

# On Type I Singularities in Ricci flow

Joerg Enders, Reto Müller and Peter M. Topping

## Abstract

We define several notions of singular set for Type I Ricci flows and show that they all coincide. In order to do this, we prove that blow-ups around singular points converge to *nontrivial* gradient shrinking solitons, thus extending work of Naber [15]. As a by-product we conclude that the volume of a finite-volume singular set vanishes at the singular time.

We also define a notion of density for Type I Ricci flows and use it to prove a regularity theorem reminiscent of White’s partial regularity result for mean curvature flow [22].

## 1 Introduction

A family  $(\mathcal{M}^n, g(t))$  of smooth complete Riemannian  $n$ -manifolds satisfying Hamilton’s Ricci flow [10],

$$\frac{\partial}{\partial t}g = -2\text{Ric}_{g(t)}, \quad (1.1)$$

on a finite time interval  $[0, T)$ ,  $T < \infty$ , is called a **Type I Ricci flow** if there exists a constant  $C > 0$  such that for all  $t \in [0, T)$

$$\sup_{\mathcal{M}} |\text{Rm}_{g(t)}|_{g(t)} \leq \frac{C}{T-t}. \quad (1.2)$$

Such a solution is said to develop a **Type I singularity** at time  $T$  (and  $T$  is called a **Type I singular time**) if it cannot be smoothly extended past time  $T$ . It is well known that this is the case if and only if

$$\limsup_{t \nearrow T} \sup_{\mathcal{M}} |\text{Rm}_{g(t)}|_{g(t)} = \infty, \quad (1.3)$$

see [10] for compact and [20] for complete flows. Here  $\text{Rm}_{g(t)}$  denotes the Riemannian curvature tensor of the metric  $g(t)$ . The main examples of Ricci flow singularities are of Type I, in particular the important neck-pinch singularity modelled on a shrinking  $n$ -dimensional cylinder (cf. [1, 2]) and singularities modelled on flows starting at a positive Einstein metric or more general at a gradient shrinking soliton with bounded curvature (see Section 2). Only very few rigorous examples of finite time singularities which are not of Type I (i.e. Type II) are known (cf. [6, 9]).

Since the manifolds  $(\mathcal{M}, g(t))$  have bounded curvatures in the Type I case (1.2), the parabolic maximum principle applied to the evolution equation satisfied by  $|\text{Rm}|^2$  shows that (1.3) is equivalent to

$$\sup_{\mathcal{M}} |\text{Rm}_{g(t)}|_{g(t)} \geq \frac{1}{8(T-t)} \quad \text{for all } t \in [0, T). \quad (1.4)$$

This motivates the following definitions.

**Definition 1.1.** A quantity  $A(t)$  is said to **blow up at the Type I rate** as  $t \rightarrow T$  if there exist constants  $C \geq c > 0$  such that  $\frac{c}{T-t} \leq A(t) \leq \frac{C}{T-t}$  for all  $t \in [T-c, T)$ .

**Definition 1.2.** A space-time sequence  $(p_i, t_i)$  with  $p_i \in \mathcal{M}$  and  $t_i \nearrow T$  in a Ricci flow is called an **essential blow-up sequence** if there exists a constant  $c > 0$  such that

$$|\text{Rm}_{g(t_i)}|_{g(t_i)}(p_i) \geq \frac{c}{T-t_i}.$$

A point  $p \in \mathcal{M}$  in a Type I Ricci flow is called a (general) **Type I singular point** if there exists an essential blow-up sequence with  $p_i \rightarrow p$  on  $\mathcal{M}$ . We denote the set of all Type I singular points by  $\Sigma_I$ .

**Remark 1.3.** If a solution to (1.1) develops a Type I singularity at time  $T$ , the existence of an essential blow-up sequence follows from (1.4). If in addition  $\mathcal{M}$  is compact, Type I singular points always exist. In the noncompact case the Type I singular set  $\Sigma_I$  may be empty if the singularity *forms at spatial infinity*. An example where this happens could be a cylinder  $S^{n-1} \times \mathbb{R}$  with radius larger than 1 in the center and tapering down to 1 at the ends. Flowing this under Ricci flow would lead to a first blow-up at spatial infinity.

A conjecture, normally attributed to Hamilton, is that a suitable blow-up sequence for a Type I singularity converges to a nontrivial gradient shrinking soliton [11]. In the case where the blow-up limit is compact, this conjecture was confirmed by Sesum [19]. In the general case, blow-up to a gradient shrinking soliton was proved by Naber [15]. However, it remained an open question whether the limit soliton Naber constructed is nontrivial, as mentioned for example in [3, Section 3.2]. In particular, one might think that the limit could be flat if all essential blow-up sequences converged “slowly” to  $p$  so that the curvature disappears at infinity after parabolically rescaling. One of the goals of this article is to rule out this possibility. More precisely, we prove the following theorem.

**Theorem 1.4.** *Let  $(\mathcal{M}^n, g(t))$  be a Type I Ricci flow on  $[0, T)$  and suppose  $p \in \Sigma_I$  is a Type I singular point as in Definition 1.2. Then for every sequence  $\lambda_j \rightarrow \infty$ , the rescaled Ricci flows  $(\mathcal{M}, g_j(t), p)$  defined on  $[-\lambda_j T, 0)$  by  $g_j(t) := \lambda_j g(T + \frac{t}{\lambda_j})$  subconverge to a normalized nontrivial gradient shrinking soliton in canonical form.*

We now turn to the relationship between the set  $\Sigma_I$  of Type I singular points and other notions of singular sets, starting with the set of *special* Type I singular points  $\Sigma_s$  defined as follows.

**Definition 1.5.** A point  $p \in \mathcal{M}$  in a Type I Ricci flow is called a **special Type I singular point** if there exists an essential blow-up sequence  $(p_i, t_i)$  with  $p_i = p$  for all  $i \in \mathbb{N}$ . The set of all such points is denoted by  $\Sigma_s$ . Moreover, we denote by  $\Sigma_{\text{Rm}} \subseteq \Sigma_s$  the set of points  $p \in \mathcal{M}$  for which  $|\text{Rm}_{g(t)}|_{g(t)}(p)$  blows up at the Type I rate as  $t \rightarrow T$ .

For mean curvature flow, Le-Sesum [13] proved that the mean curvature (rather than the second fundamental form) must be unbounded at a Type I singular time. It is not surprising and known to some Ricci flow experts that a similar result is true for the Ricci flow: if  $T$  is a Type I singular time, then the *scalar* curvature  $R_{g(t)}$  is unbounded as  $t \rightarrow T$ . We make the following definition.

**Definition 1.6.** The set  $\Sigma_R$  is defined to be the set of points  $p \in \mathcal{M}$  for which  $R_{g(t)}(p)$  blows up at the Type I rate as  $t \rightarrow T$ .

Instead of defining more restrictive singular sets, one can also think of a priori larger sets of singular points, for example the set consisting of points  $p \in \mathcal{M}$  where  $|\text{Rm}_{g(t)}|_{g(t)}(p)$  is unbounded as  $t \rightarrow T$  but possibly blows up at a rate smaller than the Type I rate, e.g. like  $\frac{1}{(T-t)^\alpha}$  for some  $\alpha < 1$ . A priori it is not clear whether (in the presence of a Type I singularity) such slowly forming singularities may exist in another part of the manifold, in particular since they cannot be observed by a blow-up argument analogous to Theorem 1.4. The following is the most general, natural definition of the singular set.

**Definition 1.7.** We call  $p \in \mathcal{M}$  a **singular point** if there does not exist any neighbourhood  $U_p \ni p$  on which  $|\text{Rm}_{g(t)}|_{g(t)}$  stays bounded as  $t \rightarrow T$ . The set of all singular points in this sense is denoted by  $\Sigma$ .

From the above definitions it is clear that

$$\Sigma_R \subseteq \Sigma_{\text{Rm}} \subseteq \Sigma_s \subseteq \Sigma_I \subseteq \Sigma. \quad (1.5)$$

For mean curvature flow with  $H > 0$ , Stone [21] showed that the (corresponding) notions of singular sets  $\Sigma_s$ ,  $\Sigma_I$  and  $\Sigma$  agree. The same is true for the Ricci flow; in fact, we show the slightly stronger result that all the singular sets defined above are identical.

**Theorem 1.8.** *Let  $(\mathcal{M}^n, g(t))$  be a Type I Ricci flow on  $[0, T)$  with singular time  $T$ . Then  $\Sigma \subseteq \Sigma_R$ , i.e. all the different notions of nested singular sets in (1.5) agree.*

In particular, this shows that for a Type I Ricci flow there cannot exist singular points where  $R_{g(t)}$  stays bounded or blows up at a rate smaller than the Type I rate as  $t \rightarrow T$ . As a corollary, we conclude that the singular set  $\Sigma$  has asymptotically vanishing volume if its volume is bounded initially.

**Theorem 1.9.** *Let  $(\mathcal{M}^n, g(t))$  be a Type I Ricci flow on  $[0, T)$  with singular time  $T$  and singular set  $\Sigma$  as in Definition 1.7. If  $\text{Vol}_{g(0)}(\Sigma) < \infty$  then*

$$\text{Vol}_{g(t)}(\Sigma) \xrightarrow{t \rightarrow T} 0.$$

**Remark 1.10.** A shrinking cylinder  $\mathbb{S}^m \times \mathbb{R}^{n-m}$ ,  $n > m \geq 2$ , shows that the condition  $\text{Vol}_{g(0)}(\Sigma) < \infty$  is necessary.

The paper is organized as follows. In Section 2, we prove Theorem 1.4. The methods we are using strongly rely on Perelman's results [17]. First, we recall Naber's result [15] that for any point  $p \in \mathcal{M}$  the rescaled flows  $g_j(t)$  as defined in Theorem 1.4 converge to a gradient shrinking soliton (Theorem 2.6). This is based on a version of Perelman's reduced length and volume based at the singular time, developed independently by the first author [7] and Naber [15]. For completeness, we sketch the main arguments of the proof. We then use Perelman's pseudolocality theorem [17] to show that the limit soliton must be nontrivial if  $p \in \Sigma_I$  is a Type I singular point. This completes the proof of Theorem 1.4. In Section 3, we prove Theorem 1.8. The argument is based again on Perelman's pseudolocality result as well as a strong rigidity result for gradient shrinking solitons, which can be found in Pigola-Rimoldi-Setti [18]. As a corollary, we obtain a proof of Theorem 1.9. Finally, in the last section

we define a **density function**  $\theta_{p,T}$  for Type I Ricci flows, related to the central density for gradient shrinking solitons defined by Cao-Hamilton-Ilmanen [4], and prove a regularity type theorem (Theorem 4.4) resembling White’s regularity result for mean curvature flow [22] and Ni’s regularity theorem for Ricci flow [16]. The proof of this result uses a gap theorem of Yokota [23].

Le and Sesum have also been studying properties of the scalar curvature at a Ricci flow singularity. In the current version of their paper [14], they observe how the arguments from our paper in fact exclude Type I singularity formation for compact manifolds under the assumption of an integral (rather than pointwise) scalar curvature bound.

**Acknowledgements:** All authors were partially supported by The Leverhulme Trust. RM was partially supported by FIRB Ideas “Analysis and Beyond”.

## 2 Blow-up to nontrivial gradient shrinking solitons

Before we start proving Theorem 1.4, let us briefly recall some basic definitions and facts about gradient shrinking solitons as well as the essential definitions and results from the first author [7] and Naber [15].

**Definition 2.1.** A triple  $(\mathcal{M}^n, g, f)$ , where  $(\mathcal{M}, g)$  is a complete  $n$ -dimensional Riemannian manifold and  $f : \mathcal{M} \rightarrow \mathbb{R}$  a smooth function, is called **gradient shrinking soliton** if

$$\text{Ric}_g + \nabla^g \nabla f = \frac{1}{2}g.$$

It is well known, that we can **normalize**  $f$  on a gradient shrinking soliton by setting

$$R_g + |\nabla f|_g^2 - f = 0. \tag{2.1}$$

It follows from (2.1) and the fact that  $R \geq 0$  (cf. e.g. [24]), that  $\nabla f$  is a complete vector field. Letting  $T > 0$  and considering the diffeomorphisms  $\phi_t$  of  $\mathcal{M}$  generated by  $\frac{1}{T-t}\nabla f$  with  $\phi_{T-1} = id$ , we obtain from the definition of gradient shrinking soliton above a corresponding Ricci flow  $g(t) = (T-t)\phi_t^*g$  on  $(-\infty, T)$  with  $(\mathcal{M}, g(T-1)) = (\mathcal{M}, g)$ . Canonically defining time-dependent functions by  $f(t) := \phi_t^*f$ , the flow satisfies

$$\text{Ric}_{g(t)} + \nabla^{g(t)} \nabla f(t) = \frac{1}{2(T-t)}g(t) \quad \text{and} \quad \frac{\partial}{\partial t} f(t) = |\nabla f(t)|_{g(t)}^2. \tag{2.2}$$

We call a Ricci flow  $(\mathcal{M}, g(t), f(t))$  on  $(-\infty, T)$  with smooth functions  $f(t) : \mathcal{M} \rightarrow \mathbb{R}$  satisfying (2.2) a **gradient shrinking soliton in canonical form**.

Let  $(\mathcal{M}^n, g(t))$  be a (connected) Type I Ricci flow on  $[0, T)$  as defined in the Section 1. For fixed  $(p, t_0) \in \mathcal{M} \times [0, T)$  and all  $(q, \bar{t}) \in \mathcal{M} \times [0, t_0]$ , Perelman’s **reduced distance** (in forward time notation) is defined by

$$l_{p,t_0}(q, \bar{t}) := \inf_{\gamma} \left\{ \frac{1}{2\sqrt{t_0 - \bar{t}}} \int_{\bar{t}}^{t_0} \sqrt{t_0 - t} (|\dot{\gamma}(t)|^2 + R_{g(t)}(\gamma(t))) dt \right\},$$

where the infimum is taken over all curves  $\gamma : [\bar{t}, t_0] \rightarrow \mathcal{M}$  with  $\gamma(t_0) = p, \gamma(\bar{t}) = q$ . The corresponding **reduced volume** is

$$\tilde{V}_{p,t_0}(\bar{t}) := \int_{\mathcal{M}} v_{p,t_0}(q, \bar{t}) d\text{vol}_{g(\bar{t})}(q),$$

where

$$v_{p,t_0}(q, \bar{t}) := (4\pi(t_0 - \bar{t}))^{-\frac{n}{2}} e^{-l_{p,t_0}(q, \bar{t})}.$$

We will use the following two results from [7] (restricted here to the Type I case):

**Lemma 2.2** (Enders [7], Theorem 3.3.1). *Let  $(\mathcal{M}^n, g(t))$  be a (connected) Type I Ricci flow on  $[0, T)$ ,  $p \in \mathcal{M}$  and  $t_k \nearrow T$ . Then there exists a locally Lipschitz function*

$$l_{p,T} : \mathcal{M} \times (0, T) \rightarrow \mathbb{R},$$

which is a subsequential limit

$$l_{p,t_k} \xrightarrow{C_{loc}^0(\mathcal{M} \times (0, T))} l_{p,T}$$

and which for all  $(q, \bar{t}) \in \mathcal{M} \times (0, T)$  satisfies

$$-\frac{\partial}{\partial \bar{t}} l_{p,T}(q, \bar{t}) - \Delta_{g(\bar{t})} l_{p,T}(q, \bar{t}) + |\nabla l_{p,T}(q, \bar{t})|_{g(\bar{t})}^2 - \text{R}_{g(\bar{t})}(q) + \frac{n}{2(T - \bar{t})} \geq 0$$

in the sense of distributions. Equivalently,

$$\square_{g(\bar{t})}^* v_{p,T}(q, \bar{t}) \leq 0,$$

where

$$\square_{g(t)}^* := -\frac{\partial}{\partial t} - \Delta_{g(t)} + \text{R}_{g(t)}$$

denotes the formal adjoint of the heat operator under the Ricci flow, and

$$v_{p,T}(q, \bar{t}) := (4\pi(T - \bar{t}))^{-\frac{n}{2}} e^{-l_{p,T}(q, \bar{t})}.$$

**Definition 2.3.** We define  $l_{p,T}$  as in Lemma 2.2 to be a **reduced distance based at the singular time**  $(p, T)$ . Moreover, the corresponding

$$\tilde{V}_{p,T}(\bar{t}) := \int_{\mathcal{M}} v_{p,T}(q, \bar{t}) d\text{vol}_{g(\bar{t})}(q)$$

is denoted a **reduced volume based at the singular time**  $(p, T)$  with  $v_{p,T}$  being a **reduced volume density based at the singular time**  $(p, T)$ .

The next result states that similarly to Perelman's reduced volume, any reduced volume based at singular time is also a monotone quantity.

**Lemma 2.4** (Enders [7], Theorem 3.4.3). *Under the assumptions as in Definition 2.3 we have*

- (i)  $\frac{d}{d\bar{t}} \tilde{V}_{p,T}(\bar{t}) \geq 0$ ,
- (ii)  $\lim_{\bar{t} \nearrow T} \tilde{V}_{p,T}(\bar{t}) \leq 1$ ,

(iii) If  $\tilde{V}_{p,T}(\bar{t}_1) = \tilde{V}_{p,T}(\bar{t}_2)$  for  $0 < \bar{t}_1 < \bar{t}_2 < T$ , then  $(\mathcal{M}, g(t), l_{p,T}(\cdot, t))$  is a normalized gradient shrinking soliton in canonical form.

Similar results to Lemma 2.2 and Lemma 2.4 have been independently obtained in [15]. We restate the estimates derived there in the following adapted form.

**Lemma 2.5** (Naber [15], Proposition 3.6). *Let  $(\mathcal{M}^n, g(t))$  be a (connected) Type I Ricci flow on  $[0, T)$ , and let  $(p, t_0) \in \mathcal{M} \times [0, T)$ . Then there exist  $K > 0$  (only dependent on  $n$  and the Type I constant  $C$ ) such that for all  $(q, \bar{t}) \in \mathcal{M} \times (0, T)$*

$$\begin{aligned} (i) \quad & \frac{1}{K} \left(1 + \frac{d_{\bar{t}}(p, q)}{\sqrt{t_0 - \bar{t}}}\right)^2 - K \leq l_{p, t_0}(q, \bar{t}) \leq K \left(1 + \frac{d_{\bar{t}}(p, q)}{\sqrt{t_0 - \bar{t}}}\right)^2, \\ (ii) \quad & |\nabla l_{p, t_0}(q, \bar{t})|_{g(\bar{t})}(q) \leq \frac{K}{\sqrt{t_0 - \bar{t}}} \left(1 + \frac{d_{\bar{t}}(p, q)}{\sqrt{t_0 - \bar{t}}}\right), \\ (iii) \quad & \left| \frac{\partial}{\partial \bar{t}} l_{p, t_0}(q, \bar{t}) \right|_{g(\bar{t})}(q) \leq \frac{K}{t_0 - \bar{t}} \left(1 + \frac{d_{\bar{t}}(p, q)}{\sqrt{t_0 - \bar{t}}}\right)^2. \end{aligned}$$

We now show that parabolic rescaling limits in a Type I Ricci flow (around any point  $p \in \mathcal{M}$  at the singular time  $T$ ) have a gradient shrinking soliton structure. For completeness, we reprove this result which was first obtained in [15].

**Theorem 2.6** (cf. Naber [15], Theorem 1.5). *Let  $(\mathcal{M}^n, g(t), p)$ ,  $t \in [0, T)$ ,  $p \in \mathcal{M}$  be a pointed Type I Ricci flow, and  $\lambda_j \nearrow \infty$ . Then any pointed Cheeger-Gromov-Hamilton limit flow  $(\mathcal{M}_\infty^n, g_\infty(t), p_\infty)$ ,  $t \in (-\infty, 0)$ , of the parabolically rescaled Ricci flows  $g_j(t) := \lambda_j g(T + \frac{t}{\lambda_j})$  is a normalized gradient shrinking soliton in canonical form.*

*Proof.* Because of the Type I curvature bound, we have at any  $x \in \mathcal{M}$  that

$$\begin{aligned} |\text{Rm}_{g_j(t)}|_{g_j(t)}(x) &= \frac{1}{\lambda_j} |\text{Rm}_{g(T + \frac{t}{\lambda_j})}|_{g(T + \frac{t}{\lambda_j})}(x) \\ &\leq \frac{C}{\lambda_j (T - (T + \frac{t}{\lambda_j}))} = \frac{C}{-t}. \end{aligned} \tag{2.3}$$

This gives a uniform curvature bound on compact subsets of  $(-\infty, 0)$ . Together with Perelman's no local collapsing theorem (which also holds for complete  $\mathcal{M}$  because of the uniform lower bound on the reduced volume as described below), we can use the Cheeger-Gromov-Hamilton Compactness Theorem [12] to extract from the sequence  $(\mathcal{M}, g_j(t), p)$  a complete pointed subsequential limit Ricci flow  $(\mathcal{M}_\infty, g_\infty(t), p_\infty)$  on  $(-\infty, 0)$ , which is still Type I.

Now, let  $l_{p,T}$  be any reduced distance based at the singular time  $(p, T)$  for the Ricci flow  $(\mathcal{M}, g(t))$  on  $[0, T)$  as defined above. For each  $(q, \bar{t}) \in \mathcal{M} \times (-\infty, 0)$ , consider for large enough  $j$

$$l_{p,0}^j(q, \bar{t}) := l_{p,T}(q, T + \frac{\bar{t}}{\lambda_j}), \tag{2.4}$$

which is a reduced distance based at the singular time  $(p, 0)$  for the rescaled Ricci flow  $(\mathcal{M}, g_j(t))$  on  $[-\lambda_j T, 0)$  because of the scaling properties of the reduced distance. The corresponding reduced volumes are then related by

$$\tilde{V}_{p,0}^j(\bar{t}) = \tilde{V}_{p,T}(T + \frac{\bar{t}}{\lambda_j}),$$

and we can conclude, using also Lemma 2.4, that

$$\tilde{V}_{p,0}^j \xrightarrow{j \rightarrow \infty} \lim_{t \nearrow T} \tilde{V}_{p,T}(t) \in (0, 1]$$

uniformly on compact subsets of  $(-\infty, 0)$ .

The uniform estimates in Lemma 2.5 hold for  $l_{p,T}$  by construction, and hence by (2.4) for each  $l_{p,0}^j$ . Note that by (2.3) they have the same Type I bound  $C$ . Hence we can conclude that there exists a locally Lipschitz function  $l_{p_\infty,0}^\infty$  on the limit manifold  $\mathcal{M}_\infty \times (-\infty, 0)$ , such that

$$l_{p,0}^j \xrightarrow{C_{loc}^0} l_{p_\infty,0}^\infty.$$

Since its corresponding formal reduced volume  $V_{p_\infty,0}^\infty$  is constant, we can conclude as in the proof of Lemma 2.4 (iii) that  $(\mathcal{M}_\infty, g_\infty(t), l_{p_\infty,0}^\infty(\cdot, t))$  is a normalized gradient shrinking soliton in canonical form.  $\square$

To obtain a complete proof of Theorem 1.4, it remains to show that for Type I singular points  $p \in \Sigma_I$  the rescaling limit flow  $(\mathcal{M}_\infty, g_\infty(t))$  in Theorem 2.6 is nontrivial and hence a suitable singularity model.

Our proof is based on Perelman's pseudolocality theorem, which states the following.

**Proposition 2.7** (Perelman [17], Theorem 10.3). *There exist  $\varepsilon, \delta > 0$  depending on  $n$  with the following property. Suppose  $g(t)$  is a complete Ricci flow with bounded curvature on an  $n$ -dimensional manifold  $\mathcal{M}^n$  for  $t \in [0, (\varepsilon r_0)^2)$ . Moreover, suppose that  $r_0 > 0$ ,  $p \in \mathcal{M}$  and assume that at  $t = 0$  we have  $|\text{Rm}_{g(0)}| \leq r_0^{-2}$  in  $B_{g(0)}(p, r_0)$  and  $\text{Vol}_{g(0)}(B_{g(0)}(p, r_0)) \geq (1 - \delta)\omega_n r_0^n$ , where  $\omega_n$  is the volume of the unit ball in  $\mathbb{R}^n$ . Then there holds the following estimate*

$$|\text{Rm}_{g(t)}|(x) \leq (\varepsilon r_0)^{-2}, \quad \text{for } 0 \leq t < (\varepsilon r_0)^2, x \in B_{g(t)}(p, \varepsilon r_0). \quad (2.5)$$

**Remark 2.8.** Note that by choosing a smaller  $\varepsilon$ , estimate (2.5) holds for  $x \in B_{g(0)}(p, \varepsilon r_0)$ . This follows directly from the following lemma, variants of which can be found elsewhere, for example [11].

**Lemma 2.9.** *Suppose that  $g(t)$  is a Ricci flow on a manifold  $\mathcal{M}^n$  for  $t \in [0, T]$ . Suppose further that for some  $p \in \mathcal{M}$  and  $r > 0$ , we have  $B_{g(t)}(p, r) \subset\subset \mathcal{M}$ , and  $|\text{Ric}| \leq M$  on  $B_{g(t)}(p, r)$  for each  $t \in [0, T]$ . Then*

$$B_{g(0)}(p, e^{-Mt}r) \subset B_{g(t)}(p, r),$$

for all  $t \in [0, T]$ .

*Proof.* Let  $\sigma \in (0, 1)$  be arbitrary. It suffices to show that

$$\overline{B_{g(0)}(p, e^{-Mt}\sigma r)} \subset B_{g(t)}(p, r),$$

for each  $t \in [0, T]$ . Clearly this is true for  $t = 0$ ; suppose it fails for some larger  $t = t_0 \in (0, T]$ . Without loss of generality,  $t_0$  is the least such time.

Pick a minimizing geodesic  $\gamma$ , with respect to  $g(0)$ , from  $p$  to a point  $y \in \mathcal{M}$  with  $d_{g(0)}(p, y) = e^{-Mt_0}\sigma r$  and  $d_{g(t_0)}(p, y) = r$ .

Then for all  $t \in [0, t_0)$ ,  $\gamma$  lies within  $B_{g(0)}(p, e^{-Mt}\sigma r) \subset B_{g(t)}(p, r)$ , and hence (by hypothesis)  $|\text{Ric}| \leq M$  on  $\gamma$  over this range of times.

But then  $\text{Length}_{g(t)}(\gamma) \leq e^{Mt} \text{Length}_{g(0)}(\gamma)$  for all  $t \in [0, t_0)$ , and hence

$$\text{Length}_{g(t_0)}(\gamma) \leq e^{Mt_0} e^{-Mt_0} \sigma r = \sigma r < r,$$

and this implies  $d_{g(t_0)}(p, y) < r$ , a contradiction.  $\square$

We are now ready to prove that the blow-up limit is nontrivial.

*Proof of Theorem 1.4.* Assume that  $g_\infty(t)$  is flat for all  $t < 0$ . In particular,  $g_\infty(t)$  is independent of time, and we denote it by  $\hat{g}$ .

Take  $r_0 > 0$  smaller than the injectivity radius of  $\hat{g}$  at  $p_\infty$  to ensure that  $B_{\hat{g}}(p_\infty, r_0)$  is a Euclidean ball. By the Cheeger-Gromov-Hamilton convergence, taking  $j$  large enough,  $B_{g_j(-(\varepsilon r_0)^2)}(p, r_0)$  is as close as we want to a Euclidean ball, where  $\varepsilon$  is chosen as in Proposition 2.7. In particular, we may fix  $j$  sufficiently large such that  $B_{g_j(-(\varepsilon r_0)^2)}(p, r_0)$  satisfies the conditions of Proposition 2.7 and hence, using also Remark 2.8,

$$|\text{Rm}_{g_j(t)}|_{g_j(t)}(x) \leq (\varepsilon r_0)^{-2} \quad \text{for } -(\varepsilon r_0)^2 \leq t < 0, x \in B_{g_j(-(\varepsilon r_0)^2)}(p, \varepsilon r_0). \quad (2.6)$$

On the other hand, since  $p \in \Sigma_I$ , there exists an essential blow-up sequence  $(p_i, t_i)$  with  $p_i \rightarrow p$  such that for a constant  $c > 0$  as in Definition 1.2

$$|\text{Rm}_{g_j(\lambda_j(t_i - T))}|_{g_j(\lambda_j(t_i - T))}(p_i) \geq \frac{c}{\lambda_j(T - t_i)}.$$

For  $i$  large enough, this contradicts (2.6). Thus  $g_\infty(t)$  cannot be flat.  $\square$

Nontrivial gradient shrinking solitons also arise as blow-down limits of certain ancient Ricci flow solutions (which are singularity models) as shown by Perelman [17] in 3 dimensions, and recently by Cao and Zhang [5] for higher dimensions in the Type I case.

### 3 Singular sets

A crucial ingredient for the theorems proved in this and the next section is the following rigidity result for gradient shrinking solitons as for example shown in [18].

**Lemma 3.1** (Pigola-Rimoldi-Setti [18], Theorem 3). *Let  $(\mathcal{M}^n, g, f)$  be a complete gradient shrinking soliton. Then the scalar curvature  $R_g$  is nonnegative, and if there exists a point  $p \in \mathcal{M}$  where  $R_g(p) = 0$ , then  $(\mathcal{M}, g, f)$  is the Gaussian soliton, i.e. isometric to flat Euclidean space  $(\mathbb{R}^n, g_{\mathbb{R}^n})$ .*

We use this lemma to prove Theorem 1.8, i.e. that  $\Sigma \subseteq \Sigma_R$ . As a first step, we show that the Type I singular set  $\Sigma_I$  is characterized by the blow-up of the scalar curvature at the Type I rate, i.e.  $\Sigma_I = \Sigma_R$ .

**Theorem 3.2.** *Let  $(\mathcal{M}^n, g(t))$  be a Type I Ricci flow on  $[0, T)$  with singular time  $T$ , Type I singular set  $\Sigma_I$  as in Definition 1.2 and  $\Sigma_R$  as in Definition 1.6. Then  $\Sigma_I = \Sigma_R$ .*

*Proof.* By definition, we know that  $\Sigma_R \subseteq \Sigma_I$ . For the converse inclusion, assume that  $p \in \mathcal{M} \setminus \Sigma_R$ . Hence there are  $c_j \searrow 0$  and  $t_j \in [T - c_j, T)$ , such that  $R_{g(t_j)}(p) < \frac{c_j}{T - t_j}$ . Let  $\lambda_j = (T - t_j)^{-1} \rightarrow \infty$  and rescale as in Theorem 2.6, i.e. let  $g_j(t) := \lambda_j g(T + \frac{t}{\lambda_j})$  on  $\mathcal{M} \times [-\lambda_j T, 0)$ . By Theorem 2.6,  $(\mathcal{M}, g_j(t), p)$  converge to a gradient shrinking soliton in canonical form  $(\mathcal{M}_\infty, g_\infty(t), p_\infty)$  on  $(-\infty, 0)$  with

$$R_{g_\infty(-1)}(p_\infty) = \lim_{j \rightarrow \infty} \lambda_j^{-1} R_{g(t_j)}(p) \leq \lim_{j \rightarrow \infty} c_j = 0.$$

By Lemma 3.1,  $(\mathcal{M}_\infty, g_\infty(-1)) = (\mathcal{M}_\infty, g_\infty(t))$  must be flat. But by Theorem 1.4, the limit soliton is nonflat for points in the Type I singular set, so  $p \in \mathcal{M} \setminus \Sigma_I$ .  $\square$

To show that  $\Sigma = \Sigma_I$ , we prove the following regularity type result.

**Theorem 3.3.** *If  $(\mathcal{M}^n, g(t))$  is a Type I Ricci flow on  $[0, T)$  with singular time  $T$  and  $p \in \mathcal{M} \setminus \Sigma_I$ , then there exists a neighbourhood  $U_p \ni p$  such that the curvature is bounded uniformly on  $U_p \times [0, T)$ . In particular,  $p \in \mathcal{M} \setminus \Sigma$ .*

*Proof.* Since  $p \in \mathcal{M} \setminus \Sigma_I$ , for any given  $\lambda_j \rightarrow \infty$  the rescaled metrics  $g_j(t) = \lambda_j g(T + \frac{t}{\lambda_j})$  converge to flat Euclidean space. As in the proof of Theorem 1.4, for large enough  $j \geq j_0$  the conditions of the pseudolocality theorem, Proposition 2.7, and Lemma 2.9 with  $r_0 = 1$  are satisfied. Let  $K := \lambda_{j_0}$  and take  $\varepsilon > 0$  to be as in the pseudolocality theorem and Remark 2.8 following it. Then we conclude for  $j = j_0$  as before

$$|\text{Rm}_{g_{j_0}(t)}|_{g_{j_0}(t)}(x) \leq \varepsilon^{-2} \quad \text{for } -\varepsilon^2 \leq t < 0, x \in B_{g_{j_0}(-\varepsilon^2)}(p, \varepsilon),$$

which is equivalent to

$$|\text{Rm}_{g(t)}|_{g(t)} \leq \frac{K}{\varepsilon^2} \quad \text{for all } t \in [T - \frac{\varepsilon^2}{K}, T)$$

on the neighbourhood  $U_p := B_{g(T - \frac{\varepsilon^2}{K})}(p, \frac{\varepsilon}{\sqrt{K}})$  of  $p$ . The bound for times  $t < T - \frac{\varepsilon^2}{K}$  follows trivially from the Type I condition (1.2).  $\square$

Combining Theorem 3.2 and Theorem 3.3, we have proved Theorem 1.8. As a corollary, we obtain Theorem 1.9, i.e. that the singular set  $\Sigma$  has asymptotically vanishing volume if  $\text{Vol}_{g(0)}(\Sigma) < \infty$ .

*Proof of Theorem 1.9.* By the bounded curvature assumption (1.2) together with the parabolic maximum principle applied to the evolution of  $R_{g(t)}$ , there exists  $\tilde{C} > 0$  such that  $\inf_{\mathcal{M}} R_{g(t)} \geq -\tilde{C}$ ,  $\forall t \in [0, T)$ . Let  $\Sigma_{R,k}$  be defined by

$$\Sigma_{R,k} := \{p \in \mathcal{M} \mid R_{g(t)}(p) \geq \frac{1/k}{T-t}, \forall t \in (T - \frac{1}{k}, T)\} \subseteq \Sigma_R = \Sigma$$

for  $k \in \mathbb{N}$  and  $\Sigma_{R,0} := \emptyset$ . We claim that on  $\Sigma_{R,k}$ , we have for all  $t \in [0, T)$

$$\int_0^t R_{g(s)} ds \geq -\tilde{C}T + \log\left(\frac{1/k}{T-t}\right)^{1/k}.$$

For  $t \leq T - \frac{1}{k}$ , this follows from  $\int_0^t R ds \geq -\tilde{C}t \geq -\tilde{C}T$  and the fact that the log-term is nonpositive in this case. For  $t \in (T - \frac{1}{k}, T)$ , we obtain by definition of  $\Sigma_{R,k}$

$$\begin{aligned} \int_0^t R_{g(s)} ds &= \int_0^{T-\frac{1}{k}} R_{g(s)} ds + \int_{T-\frac{1}{k}}^t R_{g(s)} ds \\ &\geq -\tilde{C}(T - \frac{1}{k}) + \int_{T-\frac{1}{k}}^t \frac{1/k}{T-s} ds \\ &\geq -\tilde{C}T + \log\left(\frac{1/k}{T-t}\right)^{1/k}. \end{aligned}$$

Using  $k^{1/k} \leq 2$  for all  $k \in \mathbb{N}$ , we can now bound volumes of subsets of  $\Sigma_{R,k}$  at time  $t$  in terms of their volumes at time 0 by computing

$$\begin{aligned} \text{Vol}_{g(t)}(\Sigma_{R,k} \setminus \Sigma_{R,k-1}) &= \int_{\Sigma_{R,k} \setminus \Sigma_{R,k-1}} e^{-(\int_0^t R_{g(s)} ds)} d\text{vol}_{g(0)} \\ &\leq 2e^{\tilde{C}T} (T-t)^{1/k} \text{Vol}_{g(0)}(\Sigma_{R,k} \setminus \Sigma_{R,k-1}). \end{aligned}$$

We use this last estimate to conclude

$$\begin{aligned} \limsup_{t \rightarrow T} \text{Vol}_{g(t)}(\Sigma) &= \limsup_{t \rightarrow T} \sum_{k \in \mathbb{N}} \text{Vol}_{g(t)}(\Sigma_{R,k} \setminus \Sigma_{R,k-1}) \\ &\leq 2e^{\tilde{C}T} \lim_{t \rightarrow T} \sum_{k \in \mathbb{N}} (T-t)^{1/k} \text{Vol}_{g(0)}(\Sigma_{R,k} \setminus \Sigma_{R,k-1}) \\ &= 0, \end{aligned}$$

where the last line follows easily from the fact that  $\sum_{k \in \mathbb{N}} \text{Vol}_{g(0)}(\Sigma_{R,k} \setminus \Sigma_{R,k-1}) = \text{Vol}_{g(0)}(\Sigma_R) < \infty$ .  $\square$

## 4 Density and regularity theorem

In this section, we use the reduced volume based at the singular time as reviewed in Section 2 to define a density function on the closure of space-time of Type I Ricci flows and prove a regularity theorem. We first overcome the non-uniqueness issue of the reduced distance based at the singular time. The functions  $l_{p,T}$  used in Section 2 were subsequential limits and depended on the choice of  $\{t_k\}$  and a subsequence  $\{t_{k_l}\}$ . We now denote such a choice of reduced distance based at the singular time by  $l_{p,T,\{t_{k_l}\}}$  to make the following definition.

**Definition 4.1.** Under the assumptions of Lemma 2.2, we define **the reduced distance based at the singular time** by

$$l_{p,T} := \inf_{\{t_{k_l}\}} l_{p,T,\{t_{k_l}\}},$$

where the infimum is taken over all possible subsequences of all possible sequences  $t_k \nearrow T$  used to construct a reduced distance. As in Definition 2.3, we correspondingly denote **the reduced volume density** and **the reduced volume based at the singular time** by  $v_{p,T}$  and  $\tilde{V}_{p,T}$ , respectively.

Note that Lemma 2.5 implies that  $l_{p,T}$  is well-defined and locally Lipschitz because  $l_{p,T,\{t_{k_l}\}}$  are uniformly locally Lipschitz. The monotonicity in Lemma 2.4 for the redefined  $\tilde{V}_{p,T}$  as above holds once we show the following lemma.

**Lemma 4.2.** *Under the assumptions as in Lemma 2.2, we have for  $v_{p,T}$  in Definition 4.1 that*

$$\square_{g(\bar{t})}^* v_{p,T}(q, \bar{t}) \leq 0$$

*holds in the weak sense or sense of distributions.*

*Proof.* We argue by contradiction: Assume there exists a (small) parabolic cylinder  $P = U \times [t_2, t_1] \subset \mathcal{M} \times (0, T)$ ,  $U$  open, such that for all  $0 \leq \phi \in C_{cpt}^2(\mathcal{M} \times (0, T))$  with support in  $P$

$$\iint_P v_{p,T}(q, t) \square_{g(t)} \phi(q, t) \, d\text{vol}_{g(t)}(q) dt > 0.$$

Inverting time ( $\tau := T - t$ ) implies that  $-v_{p,T}$  is strictly subparabolic in the weak sense of Friedman [8], and we will apply his (strong) maximum principle several times to derive a contradiction.

By Definition 4.1,  $v_{p,T} := \sup_{\{t_{k_l}\}} v_{p,T,\{t_{k_l}\}}$ . Let  $\{t_{k_l}\}$  be any such subsequence, then  $v_{p,T,\{t_{k_l}\}} \leq v_{p,T}$ , and we know from Lemma 2.2 that

$$\square^* v_{p,T,\{t_{k_l}\}} \leq 0$$

in the weak sense. Now let  $\Gamma := \bar{P} \setminus P$  and  $w_{\{t_{k_l}\}}$  be a weak solution to

$$\begin{cases} \square^* w_{\{t_{k_l}\}} &= 0 \text{ in } P \\ w_{\{t_{k_l}\}}|_{\Gamma} &= v_{p,T,\{t_{k_l}\}}|_{\Gamma}. \end{cases}$$

Hence,

$$\square^*(v_{p,T,\{t_{k_l}\}} - w_{\{t_{k_l}\}}) \leq 0,$$

and the maximum principle implies

$$v_{p,T,\{t_{k_l}\}} \leq w_{\{t_{k_l}\}} \text{ in } \bar{P}. \quad (4.1)$$

Similarly, let  $w$  be a weak solution to

$$\begin{cases} \square^* w &= 0 \text{ in } P \\ w|_{\Gamma} &= v_{p,T}|_{\Gamma}. \end{cases}$$

Since

$$\square^*(w_{\{t_{k_l}\}} - w) = 0$$

and  $w_{\{t_{k_l}\}}|_{\Gamma} = v_{p,T,\{t_{k_l}\}}|_{\Gamma} \leq v_{p,T}|_{\Gamma} = w|_{\Gamma}$ , the maximum principle implies that

$$w_{\{t_{k_l}\}} \leq w \text{ in } \bar{P}. \quad (4.2)$$

As  $\{t_{k_l}\}$  was arbitrary, we conclude from (4.1) and (4.2) that

$$v_{p,T} \leq w \text{ in } \bar{P}.$$

Using  $v_{p,T}|_{\Gamma} = w|_{\Gamma}$  and the maximum principle again, this contradicts that by assumption

$$\square^*(v_{p,T} - w) > 0$$

in the weak sense in  $P$ . □

We now consider points in the closure of space-time, i.e. in  $\mathcal{M} \times [0, T]$ , to include the singular time.

**Definition 4.3.** Let  $(\mathcal{M}, g(t))$  be a Type I Ricci flow on  $[0, T]$ . For any  $(p, t_0) \in \mathcal{M} \times [0, T]$  we define **the density at  $(p, t_0)$  in the Ricci flow  $(\mathcal{M}, g(t))$**  by

$$\theta_{p,t_0} := \lim_{\bar{t} \nearrow t_0} \tilde{V}_{p,t_0}(\bar{t}) \in (0, 1].$$

Note that for the special case of gradient shrinking solitons, Cao-Hamilton-Ilmanen [4] suggest a “central density of a shrinker” defined similarly.

If  $T$  is the Type I singular time and  $t_0 < T$  it follows from the properties of Perelman’s reduced volume that  $\theta_{p,t_0} = 1$  for any  $p \in \mathcal{M}$  (in fact, without the Type I assumption). At the singular time  $t_0 = T$ , the density carries information regarding the structure of the singularity, namely the corresponding gradient shrinking solitons one may obtain by taking a blow-up limit. We prove the following regularity type result similar to White’s local regularity result for mean curvature flow [22], where instead of using the Gaussian density for the mean curvature flow we use the density for Type I Ricci flows as defined above. A related result is proved by Ni [16] using a localized quantity.

**Theorem 4.4.** *Let  $(\mathcal{M}^n, g(t))$  be a Type I Ricci flow on  $[0, T]$  with singular time  $T$  and singular set  $\Sigma$  as in Definition 1.7. Then  $\theta_{p,T} = 1$  if and only if  $p \in \mathcal{M} \setminus \Sigma$ . In fact, there exists  $\eta > 0$  (only depending on  $n$ ) such that if  $\theta_{p,T} > 1 - \eta$  for a Ricci flow as above, then  $p \in \mathcal{M} \setminus \Sigma$ . Equivalently, if  $\theta_{p,T} > 1 - \eta$ , then there exists a neighbourhood  $U_p \ni p$  such that the curvature is bounded uniformly on  $U_p \times [0, T]$ .*

*Proof.* It follows from the discussion in Section 3 that any rescaling limit  $(\mathcal{M}_\infty, g_\infty(t))$  as in Theorem 2.6 around  $p \in \mathcal{M} \setminus \Sigma$  is flat, i.e.  $(\mathcal{M}_\infty, g_\infty(t))$  is isometric to the Gaussian soliton  $(\mathbb{R}^n, g_{\mathbb{R}^n})$ . Hence one easily computes  $\theta_{p,T} = 1$ .

Conversely, let  $\theta_{p,T}$  be the density of  $p \in \mathcal{M}$  and let  $(\mathcal{M}_\infty, g_\infty(t), l^\infty(t))$  be the rescaling limit flow around  $(p, T)$ . It is a normalized gradient shrinking soliton in canonical form with constant formal reduced volume

$$\theta_{p,T} = \int_{\mathcal{M}_\infty} (4\pi(T-t))^{-\frac{n}{2}} e^{-l^\infty(t)} d\text{vol}_{g_\infty(t)}$$

for any  $t$ . Note that in the proof of Theorem 2.6 we can use  $l_{p,T}$  instead of  $l_{p,T,\{t_{k_l}\}}$  as it only requires the estimates from Lemma 2.5 as well as the formal equalities on a constant reduced volume. Now we can employ [23, Corollary 1.1 (3)] to conclude that there exists  $\eta > 0$  (only depending on  $n$ ) such that if  $\theta_{p,T} > 1 - \eta$ , then the limit flow is the Gaussian soliton. In particular,  $\theta_{p,T} > 1 - \eta$  implies  $p \in \mathcal{M} \setminus \Sigma$  by Theorems 1.4 and 1.8.  $\square$

## References

- [1] S. Angenent and D. Knopf, *An example of neckpinching for Ricci flow on  $S^{n+1}$* . Math. Res. Lett. **11** (2004), no. 4, 493–518.
- [2] S. Angenent and D. Knopf, *Precise asymptotics of the Ricci flow neckpinch*. Comm. Anal. Geom. **15** (2007), no. 4, 773–844.

- [3] H.-D. Cao, *Recent progress on Ricci solitons*. ArXiv:0908.2006v1 (2009).
- [4] H.-D. Cao, R. S. Hamilton and T. Ilmanen, *Gaussian densities and stability for some Ricci solitons*. ArXiv:math/0404165v1 (2004).
- [5] X. Cao and Q. S. Zhang, *The conjugate heat equation and ancient solutions of the Ricci flow*. Arxiv:1006.0540v1 (2010).
- [6] P. Daskalopoulos and R. Hamilton, *Geometric estimates for the logarithmic fast diffusion equation*. Comm. Anal. Geom. **12** (2004), no. 1-2, 143–164.
- [7] J. Enders, *Generalizations of the reduced distance in the Ricci flow - monotonicity and applications*. Ph.D. thesis, Michigan State University (2008), <http://gradworks.umi.com/33/31/3331902.html>.
- [8] A. Friedman. *A strong maximum principle for weakly subparabolic functions*. Pacific J. Math. **11** (1961), 175–184.
- [9] H.-L. Gu and X.-P. Zhu, *The existence of type II singularities for the Ricci flow on  $S^{n+1}$* . Comm. Anal. Geom. **18** (2008), no. 3, 467–494.
- [10] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*. J. Differential Geom. **17** (1982) 255–306.
- [11] R. S. Hamilton, *The formation of singularities in the Ricci flow*. Surveys in differential geometry, Vol. II (Cambridge, MA, 1993) 7–136, Internat. Press, Cambridge, MA, 1995.
- [12] R. S. Hamilton, *A compactness property for solutions of the Ricci flow*. Amer. J. Math. **117** (1995), 545–572.
- [13] N. Q. Le and N. Sesum, *The mean curvature at the first singular time of the mean curvature flow*. ArXiv:1001.3682v1 (2010).
- [14] N. Q. Le and N. Sesum, *Remarks on curvature behavior at the first singular time of the Ricci flow*. ArXiv:1005.1220v1 (2010).
- [15] A. Naber, *Noncompact shrinking four solitons with nonnegative curvature*. J. Reine Angew. Math. **645** (2010), 125–153.
- [16] L. Ni, *Mean value theorem on manifolds*. Asian J. Math. **11** (2007), 277–304.
- [17] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*. ArXiv:math/0211159v1 (2002).
- [18] S. Pigola, M. Rimoldi and A. Setti, *Remarks on non-compact gradient Ricci solitons*. ArXiv:0905.2868v3 (2009).
- [19] N. Sesum, *Convergence of the Ricci flow toward a soliton*. Comm. Anal. Geom. **14** (2006), no. 2, 283–343.
- [20] W.-X. Shi, *Deforming the metric on complete Riemannian manifolds*. J. Diff. Geom. **30** (1989), 223–301.
- [21] A. Stone, *A density function and the structure of singularities of the mean curvature flow*. Calc. Var. Partial Diff. Equations **2** (1994), no. 4, 443–480.
- [22] B. White, *A local regularity theorem for mean curvature flow*. Ann. of Math. (2) **161** (2005), no. 3, 1487–1519.
- [23] T. Yokota, *Perelman’s reduced volume and a gap theorem for the Ricci flow*. Comm. Anal. Geom. **17** (2009), 227–263.
- [24] Z.-H. Zhang, *On the completeness of gradient Ricci solitons*. Proc. Amer. Math. Soc. **137** (2009), no. 8, 2755–2759.

Joerg Enders

INSTITUT FÜR MATHEMATIK, UNIVERSITÄT POTSDAM, 14469 POTSDAM, GERMANY

Reto Müller

DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE LONDON, LONDON, SW7 2AZ, UK

Peter M. Topping

MATHEMATICS INSTITUTE, UNIVERSITY OF WARWICK, COVENTRY, CV4 7AL, UK