GLOBAL REGULARITY AND STABILITY OF SOLUTIONS TO ELLIPTIC EQUATIONS WITH NONSTANDARD GROWTH

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Abstract. We study the regularity properties of solutions to elliptic equations similar to the $p(\cdot)$-Laplacian. Our main results are a global reverse Hölder inequality, Hölder continuity up to the boundary, and stability of solutions with respect to continuous perturbations in the variable growth exponent. We assume that the complement of the domain is uniformly fat in a capacitary sense. As technical tools, we derive a capacitary Sobolev–Poincaré inequality, and a version of Hardy's inequality.

1. Introduction

We study regularity properties of solutions to quasilinear elliptic equations

$$-\operatorname{div} \mathcal{A}(x, \nabla u) = 0 \tag{1.1}$$

with nonstandard structural conditions. These conditions involve a variable growth exponent $p(\cdot)$, and the prototype of such equations is the $p(\cdot)$-Laplacian

$$-\operatorname{div} \Big(|\nabla u(\cdot)|^{p(\cdot)-2} \nabla u(\cdot) \Big) = 0. \tag{1.2}$$

We prove global higher integrability of the gradient, Hölder continuity up to the boundary and stability of solutions with respect to continuous perturbations in the growth exponent $p(\cdot)$.

Regularity results for (1.2) do not hold without additional assumptions on the function p, see the counterexamples in [24, 43]. However, there is a condition, called logarithmic Hölder continuity, which seems to be the right one for our purposes. This condition was originally introduced by Zhikov [42] in the context of the Lavrentiev phenomenon, and it has turned out to be very useful in regularity and other applications, see, e.g., [1, 2, 6, 9, 12, 11, 10, 13, 40, 43]. In fact, there are very few regularity results that do not assume logarithmic Hölder continuity.

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In order to consider properties of solutions up to the boundary, one needs a hypothesis on the domain. We assume that the complement of the domain is uniformly fat in a capacitary sense, where the capacity involves the variable exponent $p(\cdot)$ appearing in the equation. Such an assumption is natural for a number of reasons. For instance, uniform fatness itself is self-improving, see [32]. This property is a reasonable criterion for a condition for higher integrability. One can also show that this condition is sharp for global higher integrability and the stability of solutions already in the constant exponent case, see [28, 34]. Uniform fatness is also a fairly weak assumption, since any domain whose complement satisfies a measure density condition, for instance all domains with a Lipschitz boundary, have uniformly fat complements for all exponents.

Our first result is the global higher integrability of the gradient, along the lines of [28]; see also [5, 39, 43]. This means that the gradient of a weak solution u, of which we a priori only know that $|\nabla u|^{p(\cdot)} \in L^1(\Omega)$, actually satisfies $|\nabla u|^{p(\cdot)(1+\delta)} \in L^1(\Omega)$ for a small $\delta > 0$, assuming that its boundary values are sufficiently regular. Higher integrablity is a consequence of a reverse Hölder inequality. Such an inequality follows by combining a Caccioppoli estimate with a Sobolev-Poincaré inequality. Thus to prove global higher integrabilty, one needs a Sobolev-Poincaré inequality which is applicable up to the boundary. For this purpose, we first prove a capacitary Sobolev–Poincaré inequality. Such inequalities were first considered in [26, 37]. This inequality also matches well with the capacity fatness condition, as we shall see below.

We also prove that solutions are Hölder continuous up to the boundary, provided that the boundary data is also Hölder continuous. This follows from a Maz'ya type boundary point estimate involving the natural capacity, since uniform fatness matches well with such an estimate. For the class of equations considered here, an appropriate boundary point estimate has been established in [4].

Our final result concerns the stability of solutions with respect to perturbations in the growth exponent $p(\cdot)$. More spesifically, we consider a sequence $p(\cdot)_i$ of variable exponents converging uniformly to the function $p(\cdot)$, and show that the corresponding sequence of solutions with fixed boundary values converges, up to subsequences, to the solution of the limit problem. Similar results are known for equations similar to the p-Laplacian [33, 35], for principal eigenvalues [34], for quasiminimizers [36], and for the p-parabolic equation [30]. See also [43, Lemma 3.1]. The chief problem in these stability results is the fact that perturbations in the structural exponent change the underlying Sobolev space. This difficulty can be dealt with by means of a global reverse Hölder inequality, which enables us to work in a fixed Sobolev space. Further, we derive a version of Hardy's inequality for the purpose of verifying that the limit function has the right boundary values.

2. The Dirichlet problem

In this section, we recall the definition of weak solutions to Dirichlet boundary value problems. To this end, we first introduce some notation, and then record a number of facts about the variable exponent function spaces.

We call a bounded measurable function $p : \mathbb{R}^n \to (1, \infty)$, $n \geq 2$, a variable exponent. We denote

$$p_E^- = \inf_{x \in E} p(x), \text{ and } p_E^+ = \sup_{x \in E} p(x),$$

where E is a measurable subset of $\mathbb{R}^n$. We assume that $1 < p_\Omega^- \leq p_\Omega^+ < n$, where Ω is an open, bounded subset of $\mathbb{R}^n$. In the following, for simplicity, we will often write p^- and p^+ instead of p_Ω^- and p_Ω^+ respectively.

In this article, we always assume that $p(\cdot)$ *is log-Hölder continuous with* $1 < p^- \leq p^+ < n$, *except for Section 4 where we will assume* $1 < p^- \leq p^+ < \infty$ *instead.*

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of all measurable functions f defined on Ω for which

$$\int_\Omega |f|^{p(x)} \, \mathrm{d}x < \infty.$$

The Luxemburg norm on this space is defined as

$$\|f\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{f(x)}{\lambda} \right|^{p(x)} \mathrm{d}x \leq 1 \right\}.$$

Equipped with this norm $L^{p(\cdot)}(\Omega)$ is a Banach space, see Kováčik and Rákosník [31]. For a constant function $p(\cdot)$ the variable exponent Lebesgue space coincides with the standard Lebesgue space. The conjugate exponent $p'(\cdot)$ is defined pointwise. The Hölder inequality

$$\int_\Omega fg \, \mathrm{d}x \leq C \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}$$

holds for functions $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$.

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ consists of functions $f \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇f exists and satisfies $|\nabla f| \in L^{p(\cdot)}(\Omega)$. This space is a Banach space with the norm

$$\|f\|_{1,p(\cdot)} = \|f\|_{p(\cdot)} + \|\nabla f\|_{p(\cdot)}.$$

For basic properties of the spaces $L^{p(\cdot)}$ and $W^{1,p(\cdot)}$, we refer to [31].

Smooth functions are not dense in $W^{1,p(\cdot)}(\Omega)$ without additional assumptions on the exponent $p(\cdot)$. This was observed by Zhikov [42, 43] in the context of the Lavrentiev phenomenon, which means that minimal values of variational integrals may differ depending on whether one minimises over smooth functions or Sobolev functions. Zhikov has also introduced the logarithmic Hölder continuity condition to rectify this. The condition is

$$|p(x) - p(y)| \le \frac{C}{-\log\left(|x-y|\right)} \tag{2.1}$$

for all $x, y \in \Omega$ such that $|x - y| \le 1/2$. If the exponent is bounded and satisfies (2.1), smooth functions are dense in variable exponent Sobolev spaces and we can define the Sobolev space with zero boundary values, $W_0^{1,p(\cdot)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(\cdot)}$. We refer to [6, 17, 25, 40] for density results in variable exponent Sobolev spaces.

We will use logarithmic Hölder continuity in the form

$$r^{-(p_B^+ - p_B^-)} \le C, \tag{2.2}$$

where $B = B(x_0, r)$. It is well-known that requiring (2.2) to hold for all balls is equivalent with condition (2.1); a proof of this is given in [6, Lemma 3.2]. An elementary consequence of (2.2) is the inequality

$$C^{-1} r^{-p(y)} \le r^{-p(x)} \le C\, r^{-p(y)}, \tag{2.3}$$

which holds for any points $x, y \in B(x_0, r)$ with a constant depending only on the constant of (2.2). We use phrases like "by log-Hölder continuity" when applying either (2.2) or (2.3).

We need the following assumptions, with positive constants α and β, to hold for the operator $\mathcal{A} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$.

(1) $x \mapsto \mathcal{A}(x, \xi)$ is measurable for all $\xi \in \mathbb{R}^n$,
(2) $\xi \mapsto \mathcal{A}(x, \xi)$ is continuous for almost all $x \in \Omega$,
(3) $\mathcal{A}(x, \xi) \cdot \xi \ge \alpha |\xi|^{p(x)}$ for almost all $x \in \Omega$ and $\xi \in \mathbb{R}^n$,
(4) $|\mathcal{A}(x, \xi)| \le \beta |\xi|^{p(x)-1}$ for almost all $x \in \Omega$ and $\xi \in \mathbb{R}^n$,
(5) $(\mathcal{A}(x, \eta) - \mathcal{A}(x, \xi)) \cdot (\eta - \xi) > 0$ for all $x \in \Omega$ and $\eta \neq \xi \in \mathbb{R}^n$.

We may assume that $\alpha \le \beta$ by choosing β larger if necessary. These are called the *structure conditions of* $\mathcal{A}$.

We say that a function $u \in W^{1,p(\cdot)}(\Omega)$ is a solution to the Dirichlet problem with boundary values given by a function $f \in W^{1,p(\cdot)}(\Omega)$,

$$\begin{cases} \operatorname{div} \mathcal{A}(x, \nabla u) = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

if $u - f \in W_0^{1,p(\cdot)}(\Omega)$ and

$$\int_\Omega \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, \mathrm{d}x = 0$$

for all test functions $\varphi \in C_0^\infty(\Omega)$. Due to the structural conditions, we can employ test functions $\varphi \in W_0^{1,p(\cdot)}(\Omega)$ by the usual approximation argument. Under our assumptions on $p(\cdot)$, the existence of solutions in the sense of the above definition follows from standard functional analysis results, see, e.g., [41]. Alternatively, we may assume that (2.4) is the Euler-Lagrange equation of a variational integral and use the direct method of the calculus of variations, see [21].

3. A Sobolev-Poincaré inequality

In this section, we prove a capacitary Sobolev-Poincaré inequality. We begin by recalling the definitions of the $p(\cdot)$-capacity of a condenser, the Sobolev $p(\cdot)$-capacity and the notion of quasicontinuity, see [4, 19, 20].

Definition 3.1. The relative capacity of a compact $K \subset \Omega$ is defined by setting

$$\operatorname{cap}_{p(\cdot)}(K,\Omega) = \inf_u \int_\Omega |\nabla u|^{p(x)}\,dx,$$

where the infimum is taken over all $u \in C_0(\Omega) \cap W^{1,p(\cdot)}(\Omega)$ such that $u \geq 1$ in K. The definition is then extended to open sets $U \subset \Omega$ by

$$\operatorname{cap}_{p(\cdot)}(U,\Omega) = \sup_{K \subset U \text{ compact}} \operatorname{cap}_{p(\cdot)}(K,\Omega),$$

and to arbitrary sets $E \subset \Omega$ by

$$\operatorname{cap}_{p(\cdot)}(E,\Omega) = \sup_{E \subset U \subset \Omega \text{ open}} \operatorname{cap}_{p(\cdot)}(U,\Omega).$$

If p is bounded, then the relative $p(\cdot)$-capacity is a Choquet capacity. We need the fact that

$$C^{-1} r^{n-p(y)} \leq \operatorname{cap}_{p(\cdot)}(\overline{B}(x,r), B(x,2r)) \leq C r^{n-p(y)}$$

for every $y \in B(x,2r)$. The upper estimate, that holds also if $p_{2B}^- \geq n$, can be easily derived by testing the definition of capacity with a function that equals 1 in $\overline{B}(x,r)$, $3 - 2|y-x|/r$ in $B(x,3r/2) \setminus B(x,r)$ and 0 otherwise. The upper bound then follows by log-Hölder continuity. The lower estimate can be found in [4, Proposition 5.1]

Definition 3.2. For $E \subset \mathbb{R}^n$ we denote

$$S_{p(\cdot)}(E) = \bigl\{ u \in W^{1,p(\cdot)}(\mathbb{R}^n) : u \geq 1 \text{ in an open set } U \Subset \mathbb{R}^n \text{ containing } E \bigr\}$$

and define the Sobolev $p(\cdot)$-capacity $C_{p(\cdot)}$ to be the number

$$C_{p(\cdot)}(E) = \inf_{u \in S_{p(\cdot)}(E)} \int_{\mathbb{R}^n} |u|^{p(x)} + |\nabla u|^{p(x)}\,dx.$$

Here we use the convention that $C_{p(\cdot)}(E) = \infty$ if $S_{p(\cdot)}(E) = \emptyset$.

Since smooth functions are dense in the Sobolev space, for $E \subset \Omega$ holds that $C_{p(\cdot)}(E) = 0$ if and only if $\operatorname{cap}_{p(\cdot)}(E,\Omega) = 0$. For the proofs see [19].

A function $u\colon \Omega \to [-\infty,\infty]$ is $p(\cdot)$-*quasicontinuous* if for every $\varepsilon > 0$ there exists a set E, with $C_{p(\cdot)}(E) \le \varepsilon$, so that u is continuous when restricted to $\Omega \setminus E$. Since the Sobolev capacity is an outer capacity, we can assume that E is open. Every Sobolev function has a representative that is quasicontinuous, see [20].

We are now ready to prove our capacitary Sobolev-Poincaré inequality. For a constant exponent the result is originally due to V. G. Mazya [37] and L. I. Hedberg [26]. We need the result in the modular form and since the modular is not homogenous we get an extra measure term.

Lemma 3.3 (Sobolev-Poincaré)**.** *Let B be a ball such that $|B| \le 1$. Let p be a bounded* log*-Hölder continuous exponent on $2B$ with $1 < p^- \le p^+ < n$. Then every $p(\cdot)$-quasicontinuous $u \in W^{1,p(\cdot)}(2B)$, such that $\|u\|_{W^{1,p(\cdot)}(2B)} \le 1$, satisfies*

$$\fint_B |u|^{\frac{np(x)}{n-p_B^-}}\,dx \le \left(\frac{C}{\operatorname{cap}_{p(\cdot)}(N(u),2B)}\left(\int_{2B} |\nabla u|^{p(x)}\,dx + |B|\right)\right)^{\frac{np_B^-}{p_B^+(n-p_B^-)}},$$

where $N(u) := \{x \in \overline{B} : u(x) = 0\}$. The constant C depends only on the dimension n, p^-, p^+ and the log*-Hölder constant of p.*

Proof. We write $p^*(x) := \frac{np(x)}{n-p(x)}$ and $p^\#(x) := \frac{np(x)}{n-p_B^-}$. By [7, 18] we have

$$\|u-u_B\|_{L^{p^\#(\cdot)}(B)} \le C\|u-u_B\|_{L^{p^*(\cdot)}(B)} \le C\|\nabla u\|_{L^{p(\cdot)}(B)},$$

for every $u \in W^{1,p(\cdot)}(B)$, where the constant is independent of B and u. Note that we write $u_B := \fint_B u\,dx$. Since $\|\nabla u\|_{L^{p(\cdot)}(B)} \le 1$, this yields

$$\int_B |u-u_B|^{p^\#(x)}\,dx \le C\|\nabla u\|_{L^{p(\cdot)}(B)}^{(p^\#)_B^-} \le C\left(\int_B |\nabla u|^{p(x)}\,dx\right)^{\frac{(p^\#)_B^-}{p_B^+}}$$

and furthermore

$$\begin{aligned}\fint_B |u-u_B|^{p^\#(x)}\,dx &\le C\left(|B|^{-\frac{p_B^+}{(p^\#)_B^-}}\int_B |\nabla u|^{p(x)}\,dx\right)^{\frac{(p^\#)_B^-}{p_B^+}}\\ &\le C\left(|B|^{1-\frac{p_B^+}{p_B^-}-1+\frac{p_B^+}{n}}\int_B |\nabla u|^{p(x)}\,dx\right)^{\frac{(p^\#)_B^-}{p_B^+}}.\end{aligned}\tag{3.4}$$

Since p is log-Hölder continuous, $|B|^{1-\frac{p_B^+}{p_B^-}}$ is bounded. We have

$$\operatorname{cap}_{p(\cdot)}(N(u),2B) \le \operatorname{cap}_{p(\cdot)}(\overline{B},2B) \le C|B|^{1-\frac{p_{2B}^+}{n}} \le C|B|^{1-\frac{p_B^+}{n}},$$

where the last inequality follows since p is log-Hölder continuous. Using this in (3.4), we obtain

$$\fint_B |u-u_B|^{p^\#(x)}\,dx \le C\left(\frac{1}{\operatorname{cap}_{p(\cdot)}(N(u),2B)}\int_B |\nabla u|^{p(x)}\,dx\right)^{\frac{(p^\#)_B^-}{p_B^+}}.$$

If $u_B = 0$, then the above inequality yields the claim.

Assume then that $u_B \neq 0$ and u is continuous. We have

$$\fint_B |u(x)|^{p^\#(x)}\,dx \le 2^{(p^\#)^+_B} \fint_B |u(x) - u_B|^{p^\#(x)}\,dx + 2^{(p^\#)^+_B} \fint_B |u_B|^{p^\#(x)}\,dx.$$

The first term on the right hand side is estimated as earlier. Next we estimate the second one.

We write $v := |u - u_B|$. Since u is continuous, the set $N(u)$ is compact and the function v is continuous. Let $\eta \in C_0^\infty(2B)$ be such that, $0 \le \eta \le 1$, $\eta = 1$ in $\overline{B}$ and $|\nabla \eta| \le C/\operatorname{diam}(B)$. Then $v\eta/|u_B|$ is a test function for $\operatorname{cap}_{p(\cdot)}(N(u), 2B)$. Since $\|u\|_{L^{p(\cdot)}(B)} \le 1$ and $|B| \le 1$, we obtain

$$\begin{aligned}
\operatorname{cap}_{p(\cdot)}(N(u), 2B) &\le \int_{2B} \left(\frac{|\nabla(v\eta)|}{|u_B|} \right)^{p(x)} dx \\
&\le \frac{|B|^{p_B^-}}{\left| \int_B u\,dy \right|^{p_B^+}} \int_{2B} |\nabla(v\eta)|^{p(x)}\,dx
\end{aligned}$$

and thus

$$\Big| \int_B u\,dy \Big| \le \left(\frac{|B|^{p_B^-}}{\operatorname{cap}_{p(\cdot)}(N(u), 2B)} \int_{2B} |\nabla(v\eta)|^{p(x)}\,dx \right)^{\frac{1}{p_B^+}}.$$

Using the assumptions $\|u\|_{L^{p(\cdot)}(B)} \le 1$ and $|B| \le 1$, the above estimate and log-Hölder continuity of p, we get

$$\begin{aligned}
\fint_B |u_B|^{p^\#(x)}\,dx &\le \frac{\left| \int_B u\,dy \right|^{(p^\#)^-_B}}{|B|^{(p^\#)^+_B}} \\
&\le |B|^{-(p^\#)^+_B} \left(\frac{|B|^{p_B^-}}{\operatorname{cap}_{p(\cdot)}(N(u), 2B)} \int_{2B} |\nabla(v\eta)|^{p(x)}\,dx \right)^{\frac{(p^\#)^-_B}{p_B^+}} \\
&\le |B|^{\frac{p_B^-}{p_B^+}(p^\#)^-_B - (p^\#)^+_B} \left(\frac{1}{\operatorname{cap}_{p(\cdot)}(N(u), 2B)} \int_{2B} |\nabla(v\eta)|^{p(x)}\,dx \right)^{\frac{(p^\#)^-_B}{p_B^+}} \\
&\le C \left(\frac{1}{\operatorname{cap}_{p(\cdot)}(N(u), 2B)} \int_{2B} |\nabla(v\eta)|^{p(x)}\,dx \right)^{\frac{(p^\#)^-_B}{p_B^+}}.
\end{aligned}$$

Next we estimate the integral on the right hand side. We have

$$\int_{2B} |\nabla(v\eta)|^{p(x)}\,dx \le C \int_{2B} \left(\frac{|v|}{\operatorname{diam}(B)} \right)^{p(x)} dx + C \int_{2B} |\nabla v|^{p(x)}\,dx$$

and, furthermore, for the first term on the right hand side

$$\begin{aligned}\int_{2B}\left(\frac{|v|}{\operatorname{diam}(B)}\right)^{p(x)}dx &= \int_{2B}\left(\frac{|u-u_B|}{\operatorname{diam}(B)}\right)^{p(x)}dx\\ &\le C\int_{2B}\left(\frac{|u-u_{2B}|}{\operatorname{diam}(B)}\right)^{p(x)}dx\\ &\quad + C\operatorname{diam}(B)^{-p_{2B}^+}|B|\max\left\{|u_{2B}-u_B|^{p_{2B}^-}, |u_{2B}-u_B|^{p_{2B}^+}\right\}.\end{aligned}$$

Following the proof of Lemma 3.6 in [23], we obtain

$$\int_{2B}\left(\frac{|u-u_{2B}|}{\operatorname{diam}(B)}\right)^{p(x)}dx \le C\int_{2B}|\nabla u|^{p(x)}\,dx + C|B|.$$

Since

$$\begin{aligned}|u_{2B}-u_B| &\le C\fint_B |u-u_{2B}|\,dx \le C\fint_{2B}|u-u_{2B}|\,dx\\ &\le C\operatorname{diam}(B)\fint_{2B}|\nabla u|\,dx \le C\operatorname{diam}(B)\left(\fint_{2B}|\nabla u|^{p_{2B}^-}\,dx\right)^{\frac{1}{p_{2B}^-}},\end{aligned}$$

we obtain by $\|\nabla u\|_{L^{p(\cdot)}(B)} \le 1$ and by log-Hölder continuity of p that

$$\begin{aligned}&\operatorname{diam}(B)^{-p_{2B}^+}|B|\max\left\{|u_{2B}-u_B|^{p_{2B}^-}, |u_{2B}-u_B|^{p_{2B}^+}\right\}\\ &\le C\max\left\{\operatorname{diam}(B)^{p_{2B}^- - p_{2B}^+}\int_{2B}|\nabla u|^{p_{2B}^-}\,dx, |B|^{1-\frac{p_{2B}^+}{p_{2B}^-}}\left(\int_{2B}|\nabla u|^{p_{2B}^-}\,dx\right)^{\frac{p_{2B}^+}{p_{2B}^-}}\right\}\\ &\le C\max\left\{\operatorname{diam}(B)^{p_{2B}^- - p_{2B}^+}, |B|^{1-\frac{p_{2B}^+}{p_{2B}^-}}\right\}\int_{2B}|\nabla u|^{p_{2B}^-}\,dx\\ &\le C\int_{2B}|\nabla u|^{p(x)}\,dx + C|B|.\end{aligned}$$

Collecting the above inequalities together we see that

$$\fint_B |u_B|^{p^\#(x)}\,dx \le C\left(\frac{1}{\operatorname{cap}_{p(\cdot)}(N(u),2B)}\left(\int_{2B}|\nabla u|^{p(x)}\,dx + |B|\right)\right)^{\frac{(p^\#)_B^-}{p_B^+}},$$

and hence the claim follows for continuous Sobolev functions.

Assume then that $u \in W^{1,p(\cdot)}(2B)$ is quasicontinuous. Since smooth functions are dense in the Sobolev space, we find a sequence (ϕ_i) of $C^\infty(2B)\cap W^{1,p(\cdot)}(2B)$ functions converging to u in the Sobolev sense and also $\phi_i \to u$ pointwise q.e. Thus the claim follows also for quasicontinuous functions. $\square$

4. Uniformly fat sets

In this section, we discuss uniformly fat sets, fatness being taken in a capacitary sense. The key fact here is that in domains whose complement is uniformly fat, we can employ our Sobolev-Poincaré inequality up to the boundary. This, together with the fact that uniform fatness is a self-improving property, makes it a natural assumption for the purposes of global higher integrability. In this section only, we assume that $1 < p^- \le p^+ < \infty$.

Definition 4.1. Let E be a closed subset of $\mathbb{R}^n$. We say that E is locally uniformly $p(\cdot)$-fat, if there exist a radius r_0 and a constant C such that

$$\operatorname{cap}_{p(\cdot)}(E \cap \overline{B}(x,r), B(x,2r)) \ge C \operatorname{cap}_{p(\cdot)}(\overline{B}(x,r), B(x,2r)) \tag{4.2}$$

for all $x \in E$ and all $r \le r_0$.

We start with proving that uniform $q(\cdot)$-fatness implies uniform $p(\cdot)$-fatness, provided that q is strictly below p. The constant exponent case is a simple consequence of Hölder's inequality. However, the variable exponent version of Hölder's inequality only holds with Luxemburg norms on the right hand side. Hence we give the simple proof here to show that this defect is not essential.

Proposition 4.3. *Let q and p be log-Hölder continuous variable exponents, bounded and bounded away from one. Assume that*

$$q(x) \le p(x) - \delta$$

for some $\delta > 0$. If a set E is locally uniformly $q(\cdot)$-fat, it is also locally uniformly $p(\cdot)$-fat.

Proof. Let $B = B(x_0, r)$ be a ball for which the fatness condition is to be verified. We may assume that $p^-_{2B} < n$, for otherwise there is nothing to prove. Pick a function $u \in C_0(2B) \cap W^{1,p(\cdot)}(2B)$ such that $u \ge 1$ in $\overline{B} \cap E$. The variable exponent Hölder's inequality gives

$$\int_{2B} |\nabla u|^{q(x)} \, \mathrm{d}x \le C \|1\|_{(p(\cdot)/q(\cdot))'} \| |\nabla u|^{q(\cdot)} \|_{p(\cdot)/q(\cdot)}.$$

By log-Hölder continuity,

$$\|1\|_{(p(\cdot)/q(\cdot))'} \le C |B|^{1 - q(x_0)/p(x_0)};$$

this is where we need the assumption $q(x) \le p(x) - \delta$. By the elementary relations between the Luxemburg norm and the modular, we get

$$\| |\nabla u|^{q(x)} \|_{\frac{p(\cdot)}{q(\cdot)}} \le \max \left\{ \left(\int_{2B} |\nabla u|^{p(x)} \, \mathrm{d}x \right)^{q^-_{2B}/p^+_{2B}}, \left(\int_{2B} |\nabla u|^{p(x)} \, \mathrm{d}x \right)^{q^+_{2B}/p^-_{2B}} \right\}.$$

Combining the above inequalities and taking infimum over u we obtain

$$\begin{aligned}&\operatorname{cap}_{q(\cdot)}(\overline{B}\cap E,2B)\\ &\le Cr^{n-n\frac{q(x_0)}{p(x_0)}}\max\left\{\left(\operatorname{cap}_{p(\cdot)}(\overline{B}\cap E,2B)\right)^{q_{2B}^-/p_{2B}^+},\left(\operatorname{cap}_{p(\cdot)}(\overline{B}\cap E,2B)\right)^{q_{2B}^+/p_{2B}^-}\right\}.\end{aligned}$$

Since E is locally uniformly $q(\cdot)$-fat, we have

$$r^{n-q(x_0)}\le C\operatorname{cap}_{q(\cdot)}(\overline{B},2B)\le C\operatorname{cap}_{q(\cdot)}(\overline{B}\cap E,2B).$$

Putting the estimates together, we arrive at

$$r^{(n\frac{q(x_0)}{p(x_0)}-q(x_0))s}\le C\operatorname{cap}_{p(\cdot)}(\overline{B}\cap E,2B)$$

where s is either p_{2B}^+/q_{2B}^- or p_{2B}^-/q_{2B}^+. In both cases, log-Hölder continuity allows us to replace s with $p(x_0)/q(x_0)$, and thus the claim follows since $\operatorname{cap}_{p(\cdot)}(\overline{B}\cap E,2B)\le Cr^{n-p(x_0)}$. □

It turns out that the variable exponent uniform fatness condition is self-improving. This means that there is a number $\delta>0$ such that a $p(\cdot)$-fat set is also $(p(\cdot)-\delta)$-fat. We can deduce this fact by employing a localization argument and the corresponding constant exponent result, which is is due to Lewis, see [32]. A different proof of Theorem 4.4 can be found in [38].

Theorem 4.4. *Let E be a locally p-fat set, $1<p<\infty$. Then there is an exponent $1<q<p$ such that E is also q-fat.*

The self-improving property of uniform $p(\cdot)$-fatness now follows from Proposition 4.3. and the proof of Theorem 4.4.

Corollary 4.5. *Suppose that E is is locally uniformly $p(\cdot)$-fat. Then there are positive numbers δ and r_0 such that*

(1) *The estimate (4.2) holds with the exponent $p^-_{B(x_0,3r_0)}-\delta$ for all balls $B=B(x,r)$ with $x\in E\cap B(x_0,r_0)$ and $r\le r_0$.*
(2) *E is locally uniformly $p(\cdot)-\delta$–fat with radius r_0.*

Proof. Let $x_0\in E$. Proposition 4.3 implies that the uniform fatness estimate holds with the exponent $p^+_{B(x_0,3r_0)}+2\delta$ in balls with centers in $E\cap B(x_0,r_0)$ and radii less than r_0. Using this fact, for each $x\in E\cap B(x_0,r_0)$ and $r\le r_0$ it is possible to find a compact set $F\subset E\cap B(x,r)$ which is uniformly $p^+_{B(x_0,3r_0)}+2\delta$-fat. See the proofs of Theorem 1 in [32] or Theorem 8.2 in [38] for the construction of F. With the help of the set F, the same argument as in proof of Theorem 8.2 in [38] gives us a small number $\delta>0$ such that the estimate

$$\operatorname{cap}_q(E\cap\overline{B}(x,r),B(x,2r))\ge C\operatorname{cap}_q(\overline{B}(x,r),B(x,2r))$$

holds with $q=p^+_{B(x_0,3r_0)}-2\delta$. Note that δ and C can be chosen to depend only on n, p^+, p^-, and the constant in (4.2). Choosing a smaller r_0 if necessary, we can assume that $p^+_{B(x_0,3r_0)}-p^-_{B(x_0,3r_0)}<\delta/2$. Then we can apply Proposition 4.3 to get both of the claims. □

Our next goal is a variable exponent Hardy inequality in domains whose complement is uniformly $p(\cdot)$-fat. We will use it below in connection with the next lemma in order to verify that the limit function found in Section 7 below has the right boundary values in Sobolev's sense.

Lemma 4.6. *If $u \in W^{1,p(\cdot)}(\Omega)$ and*

$$\frac{u}{\operatorname{dist}(x,\partial\Omega)} \in L^{p(\cdot)}(\mathbb{R}^n),$$

then $u \in W_0^{1,p(\cdot)}(\Omega)$

Proof. The proof of [8, Theorem 4.3, p. 223] can be adapted to cover the variable exponent case. □

Proposition 4.7. *Suppose that the complement of Ω is locally uniformly $p(\cdot)$-fat. Then the variable exponent Hardy inequality*

$$\left\| \frac{u}{\operatorname{dist}(x,\partial\Omega)} \right\|_{L^{p(\cdot)}(\Omega)} \leq C\|\nabla u\|_{L^{p(\cdot)}(\Omega)} \tag{4.8}$$

holds for all functions $u \in W_0^{1,p(\cdot)}(\Omega)$. The constant C depends only on the dimension, δ, r_0, the constant in the uniform fatness condition, $\operatorname{diam}(\Omega)/r_0$*, p^-, p^+ and* log*-Hölder constant of p. The parameter r_0 is from Corollary 4.5.*

Proof. Let $v \in C_0^\infty(\Omega)$. By homogeneity we may assume that $\|\nabla v\|_{L^{p(\cdot)}(\Omega)} \leq 1$. Let r_0 be the radius from Corollary 4.5. We can cover $\partial\Omega$ by a finite number of balls $B(x_i, r_0/4)$, $x_i \in \partial\Omega$. The number of balls depends on $\operatorname{diam}(\Omega)/r_0$. We write $B_i := B(x_i, r_0/2)$ and set $p_i^- := \inf_{x\in 2B_i} p(x)$. The proof of the pointwise Hardy inequality in [16, 29] is local in nature, so that we may localize the situation by studying $\Omega \cap 2B_i$ and multiply v a suitable cut of -function so that is belongs to $C_0^\infty(\Omega\cap 2B_i)$. Then, in view of the first conclusion in Corollary 4.5, locally uniform $p(\cdot)$-fatness of the complement implies that the pointwise Hardy inequality

$$|v(x)| \leq C \operatorname{dist}(x,\partial\Omega) \Big(M|\nabla v|^{p_i^- -\delta}(x)\Big)^{\frac{1}{p_i^- -\delta}}$$

holds for smooth functions $v \in C_0^\infty(\Omega)$ and all points $x \in B_i$, see Theorem 1 in [16] and Theorem 3.9 in [29]. The constant depends only on the dimension, p^-, δ and the constant in the uniform fatness condition. Let $\Omega_{\frac14 r_0} := \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) \leq r_0/4\}$. Thus we obtain

$$\begin{aligned} \int_{\Omega_{\frac14 r_0}} \left(\frac{|v(x)|}{\operatorname{dist}(x,\partial\Omega)}\right)^{p(x)} dx &\leq \sum_i \int_{B_i} \left(\frac{|v(x)|}{\operatorname{dist}(x,\partial\Omega)}\right)^{p(x)} dx \\ &\leq C \sum_i \int_{B_i} \Big(M|\nabla v|^{p_i^- -\delta}(x)\Big)^{\frac{p(x)}{p_i^- -\delta}} dx. \end{aligned} \tag{4.9}$$

Since $\frac{p(\cdot)}{p_i^- - \delta}$ is log-Hölder continuous, bounded and its infimum is greater than one, the maximal operator is bounded on $L^{\frac{p(\cdot)}{p_i^- - \delta}}$, see [6]. This and the fact that $\|\nabla v\|_{L^{p(\cdot)}(\Omega)} \le 1$ yields that $\int_{B_i} \left(M|\nabla v|^{p_i^- - \delta}(x) \right)^{\frac{p(x)}{p_i^- - \delta}} dx$ is uniformly bounded depending on the dimension, p^-, p^+, log-Hölder continuous constant of p and δ. Hence the right hand side of (4.9) depends also on the number of the balls i.e. $\operatorname{diam}(\Omega)/r_0$.

We obtain by the Poincaré inequality

$$\int_{\Omega \setminus \Omega_{\frac{1}{4}r_0}} \left(\frac{|v(x)|}{\operatorname{dist}(x, \partial\Omega)} \right)^{p(x)} dx \le C r_0^{p^+} \int_\Omega |v(x)|^{p(x)}\, dx \le C r_0^{p^+}$$

for every $v \in C_0^\infty(\Omega)$. Combining this with (4.9) we obtain

$$\int_\Omega \left(\frac{|v(x)|}{\operatorname{dist}(x, \partial\Omega)} \right)^{p(x)} dx \le C,$$

where the constant C depends only on the dimension, δ, r_0, the constant in the uniform fatness condition, $\operatorname{diam}(\Omega)/r_0$, p^-, p^+ and log-Hölder constant of p. This yields the claim for smooth functions. For general Sobolev functions, the claim follows by a standard approximation argument. □

Note that the above version of Hardy's inequality is rather coarse, due to the fact that the constant depends on the domain Ω. However, it is quite sufficient for our current purposes.

The next lemma generalizes [33, Lemma 3.25] to the variable exponent case.

Lemma 4.10. *Let (p_i) be a sequence of* log-*Hölder continuous variable exponents with $1 < \inf_i p_i^- \le \sup_i p_i^+ < \infty$ and with uniformly bounded* log-*Hölder constants so that $p_i \to p$ almost everywhere in Ω. Suppose that Ω is bounded and locally uniformly $p(\cdot)$-fat.*

Let $u \in W^{1,p(\cdot)}(\Omega)$ and $u_i \in W^{1,p_i(\cdot)}(\Omega)$ for every i so that $u_i \to u$ almost everywhere in Ω. If $\theta \in W^{1,p(\cdot)}(\Omega) \cap \bigcap_i W^{1,p_i(\cdot)}(\Omega)$ and $u_i - \theta \in W_0^{1,p_i(x)}(\Omega)$ for every i with

$$\int_\Omega |\nabla(u_i - \theta)|^{p_i(x)}\, dx \le M, \tag{4.11}$$

where M is finite and independent of i, then $u - \theta \in W_0^{1,p(\cdot)}(\Omega)$.

Proof. Due to the assumptions, the Hardy inequality

$$\left\| \frac{|v|}{\operatorname{dist}(x, \partial\Omega)} \right\|_{L^{p_i(\cdot)}(\Omega)} \le C \|\nabla v\|_{L^{p_i(\cdot)}(\Omega)} \tag{4.12}$$

holds for all $v \in W_0^{1,p_i(\cdot)}(\Omega)$, with the constant independent of i. By (4.11) and (4.12) we obtain

$$\int_\Omega \left(\frac{|u_i - \theta|}{\operatorname{dist}(x,\partial\Omega)}\right)^{p_i(x)} dx \le C(M)$$

and Fatou's lemma gives

$$\int_\Omega \left(\frac{|u - \theta|}{\operatorname{dist}(x,\partial\Omega)}\right)^{p(x)} \le \liminf_{i\to\infty} \int_\Omega \left(\frac{|u_i - \theta|}{\operatorname{dist}(x,\partial\Omega)}\right)^{p_i(x)} dx \le C(M).$$

Since $u - \theta \in W^{1,p(\cdot)}(\Omega)$, the claim follows by Lemma 4.6. □

5. Global higher integrability

In this section we prove a global higher integrability result for the gradient of the solution u to (2.4). We will use a proper combination of a Caccioppoli type inequality together with the capacitary Sobolev-Poincaré inequality proved in Lemma 3.3.

Theorem 5.1. *Suppose that the complement of Ω is locally uniformly $p(\cdot)$-fat, and let u be the solution to the Dirichlet boundary value problem (2.4) with $u - f \in W_0^{1,p(\cdot)}(\Omega)$, where $|\nabla f| \in L^{p(\cdot)(1+\delta)}(\Omega)$. Then there exist a positive number δ_0 and a constant C depending on n, p^-, p^+, the structure of $\mathcal{A}$, $\|\nabla f\|_{L^{p(\cdot)}(\Omega)}$, and the constant in the uniform fatness condition, such that $|\nabla u| \in L^{p(\cdot)(1+\delta)}(\Omega)$ whenever $0 < \delta < \delta_0$, and*

$$\int_\Omega |\nabla u|^{p(x)(1+\delta)}\,\mathrm{d}x \le C\left[\int_\Omega |\nabla u|^{p(x)}\,\mathrm{d}x + \int_\Omega |\nabla f|^{p(x)(1+\delta)} + 1\right] \tag{5.2}$$

Proof. Let B_0 be a ball with $\Omega \subset \frac{1}{2}B_0$. Fix $r > 0$ and let $B \equiv B(x_0, r)$ with $4B \subset B_0$. This ball $4B$ will come out in the last part of the proof.

• **Case 1:** $2B \subset \Omega$

Let $\eta \in C_0^\infty(2B)$ be a cut-off function such that $\eta = 1$ in $\bar{B}$, $0 \le \eta \le 1$ and $|\nabla\eta| \le C/\operatorname{diam}(B)$. We test (2.4) with $\varphi := \eta^{p^+_{2B}}(u - u_{2B})$, where $u_{2B} = \fint_{2B} u(x)\,dx$. We have

$$\begin{aligned} 0 &= \int_{2B} \mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,dx \\ &= \int_{2B} p^+_{2B}\,\mathcal{A}(x,\nabla u)\,\eta^{p^+_{2B}-1}(u - u_{2B})\cdot\nabla\eta\,dx \\ &\quad + \int_{2B} \mathcal{A}(x,\nabla u)\,\eta^{p^+_{2B}}\cdot\nabla(u - u_{2B})\,dx. \end{aligned}$$

We estimates the second term using the structure conditions on $\mathcal{A}$

$$\begin{aligned}\int_{2B} \mathcal{A}(x,\nabla u)\, \eta^{p^+_{2B}} \cdot \nabla(u-u_{2B})\, dx &= \int_{2B} \eta^{p^+_{2B}}\, \mathcal{A}(x,\nabla u)\ \cdot \nabla u\, dx \\ &\geq \alpha \int_{2B} \eta^{p^+_{2B}}\, |\nabla u|^{p(x)}\, dx.\end{aligned}$$

For the first term we obtain by the structure conditions on $\mathcal{A}$ and Young's inequality, $\zeta \in (0,1)$, that

$$\begin{aligned}\left|\int_{2B} \mathcal{A}(x,\nabla u)\, \eta^{p^+_{2B}-1}\, (u-u_{2B}) \cdot \nabla \eta\, dx\right| &\leq \zeta \int_{2B} \eta^{p^+_{2B}} |\nabla u|^{p(x)}\, dx \\ &\quad + c_\zeta \int_{2B} \left|\frac{u-u_{2B}}{\operatorname{diam}(B)}\right|^{p(x)} dx.\end{aligned}$$

Observe that we used the definition of p^+_{2B} to deduce

$$\tilde{p} := \frac{p(x)(p^+_{2B}-1)}{p(x)-1} \geq p^+_{2B} \qquad \forall\, x \in 2B.$$

and to estimate $\eta^{\tilde{p}} \leq \eta^{p^+_{2B}}$ in the second inequality. Now, if we choose ζ, which depends on $n, p^-, p^+, \alpha, \beta$, small enough and connect all the previous estimates, after taking the mean values, we finally deduce the following Caccioppoli type inequality

$$\fint_B |\nabla u|^{p(x)}\, dx \leq C \fint_{2B} \left|\frac{u-u_{2B}}{\operatorname{diam}(B)}\right|^{p(x)} dx,$$

where C only depends on $n, p^-, p^+, \alpha, \beta$. At this point, arguing as in [43], Theorem 1.3, or [9], Theorem 3.1, we can use a standard Sobolev-Poincaré inequality and the log-Hölder continuity which yield the following reverse Hölder estimate

$$\fint_B |\nabla u|^{p(x)}\, dx \leq C \left(\fint_{2B} |\nabla u|^{\frac{p(x)}{\bar{\theta}}}\, dx\right)^{\bar{\theta}} + C,$$

with $C \equiv C(n, p^-, p^+, \alpha, \beta, M)$, M is such that $\int_\Omega |\nabla u|^{p(x)}\, \mathrm{d}x \leq M$, and $\bar{\theta} := \min\left\{\sqrt{\frac{n+1}{n}}, p^-\right\}$. We can take $M = C \int_\Omega |\nabla f|^{p(x)}\, \mathrm{d}x$, see the arguments leading to (7.5) in the proof of Theorem 7.3 below.

• **Case 2:** $2B \setminus \Omega \neq \emptyset$**.**

This case is more complicated than the previous one, as the boundary of Ω will be involved in the analysis. We have to use the fact that the complement of Ω is locally uniformly $p(\cdot)$-fat and therefore the capacitary version of the Sobolev-Poincaré inequality established in Lemma 3.3. We divide this case in a couple of steps.

STEP 1: CACCIOPPOLI TYPE INEQUALITY. Let $\eta \in C_0^\infty(2B)$ be the cut-off function chosen at the previous step. This time we test (2.4) with $\varphi := \eta^{p_D^+}(u-f)$, where $u - f \in W_0^{1,p(\cdot)}(\Omega)$. If $D = 2B \cap \Omega$, we have

$$
\begin{aligned}
0 &= \int_D \mathcal{A}(x,\nabla u)\cdot\nabla\varphi\,dx \\
&= \int_D p_D^+\,\mathcal{A}(x,\nabla u)\,\eta^{p_D^+-1}(u-f)\cdot\nabla\eta\,dx \\
&\quad+\int_D \mathcal{A}(x,\nabla u)\,\eta^{p_D^+}\cdot\nabla(u-f)\,dx.
\end{aligned}
$$

Using the structure conditions on $\mathcal{A}$ we deduce

$$
\begin{aligned}
\alpha\int_D \eta^{p_D^+}|\nabla u|^{p(x)}\,dx &\le \int_D \eta^{p_D^+}\,\beta\,|\nabla u|^{p(x)-1}\,|\nabla f|\,dx \\
&\quad+\int_D \eta^{p_D^+}\,\beta\,|\nabla u|^{p(x)-1}|u-f|\,|\nabla\eta|\,dx.
\end{aligned}
$$

Exploiting the Young inequality and taking the mean values, we obtain the following Caccioppoli type inequality

$$
\begin{aligned}
\frac{1}{|2B|}\int_D \eta^{p_D^+}|\nabla u|^{p(x)}\,dx &\le C\,\frac{1}{|2B|}\int_D\left|\frac{u-f}{\operatorname{diam}(B)}\right|^{p(x)}dx \\
&\quad+C\,\frac{1}{|2B|}\int_{4B\cap\Omega}|\nabla f|^{p(x)}\,dx.
\end{aligned}
$$

Here the constants C only depend on $n, p^-, p^+, \alpha, \beta$. The choice to replace D with $4B\cap\Omega$ in the last integral will be clear later.

STEP 2: CHOICE OF θ AND LOCALIZATION. In order to obtain a suitable reverse Hölder inequality, we have to choose a proper value of the parameter θ. First of all, we choose $1 < \theta < n$ such that the complement of Ω is $\frac{p(\cdot)}{\theta}$-fat and, taking a smaller value of θ if necessary, we suppose that

$$
\frac{n/\theta}{n-1/\theta} > 1.
$$

Note that this last assumpion implies $1 < \theta < \frac{n+1}{n} < 2$. We choose now $\operatorname{diam}(2B) \le 1$ so small that $|2B| \le 1$, $(1+|2B|)\|u-f\|_{W^{1,p(\cdot)}(2B)} \le 1$, and further such that

$$
p_{2B}^+ < \frac{n/\theta}{n-1/\theta}\,p_{2B}^-.
$$

This implies that p_{2B}^+ is always less than the Sobolev conjugate of p_{2B}^-.

STEP 3: APPLICATION OF SOBOLEV–POINCARÉ. Since $u - f \in W_0^{1,p(\cdot)}(\Omega)$, $u - f$ has an extension belonging to $W^{1p(\cdot)}(\mathbb{R}^n)$ which is zero $p(\cdot)$-q.e. in the complement of Ω. Then we can use the capacitary version of the Sobolev-Poincaré inequality of Lemma 3.3 together with

Hölder's inequality to get

$$\begin{aligned}
&\fint_{2B} \left| \frac{u-f}{\operatorname{diam}(B)} \right|^{p(x)} dx \\
\leq\ & \operatorname{diam}(B)^{-p_{2B}^+} \left(\fint_{2B} |u-f|^{(n\frac{p(x)}{\theta})/(n-\frac{p_{2B}^-}{\theta})} \, dx \right)^{\theta \frac{n-p_{2B}^-/\theta}{n}} \\
\leq\ & \operatorname{diam}(B)^{-p_{2B}^+} \\
& \times \left[\frac{C}{\operatorname{cap}_{\frac{p(x)}{\theta}}(N(u-f), 4B)} \left(\int_{4B} |\nabla(u-f)|^{\frac{p(x)}{\theta}} \, dx + |B| \right) \right]^{\theta \frac{p_{2B}^-}{p_{2B}^+}},
\end{aligned}$$

where $N(u-f) = \{x \in \overline{2B} : u(x) = f(x)\}$. In the complement of Ω, $u - f = 0$ except for a set of $p(\cdot)$-capacity zero. Since the complement of Ω is $\frac{p(\cdot)}{\theta}$-fat and p is log-Hölder continuous, we obtain

$$\operatorname{cap}_{\frac{p(\cdot)}{\theta}}(N(u-f), 4B) \geq \operatorname{cap}_{\frac{p(\cdot)}{\theta}}(\overline{2B} \setminus \Omega, 4B) \geq C \operatorname{diam}(B)^{n-\frac{p_{2B}^+}{\theta}}.$$

We use this in the previous inequality, and obtain

$$\begin{aligned}
&\fint_{2B} \left| \frac{u-f}{\operatorname{diam}(B)} \right|^{p(x)} dx \\
\leq\ & C \operatorname{diam}(B)^{p_{2B}^- - p_{2B}^+} \left(\fint_{4B} |\nabla(u-f)|^{\frac{p(x)}{\theta}} \, dx + 1 \right)^{\theta \frac{p_{2B}^-}{p_{2B}^+}} \\
\leq\ & C \left(\fint_{4B} |\nabla(u-f)|^{\frac{p(x)}{\theta}} \, dx + 1 \right)^{\theta}
\end{aligned}$$

by log-Hölder continuity. Combining the above inequalities, we get

$$\begin{aligned}
\frac{1}{|B|} \int_{B\cap\Omega} |\nabla u|^{p(x)} \, dx \leq & C \left(\fint_{4B} |\nabla(u-f)|^{\frac{p(x)}{\theta}} \, dx + 1 \right)^{\theta} \\
& + C \frac{1}{|4B|} \int_{4B\cap\Omega} |\nabla f|^{p(x)} \, dx + C \\
\leq & C \left(\frac{1}{|4B|} \int_{4B\cap\Omega} |\nabla u|^{\frac{p(x)}{\theta}} + |\nabla f|^{\frac{p(x)}{\theta}} \, dx \right)^{\theta} \\
& + C \frac{1}{|4B|} \int_{4B\cap\Omega} |\nabla f|^{p(x)} \, dx + C \\
\leq & C \left(\frac{1}{|4B|} \int_{4B\cap\Omega} |\nabla u|^{\frac{p(x)}{\theta}} \, dx \right)^{\theta} \\
& + C \frac{1}{|4B|} \int_{4B\cap\Omega} |\nabla f|^{p(x)} \, dx + C,
\end{aligned}$$

where the constants depends on θ.

STEP 5: GEHRING LEMMA AND CONCLUSION. Summing up, we obtained in both of the above cases the following reverse Hölder estimate

$$\frac{1}{|B|}\int_{B\cap\Omega}|\nabla u|^{p(x)}\,dx \le C\left(\frac{1}{|4B|}\int_{4B\cap\Omega}|\nabla u|^{\frac{p(x)}{\theta}}\,dx\right)^{\theta} + C\,\frac{1}{|4B|}\int_{4B\cap\Omega}|\nabla f|^{p(x)}\,dx + C,$$

which holds for sufficiently small balls with constants C depending on $n, p^-, p^+, \alpha, \beta, \theta, \|\nabla f\|_{p(\cdot)}$ but independent of the radius of the ball. We therefore make the following conclusion: set $\theta_1 := \min\{\bar\theta, \theta\}$ and let

$$g(x) = \begin{cases} |\nabla u|^{\frac{p(x)}{\theta_1}}, & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

and

$$h(x) = \begin{cases} |\nabla f|^{\frac{p(x)}{\theta_1}}, & \text{if } x \in \Omega \\ 0 & \text{otherwise} \end{cases}$$

so that the previous reverse Hölder inequality reads

$$\fint_B g^{\theta_1}\,dx \le\; C\left(\fint_{4B} g\,dx\right)^{\theta_1} + C\fint_{4B} h^{\theta_1}\,dx + C,$$

whenever $4B \subset B_0$. Now we can use a standard version of Gehring's lemma (see for example [14], Chap. V, or [15], Theorem 6.6), and find a number $\delta > 0$ and a constant C such that

$$\fint_B |\nabla u|^{p(x)(1+\delta)}\,\mathrm{d}x \le C\left[\left(\fint_{4B}|\nabla u|^{p(x)}\,\mathrm{d}x\right)^{1+\delta} + \fint_{4B}|\nabla f|^{p(x)(1+\delta)}\,\mathrm{d}x + 1\right].$$

The estimate (5.2) then follows from this by noting that due to the boundedness of Ω, $\overline{\Omega}$ can be covered by a finite number of balls such that the previous inequality holds. $\square$

6. Hölder continuity up to the boundary

In this section we show that the uniform fatness condition is sufficient for Hölder continuity up to the boundary, provided that the boundary values are given by a Hölder continuous function. This is a consequence of a Maz'ya type boundary point estimate of Alkhutov and Krashenninikova, see [4]. We denote

$$\gamma(t) = \left(\frac{\mathrm{cap}_{p(\cdot)}(\overline{B}(x_0,t)\setminus\Omega, B(x_0,2t))}{\mathrm{cap}_{p(\cdot)}(\overline{B}(x_0,t), B(x_0,2t))}\right)^{1/(p(x_0)-1)},$$

and note that the $p(\cdot)$-fatness of the complement of Ω implies

$$\gamma(t) \ge \gamma_0 > 0$$

for some constant γ_0 and all $t \le r_0$.

Theorem 6.1 ([4], Theorem 1.2). *Let $f \in C(\overline{\Omega}) \cap W^{1,p(\cdot)}(\Omega)$, denote $M = \sup_{x\in\partial\Omega} |f(x)|$, and let u be the solution with the boundary value function f. Then there are constants C and θ such that for all $\rho > 0$, $0 < r \le \rho/4$ and $x_0 \in \partial\Omega$*
(6.2)
$$\sup_{x\in\Omega\cap B(x_0,r)} |u(x)-f(x_0)| \le C\left(\operatorname*{osc}_{\partial\Omega\cap B(x_0,\rho)} f + \rho + \operatorname*{osc}_{\partial\Omega} f \exp\left(-\theta\int_r^\rho \gamma(t)\frac{\mathrm{d}t}{t}\right)\right).$$

An immediate consequence of the uniform fatness condition is following refinement.

Corollary 6.3. *Assume that the complement of Ω is uniformly $p(\cdot)$-fat, with u and f as above. Then for $\rho \le r_0$, $r \le \rho/4$ we have*
$$\sup_{x\in\Omega\cap B(x_0,r)} |u(x) - f(x_0)| \le C\left(\operatorname*{osc}_{\partial\Omega\cap B(x_0,\rho)} f + \rho + \operatorname*{osc}_{\partial\Omega} f\left(\frac{r}{\rho}\right)^{\delta}\right), \tag{6.4}$$
where δ depends on the uniform fatness constant and the constant θ in (6.2).

We also record the following interior Hölder estimate. It follows, e.g., by iterating Harnack's inequality [3, 22] in a standard fashion.

Theorem 6.5. *Let u be a solution, in Ω, $B(x_0, R) \Subset \Omega$, and $0 < r < R$. Then*
$$\operatorname*{osc}_{B(x_0,r)} u \le C\left(\frac{r}{R}\right)^{\kappa}\left(\operatorname*{osc}_{B(x_0,R)} u + R\right). \tag{6.6}$$

The following elementary lemma will be used in the proof, see [27, Lemma 6.47].

Lemma 6.7. *Assume that u is a function such that the estimate* (6.6) *holds. If there are constants $L \ge 0$ and $0 < \gamma < 1$ such that*
$$|u(x) - u(x_0)| \le L|x - x_0|^{\gamma},$$
for all $x \in \Omega$ and $x_0 \in \partial\Omega$, then
$$|u(x) - u(y)| \le L_1|x - y|^{\gamma_1}$$
for all $x, y \in \overline{\Omega}$. We can choose $\gamma_1 = \min(\gamma, \kappa)$, where κ is the exponent in (6.6).

The boundary continuity result now follows from Theorem 6.1.

Theorem 6.8. *Let u be a solution with boundary values f, and assume that f is Hölder continuous, i.e.*
$$|f(x) - f(y)| \le M|x - y|^{\gamma}$$
for all $x, y \in \partial\Omega$ for some $0 < \gamma < 1$. Then
$$|u(x) - u(y)| \le M_1|x - y|^{\delta_1}$$

for all $x, y \in \overline{\Omega}$, where δ_1 depends only n, p^+, p^-, the log*-Hölder constant of p, the structural constants and the constant in the uniform fatness condition.*

Proof. By Theorem 6.5, the claim will follow by verifying the assumption of Lemma 6.7. We may assume that $r_0 \le 1$. Let $x \in \Omega$ and $x_0 \in \partial\Omega$, and consider first the case $r = |x - x_0| \le r_0^2$. Then the boundary point estimate (6.4) of Corollary 6.3 with $\rho = 4\sqrt{r}$ yields

$$\begin{aligned}|u(x) - u(x_0)| \le& CMr^{\gamma/2} + Cr^{1/2} + C \operatorname*{osc}_{\partial\Omega} f r^{\delta/2}\\ \le& Cr^{\min(\gamma/2, 1/2, \delta/2)}.\end{aligned}$$

Thus the assumption holds if $|x - x_0| \le r_0^2$.

If $|x - x_0| \ge r_0^2$, we have the trivial estimate

$$\begin{aligned}|u(x) - u(y)| \le& 2(\sup_{x\in\Omega} u(x)) r_0^{-2\gamma} |x - x_0|^\gamma\\ \le& 2(\sup_{x\in\Omega} f(x)) r_0^{-2} |x - x_0|^\gamma,\end{aligned}$$

so that the assumption holds also in this case. $\square$

7. Stability of solutions

The goal of this section is to show that solutions are stable under suitable assumptions. More precisely, let $p_i : \mathbb{R}^n \to (1, \infty)$ be continuous functions that converge pointwise to a function p, and assume that they satisfy the log-Hölder condition

$$|p_i(x) - p_i(y)| \le \frac{C}{-\log(|x - y|)}$$

with a constant independent of i. By Arzela–Ascoli's theorem, we may assume that the convergence is uniform. Further, let the vector fields $\mathcal{A}_i(x, \xi)$ have $p_i(x)$-growth with structural constants α and β independent of i, and assume they converge to $\mathcal{A}(x, \xi)$ uniformly on compact subsets of $\mathbb{R}^n$. Suppose that u_i is the solution to the Dirichlet problem

$$\begin{cases} -\operatorname{div} \mathcal{A}_i(x, \nabla u_i) = 0 \\ \quad u_i - f \in W_0^{1,p_i(\cdot)}(\Omega), \end{cases} \tag{7.1}$$

and u_0 is the solution to

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, \nabla u_0) = 0 \\ \quad u_0 - f \in W_0^{1,p(\cdot)}(\Omega). \end{cases} \tag{7.2}$$

Provided that the boundary values f are sufficiently regular, we will extract a limit function u from the sequence (u_i), and then show that $u = u_0$.

Note that for the prototype case $\mathcal{A}_i(x,\xi) = |\xi|^{p_i(x)-2}\xi$, the uniform convergence of $\mathcal{A}_i(x,\xi)$ when $|\xi| \leq M < \infty$ is a consequence of the uniform convergence of the functions p_i:

$$\begin{aligned}|\mathcal{A}_i(x,\xi) - \mathcal{A}(x,\xi)| &\leq |\xi|\Big||\xi|^{p_i(x)-2} - |\xi|^{p(x)-2}\Big| \\ &\leq \log|\xi|(|\xi|^{\max(p_i^+,p^+)-2} + |\xi|^{\min(p_i^-,p^-)-2})|p_i(x) - p(x)|,\end{aligned}$$

where the second inequality follows from the mean value theorem.

Theorem 7.3. *Let (p_i) be variable exponents with $1 < \inf_i p_i^- \leq \sup_i p_i^+ < n$, and log-Hölder continuous with a constant independent of i. Let $\mathcal{A}_i(x,\xi)$ be vector fields exhibiting $p_i(x)$-growth with constants independent of i that convergence locally uniformly to a vector field $\mathcal{A}(x,\xi)$.*

If the boundary value function f is in $W^{1,p(\cdot)(1+\gamma)}(\Omega)$ for some $\gamma > 0$, then the sequence (u_i) of solutions to (7.1) has a subsequence which converges in $W^{1,p(\cdot)(1+\delta)}(\Omega)$ for any $\delta < \gamma$, and in $C^{\alpha}_{loc}(\Omega)$, to the solution u_0 of (7.2). Further, if the boundary values are Hölder continuous, the subsequence can be taken to converge in $C^{\alpha}(\overline{\Omega})$ also.

Proof. We proceed along the lines of [33]. To get started, assume that $f \in W^{1,p(\cdot)(1+\gamma)}(\Omega)$, and choose i sufficiently large, so that $(1+\gamma/2)p_i \leq (1+\gamma)p$. Then $f \in W^{1,p_i(\cdot)(1+\gamma/2)}(\Omega)$. The first step is to prove the estimate

$$\int_\Omega |\nabla u_i|^{p(x)(1+\gamma/4)}\,\mathrm{d}x \leq C, \tag{7.4}$$

for large i, with $C < \infty$ independent of i. We use $u_i - f$ as a test function in (7.1); this gives

$$\int_\Omega \mathcal{A}_i(x,\nabla u_i)\cdot\nabla u_i\,\mathrm{d}x = \int_\Omega \mathcal{A}_i(x,\nabla u_i)\cdot\nabla f\,\mathrm{d}x.$$

The left hand side can be estimated from below by

$$\int_\Omega \mathcal{A}_i(x,\nabla u_i)\cdot\nabla u_i\,\mathrm{d}x \geq \alpha\int_\Omega |\nabla u_i|^{p(x)}\,\mathrm{d}x,$$

and the right hand side from above by means of Young's inequality

$$\begin{aligned}\int_\Omega \mathcal{A}_i(x,\nabla u_i)\cdot\nabla f\,\mathrm{d}x \leq&\beta\int_\Omega |\nabla u_i|^{p_i(x)-1}|\nabla f|\,\mathrm{d}x \\ \leq&\frac{\alpha}{2}\int_\Omega |\nabla u_i|^{p(x)}\,\mathrm{d}x + C\int_\Omega |\nabla f|^{p_i(x)}\,\mathrm{d}x,\end{aligned}$$

where C depends on α and β. Hence we have

$$\begin{aligned}\int_\Omega |\nabla u_i|^{p_i(x)}\,\mathrm{d}x &\leq C\int_\Omega |\nabla f|^{p_i(x)}\,\mathrm{d}x \\ &\leq C\left(|\Omega| + \int_\Omega |\nabla f|^{p(x)(1+\gamma)}\,\mathrm{d}x\right).\end{aligned} \tag{7.5}$$

We combine this estimate with the global reverse Hölder inequality, Theorem 5.1,

$$\left(\int_\Omega |\nabla u_i|^{p_i(x)(1+\gamma/2)}\,\mathrm{d}x\right)^{1/(1+\gamma/2)} \le C\Big[\int_\Omega |\nabla u_i|^{p_i(x)}\,\mathrm{d}x + \left(\int_\Omega |\nabla f|^{p_i(x)(1+\gamma/2)}\,\mathrm{d}x\right)^{1/(1+\gamma/2)}\Big].$$

To get (7.4) we choose i such that $(1+\gamma/4)p \le (1+\gamma/2)p_i$,

We continue by establishing the bound

$$\|u_i\|_{W^{1,p(\cdot)(1+\gamma/4)}(\Omega)} \le C. \tag{7.6}$$

Let $\kappa = \frac{n}{n-1}$; then Sobolev embedding for $u_i - f \in W_0^{1,p_i(\cdot)}(\Omega)$ and choosing i so large that $(1+\gamma/4)p \le \kappa p_i$ and $p_i \le (1+\gamma/4)p$ imply

$$\begin{aligned}\|u_i - f\|_{p(\cdot)(1+\gamma/4)} &\le C\|u_i - f\|_{\kappa p_i(\cdot)}\\ &\le C\|\nabla(u_i - f)\|_{p_i(\cdot)}\\ &\le C\left(\|\nabla u_i\|_{p(\cdot)(1+\gamma/4)} + \|\nabla f\|_{p(\cdot)(1+\gamma/4)}\right).\end{aligned}$$

Continuing the estimates, we have

$$\begin{aligned}\|u_i\|_{p(\cdot)(1+\gamma/4)} &\le C\left(\|(u_i - f)\|_{p(\cdot)(1+\gamma/4)} + \|f\|_{p(\cdot)(1+\gamma/4)}\right)\\ &\le C\left(\|\nabla u_i\|_{p(\cdot)(1+\gamma/4)} + \|\nabla f\|_{p(\cdot)(1+\gamma/4)} + \|f\|_{p(\cdot)(1+\gamma/4)}\right),\end{aligned}$$

and (7.6) follows by combining this estimate and (7.4).

Let us now simplify notation by writing $\delta = \gamma/4$. Having established (7.6), compactness arguments let us extract a subsequence, still denoted by (u_i), such that $u_i \to u$ weakly in $W^{1,p(\cdot)(1+\delta)}(\Omega)$, $u_i \to u$ in $L^{p(\cdot)(1+\delta)}(\Omega)$, and $u_i \to u$ pointwise almost everywhere in Ω. The next stage is to extract a further subsequence so that $\nabla u_i \to \nabla u$ pointwise almost everywhere in Ω. From this, it follows that $u_i \to u$ in $W^{1,p(\cdot)(1+\delta')}(\Omega)$ for all $\delta' < \delta$, and this allows us to conclude that u is the solution to (7.2) with boundary values f, i.e. $u = u_0$.

Let $G \Subset G' \Subset \Omega$ and pick $\varepsilon > 0$. We write

$$E_i^\varepsilon = \{x \in G : (\mathcal{A}_i(x, \nabla u_i) - \mathcal{A}_i(x, \nabla u)) \cdot (\nabla u_i - \nabla u) > \varepsilon\}$$

for $i = 1, 2, \ldots$. Pick a cutoff function $\eta \in C_0^\infty(G')$: $0 \le \eta \le 1$ and $\eta = 1$ on G. To estimate $|E_i^\varepsilon|$, we first note that

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{G'\cap\{|u_i-u|<\varepsilon^2\}} & (\mathcal{A}_i(x,\nabla u_i) - \mathcal{A}_i(x,\nabla u))\cdot(\nabla(u_i-u))\eta\,\mathrm{d}x \\
=&\frac{1}{\varepsilon}\int_{G'\cap\{|u_i-u|<\varepsilon^2\}} \mathcal{A}_i(x,\nabla u_i)\cdot(\nabla(u_i-u))\,\mathrm{d}x \\
&-\frac{1}{\varepsilon}\int_{G'\cap\{|u_i-u|<\varepsilon^2\}} (\mathcal{A}_i(x,\nabla u) - \mathcal{A}(x,\nabla u))\cdot(\nabla(u_i-u))\,\mathrm{d}x \\
&-\frac{1}{\varepsilon}\int_{G'\cap\{|u_i-u|<\varepsilon^2\}} \mathcal{A}(x,\nabla u)\cdot(\nabla(u_i-u))\,\mathrm{d}x \\
=&I_1^i + I_2^i + I_3^i.
\end{aligned}$$

To estimate I_1^i, we set

$$w_i = \min\{\max\{0, u+\varepsilon^2 - u_i\}, 2\varepsilon^2\}.$$

Then $\eta w_i \in W_0^{1,p(\cdot)(1+\delta)}(G')$, and using it as a test function we have

$$\begin{aligned}
\int_{G'\cap\{|u_i-u|<\varepsilon^2\}} \mathcal{A}_i(x,\nabla u_i)\cdot(\eta\nabla(u_i-u))\,\mathrm{d}x \le & \int_{G'} \mathcal{A}(x,\nabla u_i)\cdot(w_i\nabla\eta)\,\mathrm{d}x \\
\le & C\varepsilon^2\int_{G'} |\mathcal{A}_i(x,\nabla u_i)|\,\mathrm{d}x \le C\varepsilon^2\int_{G'} |\nabla u_i|^{p_i(x)-1}\,\mathrm{d}x \\
\le & C\varepsilon^2\int_{G'} 1 + |\nabla u_i|^{p_i(x)}\,\mathrm{d}x \le C\varepsilon^2,
\end{aligned}$$

with a constant C independent of i. Here (7.5) is used in the last inequality.

For I_2^i, we have

$$\begin{aligned}
|I_2^i| \le & \frac{C}{\varepsilon}\|\mathcal{A}_i(x,\nabla u) - \mathcal{A}(x,\nabla u)\|_{p_i'(\cdot)}\|\nabla(u-u_i)\|_{p_i(\cdot)} \\
\le & \frac{C}{\varepsilon}\|\mathcal{A}_i(x,\nabla u) - \mathcal{A}(x,\nabla u)\|_{p_i'(\cdot)}\||\nabla u| + |\nabla u_i|\|_{p_i(\cdot)}.
\end{aligned}$$

Hence I_2^i can be controlled by showing that

$$\int_\Omega |\mathcal{A}_i(x,\nabla u) - \mathcal{A}(x,\nabla u)|^{p_i'(x)}\,\mathrm{d}x \to 0 \tag{7.7}$$

as $i \to \infty$. By the uniform convergence assumption, $\mathcal{A}_i(x,\nabla u) \to \mathcal{A}(x,\nabla u)$ almost everywhere in Ω. Since

$$\begin{aligned}
&|\mathcal{A}_i(x,\nabla u) - \mathcal{A}(x,\nabla u)|^{p_i'(x)} \le C|\nabla u|^{p_i(x)} + C|\nabla u|^{(p(x)-1)p_i'(x)} \\
&\quad \le C + C|\nabla u|^{p(x)(1+\delta)} + C|\nabla u|^{p(x)(1+\delta)},
\end{aligned}$$

(7.7) follows by Lebesgue's dominated convergence theorem.

To estimate $|I_3^i|$, we simply note that

$$|I_3^i| \le \frac{1}{\varepsilon} \int_G' \mathcal{A}(x, \nabla u) \cdot (\nabla u_i - \nabla u) \, \mathrm{d}x,$$

and use the fact that $\nabla u_i \to \nabla u$ weakly in $L^{p(\cdot)(1+\delta)}(\Omega)$, so that $|I_3^i| \le \varepsilon$ for large i.

By the above estimates, we see that

$$\begin{aligned}
|E_i^\varepsilon| \le & |E_i^\varepsilon \cap \{|u_i - u| \ge \varepsilon^2\}| \\
& + \frac{1}{\varepsilon} \int_{E_i^\varepsilon \cap \{|u_i - u| < \varepsilon^2\}} (\mathcal{A}_i(x, \nabla u_i) - \mathcal{A}_i(x, \nabla u)) \cdot (\nabla u_i - \nabla u) \, \mathrm{d}x \\
\le & |\{|u_i - u| \ge \varepsilon^2\}| + C\varepsilon \le C\varepsilon
\end{aligned}$$

for sufficiently large i, since $u_i \to u$ almost everywhere. It follows that

$$\lim_{i\to\infty} (\mathcal{A}_i(x, \nabla u_i) - \mathcal{A}_i(x, \nabla u)) \cdot (\nabla u_i - \nabla u) = 0$$

for almost every $x \in G$. We write

$$I_i(x) = (\mathcal{A}_i(x, \nabla u_i) - \mathcal{A}_i(x, \nabla u)) \cdot (\nabla u_i - \nabla u).$$

Let x_0 be a point in G such that

(1) $I_i(x_0) \to 0$ as $i \to \infty$,
(2) $|\nabla u(x_0)| < \infty$,
(3) Structural conditions hold at x_0, and
(4) $\mathcal{A}_i(x_0, \xi) \to \mathcal{A}(x_0, \xi)$ uniformly on compact subsets of Ω.

If now $|\nabla u_i(x_0)| \to \infty$ for any subsequence, we would have a contradiction since

$$\begin{aligned}
\alpha |\nabla u_i(x_0)|^{p_i(x_0)} &\le \mathcal{A}_i(x_0, \nabla u_i(x_0)) \cdot \nabla u_i(x_0) \\
&= I_i(x_0) + \mathcal{A}_i(x_0, \nabla u(x_0)) \cdot (\nabla u_i(x_0) - \nabla u(x_0)) \\
&\quad + \mathcal{A}_i(x_0, \nabla u_i(x_0)) \cdot \nabla u(x_0) \\
&\le I_i(x_0) + \beta |\nabla u(x_0)|^{p_i(x_0)-1} (|\nabla u_i(x_0)| + |\nabla u(x_0)|) \\
&\quad + \beta |\nabla u_i(x_0)|^{p_i(x_0)-1} |\nabla u(x_0)|,
\end{aligned}$$

where the left hand side grows with the exponent $p_i(x_0)$ while the right hand side grows with the exponent $p_i(x_0) - 1$. Further, if $\nabla u_i(x_0) \to \xi \in \mathbb{R}^n$ for any subsequence, we must have $\xi = \nabla u(x_0)$. To see this, note that

$$0 = \lim_{i\to\infty} I_i(x_0) = (\mathcal{A}(x_0, \xi) - \mathcal{A}(x_0, \nabla u(x_0))) \cdot (\xi - \nabla u(x_0)) > 0,$$

where the last inequality follows by the structural conditions if $\xi \ne \nabla u(x_0)$. It follows that $\nabla u_i \to \nabla u$ almost everywhere.

Next we show that u has the boundary values given by f in Sobolev's sense. To this end, recall that $f \in W^{1,p(\cdot)(1+\delta)}(\Omega)$, $u_i - f \in W_0^{1,p_i(x)}(\Omega)$, and the sequence (u_i) is bounded in $W^{1,p(\cdot)(1+\delta)}(\Omega)$. We have

$$\int_\Omega |\nabla(u_i - f)|^{p_i(x)}\,\mathrm{d}x \le C\left(1 + \int_\Omega |\nabla u_i|^{p(x)(1+\delta)}\,\mathrm{d}x + \int_\Omega |\nabla f|^{p(x)(1+\delta)}\,\mathrm{d}x\right)$$

for i sufficiently large, such that $p_i \le p(1+\delta)$. We combine (7.4) with Lemma 4.10 to get that $u - f \in W_0^{1,p(\cdot)}(\Omega)$.

Write $u - u_0 = (u - f) + (f - u_0)$ to see that $u - u_0 \in W_0^{1,p(\cdot)}(\Omega)$. Since u_0 is a solution, it follows that

$$\int_\Omega \mathcal{A}(x, \nabla u_0)\cdot(\nabla u - \nabla u_0)\,\mathrm{d}x = 0.$$

Since the functions u_i are solutions of their respective equations, we have similarly

$$\int_\Omega \mathcal{A}_i(x, \nabla u_i)\cdot(\nabla u_0 - \nabla u_i)\,\mathrm{d}x = 0.$$

Letting $i \to \infty$ we get

$$\int_\Omega \mathcal{A}(x, \nabla u)\cdot(\nabla u_0 - \nabla u)\,\mathrm{d}x = 0,$$

since $u_i \to u$ in $W^{1,p(\cdot)(1+\delta)}(\Omega)$. It follows that $u = u_0$, since

$$0 \le \int_\Omega (\mathcal{A}(x, \nabla u) - \mathcal{A}(x, \nabla u_0))\cdot(\nabla u - \nabla u_0)\,\mathrm{d}x = 0.$$

The convergence can also be taken to be locally uniform, since Hölder regularity estimates hold for u_i with constants independent of i. Similarly, if the boundary data is Hölder continuous, we have convergence in $C^\alpha(\overline{\Omega})$. $\square$

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