INTEGRAL REPRESENTATION OF ABSTRACT FUNCTIONALS OF AUTONOMOUS TYPE

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Abstract. In this work we extend the results of [2] to a family of abstract functionals of autonomous type satisfying suitable locality and additivity properties, and general integral growth conditions of superlinear type. We single out a condition which is necessary and sufficient in order for a functional of this class to admit an integral representation, and sufficient as well to have an integral representation for its lower semicontinuous envelope. We also show that the integrand $F(x,q)$ satisfies some nice regularity properties in the q–variable, in particular a convexity–type property along lines. By adapting to the case at issue the reparametrization techniques introduced in [17], we then prove that the family of integral functionals associated to integrands of this kind do meet the condition mentioned above, in particular it is closed by Γ–convergence.

1. Introduction

1.1. **Description of the results.** In [2], the authors prove an integral representation result for a class of lower semicontinuous functionals $\mathcal{F}$ defined on $W^{1,p}(I;\mathbb{R}^N)\times\mathcal{I}(I)$ (where $p>1$, I is a bounded open interval and $\mathcal{I}(I)$ denotes the family of all open subintervals of I), satisfying suitable locality and additivity properties, and the following integral growth conditions

$$\lambda\int_a^b|\dot\gamma|^p\,\mathrm{d}s\le\mathcal{F}(\gamma,(a,b))\le\Lambda\int_a^b(1+|\dot\gamma|^p)\,\mathrm{d}s \tag{1}$$

for every $\gamma\in W^{1,p}(I;\mathbb{R}^N)$ and $(a,b)\subset I$, with Λ,λ fixed positive constants. It turns out that

$$\mathcal{F}(\gamma,(a,b))=\int_a^b f(s,\gamma(s),\dot\gamma(s))\,\mathrm{d}s\qquad\text{for every }\gamma\in W^{1,p}(I;\mathbb{R}^N)\text{ and }(a,b)\subset I,$$

with f defined as

$$f(t,x,q):=\limsup_{h\to0^+}\frac{1}{h}\inf_{\gamma\in W^{1,p}(I;\mathbb{R}^N)}\{\mathcal{F}(\gamma,(t,t+h))\ :\ \gamma(t)=x,\ \gamma(t+h)=x+h\,q\,\}.$$

Moreover, $f(t,x,\cdot)$ is proved to be continuous on $\mathbb{R}^N$, and convex for almost every $(t,x)\in I\times\mathbb{R}^N$, in agreement with previous results for similar functionals (cf. [16, 19]). As a consequence of their analysis, the authors provide an integral representation for the Γ–limit of a sequence of functionals defined as

$$\mathcal{F}_k(\gamma,(a,b)):=\int_a^b f_k(s,\gamma(s),\dot\gamma(s))\,\mathrm{d}s\qquad\text{for every }\gamma\in W^{1,p}(I;\mathbb{R}^N)\text{ and }(a,b)\subset I,$$

where $f_k:I\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ are Borel functions such that

$$\lambda\,|q|^p\le f_k(t,x,q)\le\Lambda\,(1+|q|^p)\qquad\text{for every }(t,x,q)\in I\times\mathbb{R}^N\times\mathbb{R}^N. \tag{2}$$

In particular, by taking $f_k = f$ for every $k \in \mathbb{N}$, one derives an integral representation result for the lower semicontinuous envelope of the map

$$W^{1,p}(I;\mathbb{R}^N) \ni \gamma \mapsto \int_a^b f(s,\gamma,\dot\gamma)\,\mathrm{d}s,$$

where $f(t,x,q)$ is a Borel function without continuity or convexity assumptions.

In this paper we extend the results of [2] to a wider class of abstract functionals satisfying, in place of (1), the following more general growth conditions:

$$\int_a^b \alpha(|\dot\gamma|)\,\mathrm{d}s \le \mathcal{F}(\gamma,(a,b)) \le \int_a^b \beta(|\dot\gamma|)\,\mathrm{d}s \tag{3}$$

for every $\gamma \in W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^N)$ and $(a,b) \subset \mathbb{R}$, where α, β are two superlinear functions from $[0,+\infty)$ to $\mathbb{R}$, which can be taken non–decreasing and convex as well without any loss of generality. On the other hand, we will be only concerned with functionals of autonomous type, i.e. such that

$$\mathcal{F}\big(\gamma(\cdot),\, h+(a,b)\big) = \mathcal{F}\big(\gamma(\cdot+h),\, (a,b)\big)$$

for every $\gamma \in W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^N)$, $(a,b) \subset \mathbb{R}$ and $h \in \mathbb{R}$.

Our main motivation stems from the interest to treat integral functionals of the kind

$$\mathbb{L}^t(\gamma) := \int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s, \qquad \gamma \in W^{1,1}\big((0,t),\mathbb{R}^N\big),\ t>0, \tag{4}$$

in presence of a Borel–measurable Lagrangian $L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, convex, superlinear and locally bounded in q, uniformly with respect to x. Such growth condition can be equivalently restated (cf. Lemma 2.3 in [17]) by saying that

$$\alpha(|q|) \le L(x,q) \le \beta(|q|) \qquad \text{for every } (x,q) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{5}$$

for a suitable pair of superlinear functions $\alpha, \beta : [0,+\infty) \to \mathbb{R}$, which, in general, do not grow as h^p when $h \to +\infty$, and might have different growths as well. The model examples of Lagrangians included in this class are the one of the form

$$L(x,q) = V(q) + n(x)$$

with $V(\cdot)$ convex and superlinear, and $n(\cdot)$ Borel–measurable and bounded. Through our analysis we are able to treat sequences of integral functionals of the form (4) associated to Borel–measurable Lagrangians $L_k(x,q)$, convex in q, and satisfying (5) for a fixed pair of superlinear functions α, β, and to prove that their Γ–limits admit an integral representation of the same form.([1])

In particular, we get that the lower semicontinuous envelope of (4) admits an integral representation for some Borel–measurable $\overline{L} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ (cf. [6, 9, 10, 23, 24]). This result turns out to be very useful when one is interested in proving the local Lipschitz–continuity in $(0,+\infty) \times \mathbb{R}^N$ of the *value function*

$$v(t,x) := \inf\left\{ u(\gamma(0)) + \int_0^t L(\gamma(s),\dot\gamma(s))\,\mathrm{d}s \ :\ \gamma \in W^{1,1}\big([0,t],\mathbb{R}^N\big),\ \gamma(t)=x \right\}$$

associated with a possibly discontinuous *initial cost* $u : \mathbb{R}^N \to [0,+\infty]$, and with a discontinuous Lagrangian of the kind considered above (cf. [15, 17]). According to what proved in [15], v has the desired regularity if the infimum above is attained for

[1]Throughout the paper, $W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^N)$ will be regarded as a metric space, endowed with the topology of uniform convergence on compact subsets of $\mathbb{R}$.

every $(t,x) \in (0,+\infty) \times \mathbb{R}^N$. Yet, this need not be true in this case, no matter how regular u is. In fact, the lack of continuity of L does not guarantee that the associated action functional $\mathbb{L}^t$ is lower semicontinuous on $W^{1,1}\big((0,t),\mathbb{R}^N\big)$ with respect to the suitable convergence which assures the existence of minimizers, which, in this case, turns out to be the uniform convergence (cf. Proposition 2.3). By classical results of Olech [25] and Ioffe [22], the latter is in fact assured if the Lagrangian is lower semicontinuous in x and convex in q. This difficulty can be overcome by looking for a relaxed formulation of the problem, consisting in replacing the functional $\mathbb{L}^t$ in the minimization formula above with its lower semicontinuous envelope $\overline{\mathbb{L}}^t$, but to conclude one needs to know that the latter admits an integral representation.

We now describe in a more detailed way the content of the paper. We will also underline the main technical difficulties and differences with respect to [2], which have made the extension of their results to the present setting non trivial at all.

We start by introducing, in Section 2, a family $\mathcal{A}$ of abstract functionals of autonomous type sharing the main properties enjoyed by (4) (see (F1)–(F4)). Following [2], we geodesically associate with each $\mathcal{F} \in \mathcal{A}$ a sort of distance–function $d_{\mathcal{F}}$ on $(0,+\infty) \times \mathbb{R}^N$ (see (8)), and a lenght–type functional $\mathcal{L}$ (see (10) with $d_{\mathcal{F}}$ in place of d).

When α and β have p–growth, we know by [2] that $\mathcal{L}$ is indeed the lower semicontinuous envelope of $\mathcal{F}$ and does admit an integral representation. A crucial step in the proof is showing that $d_{\mathcal{F}}\big((0,\cdot),(t,\cdot)\big)$ is continuous on $\mathbb{R}^N \times \mathbb{R}^N$, for every $t > 0$. This fact basically relies on the possibility to derive suitable *a priori* L^p–estimates on the derivatives of quasi–optimal curves. The general assumptions made here on α and β do not allow us to use the same arguments, the main obstruction being given by the fact that α and β do not have, in general, the same kind of growth at infinity.

To face these difficulties, we resort to a metric–type approach. In Section 3.1 we consider a distance function d on $(\mathbb{R} \times \mathbb{R}^N) \times (\mathbb{R} \times \mathbb{R}^N)$ satisfying the same properties as $d_{\mathcal{F}}$, and we introduce a condition on d, namely (*), under which the associated length–type functional $\mathcal{L}$ admits an integral representation. Moreover we show that the associated integrand is obtained by "differentiating" the function d, and satisfies some nice regularity properties in the q–variable, such as continuity for every $x \in \mathbb{R}^N$, and convexity for almost every $x \in \mathbb{R}^N$. An interesting aspect of this study is having singled out a condition on the integrand (cf. Theorem 3.10-(v) and (L3) in Section 4), weaker than convexity in q, which is necessary in order to have lower semicontinuity of the associated action functional. This instance is specific of the autonomous case.

In Section 3.2 we show that, when $\mathcal{F} \in \mathcal{A}$ is lower semicontinuous, condition (*) holding for $d = d_{\mathcal{F}}$ is necessary and sufficient in order for $\mathcal{F}$ to admit an integral representation (see Theorem 3.8). When $\mathcal{F}$ is any element of $\mathcal{A}$, such condition turns out to be sufficient to get an integral representation result for its lower semicontinuous envelope (cf. Theorem 3.10), and necessary as well when α and β have the same kind of growth at infinity (cf. Remark 3.11).

In Section 4 we proceed to show that the function d associated to a functional of the form (4) does satisfy condition (*) whenever L is superlinear and locally bounded in q, uniformly with respect to x, and such to satisfy the convexity–type assumption (L3). This is accomplished in Section 4, see Theorem 4.2. The consequent extension to the case of Lagrangians locally bounded in (x,q) is easily derived via a localization argument (see Theorem 4.18). Most of the section is devoted to derive suitable *a priori* estimates on the Lipschitz constants of quasi–minimizers, which is the crucial step for the proof of Theorem 4.2. We note that such constants only depend on the kind of growth conditions assumed on L. In view of the results of Section 3, this allows us to recover compactness when dealing with sequences of locally equi–bounded discontinuous Lagrangians (cf. Theorem 4.19), thus showing that the family of integrands herein considered is stable with respect to Γ–convergence of the associated action functionals.

We end this introduction by briefly describing the main ideas exploited in the proofs of Section 4. We recall that we are interested in proving that the function defined through

$$d\big((0,y),(t,x)\big) := \inf\left\{\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s \,:\, \gamma\in W^{1,1}\big((0,t),\mathbb{R}^N\big),\ \gamma(0^+)=y,\ \gamma(t^-)=x\right\}$$

for every $(y,t,x)\in\mathbb{R}^N\times(0,+\infty)\times\mathbb{R}^N$ satisfies condition (*) when L is a discontinuous Lagrangian enjoying the properties mentioned above. This issue is strictly related to the study of the regularity of Lagrangian minimizers, when they exist (cf. [1, 3, 7, 9, 10, 13, 23, 24, 28, 29]). The desired property for d can be in fact recovered via a fairly easy argument (cf. proof of Theorem 4.4 in [15], or proof of Theorem 3.8). The same argument however works as soon as we provide some *a priori* estimates on the Lipschitz constants of quasi–minimizers for $d\big((0,y),(t,x)\big)$, with some uniformity with respect to $(y,t,x)\in\mathbb{R}^N\times(0,+\infty)\times\mathbb{R}^N$.

To this aim, the idea we have followed is that, in the above minimization formula, one needs not consider all possible absolutely continuous curves connecting y to x in time t; we can restrict to consider only those having an optimal parametrization, where optimal means that we are interested in making the action of L as small as possible. This has lead us to first consider a minimization problem with fixed support: we fix a Lipschitz curve $\gamma:(0,\ell)\to\mathbb{R}^N$ parametrized by arc–length, *the support*, and we try to solve the following problem

$$\min\left\{\int_0^t L(\xi,\dot\xi)\,\mathrm{d}s \,:\, \xi\in[\gamma]_t\right\} \tag{6}$$

for every $t>0$, where $[\gamma]_t$ denotes the family of absolutely continuous curves $\xi:(0,t)\to\mathbb{R}^N$ obtained through a reparametrization of γ (see Definition 4.8). Then we introduce the notion of *a–Lagrangian parametrization* on curves (cf. Definition 4.11). When L is convex in q, it reduces to requiring that the curve satisfies the DuBois–Raymond necessary condition for optimality, i.e.

$$L(\gamma(s),\dot\gamma(s)) = \langle\dot\gamma(s),p\rangle - a \qquad \text{for every } p\in\partial_q L(\gamma(s),\dot\gamma(s))$$

for some constant $a\in\mathbb{R}$ and for almost every $s\in(0,t)$. ([2]) Then we consider the multifunction $T_\gamma(\cdot)$ defined on $\mathbb{R}$ by

$$T_\gamma(a) := \{t>0 \,:\, [\gamma]^\flat(a,t) \text{ is non empty}\} \qquad \text{for every } a\in\mathbb{R},$$

where $[\gamma]^\flat(a,t)$ denotes the subset of $[\gamma]_t$ consisting of a–Lagrangian bi–Lipschitz reparametrizations of γ, and we remark that the relation $t\in T_\gamma(a)$ implies that problem (6) admits a solution in $[\gamma]^\flat(a,t)$. Our attention is then addressed to establish the main properties of the multifunction $T_\gamma(\cdot)$, with particular interest to its

[2] Here $\partial_q L(x,q)$ denotes the subdifferential of the function $L(x,\cdot)$ at q.

range $\bigcup_{a\in\mathbb{R}} T_\gamma(a)$ (see Proposition 4.12). When this coincides with $(0,+\infty)$, we conclude that problem (6) is solvable for every $t>0$. In particular, (6) has a minimizer belonging to $[\gamma]^\flat(a,t)$ for some $a\in\mathbb{R}$, and its Lipschitz constant can be estimated by some $\kappa_a\in\mathbb{R}$ depending on a and on the kind of growth conditions assumed on L only. However, our analysis reveals that the range of $T_\gamma(\cdot)$ may actually be a bounded interval of the form $(0,T)$. In this instance, a solution to (6) exists if $t\le T$. For $t>T$, the minimum in (6) is only an infimum, in general; nevertheless, we prove that this value can be obtained by minimizing the action over the family of κ_{c_γ}–Lipschitzian reparametrizations of γ, where κ_{c_γ} is a positive constant that can be estimated in terms of the growth conditions assumed on L (see Theorem 4.14).

This information is used to get the sought *a priori* estimates on the Lipschitz constants of quasi–minimizers (see Lemma 4.16): since any absolutely continuous curve from $(0,t)$ to $\mathbb{R}^N$ belongs to $[\gamma]_t$ for a suitable choice of the Lipschitz curve $\gamma:(0,\ell)\to\mathbb{R}^N$ (cf. Lemma 4.10), a quasi–minimizer for $d\big((0,y),(t,x)\big)$ can be always assumed to be κ_a–Lipschitz continuous, for some $a\in\mathbb{R}$. By using the superlinearity of $L(x,\cdot)$, the constant a is last estimated with some uniformity with respect to (y,t,x).

The analysis outlined above relies on suitable reparametrization techniques which use in an essential way the convexity assumption (L3). The argument on which they are based was originally introduced in [20], then developed in [18] for a continuous and convex Lagrangian, and subsequently extended to the measurable case in [17]. In the present paper, we have replaced the convexity assumption of L in q with the weaker condition (L3). This gives rise to some technical difficulties (namely, the convex envelope of $L(x,\cdot)$ does not agree with $L(x,\cdot)$ any longer, cf Remark 3.18 in [17]), but the underlying idea, as well as many proofs, is the same.

2. Notation and standing assumptions

We write below a list of symbols used throughout this paper.

N	an integer number
$B_r(x)$	the open ball in $\mathbb{R}^N$ of radius r centered at x
B_r	the open ball in $\mathbb{R}^N$ of radius r centered at 0
$\langle\cdot,\cdot\rangle$	the scalar product in $\mathbb{R}^N$
$[u]$	the integer part of $u\in\mathbb{R}$
$\mathbb{R}_+$	the set of nonnegative real numbers
$\mathcal{P}(\mathbb{R}_+)$	the space of all subsets of $\mathbb{R}_+$

Given a subset U of $\mathbb{R}^k$, we denote by $\overline{U}$ its closure. If E is a Lebesgue measurable subset of $\mathbb{R}^k$, we denote by $|E|$ its k–dimensional Lebesgue measure, and we say that E is *negligible* whenever $|E|=0$. The characteristic function of E is denoted by χ_E. We say that a property holds *almost everywhere* (*a.e.* for short) on $\mathbb{R}^k$ if it holds up to a negligible subset of $\mathbb{R}^k$. The Euclidean norm of $u\in\mathbb{R}^k$ is denoted by $|u|$.

Given a measurable vector–valued function $f:E\to\mathbb{R}^m$, we write $\|f\|_\infty$ to mean $\left(\sum_{i=1}^k\|f_i\|_{L^\infty(E)}\right)^{1/2}$, where f_i and $\|f_i\|_{L^\infty(E)}$ denote the i-th component of f and the L^∞-norm of f_i, respectively. We will say that f is *transversal to* $S\subset\mathbb{R}^m$ if $|\{x\in E\,:\,f(x)\in S\}|=0$. The notation $⨍_E f\,\mathrm{d}x$ and $⨍_a^b f\,\mathrm{d}s$ stands for $\frac{1}{|E|}\int_E f\,\mathrm{d}x$ and $\frac{1}{b-a}\int_a^b f\,\mathrm{d}s$, respectively. Notice that $⨍_a^b f\,\mathrm{d}s=⨍_b^a f\,\mathrm{d}s$ for any $a\neq b$.

Let $X \subseteq \mathbb{R}^k$ and $\mathcal{B}(X)$ the family of all Borel subsets of X. A multifunction Γ from X to compact subsets of $\mathbb{R}$ is said to be *Borel–measurable* (cf. [8]) if

$$\{x \in X \,:\, \Gamma(x) \cap U \neq \emptyset\} \in \mathcal{B} \qquad \text{for every open set } U \subseteq \mathbb{R}.$$

We say that Γ is *upper semicontinuous* at x if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\Gamma(z) \subseteq \Gamma(x) + (-\varepsilon, \varepsilon) \qquad \text{for all } z \in B_\delta(x) \cap X.$$

When $k = 1$, we say that Γ is *non–decreasing on* X if

$$\sup \Gamma(x) \leq \inf \Gamma(y) \qquad \text{for every } x, y \in X \text{ with } x < y.$$

We say that Γ is *non–increasing on* X if the multifunction $-\Gamma(\cdot)$ is non–decreasing on X.

For a function $g : \mathbb{R}^k \to (-\infty, +\infty]$, we denote by $\mathrm{dom}(g)$ its effective domain; i.e., the subset of $\mathbb{R}^k$ where g is finite valued. We will say that g is superlinear if

$$\lim_{|x| \to +\infty} \frac{g(x)}{|x|} = +\infty.$$

For a convex function f from $\mathbb{R}^k$ to $\mathbb{R}$, we will denote by $\partial f(x)$ the *subdifferential* of f at x, defined as

$$\partial f(x) := \{p \in \mathbb{R}^k \,:\, f(y) \geq f(x) + \langle p, y - x\rangle \quad \text{for every } y \in \mathbb{R}^k\}.$$

The set $\partial f(x)$ is closed and convex. We furthermore have (see [26]):

Proposition 2.1. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be convex. Then f is locally Lipschitz in $\mathbb{R}^k$. More precisely, for every $x_0 \in \mathbb{R}^k$ and $r, \delta > 0$, we have*

$$|f(x) - f(y)| \leq |x - y| \, \frac{2}{\delta} \sup_{B_{r+\delta}(x_0)} f \qquad \text{for every } x, y \in B_r(x_0).$$

In particular, $\partial f(x) \subset (2 \sup_{B_{r+1}} f)\, \overline{B}_1$ for every $x \in B_r$.

Given a function $f : \mathbb{R}^k \to \mathbb{R}$, we define its conjugate $f^* : \mathbb{R}^k \to (-\infty, +\infty]$ as follows:

$$f^*(x) := \sup_{y \in \mathbb{R}^k} \{\langle x, y\rangle - f(y)\} \qquad \text{for every } x \in \mathbb{R}^k.$$

We record for later use the following well known facts (cf. [26, Theorem 23.5])

Proposition 2.2. *Let $f : \mathbb{R}^k \to \mathbb{R}$ be superlinear. Then f^* is convex and locally bounded on $\mathbb{R}^k$. Moreover*

$$f(x) \geq f^{**}(x) := \sup_{y \in \mathbb{R}^k} \{\langle x, y\rangle - f^*(y)\} \qquad \text{for every } x \in \mathbb{R}^k,$$

with equality holding for every x if and only if f is convex. When f is convex, the following conditions on $x, x^* \in \mathbb{R}^k$ are equivalent to each other:

(i) $f(x) + f^*(x^*) \leq \langle x, x^*\rangle$;

(ii) $f(x) + f^*(x^*) = \langle x, x^*\rangle$;

(iii) $x^* \in \partial f(x)$;

(iv) $x \in \partial f^*(x^*)$.

We denote by $W^{1,1}((a,b),\mathbb{R}^N)$ the space of absolutely continuous curves from the interval (a,b) to $\mathbb{R}^N$, while $W^{1,1}_{loc}(\mathbb{R},\mathbb{R}^N)$ denotes the space of locally absolutely continuous curves. We endow $W^{1,1}_{loc}(\mathbb{R},\mathbb{R}^N)$ with the metrizable topology of uniform convergence on compact subsets of $\mathbb{R}$. We recall that a curve $\gamma:(a,b)\to\mathbb{R}^N$ is said to be *parametrized by arc–length* if $|\dot\gamma(s)|=1$ for almost every $s\in(a,b)$.

Throughout the paper, α,β will always denote two functions from $\mathbb{R}_+$ to $\mathbb{R}_+$ that are convex, non–decreasing and superlinear, namely

$$\lim_{h\to+\infty}\frac{\alpha(h)}{h}=\lim_{h\to+\infty}\frac{\beta(h)}{h}=+\infty.$$

The following result is a consequence of Dunford–Pettis Theorem (cf. Theorems 2.11 and 2.12 in [7]).

Proposition 2.3. *Let I a bounded interval of $\mathbb{R}$ and $(\gamma_n)_n$ a sequence in $W^{1,1}(I,\mathbb{R}^N)$ such that*

$$\sup_n\int_I\alpha(|\dot\gamma_n|)\,\mathrm{d}s<+\infty. \tag{7}$$

Then there exists a subsequence $(\gamma_{n_k})_k$ and a curve $\gamma\in W^{1,1}(I,\mathbb{R}^N)$ such that

$$\lim_{k\to\infty}\|\gamma_{n_k}-\gamma\|_\infty=0.$$

Remark 2.4. We point out that condition (7) implies that $(\gamma_{n_k})_k$ actually converges to γ weakly in $W^{1,1}(I,\mathbb{R}^N)$.

Let $\mathcal{I}(\mathbb{R})$ be the collection of bounded, open intervals of $\mathbb{R}$. We denote by $\mathcal{A}=\mathcal{A}(\alpha,\beta)$ the class of all abstract functionals $\mathcal{F}:W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^N)\times\mathcal{I}(\mathbb{R})\to[0,+\infty]$ satisfying the following properties:

$$\mathcal{F}_a^b(\gamma)=\mathcal{F}_a^b(\xi)\qquad\text{if }\gamma,\xi\in W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^N)\text{ and }\gamma=\xi\text{ a.e. on }(a,b), \tag{F1}$$

$$\mathcal{F}_{a+h}^{b+h}\big(\gamma(\cdot)\big)=\mathcal{F}_a^b\big(\gamma(\cdot+h)\big)\qquad\text{for any }a,\,b,\,h\in\mathbb{R}\text{ and }\gamma\in W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^N), \tag{F2}$$

$$\mathcal{F}_a^c(\gamma)=\mathcal{F}_a^b(\gamma)+\mathcal{F}_b^c(\gamma)\qquad\text{for any }a<b<c\text{ and }\gamma\in W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^N), \tag{F3}$$

$$\int_a^b\alpha(|\dot\gamma|)\,\mathrm{d}t\le\mathcal{F}_a^b(\gamma)\le\int_a^b\beta(|\dot\gamma|)\,\mathrm{d}t\quad\text{for any }a,\,b\in\mathbb{R},\ \gamma\in W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^N). \tag{F4}$$

Here the notation $\mathcal{F}_a^b(\gamma)$ stands for $\mathcal{F}(\gamma,(a,b))$. When $a=0$, $\mathcal{F}_a^b$ will be more simply denoted by $\mathcal{F}^b$. Knowing the latter for any $b>0$ is sufficient to identify the functional $\mathcal{F}$, in view of property (F2).

We will say that $\mathcal{F}\in\mathcal{A}$ is lower semicontinuous on $W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^N)$ if $\mathcal{F}_a^b$ is lower semicontinuous on $W^{1,1}\big((a,b),\mathbb{R}^N\big)$ with respect to the uniform convergence, for any $(a,b)\in\mathcal{I}(\mathbb{R})$.

Given $\mathcal{F}\in\mathcal{A}$, we will denote by $\overline{\mathcal{F}}$ its *lower semicontinuous envelope* with respect to the local uniform convergence on $W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^N)$, namely the functional defined as follows:

$$\overline{\mathcal{F}}_a^b(\gamma):=\inf\left\{\liminf_n\mathcal{F}_a^b(\gamma_n)\,:\,(\gamma_n)_n\subset W^{1,1}\big((a,b),\mathbb{R}^N\big),\ \lim_n\|\gamma_n-\gamma\|_\infty=0\right\}$$

for any $a < b$. As well known, $\overline{\mathcal{F}}$ is the greatest among all lower semicontinuous functionals which are less or equal than $\mathcal{F}$ on $W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^N) \times \mathcal{I}(\mathbb{R})$ (see [14]). Note that $\overline{\mathcal{F}}$ still enjoys hypotheses (F1)–(F4).

Choose $\mathcal{F}$ in $\mathcal{A}$, and let $s, t \in \mathbb{R}$ and $x, y \in \mathbb{R}^N$. We define

$$d_{\mathcal{F}}\big((s,x),(t,y)\big) := \inf\left\{\mathcal{F}_s^t(\gamma) \,:\, \gamma \in W^{1,1}\big((s,t),\mathbb{R}^N\big),\ \gamma(s^+) = x,\ \gamma(t^-) = y\right\} \quad (8)$$

for $s < t$, while we agree that $d_{\mathcal{F}}((s,x),(s,x)) = 0$, and $d_{\mathcal{F}}((s,x),(t,y)) = +\infty$ when either $s > t$ or $s = t$ and $x \neq y$. As a simple consequence of the definitions, we derive the following facts.

Proposition 2.5. *Let $\mathcal{F} \in \mathcal{A}$ and $d = d_{\mathcal{F}}$. For any $x, y, \zeta \in \mathbb{R}^N$ and $s, t, \tau \in \mathbb{R}$, the following properties hold:*

(d1) $\quad d\big((0,x),(0,x)\big) = 0$ *and* $d\big((0,x),(t,x)\big) = +\infty$ *when* $t < 0$*;*

(d2) $\quad d\big((s,x),(t,y)\big) = d\big((0,x),(t-s,y)\big)$*;*

(d3) $\quad d\big((s,x),(t,y)\big) \leq d\big((s,x),(\tau,\zeta)\big) + d\big((\tau,\zeta),(t,y)\big)$*;*

(d4) $\quad t\,\alpha\left(\dfrac{|y-x|}{t}\right) \leq d\big((0,x),(t,y)\big) \leq t\,\beta\left(\dfrac{|y-x|}{t}\right)$ *when* $t > 0$*.*

It is apparent from the definition that

$$d_{\overline{\mathcal{F}}}\big((0,x),(t,y)\big) = \inf\left\{\liminf_n d_{\mathcal{F}}\big((0,x_n),(t,y_n)\big) \,:\, x_n \to x,\ y_n \to y\right\} \quad (9)$$

for every $x, y \in \mathbb{R}^N$ and $t > 0$. Otherwise stated, for any fixed $t > 0$ the function $d_{\overline{\mathcal{F}}}\big((0,\cdot),(t,\cdot)\big)$ is the lower semicontinuous envelope of $d_{\mathcal{F}}\big((0,\cdot),(t,\cdot)\big)$. In particular,

$$d_{\overline{\mathcal{F}}} \leq d_{\mathcal{F}} \qquad \text{on } (\mathbb{R}\times\mathbb{R}^N)\times(\mathbb{R}\times\mathbb{R}^N).$$

Remark 2.6. It is not clear whether the inequality in the above expression can be actually strict. Indeed, it is not hard to show that $d_{\mathcal{F}}\big((0,\cdot),(t,\cdot)\big)$ is upper semicontinuous on $\mathbb{R}^N \times \mathbb{R}^N$ for every fixed $t > 0$, but there is no evidence why it should be lower semicontinuous. We only know that this is true when α and β have the same kind of growth at infinity. To see this, it is enough to argue as in the proof of Theorem 3.2 in [2] by using, in place of Lemma 3.1, the fact that

$$\lim_{\eta\to 0^+} \alpha\left(\frac{|\gamma(s+\eta)-\gamma(s)|}{\eta}\right)\eta = 0 \qquad \text{for every } s \in (0,t),$$

where γ is any curve in $W^{1,1}\big((0,t),\mathbb{R}^N\big)$ such that $\displaystyle\int_0^t \alpha(|\dot\gamma|)\,\mathrm{d}s < +\infty$.

To any function d from $(\mathbb{R}\times\mathbb{R}^N)\times(\mathbb{R}\times\mathbb{R}^N)$ to $[0,+\infty]$ satisfying the statement of Proposition 2.5 we associate a functional $\mathcal{L} : W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^N)\times\mathcal{I}(\mathbb{R}) \to [0,+\infty]$ defined as follows: for any $a < b$ and $\gamma \in W^{1,1}\big((a,b),\mathbb{R}^N\big)$, we set

$$\mathcal{L}_a^b(\gamma) := \sup\left\{\sum_i d(t_i,\gamma(t_i),t_{i+1},\gamma(t_{i+1})) \,:\, a = t_0 < \cdots < t_n = b,\ n \in \mathbb{N}\right\}, \quad (10)$$

where the supremum is taken over all possible finite partitions of (a,b). The functional $\mathcal{L}$ satisfies hypotheses (F1)–(F4), as can be easily checked. According to our previous notation, in the sequel we will write $\mathcal{L}^b$ in place of $\mathcal{L}_0^b$.

3. Abstract functionals of autonomous type

3.1. **A metric–type analysis.** Let d be a function from $(\mathbb{R} \times \mathbb{R}^N) \times (\mathbb{R} \times \mathbb{R}^N)$ to $[0, +\infty]$ such to satisfy the statement of Proposition 2.5, and let $\mathcal{L}$ be the functional associated to d through (10). The purpose of this section is to show that the functional $\mathcal{L}$ admits an integral representation, provided some additional conditions are assumed on d.

We start by recording a result that will be required later in this section (cf. proof of Theorem 3.5). We have stated it at this point to emphasize its independence from any additional hypothesis we will introduce on d. The proof is omitted, for it may be easily recovered from the one provided in [2, Lemma 4.2] for the case $\beta(h) = \Lambda\,(1 + h^p)$ with $\Lambda \in \mathbb{R}_+$ and $p > 1$.

Lemma 3.1. *Let x, v_1, $v_2 \in \mathbb{R}^N$ and $\delta > 0$. For $i = 1, 2$, set*

$$\ell_i^- := \liminf_{h\to 0^+} \frac{d\big((0,x),(h, x+\xi_i(h))\big)}{h}, \qquad \ell_i^+ := \limsup_{h\to 0^+} \frac{d\big((0,x),(h, x+\xi_i(h))\big)}{h},$$

where $\xi_i : [0,\delta] \to \mathbb{R}^N$ is a function such that

$$\lim_{h\to 0^+} \frac{\xi_i(h)}{h} = v_i.$$

For any $0 < c_1 < c_2$ we have

$$\begin{aligned} c_2\ell_2^- - c_1\ell_1^- &\le (c_2 - c_1)\,\beta\left(\frac{|c_2v_2 - c_1v_1|}{c_2 - c_1}\right), \\ c_2\ell_2^+ - c_1\ell_1^+ &\le (c_2 - c_1)\,\beta\left(\frac{|c_2v_2 - c_1v_1|}{c_2 - c_1}\right). \end{aligned}$$

In particular, $\ell_1^- = \ell_2^-$ and $\ell_1^+ = \ell_2^+$ whenever $v_1 = v_2$.

We now add some hypotheses on d. Throughout the section, we assume that the following condition holds:

$$d\big((0,\cdot),(\cdot,\cdot)\big) \qquad \text{is locally Lipschitz continuous on } \mathbb{R}^N \times (0,+\infty) \times \mathbb{R}^N. \tag{*}$$

We first prove that the $\mathcal{L}^t$–length of any curve $\gamma \in W^{1,1}\big((0,t),\mathbb{R}^N\big)$ admits an integral representation in terms of its metric derivative.

Definition 3.2 (Metric derivative)**.** Given a curve $\gamma \in W^{1,1}\big((a,b),\mathbb{R}^N\big)$, we define the *metric derivative* $|\dot\gamma|_d(s)$ of γ at the point $s \in (a,b)$ as

$$|\dot\gamma|_d(s) := \limsup_{h\to 0^+} \frac{d\big((s,\gamma(s)),(s+h,\gamma(s+h))\big)}{h}. \tag{11}$$

Theorem 3.3. *Let condition (*) hold. Then, for any curve $\gamma \in W^{1,1}\big((0,t),\mathbb{R}^N\big)$,*

$$\mathcal{L}^t(\gamma) = \int_0^t |\dot\gamma|_d(s)\,\mathrm{d}s.$$

Moreover, the lim sup *at the right-hand side of* (11) *is actually a* limit *for a.e. $s \in (0,t)$ whenever $\mathcal{L}^t(\gamma) < +\infty$.*

Remark 3.4. The definition of $\mathcal{L}^t$ and the notion of metric derivative, as well as the integral representation formula provided by Theorem 3.3, are well known in classical metric spaces. In some sense, in fact, the function d can be regarded as a degenerate non–symmetric distance on the product space $\mathbb{R} \times \mathbb{R}^N$. The proof below is actually inspired to the one proposed in [4, Theorem 4.1.1].

Proof. Let us set $J := (0,t)$ and let $(t_n)_n$ be a dense sequence in J made up by differentiability points of γ. For each $n \in \mathbb{N}$, we define

$$\varphi_n(s) := \begin{cases} d\big((t_n, \gamma(t_n)), (s, \gamma(s))\,\big) & \text{if } s \in (t_n, t) \\ 0 & \text{if } s \in (0, t_n]. \end{cases}$$

From (*) we deduce that φ_n is absolutely continuous in $J \setminus \{t_n\}$, as a composition of a locally Lipschitz function with an absolutely continuous curve, hence its derivative $\dot\varphi_n(s)$ exists at almost every point $s \in J$. Let us define

$$m(s) := \sup_n \dot\varphi_n(s) \qquad \text{for a.e. } s \in J.$$

We start by proving that

$$\mathcal{L}^t(\gamma) = \int_0^t m(\varsigma)\, \mathrm{d}\varsigma \tag{12}$$

and

$$m(s) = |\dot\gamma|_d(s) \qquad \text{whenever } s \in J \text{ is Lebesgue point for } m(\cdot). \tag{13}$$

By the definition of φ_n and the triangular inequality we infer that

$$|\dot\gamma|_d(s) = \liminf_{h\to 0^+} \frac{d\big((s,\gamma(s)),(s+h,\gamma(s+h))\big)}{h} \geq \liminf_{h\to 0^+} \frac{\varphi_n(s+h) - \varphi_n(s)}{h} = \dot\varphi_n(s)$$

for a.e. $s \in J$, hence, taking the sup over $n \in \mathbb{N}$,

$$\liminf_{h\to 0^+} \frac{d\big((s,\gamma(s)),(s+h,\gamma(s+h))\big)}{h} \geq m(s) \qquad \text{for a.e. } s \in J. \tag{14}$$

On the other hand, by the properties enjoyed by d we know that

$$0 \leq d\big((t_n,\gamma(t_n)),(s,\gamma(s))\,\big) \leq (s - t_n)\,\beta\left(\frac{|\gamma(s) - \gamma(t_n)|}{|s - t_n|}\right) \qquad \text{for every } s > t_n,$$

hence $\inf_n d\big((t_n,\gamma(t_n)),(s,\gamma(s))\,\big) = 0$ whenever s is a differentiability point of γ. We derive

$$\begin{aligned} d\big((s,\gamma(s)),(\tau,\gamma(\tau))\big) &\leq \sup_n \Big(\varphi_n(\tau) - \varphi_n(s)\Big) \\ &= \sup_n \int_s^\tau \dot\varphi_n(\varsigma)\, \mathrm{d}\varsigma \leq \int_s^\tau m(\varsigma)\, \mathrm{d}\varsigma \end{aligned} \tag{15}$$

for almost every $s < \tau$. If s is a Lebesgue point for $m(\cdot)$, we obtain

$$\limsup_{h\to 0^+} \frac{d\big((s,\gamma(s)),(s+h,\gamma(s+h))\big)}{h} \leq \limsup_{h\to 0^+} \frac{1}{h}\int_s^{s+h} m(\varsigma)\, \mathrm{d}\varsigma = m(s),$$

and this inequality, combined with (14), gives (13).

We now prove (12). By (15) it follows

$$\sum_{i=0}^{k-1} d\big((t_i,\gamma(t_i)),(t_{i+1},\gamma(t_{i+1}))\big) \leq \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} m(\varsigma)\, \mathrm{d}\varsigma = \int_0^t m(\varsigma)\, \mathrm{d}\varsigma$$

for every choiche $0 = t_0 < t_1 < ... < t_k = t,\ k \in \mathbb{N}$. Taking the sup over all such partitions we obtain

$$\mathcal{L}^t(\gamma) \le \int_0^t m(\varsigma)\,\mathrm{d}\varsigma.$$

In order to prove the opposite inequality, choose $\varepsilon > 0$ and let $h := t/k,\ t_i := ih$, with $k \ge 2$ such that $h \le \varepsilon$. We observe that

$$\begin{aligned}
&\frac{1}{h}\int_0^{t-\varepsilon} d\big((\varsigma,\gamma(\varsigma)),(\varsigma+h,\gamma(\varsigma+h))\big)\,\mathrm{d}\varsigma \\
&\qquad \le \frac{1}{h}\int_0^h \sum_{i=0}^{k-2} d\big((t_i+\varsigma,\gamma(t_i+\varsigma)),(t_{i+1}+\varsigma,\gamma(t_{i+1}+\varsigma))\big)\,\mathrm{d}\varsigma \\
&\qquad \le \frac{1}{h}\int_0^h \mathcal{L}^t(\gamma)\,\mathrm{d}\varsigma = \mathcal{L}^t(\gamma).
\end{aligned}$$

From Fatou's Lemma and (14) we derive

$$\begin{aligned}
\int_0^{t-\varepsilon} m(\varsigma)\,\mathrm{d}\varsigma &\le \int_0^{t-\varepsilon} \liminf_{h\to 0^+} \frac{d\big((\varsigma,\gamma(\varsigma)),(\varsigma+h,\gamma(\varsigma+h))\big)}{h}\,\mathrm{d}\varsigma \\
&\le \liminf_{h\to 0^+} \frac{1}{h}\int_0^{t-\varepsilon} d\big((\varsigma,\gamma(\varsigma)),(\varsigma+h,\gamma(\varsigma+h))\big)\,\mathrm{d}\varsigma \le \mathcal{L}^t(\gamma),
\end{aligned}$$

and we conclude letting $\varepsilon \to 0$ by the Monotone Convergence Theorem.

It is now easy to conclude: when $\mathcal{L}^t(\gamma) = +\infty$, the statement follows by taking (12) and (14) into account; when $\mathcal{L}^t(\gamma) < +\infty$, $m(\cdot)$ is integrable and the statement is a consequence of (12) and (13) since almost every $s \in J$ is a Lebesgue point for $m(\cdot)$. $\square$

We are now ready to prove the main result of this section.

Theorem 3.5. *Assume condition (*) holds. Then $\mathcal{L}$ is lower semicontinuous on $W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^N)$ with respect to the local uniform convergence, and admits the following integral representation:*

$$\mathcal{L}^t(\gamma) = \int_0^t L^-(\gamma,\dot\gamma)\,\mathrm{d}s = \int_0^t L^+(\gamma,\dot\gamma)\,\mathrm{d}s, \qquad \gamma \in W^{1,1}\big((0,t),\mathbb{R}^N\big) \text{ and } t > 0,$$

where

$$\begin{aligned}
L^-(x,q) &:= \liminf_{h\to 0^+} \frac{d\big((0,x),(h,x+hq)\big)}{h}, \\
L^+(x,q) &:= \limsup_{h\to 0^+} \frac{d\big((0,x),(h,x+hq)\big)}{h}.
\end{aligned}$$

Moreover, the functions $L^\pm$ enjoy the following properties:

(i) $L^\pm : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ *is Borel–measurable;*

(ii) $\alpha(|q|) \le L^\pm(x,q) \le \beta(|q|)$ *for every* $(x,q) \in \mathbb{R}^N \times \mathbb{R}^N$*;*

(iii) $L^\pm(x,\cdot)$ *is continuous on* $\mathbb{R}^N$ *for every* $x \in \mathbb{R}^N$*;*

(iv) $\quad L^{\pm}(x,\cdot) \quad$ *is convex on* $\mathbb{R}^N$ *for a.e.* $x \in \mathbb{R}^N$;

(v) $\quad$ *for every* $t_1 < t_2$ *in* $\mathbb{R}$ *and* $\gamma \in W^{1,\infty}\big((t_1,t_2),\mathbb{R}^N\big)$

$$\lambda \mapsto L^{\pm}(\gamma(s), \lambda\,\dot\gamma(s)) \quad \textit{is convex on } \mathbb{R} \quad \textit{for a.e. } s \in (t_1,t_2).$$

Proof. Assumption (*) implies that the functional

$$\gamma \mapsto \sum_{i=0}^{m-1} d(t_i, \gamma(t_i), t_{i+1}, \gamma(t_{i+1}))$$

is continuous on $W^{1,1}\big((0,t),\mathbb{R}^N\big)$ with respect to the uniform convergence of curves, for any fixed partition $0 = t_0 < t_1 < \dots < t_m = t$, $m \in \mathbb{N}$. As a supremum of a family of continuous functionals, the lower semicontinuity of $\mathcal{L}^t(\cdot)$ follows.

The integral representation formula is a consequence of Theorem 3.3, since Lemma 3.1 assures that

$$L^-(\gamma(s),\dot\gamma(s)) = L^+(\gamma(s),\dot\gamma(s)) = |\dot\gamma|_d(s)$$

whenever $s \in (0,t)$ is a differentiability point of γ, that is almost everywhere.

Items (i) and (ii) are an obvious consequence of the definitions of $L^{\pm}$, together with the fact that $(x,q) \mapsto d\big((0,x),(h,x+hq)\big)/\,h$ is Borel–measurable (in fact, continuous) on $\mathbb{R}^N \times \mathbb{R}^N$ for any $h > 0$.

Items (iii) and (iv) may be proved arguing as in [2] (cf. Theorems 5.1 and 5.2, respectively).

To prove (v), we make use of a "zig–zag" argument. Let us arbitrarily fix a Lipschitz curve $\gamma : (t_1,t_2) \to \mathbb{R}^N$, and choose $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\delta \in (0,1)$ as above. Set

$$\lambda := \delta\lambda_1 + (1-\delta)\lambda_2.$$

We aim to prove that

$$L^{\pm}(\gamma(s),\lambda\,\dot\gamma(s)) \le \delta\, L^{\pm}(\gamma(s),\lambda_1\,\dot\gamma(s)) + (1-\delta)\, L^{\pm}(\gamma(s),\lambda_2\,\dot\gamma(s)) \qquad \text{for a.e. } s \in (t_1,t_2).$$

This is enough to conclude. Indeed, the statement follows by letting λ_1, λ_2 and δ vary over two sets, countable and dense in $\mathbb{R} \setminus \{0\}$ and $(0,1)$, respectively, and by using the continuity of $L^{\pm}(x,\cdot)$ for every $x \in \mathbb{R}^N$.

To prove the claimed inequality, we will actually show that

$$\int_a^b L^{\pm}(\gamma,\lambda\,\dot\gamma)\,\mathrm{d}s \le \delta \int_a^b L^{\pm}(\gamma,\lambda_1\,\dot\gamma)\,\mathrm{d}s + (1-\delta)\int_a^b L^{\pm}(\gamma,\lambda_2\,\dot\gamma)\,\mathrm{d}s \tag{16}$$

for any $[a,b] \subset (t_1,t_2)$, which is equivalent. Let us fix such an interval. Up to replacing the curve $\gamma(\cdot)$ with $\gamma(a+\cdot)$, we can assume $a = 0$. By reversing the orientation of γ if necessary, we can additionally assume $\lambda > 0$. For every fixed $n \in \mathbb{N}$, we define a picewise affine map ψ_n on $[0, b/(\lambda\, n)\,]$ as

$$\psi_n(s) := \begin{cases} \lambda_1\, s & \text{if } s \in [0, \delta\frac{b}{\lambda n}] \\[2ex] \lambda_1\,\delta\frac{b}{\lambda n} + \lambda_2\,(s - \delta\,\frac{b}{\lambda n}) & \text{if } s \in [\delta\frac{b}{\lambda n}, \frac{b}{\lambda n}]. \end{cases}$$

Consider the partition $0 = t_0^n < t_1^n < \dots < t_n^n = b/\lambda$ of the interval $[0,b/\lambda]$ obtained by choosing

$$t_i^n := i\frac{b}{\lambda\, n} \qquad \text{for each } 0 \le i \le n,$$

and define $\varphi_n : [0, b/\lambda] \to \mathbb{R}$ by setting

$$\varphi_n(s) = \lambda\, t_i^n + \psi(s - t_i^n) \qquad \text{if } s \in [t_i^n, t_{i+1}^n], \quad \text{for each } i = 0, \dots, n-1.$$

The sequence $(\varphi_n)_n$ uniformly converges to the function $\varphi(s) := \lambda\, s$ on $[0, b/\lambda]$, hence, for n sufficiently large, the curves $\xi_n := \gamma \circ \varphi_n$ are well defined and uniformly converge to $\xi := \gamma \circ \varphi$ on $[0, b/\lambda]$. By the lower semicontinuity of $\mathcal{L}$ we infer

$$\mathcal{L}^{\,b/\lambda}(\xi) \le \liminf_{n \to +\infty} \mathcal{L}^{\,b/\lambda}(\xi_n).$$

A simple computation shows that this is equivalent to saying that

$$\begin{aligned}&\int_0^b L^{\pm}(\gamma, \lambda\,\dot\gamma)\,\mathrm{d}s \\ &\quad \le \liminf_{n \to +\infty} \sum_{i=0}^{n-1} \frac{b}{n} \left(\delta \fint_{\lambda t_i^n}^{\lambda t_i^n + \frac{\delta \lambda_1}{\lambda} \frac{b}{n}} L^{\pm}(\gamma, \lambda_1\,\dot\gamma)\,\mathrm{d}s + \,(1-\delta) \fint_{\lambda t_i^n + \frac{\delta\lambda_1}{\lambda}\frac{b}{n}}^{\lambda\, t_{i+1}^n} L^{\pm}(\gamma, \lambda_2\,\dot\gamma)\,\mathrm{d}s \right),\end{aligned}$$

and (16) follows thanks to Lemma 3.6 below. $\square$

Lemma 3.6. *Let (a, b) be a bounded interval of $\mathbb{R}$ and $\mu \in (0, 1)$. For each $n \in \mathbb{N}$, let $\{I_i^n : 1 \le i \le n\}$ and $\{J_i^n : 1 \le i \le n\}$ be two collections of pairwise disjoint intervals such that:*

(a) $\displaystyle\bigcup_{i=1}^n I_i^n = (a, b)$ *and* $\displaystyle\max_{1 \le i \le n} |I_i^n| \to 0$ *as $n \to +\infty$;*

(b) $J_i^n \subseteq I_i^n$ *and* $|J_i^n| \ge \mu\, |I_i^n| > 0$ *for every $1 \le i \le n$.*

Then

$$\lim_{n \to +\infty} \sum_{i=1}^n |I_i^n| \fint_{J_i^n} g(s)\,\mathrm{d}s = \int_a^b g(s)\,\mathrm{d}s \qquad \text{for any } \ g \in L^1\big((a, b)\big). \tag{17}$$

Proof. For each $n \in \mathbb{N}$ and $1 \le i \le n$, set $\mu_i^n := \mathcal{L}^1(I_i^n) / \mathcal{L}^1(J_i^n)$ and

$$u_n(x) := \sum_{i=1}^n \mu_i^n\, \chi_{J_i^n}(x), \qquad x \in (a, b).$$

Claim (17) amounts to saying that $u_n \overset{*}{\rightharpoonup} \chi_{(a,b)}$ in $L^\infty\big((a, b)\big)$. It is easy to see that (17) holds whenever g is continuous. As $\sup_n \|u_n\|_\infty \le 1/\mu$, the statement follows by density of the continuous functions into $L^1\big((a, b)\big)$. $\square$

Remark 3.7. We record for later use that the results of this section can be easily generalized to functions d which satisfy, in place of property (d4) of Proposition 2.5, the following condition

$$\text{(d4}'\text{)} \qquad t\,\alpha\left(\frac{|y - x|}{t}\right) \le d\big((0, x), (t, y)\big) \le t\,\beta_n\left(\frac{|y - x|}{t}\right) \quad \forall\, x, y \in B_n,\ \forall\, n \in \mathbb{N},$$

where $(\beta_n)_{n \in \mathbb{N}}$ is a family of convex, non–decreasing and superlinear functions from $\mathbb{R}_+$ to $\mathbb{R}_+$. In particular, the results of Theorem 3.5 still hold, provided claim (ii) is modified as follows:

(ii)$'$ $\qquad \alpha\,(|q|) \le L^{\pm}(x, q) \le \beta_n\,(|q|) \qquad$ *for any $(x, q) \in B_n \times \mathbb{R}^N$ and $n \in \mathbb{N}$.*

3.2. **Integral representation of abstract functionals.** We now proceed to show that condition (*) holding with $d = d_{\mathcal{F}}$ characterizes all lower semicontinuous functionals $\mathcal{F} \in \mathcal{A}$ that admit an integral representation; i.e., such that

$$\mathcal{F}_{t_1}^{t_2}(\gamma) = \int_{t_1}^{t_2} F(\gamma, \dot{\gamma})\,\mathrm{d}s, \qquad \gamma \in W^{1,1}\big((t_1,t_2),\mathbb{R}^N\big) \tag{18}$$

for any $t_1 < t_2$, with $F : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ Borel–measurable. We note that, up to suitably modifying the integrand F on a subset of $\mathbb{R}^N \times \mathbb{R}^N$ which is transversal to $s \mapsto (\gamma(s), \dot{\gamma}(s))$ for any $\gamma \in W^{1,1}_{loc}(\mathbb{R}; \mathbb{R}^N)$, we can always assume that

$$\alpha(|q|) \le F(x,q) \le \beta(|q|) \quad \text{for any } (x,q) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{19}$$

Since $\mathcal{F}$ enjoys (F2), it will be enough to prove (18) for any $t_2 > 0$ and $t_1 = 0$.

Theorem 3.8. *Let $\mathcal{F}$ be a lower semicontinuous abstract functional belonging to $\mathcal{A}$. Then $\mathcal{F}$ admits an integral representation if and only if the function $d = d_{\mathcal{F}}$ associated to $\mathcal{F}$ through* (8) *satisfies condition* (*). *In this instance we have:*

$$\mathcal{F}^{\,t}(\gamma) = \int_0^t F^-(\gamma,\dot{\gamma})\,\mathrm{d}s = \int_0^t F^+(\gamma,\dot{\gamma})\,\mathrm{d}s, \qquad \gamma \in W^{1,1}\big((0,t),\mathbb{R}^N\big) \text{ and } t > 0,$$

where

$$F^-(x,q) \;:=\; \liminf_{h\to 0^+} \frac{d_{\mathcal{F}}\big((0,x),(h,x+hq)\big)}{h},$$

$$F^+(x,q) \;:=\; \limsup_{h\to 0^+} \frac{d_{\mathcal{F}}\big((0,x),(h,x+hq)\big)}{h}\,.$$

Moreover, the functions $F^\pm$ enjoy the following properties:

(i) $F^\pm : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ *is Borel–measurable;*

(ii) $\alpha(|q|) \le F^\pm(x,q) \le \beta(|q|)$ *for every* $(x,q) \in \mathbb{R}^N \times \mathbb{R}^N$*;*

(iii) $F^\pm(x,\cdot)$ *is continuous on* $\mathbb{R}^N$ *for every* $x \in \mathbb{R}^N$*;*

(iv) $F^\pm(x,\cdot)$ *is convex on* $\mathbb{R}^N$ *for a.e.* $x \in \mathbb{R}^N$*;*

(v) *for every* $t_1 < t_2$ *in* $\mathbb{R}$ *and* $\gamma \in W^{1,\infty}\big((t_1,t_2),\mathbb{R}^N\big)$

$$\lambda \mapsto F^\pm(\gamma(s), \lambda\,\dot{\gamma}(s)) \quad \text{is convex on } \mathbb{R} \quad \text{for a.e. } s \in (t_1,t_2).$$

Proof. Let us assume that $\mathcal{F}$ admits an integral representation of the form (18) for some Borel–measurable $F : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ satisfying (19). We want to show that condition (*) holds with $d = d_{\mathcal{F}}$. This basically follows by what proved in [15]. We provide a proof for the reader's convenience.

Fix $x, y \in \mathbb{R}^N$ and $t > 0$. For $0 < r < t/2$, let us denote by U_r the open set $B_r(y) \times (t-r, t+r) \times B_r(x)$. We claim that there exists a constant $K := K(t,\alpha,\beta)$ such that

$$d\big((0,\cdot),(\cdot,\cdot)\big) \qquad \text{is K–Lipschitz continuous in } U_r.$$

Choose (y_1,t_1,x_1) and (y_2,t_2,x_2) in U_r, and set

$$h := |t_1 - t_2| + |x_1 - x_2| + |y_1 - y_2|, \qquad s_0 := \frac{t_1 - t_2}{2} + h.$$

Since U_r is convex, it suffices to prove the statement locally, namely for small values of h. Choose $h < t_2/2$ so that $s_0 < t_1/2$. Let γ_1 be a curve in $W^{1,1}\big((0,t_1),\mathbb{R}^N\big)$ connecting y_1 to x_1 such that

$$d\big((0,y_1),(t_1,x_1)\big) = \int_0^{t_1} F(\gamma_1,\dot\gamma_1)\,\mathrm{d}s,$$

which does exist in force of Proposition 2.3 and of the lower semicontinuity of $\overline{\mathcal{F}}$. Theorem 2.2 in [15] asserts that γ_1 is Lipschitz continuous and $\|\dot\gamma_1\|_\infty \le \kappa$ for some constant κ depending on t, α and β only. Choose $u_1, v_1 \in \mathbb{R}^N$ so that

$$\gamma_1(s_0) = y_2 + hu_1, \qquad \gamma_1(t_1 - s_0) = x_2 + hv_1.$$

Note that $|u_1|, |v_1| < 1 + 2\kappa$. We now define a curve $\gamma_2 : [0,t_2] \to \mathbb{R}^N$ connecting y_2 to x_2 as follows:

$$\gamma_2(s) := \begin{cases} y_2 + su_1 & \text{if } s \in [0,h], \\ \gamma_1(s_0 + s - h) & \text{if } s \in [h, t_2 - h] \\ x_2 + (t_2 - s)v_1 & \text{if } s \in [t_2 - h, t_2] \end{cases}$$

Recalling that F is positive, we get

$$\begin{aligned} d\big((0,y_2),(t_2,x_2)\big) - d\big((0,y_1),(t_1,x_1)\big) &\le \int_0^{t_2} F(\gamma_2,\dot\gamma_2)\,\mathrm{d}s - \int_0^{t_1} F(\gamma_1,\dot\gamma_1)\,\mathrm{d}s \\ &\le \int_0^{h} F(\gamma_2,u_1)\,\mathrm{d}s + \int_{t_2-h}^{t_2} F(\gamma_2,u_2)\,\mathrm{d}s \le 2\,\beta(1+2\kappa)\,h, \end{aligned}$$

so, setting $\widetilde{K} := 2\,\beta(1 + 2\kappa_A)$, we obtain

$$d\big((0,y_2),(t_2,x_2)\big) - d\big((0,y_1),(t_1,x_1)\big) \le \widetilde{K}\,\left(|t_1 - t_2| + |x_1 - x_2| + |y_1 - y_2|\right).$$

The claim follows by interchanging the roles of (y_1,t_1,x_1) and (y_2,t_2,x_2) and by setting $K := \sqrt{2N+1}\,\widetilde{K}$.

Conversely, let us assume that $d_{\mathcal{F}}$ satisfies condition (*). Since $\mathcal{F}$ coincides with its lower semicontinuous envelope $\overline{\mathcal{F}}$, the assertion follows in view of Theorem 3.5 and of Proposition 3.9 below. □

Proposition 3.9. *Let $\mathcal{L}$ be defined via* (10) *with $d = d_{\mathcal{F}}$, and assume $d_{\mathcal{F}}$ satisfies condition* (*). *Then $\mathcal{L}$ is the lower semicontinuous envelope of $\mathcal{F}$, namely*

$$\mathcal{L}^t(\cdot) = \overline{\mathcal{F}}^{\,t}(\cdot) \qquad \textit{on } W^{1,1}\big((0,t),\mathbb{R}^N\big) \tag{20}$$

for every $t > 0$.

Proof. Fix $t > 0$. By Theorem 3.5, we already know that $\mathcal{L}^t(\cdot)$ is lower semicontinuous on $W^{1,1}\big((0,t),\mathbb{R}^N\big)$ with respect to the uniform convergence. This and the inequality $\mathcal{L}^t(\cdot) \le \mathcal{F}^{\,t}(\cdot)$, which is apparent from the definition, implies in particular that

$$\mathcal{L}^t(\cdot) \le \overline{\mathcal{F}}^{\,t}(\cdot) \qquad \text{on } W^{1,1}\big((0,t),\mathbb{R}^N\big).$$

To prove the opposite inequality, we will show that, for any fixed $\gamma \in W^{1,1}\big((0,t),\mathbb{R}^N\big)$, there exists a sequence of curves $(\gamma_k)_k \subset W^{1,1}\big((0,t),\mathbb{R}^N\big)$ with $\gamma_k(0) = \gamma(0)$, $\gamma_k(t) = \gamma(t)$ such that

$$\limsup_{k\to+\infty} \mathcal{F}^{\,t}(\gamma_k) \le \mathcal{L}^t(\gamma) \qquad \text{and} \qquad \lim_{k\to+\infty} \|\gamma_k - \gamma\|_\infty = 0.$$

Of course, we may assume that $\mathcal{L}^t(\gamma) < +\infty$. Fix $k \in \mathbb{N}$ and choose a finite partition $0 = t_0 < t_1 < \cdots < t_m = t$ with $|t_{i+1} - t_i| < 1/k$ for each i, such that

$$\sum_{i=0}^{m-1} d(t_i, \gamma(t_i), t_{i+1}, \gamma(t_{i+1})) < \mathcal{L}^t(\gamma) + \frac{1}{k}.$$

By the definition of d, it is easy to see that there exists a curve $\gamma_k \in W^{1,1}\big((0,t), \mathbb{R}^N\big)$ with $\gamma_k(t_i) = \gamma(t_i)$ for each i such that

$$\mathcal{F}^t(\gamma_k) < \mathcal{L}^t(\gamma) + \frac{1}{k}.$$

Let us show that $\|\gamma_k - \gamma\|_\infty \to 0$ as $k \to \infty$. Let $\alpha_1 > 0$ such that $\alpha(|q|) \geq |q| - \alpha_1$ for any $q \in \mathbb{R}^N$. For each $i = 0, \ldots, m-1$, we have

$$\int_{t_i}^{t_{i+1}} |\dot\gamma_k(s)| \,\mathrm{d}s \leq \mathcal{L}_{t_i}^{t_{i+1}}(\gamma) + \frac{1+\alpha_1}{k} =: \rho_k,$$

so $\gamma_i\left([t_i, t_{i+1}]\right) \subset \gamma\left([t_i, t_{i+1}]\right) + B_{\rho_k}$. The conclusion follows since $\rho_k \to 0$ for $k \to +\infty$ by the absolute continuity of the map $s \mapsto \mathcal{L}^s(\gamma)$. $\square$

In view of what seen so far, we also derive the following

Theorem 3.10. *Let $\mathcal{F} \in \mathcal{A}$ and assume the associated function $d_\mathcal{F}$ satisfies condition (*). Then its lower semicontinuous functional $\overline{\mathcal{F}}$ admits the following integral representation:*

$$\overline{\mathcal{F}}^t(\gamma) = \int_0^t F^-(\gamma, \dot\gamma) \,\mathrm{d}s = \int_0^t F^+(\gamma, \dot\gamma) \,\mathrm{d}s, \qquad \gamma \in W^{1,1}\big((0,t), \mathbb{R}^N\big) \textit{ and } t > 0,$$

where

$$\begin{aligned} F^-(x,q) &:= \liminf_{h \to 0^+} \frac{d_\mathcal{F}\big((0,x),(h, x+hq)\big)}{h}, \\ F^+(x,q) &:= \limsup_{h \to 0^+} \frac{d_\mathcal{F}\big((0,x),(h, x+hq)\big)}{h}. \end{aligned}$$

Moreover, the functions $F^\pm$ enjoy properties (i)–(v) in the statement of Theorem 3.8.

Proof. By assumption, $d\big((0,\cdot),(\cdot,\cdot)\big)$ is locally Lipschitz in $\mathbb{R}^N \times (0,+\infty) \times \mathbb{R}^N$, in particular it is continuous. It follows from the definitions (cf. (9)) that $d_\mathcal{F} = d_{\overline{\mathcal{F}}}$ on $(\mathbb{R} \times \mathbb{R}^N) \times (\mathbb{R} \times \mathbb{R}^N)$, and the statement follows by Theorem 3.8. $\square$

Remark 3.11. When α and β have the same kind of growth at infinity, we know by Remark 2.6 that $d_{\overline{\mathcal{F}}} = d_\mathcal{F}$ on $(\mathbb{R} \times \mathbb{R}^N) \times (\mathbb{R} \times \mathbb{R}^N)$, hence from Theorem 3.8 we derive that condition (*) holding for $d = d_\mathcal{F}$ is necessary as well in order to have an integral representation result for $\overline{\mathcal{F}}$. This might be no longer true in general. In other words, there might exist functionals $\mathcal{F} \in \mathcal{A}$ whose associated functions $d_\mathcal{F}$ are not locally Lipschitz, while $d_{\overline{\mathcal{F}}}$ are.

Remark 3.12. All results of this section can be generalized to functionals $\mathcal{F}$ which satisfy, in place of (F4), the following assumption:

for every $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$

$$\int_a^b \alpha(|\dot\gamma|)\,\mathrm{d}t \le \mathcal{F}_a^b(\gamma) \le \int_a^b \beta_n(|\dot\gamma|)\,\mathrm{d}t \qquad \text{for any } \gamma \in W^{1,1}_{loc}(\mathbb{R}, B_n). \tag{F4$'$}$$

where $(\beta_n)_{n\in\mathbb{N}}$ is a family of convex, non–decreasing and superlinear functions from $\mathbb{R}_+$ to $\mathbb{R}_+$. To prove Theorems 3.8 and 3.10, and Proposition 3.9 in this more general setting, we notice that the function $d_{\mathcal{F}}$ associated to any such $\mathcal{F}$ enjoys assertions (d1)–(d3) of Proposition 2.5, and assumption (d4$'$) of Remark 3.7, which is immediately obtained by choosing $a = 0$, $b = t$ and $\gamma(s) = x + s(y - x)/t$ in (F4$'$) for every $x, y \in B_n$ and $n \in \mathbb{N}$. By taking into account Remark 3.7, we can easily conclude via the same arguments. Of course, claim (ii) in Theorem 3.10 must be modified as follows:

(ii)$'$ $\qquad \alpha(|q|) \le L^{\pm}(x, q) \le \beta_n(|q|) \qquad$ *for any* $(x, q) \in B_n \times \mathbb{R}^N$ *and* $n \in \mathbb{N}$.

4. A class of integral functionals

In this Section we will show that condition (*) is enjoyed by a wide class of (non lower semicontinuous) integral functionals, including in particular the ones associated to integrands satisfying conditions (i)–(v) in the statement of Theorem 3.8. By this mean, we will in particular single out a family of integral functionals which is closed with respect to the Γ–convergence. More precisely, we consider an autonomous Lagrangian $L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ which satisfies the following assumptions:

(L1) L is Borel–measurable on $\mathbb{R}^N \times \mathbb{R}^N$,

(L2) $\alpha(|q|) \le L(x, q) \le \beta(|q|) \qquad$ for all $(x, q) \in \mathbb{R}^N \times \mathbb{R}^N$,

(L3) for every for every $t_1 < t_2$ in $\mathbb{R}$ and $\gamma \in W^{1,\infty}\big((t_1, t_2), \mathbb{R}^N\big)$

$$\lambda \mapsto L(\gamma(s), \lambda\,\dot\gamma(s)) \quad \text{is convex on } \mathbb{R} \quad \text{for a.e. } s \in (t_1, t_2).$$

Remark 4.1. The condition $L(x, q) \le \beta(|q|)$ for every $(x, q) \in \mathbb{R}^N \times \mathbb{R}^N$ is not so restrictive as it might appear. Indeed, it amounts to saying that (cf. Lemma 2.3 in [17])

$$\sup\{L(x, q) \,:\, (x, q) \in \mathbb{R}^N \times B_R\} < +\infty \qquad \text{for any } R > 0.$$

Condition (L3) is in particular satisfied when $L(x, \cdot)$ is convex for every $x \in \mathbb{R}^N$.

We define the associated action functional $\mathbb{L} : W^{1,1}_{loc}(\mathbb{R}; \mathbb{R}^N) \times \mathcal{I}(\mathbb{R}) \to [0, +\infty]$ by setting

$$\mathbb{L}\big(\gamma, (a, b)\big) := \int_a^b L(\gamma, \dot\gamma)\,\mathrm{d}s, \qquad \gamma \in W^{1,1}\big((a, b), \mathbb{R}^N\big), \tag{21}$$

for any $(a, b) \in \mathcal{I}(\mathbb{R})$. Clearly, $\mathbb{L}$ is a functional belonging to $\mathcal{A}$.

The main goal of this section will be proving the following result.

Theorem 4.2. *Let $L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ be an autonomous Lagrangian satisfying conditions (L1)–(L3), and $\mathbb{L}$ the integral functional defined via* (21)*. Then the associated function $d = d_{\mathbb{L}}$ defined through* (8) *satisfies condition* (*)*. More precisely, for every $M > 0$ there exists $K = K(M, \alpha, \beta)$ such that*

$$d\big((0,\cdot),(\cdot,\cdot)\big) \quad \textit{is } K\textit{–Lipschitz continuous in } \overline{C_M},$$

where $C_M := \{(y,t,x) \in \mathbb{R}^N \times (0,+\infty) \times \mathbb{R}^N \,:\, |x-y| < M\,t\}$.

Looking back at the proof of Theorem 3.8, we see that Theorem 4.2 will easily follow via a similar argument as soon as we get some *a priori* estimates on the Lipschitz constant of quasi–optimal curves parametrized in $(0,t)$ and connecting y to x, for every $(y,t,x) \in C_M$. That is precisely the content of Lemma 4.16. Its proof relies on a careful analysis on the role played by reparametrizations, which will be carried out in the next two subsections.

4.1. **Preliminary tools.** We start by introducing a piece of notation. Set

$$\Omega := \big\{(x,q) \in \mathbb{R}^N \times \mathbb{R}^N \,:\, \lambda \mapsto L(x,\lambda q) \text{ is convex on } \mathbb{R}\,\big\}$$

and

$$\mathcal{C} := \Omega \cap \big(\mathbb{R}^N \times \mathbb{S}^{N-1}\big).$$

For every $(x,q) \in \Omega$, let $f(x,q,\cdot)$ be the conjugate of the map $\lambda \to L(x,\lambda q)$, namely

$$f(x,q,u) := \max_{\lambda\in\mathbb{R}} \{u\lambda - L(x,\lambda\,q)\,\} \qquad \text{for every } u \in \mathbb{R},$$

which is convex and superlinear, due to (L2). Proposition 2.2 implies that

$$L(x,\lambda\,q) = f^*(x,q,\lambda) := \max_{u\in\mathbb{R}} \{\lambda\,u - f(x,q,u)\,\}$$

for every $\lambda \in \mathbb{R}$ and $(x,q) \in \Omega$, in particular

$$\min_{\mathbb{R}} f(x,q,\cdot) = -L(x,0) \qquad \text{for any } (x,q) \in \Omega.$$

For any $a \in \mathbb{R}$, let us define

$$\sigma_a(x,q) := \max\{\,u \,:\, f(x,q,u) \le a\,\} \qquad \text{for any } (x,q) \in \mathbb{R}^N \times \mathbb{R}^N.$$

The set appearing above is void whenever $a < -L(x,0)$. In this case, we agree that $\sigma_a(x,q) = -\infty$. The definition of σ_a can be extended to the whole $\mathbb{R}^N \times \mathbb{R}^N$ by setting $\sigma_a(x,q) = -\infty$ for every $(x,q) \notin \Omega$.

Remark 4.3. Under the additional assumption that $L(x,\cdot)$ is convex for any fixed $x \in \mathbb{R}^N$, the definition of σ_a given above reduces to

$$\sigma_a(x,q) := \max\{\langle q,p\rangle \,:\, H(x,p) \le a\} \qquad \text{for every } q \in \mathbb{R}^N,\ a \in \mathbb{R},$$

where $H : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is the Hamiltonian associated to L through the Fenchel transform, namely

$$H(x,p) := \max_{q\in\mathbb{R}^N} \{\langle p,q\rangle - L(x,q)\}\,.$$

In this case, $c(x,q)$ is actually independent of q and coincides with $\min_{\mathbb{R}^N} H(x,\cdot)$. In the above definition it is understood that $\sigma_a(x,q) = -\infty$ whenever $a < \min_{\mathbb{R}^N} H(x,\cdot)$.

Proposition 4.4. *For any $a \in \mathbb{R}$, the following properties hold:*

(i) $\quad \sigma_a(x,\lambda\,q) = \lambda\,\sigma_a(x,q) \qquad$ *for every $(x,q) \in \mathbb{R}^N \times \mathbb{R}^N$ and $\lambda > 0$;*

(ii) $\quad L(x,q) \geq \sigma_a(x,q) - a \qquad$ *for every* $(x,q) \in \mathbb{R}^N \times \mathbb{R}^N$.

Proof. To prove (i), we can assume $(x,q) \in \Omega$ and $a \geq -L(x,0)$, being the statement otherwise trivial by definition of σ_a. Then the equality $f(x,\lambda\, q,u) = f(x,q,u/\lambda)$, which holds true for every $u \in \mathbb{R}$, implies

$$\sigma_a(x,\lambda\, q) = \max\{\, u \,:\, f(x,q,u/\lambda) \leq a\,\} = \max\{\,\lambda\, u \,:\, f(x,q,u) \leq a\,\} = \lambda\, \sigma_a(x,q).$$

Let us prove (ii). Up to trivial cases, we can assume $(x,q) \in \Omega$ and $a \geq -L(x,0)$. By Proposition 2.2, we get

$$L(x,q) = \max_{u \in \mathbb{R}} \{\lambda\, u - f(x,q,u)\,\} \geq \max_{f(x,q,u) \leq a} \{\lambda\, u - a\,\} = \sigma_a(x,q) - a, \tag{22}$$

as claimed. □

For any $(x,q) \in \Omega$ and $a \in \mathbb{R}$, we set

$$\Lambda_a(x,q) := \{\,\lambda \in [0,+\infty) \,:\, L(x,\lambda\, q) = \sigma_a(x,\lambda\, q) - a\,\}, \tag{23}$$

and

$$\underline{\lambda}_a(x,q) := \inf \Lambda_a(x,q), \qquad \overline{\lambda}_a(x,q) := \sup \Lambda_a(x,q).$$

We agree that $\underline{\lambda}_a(x,q) = \overline{\lambda}_a(x,q) = 0$ whenever $\Lambda_a(x,q) = \emptyset$, that is, when either $q \neq 0$ and $a < -L(x,0)$, or $q = 0$ and $a \neq -L(x,0)$. Last, we extend the functions $\underline{\lambda}_a$, $\overline{\lambda}_a$ to the whole $\mathbb{R}^N \times \mathbb{R}^N$ by setting

$$\underline{\lambda}_a(x,q) = \overline{\lambda}_a(x,q) = -\infty \qquad \text{whenever } (x,q) \notin \Omega.$$

We define the following functions:

$$\alpha_*(u) := \max_{\lambda \in \mathbb{R}} \{u\lambda - \alpha(|\lambda|)\,\}, \quad \beta_*(u) := \max_{\lambda \in \mathbb{R}} \{u\lambda - \beta(|\lambda|)\,\} \qquad \text{for every } u \in \mathbb{R},$$

and we remark that they are convex and superlinear as $\alpha(|\cdot|)$ and $\beta(|\cdot|)$ are so. For every $a \in \mathbb{R}$, set

$$R_a := \max\{|u| \,:\, \beta_*(u) \leq a\,\} \tag{24}$$

and

$$\kappa_a := 2\max\{\alpha_*(u) \,:\, |u| \leq R_a + 1\,\}. \tag{25}$$

The following compactness result holds.

Lemma 4.5. *Let* $a \in \mathbb{R}$. *Then*

$$\{u \in \mathbb{R} \,:\, f(x,\hat{q},u) \leq a\,\} \subseteq B_{R_a}, \qquad \Lambda_a(x,\hat{q}) \subseteq B_{\kappa_a} \qquad \textit{for every } (x,\hat{q}) \in \mathcal{C}.$$

Proof. The first assertion follows at once from the fact that $\beta^*(\cdot) \leq f(x,\hat{q},\cdot)$ for every $(x,\hat{q}) \in \Omega$. To prove the second one, pick up $(x,\hat{q}) \in \mathcal{C}$ and set $f(\cdot) := f(x,\hat{q},\cdot)$. From Proposition 2.2 we infer that $\lambda \in \Lambda_a(x,\hat{q})$ if and only if $\lambda \in \partial f(u)$ for some $f(u) \leq a$, so

$$\Lambda_a(x,\hat{q}) \subseteq \{\partial f(u) \,:\, u \in B_{R_a}\,\}.$$

As $f(\cdot) \leq \alpha^*(\cdot)$, we conclude thanks to Proposition 2.1. □

Now we fix $(x,\hat{q}) \in \mathcal{C}$ and we examine the properties of the multifunction $a \mapsto \Lambda_a(x,\hat{q})$. To ease notation, we will write $g(\lambda)$ and $f(u)$ in place of $L(x,\lambda\,\hat{q})$ and $f(x,\hat{q},u)$, respectively. The duality between f and g implies, by Proposition 2.2, that

$$g(\lambda) = \lambda\, u - f(u) \qquad \text{for every } u \in \partial g(\lambda), \tag{26}$$

for any given $\lambda \in \mathbb{R}$. In view of (22), we infer that $\lambda \in \Lambda_a(x, \hat{q})$ if and only if $a \in f(\partial g(\lambda))$. We start by considering the set–valued map $A(\lambda) := f(\partial g(\lambda))$ on $[0, +\infty)$, which is the inverse of $a \mapsto \Lambda_a(x, \hat{q})$, in the sense of set–valued analysis (see [27, Chapter 5]). Indeed, note that

$$\Lambda_a(x, \hat{q}) = \{\, \lambda \in [0, +\infty) \,:\, a \in A(\lambda) \,\}. \tag{27}$$

We also remark for further use that, by classical results of non–smooth analysis (cf. [12, Theorem 2.3.10]),

$$\partial g(\lambda) = \langle \partial_q L(x, \lambda \hat{q}), \hat{q} \rangle \qquad \text{for every } \lambda \in \mathbb{R}. \tag{28}$$

Proposition 4.6. *Let $A(\cdot)$ as above. The following facts hold.*

(i) For any $\lambda \in \mathbb{R}$

$$A(\lambda) = [\underline{a}(\lambda), \overline{a}(\lambda)] \quad \textit{for some } -L(x,0) \le \underline{a}(\lambda) \le \overline{a}(\lambda) < +\infty.$$

(ii) The set–valued map $A(\cdot)$ is upper semicontinuous on $[0, +\infty)$. In particular, $\underline{a}(\cdot)$ is lower semicontinuous and $\overline{a}(\cdot)$ is upper semicontinuous on $[0, +\infty)$. Moreover

$$A(0) = \{\, -L(x,0) \,\}, \qquad \lim_{\lambda \to +\infty} \underline{a}(\lambda) = +\infty.$$

(iii) The set–valued map $\lambda \mapsto A(\lambda)$ is non–decreasing on $[0, +\infty)$.

(iv) $\displaystyle\bigcup_{\lambda \ge 0} A(\lambda) = [-L(x,0), +\infty).$

Proof. As g is convex, its subgradient $\partial g(\lambda)$ is a compact interval of $\mathbb{R}$, so the same is true for $A(\lambda)$. That proves (i). The upper semicontinuity of $A(\cdot)$ comes from the fact that the multifunction $\lambda \mapsto \partial g(\lambda)$ is upper semicontinuous and f is continuous. The equality $A(0) = \{-L(x,0)\}$ is an immediate consequence of (26) and of the fact that $g(0) = f^*(0) = L(x,0)$. The remainder of (ii) follows by definition of $\underline{a}(\cdot)$, $\overline{a}(\cdot)$ and by superlinearity of f and g.

Let us prove (iii). Since f and g are convex, the multimappings $u \mapsto \partial f(u)$ and $\lambda \mapsto \partial g(\lambda)$ are non–decreasing on $\mathbb{R}$. By superlinearity, we get in particular

$$\bigcup_{\lambda \ge 0} \partial g(\lambda) = [\underline{u}(0), +\infty) \qquad \text{with } \underline{u}(0) \in \partial g(0).$$

By duality (cf. Proposition 2.2), $0 \in \partial f(\underline{u}(0))$, so the monotonicity of $\partial f(\cdot)$ yields that f is non–decreasing on $[\underline{u}(0), +\infty)$.

Item (iv) comes from (ii) and (iii). □

We use this information to prove a result that will be crucial for our future analysis.

Proposition 4.7. *Let $(x, \hat{q}) \in \mathcal{C}$. The following facts hold.*

(i) For any $a \ge -L(x,0)$, we have

$$\Lambda_a(x,q) = [\underline{\lambda}_a(x,q), \overline{\lambda}_a(x,q)] \qquad \textit{for some} \quad 0 \le \underline{\lambda}_a(x,q) \le \overline{\lambda}_a(x,q) < +\infty.$$

Moreover,

$$\underline{\lambda}_{-L(x,0)}(x,q) = 0, \qquad \lim_{a \to +\infty} \underline{\lambda}_a(x,q) = +\infty.$$

(ii) *The set–valued map* $a \mapsto \Lambda_a(x,q)$ *is upper semicontinuous and non–decreasing on* $[-L(x,0),+\infty)$.

(iii) $$\underline{\lambda}_a(x,q) = \sup_{b<a} \overline{\lambda}_b(x,q) \qquad \text{for any } a > -L(x,0)$$

and

$$\overline{\lambda}_a(x,q) = \inf_{b>a} \underline{\lambda}_b(x,q) \qquad \text{for any } a \geq -L(x,0).$$

(iv) $$\underline{\lambda}_a(x,q) \geq \frac{a+L(x,0)}{2R_a} \quad \text{for any } a > -L(x,0), \text{ with } R_a \text{ defined by (24).}$$

Proof. We recall that $\Lambda_a(x,\hat{q}) = \{\lambda \geq 0 \,:\, a \in A(\lambda)\}$. The monotonicity property of the set–valued map $a \mapsto \Lambda_a(x,\hat{q})$ is a consequence of Proposition 4.6-(iii). In particular, $\Lambda_a(x,\hat{q})$ is a bounded interval for any $a \geq -L(x,0)$.

To prove the upper semicontinuity of $a \mapsto \Lambda_a(x,\hat{q})$, we need to show that, for each pair of sequences $(a_n)_n$ and $(\lambda_n)_n$ such that $a_n \to a \in \mathbb{R}$, $\lambda_n \to \lambda \in \mathbb{R}$ and $\lambda_n \in \Lambda_{a_n}(x,\hat{q})$ for every $n \in \mathbb{N}$, we have $\lambda \in \Lambda_a(x,\hat{q})$. That easily follows by the upper semicontinuity of $A(\cdot)$. In particular, this implies that $\Lambda_a(x,\hat{q})$ is closed for any $a \geq -L(x,0)$.

The equality $\underline{\lambda}_{-L(x,0)}(x,\hat{q}) = 0$ is a trivial consequence of definition (23). The coercivity of $a \mapsto \underline{\lambda}_a(x,\hat{q})$ comes from Proposition 4.6-(ii). Item (iii) immediately follows from the monotone and semicontinuous character of the map $a \mapsto \Lambda_a(x,\hat{q})$.

Let us prove (iv). Choose $a > -L(x,0)$ and set $\lambda := \underline{\lambda}_a(x,\hat{q})$. By Proposition 4.4-(ii) we get

$$\sigma_a(x,\lambda\hat{q}) = L(x,\lambda\hat{q}) + a \geq \sigma_{-L(x,0)}(x,\lambda\hat{q}) + a + L(x,0),$$

hence, by Lemma 4.5,

$$a + L(x,0) \leq \lambda\left(\sigma_a(x,\hat{q}) - \sigma_{-L(x,0)}(x,\hat{q})\right) \leq \lambda\left(R_a + R_{-L(x,0)}\right)|\hat{q}|,$$

and the statement follows as $R_{-L(x,0)} < R_a$ by definition. □

4.2. **Optimal reparametrizations.** Let us now consider a Lipschitz curve γ defined on a bounded interval $J := (0,\ell)$.

Definition 4.8. A curve ξ defined on a bounded interval $(0,t)$ is said to be a *reparametrization* of γ if there exists an absolutely continuous map $\varphi : [0,t] \to [0,\ell]$, surjective and non–decreasing, such that

$$\xi = \gamma \circ \varphi \qquad \text{on } (0,t).$$

We furthermore say that ξ is a *(bi–)Lipschitz reparametrization* of γ if φ is a (bi–) Lipschitz homeomorphism.

Remark 4.9. For reasons that will be clear soon, we want to allow a reparametrization to stop at a point for some time. This accounts for the choice of the unusual definition given above.

We introduce the following notation:

$$[\gamma]_t \;:=\; \{\xi \in W^{1,1}\big((0,t),\mathbb{R}^N\big) \,:\, \xi \text{ is a reparametrization of } \gamma\}$$
$$[\gamma]_t^\flat \;:=\; \{\xi \in W^{1,1}\big((0,t),\mathbb{R}^N\big) \,:\, \xi \text{ is a bi–Lipschitz reparametrization of } \gamma\}.$$

The following lemma comes from classical results of analysis in metric spaces (see e.g. Section VII.2 in [21]).

Lemma 4.10. *Let $\xi \in W^{1,1}\big((0,t),\mathbb{R}^N\big)$. Then there exists a Lipschitz curve γ, defined on a bounded interval $(0,\ell)$, such that $\xi \in [\gamma]_t$. We can furthermore assume that γ is parametrized by arc–length.*

A further step in the analysis is carried out by picking up some special reparametrizations of the curve γ.

Definition 4.11. A curve ξ defined on a bounded interval $(0,t)$ is said to have an *a–Lagrangian parametrization* if

$$L(\xi(s),\dot\xi(s)) = \sigma_a(\xi(s),\dot\xi(s)) - a \qquad \text{for a.e. } s \in (0,t),\ a \in \mathbb{R}.$$

For any $a \in \mathbb{R}$ and $t > 0$, we define

$$\begin{aligned} [\gamma](a,t) &:= \{\xi \in [\gamma]_t \,:\, \xi \text{ has an } a\text{–Lagrangian parametrization}\,\},\\ [\gamma]^\flat(a,t) &:= \{\xi \in [\gamma]^\flat_t \,:\, \xi \text{ has an } a\text{–Lagrangian parametrization}\,\}. \end{aligned}$$

Now assume γ parametrized by arc–length, and let

$$c_\gamma := \operatorname{ess\,sup}_{s\in J} \ -L\,(\gamma(s),0)$$

We define a multifunction $T_\gamma : (c_\gamma,+\infty) \to \mathcal{P}(\mathbb{R}_+)$ by setting

$$T_\gamma(a) := \{t > 0 \,:\, [\gamma]^\flat(a,t) \text{ is non–empty}\,\}.$$

The properties of the multifunction $T_\gamma(\cdot)$ are stated below.

Proposition 4.12. *Let γ and $T(\cdot) := T_\gamma(\cdot)$ as above. The following facts hold.*

(i) For any $a > c_\gamma$, $T(a)$ is a compact interval in $(0,+\infty)$, namely

$$T(a) := [\underline{T}\,(a),\overline{T}\,(a)] \qquad \textit{for some} \quad \overline{T}\,(a) \geq \underline{T}\,(a) > 0.$$

(ii) The multifunction $T(\cdot)$ is non–increasing and upper semicontinuous on $(c_\gamma,+\infty)$. Moreover $\quad \inf_{a>c_\gamma} \overline{T}\,(a) = 0$.

(iii) Let $\underline{T}(c_\gamma) := \sup_{a>c_\gamma} \overline{T}(a)$. If $\underline{T}(c_\gamma)$ is finite, then $[\gamma](c_\gamma,\underline{T}(c_\gamma)) \neq \emptyset$.

In particular, for any $0 < t \leq \underline{T}(c_\gamma)$ with $t < +\infty$, there exists $a \geq c_\gamma$ such that γ admits an a–Lagrangian Lipschitz reparametrization on $(0,t)$.

We first prove an auxiliary lemma.

Lemma 4.13. *Let $\gamma : (0,\ell) \to \mathbb{R}^N$ be a Lipschitz curve parametrized by arc–length and $a \in \mathbb{R}$. The following facts hold true.*

(i) For every $t > 0$ and $\xi \in [\gamma]_t$, the map $\sigma_a(\xi(\cdot),\dot\xi(\cdot))$ is Lebesgue–measurable on $(0,t)$, and

$$\int_0^t \sigma_a(\xi(s),\dot\xi(s))\,\mathrm{d}s = \int_0^\ell \sigma_a(\gamma(s),\dot\gamma(s))\,\mathrm{d}s. \tag{29}$$

(ii) The maps $\underline{\lambda}_{\,a}(\gamma(\cdot),\dot\gamma(\cdot))$, $\overline{\lambda}_{\,a}(\gamma(\cdot),\dot\gamma(\cdot))$ are Lebesgue–measurable on $(0,\ell)$.

Proof. Let $\Omega := \{(x, \lambda\,\hat{q}) \,:\, (x, \hat{q}) \in \mathcal{C},\, \lambda \in \mathbb{R}\}$. Take $t > 0$ and $\xi \in [\gamma]_t$. In order to prove (i), it suffices to define a function $\tilde{\sigma}_a$ such to be Borel–measurable on $\mathbb{R}^N \times \mathbb{R}^N$ and coinciding with σ_a on Ω. Indeed the map $s \mapsto (\xi(s), \dot{\xi}(s))$ is Lebesgue measurable and takes values in Ω for a.e. $s \in (0, t)$ by assumption (L3), hence the function $\sigma_a(\xi(\cdot), \dot{\xi}(\cdot))$ coincides, almost everywhere on $(0, \ell)$, with $\tilde{\sigma}_a(\gamma(\cdot), \dot{\gamma}(\cdot))$, which is Lebesgue–measurable as a composition of a Borel–measurable map with a Lebesgue–measurable one.

To this aim, let us denote by $(\lambda_n)_n$ and $(u_n)_n$ two dense sequences in $\mathbb{R}$ with $0 \in (u_n)_n$, and define, for every $(x, q) \in \mathbb{R}^N \times \mathbb{R}^N$ and $a \in \mathbb{R}$,

$$\tilde{f}(x, q, u) := \max_{n \in \mathbb{N}} \{u\lambda_n - L(x, \lambda_n\, q)\} \quad \text{for every } u \in \mathbb{R},$$

and

$$\tilde{\sigma}_a(x, q) := \inf_k \left(\sup_n \{u_n\, \vartheta_{E_n^k}(x, q)\} \right),$$

where $E_n^k := \{(x, q) \in \mathbb{R}^N \times \mathbb{R}^N \,:\, \tilde{f}(x, q, u_n) \le a + 1/k\}$ and $\vartheta_{E_n^k}$ denotes the function identically 1 on E_n^k and $-\infty$ elsewhere. The function $\tilde{\sigma}_a$ is Borel–measurable on $\mathbb{R}^N \times \mathbb{R}^N$ for any $a \in \mathbb{R}$ by construction. Moreover

$$\sigma_a(x, q) = \tilde{\sigma}_a(x, q) \quad \text{for every } (x, q) \in \Omega,$$

which holds true since $\tilde{f}(x, q, \cdot) = f(x, q, \cdot)$ by the continuity of the map $\lambda \mapsto L(x, \lambda\, q)$. The equality (29) is a consequence of the fact that $\sigma_a(x, \cdot)$ is positively 1–homogeneous.

To prove (ii), we notice that the map $s \mapsto (\gamma(s), \dot{\gamma}(s))$ takes values in $\mathcal{C}$ for a.e. $s \in (0, \ell)$, due to assumption (L3) and to the fact that it is parameterized by arc–length, hence it suffices to define two functions $\underline{\tilde{\lambda}}_a$, $\overline{\tilde{\lambda}}_a$ such to be Borel–measurable on $\mathbb{R}^N \times \mathbb{S}^{N-1}$ and coinciding on $\mathcal{C}$ with $\underline{\lambda}_a$, $\overline{\lambda}_a$, respectively.

For each $n \in \mathbb{N}$, let

$$F_n := \{(x, q) \in \mathbb{R}^N \times \mathbb{R}^N \,:\, \tilde{f}(x, q, \langle \partial_q L(x, \lambda_n\, q), q\rangle) \cap (-\infty, a) \neq \emptyset\},$$

which is Borel measurable for the multifunction $(x, q) \mapsto \tilde{f}(x, q, \langle \partial_q L(x, \lambda_n\, q), q\rangle)$ is so. Set

$$\underline{\tilde{\lambda}}_a(x, q) := \sup_n \lambda_n\, \chi_{F_n}(x, q) \qquad \text{for every } (x, q) \in \mathbb{R}^N \times \mathbb{S}^{N-1}.$$

Since $\tilde{f}(x, q, \cdot) = f(x, q, \cdot)$ for every $(x, q) \in \mathcal{C}$, we conclude that $\underline{\tilde{\lambda}}_a(x, q) = \underline{\lambda}_a(x, q)$ on $\mathcal{C}$ in view of (27), (28) and of Proposition 4.7. The analogous statement for $\overline{\lambda}_a$ can be proved in a similar way. $\square$

Proof of Proposition 4.12. (i) Fix $a > c_\gamma$, and set

$$\underline{\lambda}_a(\varsigma) := \underline{\lambda}_a(\gamma(\varsigma), \dot{\gamma}(\varsigma)), \quad \overline{\lambda}_a(\varsigma) := \overline{\lambda}_a(\gamma(\varsigma), \dot{\gamma}(\varsigma)) \qquad \text{for a.e. } \varsigma \in (0, \ell).$$

Let

$$\underline{T}(a) := \int_0^\ell \frac{1}{\overline{\lambda}_a(\varsigma)}\, \mathrm{d}\varsigma, \quad \overline{T}(a) := \int_0^\ell \frac{1}{\underline{\lambda}_a(\varsigma)}\, \mathrm{d}\varsigma.$$

Such quantities are well defined, positive real values, thanks to Proposition 4.7-(iv) and to the measurable character of $\underline{\lambda}_a(\cdot)$, $\overline{\lambda}_a(\cdot)$. To show that they belong to $T(a)$, we will prove the existence of two curves $\underline{\gamma}_a$, $\overline{\gamma}_a$, defined on $\big(0, \underline{T}(a)\big)$ and $\big(0, \overline{T}(a)\big)$, respectively, which are a–Lagrangian bi–Lipschitz reparametrizations of γ. To this aim, let us define

$$\underline{f}_a(s) := \int_0^s \frac{1}{\overline{\lambda}_a(\varsigma)}\,\mathrm{d}\varsigma, \quad \overline{f}_a(s) := \int_0^s \frac{1}{\underline{\lambda}_a(\varsigma)}\,\mathrm{d}\varsigma \qquad \text{for any } s \in (0,\ell),$$

and set

$$\underline{\varphi}_a := (\underline{f}_a)^{-1}, \qquad \overline{\varphi}_a := \left(\overline{f}_a\right)^{-1},$$

defined on $\big(0, \underline{T}(a)\big)$ and $\big(0, \overline{T}(a)\big)$, respectively. As

$$\dot{\underline{\varphi}}_a(\tau) = \overline{\lambda}_a(\underline{\varphi}_a(\tau)), \qquad \dot{\overline{\varphi}}_a(\tau) = \underline{\lambda}_a(\overline{\varphi}_a(\tau)) \qquad \text{for a.e. } \tau,$$

we immediately derive that $\underline{\varphi}_a$ and $\overline{\varphi}_a$ are order–preserving bi–Lipschitz diffeomorphisms. Let us set

$$\underline{\gamma}_a := \gamma \circ \underline{\varphi}_a \quad \text{on } \big(0, \underline{T}(a)\big), \qquad \overline{\gamma}_a := \gamma \circ \overline{\varphi}_a \quad \text{on } \big(0, \overline{T}(a)\big).$$

Since

$$\dot{\underline{\gamma}}_a(\cdot) := \overline{\lambda}_a(\underline{\varphi}_a(\cdot))\ \dot{\gamma}(\underline{\varphi}_a(\cdot)) \qquad \text{a.e. on } \big(0, \underline{T}(a)\big)$$

and

$$\dot{\overline{\gamma}}_a(\cdot) := \underline{\lambda}_a(\overline{\varphi}_a(\cdot))\ \dot{\gamma}(\overline{\varphi}_a(\cdot)) \qquad \text{a.e. on } \big(0, \overline{T}(a)\big),$$

we conclude that the curves $\underline{\gamma}_a$ and $\overline{\gamma}_a$ has an a–Lagrangian parametrization by the very definition of $\overline{\lambda}_a$ and $\underline{\lambda}_a$.

In order to prove that $[\underline{T}(a), \overline{T}(a)] \subseteq T(a)$, we will show that

$$\delta\,\underline{T}(a) + (1-\delta)\,\overline{T}(a) \in T(a) \qquad \text{for any } \delta \in (0,1). \tag{30}$$

Fix $\delta \in (0,1)$, and set

$$\delta(\varsigma) := \frac{\delta\,\overline{\lambda}_a(\varsigma)}{\delta\,\overline{\lambda}_a(\varsigma) + (1-\delta)\,\underline{\lambda}_a(\varsigma)}, \qquad \lambda(\varsigma) := \delta(\varsigma)\,\underline{\lambda}_a(\varsigma) + (1-\delta(\varsigma))\,\overline{\lambda}_a(\varsigma)$$

for almost every $\varsigma \in (0,\ell)$, and

$$f(s) := \int_0^s \frac{1}{\lambda(\varsigma)}\,\mathrm{d}\varsigma \quad \text{for } s \in (0,\ell), \qquad \varphi := f^{-1} \quad \text{on } [0, f(\ell)].$$

Since $\delta(\varsigma) \in [0,1]$ for almost every $\varsigma \in (0,\ell)$, we get that $\lambda_a(\varsigma) \in \Lambda_a(\gamma(\varsigma), \dot{\gamma}(\varsigma))$ for almost every $\varsigma \in (0,\ell)$, in particular φ is an order–preserving bi–Lipschitz diffeomorphism. Arguing as above, we see that the curve $\gamma_a := \gamma \circ \varphi$ is an a–Lagrangian bi–Lipschitz reparametrization of γ on $\big(0, f(\ell)\big)$, so $f(\ell) \in T(a)$. Now it is easy to check, by definition of $\delta(\cdot)$, that $f(\ell) = \delta\,\overline{T}(a) + (1-\delta)\,\underline{T}(a)$. That proves (30) as δ was arbitrarily chosen in $(0,1)$.

Let us now prove that $T(a) \subseteq [\underline{T}(a), \overline{T}(a)]$. Let $T \in T(a)$ and $\widetilde{\gamma} := \gamma \circ \varphi$ be an a–Lagrangian reparametrization of γ for some order–preserving bi–Lipschitz diffeomorphism $\varphi : (0,T) \to (0,\ell)$. Then

$$\dot{\varphi}(\tau) \in \Lambda_a\left(\gamma(\varphi(\tau)), \dot{\gamma}(\varphi(\tau))\right) \qquad \text{for a.e. } \tau \in (0,T).$$

Let $f := \varphi^{-1}$. We have

$$T = f(\ell) = \int_0^\ell \dot{f}(\varsigma)\,\mathrm{d}\varsigma = \int_0^\ell \frac{1}{\dot{\varphi}(f(\varsigma))}\,\mathrm{d}\varsigma,$$

and since $\dot{\varphi}(f(\varsigma)) \in \Lambda_a(\gamma(\varsigma), \dot{\gamma}(\varsigma)) = [\underline{\lambda}_a(\varsigma), \overline{\lambda}_a(\varsigma)]$ for a.e. $\varsigma \in (0,\ell)$, we clearly get $T \in [\underline{T}(a), \overline{T}(a)]$.

(ii) Let $b > a > c_\gamma$. Then $\underline{\lambda}_b(\varsigma) \geq \overline{\lambda}_a(\varsigma)$ for almost every $\varsigma \in (0,\ell)$, hence $\overline{T}(b) \leq \underline{T}(a)$. That proves that $T(\cdot)$ is a non–increasing multifunction. To prove that $T(\cdot)$ is u.s.c. on $(c_\gamma,+\infty)$, it will be enough to show that

$$\underline{T}(a) = \sup_{b>a} \overline{T}(b), \qquad \overline{T}(a) = \inf_{b<a} \underline{T}(b) \qquad \text{for any } a > c_\gamma.$$

This actually follows as a simple application of the Monotone Convergence Theorem and by the monotonicity poperties of $\underline{\lambda}_a$, $\overline{\lambda}_a$ (cf. Proposition 4.7-(iii)). The last assertion holds by definition of $\underline{T}(a)$ since $\sup_{a>c_\gamma} \overline{\lambda}_a(\varsigma) = +\infty$ for almost every $\varsigma \in (0,\ell)$.

(iii) Let $\underline{T}(c_\gamma)$ be finite. Arguing as in (i), we may find a non–increasing sequence of Borel–measurable maps $\lambda_n : (0,\ell) \to [0,+\infty)$ such that, for each $n \in \mathbb{N}$,

$$T_n = \int_0^\ell \frac{1}{\lambda_n(\varsigma)}\,\mathrm{d}\varsigma \qquad \text{and} \qquad \lambda_n(\varsigma) \in \Lambda_{c_\gamma+1/n}(\gamma(\varsigma),\dot\gamma(\varsigma)) \qquad \text{for a.e. } \varsigma \in (0,\ell),$$

with $\sup_n T_n = \underline{T}(c_\gamma)$. Set

$$\lambda(\varsigma) = \inf_n \lambda_n(\varsigma) \qquad \text{for every } \varsigma \in (0,\ell).$$

Then $\lambda(\cdot)$ is measurable and $\lambda(\varsigma) \in \Lambda_{c_\gamma}(\gamma(\varsigma),\dot\gamma(\varsigma))$ for almost every $\varsigma \in (0,\ell)$. Moreover the Monotone Convergence Theorem yields

$$\underline{T}(c_\gamma) = \sup_{n\in\mathbb{N}} T_n = \sup_{n\in\mathbb{N}} \int_0^\ell \frac{1}{\lambda_n(\varsigma)}\,\mathrm{d}\varsigma = \int_0^\ell \frac{1}{\lambda(\varsigma)}\,\mathrm{d}\varsigma,$$

in particular the map

$$f(s) := \int_0^s \frac{1}{\lambda(\varsigma)}\,\mathrm{d}\varsigma$$

is increasing and absolutely continuous on $(0,\ell)$. A c_γ–Lagrangian Lipschitz reparametrization of γ defined on $(0,\underline{T}(c_\gamma))$ can be now obtained by setting $\widetilde{\gamma} := \gamma \circ \varphi$ with $\varphi := (f)^{-1}$ on $(0,\underline{T}(c_\gamma))$.

Last, the fact that the multifunction is upper semicontinuous, monotone and convex–set–valued implies that

$$\bigcup_{a>c_\gamma} T(c_\gamma) = \big(0,\underline{T}(c_\gamma)\big)$$

and this is enough to obtain the remainder of the statement. □

We now seek for an optimal reparametrization of γ on the interval $(0,t)$, for any given $t \in (0,+\infty)$. Such a reparametrization does not exist in general, as we will see. In any case, however, we are able to derive an estimate on the Lipschitz constants of quasi–optimal reparametrizations. This is a crucial step for our study.

Theorem 4.14. *Let $\gamma : (0,\ell) \to \mathbb{R}^N$ be a Lipschitz curve parametrized by arc–length. Then, for every $t \in (0,+\infty)$, there exists $a \geq c_\gamma$ such that*

$$\inf_{\xi\in[\gamma]_t} \int_0^t L(\xi,\dot\xi)\,\mathrm{d}s \;=\; \inf_{\xi\in[\gamma]_t} \left\{ \int_0^t L(\xi,\dot\xi)\,\mathrm{d}s \;:\; \|\dot\xi\|_\infty \leq \kappa_a \right\} = \int_0^\ell \sigma_a(\gamma,\dot\gamma)\,\mathrm{d}s - a\,t,$$

with κ_a given by (25). The above infimum is a minimum whenever $t \leq \underline{T}_\gamma(c_\gamma)$, and is in particular attained by some curve belonging to $[\gamma]^\flat(a,t)$ with $a > c_\gamma$ when $t < \underline{T}_\gamma(c_\gamma)$.

Proof. By Proposition 4.4 and Lemma 4.13, we get

$$\int_0^t L(\xi,\dot\xi)\,\mathrm{d}s \geq \int_0^t \Big(\sigma_a(\xi,\dot\xi)-a\Big)\,\mathrm{d}s = \int_0^\ell \sigma_a(\gamma,\dot\gamma)\,\mathrm{d}s - a\,t \tag{31}$$

for any $a \geq c_\gamma$ and $\xi \in [\gamma]_t$, and (31) is an equality whenever $\xi \in [\gamma](a,t)$. The assertion for $t \leq \underline{T}_\gamma(c_\gamma)$ hence follows in force of Proposition 4.12 and Lemma 4.5.

Let us now assume $t > \underline{T}_\gamma(c_\gamma)$ and set $h := t - \underline{T}_\gamma(c_\gamma)$. Let $\xi \in [\gamma](c_\gamma, \underline{T}_\gamma(c_\gamma))$. By definition of c_γ, there exists, for each $n \in \mathbb{N}$, $s_n \in \big(0, \underline{T}_\gamma(c_\gamma)\big)$ such that

$$c_\gamma + L(\gamma(s_n),0) < \frac{1}{n}.$$

To ease notation, we will write c_n in place of $-L(\gamma(s_n),0)$. We define

$$\xi_n(s) := \begin{cases} \xi(s) & \text{if } s \in (0,s_n] \\ \xi(s_n) & \text{if } s \in [s_n, s_n+h] \\ \xi(s-h) & \text{if } s \in [s_n+h, t). \end{cases}$$

We have

$$\begin{aligned}\int_0^t L(\xi_n,\dot\xi_n)\,\mathrm{d}s &= \int_0^{\underline{T}_\gamma(c_\gamma)} L(\xi,\dot\xi)\,\mathrm{d}s - h\,c_n = \int_0^{\underline{T}_\gamma(c_\gamma)} \sigma_{c_\gamma}(\xi,\dot\xi)\,\mathrm{d}s - \underline{T}_\gamma(c_\gamma)\,c_\gamma \\ -h\,c_n &= \int_0^\ell \sigma_{c_\gamma}(\gamma,\dot\gamma)\,\mathrm{d}s - c_\gamma\,t + h\,\big(c_\gamma - c_n\big) < \int_0^\ell \sigma_{c_\gamma}(\gamma,\dot\gamma)\,\mathrm{d}s - c_\gamma\,t + \frac{h}{n}.\end{aligned}$$

Taking (31) into account, we derive

$$\int_0^\ell \sigma_{c_\gamma}(\gamma,\dot\gamma)\,\mathrm{d}s - c_\gamma\,t \leq \int_0^t L(\xi_n,\dot\xi_n)\,\mathrm{d}s < \int_0^\ell \sigma_{c_\gamma}(\gamma,\dot\gamma)\,\mathrm{d}s - c_\gamma\,t + \frac{h}{n},$$

and we conclude letting $n \to +\infty$. □

Remark 4.15. If in Theorem 4.14 the Lagrangian is assumed lower semicontinuous in x, we can furthermore say that, for every $t > 0$,

$$\inf_{\xi\in[\gamma]_t} \int_0^t L(\xi,\dot\xi)\,\mathrm{d}s \;= \min\left\{ \int_0^t L(\xi,\dot\xi)\,\mathrm{d}s \;:\; \xi \in [\gamma](a,t) \right\}$$

for some constant $a \geq c_\gamma$. This can be proved by considering, in place of $T_\gamma(\cdot)$, the set–valued map defined as

$$T^*_\gamma(a) := \{t > 0 \,:\, [\gamma](a,t) \text{ is non–empty}\,\}$$

for every $a \geq c^*_\gamma := \sup_{s\in J} -L\,(\gamma(s),0)$. The multifunction $T^*_\gamma(\cdot)$ agrees with $T_\gamma(\cdot)$ on $(c^*_\gamma, +\infty)$. Indeed, the inequality

$$\underline{\lambda}_{\,a}(\gamma(s),\dot\gamma(s)) \geq \frac{a - c^*_\gamma}{2R_a} \qquad \text{for a.e. } s \in (0,\ell),$$

which holds true by Proposition 4.7, implies that $[\gamma](a,t) = [\gamma]^\flat(a,t)$ for every $a > c^*_\gamma$ and $t > 0$ (cf. the argument showing that $T(a) \subseteq [\underline{T}\,(a), \overline{T}\,(a)]$ in the proof of Proposition 4.12). On the other hand, we always have

$$T^*_\gamma(c^*_\gamma) = [\underline{T}_\gamma(c^*_\gamma), +\infty) \tag{32}$$

when $\underline{T}_\gamma(c^*_\gamma)$ is finite, and that is enough to get the statement in view of (31).

To prove (32), let ξ be a curve belonging to $[\gamma](c^*_\gamma, \underline{T}(c^*_\gamma))$ (which does exist in force of Proposition 4.12) and take $s_0 \in [0, \underline{T}_\gamma(c^*_\gamma)]$ such that $L(\xi(s_0), 0) = -c^*_\gamma$. Such an s_0 always exists by the upper semicontinuity of $-L(\gamma(\cdot), 0)$ on $[0, \ell]$. For every $h > 0$, define $\xi_h : [0, \underline{T}(c^*_\gamma) + h] \to \mathbb{R}^N$ as

$$\xi_h(s) := \begin{cases} \xi(s) & \text{if } s \in [0, s_0] \\ \xi(s_0) & \text{if } s \in [s_0, s_0 + h] \\ \xi(s - h) & \text{if } s \in [s_0 + h, \underline{T}_\gamma(c^*_\gamma) + h]. \end{cases}$$

It is easily seen that ξ_h is a c^*_γ–Lagrangian reparametrization of γ. This shows that

$$\underline{T}_\gamma(c^*_\gamma) + h \in T^*_\gamma(c^*_\gamma) \qquad \text{for every } h > 0,$$

as claimed.

4.3. **Proof of Theorem 4.2 and further extensions.** With the aid of the results obtained so far, we can now establish the following key lemma.

Lemma 4.16. *Let $x, y \in \mathbb{R}^N$ and $t > 0$ such that $d\big((0, y), (t, x)\big) < M\, t$. Then there exists a constant $\kappa = \kappa(M, \alpha, \beta)$ such that*

$$d\big((0, y), (t, x)\big) = \inf\left\{ \int_0^t L(\xi, \dot\xi)\,\mathrm{d}s \;:\; \xi(0) = y,\ \xi(t) = x,\ \|\dot\xi\|_\infty \le \kappa \right\}.$$

Proof. Choose $\overline{n} \in \mathbb{N}$ such that $M/\alpha(\overline{n}) < 1/2$ and set

$$A = A(\overline{n}) := \max\{\alpha_*(u) \;:\; |u| \le 2\,\beta(\overline{n} + 1)\,\},$$

where we recall that $\alpha_*(u) := \max_{\lambda \in \mathbb{R}}\{\lambda u - \alpha(|\lambda|)\,\}$. We claim that the statement holds true with $\kappa := \kappa_A$ defined according to (25).

Indeed, pick up a curve $\xi \in W^{1,1}\big((0, t), \mathbb{R}^N\big)$ such that

$$\int_0^t L(\xi, \dot\xi)\,\mathrm{d}s < M\, t,$$

and let $\gamma : (0, \ell) \to \mathbb{R}^N$ be a Lipschitz curve, parametrized by arc–length, such that $\xi \in [\gamma]_t$, according to Lemma 4.10. In view of Theorem 4.14, up to choosing a different ξ in $[\gamma]_t$ without increasing the action, we can always assume that either $\|\dot\xi\|_\infty \le \kappa_{c_\gamma}$, or $\xi \in [\gamma]^\flat(a, t)$ for some $a > c_\gamma$. In the first case, we note that

$$c_\gamma \le \alpha_*(0),$$

for $-L(x, 0) \le -\alpha(0) \le \alpha_*(0)$ for every $x \in \mathbb{R}^N$. The claim follows by definition of κ_A as $\alpha_*(0) \le A$.

Let us instead assume that ξ belongs to $[\gamma]^\flat(a, t)$ for some $a > c_\gamma$. In particular, $|\dot\xi(s)| \ne 0$ almost everywhere on $(0, t)$. Set $J := \{s \in (0, t) \;:\; 0 < |\dot\xi(s)| < \overline{n}\,\}$. We have

$$M\, t > \int_0^t L(\xi, \dot\xi)\,\mathrm{d}s \ge \int_0^t \alpha(|\dot\xi|)\,\mathrm{d}s \ge \alpha(\overline{n})\,\big|(0, t) \setminus J\,\big|,$$

hence $|J| > t/2$. Pick up a differentiability point $\overline{s} \in J$ for ξ, and denote by f and g the functions $u \mapsto f(\xi(\overline{s}), \dot\xi(\overline{s})/|\dot\xi(\overline{s})|, u)$, $\lambda \mapsto L(x, \lambda\,\dot\xi(\overline{s})/|\dot\xi(\overline{s})|)$, respectively. By the fact that ξ has an a–Lagrangian parametrization we derive that

$$a \in f\left(\partial g(\,|\dot\xi(\overline{s})|\,)\right),$$

in particular $a \le A$ by Proposition 2.1. As $|\dot\xi(s)| \in \Lambda_a(\xi(s), \dot\xi(s)/|\dot\xi(s)|)$ for a.e. $s \in (0, t)$, the claim follows in force of Lemma 4.5 since $\kappa_a \le \kappa_A$ by (25). $\square$

Proof of Theorem 4.2. For a fixed $M > 0$, choose (y_1, t_1, x_1) and (y_2, t_2, x_2) in C_M, and set

$$h := |t_1 - t_2| + |x_1 - x_2| + |y_1 - y_2|, \qquad s_0 := \frac{t_1 - t_2}{2} + h.$$

Since C_M is convex, it suffices to prove the statement locally, namely for $h \ll 1$. Choose $h < t_2/2$ so that $s_0 < t_1/2$. Fix $\varepsilon > 0$ and let $\gamma_1 \in W^{1,1}\big((0,t_1), \mathbb{R}^N\big)$ be an ε–minimizer connecting y_1 to x_1. Thanks to Lemma 4.16, we can assume $\|\dot\gamma\|_\infty \le \kappa_A$ for some constant $A = A(M, \alpha, \beta)$. Choose $u_1, v_1 \in \mathbb{R}^N$ so that

$$\gamma_1(s_0) = y_2 + h u_1, \qquad \gamma_1(t_1 - s_0) = x_2 + h v_1,$$

and note that $|u_1|, |v_1| < 1 + 2\kappa_A$. Define a curve $\gamma_2 : [0, t_2] \to \mathbb{R}^N$ connecting y_2 to x_2 as in the proof of Theorem 3.8. By arguing similarly, we get

$$d\big((0, y_2), (t_2, x_2)\big) - d\big((0, y_1), (t_1, x_1)\big) \le 2\,\beta(1 + 2\kappa_A)\, h + \varepsilon,$$

so the statement follows by interchanging the roles of (y_1, t_1, x_1) and (y_2, t_2, x_2) and since ε is arbitrary. □

Let us now consider a Lagrangian $L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ which satisfies, in place of (L2), the following condition:

$$\text{(L2}'\text{)} \qquad \alpha\,(|q|) \le L(x,q) \le \beta_n\,(|q|) \qquad \text{for all } (x,q) \in B_n \times \mathbb{R}^N \text{ and } n \in \mathbb{N},$$

where $(\beta_n)_{n\in\mathbb{N}}$ is a family of convex, non–decreasing and superlinear functions from $\mathbb{R}_+$ to $\mathbb{R}_+$.

Remark 4.17. We point out that condition (L2′) amounts to requiring that L is uniformly superlinear in q, and locally bounded in $\mathbb{R}^N \times \mathbb{R}^N$ (cf. Lemma 2.3 in [17]).

In view of the results proved so far, it is now easy to generalize Theorem 4.2 as follows.

Theorem 4.18. *Let $L : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}_+$ be an autonomous Lagrangian satisfying conditions (L1), (L2′),(L3), and let $\mathbb{L}$ be the integral functional defined via* (21). *Then the associated function $d = d_{\mathbb{L}}$ defined through* (8) *satisfies condition* (*). *More precisely, for every $M, r > 0$ there exist a constant $K = K\big(M, r, \alpha, (\beta_n)_n\big)$ such that*

$$d\big((0,\cdot),(\cdot,\cdot)\big) \quad \textit{is } K\textit{–Lipschitz continuous in } \overline{C_M(r)},$$

where $C_M(r) := \{(y,t,x) \in B_r \times (0,r) \times B_r \,:\, |x - y| < M\,t\}$.

Proof. For every $n \in \mathbb{N}$, let us denote by $\mathbb{L}_n$ the integral functional associated through (21) to the Lagrangian $L_n(x,q) := L(x,q)\,\chi_{B_n}(x) + \beta_n(q)\,\chi_{\mathbb{R}^N\setminus B_n}(x)$. We claim that, for every $M, r > 0$, there exists an index $k = k\big(M, r, \alpha, (\beta_n)_n\big)$ such that

$$d = d_{\mathbb{L}_k} \quad \text{on } C_M(r). \tag{33}$$

Clearly, that is enough to conclude in force of Theorem 4.2.

Let us fix $M, r > 0$. We first notice that d enjoys assertions (d1)–(d3) of Proposition 2.5, and assumption (d4′) of Remark 3.7; in particular,

$$d\big((0,y),(t,x)\big) < r\,\beta_m\,(M) \qquad \text{for any } (y,t,x) \in C_M(r), \tag{34}$$

where $m := [r] + 1$. Let γ be a curve in $W^{1,1}\big((0,t),\mathbb{R}^N\big)$ connecting y to x such to be quasi–optimal for $d\big((0,y),(t,x)\big)$. By (34), it is not restrictive to assume that

$$\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s < r\,\beta_m\,(M)\,,$$

in particular

$$\int_0^t |\dot\gamma|\,\mathrm{d}s < r\Big(\alpha_1 + \beta_m(M)\Big),$$

with $\alpha_1 > 0$ such that $\alpha(|q|) \geq |q| - \alpha_1$ for any $q \in \mathbb{R}^N$. As γ has end–points lying in B_r, we deduce that γ is entirely contained in the open ball B_k with

$$k := \big[r\big(1 + \alpha_1 + \beta_m(M)\big)\big] + 1.$$

Thus

$$d\big((0,y),(t,x)\big) = \inf\left\{\int_0^t L(\gamma,\dot\gamma)\,\mathrm{d}s\;:\;\gamma(0)=y,\;\gamma(t)=x,\;\gamma([0,t])\subset B_k\right\}$$

for every $(y,t,x) \in C_M(r)$, and claim (33) follows at once as L coincide with L_k on $B_k \times \mathbb{R}^N$. □

4.4. **Compactness with respect to Γ–convergence.** As a simple consequence of the analysis carried out in the preceding sections, we derive a compactness result for a class of locally bounded, discontinuous autonomous Lagrangians.

We will say that a sequence of functionals $\mathcal{F}_k : W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^N) \times \mathcal{I}(\mathbb{R}) \to [0,+\infty]$ Γ*–converges* to a functional $\overline{\mathcal{F}} : W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^N) \times \mathcal{I}(\mathbb{R}) \to [0,+\infty]$ if, for every $\gamma \in W^{1,1}_{loc}(\mathbb{R};\mathbb{R}^N)$ and $(a,b) \in \mathcal{I}(\mathbb{R})$, the following two conditions hold:

1) for every $(\gamma_k)_k \subset W^{1,1}\big((a,b),\mathbb{R}^N\big)$ such that $\lim_k \|\gamma_k - \gamma\|_\infty = 0$ we have
$$\liminf_k\, \mathcal{F}_k\big(\gamma_k,(a,b)\big) \geq \overline{\mathcal{F}}\big(\gamma,(a,b)\big);$$
2) there exists $(\gamma_k)_k \subset W^{1,1}\big((a,b),\mathbb{R}^N\big)$ with $\lim_k \|\gamma_k - \gamma\|_\infty = 0$ such that
$$\limsup_k\, \mathcal{F}_k\big(\gamma_k,(a,b)\big) \leq \overline{\mathcal{F}}\big(\gamma,(a,b)\big).$$

The Γ–limit of a sequence of functionals is lower semicontinuous with respect to the convergence at issue. It is furthermore known (see [14, Theorem 8.5]) that the Γ–convergence is sequentially compact, that is, any sequence $(\mathcal{F}_k)_k$ admits a Γ–convergent subsequence. For a general survey on the theory of Γ–convergence, we refer to [5, 14].

Let now $(\beta_n)_{n\in\mathbb{N}}$ a sequence of convex, non–decreasing and superlinear functions from $\mathbb{R}_+$ to $\mathbb{R}_+$, and denote by $\mathcal{N} := \mathcal{N}\big(\alpha,(\beta_n)_n\big)$ the family of Lagrangians L satisfying conditions (L1), (L2)′, (L3). With a slight abuse of notation, we will say that a sequence $(L_k)_k$ Γ–converges to L in $\mathcal{N}$ if the associated action functionals Γ–converge to the integral functional associated to L.

Theorem 4.19. *Let $\mathcal{N} := \mathcal{N}\big(\alpha,(\beta_n)_n\big)$ as above. Then $\mathcal{N}$ is sequentially compact with respect to Γ–convergence; i.e., any sequence in $\mathcal{N}$ admits a Γ–converging subsequence.*

Proof. Let $(L_k)_k$ be a sequence in $\mathcal{N}$, and denote by $\mathbb{L}_k$ the integral functional associated to each L_k through (21). By what remarked above, we already know that, up to subsequences, $(\mathbb{L}_k)_k$ Γ–converges to some lower semicontinuous functional $\overline{\mathbb{L}}$ satisfying assumptions (F1)–(F3) and (F4$'$). In order to conclude, we need to show that $\overline{\mathbb{L}}$ admits an integral representation for some $L \in \mathcal{N}$.

For every $x, y \in \mathbb{R}^N$ and $t > 0$, set

$$\begin{aligned} X^t(y,x) &:= \left\{ \gamma \in W^{1,1}\big((0,t), \mathbb{R}^N\big) \,:\, \gamma(0) = y,\, \gamma(t) = x \,\right\}, \\ Y^t(y,x) &:= \left\{ \gamma \in X^t(y,x) \,:\, \int_0^t \alpha(|\dot\gamma|)\,\mathrm{d}s \le \beta_n\left(\frac{|x-y|}{t}\right) t \right\}, \end{aligned}$$

where $n = n(x,y)$ is a sufficiently large integer number such that $x, y \in B_n$. The growth assumptions (L2) implies that

$$d_{\mathbb{L}_k}\big((0,y),(t,x)\big) = \inf\left\{ \mathbb{L}_k(\gamma) \,:\, \gamma \in Y^t(y,x) \,\right\}$$

for every $x, y \in \mathbb{R}^N$, $t > 0$ and $k \in \mathbb{N}$. By Proposition 2.3 we know that $Y^t(y,x)$ is compact in $X^t(y,x)$ with respect to the uniform convergence, hence by the crucial result of Γ–convergence (cf. [14, Theorem 7.4]) we infer that

$$d_{\overline{\mathbb{L}}}\big((0,y),(t,x)\big) = \lim_{k\to+\infty} d_{\mathbb{L}_k}\big((0,y),(t,x)\big) \quad \text{for every } x, y \in \mathbb{R}^N \text{ and } t > 0.$$

Moreover, for any $M, r > 0$ the functions $d_{\mathbb{L}_k}\big((0,\cdot),(\cdot,\cdot)\big)$ are equi–Lipschitz continuous on $\overline{C_M(r)}$ by Theorem 4.18, so the same is true for $d_{\overline{\mathbb{L}}}\big((0,\cdot),(\cdot,\cdot)\big)$. The conclusion now follows in force of Theorem 3.8 and Remark 3.12. $\square$

Remark 4.20. We note that the convergence of $d_{\mathbb{L}_k}\big((0,\cdot),(\cdot,\cdot)\big)$ to $d_{\overline{\mathbb{L}}}\big((0,\cdot),(\cdot,\cdot)\big)$ is actually uniform in $\overline{C_M(r)}$, for any $M, r > 0$.

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