Regularity and nonexistence results for anisotropic quasilinear elliptic equations in convex domains

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Abstract

For a class of anisotropic elliptic problems in bounded domains Ω we show that the convexity of Ω plays an important role in regularity and nonexistence results. Using recent results in [9] we improve some statements in [3].

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1 Introduction

Let $\Omega \subset \mathbf{R}^n$ $(n \geq 3)$ be a smooth bounded domain, consider n numbers $m_i \geq 2$ for all $i = 1, ..., n$, take $\lambda > 0$ and $p > 1$. In a recent paper [3], it was shown that existence and nonexistence results for nontrivial solutions to the following anisotropic quasilinear elliptic problem

$$\begin{cases} -\sum_{i=1}^{n} \partial_i \left(|\partial_i u|^{m_i - 2} \partial_i u \right) = \lambda u^{p-1} & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

are in fact related to the regularity of the solutions to the following "coercive regularized" problem

$$\begin{cases} -\sum_{i=1}^{n} \partial_i \left[(|\partial_i w|^{m_i-2} + \varepsilon(1+|Dw|^2)^{(m_- -2)/2})\partial_i w \right] + \lambda |w|^{p-2} w = f & \text{in } \Omega \\ w = 0 \qquad \text{on } \partial\Omega \end{cases} \tag{2}$$

where $\varepsilon > 0$, $m_- := \min\{m_1, \ldots, m_n\}$ and f is a smooth function; here and in the sequel, $\partial_i = \partial/\partial x_i$ for $i = 1, ..., n$.
If $m_i = 2$ for all i, then (1) reduces to the widely studied semilinear equation $-\Delta u = \lambda u^{p-1}$. In recent years, an increasing interest has turned towards anisotropic problems. With no hope of being complete, let us mention the pioneering works on anisotropic Sobolev spaces [6, 11, 14, 15, 16] and more recent regularity results for anisotropic problems [1, 2, 4, 8, 9, 10, 19].
Since the anisotropy in (1) weights differently each single partial derivative, one expects the *geometry* of the domain Ω to play a crucial role in related results. This is precisely what happens for optimal Sobolev embedding theorems which only hold under suitable assumptions on the geometry of Ω, see examples and counter-examples in [6, 14, 15].
In view of these facts, also results concerning (1) and (2) should strongly depend on the geometry of Ω. In the present note, we show that the regularity and nonexistence results in [3] can be strengthened provided the domain Ω is *convex*. In [3] it was shown that any weak solution of (1) (see Definition 1 below) is in fact bounded whenever p is at most critical in a suitable sense; by taking advantage of recent results in [9] (see also [8]), in Theorem 1 we prove that if Ω is convex then the solutions are also globally Lipschitz continuous. Similarly, the full $C^{2,\gamma}$ regularity of the solution of (2) was obtained in [3] assuming that the exponents m_i's are not too spread; it was also suggested in [3, Problem 3] that this condition could possibly be removed. In Theorem 2, we show that this assumption can indeed be dropped, whenever Ω is convex. By combining a nice idea by Otani [12] with the celebrated Pohožaev identity [13], one can show that this regularity result for (2) is related to nonexistence results for (1) when the exponent p is *critical* with respect to Sobolev inequality. As a consequence, for convex domains Ω we can drop again the assumption on the m_i's in [3]: in Theorem 3 we obtain a nonexistence result for (1) for at least critical exponents p in convex α-starshaped domains Ω (see Definition 2).
This paper is organized as follows. In next section we introduce the basic tools needed to study (1) and (2) and we state our results. The proofs are postponed to the last section.

2 Results

Throughout the paper we assume that Ω is an open bounded domain of $\mathbf{R}^n$. Further, we always require that the exponents p and m_i's appearing in (1) satisfy

$$p > 1 \ , \qquad m_i \geq 2 \qquad \forall i = 1, ..., n \ , \qquad\qquad \sum_{i=1}^{n} \frac{1}{m_i} > 1 \tag{3}$$

(notice that, as a consequence, we necessarily have $n \geq 3$). Set

$$m_- := \min\{m_1, \dots, m_n\} \ , \qquad m_+ := \max\{m_1, \dots, m_n\} \ ,$$

$$m^* = \frac{n}{\sum_{i=1}^{n} \frac{1}{m_i} - 1} \ , \qquad m_\infty = \max\{m_+, m^*\} \ .$$

For every $q \in [1, +\infty]$ we denote by $q' := q/(q-1)$ its conjugate exponent. Let $m = (m_1, ..., m_n)$, and denote by $W_0^{1,m}(\Omega)$ the closure of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,m} = \sum_{i=1}^{n} \|\partial_i u\|_{m_i} \ .$$

When the exponents m_i's are not "too far apart", the critical exponent for the embedding $W_0^{1,m}(\Omega) \subset L^q(\Omega)$ is m^* (which coincides with $n\overline{m}/(n - \overline{m})$, the usual critical exponent for the harmonic mean $\overline{m}$ of the m_i's). On the other hand, if the m_i's are "too much spread out" it coincides with m_+. Therefore, since possibly $m_+ > m^*$, the effective critical exponent is in fact m_∞. In [3] it was shown that existence results for (1) are quite different according to whether m_∞ equals m^* or m_+.
Let us first make clear what we mean by *solution* of (1):

Definition 1. We say that $u \in W_0^{1,m}(\Omega) \cap L^{(p-1)m'_\infty}(\Omega)$ is a *weak* solution of (1) if $u \geq 0$ a.e. in Ω and

$$\sum_{i=1}^{n} \int_\Omega |\partial_i u|^{m_i - 2} \partial_i u \partial_i v = \lambda \int_\Omega u^{p-1} v \qquad \forall v \in W_0^{1,m}(\Omega) \ . \tag{4}$$

If in addition $u \in L^{(p-1)m'_-}(\Omega)$, we say that u is a *mild* solution. If $u \in L^\infty(\Omega)$ we say that u is a *strong* solution.

In [3, Theorem 2] it was proved that every weak solution to (1) is actually a strong solution provided one of the following situations occurs

$$(i)\, p < m_\infty \qquad (ii)\, p = m_\infty \ \text{ and } \ m_\infty > m_+ \ .$$

Here we strengthen such result with the following (note that no regularity is assumed on the boundary):

Theorem 1. *Assume that the exponents m_i's and p satisfy* (3)*, and that one of the above conditions* (*i*) *or* (*ii*) *holds. If u is a weak solution to* (1)*, then, $u \in W^{1,\infty}_{\rm loc}(\Omega)$. Moreover, if Ω is convex, then $u \in W^{1,\infty}(\overline{\Omega})$.*

In a completely different fashion we also prove a regularity result for the coercive problem (2):

Theorem 2. *Assume that Ω is convex with $\partial\Omega \in C^{2,\gamma}$, and that the exponents m_i's and p satisfy* (3)*. Let $\lambda > 0$ and $f \in C^\infty_c(\Omega)$. Then, for all $\varepsilon > 0$, problem* (2) *admits a unique (classical) solution $w \in C^{2,\gamma}(\overline{\Omega})$.*

This statement should be compared with [3, Theorem 5], where the same thesis was obtained on possibly non convex domains under the additional assumption $m_+/m_- < (n+2)/n$, here dropped.
If we keep the convexity assumption on the domain Ω, thanks to Theorem 2 we may also improve the nonexistence result obtained in [3, Theorem 6] for problem (1) in the at least critical case $p \geq m^*$. Such result is stated on α-starshaped domains, according to the definition below applied with

$$\alpha_i = n\left(\frac{1}{m_i} - \frac{1}{m^*}\right) \; ; \tag{5}$$

notice that the above α_i are all strictly positive provided

$$m_+ < m^* \; . \tag{6}$$

Definition 2. Let $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{R}^n$ with $\alpha_i > 0$ for all i. We say that a bounded smooth domain $\Omega \subset \mathbf{R}^n$ is α-starshaped with respect to the origin if

$$\sum_{i=1}^{n} \alpha_i x_i \nu_i \geq 0 \qquad \text{on } \partial\Omega \; , \tag{7}$$

with $\nu = (\nu_1, \ldots, \nu_n)$ denoting the outer normal to $\partial\Omega$. We say that Ω is *strictly* α-starshaped with respect to the origin if (7) holds with strict inequality. If these inequalities hold after replacing x_i by $x_i - P_i$, we say that Ω is (strictly) α-starshaped with respect to the center $P = (P_1, \ldots P_n)$. If Ω is (strictly) α-starshaped with respect to some of its points, we simply say that Ω is (strictly) α-starshaped.

We refer to [3] for some properties of α-starshaped domains. Let us just mention that a convex domain *is not* necessarily α-starshaped and that a simple example of a convex α-starshaped domain (for any α) is a ball.
We can now state our main nonexistence result:

Theorem 3. *Assume that Ω is convex with $\partial\Omega \in C^{2,\gamma}$, and that the exponents m_i's and p satisfy* (3) *and* (6)*. Let $\alpha = (\alpha_1, ..., \alpha_n)$ with α_i as in* (5)*. Assume that either $p > m^*$ and Ω is α-starshaped, or $p = m^*$ and Ω is strictly α-starshaped. Then, for every $\lambda > 0$, the unique mild solution of* (1) *is $u \equiv 0$.*

3 Proofs

We first prove Theorem 2. To this end, we establish a uniform boundary gradient estimate; it can essentially be drawn from [5, Chapter XIV], but for the sake of completeness we enclose a quick proof via a barrier argument.

Lemma 1. *Assume that Ω is convex, and let $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ be a solution to the boundary value problem*

$$\begin{cases} Qu := a_{ij}(Du)D_{ij}u + b(x,u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \ , \end{cases}$$

where the operator Q is elliptic, the function $b = b(x,z)$ is nonincreasing in z, and the following structure condition holds for some nondecreasing function μ on $\mathbf{R}^+$:

$$|b(x,z)| \leq \mu(|z|)a_{ij}(\xi)\xi_i\xi_j \qquad \text{for } |\xi| \geq \mu(|z|) \ . \tag{8}$$

Then there exists a constant $C = C(M, \mu(M))$, with $M = \sup_\Omega |u|$, such that $|Du| \leq C$ on $\partial\Omega$.

Proof. For $x_0 \in \partial\Omega$, let $\mathcal{P} = \mathcal{P}(x_0)$ be a hyperplane with $x_0 \in \mathcal{P} \cap \overline{\Omega} = \mathcal{P} \cap \partial\Omega$, and set $d(x) := \mathrm{dist}(x, \mathcal{P})$. Define the parameters k and a by

$$k := (\mu(M))^2 e^{M\mu(M)} \qquad \text{and} \qquad a := \frac{e^{M\mu(M)} - 1}{k} \ .$$

We claim that the functions

$$w^\pm := \pm\psi(d) = \pm\frac{1}{\mu(M)}\log(1 + kd) \ ,$$

are respectively a so-called *upper* and *lower barrier* at x_0 for the function u and the operator Q on the neighborhood $\mathcal{N}$ of x_0 given by $\mathcal{N} := \{x \in \overline{\Omega} \ : \ d(x) < a\}$. Were this claim proved, the lemma would follow at once. Indeed recall that, by definition of upper and lower barrier, one has

(i) $\pm Qw^\pm < 0$ in $\mathcal{N} \cap \Omega$;

(ii) $w^\pm(x_0) = 0$;

(iii) $w^- \leq u \leq w^+$ on $\partial(\mathcal{N} \cap \Omega)$.

Then the comparison principle ensures that $w^- \le u \le w^+$ in $\mathcal{N} \cap \Omega$; using (ii), it follows $\frac{\partial w}{\partial \nu}^-(x_0) \le \frac{\partial u}{\partial \nu}(x_0) \le \frac{\partial w}{\partial \nu}^+(x_0)$, and so

$$|Du(x_0)| \le \psi'(0) = \mu(M)e^{M\mu(M)} =: C \ .$$

It remains to show that $w^\pm$ are actually barriers. Let us check that w^+ satisfies conditions (i)-(iii) above (the check for w^- being completely analogous). One has

$$\begin{aligned} Qw^+ &= \psi'(d)a_{ij}(Dw^+)D_{ij}d + \psi''(d)a_{ij}(Dw^+)D_i d D_j d + b(x,w^+) \\ &= \frac{\psi''(d)}{(\psi'(d))^2}a_{ij}(Dw^+)D_i w^+ D_j w^+ + b(x,w^+) \ . \end{aligned}$$

Now observe that, for all $x \in \mathcal{N} \cap \Omega$, there holds

$$|Dw^+(x)| = \psi'(d(x)) = \frac{k}{\mu(M)(1+kd(x))} \ge \frac{k}{\mu(M)(1+ka)} = \mu(M) \ge \mu(w^+(x)) \ ,$$

where in the last inequality we have used the fact that μ is nondecreasing and $w^+(x) \le \psi(a) = M$ in $\mathcal{N} \cap \Omega$. Hence, by (8) and using again the monotonicity of μ, we get

$$Qw^+ \le \left\{ \frac{\psi''(d)}{(\psi'(d))^2} + \mu(M) \right\} a_{ij}(Dw^+)D_i w^+ D_j w^+ = 0 \ ,$$

and (i) is proved. Condition (ii) is immediately satisfied. Finally, also (iii) is fulfilled since, for $x \in \partial(\mathcal{N} \cap \Omega)$, there holds $w^+(x) \ge 0 = u(x)$ if $x \in \partial\Omega \cap \mathcal{N}$, and $w^+(x) = M \ge u(x)$ otherwise. □

Another tool needed for the proof of Theorem 2 is the following Leray-Schauder principle (see [7]):

Lemma 2. *Let X be a Banach space and let $T : X \to X$ be a compact operator. Assume that there exists $K > 0$ such that $\|x\|_X \le K$ for all $x \in X$ satisfying $x = \sigma T x$ for some $\sigma \in [0,1]$. Then, T has a fixed point.*

Proof of Theorem 2. It is not restrictive to assume that $\lambda = 1$. Fix $\gamma \in (0,1)$. For all $v \in C^{1,\gamma}(\overline{\Omega})$ define

$$b(x,v) := f(x) - |v|^{p-2}v \ ,$$

$$a_i(x,v) := |\partial_i v|^{m_i-2} + \varepsilon(1 + |Dv|^2)^{(m_- -2)/2} \quad (i = 1,...,n) \ ,$$

and consider the following linear problem

$$\begin{cases} -\sum_{i=1}^n \partial_i \Big(a_i(x,v)\partial_i u \Big) = b(x,v) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \ . \end{cases} \tag{9}$$

By Theorem 6.16 in [5], problem (9) admits a unique solution $u \in C^{2,\gamma}(\overline{\Omega})$. Hence, together with the compact embedding $C^{2,\gamma}(\overline{\Omega}) \subset C^{1,\gamma}(\overline{\Omega})$, (9) defines a compact operator $T : C^{1,\gamma}(\overline{\Omega}) \to C^{1,\gamma}(\overline{\Omega})$ such that $Tv = u =$ unique solution of (9).
Take $\sigma \in [0,1]$ and assume that $u \in C^{1,\gamma}(\overline{\Omega})$ solves $u = \sigma Tu$, namely

$$\begin{cases} -\sum_{i=1}^{n} \partial_i \Big(a_i(x,u)\partial_i u \Big) = \sigma b(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \ ; \end{cases} \tag{10}$$

then, we just said that $u \in C^{2,\gamma}(\overline{\Omega})$.
We want now to prove uniform (w.r.t. σ) boundedness for solutions of (10). Set $k := (\sup|f|)^{1/(p-1)}$ and $\Omega_k := \{x \in \Omega \, : \, |u(x)| > k\}$. By multiplying the equation in (10) with $\varphi = (\operatorname{sign} u) \max\{|u| - k, 0\}$ and integrating over Ω, we have

$$\begin{aligned} \int_{\Omega_k} \sum_{i=1}^{n} |\partial_i u|^{m_i} \ &\leq \int_{\Omega_k} \Big[\sum_{i=1}^{n} |\partial_i u|^{m_i} + \varepsilon (1+|Du|^2)^{(m_- -2)/2} |Du|^2 \Big] \\ &= \sigma \int_{\Omega_k} (|u|-k)[f \cdot (\operatorname{sign} u) - |u|^{p-1}] \leq 0 \ , \end{aligned}$$

which shows that

$$\|u\|_\infty \leq k \qquad \forall \sigma \in [0,1] \ . \tag{11}$$

Now, observe that the boundary value problem (10) satisfies the assumptions of Lemma 1. Indeed, the first equation in (10) can be written under the form

$$a_{ij}(Dv) D_{ij} v + b(x,v) = 0 \ ,$$

where the coefficients a_{ij} satisfy the ellipticity condition $a_{ij}(\xi)\xi_i\xi_j \geq \varepsilon|\xi|^2$, and the structure condition (8) is fulfilled: in fact, for $|\xi| \geq \mu(|z|) := \big\{ (\|f\|_\infty + |z|^{p-1})/\varepsilon \big\}^{1/3}$, there holds

$$\begin{aligned} |b(x,z)| &= \big| f(x) - |z|^{p-2} z \big| \leq \|f\|_\infty + |z|^{p-1} \\ &= \varepsilon \mu^3(|z|) \leq \varepsilon \mu(|z|)|\xi|^2 \leq \mu(|z|) a_{ij}(\xi)\xi_i\xi_j \ . \end{aligned}$$

Then, taking (11) into account, Lemma 1 gives a uniform boundary gradient bound:

$$\exists C > 0 \qquad \text{such that} \qquad |Du| \leq C \quad \text{on } \partial\Omega \qquad \forall \sigma \in [0,1] \ . \tag{12}$$

In turn, together with Theorems 15.6 and 13.2 in [5], (12) gives a uniform $C^{1,\gamma}$-bound, namely

$$\exists C > 0 \qquad \text{such that} \qquad \|u\|_{C^{1,\gamma}(\overline{\Omega})} \leq C \qquad \forall \sigma \in [0,1] \ . \tag{13}$$

This enables us to apply Lemma 2 and ensures the existence of a solution $u \in C^{2,\gamma}(\overline{\Omega})$ of problem (2). On the other hand, a solution of (2) is a critical point of the integral functional

$$J(u) = \int_\Omega \left[j(Du) + \frac{1}{p}|u|^p - fu \right] ,$$

with $j(\xi) := \sum_{i=1}^n \frac{|\xi_i|^{m_i}}{m_i} + \frac{\varepsilon}{m_-}\big(1 + |\xi|^2\big)^{m_-/2}$. Therefore, the uniqueness of the solution follows by the strict convexity of the functional J. □

Proof of Theorem 1. By [3, Theorem 2], we have $u \in L^\infty(\Omega)$. The local Lipschitz continuity follows from Example 4 of [8] with the choices $g_i(t) = t^{m_i-1}$ and $a_i \equiv 1$. (See also Example 7 of [9]).

The global Lipschitz continuity is only a little harder. We follow the argument in [17, Theorem 5] (or [18, Theorem 7.2]) except that those references assume that the solution is continuously differentiable in Ω; in other words, since we already have an interior gradient bound, we must construct a barrier to deduce a global gradient bound. This may be done as in the proof of Theorem 2. Indeed, (8) may be satisfied recalling that u is bounded and that by (3)

$$\delta := \min_{|\xi|\geq 1} \sum_{i=1}^n \frac{|\xi_i|^{m_i}}{m_i} > 0 \ .$$

To deduce the claimed global bound, fix a point $x_0 \in \Omega$ and set $R = \mathrm{dist}(x_0, \partial\Omega)$. It follows from the just mentioned barrier argument that $|u(x)| \leq CR$ in $B(x_0, R)$. Then, the form of the gradient estimate in [8] (with ρ there equal to R here, so the quantity σ/ρ in the last display on page 518 of [8] is bounded by a known constant) implies that Du is bounded in $B(x_0, R/2)$ by a constant independent of R. Hence u is globally Lipschitz. □

Proof of Theorem 3. Since the solution in Theorem 2 is smooth up to the boundary, we may write the corresponding Pohožaev identity [13]. Then, repeating the same steps in the proof of [3, Theorem 6] (a suitable double passage to the limit, inspired by a nice paper of Otani [12]), one obtains Theorem 3. □

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