Discrete approximation of functionals with jumps and creases

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Let $\mathcal{H}^2(0,L)$ denote the space of piecewise-H^2 functions on the interval $(0,L)$; if $u \in \mathcal{H}^2(0,L)$ then u, u' and u'' are regarded as defined on the whole interval $(0,L)$, and u and u' are piecewise-continuous functions. Let $S(u)$ be the set of the discontinuity points (*jump points*) of the function $u \in \mathcal{H}^2(0,L)$ and, with a slight abuse of notation, denote by $S(u')$ the set of *crease points* of u (i.e., those points where u is continuous but u' is discontinuous). Let $\alpha, \beta > 0$. The functional $\mathcal{F} : \mathcal{H}^2(0,L) \to [0,+\infty)$ defined by

$$\mathcal{F}(u) = \int_0^L |u''|^2\,dt + \alpha\#(S(u')) + \beta\#(S(u)) \tag{1}$$

has been introduced by Blake and Zisserman ([2]) to model some signal reconstruction problems,which can then be reduced to the solution of the minimum problems

$$m = \min\Big\{\mathcal{F}(u) + \int_0^L |u-g|^2\,dt :\ u \in \mathcal{H}^2(0,L)\Big\}, \tag{2}$$

where g is the input (distorted) signal. The theoretical study of these problems has been performed by Coscia [9], who interpreted the functional $\mathcal{F}$ in the spirit of free-discontinuity problems as introduced by De Giorgi and Ambrosio (see [1], [4]). The key point is to notice that if $\alpha \le \beta \le 2\alpha$ then $\mathcal{F}$ is lower semicontinuous with respect to the $L^1(0,L)$ convergence. At this point the direct methods of the Calculus of Variations may be applied to obtain existence of the solutions to the abovementioned problems. A characterization of lower semicontinuity for general functionals on jumps and creases is given by Braides [3].

Here we show an approximation result by Γ-convergence of the functional above in the same spirit of that proved by Chambolle [8] for the Mumford-Shah functional (see [12], [11], [1])

$$\int_0^L |u'|^2\,dt + \gamma\#(S(u))$$

defined on piecewise-H^1 functions. In that case, the approximating discrete functionals take the form

$$E_\varepsilon(u) = \sum_{x\in\varepsilon\mathbb{Z}\cap(0,L)} \varepsilon\Phi_\varepsilon\Big(\frac{u(x+\varepsilon)-u(x)}{\varepsilon}\Big), \tag{3}$$

defined on discrete functions $u:\varepsilon\mathbb{Z}\cap(0,L)\to\mathbb{R}$, where Φ_ε are suitable functions, whose crucial property is to satisfy

$$\lim_\varepsilon \Phi_\varepsilon(z) = z^2 \text{ on } \mathbb{R} \qquad \lim_\varepsilon \varepsilon\Phi_\varepsilon\Big(\frac{z}{\varepsilon}\Big) = \gamma$$

($z \neq 0$). The model case is with $\Phi_\varepsilon(z) = \min\{z^2, \gamma/\varepsilon\}$. The description of the behaviour of general difference schemes of the form (3) with minimal hypotheses on Φ_ε can be found in [7]. In our case we can define suitable Ψ_ε such that, setting

$$E_\varepsilon(u) = \sum_{x\in\varepsilon\mathbb{Z}\cap(0,L)} \varepsilon\Psi_\varepsilon\Big(\frac{u(x+\varepsilon)+u(x-\varepsilon)-2u(x)}{\varepsilon^2}\Big)$$

(i.e., using a discretization of u'' in place of the difference quotients) these discrete energies converge to $\mathcal{F}$ as $\varepsilon\to 0$ in the sense of Γ-convergence (see below). As a result, we immediately obtain approximate minimum problems for (2) of the form

$$m_\varepsilon = \min\Big\{E_\varepsilon(u) + \sum_{x\in\varepsilon\mathbb{Z}\cap(0,L)} \varepsilon|u(x)-g_\varepsilon(x)|^2 :\ u:\varepsilon\mathbb{Z}\cap(0,L)\to\mathbb{R}\Big\}, \tag{4}$$

where g_ε are suitable discretizations of g. The main requirement on Ψ_ε is now that

$$\lim_\varepsilon \Psi_\varepsilon(z) = z^2 \text{ on } \mathbb{R} \qquad \lim_\varepsilon \varepsilon\Psi_\varepsilon\Big(\frac{z}{\varepsilon}\Big) = \alpha, \qquad \lim_\varepsilon 2\varepsilon\Psi_\varepsilon\Big(\frac{z}{\varepsilon^2}\Big) = \beta,$$

thus highlighting an interesting double-scale effect. Note in particular that the choice of $\Psi_\varepsilon(z) = \min\{z^2,\gamma/\varepsilon\}$ gives $\alpha=\gamma$ and $\beta = 2\gamma$.

Before proceeding in the exact statement and proof of the result, for notational convenience we replace the continuous small parameter ε by a discrete parameter λ_n. Let $L>0$ and $n\in\mathbb{N}$. We set

$$\lambda_n = \frac{L}{n}, \qquad x_n^i = i\lambda_n \quad (i = 0,\dots,n).$$

Let $0<\alpha\le\beta\le 2\alpha$, $c_1, c_2 > 0$ and define

$$\Psi_n(z) = \begin{cases} z^2 & \text{if } |z|\le c_1/\sqrt{\lambda_n} \\ \dfrac{\alpha}{\lambda_n} & \text{if } c_1/\sqrt{\lambda_n} < |z| \le c_2/\lambda_n\sqrt{\lambda_n} \\ \dfrac{\beta}{2\lambda_n} & \text{if } c_2/\lambda_n\sqrt{\lambda_n} < |z|. \end{cases} \tag{5}$$

Note that the terms $c_1/\sqrt{\lambda_n}$ and $c_2/\lambda_n\sqrt{\lambda_n}$ may be replaced by any other pair of sequences $(T_n^1), (T_n^2)$ such that $1 << T_n^1 << 1/\lambda_n << T_n^2 << 1/{\lambda_n}^2$ as $n \to +\infty$. Again, we have made this particular choice for notational convenience.

Let

$$\mathcal{A}_n = \{u : \lambda_n\mathbb{Z} \cap [0, L] \to \mathbb{R}\}$$

and let

$$E_n(u) = \sum_{i=1}^{n-1} \lambda_n \Psi_n\Big(\frac{u(x_n^{i+1}) + u(x_n^{i-1}) - 2u(x_n^i)}{{\lambda_n}^2}\Big) \tag{6}$$

be defined for $u \in \mathcal{A}_n$.

We identify $\mathcal{A}_n$ with a subspace of $L^2(0, L)$ by regarding each function u as defined by the value $u(x_n^i)$ on $[x_n^i, x_n^{i+1})$. We then have the following approximation result by Γ-convergence. For a general introduction to Γ-convergence we refer to [5], [10], or [6] Part II. For the application of Γ-convergence to the approximation of free-discontinuity problems we refer to [4]

Theorem 0.1 *The functionals E_n Γ-converge to $\mathcal{F}$ with respect to the $L^1(0, L)$ convergence on bounded sets of $L^2(0, L)$. Namely, we have*

(i) *if (u_n) is bounded in $L^2(0, L)$ and $\sup_n E_n(u_n) < +\infty$ then (u_n) is precompact in $L^1(0, L)$;*

(ii) *if (u_n) is bounded in $L^2(0, L)$, $u_n \to u$ in $L^1(0, L)$ and $\liminf_n E_n(u_n) < +\infty$ then $u \in \mathcal{H}^2(0, L)$ and $\mathcal{F}(u) \leq \liminf_n E_n(u_n)$;*

(iii) *for all $u \in \mathcal{H}^2(0, L)$ there exist $u_n \in \mathcal{A}_n$ with $u_n \to u$ in $L^2(0, L)$ and $\mathcal{F}(u) \geq \limsup_n E_n(u_n)$.*

As a consequence of (i)–(iii)*, if $g \in L^2(0, L)$ then the minimum values*

$$m_n = \min\Big\{E_n(u) + \int_0^L |g - u|\, dt : u \in \mathcal{A}_n\Big\}$$

converge to the minimum value

$$m = \min\Big\{\mathcal{F}(u) + \int_0^L |g - u|^2\, dt : u \in \mathcal{H}^2(0, L)\Big\}$$

and from every sequence (u_n) of solutions of the first problems we can extract a converging subsequence to a solution of the latter.

Proof. Statements (i) and (ii) will be proven by using the compactness properties of $\mathcal{F}$ and by comparing E_ε with a suitable family (F_ε) Γ-converging to $\mathcal{F}$. Let $u_n \in \mathcal{A}_n$ be such that $\liminf_n E_n(u_n) < +\infty$. Upon extracting a subsequence, we can suppose that this liminf is actually a limit, so that in particular $\sup_n E_n(u_n) < +\infty$. We will modify u_n so as to obtain a comparison sequence (v_n) in $\mathcal{H}^2(0, L)$.

We first modify u_n to obtain piecewise-affine functions w_n. On $[x_n^{i-1}, x_n^i]$ the function w_n is defined as

$$w_n(x) = u_n(x_n^{i-1}) + (x - x_n^{i-1})\Big(\frac{u_n(x_n^i) - u_n(x_n^{i-1})}{\lambda_n}\Big).$$

Let

$$I_n = \{i \in \{1, \dots, n-1\} : \ u_n(x_n^{i+1}) + u_n(x_n^{i-1}) - 2u_n(x_n^i) > c_1\lambda_n\sqrt{\lambda_n}\}.$$

For all $i \in \{1, \dots, n-1\} \setminus I_n$ we define v_n on the interval $(x_n^i - (\lambda_n/2), x_n^i + (\lambda_n/2))$ as the polynomial of degree 2 satisfying

$$v_n(x_n^i - \frac{\lambda_n}{2}) = w_n(x_n^i - \frac{\lambda_n}{2}) = \frac{u_n(x_n^i) + u_n(x_n^{i-1})}{\lambda_n},$$

$$v_n(x_n^i + \frac{\lambda_n}{2}) = w_n(x_n^i + \frac{\lambda_n}{2}) = \frac{u_n(x_n^{i+1}) + u_n(x_n^i)}{\lambda_n},$$

$$v_n'(x_n^i - \frac{\lambda_n}{2}) = w_n'(x_n^i - \frac{\lambda_n}{2}) = \frac{u_n(x_n^i) - u_n(x_n^{i-1})}{\lambda_n},$$

$$v_n'(x_n^i + \frac{\lambda_n}{2}) = w_n'(x_n^i + \frac{\lambda_n}{2}) = \frac{u_n(x_n^{i+1}) - u_n(x_n^i)}{\lambda_n}.$$

For such v_n the constant second derivative is given on $(x_n^i - (\lambda_n/2), x_n^i + (\lambda_n/2))$ by

$$v_n''(y) = \frac{1}{\lambda_n}(w_n'(x_n^i+) - w_n'(x_n^i-)) = \frac{u_n(x_n^{i+1}) + u_n(x_n^{i-1}) - 2u_n(x_n^i)}{{\lambda_n}^2}.$$

On the remaining part of $[0, l]$ we simply set

$$v_n(x) = w_n(x).$$

Note that $v_n \in \mathcal{H}^2(0, L)$ but $S(v_n) = \emptyset$. Moreover,

$$S(v_n') = \{x_n^i : \ i \in I_n\}.$$

If $x_n^i \in S(v_n')$ then

$$\frac{u_n(x_n^{i+1}) + u_n(x_n^{i-1}) - 2u_n(x_n^i)}{{\lambda_n}^2} = \frac{1}{\lambda_n}(v_n'(x_n^i+) - v_n'(x_n^i-)). \tag{7}$$

Finally, to complete the description of v_n, note that on the two intervals $(0, \lambda_n/2)$ and $(L - \lambda_n/2, L)$ v_n is affine.

Let

$$F_n(v) = \int_0^L |v''|^2\, dt + \sum_{t \in S(v)} \varphi_n(v'(t+) - v'(t-)) + \beta\#(S(v)).$$

be defined on $\mathcal{H}^2(0,L)$, where

$$\varphi_n(z) = \begin{cases} \alpha & \text{if } |z| < c_2/\sqrt{\lambda_n} \\ \beta/2 & \text{otherwise.} \end{cases}$$

By construction, we immediately obtain that

$$E_n(u_n) \geq F_n(v_n).$$

In fact, recalling (7), we have

$$\begin{aligned} E_n(u_n) &= \sum_{i\in\{1,\dots,n-1\}\setminus I_n} \lambda_n \Big| \frac{u_n(x_n^{i+1}) + u_n(x_n^{i-1}) - 2u_n(x_n^i)}{\lambda_n{}^2} \Big|^2 \\ &\quad + \alpha\,\#\{i \in I_n :\ |u_n(x_n^{i+1}) + u_n(x_n^{i-1}) - 2u_n(x_n^i)| < c_2\sqrt{\lambda_n}\} \\ &\quad + \frac{\beta}{2}\,\#\{i \in I_n :\ |u_n(x_n^{i+1}) + u_n(x_n^{i-1}) - 2u_n(x_n^i)| \geq c_2\sqrt{\lambda_n}\} \\ &= \int_0^L |v_n''|^2\,dt + \alpha\,\#\{t \in S(v_n') :\ |v_n(t+) - v_n(t-)| < c_2/\sqrt{\lambda_n}\} \\ &\quad + \frac{\beta}{2}\,\#\{t \in S(v_n') :\ |v_n(t+) - v_n(t-)| \geq c_2/\sqrt{\lambda_n}\} \\ &= \int_0^L |v_n''|^2\,dt + \sum_{t\in S(u)} \varphi_n(v_n'(t+) - v_n'(t-)) \\ &= F_n(v_n). \end{aligned}$$

It is easily checked that the Γ-limit of F_n is $\mathcal{F}$.

Note that $u_n - v_n$ tends to 0 in $L^1(0,L)$. Since $F_n(v_n) \geq \frac{\alpha}{\beta}\mathcal{F}(v_n)$, by the coerciveness properties of $\mathcal{F}$ we obtain that (v_n) is precompact in $L^1(0,L)$ and each limit of a converging subsequence belongs to $\mathcal{H}^2(0,L)$ (see [9]). Moreover, if $u_n \to u$ then also $v_n \to u$, and by the Γ-convergence of F_n to $\mathcal{F}$ we obtain

$$\liminf_n E_n(u_n) \geq \liminf_n F_n(v_n) \geq \mathcal{F}(u),$$

and (ii) is proved.

To prove (iii), by a density argument it suffices to show it for u piecewise C^2 and with bounded second derivative. Then, upon a slight piecewise-affine change of variable tending uniformly to the identity as $n \to +\infty$, we can reason as if for all n we have $S(u) \cup S(u') \subset \lambda_n\mathbb{Z}$. We can then take $u_n(x) = u(x-)$ on $\lambda_n\mathbb{Z}$ (except at 0, where we set $u_n(0) = u(0+)$); i.e., we choose as u_n in (iii) the piecewise-constant interpolation of u. We have:

(a) if $x_n^i = t \in S(u)$ or $x_n^{i-1} = t \in S(u)$ then

$$\frac{u_n(x_n^{i+1}) + u_n(x_n^{i-1}) - 2u_n(x_n^i)}{\lambda_n{}^2} = \frac{u(t+) - u(t-) + o(1)}{\lambda_n{}^2};$$

(b) if $x_n^i \in S(u')$ then

$$\frac{u(x_n^{i+1}) + u(x_n^{i-1}) - 2u(x_n^i)}{{\lambda_n}^2} = \frac{u'(x_n^i+) - u'(x_n^i-) + o(1)}{\lambda_n};$$

(c) in all other cases,

$$\frac{u(x_n^{i+1}) + u(x_n^{i-1}) - 2u(x_n^i)}{{\lambda_n}^2} = u''(x_n^i) + o(1),$$

with all the rests tending to 0 uniformly as $n \to +\infty$.

Note that we have, in case (a),

$$\Big|\frac{u(x_n^{i+1}) + u(x_n^{i-1}) - 2u(x_n^i)}{{\lambda_n}^2}\Big| >> \frac{c_2}{\lambda_n\sqrt{\lambda_n}},$$

while in case (b)

$$\frac{c_1}{\sqrt{\lambda_n}} << \Big|\frac{u(x_n^{i+1}) + u(x_n^{i-1}) - 2u(x_n^i)}{{\lambda_n}^2}\Big| << \frac{c_2}{\lambda_n\sqrt{\lambda_n}}$$

as $n \to +\infty$. Taking into account (a)–(c) and the remark above, we immediately obtain

$$\limsup_n E_n(u) \le \mathcal{F}(u)$$

so that (iii) is proved.

The final statement of the theorem follows easily from the well-known property of convergence of minima of Γ-limits. In fact, from (ii) and the lower semicontinuity of $u \mapsto \int_0^L |u-g|^2\,dt$ with respect to the $L^1(0,L)$ convergence, we obtain $m \le \liminf_n m_n$. On the other hand, if u is a solution of m, by (iii) we can find (u_n) as in (iii) so that

$$m = \lim_n \Big(E_n(u_n) + \int_0^L |u_n - g|^2\,dt \Big) \ge \limsup_n m_n.$$

Finally, the pre-compactness property of minimizing sequences follows from (i).

References

[1] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, Oxford, 2000.

[2] A. Blake and A. Zisserman, *Visual Reconstruction*, MIT Press, Cambridge, Massachussets, 1987.

[3] A. Braides. Lower semicontinuity conditions for functionals on jumps and creases. *SIAM J. Math Anal.* **26** (1995), 1184–1198.

[4] A. Braides, *Approximation of Free-Discontinuity Problems*, Lecture Notes in Mathematics, Springer Verlag, Berlin, 1998.

[5] A. Braides, Γ-*convergence for Beginners*, book in preparation.

[6] A. Braides and A. Defranceschi, *Homogenization of Multiple Integrals*, Oxford University Press, Oxford, 1998.

[7] A. Braides and M.S. Gelli, Limits of discrete systems without convexity hypotheses, Preprint SISSA, Trieste, 1999.

[8] A. Chambolle Un theoreme de Γ-convergence pour la segmentation des signaux. *C. R. Acad. Sci., Paris, Ser. I* **314** (1992), 191-196.

[9] A. Coscia, Existence Results for a New Variational Problem in One-dimensional Segmentation Theory, *Ann. Univ. Ferrara, Sez. VII*, 37 (1991), 185–203.

[10] G. Dal Maso, *An Introduction to* Γ-*convergence*, Birkhäuser, Boston, 1993.

[11] J. M. Morel and S. Solimini, *Variational Models in Image Segmentation*, Birkhäuser, Boston, 1995.

[12] D. Mumford and J. Shah, Optimal approximation by piecewise smooth functions and associated variational problems, *Comm. Pure Appl. Math.*, **42** (1989), 577-685.