On sequences of maps into a manifold with equibounded $\mathbf{W^{1/2}}$-energies

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We characterize weak limits of sequences of smooth functions from B^n into suitable manifolds $\mathcal{Y}$ with equibounded $W^{1/2}$-energies, the relaxed $W^{1/2}$-energy and we prove strong density of smooth maps. We then obtain the weak sequential density of smooth maps in $W^{1/2}(B^n,\mathcal{Y})$ and a criterion for strong density of smooth maps.

Let B^n be the unit ball in $\mathbb{R}^n$ and let $\mathcal{Y}$ be a smooth oriented Riemannian manifold of dimension $M\geq 1$, isometrically embedded in $\mathbb{R}^N$ for some $N\geq 2$. We shall assume that $\mathcal{Y}$ is compact, connected, without boundary and, moreover, that its integral 1-homology group $H_1(\mathcal{Y}):=H_1(\mathcal{Y},\mathbb{Z})$ has no torsion, so that $H_1(\mathcal{Y},X)=H_1(\mathcal{Y})\otimes X$ for $X=\mathbb{R},\mathbb{Q}$. Motivated by the recent papers [3] [4] [5] [6] [15] [18] [22] we shall characterize weak limits of sequences of graphs of smooth functions from B^n into $\mathcal{Y}$ with equibounded $W^{1/2}$-energies, $\operatorname{cart}^{1/2}(B^n\times\mathcal{Y})$, and deal with the corresponding relaxed $W^{1/2}$-energy. A key point will be a strong density result of smooth maps in $\operatorname{cart}^{1/2}(B^n\times\mathcal{Y})$.

Definition 0.1 *We say that an integer multiplicity (say i.m.)* 1*-cycle* $C\in\mathcal{Z}_1(\mathcal{Y})$ *is an* integral flat cycle *if there exists an i.m. rectifiable current* $R\in\mathcal{R}_2(\mathbb{R}^N)$ *such that* $\partial R=C$.

We recall that for every $m=0,\dots,N$ the class of *integral flat chains* in $\mathbb{R}^N$ is defined by

$$\mathcal{F}_m(\mathbb{R}^N):=\{R+\partial S\mid R\in\mathcal{R}_m(\mathbb{R}^N)\,,\ S\in\mathcal{R}_{m+1}(\mathbb{R}^N)\}\,.$$

Therefore, an integral flat cycle C in $\mathcal{Y}$ is an element of $\mathcal{F}_1(\mathbb{R}^N)$ such that $\partial C=0$ and $\operatorname{spt}C\subset\mathcal{Y}$.

To our purposes, we would like to consider homology classes defined not only by means of merely integral rectifiable cycles (i.e., $C\in\mathcal{R}_1(\mathcal{Y})$ with $\partial C=0$), but for which the representatives belong to the wider class of integral flat cycles. This can be done as follows, compare [8] and [9].

Relative integral homology. If A, B are two locally Lipschitz neighborhood retracts in $\mathbb{R}^N$, with $B\subset A$ and B relatively closed in A, then the *relative integral cycles* are defined by

$$Z_m(A,B,\mathbb{Z}):=\{T\in\mathcal{F}_m(\mathbb{R}^N)\mid \operatorname{spt}T\subset A\,,\ \operatorname{spt}\partial T\subset B\}\,,$$

the *relative integral boundaries* by

$$B_m(A,B,\mathbb{Z}):=\{Q+\partial W\mid Q\in\mathcal{F}_m(\mathbb{R}^N)\,,\ \operatorname{spt}Q\subset B\,,\ W\in\mathcal{F}_{m+1}(\mathbb{R}^N)\,,\ \operatorname{spt}W\subset A\}$$

and the *relative integral homology* by

$$H_m(A,B,\mathbb{Z}):=Z_m(A,B,\mathbb{Z})/B_m(A,B,\mathbb{Z})\,.$$

Definition 0.1 has to do with the relative integral homology in the case $A=\mathcal{Y}$ and $B=\partial\mathcal{Y}=\emptyset$. In fact we easily infer that

$$Z_1(\mathcal{Y},\emptyset,\mathbb{Z})=\{C\in\mathcal{Z}_1(\mathcal{Y})\mid C\text{ is an integral flat 1-cycle}\}$$

whereas, taking into account Federer's flatness theorem,

$$B_1(\mathcal{Y},\emptyset,\mathbb{Z})=\{\partial W\mid W\in\mathcal{R}_2(\mathcal{Y})\}\,.$$

Therefore, an element q in $H_1(\mathcal{Y},\emptyset,\mathbb{Z})$ is an equivalence class of integral flat 1-cycles of $\mathcal{Y}$, see Definition 0.1, where

$$C\sim Z\qquad\Longleftrightarrow\qquad \exists\,W\in\mathcal{R}_2(\mathcal{Y})\,:\,C-Z=\partial W\,.$$

Remark 0.2 In these circumstances, see e.g. [8], [9] and [14, Vol. I, Sec. 5.4.1], in each homology class γ in $H_1(\mathcal{Y},\emptyset,\mathbb{Z})$ there exists a homological mass minimizer, i.e., an integral cycle $\widetilde{C}\in\mathcal{Z}_1(\mathcal{Y})$ with finite mass such that

$$\mathbf{M}(\widetilde{C})=\inf\{\mathbf{M}(C)\mid C\in Z_1(\mathcal{Y},\emptyset,\mathbb{Z})\,,\ [C]=\gamma\}<+\infty\,.$$

Moreover we actually have

$$H_1(\mathcal{Y},\emptyset,\mathbb{Z})\simeq H_1(\mathcal{Y})\,.$$

Finally, there are generators $[\gamma_1],\dots,[\gamma_{\overline{s}}]$, i.e. integral cycles (with finite mass) in $\mathcal{Z}_1(\mathcal{Y})$, such that

$$H_1(\mathcal{Y})=\left\{\sum_{s=1}^{\overline{s}}n_s\,[\gamma_s]\mid n_s\in\mathbb{Z}\right\}.$$

By the De Rham theorem the first real homology group is in duality with the first cohomology group $H^1_{dR}(\mathcal{Y})$, the duality being given by the natural pairing

$$<[\gamma],[\omega]>:=\gamma(\omega)=\int_\gamma\omega\,,\qquad [\gamma]_{\mathbb{R}}\in H_1(\mathcal{Y},\mathbb{R})\,,\quad [\omega]\in H^1_{dR}(\mathcal{Y})\,.$$

We will then denote by $[\omega^1],\dots,[\omega^{\overline{s}}]$ a dual basis in $H^1_{dR}(\mathcal{Y})$ so that, δ_{sr} being the Kronecker symbols,

$$\gamma_s(\omega^r)=\delta_{sr}\,. \tag{0.1}$$

Also, we may and do assume that ω^s is the harmonic form in its cohomology class.

Definition 0.3 *We say that an integral flat* 1*-cycle* $C\in\mathcal{Z}_1(\mathcal{Y})$ *is of* S^1-type *if its homology class in* $\mathcal{Y}$ *contains a Lipschitz image of the unit circle* S^1*; more precisely, if there exists* $Z\in\mathcal{Z}_1(\mathcal{Y})$, $W\in\mathcal{R}_2(\mathcal{Y})$ *and a Lipschitz function* $\phi:S^1\to\mathcal{Y}$ *such that*

$$C-Z=\partial W\qquad\textit{and}\qquad Z=\phi_\#[\![\,S^1\,]\!]\,.$$

Denoting then

$$H_1^{sph}(\mathcal{Y}):=\{[\gamma]\in H_1(\mathcal{Y})\mid \text{there exists }\ \phi\in\mathrm{Lip}(S^1,\mathcal{Y})\ \text{ such that }\ \phi_\#[\![\,S^1\,]\!]\in[\gamma]\}\,, \tag{0.2}$$

we also may and do choose the γ_i's in such a way that $[\gamma_1],\dots,[\gamma_{\widetilde{s}}]$ generate the S^1-type homology classes in $H_1^{sph}(\mathcal{Y})$ for some $\widetilde{s}\le\overline{s}$.

The class $W^{1/2}(B^n)$. We recall, see e.g. [1], that the fractional Sobolev space $W^{1/2}(B^n)$ is the Hilbert space of real valued functions u which have finite $W^{1/2}$-seminorm

$$|u|^2_{1/2,B^n}:=\int_{B^n}\int_{B^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+1}}\,dx\,dy \tag{0.3}$$

endowed with the norm

$$\|u\|^2_{1/2,B^n}:=\|u\|^2_{L^2(B^n)}+|u|^2_{1/2,B^n}\,.$$

Moreover $W^{1/2}(B^n,\mathbb{R}^N)$ is the space of vector valued functions $u=(u^1,\dots,u^N)$ such that $u^j\in W^{1/2}(B^n)$ for every $j=1,\dots,N$. Also, let

$$W^{1/2}(B^n,\mathcal{Y}):=\{u\in W^{1/2}(B^n,\mathbb{R}^N)\mid u(x)\in\mathcal{Y}\quad\text{for a.e. }x\in B^n\}\,.$$

Results. In this paper we consider sequences of smooth maps $u_k:B^n\to\mathcal{Y}$ with equibounded $W^{1/2}$-energy

$$\sup_k\|u_k\|_{1/2,B^n}<+\infty\,.$$

Modulo passing to a subsequence, the $(n,1)$-currents G_{u_k}, integration over the graphs of u_k of n-forms with at most one vertical differential, converge to a current $T\in\mathrm{cart}^{1/2}(B^n\times\mathcal{Y})$, see Sec. 2 below. Those currents will be shown to have the form

$$T=G_{u_T}+\sum_{q\in H_1^{sph}(\mathcal{Y})}\mathbb{L}_q\times C_q\,,$$

where u_T is the weak $W^{1/2}$-limit of the u_k's, $\mathbb{L}_q$ is an integer multiplicity rectifiable current of dimension $n-1$ in B^n and C_q is a 1-cycle in q.

In the sequel, instead of working with the $W^{1/2}$-energy, given by (0.3), we shall work with the equivalent energy $\mathcal{E}_{1/2}(u)$ defined as follows. For $u \in W^{1/2}(B^n, \mathcal{Y})$ we define the *extension* of u

$$U := \mathrm{Ext}(u) \in W^{1,2}(\mathcal{C}^{n+1}, \mathbb{R}^N)\,,$$

where $\mathcal{C}^{n+1}$ is the cylinder $\mathcal{C}^{n+1} := B^n \times I$, $I := [0,1]$, as the harmonic function U which minimizes the *Dirichlet integral*

$$\mathbf{D}(U) := \frac{1}{2}\int_{\mathcal{C}^{n+1}} |DU(x,t)|^2\, dx\, dt$$

among all functions that agree with u on $B^n \times \{0\}$, and we set

$$\mathcal{E}_{1/2}(u) := \mathbf{D}(\mathrm{Ext}(u))\,.$$

For every $T \in \mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$ the $W^{1/2}$-energy of T, $\mathcal{E}_{1/2}(T)$, turns out to be well defined, see Sec. 2, and agrees with $\mathcal{E}_{1/2}(u)$ if $T = G_u$ with $u \in W^{1/2}(B^n, \mathcal{Y})$.

We first prove, if $n = 1$, that every $T \in \mathrm{cart}^{1/2}(B^1 \times \mathcal{Y})$ is the weak limit of a sequence of graphs of smooth maps $u_k : B^1 \to \mathcal{Y}$, i.e., $G_{u_k} \rightharpoonup T$; moreover

$$\mathcal{E}_{1/2}(T) = \inf\{\liminf_{k\to+\infty} \mathcal{E}_{1/2}(u_k) \mid \{u_k\} \subset \mathrm{Lip}(B^1, \mathcal{Y})\,, \quad G_{u_k} \rightharpoonup T\}$$

and actually there is a sequence of smooth maps $u_k : B^1 \to \mathcal{Y}$ such that

$$G_{u_k} \rightharpoonup T \qquad \text{and} \qquad \mathcal{E}_{1/2}(u_k) \to \mathcal{E}_{1/2}(T)\,.$$

Assume now that the first homotopy group $\pi_1(\mathcal{Y})$ is commutative, so that in particular it is isomorphic to the first homology group

$$\pi_1(\mathcal{Y}) \cong H_1(\mathcal{Y})\,.$$

We then extend the previous density result to any dimension $n \geq 2$.

As a consequence we infer the weak sequential density of smooth maps in $W^{1/2}(B^n, \mathcal{Y})$ and a criterion for strong density of smooth maps, too.

We also show that the commutativity hypothesis of the first homotopy group $\pi_1(\mathcal{Y})$ cannot be avoided. If the first homotopy group is not commutative, in fact, we exhibit a current T in $\mathrm{cart}^{1/2}(B^2 \times \mathcal{Y})$ of the type $T = G_u$ for which there is a positive constant C such that

$$\mathcal{E}_{1/2}(u) + C \leq \liminf_{k\to+\infty} \mathcal{E}_{1/2}(u_k)$$

for any sequence of smooth maps u_k with $G_{u_k} \rightharpoonup G_u$.

1 Graphs of maps with finite $\mathbf{W^{1/2}}$-energy

In this section we deal with properties of currents carried by graphs of $W^{1/2}$-maps.

$\mathcal{D}_{k,p}$-currents. For $k = 1, \dots, n$, every differential k-form $\omega \in \mathcal{D}^k(B^n \times \mathcal{Y})$ splits as a sum $\omega = \sum_{j=0}^{\underline{j}} \omega^{(j)}$, $\underline{j} := \min(k, M)$, where the $\omega^{(j)}$'s are the k-forms that contain exactly j differentials in the vertical $\mathcal{Y}$ variables. For fixed $p = 1, \dots, \underline{j}$ we denote by $\mathcal{D}^{k,p}(B^n \times \mathcal{Y})$ the subspace of $\mathcal{D}^k(B^n \times \mathcal{Y})$ of k-forms of the type $\omega = \sum_{j=0}^{p} \omega^{(j)}$, and by $\mathcal{D}_{k,p}(B^n \times \mathcal{Y})$ the dual space of $\mathcal{D}^{k,p}(B^n \times \mathcal{Y})$. Of course we have $\mathcal{D}_{k,k} = \mathcal{D}_k$. For example, if $u \in W^{1,2}(B^n, \mathbb{R}^N)$, then G_u is an $(n,2)$-current in $\mathcal{D}_{n,2}(B^n \times \mathbb{R}^N)$, where in an approximate sense $G_u := (Id \bowtie u)_\# [\![B^n]\!]$, $(Id \bowtie u)(x) := (x, u(x))$, compare [14].

Weak $\mathcal{D}_{k,p}$-convergence. If $\{T_k\} \subset \mathcal{D}_{k,p}(B^n \times \mathcal{Y})$, we say that $\{T_k\}$ converges weakly in $\mathcal{D}_{k,p}(B^n \times \mathcal{Y})$, $T_k \rightharpoonup T$, if $T_k(\omega) \to T(\omega)$ for every $\omega \in \mathcal{D}^{k,p}(B^n \times \mathcal{Y})$. The class $\mathcal{D}_{k,p}(B^n \times \mathcal{Y})$ is closed under weak convergence in $\mathcal{D}_{k,p}$.

Boundaries. The exterior differential d splits into a horizontal and a vertical differential $d = d_x + d_y$. Of course $\partial_x T(\omega) := T(d_x\omega)$ defines a horizontal boundary operator $\partial_x : \mathcal{D}_{k,p}(B^n \times \mathcal{Y}) \to \mathcal{D}_{k-1,p}(B^n \times \mathcal{Y})$ and for any $\omega \in \mathcal{D}^{k-1,p}(B^n \times \mathcal{Y})$, $d_y\omega$ belongs to $\mathcal{D}^{k,p}(B^n \times \mathcal{Y})$ if and only if $d_y\omega^{(p)} = 0$. Therefore, for every $T \in \mathcal{D}_{k,p}(B^n \times \mathcal{Y})$, the vertical boundary operator $\partial_y T$ makes sense only as an element of the dual space of $\mathcal{Z}^{k-1,p}(B^n \times \mathcal{Y})$, where

$$\mathcal{Z}^{k-1,p}(B^n \times \mathcal{Y}) := \{\omega \in \mathcal{D}^{k-1,p}(B^n \times \mathcal{Y}) \mid d_y\omega^{(p)} = 0\}\,.$$

Extension of $\mathbf{W^{1/2}}$-maps. For any $u \in W^{1/2}(B^n, \mathcal{Y})$ we define the *extension* of u

$$U := \mathrm{Ext}(u) \in W^{1,2}(\mathcal{C}^{n+1}, \mathbb{R}^N)\,,$$

where $\mathcal{C}^{n+1}$ is the cylinder

$$\mathcal{C}^{n+1} := B^n \times I\,, \qquad I := [0,1]\,,$$

as the harmonic function U which minimizes the *Dirichlet integral*

$$\mathbf{D}(U) := \frac{1}{2}\int_{\mathcal{C}^{n+1}} |DU(x,t)|^2\, dx\, dt$$

among all functions that agree with u on $B^n \times \{0\}$. It is well-known that

$$\mathbf{D}(U) \simeq |u|_{1/2,B^n}\,.$$

We will also write

$$\mathbf{T}(U) = u$$

if $u \in W^{1/2}(B^n, \mathbb{R}^N)$ is the trace of a function $U \in W^{1,2}(\mathcal{C}^{n+1}, \mathbb{R}^N)$ on $B^n \times \{0\}$.

Graphs of $\mathbf{W^{1/2}}$ maps. To any map $u \in W^{1/2} \cap L^\infty(B^n, \mathbb{R}^N)$ we can associate an $(n,1)$-current G_u in $\mathcal{D}_{n,1}(B^n \times \mathbb{R}^N)$, compare [15]. For this we recall the following facts:

(i) $W^{1/2} \cap L^\infty(B^n)$ is an algebra, since trivially

$$|uv|_{1/2} \le \|u\|_\infty |v|_{1/2} + \|v\|_\infty |u|_{1/2}\,;$$

(ii) if $u \in W^{1/2}(B^n)$, then $D_i u$ belongs to the dual space $W^{-1/2}(B^n)$ of $W^{1/2}(B^n)$ for every $i = 1, \dots, n$ and

$$|< Du, v >| \le c\, |u|_{1/2} |v|_{1/2} \qquad \forall\, u, v \in W^{1/2}(B^n)\,;$$

(iii) if $u \in W^{1/2}(B^n)$ and $v \in W^{1/2} \cap L^\infty(B^n)$, then $v\, Du$ defines a distribution in B^n, indeed a linear continuous functional on $W^{1/2} \cap L^\infty$ by

$$< v\, Du, \varphi > := < Du, v\,\varphi >\,, \qquad \varphi \in W^{1/2} \cap L^\infty(B^n)\,;$$

in fact

$$|< v\, Du, \varphi >| \le c\, |u|_{1/2}\Big(\|v\|_\infty |\varphi|_{1/2} + \|\varphi\|_\infty |v|_{1/2}\Big)\,.$$

Every $(n,1)$-form ω in $\mathcal{D}^{n,1}(B^n \times \mathbb{R}^N)$ can be written as

$$\omega := \omega_0(x,y)\, dx + \sum_{i=1}^{n}\sum_{j=1}^{N} \omega_{i,j}(x,y)\, \widehat{dx_i} \wedge dy^j$$

where x and y are the variables in $\mathbb{R}^n$ and $\mathbb{R}^N$, respectively, ω_0 and $\omega_{i,j}$ are functions in $C^\infty_c(B^n \times \mathbb{R}^N)$, $dx := dx^1 \wedge \cdots \wedge dx^n$ and $\widehat{dx_i} := dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n$. If $u \in W^{1/2} \cap L^\infty(B^n, \mathbb{R}^N)$, then it is easily seen that also $\varphi(x,u(x))$ belongs to $W^{1/2}\cap L^\infty(B^n)$ for every $\varphi\in C^\infty_c(B^n\times\mathbb{R}^N)$ and hence, since $Du\in W^{-1/2}$, we can define the *graph current* associated to u in a distributional sense as

$$G_u(\omega):=\int_{B^n}\omega_0(x,u(x))\,dx+\sum_{i=1}^n\sum_{j=1}^N(-1)^{n-i}<\omega_{i,j}(x,u(x)),\,D_iu^j(x)>,$$

compare [15]. Notice that, if u is smooth, we have $G_u=(\mathrm{Id}\bowtie u)_\#[\![B^n]\!]$, i.e.

$$G_u(\omega)=\int_{B^n}(\mathrm{Id}\bowtie u)^\#\omega\,,\qquad \omega\in\mathcal{D}^{n,1}(B^n\times\mathbb{R}^N)\,.$$

Every $u\in W^{1/2}(B^n,\mathcal{Y})$ is in L^∞, since $\mathcal{Y}$ is compact, hence G_u is well defined as an $(n,1)$-current in $\mathcal{D}_{n,1}(B^n\times\mathcal{Y})$. However, even in dimension $n=1$ in general G_u is not an i.m. rectifiable current in $B^n\times\mathcal{Y}$. Moreover, if u_k converges to u strongly in $W^{1/2}$, then G_{u_k} converges to G_u weakly in the sense of currents in $\mathcal{D}_{n,1}$. Finally, since $\mathrm{Ext}(u_k)\to U:=\mathrm{Ext}(u)$ strongly in $W^{1,2}(\mathcal{C}^{n+1},\mathbb{R}^N)$, as in [15] we obtain that

$$(-1)^{n-1}\partial G_U=G_u\qquad\text{on}\quad\mathcal{D}^{n,1}(B^n\times\{0\}\times\mathcal{Y})\,.\tag{1.1}$$

Boundary data. In the sequel we will denote by $\widetilde{B}^n$ a bounded domain in $\mathbb{R}^n$ such that $B^n\subset\subset\widetilde{B}^n$ and we let $\varphi:\widetilde{B}^n\to\mathcal{Y}$ be a given $W^{1/2}$-function, which will always be assumed to be smooth on $\widetilde{B}^n$, and we will denote

$$\begin{aligned}W^{1/2}_\varphi(\widetilde{B}^n,\mathcal{Y})&:=\{u\in W^{1/2}(\widetilde{B}^n,\mathcal{Y})\mid u=\varphi\quad\text{on}\ \widetilde{B}^n\setminus\overline{B}^n\}\\ C^\infty_\varphi(\widetilde{B}^n,\mathcal{Y})&:=\{u\in C^\infty(\widetilde{B}^n,\mathcal{Y})\mid u=\varphi\quad\text{on}\ \widetilde{B}^n\setminus\overline{B}^n\}\,.\end{aligned}$$

Density results for $\mathbf{W^{1/2}}$-maps. If $n\geq 2$, let $R^\infty_{1/2}(B^n,\mathcal{Y})$, respectively $R^0_{1/2}(B^n,\mathcal{Y})$, be the set of all maps $u\in W^{1/2}(B^n,\mathcal{Y})$ which are smooth, respectively continuous, except on a singular set $\Sigma(u)$ of the type

$$\Sigma(u)=\bigcup_{i=1}^r\Sigma_i\,,\qquad r\in\mathbb{N}\,,$$

where Σ_i is a smooth $(n-2)$-dimensional subset of B^n with smooth boundary, if $n\geq 3$, and Σ_i is a point if $n=2$. Moreover, let

$$R^\infty_{1/2,\varphi}(\widetilde{B}^n,\mathcal{Y}):=\{u\in R^\infty_{1/2}(\widetilde{B}^n,\mathcal{Y})\mid u=\varphi\quad\text{on}\ \widetilde{B}^n\setminus\overline{B}^n\}\,.$$

It is well-known that if $n=1$ maps in $C^1(B^1,\mathcal{Y})$ are dense in $W^{1/2}(B^1,\mathcal{Y})$, compare e.g. [3]. The following density result was proved in [18].

Theorem 1.1 *For every $n\geq 2$ the class $R^\infty_{1/2,\varphi}(\widetilde{B}^n,\mathcal{Y})$ is dense in $W^{1/2}_\varphi(\widetilde{B}^n,\mathcal{Y})$, and $R^\infty_{1/2}(B^n,\mathcal{Y})$ is dense in $W^{1/2}(B^n,\mathcal{Y})$. Moreover, the class $C^\infty_\varphi(\widetilde{B}^n,\mathcal{Y})$ is dense in $W^{1/2}_\varphi(\widetilde{B}^n,\mathcal{Y})$ provided that $\pi_1(\mathcal{Y})=0$.*

Structure of graphs of $\mathbf{W^{1/2}}$-maps. Let $u\in W^{1/2}(B^n,\mathcal{Y})$. We have $\partial G_u(\omega)=0$ if $\omega\in\mathcal{D}^{n-1,1}(B^n\times\mathcal{Y})$ with $\omega^{(1)}=0$ or $d_y\omega=0$. Setting then

$$\mathcal{B}^{k,p}(B^n\times\mathcal{Y}):=\{\omega\in\mathcal{D}^{k,p}(B^n\times\mathcal{Y})\mid\exists\eta\in\mathcal{D}^{k-1,p-1}(B^n\times\mathcal{Y})\ :\ \omega=d_y\eta\}$$

and

$$\mathcal{H}^{k,p}(B^n\times\mathcal{Y}):=\frac{\mathcal{Z}^{k,p}(B^n\times\mathcal{Y})}{\mathcal{B}^{k,p}(B^n\times\mathcal{Y})}\,,$$

we have $\partial G_u=0$ on $\mathcal{B}^{n-1,1}(B^n\times\mathcal{Y})$ and $\partial_y\partial G_u=0$, whence $\partial G_u(\omega)$ depends only on the cohomology class of $\omega\in\mathcal{Z}^{n-1,1}(B^n\times\mathcal{Y})$. As a consequence ∂G_u induces a functional $(\partial G_u)_\star$ on $\mathcal{H}^{n-1,1}(B^n\times\mathcal{Y})$ given by

$$(\partial G_u)_\star(\omega+\mathcal{B}^{n-1,1}):=\partial G_u(\omega+\mathcal{B}^{n-1,1})=\partial G_u(\omega)\,,\qquad\omega\in\mathcal{Z}^{n-1,1}\,,$$

compare [14, Vol. II, Sec. 5.4.1]. Since

$$\mathcal{H}^{k,p}(B^n\times\mathcal{Y})\simeq\mathcal{D}^{k-p}(B^n)\otimes H^p_{dR}(\mathcal{Y})\,,$$

taking $k = n-1$ and $p=1$ we infer that the homology map $(\partial G_u)_\star$ is uniquely represented as an element of $\mathcal{D}_{n-2}(B^n; H_1(\mathcal{Y},\mathbb{R}))$. More explicitly, if $\phi \in \mathcal{D}^{n-2}(B^n)$, we have $[(\partial G_u)_\star(\phi)] \in H_1(\mathcal{Y},\mathbb{R})$ and for $s = 1,\dots,\overline{s}$

$$< (\partial G_u)_\star(\phi), [\omega^s] >= \partial G_u(\pi^\# \phi \wedge \widehat{\pi}^\# \omega^s)\,,$$

$<,>$ denoting the De Rham duality between $H_1(\mathcal{Y},\mathbb{R})$ and $H^1_{dR}(\mathcal{Y})$.

Interior multiplication. The *interior multiplication of p-vectors by q-covectors*, $0 \le q \le p \le n+N$, is defined as the bilinear form $\llcorner : \Lambda_p\mathbb{R}^{n+N} \times \Lambda^q\mathbb{R}^{n+N} \to \Lambda_{p-q}\mathbb{R}^{n+N}$ given for $\xi \in \Lambda_p\mathbb{R}^{n+N}$ and $\phi \in \Lambda^q\mathbb{R}^{n+N}$ by

$$< \xi \llcorner \phi, \omega >:=< \xi, \phi \wedge \omega > \qquad \forall\, \omega \in \Lambda^{p-q}\mathbb{R}^{n+N}.$$

The *interior multiplication of p-currents by q-forms* is defined for every current $T \in \mathcal{D}_p(\mathcal{U})$, where $\mathcal{U}$ is open in $\mathbb{R}^{n+N}$, and every smooth form $\phi \in \mathcal{D}^q(\mathcal{U})$ by the $(p-q)$-current $T \llcorner \phi \in \mathcal{D}_{p-q}(\mathcal{U})$ given by

$$(T \llcorner \phi)(\omega) := T(\phi \wedge \omega) \qquad \forall\, \omega \in \mathcal{D}^{p-q}(\mathcal{U})\,.$$

The currents $\mathbb{P}(u)$ and $\mathbb{D}(u)$. We shall denote by $\pi : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^n$ and $\widehat{\pi} : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^N$ the orthogonal projection onto the first and the second factor, respectively. Following [14, Vol. II, Sec. 5.4.2], we now set

$$\mathbb{P}(u) := (-1)^{n-1}(\partial G_u)_\star \in \mathcal{D}_{n-2}(B^n; H_1(\mathcal{Y};\mathbb{R}))$$

and, for each $\omega \in [\omega] \in H^1_{dR}(\mathcal{Y})$, we define the current $\mathbb{P}(u;\omega) \in \mathcal{D}_{n-2}(B^n)$ by $\mathbb{P}(u;\omega) := \pi_\#(\partial G_u) \llcorner \widehat{\pi}^\# \omega$, so that

$$\mathbb{P}(u;\omega)(\phi) = (-1)^{n-1}\partial G_u(\pi^\# \phi \wedge \widehat{\pi}^\# \omega) \qquad \forall\, \phi \in \mathcal{D}^{n-2}(B^n)\,.$$

Also, for every $\omega \in \mathcal{Z}^1(\mathcal{Y})$ we define the current $\mathbb{D}(u;\omega) \in \mathcal{D}_{n-1}(B^n)$ as follows. We choose a smooth extension $\widetilde{\omega} \in \mathcal{D}^1(\mathbb{R}^N)$ of ω and a function $\eta \in C^\infty([0,1],[0,1])$ with $\eta \equiv 1$ and $\eta \equiv 0$ respectively in a neighborhood of 0 and 1 and we let $U = \mathrm{Ext}(u)$. Finally for every $\phi \in \mathcal{D}^{n-1}(B^n)$ we let $\widetilde{\phi} \in \mathcal{D}^{n-1}(\mathcal{C}^{n+1})$ be given by $\widetilde{\phi}(x,t) := \phi(x)\,\eta(t)$. We now define

$$\mathbb{D}(u;\omega)(\phi) := \pi_\#(G_U \llcorner \widehat{\pi}^\# d\widetilde{\omega})(\widetilde{\phi}) = G_U(\pi^\# \widetilde{\phi} \wedge \widehat{\pi}^\# d\widetilde{\omega})\,, \qquad \phi \in \mathcal{D}^{n-1}(B^n)\,.$$

Proposition 1.2 *The following facts hold true:*

(i) *for* $s = 1,\dots,\overline{s}$

$$\mathbb{P}(u;\omega^s)(\phi) =< \mathbb{P}(u)(\phi), [\omega^s] >\,,$$

i.e., $\mathbb{P}(u;\omega^s)$ *does not depend on the representative in the cohomology class* $[\omega^s]$*;*

(ii) $\partial\,\mathbb{P}(u) = 0$ *and* $\mathbb{P}(u) = \displaystyle\sum_{s=1}^{\overline{s}} \mathbb{P}(u;\omega^s) \otimes [\gamma_s]$*, hence it does not depend on the choice of* $\gamma_1,\dots,\gamma_{\overline{s}}$*;*

(iii) $(-1)^{n-1}\partial\,\mathbb{D}(u;\omega^s) = \mathbb{P}(u;\omega^s)$ *for each representative* ω^s *in* $[\omega^s]$.

PROOF: Properties (i) and (ii) are proved as in [14, Vol. II, Sec. 5.4.2]. To prove (iii), we observe that by (1.1) we have

$$\begin{aligned}\partial G_U(\pi^\# d\widetilde{\phi} \wedge \widehat{\pi}^\# \widetilde{\omega}) &= (-1)^{n-1} G_u(\pi^\#(d_x\widetilde{\phi} + d_t\widetilde{\phi})_{|t=0} \wedge \widehat{\pi}^\# \omega) \\ &= (-1)^{n-1} G_u(\pi^\# d\phi \wedge \widehat{\pi}^\# \omega)\end{aligned}$$

so that, since $d\omega^s = 0$, we compute for every $\phi \in \mathcal{D}^{n-2}(B^n)$

$$\begin{aligned}(-1)^{n-1}\,\mathbb{P}(u;\omega^s)(\phi) &= \partial G_u(\pi^\# \phi \wedge \widehat{\pi}^\# \omega^s) \\ &= G_u(\pi^\# d\phi \wedge \widehat{\pi}^\# \omega^s) = (-1)^{n-1}\partial G_U(\pi^\# d\widetilde{\phi} \wedge \widehat{\pi}^\# \widetilde{\omega}^s) \\ &= G_U(\pi^\# d\widetilde{\phi} \wedge \widehat{\pi}^\# d\widetilde{\omega}^s) = (G_U \llcorner \widehat{\pi}^\# d\widetilde{\omega}^s)(\pi^\# d\widetilde{\phi}) \\ &= \partial \pi_\#(G_U \llcorner \widehat{\pi}^\# d\widetilde{\omega}^s)(\widetilde{\phi}) = \partial\,\mathbb{D}(u;\omega^s)(\phi)\,.\end{aligned}$$

□

As a consequence we set

$$\mathbb{D}_s(u) := \mathbb{D}(u;\omega^s)\,, \qquad \mathbb{P}_s(u) := \mathbb{P}(u;\omega^s)\,, \qquad s=1,\dots,\overline{s}\,,$$

so that

$$\mathbb{P}_s(u) = (-1)^{n-1}\partial\,\mathbb{D}_s(u) \qquad \text{in} \quad \mathcal{D}_{n-2}(B^n)\,.$$

We remark that $\mathbb{D}_s(u)$ is a current of finite mass in $\mathcal{D}_{n-1}(B^n)$ since $\mathrm{Ext}(u)$ is a $W^{1,2}$ function and $d\widetilde{\omega} \in \mathcal{D}^2(\mathbb{R}^N)$. Also, clearly $\mathbb{P}(u) = 0$ if u is smooth, say Lipschitz. Moreover, taking into account Theorem 1.1, exactly as in [14, Vol. II, Sec. 5.4.2], where we take $p=1$, we obtain the following

Proposition 1.3 *Let $u \in W^{1/2}_\varphi(\widetilde{B}^n, \mathcal{Y})$ and let $\{u_k\} \subset R^\infty_{1/2,\varphi}(\widetilde{B}^n, \mathcal{Y})$ converge strongly in $W^{1/2}$ to u. Then $\mathbb{P}(u)$ is an $(n-2)$-dimensional real flat chain, actually, is the real flat limit of the currents $\mathbb{P}(u_k)$ in $\mathcal{R}_{n-2}(\widetilde{B}^n; H_1^{sph}(\mathcal{Y}))$ and*

$$\mathbb{P}(u)(\phi) = \sum_{s=1}^{\widetilde{s}} \mathbb{P}_s(u)(\phi)\,[\gamma^s] \in H_1^{sph}(\mathcal{Y},\mathbb{R}) \qquad \forall\, \phi \in \mathcal{D}^{n-2}(\widetilde{B}^n)\,, \tag{1.2}$$

where $H_1^{sph}(\mathcal{Y},\mathbb{R}) := H_1^{sph}(\mathcal{Y}) \otimes \mathbb{R}$. Moreover

(i) *$\mathbb{P}(u_k)$ is a finite combination*

$$\mathbb{P}(u_k) = \sum_{s=1}^{\widetilde{s}} R_s \otimes [\gamma_s]\,;$$

where R_s is an i.m. rectifiable current in $\mathcal{R}_{n-2}(\overline{B}^n)$; in particular in case $n=2$ we have

$$R_s = \sum_i d_{i,s}\delta_{a_i}\,,$$

where $d_{i,s} \in \mathbb{Z}$ are integer coefficients and the δ_{a_i}'s are Dirac unit measures at points $a_i \in \overline{B}^2$;

(ii) *since the boundary data φ has a smooth extension from $\widetilde{B}^n$ into $\mathcal{Y}$, each $\mathbb{P}(u_k)$ is the boundary of an i.m. rectifiable current.*

We now show that $\mathbb{P}_s(u)$ is an integral flat chain.

Proposition 1.4 *Let $u \in W^{1/2}_\varphi(\widetilde{B}^n, \mathcal{Y})$ and $\{u_k\} \subset R^\infty_{1/2,\varphi}(\widetilde{B}^n, \mathcal{Y})$ converge strongly in $W^{1/2}$ to u. Then*

(i) *$\mathbf{M}(\mathbb{D}_s(u_k) - \mathbb{D}_s(u)) \to 0$ as $k \to +\infty$;*

(ii) *there exists $L \in \mathcal{R}_{n-1}(\widetilde{B}^n; H_1^{sph}(\mathcal{Y}))$, $\mathbf{M}(L) < +\infty$, such that $\mathbb{P}(u) = \partial L$; in particular $\mathbb{P}(u)$ is an integral flat chain;*

(iii) *if $L^s_{u_k,u}$ denotes an $(n-1)$-dimensional i.m. rectifiable current of least mass with support in $\overline{B}^n$ such that*

$$\partial L^s_{u_k,u} = \mathbb{P}_s(u) - \mathbb{P}_s(u_k)\,, \tag{1.3}$$

then

$$\mathbf{M}(L^s_{u_k,u}) \to 0 \qquad as \quad k \to +\infty\,.$$

(iv) *if $n=2$, then for each $s=1,\dots,\widetilde{s}$ there exist points $a_i, b_i \in \overline{B}^2$ such that*

$$\mathbb{P}_s(u) = \sum_{i=1}^{+\infty}(\delta_{a_i} - \delta_{b_i})\,, \qquad \sum_{i=1}^{+\infty}|a_i - b_i| < +\infty\,.$$

Proof: To prove property (i), observe that

$$\mathbb{D}(u;\omega)(\phi) = G_U(\pi^{\#}\widetilde{\phi} \wedge \widehat{\pi}^{\#} d\widetilde{\omega}) = \int_{z\in\widetilde{B}^n\times I} \sum_{\substack{|\alpha| = n-1 \\ |\beta| = 2}} \phi_\alpha(z) dz^\alpha \wedge \sigma_\beta(U) M^{\beta}_{\overline{\alpha}}(DU)(z)\, dz^{\overline{\alpha}} \,,$$

where

$$\widetilde{\phi} = \sum_{|\alpha|=n-1} \phi_\alpha(z)\, dz^\alpha \,, \qquad d\widetilde{\omega} = \sum_{|\beta|=2} \sigma_\beta(y)\, dy^\beta$$

and $M^{\beta}_{\overline{\alpha}}(DU)$ is the determinant of the 2×2 minor of DU of rows β and columns $\overline{\alpha}$, where $\overline{\alpha}$ is the complement of the multi-index α in $(1,\dots,n+1)$. Since $u_k \to u$ strongly in $W^{1/2}$, then $U_k := \mathrm{Ext}(u_k) \to \mathrm{Ext}(u) =: U$ strongly in $W^{1,2}$ and hence, by Lebesgue theorem, we infer that $M^{\beta}_{\overline{\alpha}}(DU_k) \to M^{\beta}_{\overline{\alpha}}(DU)$ strongly in $L^1(\widetilde{B}^n\times I)$ for every α and β with $|\alpha| = n-1$ and $|\beta| = 2$. Therefore (i) holds.

The rest of the theorem is proved as in [14, Vol. II, Sec. 4.2.5]. In fact, if Γ is an $(n-1)$-dimensional i.m. rectifiable current with compact support in $\mathbb{R}^n$ and

$$\begin{array}{rcl} m_i(\Gamma) & := & \inf\{\mathbf{M}(T) \mid T\in\mathcal{R}_n(\mathbb{R}^n)\,, \quad \partial T = R\} \\ m_r(\Gamma) & := & \inf\{\mathbf{M}(T) \mid T\in\mathcal{D}_n(\mathbb{R}^n)\,, \quad \partial T = R\}\,, \end{array}$$

by Hardt-Pitts' theorem [19] we have that $m_i(\Gamma) = m_r(\Gamma)$. Therefore by Rmk. 1 in [14, Vol. II, Sec. 5.4.2] the claims follow. □

2 Cartesian currents with finite $\mathbf{W^{1/2}}$-energy

In this section we introduce the class of Cartesian currents with finite $W^{1/2}$-energy, see Definitions 2.1, 2.7 and 2.8, collecting the main properties. For the sake of clearness, all the proofs are postponed to the next section except for the proof of the closure theorem, Theorem 2.12.

Definition 2.1 *Let $T\in\mathcal{D}_{n,1}(B^n\times\mathcal{Y})$. We say that T is a current in $\mathcal{E}_{1/2}$-*graph$(B^n\times\mathcal{Y})$ *if*

$$\partial_x T = 0 \quad on \quad \mathcal{D}^{n-1,1}(B^n\times\mathcal{Y})\,, \qquad \partial_y T = 0 \quad on \quad \mathcal{Z}^{n-1,1}(B^n\times\mathcal{Y}) \tag{2.1}$$

and T can be decomposed as

$$T = G_{u_T} + S_T\,, \qquad S_T := \sum_{q\in H_1^{sph}(\mathcal{Y})} \mathbb{L}_q \times C_q\,, \qquad on \quad \mathcal{Z}^{n,1}(B^n\times\mathcal{Y}) \tag{2.2}$$

where $u_T\in W^{1/2}(B^n,\mathcal{Y})$ and $\mathbb{L}_q$ is an i.m. rectifiable current in $\mathcal{R}_{n-1}(B^n)$, $\mathbb{L}_q := \tau(\mathcal{L}_q, 1, \overrightarrow{\mathcal{L}}_q)$, where the $(n-1)$-rectifiable sets $\mathcal{L}_q$ are pairwise disjoint. Finally, C_q is an integral flat cycle in the homology class q, according to Definition 0.1 *and to* (0.2).

Note that S_T is completely vertical, i.e. $S_T(\omega) = 0$ if $\omega = \omega^{(0)} \in \mathcal{D}^n(B^n\times\mathcal{Y})$. Moreover, the graph G_u of a $W^{1/2}$-map u is in $\mathcal{E}_{1/2}$-graph$(B^n\times\mathcal{Y})$ if it has no inner boundary, i.e.,

$$\partial_x G_u = 0 \quad \text{on} \quad \mathcal{D}^{n-1,1}(B^n\times\mathcal{Y})\,, \qquad \partial_y G_u = 0 \quad \text{on} \quad \mathcal{Z}^{n-1,1}(B^n\times\mathcal{Y})\,, \tag{2.3}$$

condition which is automatically satisfied in case of dimension $n=1$.

Remark 2.2 Note that currents in $\mathcal{E}_{1/2}$-graph$(B^1\times\mathcal{Y})$ are defined in a homological sense. More precisely, the decomposition (2.2) does not depend on the choice of the representative C_q in the homology class q. In fact, if $\widetilde{C}_q - C_q = \partial W$ for some $W\in\mathcal{R}_2(\mathcal{Y})$, then for every $\phi\in\mathcal{D}^{n-1}(B^n)$ and $\omega\in\mathcal{D}^1(\mathcal{Y})$

$$(\mathbb{L}_q\times\widetilde{C}_q)(\phi\wedge\omega) - (\mathbb{L}_q\times C_q)(\phi\wedge\omega) = \mathbb{L}_q(\phi)\cdot W(d\omega) \neq 0$$

if and only if $d_y\omega \neq 0$, i.e., if and only if $\phi\wedge\omega \notin \mathcal{Z}^{n-1,1}(B^1\times\mathcal{Y})$.

Extension of $\mathcal{E}_{1/2}$-graphs. Following an idea from [15], we now extend currents in $\mathcal{E}_{1/2}\text{-graph}(B^n \times \mathcal{Y})$ to suitable currents in $\mathcal{D}_{n+1,2}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$.

Definition 2.3 *Let $T \in \mathcal{E}_{1/2}\text{-graph}(B^n \times \mathcal{Y})$ be such that* (2.2) *holds. Then the* extension $\widetilde{T} := \mathrm{Ext}(T)$ *is the current $\widetilde{T} \in \mathcal{D}_{n+1,2}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$ defined by*

$$\widetilde{T} = (-1)^{n-1}\Big(G_{U_T} + \sum_{q\in H_1^{sph}(\mathcal{Y})} \mathbb{L}_q \times R_q\Big)\,, \tag{2.4}$$

where $U_T := \mathrm{Ext}(u_T)$ and R_q is the i.m. rectifiable current of least mass among all currents in $\mathcal{R}_2(\mathbb{R}^N)$ such that $Z_q = \partial R_q$ is in the homology class q, see Remark 2.2.

Remark 2.4 Note that even if R_q minimizes the mass among all currents in $\mathcal{R}_2(\mathbb{R}^N)$ such that ∂R_q is in the homology class q, then a priori the mass of ∂R_q is not finite, compare Remark 0.2.

Remark 2.5 Note that from Definition 2.3 and (1.1) we infer that the *boundary* of $\widetilde{T}$ over $B^n \times \{0\} \times \mathcal{Y}$ is equal to T on $\mathcal{Z}^{n,1}(B^n \times \mathcal{Y})$. In fact, for every $q \in H_1^{sph}(\mathcal{Y})$

$$\partial(\mathbb{L}_q \times R_q) = \partial\,\mathbb{L}_q \times R_q + (-1)^{n-1}\,\mathbb{L}_q \times \partial R_q$$

and hence, since $\partial R_q \in q$, whereas $\partial\,\mathbb{L}_q \times R_q = 0$ on $\mathcal{D}^{n,1}(B^n \times \mathbb{R}^N)$, we have

$$(-1)^{n-1}\partial(\mathbb{L}_q \times R_q) = \mathbb{L}_q \times C_q \qquad \text{on} \qquad \mathcal{Z}^{n,1}(B^n \times \mathcal{Y})\,.$$

The $\mathcal{E}_{1/2}$-energy. We recall from [14] that the *Dirichlet energy* of a current T in the so called class $\mathbf{D}\text{-graph}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$ is defined by the integral

$$\mathbf{D}(T) := \int_{\mathcal{C}^{n+1}\times\mathbb{R}^N} F(y, \overrightarrow{T}\,)\, d\|T\|_{\mathbf{D}}\,,$$

where $\overrightarrow{T}$ is the Radon-Nikodym derivative $dT/d\|T\|_{\mathbf{D}}$, $\|T\|_{\mathbf{D}}$ being the $\mathbf{D}$-*norm*, and the function $F(y,\xi)$ of $y \in \mathcal{Y}$ and $\xi \in \bigwedge_{n+1} \mathbb{R}^{n+1+N}$ is the *Dirichlet density* of T. Now, if $\widetilde{T} \in \mathcal{D}_{n+1,2}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$ is given by (2.4), then it is not difficult to prove that its Dirichlet energy is

$$\mathbf{D}(\widetilde{T}) = \frac{1}{2}\int_{\mathcal{C}^{n+1}} |DU_T|^2\, dx\, dt + \sum_{q\in H_1^{sph}(\mathcal{Y})} \mathbf{M}(\mathbb{L}_q)\cdot \mathbf{M}(R_q)\,. \tag{2.5}$$

In particular, if $T = G_U$ for some $U \in W^{1,2}(\mathcal{C}^{n+1}, \mathbb{R}^N)$, then $\mathbf{D}(G_U) = \mathbf{D}(U)$.

Definition 2.6 *Let T be in $\mathcal{E}_{1/2}\text{-graph}(B^n \times \mathcal{Y})$, so that* (2.2) *holds. Then the $\mathcal{E}_{1/2}$-energy $\mathcal{E}_{1/2}(T)$ of T is defined as the Dirichlet energy $\mathbf{D}(\widetilde{T})$ of the extension $\widetilde{T} := \mathrm{Ext}(T)$, see* (2.4) *and* (2.5).

If $T = G_u$ for some $u \in W^{1/2}(B^n, \mathcal{Y})$ and $U = \mathrm{Ext}(u)$, we also define

$$\mathrm{Ext}(G_u) := (-1)^{n-1} G_U\,, \qquad \mathcal{E}_{1/2}(G_u) := \mathbf{D}(G_U) = \mathbf{D}(U) \simeq |u|_{1/2}\,.$$

Finally, if $A \subset B^n$ is a Borel set, and $T \in \mathcal{E}_{1/2}\text{-graph}(B^n \times \mathcal{Y})$, we will denote

$$\mathcal{E}_{1/2}(T, A \times \mathcal{Y}) := \mathbf{D}(\mathrm{Ext}(T \llcorner A \times \mathbb{R}^N))$$

and if $u \in W^{1/2}(B^n, \mathcal{Y})$

$$\mathcal{E}_{1/2}(u, A) := \mathbf{D}(\mathrm{Ext}(u_{|A}), A \times I) = \frac{1}{2}\int_{A\times I} |D\,\mathrm{Ext}(u_{|A})|^2\, dx\, dt\,.$$

We now give the following

Definition 2.7 Let $T \in \mathcal{D}_{n,1}(B^n, \mathcal{Y})$. We say that T is a *Cartesian current* in $\mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$ if T belongs to $\mathcal{E}_{1/2}$-graph$(B^n \times \mathcal{Y})$ and the $\mathcal{E}_{1/2}$-energy $\mathcal{E}_{1/2}(T)$ of T is finite, see Definitions 2.1 and 2.6.

Definition 2.8 *We say that a map $u \in W^{1/2}(B^n, \mathcal{Y})$ is in $\mathrm{cart}^{1/2}(B^n, \mathcal{Y})$ if the current G_u associated to its graph is in $\mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$.*

Therefore, a $W^{1/2}$-map u is in $\mathrm{cart}^{1/2}(B^n, \mathcal{Y})$ if the graph G_u has no inner boundary, i.e. (2.3) holds true. In particular, any smooth map $u : B^n \to \mathcal{Y}$ with finite $W^{1/2}$-energy belongs to $\mathrm{cart}^{1/2}(B^n, \mathcal{Y})$.

Finally, if $\varphi : \widetilde{B}^n \to \mathcal{Y}$ is a given $W^{1/2}$-function, which is assumed to be smooth on $\widetilde{B}^n$, where $\widetilde{B}^n$ is a bounded domain in $\mathbb{R}^n$ such that $B^n \subset\subset \widetilde{B}^n$, in the sequel we will denote

$$
\begin{aligned}
\mathrm{cart}^{1/2}_\varphi(\widetilde{B}^n, \mathcal{Y}) &:= \{u \in \mathrm{cart}^{1/2}(\widetilde{B}^n, \mathcal{Y}) \mid u = \varphi \quad \text{on } \widetilde{B}^n \setminus \overline{B}^n\} \\
\mathrm{cart}^{1/2}_\varphi(\widetilde{B}^n \times \mathcal{Y}) &:= \{T \in \mathrm{cart}^{1/2}(\widetilde{B}^n \times \mathcal{Y}) \mid (T - G_\varphi) \llcorner (\widetilde{B}^n \setminus \overline{B}^n) \times \mathbb{R}^N = 0\}\,.
\end{aligned}
$$

The weak convergence. We say that $\{T_k\} \subset \mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$ converges to $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ weakly in $\mathrm{cart}^{1/2}$ if $T_k(\omega) \to T(\omega)$ for every $\omega \in \mathcal{Z}^{n,1}(B^n \times \mathcal{Y})$ and $\sup_k \mathcal{E}_{1/2}(T_k) < +\infty$.

The 1-dimensional case. Definition 2.7 is motivated by the following

Theorem 2.9 *Let $\{u_k\} \subset C^1(B^1, \mathcal{Y})$ be a sequence of smooth maps with $\sup_k |u_k|_{1/2} < +\infty$. Then, possibly passing to a subsequence, $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_1(B^1 \times \mathcal{Y})$ to some current $T \in \mathrm{cart}^{1/2}(B^1 \times \mathcal{Y})$.*

Therefore the class $\mathrm{cart}^{1/2}(B^1 \times \mathcal{Y})$ contains the weak limits in $\mathrm{cart}^{1/2}$ of sequences of graphs of smooth maps with equibounded $\mathcal{E}_{1/2}$-energy. Moreover we have the following lower semicontinuity property.

Proposition 2.10 *Let $\{u_k\} \subset C^1(B^1, \mathcal{Y})$ be a sequence of smooth maps with $\sup_k |u_k|_{1/2} < +\infty$ and such that $G_{u_k} \rightharpoonup T$ weakly in $\mathcal{D}_1(B^1 \times \mathcal{Y})$ to some current $T \in \mathcal{E}_{1/2}$-graph$(B^1 \times \mathcal{Y})$. Then $T \in \mathrm{cart}^{1/2}(B^1 \times \mathcal{Y})$ and*

$$\mathcal{E}_{1/2}(T) \le \liminf_{k \to +\infty} \mathcal{E}_{1/2}(G_{u_k})\,.$$

Finally the following closure theorem holds true.

Theorem 2.11 *The classes $\mathrm{cart}^{1/2}(B^1 \times \mathcal{Y})$ and $\mathrm{cart}^{1/2}_\varphi(\widetilde{B}^1 \times \mathcal{Y})$ are closed under weak convergence in $\mathrm{cart}^{1/2}$.*

The n-dimensional case. Taking into account Theorem 2.11 and the approximation theorems in Sec. 6 and Sec. 7 we shall prove in any dimension $n \ge 2$ the following

Theorem 2.12 (Closure theorem). *We have:*

- *(i) if $n = 2$, the class $\mathrm{cart}^{1/2}_\varphi(\widetilde{B}^2 \times \mathcal{Y})$ is closed under weak convergence in $\mathrm{cart}^{1/2}$;*
- *(ii) in any dimension, $\mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$ and $\mathrm{cart}^{1/2}_\varphi(\widetilde{B}^n \times \mathcal{Y})$ are closed under weak convergence in $\mathrm{cart}^{1/2}$, provided that $\pi_1(\mathcal{Y})$ is commutative.*

We will also prove that the following lower semicontinuity and compactness properties hold in any dimension $n \ge 1$.

Proposition 2.13 *Let $\{T_k\} \subset \mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$ be such that $T_k \rightharpoonup T$ weakly in $\mathrm{cart}^{1/2}$ to some current $T \in \mathcal{E}_{1/2}$-graph$(B^n \times \mathcal{Y})$. Then $T \in \mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$ and*

$$\mathcal{E}_{1/2}(T) \le \liminf_{k \to +\infty} \mathcal{E}_{1/2}(T_k)\,. \tag{2.6}$$

Proposition 2.14 *Let $\{T_k\} \subset \mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$ be such that $\sup_k \mathcal{E}_{1/2}(T_k) < +\infty$. Then, possibly passing to a subsequence we have that $T_k \rightharpoonup T$ weakly in $\mathrm{cart}^{1/2}$ to some current $T \in \mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$.*

The rest of this section is dedicated to outline the proof of Theorem 2.12.

Let $\{T_k\} \subset \mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$ be such that $T_k \rightharpoonup T$ weakly in $\mathrm{cart}^{1/2}$ to some current $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$. We have to show that $T \in \mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$. To this aim, we first write T_k as

$$T_k = G_{u_k} + \sum_{q \in H_1^{sph}(\mathcal{Y})} \mathbb{L}_q^{(k)} \times C_q^{(k)} \qquad \text{on} \quad \mathcal{Z}^{n,1}(B^n \times \mathcal{Y}) , \tag{2.7}$$

where $u_k \in W^{1/2}(B^n, \mathcal{Y})$, $\mathbb{L}_q^{(k)} \in \mathcal{R}_{n-1}(B^n)$ and $C_q^{(k)}$ is an integral flat cycle in the homology class q, for every q and k. If $\widetilde{T}_k := \mathrm{Ext}(T_k)$, we have

$$\widetilde{T}_k = (-1)^{n-1} \Big(G_{U_k} + \sum_{q \in H_1^{sph}(\mathcal{Y})} \mathbb{L}_q^{(k)} \times R_q^{(k)} \Big) , \tag{2.8}$$

where $U_k = \mathrm{Ext}(u_k) \in W^{1,2}(\mathcal{C}^n, \mathbb{R}^N)$ and $R_q^{(k)} \in \mathcal{R}_2(\mathbb{R}^N)$ is mass minimizing with $\partial R_q^{(k)} = Z_q^{(k)}$, where $Z_q^{(k)} \in \mathcal{Z}_1(\mathcal{Y})$ is in the homology class q. Moreover, since $\sup_k \mathcal{E}_{1/2}(T_k) < +\infty$ we have

$$\sup_k \mathbf{D}(U_k) < +\infty , \qquad \sup_k \sum_{q \in H_1^{sph}(\mathcal{Y})} \mathbf{M}(\mathbb{L}_q^{(k)}) \cdot \mathbf{M}(R_q^{(k)}) < +\infty . \tag{2.9}$$

Therefore, from [14], possibly passing to a subsequence we have $\widetilde{T}_k \rightharpoonup \widetilde{T}$ weakly in $\mathcal{D}_{n+1,2}$ to some current $\widetilde{T} \in \mathcal{D}_{n+1,2}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$ such that

$$\widetilde{T} = (-1)^{n-1} \Big(G_U + \widetilde{S}_T \Big) \qquad \text{on} \quad \mathcal{Z}^{n+1,2}(\mathcal{C}^{n+1} \times \mathbb{R}^N) ,$$

where $U \in W^{1,2}(\mathcal{C}^n, \mathbb{R}^N)$. Now we check that

$$T = \partial \widetilde{T} = G_u + S_T \qquad \text{on} \quad \mathcal{Z}^{n,1}(B^n \times \{0\} \times \mathbb{R}^N) ,$$

where $u \in W^{1/2}(B^n \times \mathcal{Y})$ is the trace of U on $B^n \times \{0\}$ and $S_T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ is completely vertical, i.e. $S_T(\omega) = 0$ if $\omega = \omega^{(0)} \in \mathcal{D}^n(B^n \times \mathcal{Y})$. Moreover, due to the weak convergence we also infer that T satisfies (2.1). To show that T decomposes as in (2.2), we argue as follows.

Structure of the weak limit current. According to [14], similarly to e.g. Thm. 1 of Sec. 5.3.1 in Vol. II, the weak limit current $T \in \mathcal{D}_{n,1}(B^n \times \mathcal{Y})$ decomposes as

$$T = G_u + \sum_{s=1}^{\overline{s}} \mathbb{L}_s(T) \times \gamma_s \qquad \text{on} \quad \mathcal{Z}^{n,1}(B^n \times \mathcal{Y}) , \tag{2.10}$$

where $\mathbb{L}_s(T) \in \mathcal{D}_{n-1}(B^n)$ only depends on the cohomology class of ω^s and is defined by

$$\mathbb{L}_s(T)(\phi) := S_T(\pi^{\#}\phi \wedge \widehat{\pi}^{\#}\omega^s) , \qquad \phi \in \mathcal{D}^{n-1}(B^n) .$$

Following an idea from [11], we show that the $\mathbb{L}_s(T)$'s are i.m. rectifiable currents in $\mathcal{R}_{n-1}(B^n)$. To this aim, taking into account the density of smooth graphs in $\mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$, see Theorems 4.1, 6.1, 6.2 and 7.1 below, in the next section we will prove the following

Theorem 2.15 *The current T is a flat chain in $B^n \times \mathcal{Y}$. Moreover, the $\mathbb{L}_s(T)$'s are flat chains in B^n.*

As a consequence, the following slicing argument holds true.

Slicing by lines. Let P be an oriented straight line in $\mathbb{R}^n$ and $P_\lambda := P + \sum_{i=1}^{n-1} \lambda_i \nu_i$ the family of oriented lines parallel to P, where $\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in \mathbb{R}^{n-1}$, $\mathrm{span}(\nu_1, \ldots, \nu_{n-1})$ being the orthogonal hyperspace to P. Similarly to the case of normal currents, since T is a flat chain, for $\mathcal{H}^{n-1}$-a.e. λ the *slice* $T \llcorner \pi^{-1}(P_\lambda)$

of T over $\pi^{-1}(P_\lambda)$ is a well defined 1-dimensional flat chain in $(B^n \cap P_\lambda) \times \mathcal{Y}$ and $T_k \llcorner \pi^{-1}(P_\lambda)$ belongs to $\mathrm{cart}^{1/2}((B^n \cap P_\lambda) \times \mathcal{Y})$ for every k. Moreover, since $T_k \rightharpoonup T$ weakly in $\mathrm{cart}^{1/2}$, for $\mathcal{H}^{n-1}$-a.e. λ, passing to a subsequence we have $T_k \llcorner \pi^{-1}(P_\lambda) \rightharpoonup T \llcorner \pi^{-1}(P_\lambda)$ weakly in $\mathrm{cart}^{1/2}$.

Therefore, from the closure result of Theorem 2.11 we infer that the slice $T \llcorner \pi^{-1}(P_\lambda) \in \mathrm{cart}^{1/2}((B^n \cap P_\lambda) \times \mathcal{Y})$ and hence that $\mathbb{L}_s(T) \llcorner \pi^{-1}(P_\lambda) = \mathbb{L}_s(T \llcorner \pi^{-1}(P_\lambda))$ is 0-dimensional and rectifiable, with moreover $\mathbb{L}_s(T \llcorner \pi^{-1}(P_\lambda)) = 0$ for $s = \widetilde{s}+1, \dots, \overline{s}$. Since the $\mathbb{L}_s(T)$'s are flat chains, the rectifiability criterion of B. White [25] yields that the $\mathbb{L}_s(T)$'s are i.m. rectifiable currents in $\mathcal{R}_{n-1}(B^n)$, whereas $\mathbb{L}_s(T) = 0$ for $s = \widetilde{s}+1, \dots, \overline{s}$. Similarly to [11], we then conclude that T decomposes as in (2.2) and hence that $T \in \mathcal{E}_{1/2}$-graph$(B^n \times \mathcal{Y})$. Finally, by lower semicontinuity, Proposition 2.13, we have $\mathcal{E}_{1/2}(T) < +\infty$ and hence $T \in \mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$. The closure of the class $\mathrm{cart}^{1/2}_\varphi(\widetilde{B}^n \times \mathcal{Y})$ is obtained in a similar way.

3 Proofs

In this section we collect the proofs of the results stated in Sec. 2.

Proof of Theorem 2.9: Let $\mathbb{R}^N \simeq \mathbb{R}^N \times \{0\} \subset \mathbb{R}^{N+1}$ and let $\widetilde{\mathcal{Y}}^\pm$ be the compact manifolds in $\mathbb{R}^{N+1} \simeq \mathbb{R}^N \times \mathbb{R}$, with boundary $\partial \widetilde{\mathcal{Y}}^\pm = \mathcal{Y}$, given by

$$\widetilde{\mathcal{Y}}^\pm := \{\widetilde{y} \in \mathbb{R}^{N+1} \,:\, \widetilde{y} = (\lambda y, \pm(1-\lambda))\,, \quad y \in \mathcal{Y}\,, \quad \lambda \in [0,1]\}\,.$$

Also let $\widetilde{\mathcal{Y}}$ be the boundaryless compact manifold

$$\widetilde{\mathcal{Y}} := \widetilde{\mathcal{Y}}^+ \cup \widetilde{\mathcal{Y}}^-$$

and let

$$Q^+ := B^1 \times]0,1[\,, \qquad Q^- := B^1 \times]-1,0[\,, \qquad Q := B^1 \times]-1,1[\,.$$

Moreover, for every k, let $U_k^+ : Q^+ \to \widetilde{\mathcal{Y}}^+$ be the energy minimizing map with boundary condition $U^+_{k\,|B^1 \times \{0\}} = u_k$. Of course

$$\mathbf{D}(U_k^+, Q^+) \simeq |u_k|_{1/2}\,.$$

Also, let $U_k^- : Q^- \to \widetilde{\mathcal{Y}}^-$ be given by

$$U_k^-(x,t) := \Phi \circ U_k^+(x,-t)\,, \qquad \Phi(y,\lambda) := (y, -\lambda)\,.$$

Finally let $U_k : Q \to \widetilde{\mathcal{Y}}$ be defined by $U_k(x,t) := U_k^\pm(x,t)$ if $\pm t \in [0,1[$. We notice that, being $\widetilde{\mathcal{Y}}$ a Lipschitz neighborhood retract, $U_k^\pm$ is Hölder continuous because of Morrey's theorem, see [21] or e.g. [14, Vol. II, Sec. 3.2.2].

Since $\sup_k |u_k|_{1/2} < +\infty$, then we have that $\sup_k \mathbf{D}(U_k, Q) < +\infty$. Moreover G_{U_k} belongs to the class $\mathrm{cart}^{2,1}(Q \times \widetilde{\mathcal{Y}})$. Finally, since $\widetilde{\mathcal{Y}}$ is the *suspension* of the manifold $\mathcal{Y}$, we have

$$H_q(\widetilde{\mathcal{Y}}) = H_{q-1}(\mathcal{Y}) \qquad \forall\, q \geq 1$$

and hence $H_2(\widetilde{\mathcal{Y}})$ has no torsion.

Therefore, possibly passing to a subsequence, from [14] we infer that $G_{U_k} \rightharpoonup \widetilde{T}$ weakly in $\mathcal{D}_2(Q \times \widetilde{\mathcal{Y}})$ to some current $\widetilde{T} \in \mathrm{cart}^{2,1}(Q \times \widetilde{\mathcal{Y}})$. As a consequence, $\widetilde{T}$ may be decomposed as

$$\widetilde{T} = G_U + \sum_{i=1}^{j_0} \delta_{x_i} \times R_i \qquad \text{on} \quad \mathcal{Z}^{2,2}(Q \times \widetilde{\mathcal{Y}})\,, \tag{3.1}$$

where $U \in W^{1,2}(Q, \widetilde{\mathcal{Y}})$, δ_{x_i} is the unit Dirac mass at the point $x_i \in Q$ and $R_i \in \mathcal{Z}_2(\widetilde{\mathcal{Y}})$ is an integral 2-cycle of spherical type, being $\widetilde{\mathcal{Y}}^\pm$ Lipschitz neighborhood retracts. In other words, the homology class of

R_i contains a Lipschitz image of the 2-sphere S^2, compare [14]. More precisely, there exist $Z_i \in \mathcal{Z}_2(\widetilde{\mathcal{Y}})$, $W_i \in \mathcal{R}_3(\widetilde{\mathcal{Y}})$ and a Lipschitz function $\phi_i : S^2 \to \widetilde{\mathcal{Y}}$ such that

$$R_i - Z_i = \partial W_i \qquad \text{and} \qquad \phi_{i\#}[\![S^2]\!] = Z_i \,.$$

Moreover, since $U_k \rightharpoonup U$ weakly in $W^{1,2}(Q,\widetilde{\mathcal{Y}})$, and $\partial G_{U_k} \llcorner B^1 \times \{0\} = G_{u_k}$, then the trace of U on $B^1 \times \{0\}$ is a function u in $W^{1/2}(B^1,\mathcal{Y})$ and $\partial G_U \llcorner (B^1 \times \{0\} \times \mathcal{Y}) = G_u$.

We now show that the slice of $G_{U_k^+}$ w.r.t. $B^1 \times \{0\}$ weakly converges to a current $T \in \mathrm{cart}^{1/2}(B^1,\mathcal{Y})$. To this aim, possibly reordering the indices, we may and will suppose that the points x_i in (3.1) belong to $B^1 \times \{0\}$ if and only if $i \in \{1,\dots,i_0\}$ for some given $i_0 \le j_0$. Now we have

$$G_{u_k} = \partial G_{U_k^+} \llcorner (B^1 \times \{0\} \times \mathcal{Y}) \rightharpoonup G_u + \partial\Big(\sum_{i=1}^{i_0} \delta_{x_i} \times \widetilde{R}_i\Big) \qquad \text{on} \quad \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y})$$

weakly in $D_1(B^1 \times \mathcal{Y})$, where, due to the symmetry of the functions U_k, we infer that each $\widetilde{R}_i \in \mathcal{R}_2(\widetilde{\mathcal{Y}}^+)$ is the weak limit of Lipschitz images $\widetilde{\phi}_{i\#}[\![B^2]\!]$ of the unit disk B^2, with $\widetilde{\phi}_i(\partial B^2) \subset \mathcal{Y}$. As a consequence we have

$$T = G_u + \sum_{i=1}^{i_0} \delta_{x_i} \times C_i \qquad \text{on} \quad \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y})\,,$$

where $C_i = \partial \widetilde{R}_i$ is an integral flat cycle in $\mathcal{Z}_1(\mathcal{Y})$ of S^1-type, see Definition 0.3, so that $T \in \mathcal{E}_{1/2}$-graph$(B^1 \times \mathcal{Y})$, see Definition 2.1. To conclude that $\mathcal{E}_{1/2}(T) < +\infty$, and hence that $T \in \mathrm{cart}^{1/2}(B^1 \times \mathcal{Y})$ according to Definition 2.7, we argue as in the proof of Proposition 2.10. $\square$

Proof of Proposition 2.10: As in the proof of Proposition 2.13, with $T_k = G_{u_k}$. $\square$

Proof of Theorem 2.11: Let $\{T_k\}$ be a sequence in $\mathrm{cart}^{1/2}(B^1 \times \mathcal{Y})$ such that $T_k \rightharpoonup T$ weakly in $\mathrm{cart}^{1/2}$ to some $T \in \mathcal{D}_1(B^1 \times \mathcal{Y})$. If $T_k = G_{u_k}$ for some $u_k \in W^{1,2}(B^1,\mathcal{Y})$, since $n = 1$, for every k we find a sequence of smooth maps $\{v_h^{(k)}\} \subset W^{1/2}(B^1,\mathcal{Y})$ such that $v_h^{(k)} \to u_k$ strongly in $W^{1/2}$ as $h \to +\infty$, compare [3]. As a consequence, taking $v_h := v_h^{k(h)}$, we have that $G_{v_h} \rightharpoonup T$ weakly in $\mathcal{D}_1(B^1 \times \mathcal{Y})$, with $\sup_h |v_h|_{1/2} < +\infty$, so that by Theorem 2.9 we conclude that $T \in \mathrm{cart}^{1/2}(B^1 \times \mathcal{Y})$.

In the general case, from Definition 2.7 we may write

$$T_k = G_{u_k} + \sum_{i=1}^{i_k} \delta_{x_i^{(k)}} \times C_i^{(k)} \qquad \text{on} \quad \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y})\,,$$

for some integral flat cycles $C_i^{(k)} \in \mathcal{Z}_1(\mathcal{Y})$ and some points $x_i^{(k)} \in B^1$. Note that $\{i_k\}_k \subset \mathbb{N}$ is a priori not bounded. If $\widetilde{T}_k := \mathrm{Ext}(T_k)$, we have

$$\widetilde{T}_k = G_{U_k} + \sum_{i=1}^{i_k} \delta_{x_i^{(k)}} \times R_i^{(k)}\,,$$

where $U_k = \mathrm{Ext}(u_k)$ and $R_i^{(k)} \in \mathcal{R}_2(\mathbb{R}^N)$ is mass minimizing with $\partial R_i^{(k)} = Z_i^{(k)}$, where $Z_i^{(k)} \in \mathcal{Z}_1(\mathcal{Y})$ is in the homology class of $C_i^{(k)}$, see Definition 2.3.

From condition $\sup_k \mathcal{E}_{1/2}(T_k) < +\infty$ we infer that

$$\sup_k \mathbf{D}(U_k) < +\infty \qquad \text{and} \qquad \sup_k \sum_{i=1}^{i_k} \mathbf{M}(R_i^{(k)}) < +\infty\,.$$

Therefore, from the proof of Theorem 2.9, possibly passing to a subsequence we deduce that

$$G_{u_k} \rightharpoonup \widehat{T} = G_u + \sum_{i=1}^{\widehat{I}} \delta_{\widehat{x}_i} \times \widehat{C}_i \qquad \text{on} \quad \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y})\,,$$

where $[\widehat{C}_i] \in H_1^{sph}(\mathcal{Y})$ and $u \in W^{1/2}(B^1, \mathcal{Y})$. Moreover, since all the $C_i^{(k)}$'s are non trivial S^1-type cycles in $\mathcal{Y}$, we infer that there is a positive constant δ such that $\mathbf{M}(R_i^{(k)}) \geq \delta$ for all i and k. Consequently $\overline{I} := \sup_k i_k < +\infty$ and then

$$\sum_{i=1}^{i_k} \delta_{x_i^{(k)}} \times R_i^{(k)} \rightharpoonup \sum_{i=1}^{\overline{I}} \delta_{\overline{x}_i} \times \overline{R}_i \qquad \text{on} \quad \mathcal{Z}^{2,2}(B^1 \times \mathcal{Y}),$$

where $[\partial \overline{R}_i] \in H_1^{sph}(\mathcal{Y})$, since all $[\partial R_i^{(k)}] \in H_1^{sph}(\mathcal{Y})$ and the homology classes are closed w.r.t. the weak convergence. Therefore, taking into account Proposition 2.10, we easily conclude that the weak limit

$$T = G_u + \sum_{i=1}^{\widehat{I}} \delta_{\widehat{x}_i} \times \widehat{C}_i + \sum_{i=1}^{\overline{I}} \delta_{\overline{x}_i} \times \partial \overline{R}_i \qquad \text{on} \quad \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y})$$

belongs to $\text{cart}^{1/2}(B^1 \times \mathcal{Y})$. □

Proof of Proposition 2.13: If T_k and T are given by (2.7) and (2.2), and $\widetilde{T}_k := \text{Ext}(T_k)$ and $\widetilde{T} := \text{Ext}(T)$ by (2.8) and (2.4), respectively, possibly passing to a subsequence we may and will suppose that, on one side, the lower limit in (2.6) is a finite limit and, on the other side, that $\widetilde{T}_k \rightharpoonup \widehat{T}$ weakly in $\mathcal{D}_{n+1,2}$ to some current $\widehat{T} \in \mathcal{D}_{n+1,2}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$. Due to weak convergence $T_k \rightharpoonup T$, we infer that $T = \partial \widehat{T}$ on $\mathcal{Z}^{n,1}(B^n \times \{0\} \times \mathcal{Y})$. Moreover, since U_k is the harmonic extension of u_k, the energy of U_k does not concentrate in the interior of $\mathcal{C}^{n+1}$ as $k \to +\infty$. As a consequence we obtain

$$\widehat{T} = G_{U_T} + \sum_{q \in H_1^{sph}(\mathcal{Y})} \mathbb{L}_q \times \widehat{R}_q \qquad \text{on} \quad \mathcal{Z}^{n+1,2}(\mathcal{C}^{n+1} \times \mathbb{R}^N),$$

where $U_T = \text{Ext}(u_T)$ and $\widehat{R}_q \in \mathcal{R}_2(\mathbb{R}^N)$ is such that $\partial \widehat{R}_q = C_q$ for every q. Moreover

$$\mathbf{D}(\widehat{T}) \geq \mathbf{D}(U_T) + \sum_{q \in H_1^{sph}(\mathcal{Y})} \mathbf{M}(\mathbb{L}_q) \cdot \mathbf{M}(\widehat{R}_q)$$

so that, from the definition of $\mathcal{E}_{1/2}$-energy, we infer $\mathcal{E}_{1/2}(T) := \mathbf{D}(\widetilde{T}) \leq \mathbf{D}(\widehat{T})$. Finally, by lower semicontinuity of the Dirichlet energy w.r.t. the weak convergence in $\mathcal{D}_{n+1,2}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$, we get $\mathbf{D}(\widehat{T}) \leq \liminf_k \mathbf{D}(\widetilde{T}_k)$ and hence the assertion, as $\mathbf{D}(\widetilde{T}_k) = \mathcal{E}_{1/2}(T_k)$. □

Proof of Proposition 2.14: If $\widetilde{T}_k := \text{Ext}(T_k)$ then, as in the proof of Theorem 2.12, possibly passing to a subsequence we infer that $\widetilde{T}_k \rightharpoonup \widetilde{T}$ weakly in $\mathcal{D}_{n+1,2}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$ to some current $\widetilde{T}$ such that if $T := \partial \widetilde{T}$ on $\mathcal{Z}^{n,1}(B^n \times \{0\} \times \mathcal{Y})$, then $T \in \text{cart}^{1/2}(B^n \times \mathcal{Y})$ and $T_k \rightharpoonup T$ weakly in $\mathcal{D}_{n,1}(B^1 \times \mathcal{Y})$, as required. □

Proof of Theorem 2.15: In the following sections we will show that for every T_k in $\text{cart}^{1/2}(B^n \times \mathcal{Y})$, respectively in $\text{cart}^{1/2}_\varphi(\widetilde{B}^n \times \mathcal{Y})$, there exists a sequence of smooth maps $\{u_h^{(k)}\}$ in $C^\infty(B^n, \mathcal{Y})$, respectively in $C^\infty_\varphi(\widetilde{B}^n, \mathcal{Y})$, such that $G_{u_h^{(k)}} \rightharpoonup T_k$ weakly in $\text{cart}^{1/2}$ and $\mathcal{E}_{1/2}(G_{u_h^{(k)}}) \to \mathcal{E}_{1/2}(T_k)$ as $h \to +\infty$. Therefore, since T is the weak limit in $\text{cart}^{1/2}$, i.e. with equibounded $\mathcal{E}_{1/2}$-energy, of a sequence $\{T_k\}$ in $\text{cart}^{1/2}(B^n \times \mathcal{Y})$, by a diagonal argument we may and will assume that $T_k = G_{u_k}$ for some smooth map $u_k \in W^{1/2}(B^n, \mathcal{Y})$.

Now, since $u_k \in W^{1/2}(B^n, \mathcal{Y})$ is smooth, we infer that $\mathbb{L}_s(G_{u_k}) := \pi_\#(G_{u_k} \llcorner \widehat{\pi}^\# \omega^s)$ is a flat chain. In fact, since $du_k^\# \omega^s = u_k^\# d\omega^s = 0$, then $u_k^\# \omega^s$ is a closed 1-form in $\mathcal{D}^1(B^n)$ and hence $u_k^\# \omega^s = dg_k^s$ for some $g_k^s \in C^\infty_c(B^n)$. As a consequence, since for every $\phi \in \mathcal{D}^{n-1}(B^n)$ one has $d(\phi \wedge g_k^s) = d\phi \wedge g_k^s + (-1)^{n-1} \phi \wedge dg_k^s$, whereas $G_{u_k} = (Id \bowtie u_k)_\# [\![B^n]\!]$, we infer

$$\begin{aligned} (-1)^{n-1} \mathbb{L}_s(G_{u_k})(\phi) &= G_{u_k}(\pi^\# \phi \wedge \widehat{\pi}^\# \omega^s) = \int_{B^n} \phi \wedge u_k^\# \omega^s \\ &= \int_{B^n} \phi \wedge dg_k^s = (-1)^n \int_{B^n} d\phi \wedge g_k^s \end{aligned}$$

and finally, by the definition of *flat norm*,

$$\mathbf{F}\big(\mathbb{L}_s(G_{u_k})\big) := \inf\{\mathbb{L}_s(G_{u_k})(\phi) \mid \phi \in \mathcal{D}^{n-1}(B^n)\,,\ \mathbf{F}(\phi) \le 1\} \le \int_{B^n} |g_k^s|\,dx < +\infty\,,$$

where

$$\mathbf{F}(\phi) := \max\{\sup_{x\in B^n} \|\phi(x)\|\,,\ \sup_{x\in B^n} \|d\phi(x)\|\}\,.$$

We now show that $\{\mathbb{L}_s(G_{u_k})\}_k$ is a Cauchy sequence w.r.t. the flat norm, i.e., that

$$\mathbf{F}\big(\mathbb{L}_s(G_{u_k}) - \mathbb{L}_s(G_{u_h})\big) := \inf\{\big(\mathbb{L}_s(G_{u_k}) - \mathbb{L}_s(G_{u_h})\big)(\phi) \mid \phi \in \mathcal{D}^{n-1}(B^n)\,,\ \mathbf{F}(\phi) \le 1\}$$

is small for k, h large. Similarly to Sec. 1, we choose a smooth extension $\widetilde{\omega}^s \in \mathcal{D}^1(\mathbb{R}^N)$ of ω^s and a function $\eta \in C^\infty([0,1],[0,1])$ with $\eta \equiv 1$ and $\eta \equiv 0$ respectively in a neighborhood of 0 and 1, and we let $U_k = \mathrm{Ext}(u_k)$. Also, for every $\phi \in \mathcal{D}^{n-1}(B^n)$ with $\mathbf{F}(\phi) \le 1$, we let $\widetilde{\phi} \in \mathcal{D}^{n-1}(\mathcal{C}^{n+1})$ be given by $\widetilde{\phi}(x,t) := \phi(x)\,\eta(t)$. We have

$$\mathbb{L}_s(G_{u_k})(\phi) = \partial G_{U_k}(\pi^{\#}\widetilde{\phi} \wedge \widehat{\pi}^{\#}\widetilde{\omega}^s) = \int_{\mathcal{C}^{n+1}} \big(d\widetilde{\phi} \wedge U_k^{\#}\widetilde{\omega}^s + (-1)^{n-1}\widetilde{\phi} \wedge U_k^{\#} d\widetilde{\omega}^s\big)\,.$$

Since $\{U_k\}$ is equibounded in $W^{1,2}$, possibly passing to a subsequence $U_k^{\#}\widetilde{\omega}^s \to U^{\#}\widetilde{\omega}^s$ and $U_k^{\#}d\widetilde{\omega}^s \rightharpoonup U^{\#}d\widetilde{\omega}^s$ weakly in L^1 to some $U \in W^{1,2}(\mathcal{C}^{n+1},\mathbb{R}^N)$, so that we infer that $\{\mathbb{L}_s(G_{u_k})(\phi)\}_k$ is a Cauchy sequence. As a consequence, if $\mathcal{F}^{n-1}(B^n)$ denotes a countable dense subset of smooth forms ϕ in $\mathcal{D}^{n-1}(B^n)$ satisfying $\mathbf{F}(\phi) \le 1$, by a diagonal argument we infer that

$$\inf\{\big(\mathbb{L}_s(G_{u_k}) - \mathbb{L}_s(G_{u_h})\big)(\phi) \mid \phi \in \mathcal{F}^{n-1}(B^n)\}$$

is small for k, h large. By density of $\mathcal{F}^{n-1}(B^n)$, we obtain that $\{\mathbb{L}_s(G_{u_k})\}_k$ is a Cauchy sequence w.r.t. the flat norm and hence, due to weak convergence of G_{u_k} to T, that $R_s := \pi_{\#}(T \llcorner \widehat{\pi}^{\#}\omega^s)$ is a flat chain.

As a consequence we infer that T is a flat chain, too. Moreover, since $T = G_u + S_T$, where $u \in W^{1/2}(B^n,\mathcal{Y})$ and $S_{T(0)} = 0$, as a consequence of the strong density result for Cartesian currents in $\mathrm{cart}^{1/2}$, in the next sections we will obtain the sequential weak density of smooth maps in $W^{1/2}(B^n,\mathcal{Y})$, see Corollary 6.3 and 7.2. Therefore, by repeating the previous argument we easily infer that $D_s := \pi_{\#}(G_u \llcorner \widehat{\pi}^{\#}\omega^s)$ is a flat chain and hence, being $\mathbb{L}_s(T) = R_s - D_s$, that $\mathbb{L}_s(T)$ is a flat chain, too. $\square$

4 A density result in dimension 1

In this section we prove the following strong density result for the $W^{1/2}$-energy of graphs of maps from 1-dimensional domains into general target manifolds $\mathcal{Y}$, as described in the introduction. We remark that no condition on the first homotopy group $\pi_1(\mathcal{Y})$ is required.

Theorem 4.1 *Let $n = 1$. For every $T \in \mathrm{cart}^{1/2}(B^1 \times \mathcal{Y})$ there exists a sequence of smooth maps $u_k : B^1 \to \mathcal{Y}$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathrm{cart}^{1/2}$ and*

$$\lim_{k\to+\infty} \mathcal{E}_{1/2}(u_k, B^1) = \mathcal{E}_{1/2}(T, B^1 \times \mathcal{Y})\,. \tag{4.1}$$

As a consequence, we have a similar density result in the case with boundary data.

Corollary 4.2 *For every $T \in \mathrm{cart}^{1/2}_{\varphi}(\widetilde{B}^1 \times \mathcal{Y})$ there exists a sequence of smooth maps $\{u_k\} \subset C^\infty_{\varphi}(\widetilde{B}^1,\mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathrm{cart}^{1/2}$ and $\mathcal{E}_{1/2}(u_k,\widetilde{B}^1) \to \mathcal{E}_{1/2}(T,\widetilde{B}^1 \times \mathcal{Y})$ as $k \to +\infty$.*

We recall by Sec. 2 that every $T \in \mathrm{cart}^{1/2}(B^1 \times \mathcal{Y})$ has the form

$$T = G_{u_T} + \sum_{i=1}^{i_0} \delta_{x_i} \times C_i \qquad \text{on} \qquad \mathcal{Z}^{1,1}(B^1 \times \mathcal{Y})\,, \tag{4.2}$$

where δ_x is the Dirac mass in x, $x_i \in B^1$, and the $C_i \in \mathcal{Z}_1(\mathcal{Y})$ are integral flat 1-cycles with non trivial homology of S^1-type. Moreover, without loss of generality, we may and will choose the representative C_i in such a way that if $R_i \in \mathcal{R}_2(\mathbb{R}^N)$ minimizes the mass among all the i.m. rectifiable currents $R \in \mathcal{R}_2(\mathbb{R}^N)$ such that $Z = \partial R$ is in the homology class of C_i, then

$$C_i = \partial R_i \, .$$

The $\mathcal{E}_{1/2}$-energy of T is then defined as the Dirichlet energy of its extension $\widetilde{T}$

$$\mathcal{E}_{1/2}(T) := \mathbf{D}(\widetilde{T}) = \frac{1}{2}\int_{\mathcal{C}^2} |Du_T|^2 \, dx + \sum_{i=1}^{i_0} \mathbf{M}(R_i) \, ,$$

where $\widetilde{T} = \mathrm{Ext}(T) := G_{U_T} + \sum_{i=1}^{i_0} \delta_{x_i} \times R_i$, with $U_T := \mathrm{Ext}(u_T) \in W^{1,2}(\mathcal{C}^2, \mathbb{R}^N)$, compare Definition 2.3.

In the sequel we will denote

$$B_r^+ := \overline{B}_r^2 \cap \mathcal{C}^2 \, , \qquad \partial^+ B_r := \partial B_r^2 \cap \{(x,t) \in \mathcal{C}^2 \mid t > 0\} \, , \qquad J_r := \partial B_r^+ \setminus \partial^+ B_r = [-r, r] \times \{0\} \, , \tag{4.3}$$

where $B_r^2 := \{(x,t) \in \mathbb{R}^2 \mid x^2 + t^2 < r^2\}$, and

$$B^+ := B_1^+ \, , \qquad \partial^+ B := \partial^+ B_1 \, , \qquad J := J_1 \, .$$

Remark 4.3 For future use, we denote

$$\mathcal{Y}_\varepsilon := \{y \in \mathbb{R}^N \mid \mathrm{dist}(y, \mathcal{Y}) \leq \varepsilon\}$$

the ε-neighborhood of $\mathcal{Y}$ and we observe that, since $\mathcal{Y}$ is smooth, there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$ the nearest point projection Π_ε of $\mathcal{Y}_\varepsilon$ onto $\mathcal{Y}$ is a well defined Lipschitz map with Lipschitz constant $L_\varepsilon \to 1^+$ as $\varepsilon \to 0^+$. Note that for $0 < \varepsilon \leq \varepsilon_0$ the set $\mathcal{Y}_\varepsilon$ is topologically equivalent to $\mathcal{Y}$. In particular, we have that

$$\pi_1(\mathcal{Y}_\varepsilon) = \pi_1(\mathcal{Y}) \, .$$

The proof of Theorem 4.1 relies on the following result.

Proposition 4.4 (Approximation of S^1-type cycles) *Let U be a smooth $W^{1,2}$ map from $\mathcal{C}^2$ into $\mathbb{R}^N$ with trace $\mathbf{T}(U) \in W^{1/2}(B^1, \mathcal{Y}_\varepsilon)$. Let $C \in \mathcal{Z}_1(\mathcal{Y})$ be a 1-cycle of S^1-type. Also, let $R \in \mathcal{R}_2(\mathbb{R}^N)$ be the i.m. rectifiable current of least mass in $\mathbb{R}^N$ such that $\partial R = C$. Then there exist a sequence $\{U_k\}$ of smooth maps from $\mathcal{C}^2$ into $\mathbb{R}^N$, with traces $u_k := T(U_k) \in W^{1/2}(B^1, \mathcal{Y}_{2\varepsilon})$ for every k, and a sequence of radii $\delta_k \searrow 0$ such that $U_k = U$ outside $B_{\delta_k}^+$ and $G_{U_k} \rightharpoonup G_U + \delta_0 \times R$ weakly in $\mathcal{D}_2(\mathcal{C}^2 \times \mathbb{R}^N)$ with*

$$\lim_{k \to +\infty} \mathbf{D}(U_k, \mathcal{C}^2) = \mathbf{D}(U, \mathcal{C}^2) + \mathbf{M}(R) \, .$$

We postpone the proof of Proposition 4.4 and first prove Theorem 4.1 and Corollary 4.2.

Proof of Theorem 4.1: Since $n = 1$, adapting an argument by [23], as in [3, 2.1] we can find a sequence of smooth maps $U_k : \mathcal{C}^2 \to \mathbb{R}^N$ such that $U_k \to \mathrm{Ext}(u)$ strongly in $W^{1,2}(\mathcal{C}^2, \mathbb{R}^N)$ and for which there exists a positive number $t_0 > 0$ such that $U_k(B^1 \times [0, t_0]) \subset \mathcal{Y}_{\varepsilon_0}$ for every k. In particular we have that the traces $u_k := \mathbf{T}(U_k) \in W^{1/2}(B^1, \mathcal{Y}_{\varepsilon_0})$ and $u_k \to u$ in $W^{1/2}(B^1, \mathcal{Y}_{\varepsilon_0})$.

On small half-disks $x_i + B_{r_{k,h}}^+$ around each x_i and contained in $\mathcal{C}^2$, we then apply Proposition 4.4 to each U_k and find a sequence of smooth maps $\{U_{k,h}\}_h$ from $\mathcal{C}^2$ into $\mathbb{R}^N$, with traces $u_{k,h} := \mathbf{T}(U_{k,h}) \in W^{1/2}(B^1, \mathcal{Y}_{\varepsilon_0})$ for every h, and a sequence of radii $r_{k,h} \searrow 0$ as $h \to +\infty$ such that $U_{k,h} = U_k$ outside $x_i + B_{r_{k,h}}^+$,

$$G_{U_{k,h}} \rightharpoonup G_{U_k} + \sum_{i=1}^{i_0} \delta_{x_i} \times R_i$$

weakly in $\mathcal{D}_2(\mathcal{C}^2 \times \mathbb{R}^N)$ and

$$\lim_{h\to+\infty} \mathbf{D}(U_{k,h}, \mathcal{C}^2) = \mathbf{D}(U_k, \mathcal{C}^2) + \sum_{i=1}^{i_0} \mathbf{M}(R_i)\,.$$

By a diagonal procedure we therefore find a smooth sequence $\{V_k\} \subset C^1(\mathcal{C}^2, \mathbb{R}^N)$, again with traces $v_k := \mathbf{T}(V_k) \in W^{1/2}(B^1, \mathcal{Y}_{\varepsilon_0})$, such that $G_{V_k} \rightharpoonup \widetilde{T}$ weakly in $\mathcal{D}_2(\mathcal{C}^2 \times \mathbb{R}^N)$ and $\mathbf{D}(V_k, \mathcal{C}^2) \to \mathbf{D}(\widetilde{T})$ as $l \to +\infty$. Finally, setting $u_k := \Pi_{2\varepsilon_0} \circ v_k$, compare Remark 4.3, we obtain the assertion. $\square$

PROOF OF COROLLARY 4.2: Since u is smooth on $\widetilde{B}^1 \setminus B^1$, we may and do define the sequence $U_k : \mathcal{C}^2 \to \mathbb{R}^N$ so that in particular $(G_{u_k} - G_\varphi) \llcorner (\widetilde{B}^1 \setminus \overline{B}^1) \times \mathbb{R}^N = 0$ for every k. Moreover, since the points x_i in (4.2) can be taken distant from the boundary of B^1, we apply Proposition 4.4 by taking the radii $r_{k,h}$ small so that in particular $U_{k,h}$ coincides with U_k in a small neighborhood of $\partial B^1 \times I$, as required. $\square$

To prove Proposition 4.4 we make use of the following result.

Proposition 4.5 *Let $C \in \mathcal{Z}_1(\mathcal{Y})$ be an integral flat 1-cycle of S^1-type and let $R \in \mathcal{R}_2(\mathbb{R}^N)$ be the i.m. rectifiable current of least mass in $\mathbb{R}^N$ such that $\partial R = C$. Let $P \in \mathcal{Y}$ be a given point. Then there exists a sequence of Lipschitz functions $f_\varepsilon : B^+ \to \mathbb{R}^N$ such that $f_{\varepsilon|\partial^+ B} \equiv P$, $f_\varepsilon(J) \subset \mathcal{Y}_{2\varepsilon}$, $f_{\varepsilon\#}[\![B^+]\!] \rightharpoonup R$ and $f_{\varepsilon\#}[\![J]\!] \rightharpoonup C$ weakly in $\mathcal{D}_2(\mathbb{R}^N)$ and $\mathcal{D}_1(\mathcal{Y}_{2\varepsilon})$, respectively, and*

$$\lim_{\varepsilon\to 0} \mathbf{D}(f_\varepsilon, B^+) = \mathbf{M}(R)\,.$$

We postpone its proof and we give the

PROOF OF PROPOSITION 4.4: If $P := u(0)$, we let $\mathcal{U}_\varepsilon$ be a convex open subset of $\mathcal{Y}_\varepsilon$ containing P and let $r_\varepsilon \in (0,1)$ be such that $U(B^+_{r_\varepsilon}) \subset \mathcal{U}_\varepsilon$. We define, for $k \in \mathbb{N}$ and $r \in (0, r_\varepsilon)$,

$$U_{k,r}(z) := \begin{cases} U(z) & \text{if} \quad |z| > r \\ v_r(z) & \text{if} \quad r/2 \le |z| \le r \\ f_k(2z/r) & \text{if} \quad |z| < r/2 \end{cases} \qquad z \in \mathcal{C}^2\,,$$

where f_k is given by Proposition 4.5, with $P = U(0)$, and

$$v_r(z) := \left(\frac{2}{r}\,|z| - 1\right) \cdot U\left(r\,\frac{z}{|z|}\right) + \left(2 - \frac{2}{r}\,|z|\right) \cdot U(0)\,.$$

Now, since $v_r(z) = U(z)$ for $|z| = r$ and $v_r(z) \equiv P$ for $|z| = r/2$, it follows that $U_{k,r}$ is Lipschitz continuous. Moreover by Proposition 4.5 and a change of variables

$$\mathbf{D}(U_{k,r}, B^+_{r/2}) = \mathbf{D}(f_k, B^+) \to \mathbf{M}(R)$$

as $k \to +\infty$, so that the claim holds if we show that

$$\liminf_{r\to 0^+} \mathbf{D}(v_r, B^+_r \setminus B^+_{r/2}) = 0\,, \tag{4.4}$$

by taking $U_k := U_{k,r_k}$ for a suitable sequence $r_k \searrow 0$. Now we estimate

$$\mathbf{D}(v_r, B^+_r \setminus B^+_{r/2}) \le c\left(\|U(z) - U(0)\|^2_{\infty,\partial B^+_r} + r \int_{\partial B^+_r} |D_\tau U|^2 \, d\mathcal{H}^1 \right),$$

where $c > 0$ is an absolute constant and τ is the tangential direction to ∂B^+_r. By continuity we have $\|U(z) - U(0)\|^2_{\infty,\partial B^+_r} \to 0$ as $r \to 0^+$. Moreover, if $F(r) := \int_{\partial B^+_r} |D_\tau U|^2 \, d\mathcal{H}^1$, by the coarea formula [8]

$$\int_0^{r_0} F(r)\, dr \le \int_{B^+_{r_0}} |DU|^2 \, dx < +\infty,$$

so that F is a non-negative function in $L^1(0,r_0)$. As a consequence, $\liminf_{r\to 0^+} r\,F(r)=0$ and then (4.4) holds, as required. $\square$

Proof of Proposition 4.5: The proof is an adaptation of [16, Prop. 4.5], to which we refer for further details. We divide the proof in four steps. According to Definition 0.3, $C-Z=\partial W$ and $Z=\varphi_\#[\![\, S^1 \,]\!]$. First, we approximate R, C, Z and W with polyhedral chains R_ε, C_ε, Z_ε and W_ε so that (4.5) and (4.6) hold for some Lipschitz function ϕ_ε of S^1; secondly, we find a Lipschitz map f_ε of B^2 with $f_{\varepsilon\#}[\![\, B^2 \,]\!]=R_\varepsilon$; then we show that we can choose f_ε so that in particular its mapping area is equal to the mass of R_ε; finally, we modify f_ε and prove the assertion by means of Morrey's ε-conformality theorem.

Step 1: For every $\varepsilon>0$ we find an open set $\mathcal{U}_\varepsilon\subset\mathbb{R}^N$ such that $\mathbf{M}(R\llcorner(\mathbb{R}^N\setminus\mathcal{U}_\varepsilon))<\varepsilon$, $\mathbf{M}(\partial(R\llcorner\mathcal{U}_\varepsilon))<+\infty$ and $\operatorname{spt}(R\llcorner(\mathbb{R}^N\setminus\mathcal{U}_\varepsilon))\subset\mathcal{Y}_{\varepsilon/2}$. We then apply Federer's strong approximation theorem [8, 4.2.20] to the integral current $R\llcorner\mathcal{U}_\varepsilon\in\mathcal{R}_2(\mathbb{R}^N)$ and find an integral polyhedral 2-current R_ε in $\mathbb{R}^N$ and a C^1-diffeomorphism $\psi_\varepsilon:\mathbb{R}^N\to\mathbb{R}^N$, with $\operatorname{Lip}\psi_\varepsilon,\operatorname{Lip}\psi_\varepsilon{}^{-1}\le 1+\varepsilon$, such that

$$\mathbf{M}(R_\varepsilon-\psi_{\varepsilon\#}(R\llcorner\mathcal{U}_\varepsilon))\le\varepsilon \qquad\text{and}\qquad \mathbf{M}(\partial R_\varepsilon-\partial\psi_{\varepsilon\#}(R\llcorner\mathcal{U}_\varepsilon))\le\varepsilon\,.$$

As a consequence we obtain that

$$R_\varepsilon\rightharpoonup R \qquad\text{and}\qquad \mathbf{M}(R_\varepsilon)\to\mathbf{M}(R) \tag{4.5}$$

as $\varepsilon\to 0^+$. Moreover, if $C_\varepsilon:=\partial R_\varepsilon$, we also infer that $\operatorname{spt} C_\varepsilon\subset\mathcal{Y}_\varepsilon$ and hence, due to weak convergence of $C_\varepsilon\rightharpoonup C$, for ε small C_ε is an integral 1-cycle with finite mass and of S^1-type in $\mathcal{Y}_\varepsilon$. Then, there exist $\widetilde{Z}_\varepsilon\in\mathcal{Z}_1(\mathcal{Y}_\varepsilon)$, with finite mass $\mathbf{M}(\widetilde{Z}_\varepsilon)<+\infty$, and $\widetilde{W}_\varepsilon\in\mathcal{R}_2(\mathcal{Y}_\varepsilon)$ such that

$$C_\varepsilon-\widetilde{Z}_\varepsilon=\partial\widetilde{W}_\varepsilon \qquad\text{and}\qquad \widetilde{Z}_\varepsilon=\widetilde{\phi}_{\varepsilon\#}[\![\, S^1 \,]\!]$$

for some Lipschitz function $\widetilde{\phi}_\varepsilon:S^1\to\mathcal{Y}_\varepsilon$. Moreover, as in Step 1 of [16, Prop. 4.5], using an argument by [24] and [8, 4.2.19], and regarding $\mathbb{R}^N$ as the subspace $\mathbb{R}^N\times\{0_{\mathbb{R}^2}\}$ of $\mathbb{R}^{N+2}$, we find a Lipschitz embedding $\phi_\varepsilon:S^1\to\widetilde{\mathcal{Y}}_\varepsilon$, where $\widetilde{\mathcal{Y}}_\varepsilon:=\mathcal{Y}_{2\varepsilon}\times\mathbb{R}^2$, and an integral polyhedral chain $W_\varepsilon\in\mathcal{R}_2(\mathbb{R}^{N+2})$ such that

$$C_\varepsilon-Z_\varepsilon=\partial W_\varepsilon \qquad\text{and}\qquad \phi_{\varepsilon\#}[\![\, S^1 \,]\!]=Z_\varepsilon\,. \tag{4.6}$$

Step 2: As in Step 2 of [16, Prop. 4.5], by "covering" the 2-faces of W_ε we first construct a Lipschitz embedding $\varphi_\varepsilon:S^1\to\mathbb{R}^{N+2}$ such that $\varphi_{\varepsilon\#}[\![\, S^1 \,]\!]=C_\varepsilon$. Secondly, as in [24], by "covering" the 2-faces of R_ε we find a Lipschitz extension $\Phi_\varepsilon:B^2\to\mathbb{R}^{N+2}$ such that $\Phi_{\varepsilon|S^1}=\varphi_\varepsilon$ and $\Phi_{\varepsilon\#}[\![\, B^2 \,]\!]=R_\varepsilon$. Moreover, we are able to define Φ_ε so that

$$\begin{array}{c}\text{There is a simplicial decomposition of } B^2\text{, say } \widetilde{\Delta}\text{, such that}\\ \Phi_\varepsilon \text{ maps each curvilinear 2-simplex } D \text{ of } \widetilde{\Delta} \text{ bijectively onto a 2-face of the 2-skeleton of } R_\varepsilon.\end{array} \tag{4.7}$$

Step 3: We now construct a Lipschitz map $g_\varepsilon:B^2\to\mathbb{R}^N$ such that g_ε maps B^2 into R_ε with mapping area equal to the mass of R_ε, i.e.,

$$g_{\varepsilon\#}[\![\, B^2 \,]\!]=R_\varepsilon \qquad\text{and}\qquad A(g_\varepsilon,B^2)=\mathbf{M}(R_\varepsilon)\,, \tag{4.8}$$

and such that, if $h_\varepsilon:=g_{\varepsilon|S^1}$, then

$$h_\varepsilon(S^1)\subset\widetilde{\mathcal{Y}}_\varepsilon \qquad\text{and}\qquad h_{\varepsilon\#}[\![\, S^1 \,]\!]=C_\varepsilon\,. \tag{4.9}$$

By (4.7) let $\{\widetilde{D}_i\}$ be a subfamily of the simplices of $\widetilde{\Delta}$ such that Φ_ε maps each $\widetilde{D}_i$ bijectively onto a 2-face of R_ε (with multiplicity and orientation), so that if $\widetilde{\mathcal{U}}:=\cup\widetilde{D}_i$ we have

$$\Phi_{\varepsilon\#}[\![\,\widetilde{\mathcal{U}}\,]\!]=R_\varepsilon \qquad\text{and}\qquad A(\Phi_\varepsilon,\widetilde{\mathcal{U}})=\mathbf{M}(R_\varepsilon)\,.$$

For every i, let D_i be the 2-simplex obtained by contracting $\widetilde{D}_i$ from its barycenter with an homothetic factor $1/2$, so that $\operatorname{dist}(D_{i_1},D_{i_2})>0$ if $i_1\ne i_2$. Finally, let $\mathcal{U}:=\cup D_i$. We first define g_ε on each D_i by contracting

$\varphi_{\varepsilon|\widetilde{D}_i}$ from the barycenter of $\widetilde{D}_i$. Then, since the mapping area is invariant under reparametrizations of the domain, (4.8) clearly holds if we are able to find a Lipschitz extension of g_ε to the whole disk B^2 so that the image of $g_{\varepsilon|B^2\setminus\mathcal{U}}$ is 1-dimensional.

To do this, we first make a list of the 1-simplices of the 1-skeleton of R_ε, each one with a fixed orientation. Then, for every i, we label each 1-face F of the boundary of D_i with $\pm j$, according to the property that g_ε maps F, with the orientation induced by D_i, onto the j^{th} 1-simplex of R_ε with orientation $\pm$.

Let now R^k_ε, $k = 1,\dots,l$, be the connected components of R_ε, so that $R_\varepsilon = \sum_{k=1}^l R^k_\varepsilon$, $\mathbf{M}(R_\varepsilon) = \sum_{k=1}^l \mathbf{M}(R^k_\varepsilon)$ and $\operatorname{spt} R^k_\varepsilon$ is connected. At the first step, we consider the simplices D_i corresponding to the faces of R^1_ε, say $D_1,\dots,D_m$. Possibly reordering the D_i's, for every $i = 1,\dots,m-1$ we connect D_i with D_{i+1} by a rectifiable arc γ_i with suitably chosen initial and final points IP_i and FP_i in the 0-skeleton of D_i and D_{i+1}, respectively, so that $g_\varepsilon(IP_i) = g_\varepsilon(FP_i)$, the interior of γ_i lies in $B^2\setminus\mathcal{U}$ and γ_i does not intersect γ_j for $j = 1,\dots,i-1$. Also, we slightly modify g_ε on the D_i's so that it is constant near the end points of γ_i. Then, by taking a small tubular neighborhood Γ_i of γ_i in B^2, we define g_ε on Γ_i as the constant map equal to $g_\varepsilon(IP_i) = g_\varepsilon(FP_i)$.

As a consequence, if $O_1 := \bigcup_{i=1}^m (D_i\cup\Gamma_i)$, with $\Gamma_m = \emptyset$, then

(i) O_1 has positive distance from each of the remaining simplices D_i;

(ii) $g_{\varepsilon\#}[\![O_1]\!] = R^1_\varepsilon$ and $A(g_\varepsilon, O_1) = \mathbf{M}(R^1_\varepsilon)$.

Actually, since O_1 is a topological disk in B^2, by using the labels $\pm j$ of the 1-faces of the D_i's, everything can be done in such a way that

(iii) $g_{\varepsilon|\partial O_1}$ is homotopic on the 1-skeleton of R^1_ε to a map $\widetilde{g}_\varepsilon$ such that $\widetilde{g}_\varepsilon(\partial O_1)\subset\widetilde{\mathcal{Y}}_\varepsilon$ and $\widetilde{g}_\varepsilon[\![\partial O_1]\!] = \partial R^1_\varepsilon$.

By induction on the connected components of R_ε, for $k = 2,\dots,l$, at the k^{th} step we repeat the previous argument for R^k_ε, defining g_ε on O_k so that conditions (i), (ii) and (iii) hold true, with k instead of 1. Moreover, we define the arcs γ_i so that in particular the Γ_i's do not intersect any of the O_j's, for $j = 1,\dots,k-1$. Then we can also require that

(iv) O_k has positive distance from O_j, for every $j = 1,\dots,k-1$.

As a consequence, by (iii) and (iv) we can find for every k a small neighborhood $\widetilde{O}_k$ of O_k in B^2, with $\operatorname{dist}(\widetilde{O}_{k_1},\widetilde{O}_{k_2}) > 0$ if $k_1\neq k_2$, and a Lipschitz extension of $g_{\varepsilon|O_k}$ to $\widetilde{O}_k$, so that the image of $\widetilde{O}_k\setminus O_k$ is a 1-dimensional subset of $\widetilde{\mathcal{Y}}_\varepsilon$ and g_ε maps the boundary of $\widetilde{O}_k$ onto $\widetilde{\mathcal{Y}}_\varepsilon$ with $g_\varepsilon[\![\partial\widetilde{O}_k]\!] = \partial R^k_\varepsilon$.

Choose a point p_k of the boundary of $\widetilde{O}_k$ and let $P_k := g_\varepsilon(p_k)$. For every $k = 1,\dots,l-1$, we connect p_k with p_{k+1} by means of a rectifiable arc $\widetilde{\gamma}_k$ such that the interior of $\widetilde{\gamma}_k$ lies inside $B^2\setminus\cup_{k=1}^l\widetilde{O}_k$ and $\widetilde{\gamma}_k$ does not intersect any of the $\widetilde{\gamma}_j$, for $j = 1,\dots,k-1$. Then define g_ε on each $\widetilde{\gamma}_k$ by parameterizing a Lipschitz continuous arc connecting the points P_k and P_{k+1}, so that $g_\varepsilon(\widetilde{\gamma}_k)\subset\widetilde{\mathcal{Y}}_\varepsilon$. By the construction we then find a Lipschitz extension of g_ε to B^2 such that the image of $B^2\setminus\bigcup_{k=1}^l((\widehat{O}_k\cup\widetilde{\gamma}_k)\setminus\widehat{O}_k)$ is a 1-dimensional subset of $\widetilde{\mathcal{Y}}_\varepsilon$ and finally (4.8) and (4.9) hold true.

Step 4: We first slightly modify the function g_ε near the boundary of B^2 in such a way that the conclusion of Step 3 still holds and g_ε maps the upper part ∂^+B of the boundary of B^2 constantly onto the given point $P\in\mathcal{Y}$. This can be done by means of an homotopy of $g_{\varepsilon|S^1}$ with values into a 1-dimensional subset of $\widetilde{\mathcal{Y}}_\varepsilon$.

Let $\psi_\varepsilon : B^+\to\overline{B}^2$ be a bilipschitz homeomorphism such that ψ_ε is the identity on ∂^+B. Moreover let $\Pi : \mathbb{R}^{N+2}\to\mathbb{R}^N$ be the orthogonal projection onto the first N coordinates.

If $f_\varepsilon := \Pi\circ g_\varepsilon\circ\psi_\varepsilon : B^+\to\mathbb{R}^N$, for $\varepsilon > 0$ small f_ε is a Lipschitz continuous function, with $f_{\varepsilon|\partial^+B}\equiv P$, and by (4.8) and (4.5), since $\Pi_\# R = R$,

$$f_{\varepsilon\#}[\![B^+]\!] = (\Pi\circ g_\varepsilon)_\#[\![B^2]\!] = \Pi_\# R_\varepsilon \rightharpoonup R$$

weakly in $\mathcal{D}_2(\mathbb{R}^N)$, as $\varepsilon\to 0^+$. Moreover, since $\operatorname{Lip}\Pi = 1$, we also have

$$A(f_\varepsilon, B^+) = A(\Pi\circ g_\varepsilon, B^2)\le A(g_\varepsilon, B^2)\,. \tag{4.10}$$

We now apply Morrey's ε-conformality theorem [21, Thm. 2.1] and define an orientation preserving diffeomorphism $\Psi_\varepsilon : B^+ \to B^+$ such that, if $\widetilde{f}_\varepsilon := f_\varepsilon \circ \Psi_\varepsilon$, then

$$\mathbf{D}(\widetilde{f}_\varepsilon, B^+) \leq (1+\varepsilon)\, A(\widetilde{f}_\varepsilon, B^+) = (1+\varepsilon)\, A(f_\varepsilon, B^+)\,.$$

Then, by (4.10), (4.8) and (4.5) we have $\lim_{\varepsilon\to 0} \mathbf{D}(\widetilde{f}_\varepsilon, B^+) = \mathbf{M}(R)$. Finally, due to the *three points condition*, we may and do define Ψ_ε so that it maps $\partial^+ B$ onto $\partial^+ B$ and J onto J. As a consequence, by (4.9) we infer that, for $\varepsilon > 0$ small, $\widetilde{f}_{\varepsilon|\partial^+ B} \equiv P$, $\widetilde{f}_\varepsilon(J) \subset \mathcal{Y}_{2\varepsilon}$ and $\widetilde{f}_{\varepsilon\#}[\![J]\!] \rightharpoonup C$ weakly in $\mathcal{D}_1(\mathcal{Y}_{2\varepsilon})$. $\square$

5 Approximate dipoles

In this section we provide the approximation of dipoles, see [7] [12] [14, Vol. II, Sec. 4.2.3], for $W^{1/2}$-maps with values in $\mathcal{Y}$. Using an argument similar to [18], we first show how to remove *homologically trivial* point singularities.

Proposition 5.1 (Removing point singularities). *Let $\mathcal{Y}$ be such that $\pi_1(\mathcal{Y})$ is commutative. Let $u \in R^\infty_{1/2,\varphi}(B^2, \mathcal{Y})$ be in $\mathrm{cart}^{1/2}(B^2, \mathcal{Y})$, so that (2.3) holds, with $n = 2$. Then there exists a sequence of smooth maps $u_k \subset C^\infty(B^2, \mathcal{Y})$ which converges to u strongly in $W^{1/2}$.*

Proof: Since we use a local argument, we may assume that u has only one singularity at the origin, i.e., $u \in C^\infty(B^2 \setminus \{0\}, \mathcal{Y})$. For $0 < r < 1$ we denote

$$Q_r := B^3_r \cap \mathcal{C}^3\,, \qquad \partial^+ Q_r := \partial B^3_r \cap \{z = (x,t) \in \mathcal{C}^3 \mid t > 0\}\,, \qquad F_r := Q_r \cap (B^2 \times \{0\})\,.$$

Let $U \in W^{1,2}(\mathcal{C}^3, \mathbb{R}^N)$ be the harmonic extension of u. For every fixed $\varepsilon > 0$ let $0 < R = R(\varepsilon) \ll 1$ be such that

$$\mathbf{D}(U, Q_R) \leq \varepsilon\,.$$

Since

$$\mathbf{D}(U, Q_R \setminus Q_{R/2}) = \frac{1}{2} \int_{R/2}^{R} dr \int_{\partial^+ Q_r} |DU|^2 \, d\mathcal{H}^2\,,$$

there exists $r = r_\varepsilon \in [R/2, R]$ such that

$$\mathbf{D}(U, \partial^+ Q_r) := \frac{1}{2} \int_{\partial^+ Q_r} |DU|^2 \, d\mathcal{H}^2 \leq \frac{4}{R} \mathbf{D}(U, Q_R \setminus Q_{R/2}) \leq \frac{4\varepsilon}{R}\,. \tag{5.1}$$

To remove the singularity of u, we have to show that

$$\{w \in W^{1/2}(B^2_r, \mathbb{R}^N) \cap C^0(\overline{B}^2_r, \mathcal{Y}) \mid w_{|\partial B^2_r} = u_{|\partial B^2_r}\} \neq \emptyset\,, \tag{5.2}$$

i.e., $u_{|\partial B^2_r}$ is homotopic to a constant map in $\mathcal{Y}$. Since by assumption the first homotopy group $\pi_1(\mathcal{Y})$ is commutative, it suffices to show that for every closed 1-form ω in $\mathcal{Y}$, or in a basis $\{\omega^s\}_{s=1}^{\overline{s}}$ of $\mathcal{Z}^1(\mathcal{Y})$, we have $d\, u_{|\partial B^2_r}{}^\# \omega = 0$. This follows from condition (2.3), as

$$\int_{\partial B^2_r} u_{|\partial B^2_r}{}^\# \omega^s = G_{u_{|\partial B^2_r}}(\widehat{\pi}^\# \omega^s) = \partial G_{u_{|B^2_r}}(\widehat{\pi}^\# \omega^s) = G_{u_{|B^2_r}}(d\widehat{\pi}^\# \omega^s) = G_{u_{|B^2_r}}(\widehat{\pi}^\# d\omega^s) = 0\,.$$

As a consequence there exists a smooth extension $u_r : B^2_r \to \mathcal{Y}$ of $u_{|\partial B^2_r}$ with finite $W^{1/2}$-energy.

Let now $V_r : Q_r \to \mathbb{R}^N$ be the solution of the Dirichlet problem on Q_r with boundary condition

$$\begin{cases} V_r = U & \text{on} \quad \partial^+ Q_r \\ V_r = u_r & \text{on} \quad F_r\,. \end{cases}$$

Let $0<\delta<r$ to be fixed later. Define $U_r:\mathcal{C}^3\to\mathbb{R}^N$ by

$$U_r(z):=\begin{cases} V_r\left(\dfrac{r}{\delta}z\right) & \text{if}\quad |z|\le\delta \\ U\left(r\dfrac{z}{|z|}\right) & \text{if}\quad \delta\le|z|\le r \\ U(z) & \text{if}\quad |z|\ge r \end{cases}$$

so that $U_r\in W^{1,2}(\mathcal{C}^3,\mathbb{R}^N)$ is continuous and with trace $\mathbf{T}(U_r)\in W^{1/2}(B^2,\mathcal{Y})$. We easily estimate

$$\mathbf{D}(U_r,\mathcal{C}^3)\le\mathbf{D}(U,\mathcal{C}^3)+c\,r\,\mathbf{D}(U,\partial^+Q_r)+\frac{\delta}{r}\mathbf{D}(V_r,Q_r)$$

for some absolute constant $c>0$, so that, by (5.1) and since $r<R$,

$$\mathbf{D}(U_r,\mathcal{C}^3)\le\mathbf{D}(U,\mathcal{C}^3)+4c\varepsilon+\frac{\delta}{r}\mathbf{D}(V_r,Q_r)\le\mathbf{D}(U,\mathcal{C}^3)+(4c+1)\,\varepsilon\,,$$

taking $\delta=\delta(\varepsilon)$ sufficiently small. Letting $\varepsilon\to0$ we infer that $U_{r_\varepsilon}\to U$ in $W^{1,2}(\mathcal{C}^3,\mathbb{R}^N)$ and finally that $\mathbf{T}(U_{r_\varepsilon})\to u$ in $W^{1/2}(B^2,\mathcal{Y})$, with $\mathbf{T}(U_{r_\varepsilon})\in W^{1/2}(B^2,\mathcal{Y})$ continuous. By a standard argument, we now approximate $\mathbf{T}(U_{r_\varepsilon})$ by smooth functions, as required. $\square$

Remark 5.2 In Proposition 6.6 we will extend Proposition 5.1 to remove the point singularities of any map $u\in R^\infty_{1/2,\varphi}(\widetilde{B}^2,\mathcal{Y})$, even if $\pi_1(\mathcal{Y})$ is not commutative, taking advantage of the smoothness of the boundary datum $\varphi:\widetilde{B}^2\to\mathcal{Y}$ on the whole of $\widetilde{B}^2$.

The dipole construction. We will adapt some results from [17], to which we refer for further details. To fix the notation, let $a_+,a_-\in B^2\times\{0\}$ and $L\in\mathcal{R}_1(\mathbb{R}^3)$ be the 1-current integration over the segment joining a_- to a_+, oriented so that the boundary $\partial L=\delta_{a_+}-\delta_{a_-}$ and with mass $\mathbf{M}(L)=l:=|a_+-a_-|\in(0,1)$. We assume

$$a_+:=(l,0,0)\,,\qquad a_-:=(0,0,0)\,.$$

Also, if $P\in\mathcal{Y}$ we let G_P denote the current carried by the graph of the map constantly equal to P. Moreover, we let $C\in\mathcal{Z}_1(\mathcal{Y})$ denote a nontrivial integral flat 1-cycle of S^1-type, see Definition 0.3, and $R\in\mathcal{R}_2(\mathbb{R}^N)$ the i.m. rectifiable current of least mass such that $\partial R=C$.

We first notice that in Proposition 4.5 we have in fact proved the following

Proposition 5.3 *For every $P\in\mathcal{Y}$ there exists a family $\{f^P_\varepsilon\}_{\varepsilon>0}$ of Lipschitz functions $f^P_\varepsilon:B^+\to\mathbb{R}^N$ such that $f^P_{\varepsilon|\partial^+B}\equiv P$, $f^P_\varepsilon(J)\subset\mathcal{Y}_\varepsilon$ and*

$$\mathbf{D}(f^P_\varepsilon,B^+)\le\mathbf{M}(R)+\frac{\varepsilon}{2}\,,$$

compare (4.3). *Moreover $R_\varepsilon:=f^P_{\varepsilon\#}[\![\,B^+\,]\!]\rightharpoonup R$ and $C_\varepsilon:=\partial R_\varepsilon=f^P_{\varepsilon\#}[\![\,J\,]\!]\rightharpoonup C$ weakly in $\mathcal{D}_2(\mathbb{R}^N)$ and $\mathcal{D}_1(\mathcal{Y}_\varepsilon)$, respectively, with $\mathbf{M}(R_\varepsilon)\to\mathbf{M}(R)$ as $\varepsilon\to0$. Finally R_ε and C_ε do not depend on the choice of $P\in\mathcal{Y}$.*

We shall need the following result from [17], compare [14, Vol. II, Sec. 4.2.3]. For $0<\delta<1$ and $0<m\le1$, let

$$\varphi^m_\delta(y):=\min\{my,m(l-y),\delta\}\,,\qquad 0\le y\le l\,.$$

Consider the map $\phi^m_\delta:(0,l)\times B^+\to\mathcal{C}^3$

$$\phi^m_\delta(z):=\big(x_1,\varphi^m_\delta(x_1)x_2,\varphi^m_\delta(x_1)t\big)\,,\qquad z=(x_1,x_2,t)$$

and define

$$\Omega^m_\delta:=\phi^m_\delta((0,l)\times B^+)=\{z=(x_1,x_2,t)\in\mathcal{C}^3\mid(x_2)^2+t^2<\varphi^m_\delta(x_1)^2\,,\quad 0<x_1<l\}\,.\tag{5.3}$$

Lemma 5.4 *Let $V:(0,l)\times B^+\to\mathbb{R}^N$ be a $W^{1,2}$ function and let*

$$V^m_\delta(z):=V\circ(\phi^m_\delta)^{-1}(z)\,,\qquad z\in\Omega^m_\delta\,.$$

Then there exists an absolute constant $c>0$ such that

$$\begin{aligned}\int_{\Omega^m_\delta}|DV^m_\delta|^2\,dz &\le \int_{(0,l)\times B^+}|D_{(x_2,t)}V|^2\,dz\\ &+ c\,m^2\int_{((0,\delta/m)\cup(l-\delta/m,l))\times B^+}|D_{(x_2,t)}V|^2\,dz+c\,\delta^2\int_{(0,l)\times B^+}|D_{x_1}V|^2\,dz\,.\end{aligned}\tag{5.4}$$

We then obtain the following

Theorem 5.5 (Approximate dipoles) *For every $P\in\mathcal{Y}$, there exists a family of maps $\{u_\varepsilon\}\subset C^1(B^2\setminus\{a_-,a_+\},\mathcal{Y})$ such that*

$$G_{u_\varepsilon}\rightharpoonup G_P+L\times C\qquad\text{weakly in }\mathcal{D}_{2,1}(B^2\times\mathcal{Y})\tag{5.5}$$

as $\varepsilon\to 0$ and

$$\mathcal{E}_{1/2}(u_\varepsilon,B^2)\to l\cdot\mathbf{M}(R)\,,\qquad l:=|a_+-a_-|\,.$$

Proof: First define $W_\varepsilon:(0,l)\times B^+\to\mathbb{R}^N$ by $W_\varepsilon(x_1,x_2,t):=f^P_\varepsilon(x_2,t)$, where f^P_ε is given by Proposition 5.3, so that W_ε is a smooth function. Then apply Lemma 5.4 with $V:=W_\varepsilon$ and $m=1$, to obtain a map $W_{\varepsilon,\delta}:=W_\varepsilon\circ(\phi^1_\delta)^{-1}$. Finally set

$$U_\varepsilon(z):=\begin{cases}W_{\varepsilon,\delta}(z) & \text{if}\quad z\in\Omega^1_\delta\\ P & \text{otherwise}\,,\end{cases}$$

so that U_ε belongs to $C^1(\mathcal{C}^3\setminus\{a_-,a_+\},\mathbb{R}^N)$, the trace $u_\varepsilon:=\mathbf{T}(U_\varepsilon)$ belongs to $W^{1/2}(B^2,\mathcal{Y}_\varepsilon)$ and by (5.4)

$$\mathbf{D}(U_\varepsilon,\mathcal{C}^3)\le l\cdot\mathbf{D}(f^P_\varepsilon,B^+)+\frac{\varepsilon}{2}$$

if δ is sufficiently small in dependence of ε and of the Lipschitz constant of W_ε. The assertion then follows from Proposition 5.3, projecting into $\mathcal{Y}$ and letting $\varepsilon\searrow 0$, see Remark 4.3. $\square$

We now extend Theorem 5.5 to Theorem 5.6, in which we connect an approximate dipole with a map $u\in W^{1/2}(B^2,\mathcal{Y})$ smooth outside a_+ and a_-.

Theorem 5.6 *Suppose that $\pi_1(\mathcal{Y})$ is commutative. Let $u:B^2\to\mathcal{Y}$ be a $W^{1/2}$-map which is smooth except at a_- to a_+. Suppose that*

$$\partial G_u=-\delta_{a_+}\times C+\delta_{a_-}\times C\qquad on\quad \mathcal{Z}^{1,1}(B^2\times\mathcal{Y})\,,\tag{5.6}$$

where $C\in\mathcal{Z}_1(\mathcal{Y})$ is an integral flat 1-cycle of S^1-type. Then there exists a smooth map $u_\varepsilon:B^2\to\mathcal{Y}$ such that, if $U_\varepsilon=\mathrm{Ext}(u_\varepsilon)$ and $U=\mathrm{Ext}(u)$, then $U_\varepsilon=U$ outside a neighborhood $\mathcal{U}^m_\delta$ of $\overline{\Omega}^m_\delta$, for some arbitrarily small $\delta,m>0$, see (5.3). Moreover, $G_{U_\varepsilon}\rightharpoonup G_U+L\times R$ weakly in $\mathcal{D}_{3,2}(\mathcal{C}^3\times\mathbb{R}^N)$ and

$$\mathbf{D}(U_\varepsilon,\mathcal{U}^m_\delta)\le l\cdot\mathbf{M}(R)+\varepsilon\,,\tag{5.7}$$

where $R\in\mathcal{R}_2(\mathbb{R}^N)$ the i.m. rectifiable current of least mass such that $\partial R=C$. In particular $G_{u_\varepsilon}\rightharpoonup G_u+L\times C$ weakly in $\mathcal{D}_{2,1}(B^2\times\mathcal{Y})$ and $\mathcal{E}_{1/2}(G_{u_\varepsilon})\to\mathcal{E}_{1/2}(G_u+L\times C)$.

In the proof of the previous theorem, we will make use of the following facts, which we state separately for future use.

Remark 5.7 Similarly to [2], for any given point $p=(x_0,0)$ in $B^2\times\{0\}$ we can choose small half-balls of radius r around p and replace U there by the radial maps

$$U_r(z):=U\Big(p+r\frac{z-p}{|z-p|}\Big)\tag{5.8}$$

so that

$$\mathbf{D}(U_r, B^3_r(p)\cap\mathcal{C}^3) = \frac{r}{2}\int_{\partial B^3_r(p)\cap\mathcal{C}^3} |D_\tau U|^2\, d\mathcal{H}^2 = O(r)\,,$$

where τ is an orthonormal frame of $\partial B^3_r(p)$ and $O(r_j)\to 0$ for some sequence $r_j\searrow 0$. In particular, if U is smooth out of the point p and $u=\mathbf{T}(U)\in W^{1/2}(B^2,\mathcal{Y})$, setting $u_r=\mathbf{T}(U_r)\in W^{1/2}(B^2_r(x_0),\mathcal{Y})$ we have

$$\partial G_{u_r} = -\delta_{x_0}\times C_r \qquad \text{on} \quad \mathcal{Z}^{1,1}(B^2_r(x_0)\times\mathcal{Y})\,,$$

where

$$C_r := u_{|\partial B^2_r(x_0)\#}[\![\,\partial B^2_r(x_0)\,]\!]\,. \tag{5.9}$$

Remark 5.8 Note that, if u is smooth near the point x_0, by (5.9) we have that C_r is a boundary on $\mathcal{Y}$, being $C_r=\partial R_r$ with $R_r := u_{|B^2_r(x_0)\#}[\![\,B^2_r(x_0)\,]\!]\in\mathcal{D}_2(\mathcal{Y})$. On the other hand, if $\partial G_u=\delta_{x_0}\times C$ on $\mathcal{Z}^{1,1}(B^2_r(x_0)\times\mathcal{Y})$ for some 1-cycle $C\in\mathcal{Z}_1(\mathcal{Y})$, then $C_r-C=\partial R_r$.

Lemma 5.9 *Suppose that $\pi_1(\mathcal{Y})$ is commutative. Let $C\in\mathcal{Z}_1(\mathcal{Y})$ and $R\in\mathcal{R}_2(\mathbb{R}^N)$ be such that $\partial R=C$. Let $u_\varepsilon: B^2\to\mathcal{Y}_\varepsilon$ be a sequence of $W^{1/2}$-functions smooth outside the origin such that*

$$\partial G_{u_\varepsilon} \llcorner B^2\times\mathcal{Y}_\varepsilon = \delta_0\times(C_\varepsilon - Z_\varepsilon) \qquad \textit{on} \quad \mathcal{Z}^{1,1}(B^2\times\mathcal{Y}_\varepsilon)\,, \tag{5.10}$$

where $Z_\varepsilon\in\mathcal{Z}_1(\mathcal{Y}_\varepsilon)$ is such that $Z_\varepsilon - C=\partial W_\varepsilon$ for some $W_\varepsilon\in\mathcal{D}_2(\mathcal{Y}_\varepsilon)$ and $C_\varepsilon\in\mathcal{Z}_1(\mathcal{Y}_\varepsilon)$ is the boundary $C_\varepsilon=\partial R_\varepsilon$ of an i.m. rectifiable current $R_\varepsilon\in\mathcal{R}_2(\mathbb{R}^N)$ such that $R_\varepsilon\rightharpoonup R$ weakly in $\mathcal{D}_2(\mathbb{R}^N)$ with $\mathbf{M}(R_\varepsilon)\to\mathbf{M}(R)$ as $\varepsilon\to 0$. Then there exists a sequence of smooth maps $v_\varepsilon: B^2\to\mathcal{Y}_\varepsilon$ such that $v_\varepsilon=u_\varepsilon$ in a neighborhood of the boundary ∂B^2, $(G_{v_\varepsilon}-G_{u_\varepsilon})\rightharpoonup 0$ weakly in $\mathcal{D}_{2,1}(B^2\times\mathcal{Y}_\varepsilon)$ as $\varepsilon\to 0$ and

$$\mathcal{E}_{1/2}(v_\varepsilon, B^2)\le\mathcal{E}_{1/2}(u_\varepsilon,B^2)+\varepsilon\,.$$

Proof of Theorem 5.6: Applying Remarks 5.7 and 5.8 we may suppose

$$U(z)=U\left(a_\pm + r_\pm\frac{z-a_\pm}{|z-a_\pm|}\right) \qquad \text{if} \quad 0<|z-a_\pm|<r_\pm \tag{5.11}$$

for some small radius $0<r_\pm\ll l$. Moreover by (5.6) we have

$$\partial G_u = -\delta_{a_+}\times C^+ + \delta_{a_-}\times C^- \qquad \text{on} \quad \mathcal{Z}^{1,1}(B^2\times\mathbb{R}^N)\,, \tag{5.12}$$

where $C^\pm\in\mathcal{Z}_1(\mathcal{Y})$ are such that $C^\pm - C=\partial R^\pm$ for some $R^\pm\in\mathcal{D}_2(\mathcal{Y})$.

We divide the cylinder $(0,l)\times B^+_r$ so that the oscillation of U is smaller than ε. To this aim, let $K^{\delta,m}_\pm$ be the intersection of $\mathcal{C}^3\cap B^3(a_\pm, r_{\delta,m})$ with the "left" and "right" interior of the half cones of center $a_\pm$, x_1-axis, and angle $\arctan m$, see (5.13) and (5.14). By (5.11) we infer that we may find $0<m=m(\varepsilon)\ll 1$ so that U maps the $K^{\delta,m}_\pm$'s into sets of diameter ε.

More precisely, we introduce the cylindrical coordinates w.r.t. the x_1-axis

$$z=(x_1,x_2,t)=F(\rho,\theta,y):=(y,\rho\cos\theta,\rho\sin\theta)\,, \qquad \rho>0, \quad \theta\in[0,\pi]\,,$$

so that $\rho=\sqrt{{x_2}^2+t^2}$, and define

$$\begin{aligned} K^{\delta,m}_+ &:= \{z\in\mathcal{C}^3 \,:\, 0<|z-a_+|<r_{\delta,m}\,, \quad l-\delta/m<x_1<l\,, \quad \rho<m\,(l-x_1)\} \\ K^{\delta,m}_- &:= \{z\in\mathcal{C}^3 \,:\, 0<|z-a_-|<r_{\delta,m}\,, \quad 0<x_1<\delta/m\,, \quad \rho<m\,x_1\}\,, \end{aligned} \tag{5.13}$$

where

$$r_{\delta,m}:=\delta\,\frac{\sqrt{1+m^2}}{m}\,. \tag{5.14}$$

We first choose $0<m=m(\varepsilon)\ll 1$ and then $\delta=\delta(m,r_\pm)$ small so that U does not depend on $|z-a_\pm|$ on the cones $K^{\delta,m}_\pm$ and U maps the cones $K^{\delta,m}_\pm$ into sets of diameter ε. Note that by (5.11) the first

condition is satisfied if $r_{\delta,m} < \min\{r_+, r_-\}$. Moreover, since U is smooth outside the points $a_\pm$, if ϕ^m_δ is given by Lemma 5.4, we may and will choose δ small enough so that we find a finite number of points $0 = y_0 < y_1 < \cdots < y_n = l$ such that U maps every set $\phi^m_\delta((y_{i-1}, y_i) \times B^+)$ into a set of diameter ε. Let now

$$p_i = (y_i, 0, 0)\,, \qquad i = 0, \dots, n\,,$$

so that $p_0 = a_-$ and $p_n = a_+$. By using the coarea formula w.r.t. the angle coordinate $\phi := \arctan(\rho/|y - y_i|)$, due to the smoothness of U, without loss of generality we may and do choose m so that

$$\int_{K^m_{p_i}} |DU|^2 \, d\mathcal{H}^2 < +\infty \qquad \forall\, i = 0, \dots, n\,, \tag{5.15}$$

where $K^m_{p_i}$ is the cone of vertex p_i and angle $\arctan m$

$$K^m_{p_i} := \{z = F(\rho, \theta, y) \in \mathcal{C}^3 \,:\, 0 < \rho = m\,|y - y_i|\}\,.$$

For every $i = 1, \dots, n$ we define $W^i_\varepsilon : (y_{i-1}, y_i) \times B^+ \to \mathbb{R}^N$ by $W^i_\varepsilon(x_1, x_2, t) := f^{P_i}_\varepsilon(x_2, t)$, where $f^{P_i}_\varepsilon$ is given by Proposition 5.3 in correspondence to C and to the point $P_i := u(q_i) \in \mathcal{Y}$ for some given q_i inside the segment $[p_{i-1}, p_i]$. Moreover we let

$$\varphi^{m,i}_\delta(y) := \min\{m(y - y_{i-1}), m(y_i - y), \delta\}\,, \qquad y_{i-1} \le y \le y_i\,,$$

consider the map $\phi^{m,i}_\delta : (y_{i-1}, y_i) \times B^+ \to \mathbb{R}^3$

$$\phi^{m,i}_\delta(x_1, x_2, t) := \big(x_1, \varphi^{m,i}_\delta(x_1) x_2, \varphi^{m,i}_\delta(x_1) t\big)$$

and define

$$V^i_\varepsilon(z) := W^i_\varepsilon \circ (\phi^{m,i}_\delta)^{-1}(z)\,, \qquad z \in \phi^{m,i}_\delta((y_{i-1}, y_i) \times B^+)\,.$$

As in Theorem 5.5 we estimate

$$\mathbf{D}(V^i_\varepsilon, \phi^{m,i}_\delta((y_{i-1}, y_i) \times B^+)) \le |y_i - y_{i-1}| \cdot \mathbf{D}(f^{P_i}_\varepsilon, B^+) + \frac{\varepsilon}{5n} \tag{5.16}$$

if we choose $\delta = \delta(W^i_\varepsilon, \varepsilon, m, n)$ sufficiently small.

In the sequel we will denote by

$$\widetilde{W}(\rho, \theta, y) := W(F(\rho, \theta, y)) = W(y, \rho\cos\theta, \rho\sin\theta)$$

a function in cylindrical coordinates. We then define U^i_ε on $\phi^m_\delta((y_{i-1}, y_i) \times B^+)$ by

$$\widetilde{U}^i_\varepsilon(\rho, \theta, y) := \begin{cases} \widetilde{V}^i_\varepsilon(2\rho, \theta, y) & \text{if} \quad 0 \le \rho < \varphi^{m,i}_\delta(y)/2 \\ \widetilde{\Psi}^{m,i}_\delta(\rho, \theta, y) & \text{if} \quad \varphi^{m,i}_\delta(y)/2 \le \rho < \varphi^{m,i}_\delta(y) \end{cases} \qquad \theta \in [0, \pi]\,, \quad y \in (y_{i-1}, y_i)\,,$$

where

$$\widetilde{\Psi}^{m,i}_\delta(\rho, \theta, y) := \left(\frac{2\rho}{\varphi^{m,i}_\delta(y)} - 1\right) \cdot \widetilde{U}(\varphi^{m,i}_\delta(y), \theta, y) + \left(2 - \frac{2\rho}{\varphi^{m,i}_\delta(y)}\right) \cdot P_i\,.$$

Setting $U_\varepsilon := U^i_\varepsilon$ on $\phi^m_\delta((y_{i-1}, y_i) \times B^+)$ for $i = 1, \dots, n$, and $U_\varepsilon = U$ elsewhere, we find that U_ε is smooth outside the set $\{p_i \mid i = 0, \dots, n\}$. Moreover, we infer that the trace $u_\varepsilon = \mathbf{T}(U_\varepsilon)$ belongs to $W^{1/2}(B^2, \mathcal{Y}_\varepsilon)$ whereas, by the construction, u_ε satisfies

$$\partial G_{u_\varepsilon} = \delta_{a_+} \times (C_\varepsilon - C^+_\varepsilon) - \delta_{a_-} \times (C_\varepsilon - C^-_\varepsilon) \qquad \text{on} \quad \mathcal{Z}^{1,1}(B^2 \times \mathcal{Y}_\varepsilon)\,, \tag{5.17}$$

where $C^\pm_\varepsilon \in \mathcal{Z}_1(\mathcal{Y}_\varepsilon)$ is such that $C^\pm_\varepsilon - C = \partial W^\pm_\varepsilon$ for some $W^\pm_\varepsilon \in \mathcal{D}_2(\mathcal{Y}_\varepsilon)$ and $C_\varepsilon := f_{\varepsilon\#}[\![J]\!] \in \mathcal{Z}_1(\mathcal{Y}_\varepsilon)$, compare Proposition 5.3, so that $C_\varepsilon = \partial R_\varepsilon$, where $R_\varepsilon := f^P_{\varepsilon\#}[\![B^+]\!] \rightharpoonup R$ weakly in $\mathcal{D}_2(\mathbb{R}^N)$ with $\mathbf{M}(R_\varepsilon) \to \mathbf{M}(R)$ as $\varepsilon \to 0$. In particular we infer that the graph of u_ε has no boundary near the points $(y_i, 0)$, for $i = 1, \dots, n-1$. Therefore, since $\pi_1(\mathcal{Y}_\varepsilon)$ is commutative, compare Remark 4.3, these singularities can be removed by means of Proposition 5.1, by paying a small amount of energy. Similarly, using Lemma 5.9 we remove the singularities at $a_\pm$. Consequently, u_ε is smooth everywhere.

We now show that if δ is small then (5.7) holds. To this aim, in the sequel $c>0$ will denote an absolute constant, possibly varying from line to line.

For fixed $i=1,\ldots,n$, if $y_{i-1}+\delta/m<y<y_i-\delta/m$, then $\varphi_\delta^{m,i}(y)\equiv\delta$ and hence, since U maps $\phi_\delta^m((y_{i-1},y_i)\times B^+)$ into a set of diameter ε, we estimate

$$|DU_\varepsilon^i|\leq\frac{2\varepsilon}{\delta}+c\left(\frac{1}{\rho}\,|\widetilde{U}_{,\theta}(\delta,\theta,y)|+|\widetilde{U}_{,y}(\delta,\theta,y)|\right)$$

for a.e. $\rho\in(\delta/2,\delta)$, so that

$$\begin{aligned}\mathbf{D}(U_\varepsilon^i,\{\rho\in(\varphi_\delta^{m,i}(y)/2,\varphi_\delta^{m,i}(y))\,,\ y\in(y_{i-1}+\delta/m,y_i-\delta/m)\,,\ \theta\in[0,\pi]\})\leq\\ \leq 4\,|y_i-y_{i-1}|\,\varepsilon^2+c\,\delta\int\limits_{(y_{i-1},y_i)\times\partial B_\delta^2}|DU|^2\,d\mathcal{H}^2\,.\end{aligned}$$

If $y_i-\delta/m<y<y_i$, then $\varphi_\delta^{m,i}(y)=m(y_i-y)$ and hence, since $m\in(0,1)$,

$$|DU_\varepsilon^i(z)|\leq c\left(\frac{\varepsilon}{m\,|z|}+|DU\circ F(m(y_i-y),\theta,y)|\right)$$

for a.e. $\rho\in(\varphi_\delta^{m,i}(y)/2,\varphi_\delta^{m,i}(y))$, so that

$$\begin{aligned}\mathbf{D}(U_\varepsilon^i,\{\rho\in(\varphi_\delta^{m,i}(y)/2,\varphi_\delta^{m,i}(y))\,,\ y\in(y_i-\delta/m,y_i)\,,\ \theta\in[0,\pi]\})\leq\\ c\left(\frac{\delta}{m}\,\varepsilon^2+m\int\limits_{K_{p_i}^m\cap B^3_{r_{\delta,m}}(p_i)}|DU|^2\,d\mathcal{H}^2\right),\end{aligned}$$

see (5.14) and (5.15). Moreover a similar estimate holds if $y_{i-1}<y<y_{i-1}+\delta/m$. Finally we clearly have

$$\mathbf{D}(U_\varepsilon^i,\{\rho\in(0,\varphi_\delta^{m,i}(z)/2)\,,\ y\in(y_{i-1},y_i)\,,\ \theta\in[0,\pi]\})=\mathbf{D}(V_\varepsilon^i,\phi_\delta^{m,i}((y_{i-1},y_i)\times B^+))\,.$$

Now, if $\Phi(\rho):=\int_{(0,l)\times\partial B_\rho^+}|DU|^2\,d\mathcal{H}^2$, since $\int_0^1\Phi(\rho)\,dr<+\infty$, we obtain that $\liminf_{\rho\to0^+}\rho\,\Phi(\rho)=0$. Therefore, if we first choose m small, and then $\delta:=\delta_\varepsilon=\delta(W_\varepsilon^i,\varepsilon,m,n)$ sufficiently small so that, by (5.15),

$$c\left(\delta\int\limits_{(y_{i-1},y_i)\times\partial B_\delta^2}|DU|^2\,d\mathcal{H}^2+\frac{\delta}{m}\,\varepsilon^2+m\int\limits_{K_{p_i}^m\cap B^3_{r_{\delta,m}}(p_i)}|DU|^2\,d\mathcal{H}^2\right)\ll\varepsilon\qquad\forall\,i=0,\ldots,n\,,$$

then by (5.16) for every i we obtain

$$\mathbf{D}(U_\varepsilon^i,\phi_\delta^{m,i}((y_{i-1},y_i)\times B^+))\leq|y_i-y_{i-1}|\left(\mathbf{D}(f_\varepsilon^{P_i},B^+)+4\,\varepsilon^2\right)+\frac{\varepsilon}{4n}\,.$$

By Proposition 5.3, and by taking the sum on i, we then infer

$$\mathbf{D}(U_\varepsilon,\{U_\varepsilon\neq U\})\leq l\cdot\mathbf{M}(R)+4\,l\,\varepsilon^2+\frac{3}{4}\,\varepsilon\,,$$

so that (5.7) follows for ε small. Finally, projecting into the manifold $\mathcal{Y}$, and letting $\varepsilon\searrow0$, we conclude taking into account Proposition 5.3. □

Proof of Lemma 5.9: Arguing as in the proof of Proposition 5.1, we have to show that (5.2) holds true for a.e. small $r>0$. By Remark 4.3 we infer that $\pi_1(\mathcal{Y}_\varepsilon)$ is commutative. Therefore it suffices to show that $d\,u_{|\partial B_r^2}{}^\#\omega^s=0$ for every closed 1-form in a basis $\{\omega^s\}_{s=1}^{\overline{s}}$ of $\mathcal{Z}^1(\mathcal{Y}_\varepsilon)$. By (5.10) we have

$$\partial G_{u_\varepsilon|B_r^2}=\partial(G_{u_\varepsilon}\llcorner B_r^2\times\mathcal{Y}_\varepsilon)=G_{u_{\varepsilon|\partial B_r^2}}+\delta_0\times(C_\varepsilon-Z_\varepsilon)\qquad\text{on}\quad\mathcal{Z}^{1,1}(B^2\times\mathcal{Y}_\varepsilon)\,,$$

so that we infer

$$\int_{\partial B_r^2} {u_{\varepsilon|\partial B_r^2}}^{\#}\omega^s = G_{u_{\varepsilon|\partial B_r^2}}(\widehat{\pi}^{\#}\omega^s) = G_{u_{\varepsilon|B_r^2}}(d\widehat{\pi}^{\#}\omega^s) - C_\varepsilon(\omega^s) + Z_\varepsilon(\omega^s)\,.$$

Since $Z_\varepsilon - C = \partial W_\varepsilon$ for some $W_\varepsilon \in \mathcal{D}_2(\mathcal{Y}_\varepsilon)$, we have

$$Z_\varepsilon(\omega^s) = W_\varepsilon(d\omega^s) + C(\omega^s) = C(\omega^s)\,.$$

Let now $j : \mathcal{Y}_{\varepsilon_0} \to \mathbb{R}^N$ be the inclusion map, and let $\widetilde{\omega}^s \in \mathcal{D}_1(\mathbb{R}^N)$ be such that $\omega^s = j^{\#}\widetilde{\omega}^s$. Since $j_{\#}C = C$ and $j_{\#}C_\varepsilon = C_\varepsilon$, due to strong convergence of R_ε to R we have

$$C_\varepsilon(\omega^s) = C_\varepsilon(\widetilde{\omega}^s) = R_\varepsilon(d\widetilde{\omega}^s) \to R(d\widetilde{\omega}^s) = C(\widetilde{\omega}^s) = C(\omega^s)\,,$$

so that

$$C_\varepsilon(\omega^s) - Z_\varepsilon(\omega^s) = C_\varepsilon(\omega^s) - C(\omega^s) \to 0$$

as $\varepsilon \to 0$, whereas

$$G_{u_{\varepsilon|B_r^2}}(d\widehat{\pi}^{\#}\omega^s) = G_{u_{\varepsilon|B_r^2}}(\widehat{\pi}^{\#}d\omega^s) = 0\,.$$

Taking ε small, and $r = r(\varepsilon)$ small with ε, we conclude that $d\,{u_{\varepsilon|\partial B_r^2}}^{\#}\omega^s = 0$, so that the assertion follows similarly to Proposition 5.1, by a diagonal argument. $\square$

6 A density result in dimension 2

In this section we prove a strong density result for the $W^{1/2}$-energy for graphs of maps from B^2 into a manifold $\mathcal{Y}$ as described in the introduction.

Theorem 6.1 *Assume that the first homotopy group $\pi_1(\mathcal{Y})$ is commutative. For every $T \in \mathrm{cart}^{1/2}(B^2 \times \mathcal{Y})$ there exists a sequence of smooth maps $\{u_k\} \subset C^\infty(B^2, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathrm{cart}^{1/2}$ and*

$$\lim_{k\to+\infty} \mathcal{E}_{1/2}(u_k, B^2) = \mathcal{E}_{1/2}(T, B^2 \times \mathcal{Y})\,.$$

Theorem 6.1 holds true even if the first homotopy group $\pi_1(\mathcal{Y})$ is not commutative, provided that the boundary data are smooth.

Theorem 6.2 *Let $T \in \mathrm{cart}_\varphi^{1/2}(\widetilde{B}^2 \times \mathcal{Y})$, where $\varphi : \widetilde{B}^2 \to \mathcal{Y}$ is a smooth map. Then there exists a sequence of smooth maps $\{u_k\} \subset C_\varphi^\infty(\widetilde{B}^2, \mathcal{Y})$ such that $G_{u_k} \rightharpoonup T$ weakly in $\mathrm{cart}^{1/2}$ and*

$$\lim_{k\to+\infty} \mathcal{E}_{1/2}(u_k, \widetilde{B}^2) = \mathcal{E}_{1/2}(T, \widetilde{B}^2 \times \mathcal{Y})\,.$$

As a consequence of Theorems 6.1 and 6.2, compare [14], Vol. II, Sec. 4.2.5, we infer the following density results for $W^{1/2}$-maps.

Corollary 6.3 *Let $\mathcal{Y}$ and φ be as in Theorem 6.1, respectively in Theorem 6.2. Then every map u in $W^{1/2}(B^2, \mathcal{Y})$, respectively in $W_\varphi^{1/2}(\widetilde{B}^2, \mathcal{Y})$, can be approximated weakly in $W^{1/2}$ by a sequence of maps in $C^1(B^2, \mathcal{Y}) \cap W^{1/2}(B^2, \mathcal{Y})$, respectively in $C_\varphi^1(\widetilde{B}^2, \mathcal{Y}) \cap W^{1/2}(\widetilde{B}^2, \mathcal{Y})$. Moreover, u can be approximated strongly in $W^{1/2}$ by a sequence of smooth maps provided that $\mathbb{P}_s(u) = 0$ for every $s = 1, \ldots, \overline{s}$.*

Remark 6.4 Similarly to [17], if the first homotopy group $\pi_1(\mathcal{Y})$ is not commutative there exist currents T in $\mathrm{cart}^{1/2}(B^2 \times \mathcal{Y})$ of the type $T = G_u$ which cannot be approximated weakly in $\mathrm{cart}^{1/2}$ by graphs of smooth maps $u_k : B^2 \to \mathcal{Y}$ such that $\mathcal{E}_{1/2}(G_{u_k}) \to \mathcal{E}_{1/2}(G_u)$. It suffices to take $u = \varphi(x/|x|)$ for some Lipschitz function $\varphi : S^1 \to \mathcal{Y}$ with $\varphi_{\#}[\![S^1]\!] = 0$, so that $u \in \mathrm{cart}^{1/2}(B^2, \mathcal{Y})$, but such that φ is not homotopic to a constant map in $\mathcal{Y}$. In fact, if u were approximable by smooth maps from B^2 to $\mathcal{Y}$ strongly in $W^{1/2}$, then from [3, Lemma 1] we would obtain that φ is homotopically trivial.

As we have seen, every $T = G_{u_T} + S_T \in \mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$ decomposes as

$$T = G_{u_T} + \sum_{s=1}^{\widetilde{s}} \mathbb{L}_s(T) \times \gamma_s \qquad \text{on} \qquad \mathcal{Z}^{n,1}(B^n \times \mathcal{Y}) \,, \tag{6.1}$$

compare Sec. 2, where $\mathbb{L}_s(T) \in \mathcal{R}_{n-1}(B^n)$ only depends on the cohomology class of ω^s and is defined by

$$\mathbb{L}_s(T)(\phi) := S_T(\pi^{\#}\phi \wedge \widehat{\pi}^{\#}\omega^s) \,, \qquad \phi \in \mathcal{D}^{n-1}(B^n) \,.$$

We explicitly state the following easy consequence of the results proved in Sec. 2.

Proposition 6.5 *Let $n \geq 2$ and $T \in \mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$, respectively $T \in \mathrm{cart}_{\varphi}^{1/2}(\widetilde{B}^n \times \mathcal{Y})$. There exists a sequence $\{u_k\}$ in $R^{\infty}_{1/2}(B^n, \mathcal{Y})$, respectively in $R^{\infty}_{1/2,\varphi}(\widetilde{B}^n, \mathcal{Y})$, strongly converging to u_T in $W^{1/2}$, such that if $L^s_{u_k,u_T}$ is given by* (1.3)*, then we have*

$$T_k := G_{u_k} + \sum_{s=1}^{\widetilde{s}} ((-1)^{n-1} L^s_{u_k,u_T} + \mathbb{L}_s(T)) \times \gamma_s$$

belongs to $\mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$, respectively to $\mathrm{cart}_{\varphi}^{1/2}(\widetilde{B}^n \times \mathcal{Y})$, the masses $\mathbf{M}(\partial((-1)^{n-1} L^s_{u_k,u_T} + \mathbb{L}_s(T)))$ are finite for every s and k, $T_k \rightharpoonup T$ weakly in $\mathcal{D}_{n,1}$ and $\mathcal{E}_{1/2}(T_k) \to \mathcal{E}_{1/2}(T)$ as $k \to +\infty$.

Proof of Theorem 6.1: We divide the proof in five steps.

Step 1: Reduction to finite singularities. By applying Proposition 6.5, we can assume that u_T has only a finite number of singularities or, equivalently, that $\mathbb{P}_s(u_T)$ is rectifiable for every $s = 1, \ldots, \widetilde{s}$. This yields that we may and do suppose

$$T = G_{u_T} + \sum_{q \in H_1^{sph}(\mathcal{Y})} \mathbb{L}_q \times C_q \,, \qquad \widetilde{T} := \mathrm{Ext}(T) = (-1)^{n-1}\Big(G_{U_T} + \sum_{q \in H_1^{sph}(\mathcal{Y})} \mathbb{L}_q \times R_q\Big) \,,$$

where $U_T := \mathrm{Ext}(u_T)$ and the $\mathbb{L}_q$'s are i.m. rectifiable currents in $\mathcal{R}_1(B^2)$ with multiplicity 1, pairwise disjoint supports and finite boundary mass. Moreover $C_q \in q$ and $R_q \in \mathcal{R}_2(\mathbb{R}^N)$ is the i.m. rectifiable current of least mass in $\mathbb{R}^N$ such that $\partial R_q \in q$, so that

$$\mathcal{E}_{1/2}(T) = \mathbf{D}(\widetilde{T}) = \frac{1}{2} \int_{\mathcal{C}^3} |DU_T|^2 \, dx \, dt + \sum_{q \in H_1^{sph}(\mathcal{Y})} \mathbf{M}(\mathbb{L}_q) \cdot \mathbf{M}(R_q) \,.$$

Actually, since T is defined in a homological sense, compare Remark 2.2, we may and will choose $C_q \in q$ in such a way that $C_q = \partial R_q$. Finally, since

$$\sum_{q \in H_1^{sph}(\mathcal{Y})} \mathbf{M}(\partial \, \mathbb{L}_q) < +\infty \,,$$

by the boundary rectifiability theorem [8] we infer that the $\partial \, \mathbb{L}_q$'s are i.m. rectifiable currents.

Step 2: Approximation by polyhedral chains. Since the $\mathbb{L}_q$'s are integral currents with pairwise disjoint supports, using Federer's polyhedral approximation theorem [8], for every $q \in H_1^{sph}(\mathcal{Y})$ we find an integral polyhedral chain P^{ε}_q with support contained in a small neighborhood of radius $c\,\varepsilon$ of the support of $\mathbb{L}_q$, and a function $U_\varepsilon \in C^1(\mathcal{C}^3, \mathbb{R}^N)$, with trace $u_\varepsilon := \mathbf{T}(U_\varepsilon) \in R^{\infty}_{1/2}(B^2, \mathcal{Y})$, such that if

$$\widetilde{T}_\varepsilon := G_{U_\varepsilon} + \sum_{q \in H_1^{sph}(\mathcal{Y})} P^{\varepsilon}_q \times R_q \,,$$

$\widetilde{T}_\varepsilon$ converges weakly in $\mathcal{D}_{3,2}(\mathcal{C}^3 \times \mathbb{R}^N)$ to $\widetilde{T}$ as $\varepsilon \to 0$ and

$$\mathbf{D}(U_\varepsilon, \mathcal{C}^3) + \sum_{q \in H_1^{sph}(\mathcal{Y})} \mathbf{M}(P_q^\varepsilon)\, \mathbf{M}(R_q) \to \mathbf{D}(U_T, \mathcal{C}^3) + \sum_{q \in H_1^{sph}(\mathcal{Y})} \mathbf{M}(\mathbb{L}_q)\, \mathbf{M}(R_q)\,,$$

i.e. $\mathbf{D}(\widetilde{T}_\varepsilon) \to \mathbf{D}(\widetilde{T})$ as $\varepsilon \to 0$. Moreover, since the $\mathbb{L}_q$'s have disjoint supports, we may and do choose the P_q^ε's so that for every small $\varepsilon > 0$ they have pairwise disjoint supports.

Step 3: Approximation by well-intersecting polyhedral chains. By Step 2 we may suppose

$$T := G_{u_T} + \sum_{q \in H_1^{sph}(\mathcal{Y})} P_q \times C_q \in \mathrm{cart}^{1/2}(B^2 \times \mathcal{Y})\,, \tag{6.2}$$

where the P_q's are polyhedral 1-chains, with multiplicity 1 and pairwise disjoint supports $\mathrm{spt}\, P_q \subset B^2$, and $u_T \in R^0_{1/2}(B^2, \mathcal{Y})$ is locally Lipschitz on $B^2 \setminus \bigcup_q \mathrm{spt}\, \partial P_q$. Moreover, possibly dividing the segments of the P_q's, we may and will suppose that every P_q is the union of a finite number of line segments S which only intersect at boundary points.

Step 4: Approximating the dipoles. Let T be as in (6.2), where the P_q's are polyhedral 1-chains as in the conclusion of Step 3. Let us fix $\varepsilon > 0$ small. If S is any of the segments of the P_q's, and $S := [\![(n_i, p_i)]\!]$, by a suitable change of coordinates we may and do assume that $n_i = a_-$ and $p_i = a_+$. We then apply Theorem 5.6 to each of the segments S. Interior singularities, i.e. boundary points of S which are not boundary points of P_q, are removed by Proposition 5.1. Singularities at boundary points of P_q are finally removed by Lemma 5.9. □

Proof of Theorem 6.2: Similarly to Theorem 6.1, we can approximate T by a sequence of graphs of smooth maps $\{u_\varepsilon\} \subset R^\infty_\varphi(\widetilde{B}^2, \mathcal{Y})$, i.e., which are smooth out of a discrete set of points Σ_ε of B^2. We then remove the point singularities by means of the following

Proposition 6.6 *Let $\mathcal{Y}$ and φ be as in Theorem 6.2. Then for every $u \in R^\infty_{1/2,\varphi}(\widetilde{B}^2, \mathcal{Y})$ there exists a sequence of smooth maps $u_k \subset C^\infty_\varphi(\widetilde{B}^2, \mathcal{Y})$ which converges to u strongly in $W^{1/2}$.*

Proof of Proposition 6.6: Let γ be a smooth, simple, rectifiable arc connecting the points of the singular set of u with γ contained in B^2 and with curvature radius uniformly bounded from below. Moreover, we let $\widetilde{\gamma}$ denote a line segment of length $\mathcal{L}(\widetilde{\gamma}) = \mathcal{L}(\gamma)$, with $\widetilde{\gamma} \subset \mathbb{R}^2 \times \{0\} \subset \mathbb{R}^2 \times I$. Finally, for any $\delta > 0$ we set

$$\Gamma_\delta := \{z \in \mathbb{R}^2 \times I \mid \mathrm{dist}(z, \gamma) < \delta\}\,, \quad \partial^+\Gamma_\delta := \{z = (x,t) \in \partial\Gamma_\delta \mid t > 0\}\,, \quad F_\delta := \Gamma_\delta \cap (\mathbb{R}^2 \times \{0\})$$

and similarly

$$\widetilde{\Gamma}_\delta := \{z \in \mathbb{R}^2 \times I \mid \mathrm{dist}(z, \widetilde{\gamma}) < \delta\}\,, \quad \partial^+\widetilde{\Gamma}_\delta := \{z = (x,t) \in \partial\widetilde{\Gamma}_\delta \mid t > 0\}\,, \quad \widetilde{F}_\delta := \widetilde{\Gamma}_\delta \cap (\mathbb{R}^2 \times \{0\})\,.$$

By the smoothness of γ we infer the existence of $\delta_0 > 0$ for which the following holds:

(i) for any $\delta \in (0, \delta_0)$, F_δ is an open subset of $B^2 \times \{0\}$ bilipschitz homeomorphic to the unit ball B^2;

(ii) there exists a bilipschitz homeomorphism $\Psi : \Gamma_{\delta_0} \to \widetilde{\Gamma}_{\delta_0}$, with Lipschitz constants $\mathrm{Lip}\,\Psi, \mathrm{Lip}\,\Psi^{-1} \le K$, such that, for every $\delta \in (0, \delta_0)$, Ψ maps $\partial^+\Gamma_\delta$ and F_δ onto $\partial^+\widetilde{\Gamma}_\delta$ and $\widetilde{F}_\delta$, respectively.

Let $U = \mathrm{Ext}(u)$ and define $V := U \circ \Psi^{-1}$ and $v = \mathbf{T}(V) := V_{|\mathbb{R}^2 \times \{0\}}$. By applying the coarea formula w.r.t. the Lipschitz function $d(z) := \mathrm{dist}(z, \widetilde{\gamma})$, we find that

$$\int_{\widetilde{\Gamma}_{\delta_0}} |DV|^2\, dz = \int_0^{\delta_0} d\rho \int_{\partial^+\widetilde{\Gamma}_\rho} |DV|^2\, d\mathcal{H}^2 < +\infty\,.$$

Therefore, similarly to Proposition 5.1, for every $\varepsilon > 0$ we find $0 < R < \delta_0$ such that $\mathbf{D}(V, \widetilde{\Gamma}_R) \le \varepsilon$ and then $r = r_\varepsilon \in [R/2, R]$ for which

$$\mathbf{D}(V, \partial^+ \widetilde{\Gamma}_r) := \frac{1}{2} \int_{\partial^+ \widetilde{\Gamma}_r} |DV|^2 \, d\mathcal{H}^2 \le \frac{4\varepsilon}{R} \, .$$

Since $u = \varphi$ on $\widetilde{B}^2 \setminus \overline{B}^2$, where $\varphi : \widetilde{B}^2 \to \mathcal{Y}$ is a smooth map, condition (i) above yields that for a.e. $\delta > 0$ small the restriction $u_{|\partial F_\delta}$ of u to the boundary of F_δ is topologically trivial on $\mathcal{Y}$. Therefore, we may and will choose r so that there exists a continuous extension $w_r \in W^{1,2}(\widetilde{F}_r, \mathcal{Y})$ of $v_{|\partial \widetilde{F}_r}$.

Let $W_r : \widetilde{\Gamma}_r \to \mathbb{R}^N$ be the solution of the Dirichlet problem on $\widetilde{\Gamma}_r$ with boundary condition

$$\begin{cases} W_r = V & \text{on} \quad \partial^+ \widetilde{\Gamma}_r \\ W_r = w_r & \text{on} \quad \widetilde{F}_r \end{cases}$$

and, for $0 < \delta < r$ to be fixed later, define $V_\varepsilon : \widetilde{\Gamma}_{\delta_0} \to \mathbb{R}^N$ by

$$V_\varepsilon(z) := \begin{cases} W_r \Big(p(z) + \dfrac{r}{\delta} \big(z - p(z) \big) \Big) & \text{if} \quad d(z) \le \delta \\ V \Big(p(z) + r \dfrac{z - p(z)}{|z - p(z)|} \Big) & \text{if} \quad \delta \le d(z) \le r \\ V(z) & \text{if} \quad r \le d(z) < \delta_0 \, , \end{cases}$$

where $z \mapsto p(z)$ is the nearest point projection of $\widetilde{\Gamma}_{\delta_0}$ onto $\widetilde{\gamma}$, so that $d(z) := \operatorname{dist}(z, \widetilde{\gamma}) = |z - p(z)|$. As in Proposition 5.1, choosing $\delta = \delta(\varepsilon)$ sufficiently small we find

$$\mathbf{D}(V_\varepsilon, \widetilde{\Gamma}_r) \le c \, \varepsilon$$

so that, taking $U_\varepsilon = V_\varepsilon \circ \Psi$ on Γ_{δ_0}, and $U_\varepsilon = U$ elsewhere, we deduce that

$$\mathbf{D}(U_\varepsilon, \widetilde{B}^2 \times I) \le \mathbf{D}(U, \widetilde{B}^2 \times I) + K \, c \, \varepsilon$$

and finally the assertion, by letting $\varepsilon \searrow 0$. □

Proof of Proposition 6.5: Suppose $T \in \operatorname{cart}^{1/2}_\varphi(\widetilde{B}^n, \mathcal{Y})$. First note that if T satisfies (6.1), then for every $\phi \in \mathcal{D}^{n-2}(\widetilde{B}^n)$

$$S_T(\pi^\# d\phi \wedge \widehat{\pi}^\# \omega^s) = S_T(d(\pi^\# \phi \wedge \widehat{\pi}^\# \omega^s)) = -\partial G_{u_T}(\pi^\# \phi \wedge \widehat{\pi}^\# \omega^s) \, .$$

As a consequence we infer that

$$\partial \, \mathbb{L}_s(T) = (-1)^n \, \mathbb{P}_s(u_T) \qquad \forall \, s = 1, \ldots, \widetilde{s} \, .$$

Due to Theorem 1.1 and Proposition 1.4, by (1.3) we then have

$$\partial((-1)^{n-1} L^s_{u_k, u_T} + \mathbb{L}_s(T)) = (-1)^n \, \mathbb{P}_s(u_k) \, ,$$

which is a finite sum of i.m. rectifiable $(n-2)$-currents (Dirac masses if $n = 2$), see Proposition 1.3. Moreover

$$\partial G_{u_k} = (-1)^{n-1} \sum\nolimits_{s=1}^{\widetilde{s}} \mathbb{P}_s(u_k) \times \gamma_s \qquad \text{on} \qquad \mathcal{Z}^{n-1,1}(\widetilde{B}^n \times \mathcal{Y}) \tag{6.3}$$

whence $T_k \in \operatorname{cart}^{2,1}_\varphi(\widetilde{B}^n \times \mathcal{Y})$, with $T_k \rightharpoonup T$ weakly in $D_{2,1}(\widetilde{B}^n \times \mathcal{Y})$. Now, writing

$$T_k = G_{u_k} + \sum\nolimits_{s=1}^{\widetilde{s}} \mathbb{L}_s(T_k) \times \gamma_s \qquad \text{on} \qquad \mathcal{Z}^{n,1}(\widetilde{B}^n \times \mathcal{Y}) \, ,$$

we clearly have $\mathbb{L}_s(T_k) = (-1)^{n-1} L^s_{u_k, u_T} + \mathbb{L}_s(T)$, whence $\mathbf{M}(\mathbb{L}_s(T_k) - \mathbb{L}_s(T)) = \mathbf{M}(L^s_{u_k, u_T}) \to 0$, which yields that $T_k \rightharpoonup T$ and $\mathcal{E}_{1/2}(T_k) \to \mathcal{E}_{1/2}(T)$. □

7 A density result in higher dimension

In this section we extend Theorem 6.1 to any dimension $n \geq 3$, by proving the following

Theorem 7.1 *Let $n \geq 3$. Suppose that the first homotopy group $\pi_1(\mathcal{Y})$ is commutative. Also, let $\varphi : \widetilde{B}^n \to \mathcal{Y}$ be a given smooth $W^{1/2}$-function. For every $T \in \mathrm{cart}^{1/2}(B^n \times \mathcal{Y})$, respectively $T \in \mathrm{cart}^{1/2}_{\varphi}(\widetilde{B}^n \times \mathcal{Y})$, there exists a sequence of smooth maps $\{u_k\}$ in $C^\infty(B^n, \mathcal{Y})$, respectively in $C^\infty_\varphi(\widetilde{B}^n, \mathcal{Y})$, such that $G_{u_k} \rightharpoonup T$ weakly in $\mathrm{cart}^{1/2}$ and*

$$\lim_{k \to +\infty} \mathcal{E}_{1/2}(u_k) = \mathcal{E}_{1/2}(T)\,.$$

We also extend Corollary 6.3 by the following

Corollary 7.2 *Let $\mathcal{Y}$ and φ be as in Theorem 7.1. Then every map u in $W^{1/2}(B^n, \mathcal{Y})$, respectively in $W^{1/2}_\varphi(\widetilde{B}^n, \mathcal{Y})$, can be approximated weakly in $W^{1/2}$ by a sequence of maps in $C^1(B^n, \mathcal{Y}) \cap W^{1/2}(B^n, \mathcal{Y})$, respectively in $C^1_\varphi(\widetilde{B}^n, \mathcal{Y}) \cap W^{1/2}(\widetilde{B}^n, \mathcal{Y})$. Moreover, u can be approximated strongly in $W^{1/2}$ by a sequence of smooth maps provided that $\mathbb{P}_s(u) = 0$ for every $s = 1, \ldots, \overline{s}$.*

For the sake of simplicity, we omit to write all the details and we sketch the main steps of the proof in the case with the boundary data. Since Proposition 6.5 and Federer's polyhedral approximation theorem [8] hold in any dimension, we infer that Steps 1-2 in Theorem 6.1 hold again. Therefore, as in Step 3, we may and will suppose

$$T := G_{u_T} + \sum_{q \in H_1^{sph}(\mathcal{Y})} P_q \times C_q \in \mathrm{cart}^{1/2}_\varphi(\widetilde{B}^n \times \mathcal{Y}) \tag{7.1}$$

where the P_q's are polyhedral $(n-1)$-chains with multiplicity 1 and pairwise disjoint supports $\mathrm{spt}\, P_q \subset B^n$, and $u_T \in R^0_{1/2,\varphi}(\widetilde{B}^n, \mathcal{Y})$ is locally Lipschitz on $\widetilde{B}^n \setminus \bigcup_q \mathrm{spt}\, \partial P_q$. Possibly dividing the simplices of a triangulation of P_q, we may and will suppose that every P_q is the union of a finite number of $(n-1)$-simplices Δ which only intersect at the boundary points.

Now, as in Step 4 of Theorem 6.1, we first approximate the dipoles $\Delta \times P_q$, see Proposition 7.3, and then we remove the singularities produced in this process of approximation, Proposition 7.4.

First we fix some notation. We set

$$\widetilde{\mathcal{C}}^{n+1} := \widetilde{B}^n \times I\,, \qquad I = [0,1]\,.$$

Let Δ denote the $(n-1)$-simplex in B^n given by the convex hull

$$\Delta := \mathrm{coh}(\{0_{\mathbb{R}^n}, l\, e_1, l\, e_2, \ldots, l\, e_{n-1}\})\,, \qquad 0 < l < 1\,,$$

$(e_1, \ldots, e_n)$ being the standard basis in $\mathbb{R}^n$. We will denote by

$$z = (x,t) = (\widetilde{x}, x_n, t)\,, \qquad \widetilde{x} = (x_1, \ldots, x_{n-1})\,,$$

a generic point z in $\widetilde{\mathcal{C}}^{n+1}$. Also, for $\delta > 0$ and $0 < m \ll 1$, in the sequel we let

$$\varphi^m_\delta(y) := \min\{my, \delta\}\,, \qquad y \geq 0\,,$$

we denote by

$$y(\widetilde{x}) := \mathrm{dist}(\widetilde{x}, \partial\Delta)$$

the distance of $\widetilde{x}$ from the boundary of the $(n-1)$-simplex Δ and we set

$$\phi^m_\delta(z) := (\widetilde{x}, \varphi^m_\delta(y(\widetilde{x}))x_n, \varphi^m_\delta(y(\widetilde{x}))t)\,,$$

so that if

$$\Omega^m_\delta := \phi^m_\delta(\Delta \times B^+)\,, \qquad B^+ := \{(x_n, t) \in B^2 \mid t > 0\}\,,$$

then Ω^m_δ is a small "neighbor" of the simplex Δ in $\mathcal{C}^{n+1}$, compare (5.3) and (4.3).

Proposition 7.3 *Let $U : \mathcal{C}^{n+1} \to \mathbb{R}^N$ be a $W^{1,2}$ map which is smooth in the interior of $\Omega^{m_0}_{\delta_0}$, for some fixed small $m_0, \delta_0 > 0$, and such that $u := \mathbf{T}(U) \in W^{1/2}(B^n, \mathcal{Y})$. Let $C \in \mathcal{Z}_1(\mathcal{Y})$ be an integral flat 1-cycle of S^1-type. Then for every $\varepsilon > 0$, $0 < \delta < \delta_0$ and $0 < m < m_0$ there exists a map $U_\varepsilon : \mathcal{C}^{n+1} \to \mathbb{R}^N$ with trace $\mathbf{T}(U_\varepsilon) \in W^{1/2}(B^n, \mathcal{Y}_\varepsilon)$ such that U_ε is smooth in the closure of Ω^m_δ, except for the $(n-2)$-skeleton of a triangulation of Δ. Moreover $G_{U_\varepsilon} \rightharpoonup G_U + [\![\Delta]\!] \times R$ weakly in $\mathcal{D}_{n+1,2}(\mathcal{C}^{n+1} \times \mathbb{R}^N)$ as $\varepsilon \to 0^+$ and*

$$\mathbf{D}(U_\varepsilon, \mathcal{C}^{n+1}) \le \mathbf{D}(U, \mathcal{C}^{n+1}) + \mathcal{H}^{n-1}(\Delta) \cdot \mathbf{M}(R) + \varepsilon \,, \tag{7.2}$$

where $R \in \mathcal{R}_2(\mathbb{R}^N)$ the i.m. rectifiable current of least mass such that $\partial R = C$.

Once we have applied Proposition 7.3 to approximate the dipoles $P_q \times C_q$, by a diagonal argument we find a sequence $\{U_\varepsilon\}$ such that $u_\varepsilon := \mathbf{T}(U_\varepsilon) \in R^\infty_{1/2,\varphi}(\widetilde{B}^n \times \mathcal{Y}_\varepsilon)$ and the graphs G_{u_ε} weakly converge to T with $\mathcal{E}_{1/2}(G_{u_\varepsilon}) \to \mathcal{E}_{1/2}(T)$. However, u_ε is smooth except on a singular set Σ_ε of B^n given by the $(n-2)$-skeleton of a triangulation of the union of the polyhedral $(n-1)$-chains P_q. To remove the singular set Σ_ε, we finally make use of the following variant of a result from [18].

Proposition 7.4 *Under the previous hypotheses, for $\varepsilon > 0$ small enough there exists a sequence of smooth maps $\{u^{(\varepsilon)}_m\} \subset C^\infty_\varphi(\widetilde{B}^n, \mathcal{Y}_\varepsilon)$ which converges to u_ε strongly in $W^{1/2}$ as $m \to +\infty$.*

Proof of Proposition 7.3: Let $\psi : B^n \to B^n$ be a bilipschitz homeomorphism which takes the $(n-1)$-simplex Δ onto the $(n-1)$-disk D of diameter l

$$D := \{x = (\widetilde{x}, x_n) \in B^n \,:\, |x| \le l/2 \quad \text{and} \quad x_n = 0\} \,,$$

with Lipschitz constant

$$\operatorname{Lip} \psi \le K \,, \qquad \operatorname{Lip} \psi^{-1} \le K \,, \tag{7.3}$$

where $K = K(n)$ does not depend on l, but possibly on the distance of Δ from ∂B^n. Also, let $V : \mathcal{C}^{n+1} \to \mathbb{R}^N$ be given by

$$V(z) := U \circ \Psi^{-1}(z) \,, \qquad \Psi(z) = \Psi(x,t) := (\psi(x), t) \,.$$

Finally set

$$\begin{aligned} W_\rho &:= \{z \in \mathcal{C}^{n+1} \mid \operatorname{dist}(z, \partial D) < \rho\} \\ \partial^+ W_\rho &:= \{z \in \mathcal{C}^{n+1} \mid \operatorname{dist}(z, \partial D) = \rho\} \,, \end{aligned}$$

fix $0 < R < l/2$ and let $p : W_R \to \partial D$ denote the nearest point projection, so that for every $z \in W_R$

$$p(z) \in \partial D \quad \text{and} \quad |z - p(z)| = \operatorname{dist}(z, \partial D) \,.$$

By applying the coarea formula w.r.t. "cylindrical type" coordinates defined around the $(n-2)$-sphere ∂D, since property

$$\int_{W_R} |DV|^2 \, dz = \int_0^R d\rho \int_{\partial^+ W_\rho} |DV|^2 \, d\mathcal{H}^n < +\infty$$

yields

$$\liminf_{\rho \to 0^+} \rho \int_{\partial^+ W_\rho} |DV|^2 \, d\mathcal{H}^n = 0 \,,$$

we can choose a small radius $r > 0$ and replace V on W_r by the map

$$V_r(z) := V\Big(p(z) + r \frac{z - p(z)}{|z - p(z)|}\Big) \,, \tag{7.4}$$

compare (5.8), so that

$$\mathbf{D}(V_r, W_r) \le c_n \cdot r \int_{\partial^+ W_r} |DV|^2 \, d\mathcal{H}^n = O(r) \,,$$

where $O(r_j) \to 0$ along a sequence $r_j \searrow 0$. We set

$$\widetilde{y}(\widetilde{x}) := \operatorname{dist}(\widetilde{x}, \partial D) \,,$$

$$\widetilde{\phi}^m_\delta(z) := (\widetilde{x}, \varphi^m_\delta(\widetilde{y}(\widetilde{x}))x_n, \varphi^m_\delta(\widetilde{y}(\widetilde{x}))t)\,,$$

moreover,

$$\widetilde{\Omega}^m_\delta := \widetilde{\phi}^m_\delta(D \times B^+)$$

and finally

$$K^m_\delta := \{z \in \mathcal{C}^{n+1} \quad | \quad 0 < \operatorname{dist}(z, \partial D) < r_{\delta.m}\,, \\ 0 < \operatorname{dist}(\widetilde{x}, \partial D) < \delta/m \text{ and } \sqrt{{x_n}^2 + t^2} < m \cdot \operatorname{dist}(\widetilde{x}, \partial D)\}\,,$$

where $r_{\delta,m}$ is given by (5.14), so that if $r_{\delta,m} < r$, by (7.4) we infer that V does not depend on the distance of z from ∂D on K^m_δ.

We now wish that the following conditions hold true:

(i) V maps K^m_δ into a set of diameter ε;

(ii) V maps $\widetilde{\Omega}^m_\delta$ into a set of diameter ε.

If it is not the case, we let $\{\Delta_i\}_{i=1}^{c(n)}$ be a barycentric-type subdivision of Δ into smaller simplices of side $l/2$. Possibly slightly moving the center of the faces of Δ, without loss of generality we may and will suppose that V has finite energy on the boundary of the Δ_i's for every i. We then apply the previous construction to each Δ_i, where K, see (7.3), is an upper bound for the Lipschitz constants of the homeomorphisms of B^n which map Δ_i onto D_i, the $(n-1)$-disk of diameter $l/2$, for every i.

If V does not satisfy conditions (i) and (ii) on the sets $K^m_{\delta,i}$ and $\Omega^m_{\delta,i}$ corresponding to D_i, we start again with the previous procedure, by taking a barycentric subdivision of Δ_i.

Notice that V is smooth on the interior of Ω^m_δ, for δ and m sufficiently small, and, by paying a small amount of energy, we may suppose that V does not depend on the distance of z from ∂D_i on $K^m_{\delta,i}$. We then infer that conditions (i) and (ii) above are obtained after a finite number of barycentric subdivisions, by first taking $0 < m = m(\varepsilon) \ll 1$ and then $\delta = \delta(m, r) > 0$ small. Therefore, in the sequel we omit to write the index i corresponding to the simplex Δ_i of the given subdivision of Δ.

Let now $W_\varepsilon : D \times B^+ \to \mathbb{R}^N$ be given by

$$W_\varepsilon(\widetilde{x}, x_n, t) := f^P_\varepsilon(x_n, t)\,, \tag{7.5}$$

where f^P_ε is given by Proposition 5.3 in correspondence to C and to the point $P := U(q)$ for some given $q \in \operatorname{int}(\Delta)$. Setting

$$\Phi_\varepsilon(z) := W_\varepsilon \circ (\widetilde{\phi}^m_\delta)^{-1}(z)\,, \qquad z \in \widetilde{\Omega}^m_\delta\,,$$

as in Theorem 5.5, compare (5.16), we estimate

$$\mathbf{D}(\Phi_\varepsilon, \widetilde{\Omega}^m_\delta) \le \mathcal{H}^{n-1}(\Delta) \cdot \mathbf{D}(f^P_\varepsilon, B^+) + \frac{\varepsilon}{2\mu K^2}$$

if we choose $\delta = \delta(W_\varepsilon, \varepsilon, m, K, \mu)$ sufficiently small. Here μ is the number of the Δ_i's in the given subdivision of Δ. We now introduce the cylindrical coordinates

$$z = (\widetilde{x}, x_n, t) = F(\rho, \theta, \widetilde{x}) := (\widetilde{x}, \rho\cos\theta, \rho\sin\theta)\,, \qquad \rho > 0, \quad \theta \in [0, \pi]\,,$$

so that $\rho = \sqrt{{x_n}^2 + t^2}$, denote

$$\widehat{W}(\rho, \theta, \widetilde{x}) := W(F(\rho, \theta, \widetilde{x}))$$

and define $V_\varepsilon : \widetilde{\Omega}^m_\delta \to \mathbb{R}^N$ by

$$\widehat{V}_\varepsilon(\rho, \theta, \widetilde{x}) := \begin{cases} \widehat{\Phi}_\varepsilon(2\rho, \theta, \widetilde{y}) & \text{if} \quad 0 \le \rho < \varphi^m_\delta(\widetilde{y})/2 \\ \widehat{\Psi}^m_\delta(\rho, \theta, \widetilde{y}) & \text{if} \quad \varphi^m_\delta(\widetilde{y})/2 \le \rho < \varphi^m_\delta(\widetilde{y}) \end{cases} \qquad \theta \in [0, \pi]\,, \quad \widetilde{x} \in \operatorname{int}(\Delta)\,,$$

where $\widetilde{y} = \widetilde{y}(\widetilde{x}) := \operatorname{dist}(\widetilde{x}, \partial D)$ and

$$\widehat{\Psi}^m_\delta(\rho, \theta, \widetilde{y}) := \left(\frac{2\rho}{\varphi^m_\delta(\widetilde{y})} - 1\right) \cdot \widehat{V}(\varphi^m_\delta(\widetilde{y}), \theta, \widetilde{y}) + \left(2 - \frac{2\rho}{\varphi^m_\delta(\widetilde{y})}\right) \cdot P\,.$$

We also extend $V_\varepsilon \equiv V$ outside $\widetilde{\Omega}^m_\delta$. Similarly to Theorem 5.6, by conditions (i) and (ii) above we estimate

$$\mathbf{D}(V_\varepsilon, \widetilde{\Omega}^m_\delta) \le \mathcal{H}^{n-1}(\Delta) \cdot (\mathbf{M}(R) + 4\varepsilon^2) + \frac{\varepsilon}{2\mu K^2} \,. \tag{7.6}$$

We finally define

$$U_\varepsilon(z) := V_\varepsilon \circ \Psi(z) \,.$$

Possibly repeating the argument for each simplex Δ_i of the given subdivision of Δ, by (7.6) and (7.3) we easily estimate

$$\mathbf{D}(U_\varepsilon, \mathcal{C}^{n+1}) \le \mathbf{D}(U, \mathcal{C}^{n+1}) + \mathcal{H}^{n-1}(\Delta) \cdot (\mathbf{M}(R) + 4K^2\varepsilon^2) + \frac{\varepsilon}{2} \,,$$

so that (7.2) follows for $\varepsilon > 0$ small. $\square$

Proof of Proposition 7.4: Let U_ε be the harmonic extension of u_ε to $\widetilde{B}^n \times]-1,1[$. For $m \in \mathbb{N}^*$ and $a \in [1/4m, 3/4m]^{n+1}$ we denote by $\mathcal{L}_m = \mathcal{L}_m(a)$ the grid of $\mathbb{R}^{n+1}$

$$\mathcal{L}_m := a + \bigcup_{z \in \mathbb{Z}^{n+1}} \frac{1}{m} \cdot z$$

and by $\mathcal{L}^{(k+1)}_m$ the family of all the $(k+1)$-faces Q of the $(n+1)$-cubes of $\mathcal{L}_m$ which intersect the n-disk $B^n \times \{0\}$. Moreover, we let $\mathcal{F}^{(k)}_m$ denote the set of k-faces F obtained by intersecting the $(k+1)$-faces Q of $\mathcal{L}^{(k)}_m$ with the n-disk $B^n \times \{0\}$

$$F = Q \cap (\widetilde{B}^n \times \{0\}) \,. \tag{7.7}$$

We finally set

$$G_m := \widetilde{B}^n \times]-10m^{-1}, 10m^{-1}[\,.$$

Similarly to [18], we may and do choose $a = a(m, U_\varepsilon)$ so that the following conditions hold:

(i) for every $k = 1, \dots n-1$, the restriction of U_ε to every $(k+1)$-face of $\mathcal{L}^{(k+1)}_m$ is a $W^{1,2}$ function;

(ii) there exists an absolute constant $c > 0$ such that

$$\mathbf{D}(U_\varepsilon, \cup \mathcal{L}^{(k+1)}_m) \le c\, m^{n-k}\, \mathbf{D}(U_\varepsilon, G_m) \qquad \forall\, k = 1, \dots, n-1 \,. \tag{7.8}$$

Moreover, since the singular set Σ_ε is given by the $(n-2)$-skeleton of some fixed triangulation of the P_q's, by a slicing argument, for m sufficiently large we can also require that

(iii) Σ_ε does not intersect the 1-faces of $\mathcal{F}^{(1)}_m$;

(iv) every 2-face F of $\mathcal{F}^{(2)}_m$ intersects Σ_ε at almost one interior point $p_F \in \mathrm{int}(F)$, which does not belong to the $(n-3)$-skeleton of the triangulation of P_q's;

(v) the restriction $u_{\varepsilon|F}$ of u_ε to any 2-face F of $\mathcal{F}^{(2)}_m$ is continuous, possibly except at the point p_F;

(vi) in this case, if $p_F \in \mathrm{spt}\, P_q$ and $C_q \in \mathcal{Z}_1(\mathcal{Y})$ is given by (7.1), we have

$$\partial G_{u_{\varepsilon|F}} \llcorner F \times \mathcal{Y}_\varepsilon = \delta_{p_F} \times (C_\varepsilon - Z_\varepsilon) \qquad \text{on} \quad \mathcal{Z}^{1,1}(F \times \mathcal{Y}_\varepsilon) \,, \tag{7.9}$$

where $Z_\varepsilon \in \mathcal{Z}_1(\mathcal{Y}_\varepsilon)$ is such that $Z_\varepsilon - C_q = \partial W_\varepsilon$ for some $W_\varepsilon \in \mathcal{D}_2(\mathcal{Y}_\varepsilon)$ and $C_\varepsilon \in \mathcal{Z}_1(\mathcal{Y}_\varepsilon)$ is the boundary $C_\varepsilon = \partial R_\varepsilon$ of an i.m. rectifiable current $R_\varepsilon \in \mathcal{R}_2(\mathbb{R}^N)$ such that $R_\varepsilon \rightharpoonup R_q$ weakly in $\mathcal{D}_2(\mathbb{R}^N)$ with $\mathbf{M}(R_\varepsilon) \to \mathbf{M}(R_q)$ as $\varepsilon \to 0$, where $R_q \in \mathcal{R}_2(\mathbb{R}^N)$ is mass minimizing under the condition $\partial R_q = C_q$.

Note that condition (vi) follows from the fact that the restriction of u_ε to F behaves similarly as in the problem of the dipole insertion in dimension $n=2$, described by Theorem 5.6. As a consequence, arguing as in Lemma 5.9, by (7.9) and by the commutativity of $\pi_1(\mathcal{Y})$, for $\varepsilon>0$ small enough we infer that

$$\{w\in W^{1/2}(F,\mathbb{R}^N)\cap C^0(F,\mathcal{Y}_\varepsilon)\mid w_{|\partial F}=u_{\varepsilon|\partial F}\}\neq\emptyset \tag{7.10}$$

holds true for every 2-face F of $\mathcal{F}_m^{(2)}$.

To remove the singular set Σ_ε of u_ε we make use of an argument taken from [18]. To this aim, at the 1^{st} step we set $U_m^{(\varepsilon)}\equiv U_\varepsilon$ on $\cup\mathcal{L}_m^{(2)}$ and on every $Q\in\mathcal{L}_m^{(k+1)}$ which does not intersect the n-disk $\widetilde{B}^n\times\{0\}$. We then argue by induction on the dimension $k=2,\dots,n$ and, at the k^{th} step, we define $U_m^{(\varepsilon)}$ on every $Q\in\mathcal{L}_m^{(k+1)}$ which intersects $\widetilde{B}^n\times\{0\}$ by means of a "cone" construction starting from the restriction $U_{m|\partial Q}^{(\varepsilon)}$ of $U_m^{(\varepsilon)}$ to the boundary ∂Q. To do this, if $F\in\mathcal{F}_m^{(k)}$ is given by (7.7), it suffices to require that the trace $\varphi_F:=\mathbf{T}(U_{m|\partial Q}^{(\varepsilon)})$ of $U_{m|\partial Q}^{(\varepsilon)}$ on the boundary of F has a continuous extension $\Phi_F\in W^{1/2}(F,\mathcal{Y}_\varepsilon)$.

Note that at the 2^{nd} step this last condition is given by (7.10). To extend this condition to the case $k\geq 3$, due to the fact that $u_\varepsilon\equiv\varphi$ on $\widetilde{B}^n\setminus\overline{B}^n$, where $\varphi:\widetilde{B}^n\to\mathcal{Y}$ is smooth on the whole of $\widetilde{B}^n$, for every $k\geq 2$ at the k^{th} step we first modify the definition of $u_m^{(\varepsilon)}:=\mathbf{T}(U_m^{(\varepsilon)})$ on $\mathcal{F}_m^{(k)}$ in a suitable way, see [18] for the details. We secondly extend $U_m^{(\varepsilon)}$ to every $Q\in\mathcal{L}_m^{(k+1)}$ in a continuous way, so that its trace $u_m^{(\varepsilon)}$ belongs to $W^{1/2}(F,\mathcal{Y}_\varepsilon)$ and

$$\mathbf{D}(U_m^{(\varepsilon)},Q)\leq\frac{c}{m}\,\mathbf{D}(U_\varepsilon,\partial Q)\,. \tag{7.11}$$

More precisely, let $v_Q:Q\to\mathbb{R}^N$ be defined by $v_Q(z)=v_Q^\pm(z)$ if $z\in Q^\pm$, where $Q^\pm:=\{z=(x,t)\in Q\mid \pm t\geq 0\}$ and $v_Q^\pm:Q^\pm\to\mathbb{R}^N$ is the solution of the Dirichlet problem on $Q^\pm$ with boundary condition

$$\begin{cases} v_Q^\pm=U_m^{(\varepsilon)} & \text{on}\quad \partial Q^\pm\cap\{(x,t)\mid \pm t>0\}\\ v_Q^\pm=\Phi_F & \text{on}\quad F\end{cases}$$

where $\Phi_F:F\to\mathcal{Y}_\varepsilon$ is the continuous $W^{1/2}$-extension of the boundary datum $\varphi_F:=\mathbf{T}(U_{m|\partial Q}^{(\varepsilon)})$. Assuming e.g. that the center of Q is the origin $0_{\mathbb{R}^{n+1}}$, we define $U_m^{(\varepsilon)}$ on Q by setting, for $0<\delta\ll 1/2m$,

$$U_m^{(\varepsilon)}(z):=\begin{cases} v_Q\Big(\dfrac{z}{2m\delta}\Big) & \text{if}\ \ \|z\|\leq\delta\\ U_m^{(\varepsilon)}\Big(\dfrac{z}{2m\|z\|}\Big) & \text{if}\ \ \delta\leq\|z\|\leq\dfrac{1}{2m}\end{cases}\qquad z\in Q$$

where $\|z\|:=\sup_i|z_i|$ if $z=(z^1,\dots,z^{n+1})$, so that $\|z\|=1/2m$ if $z\in\partial Q$. A similar definition works in the general case, so that (7.11) holds true and $\mathbf{T}(U_m^{(\varepsilon)})\in W^{1/2}(F,\mathcal{Y}_\varepsilon)$.

Repeating the argument for $k=2,\dots,n$, from (7.11) we estimate

$$\mathbf{D}(U_m^{(\varepsilon)},\cup\mathcal{L}_m^{(n+1)})\leq C(n)\sum_{k=1}^{n-1}\frac{1}{m^{n-k}}\mathbf{D}(U_\varepsilon,\mathcal{L}_m^{(k+1)})$$

and hence, by (7.8), we obtain

$$\mathbf{D}(U_m^{(\varepsilon)},\cup\mathcal{L}_m^{(n+1)})\leq C(n)\,\mathbf{D}(U_\varepsilon,G_m)\to 0$$

as $m\to+\infty$, since $|G_m|\to 0$. We finally set $U_m^{(\varepsilon)}=U_\varepsilon$ on $\mathcal{C}^{n+1}\setminus\cup\mathcal{L}_m^{(n+1)}$, as required. $\square$

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