prove that the integral functional $\Phi : X_\psi(\Omega) \to \overline{I\!R}$ defined by

$$\Phi(g) = -\int_\Omega g\,dx + c$$

is $w\gamma$-lower semicontinuous.

Proposition 5.6 *Let $g_n, g \in X_\psi(\Omega)$ with $g_n \overset{w\gamma}{\rightharpoonup} g$. Then*

$$\int_\Omega g\,dx \ge \limsup_{n\to+\infty} \int_\Omega g_n\,dx.$$

Proof ¿From the definition of $w\gamma$-convergence we obtain that there exists a dense subset D of $I\!R$, that we may suppose countable, such that

$$\{g_n < l\} \overset{w\gamma}{\rightharpoonup} \{g < l\} \quad \forall\, l \in D.$$

Then, as done in the proof of Proposition 3.6, we deduce

$$1_{\{g<l\}} \le \liminf_{n\to+\infty} 1_{\{g_n<l\}} \quad \text{a.e. in } \Omega,\ \forall\, l \in D.$$

This gives, for a.e. $x \in \Omega$ and for all $l \in D$

$$g_n(x) < l \quad \Rightarrow \quad \limsup_{n\to+\infty} g_n(x) \le l$$

that is

$$g(x) \ge \limsup_{n\to+\infty} g_n(x) \text{ a.e. in } \Omega.$$

The conclusion now follows by Fatou's lemma. □

The abstract framework is now completely fulfilled, and as a direct consequence of the abstract Theorem 2.2 we obtain the following existence theorem.

Theorem 5.7 *The minimization problem*

$$\min\{F(g) \,:\, g \in X_\psi(\Omega),\ \Phi(g) \le 0\} \tag{11}$$

admits at least a solution, provided the functional F verifies the following assumptions:

(i) F is γ-lower semicontinuous;

(ii) F is monotone increasing (i.e. for all $g_1, g_2 \in X_\psi(\Omega)$ with $g_1 \le g_2$ we have $F(g_1) \le F(g_2)$).

Remark 5.8 If problem (11) admits a solution, it is easy to prove that problem (9) admits a solution too. In fact, if $\Phi(g) \leq 0$, we may replace the optimal solution g of problem (11) by the function

$$\tilde{g} = g + \frac{1}{\mathrm{meas}(\Omega)}(-\int_\Omega g\,dx + c).$$

We have $\tilde{g} \leq g$, then $\tilde{g} \in X_\psi(\Omega)$, and $\Phi(\tilde{g}) = 0$. Moreover, by the monotonicity of F we obtain $F(\tilde{g}) \leq F(g)$, so that $\tilde{g}$ is a solution of problem (9).

Many cost functionals are known to be γ-lower semicontinuous and monotone increasing; for instance (see Bucur, Buttazzo and Trebeschi [5]) this occurs in the following cases.

Example 5.9 Integral functionals like

$$F(g) = \int_\Omega j(x, u_g)\,dx$$

provided $j(x,u)$ is a Charathéodory nonnegative function, monotone increasing with respect to the second variable, and u_g is the solution of the obstacle problem with g, given as the minimizer of the state functional (8).

Example 5.10 Variational functionals like

$$F(g) = \min_{u \in K_g} \tilde{F}(u)$$

where $\tilde{F} : H_0^{1,p}(\Omega) \to \overline{I\!R}$ is given, and where K_g is the convex set

$$K_g = \{u \in H_0^{1,p}(\Omega) \ : \ u \geq g\}.$$

The monotonicity of F is a direct consequence of the definition and the properties of the solutions of variational inequalities. For the γ-lower semicontinuity of F is sufficient to suppose that $\tilde{F}$ is weakly sequentially lower semicontinuous and coercive on $H_0^{1,p}(\Omega)$.

We conclude this section observing that the existence result obtained for the class of upper dominate obstacles $X_\psi(\Omega)$ can be extended to the case of lower dominate obstacles

$$\tilde{X}_\psi(\Omega) = \{g : \Omega \to \overline{I\!R} \ : g \ \text{quasi-lower semicontinuous} \ , \ g \leq \psi\}$$

with the γ and $w\gamma$ defined, in this case, in terms of the behavior of the upper level set $\{g > t\}$; we denote them by $\tilde{\gamma}$ and $\tilde{w\gamma}$. If assumption (A) is changed in the following one:

Assumption (A') *For every* $x_n \overset{w\gamma}{\rightharpoonup} x$ *there exist a sequence of integers* (n_k) *and a sequence* (y_{n_k}) *in* X *such that* $y_{n_k} \geq x_{n_k}$ *and* $y_{n_k} \overset{\gamma}{\to} x$,

then the abstract Theorem 2.2 still holds provided, in this case, that F is monotone decreasing with respect to the second variable and sequentially $\tilde{\gamma}$-lower semicontinuous.

More in general, the existence result can be extended to *bilateral problems* of the form

$$\min\{F(g_1, g_2) \,:\, (g_1, g_2) \in \widehat{X}_\psi(\Omega),\ \int_\Omega g_1 dx = c,\ \int_\Omega g_2 dx = \tilde{c}\} \tag{12}$$

where $F : \widehat{X}_\psi(\Omega) \to \overline{I\!R}$ is the cost functional defined on the class of admissible obstacles

$$\widehat{X}_\psi(\Omega) = \{(g_1, g_2) \,:\, g_i : \Omega \to \overline{I\!R},\ g_1 \text{ quasi-u.s.c.},\ g_2 \text{ quasi-l.s.c.},\ g_1 \leq \psi \leq g_2\}$$

and where $c \leq \tilde{c}$ are two fixed real constants.

The abstract Theorem 2.2 is generalized to the bilateral case as follows.

Theorem 5.11 *Let us suppose that assumptions (A) and (A') are fulfilled. Let* $F : \widehat{X}_\psi(\Omega) \to \overline{I\!R}$ *such that:*

(*i*) $F(\cdot,\cdot)$ *is lower semicontinuous with respect to* $(\gamma, \tilde{\gamma})$*;*

(*ii*) F *is monotone increasing (decreasing) with respect to the first (second) variable.*

Then the minimum problem (12) admits a solution.

Remark 5.12 If we choose $g_1 = -\infty 1_A, g_2 = +\infty 1_A$, with A a quasi-open subset of Ω, the bilateral obstacle $(-\infty 1_A, +\infty 1_A)$ is identified with the domain A. Moreover, if $c = \tilde{c}$, then the integral constraints in problem (12) becomes meas$(A) = c$, and the bilateral problem (12) becomes the shape optimization problem presented in Remark 3.8.

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