Stepanov's Theorem in Wiener spaces

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1 Introduction and main results.

One important generalization of Rademacher's theorem due to Stepanov [S], [S1], [F, 3.1.8], states that a function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a.e. $x \in S(f)$ with respect to the Lebesgue measure, where

$$S(f) := \Big\{ x \in \mathbb{R}^n : \operatorname{Lip} f(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|} < \infty \Big\}.$$

The proof of the theorem is traditionally based on the well-known fact that, for every Lebesgue measurable set $A \subset \mathbb{R}^n$, almost every point of A is point of density, that is,

$$\lim_{r \to 0^+} \frac{\mathscr{L}^n(A \cap B(x,r))}{\mathscr{L}^n(B(x,r))} = 1 \quad \text{for almost all } x \in A.$$

This result has found many applications in geometric measure theory [F] and in the differentiability of quasiconformal and quasiregular mappings [V].

One generalization of this result has been given in [BRZ], when the domain X is a metric measure space. In this case, the authors deal with metric spaces endowed with a doubling measure and supporting Poincaré-type inequalities. Both conditions ensure the existence of a strong measurable differentiable structure on the space X which provides a differentiation theory for Lipschitz functions in this setting [Ch]. It turns out that for doubling measures, Lebesgue differentiation theorem still holds (see for instance [H, Thm. 1.8]), and so, it is an useful tool when proving Stepanov's theorem as in the classical setting.

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In [Bo], [D] Stepanov's theorem has been generalized to mappings between Banach spaces, when the domain is separable, using the concept of *Aronszajn null* sets. Moreover, the proof of Stepanov's theorem for separable Banach spaces domains is based on Aronszajn's theorem on differentiability of Lipschitz functions, applied to the distance function from the set; this concept is used as a replacement of the density theorem, because no longer a measure is needed to define negligible sets.

Our aim is to obtain an Stepanov-type theorem in the context of abstract Wiener spaces. Recall that an *abstract Wiener space* $(E, \mathcal{H}, \gamma)$ consists of a separable Banach space E, a gaussian measure γ on E and the Cameron Martin space $\mathcal{H} = \mathcal{H}(\gamma)$ associated to (E, γ). A *Gaussian measure* γ on E equipped with its Borel $\sigma-$algebra $\mathscr{B}$ is a probability measure on $(E, \mathscr{B})$ such that the law (pushforward measure) of each continuous linear functional on E is Gaussian, that is, $\gamma \circ (e^*)^{-1}$ is a Gaussian measure on $\mathbb{R}$ for each $e^* \in E^* \setminus \{0\}$, possibly a Dirac mass. If we assume, as we shall do, that γ is not supported in a proper subspace of E, then all such measures are Gaussian measures with positive variance. The Cameron Martin space $\mathcal{H} = \mathcal{H}(\gamma)$ is a separable Hilbert space, whose norm is denoted by $\| \cdot \|_{\mathcal{H}}$, compactly and densely embedded into E and uniquely determined by (E, γ). Whenever E is infinite-dimensional, $\mathcal{H}$ is much smaller than E, since $\gamma(H) = 0$. The precise characterization of $\mathcal{H}$ and its norm are given by the Cameron-Martin theorem (see [B, 2.4.3]): if $v \in E$ and $T_v\gamma(B) = \gamma(B + v)$ is the shifted measure, then $T_v\gamma \ll \gamma$ if and only if $v \in \mathcal{H}$; in addition $\|v\|_{\mathcal{H}}$ can be computed in terms of the densities of $T_{th}\gamma$ with respect to γ, for $t \in \mathbb{R}$. For a complete description of abstract Wiener spaces we refer the reader to [B].

D. Preiss proved in [P] that the density theorem for gaussian measures is no longer true, at least if balls for the norm of E are involved; on the other hand, these balls are not natural in the differential calculus (Sobolev and BV functions, integration by parts, etc.) in Wiener spaces, that involves only directions in $\mathcal{H}$. For these reasons, we use $\mathcal{H}-$Gâteaux differentiability (i.e. Gâteaux differentiability, along directions in $\mathcal{H}$) of $\mathcal{H}-$distance functions, in the same spirit of [Bo],[D].

The following definition arises naturally in this context.

Definition 1 Let $(E, \mathcal{H}, \gamma)$ be an abstract Wiener space. We say that a function $f : E \to \mathbb{R}$ is $\mathcal{H}-$ *pointwise Lipschitz* at x if

$$\mathrm{Lip}_{\mathcal{H}} f(x) := \limsup_{\|h\|_{\mathcal{H}} \to 0} \frac{|f(x+h) - f(x)|}{\|h\|_{\mathcal{H}}} < \infty.$$

Let us denote

$$\begin{aligned} S(f) :=& \big\{x \in E : f \text{ is } \mathcal{H}- \text{ pointwise Lipschitz at } x\big\} \\ =& \big\{x \in E : \exists\, \delta_x > 0, \exists\, C_x \geq 0 \text{ s.t. } |f(x+h) - f(x)| \leq C_x \|h\|_{\mathcal{H}},\ \forall h \in \mathcal{H},\ \|h\|_{\mathcal{H}} < \delta_x\big\}. \end{aligned}$$

We will use an useful decomposition of the set $S(f)$ as follows. For each natural number $m \in \mathbb{N}$ consider the set

$$A_m := \big\{x \in E : |f(x+h) - f(x)| \leq m\|h\|_{\mathcal{H}}, \text{ for all } h \in \mathcal{H} \text{ with } \|h\|_{\mathcal{H}} < \frac{1}{m}\big\}. \tag{1}$$

Then we have that $S(f) = \bigcup_{m \in \mathbb{N}} A_m$ and $A_n \subset A_m$ if $n \leq m$.

Proposition 2 *Let $f : E \to \mathbb{R}$ be a Borel function. Then the sets A_m in (1) are γ-measurable. In particular the set $S(f)$ is γ-measurable.*

Proof. We claim that the vector space

$$\{g : E \to \mathbb{R} : g(x+h) - g(x) \text{ is Borel in } E \times \mathcal{H}\}$$

contains all Borel functions: indeed, it contains bounded continuous functions and it is stable under monotone equibounded limits. Hence, it contains all Borel functions.

The claim implies that the set

$$\Lambda_m := \left\{ (x,h) \in E \times \mathcal{H} : 0 < \|h\|_{\mathcal{H}} < \frac{1}{m}, \ \frac{|f(x+h) - f(x)|}{\|h\|_{\mathcal{H}}} \geq m \right\}$$

is Borel in $E \times \mathcal{H}$. By [F, 2.2.10] we obtain that Λ_m is Suslin. Since $E \setminus A_m$ is the projection of Λ_m on the first variable, and since Suslin sets are stable under continuous projections, we get that $E \setminus A_m$ is Suslin. By [F, 2.2.12] we conclude that $E \setminus A_m$, and hence A_m, is γ-measurable.
□

Note that $\mathcal{H}-$Lipschitzian functions are $\mathcal{H}-$ pointwise Lipschitz $\gamma-$a.e. Recall that a Borel mapping $f : E \to \mathbb{R}$ is said to be *$\mathcal{H}-$Lipschitzian at x* with constant C if

$$|f(x+h) - f(x)| \leq C\|h\|_{\mathcal{H}} \qquad \forall h \in \mathcal{H},$$

and that f is $\mathcal{H}-$Lipschitzian with constant C if f is $\mathcal{H}-$Lipschitzian at x with constant C for $\gamma-$a.e. x.

We state in the next theorem two properties of $\mathcal{H}-$Lipschitzian functions; the first one corresponds, in this context, to Rademacher's theorem.

Theorem 3 [ES],[B, 5.11.8] *Let $f : E \to \mathbb{R}$ be $\mathcal{H}$-Lipschitzian. Then*

(*i*) *there exists a Borel $\gamma-$negligible set $N \subset E$ such that, for all $x \in E \setminus N$, the map $h \mapsto f(x+h)$ is Gâteaux differentiable at 0;*

(*ii*) *there exists a Borel modification $\tilde{f}$ of f in a $\gamma-$negligible set which is $\mathcal{H}-$Lipschitzian at all $x \in E$.*

Now we state the main result of the paper, a generalization of Stepanov's theorem in the context of abstract Wiener spaces.

Theorem 4 (Stepanov's theorem) *Let (E, H, γ) be an abstract Wiener space and let $f : E \to \mathbb{R}$ be a Borel function. Then the Gâteaux derivative exists $\gamma-$a.e. in $S(f)$.*

Before proving the main theorem we need to introduce some terminology and preliminary results.

First of all, we recall the definition of $\mathcal{H}-$*distance function* which will be useful in the following: for $K \subset E$ Borel we define

$$d_{\mathcal{H}}(x, K) := \begin{cases} \inf\{\|h\|_{\mathcal{H}} : \ x + h \in K, \, h \in \mathcal{H}\} & \text{if } (x + \mathcal{H}) \cap K \neq \emptyset \\ \infty & \text{otherwise,} \end{cases}$$

which is easily seen to be $\mathcal{H}-$Lipschitzian with constant 1, i.e.

$$d_{\mathcal{H}}(x + h, K) \leq d_{\mathcal{H}}(x, K) + \|h\|_{\mathcal{H}} \qquad \forall x \in X, \ h \in \mathcal{H}.$$

The measurability of $d_{\mathcal{H}}(\cdot, K)$ is proved in [B, 5.4.10]. Even though the Cameron-Martin space is small in X, we will see that $d_{\mathcal{H}}(\cdot, B)$ is finite $\gamma-$a.e. x, provided $\gamma(K) > 0$; if this happens, then Theorem 3 yields that $d_{\mathcal{H}}(\cdot, K)$ is $\mathcal{H}-$Gâteaux differentiable $\gamma-$a.e.

In order to explain why $d_{\mathcal{H}}(\cdot, K)$ is finite γ-a.e. we briefly recall some basic results on the Monge-Kantorovich problem in abstract Wiener spaces, with a singular quadratic cost. Let $\mathcal{P}(E)$ denote the space of probability measures on E. We define a cost function $c : E \times E \to \mathbb{R}_+ \cup \{+\infty\}$ by $c(x, y) = |x - y|^2_{\mathcal{H}}$ if $x - y \in \mathcal{H}$ and $c(x, y) = +\infty$ if $x - y \notin \mathcal{H}$. Observe that, by the continuity of the embedding of $\mathcal{H}$ in E, the function c is lower semicontinuous in $E \times E$.

Monge problem of quadratic cost on E is the following: given μ, $\nu \in \mathcal{P}(E)$, consider the minimization problem

$$\inf_{T_\sharp \mu = \nu} \int_E c(x, T(x)) \, d\mu(x).$$

Here the infimum is taken over all Borel maps $T : E \to E$, called transports of μ to ν, such that the push-forward $T_\sharp \mu$ of μ by T coincides with ν, i.e. $\nu(A) = \mu(T^{-1}(A))$ for all Borel subsets A of E. This problem has a nice interpretation: if we interpret $c(x, y)$ as the cost of moving a unit mass from x to y, then the above minimization problem simply consists in minimizing the total cost (work) by optimizing the destination $T(x)$ for each x.

The *Kantorovich problem* is a kind of relaxation of Monge's problem: it consists in finding a probability measure on $E \times E$ which minimizes the function

$$\int_{E \times E} c(x, y) \, d\beta(x, y)$$

among all probability measures β in $E \times E$ whose first and second marginal are respectively μ and ν, i.e., $\mu(A) = \beta(A \times E)$ and $\nu(A) = \beta(E \times A)$ for all Borel sets $A \subset E$. Since c is lower semicontinuous it is not hard to show that the infimum is attained, and its value shall be denoted by $W_2^2(\mu, \nu)$ (the square of the so-called quadratic optimal transportation distance). In [FeU, 3.1,4.1], passing to the limit in the celebrated Talagrand's optimal transportation inequality in finite-dimensional Gaussian spaces [T], the following estimate is proved:

$$W_2^2(\gamma, \rho\gamma) \leq 2 \int \rho \ln \rho \, d\gamma. \tag{2}$$

Here ρ is any nonnegative function in $L^1(\gamma)$ with $\int \rho\, d\gamma = 1$ and $\rho\gamma \in \mathcal{P}(E)$ is defined by $\rho\gamma(A) = \int_A \rho d\gamma$ for all Borel subsets A of E. In addition, the authors show that the optimal β in Kantorovich problem corresponds to a map T, and therefore also Monge's problem has a solution, but we won't need this fact.

Now, we are in a position to prove that $d_{\mathcal{H}}(\cdot, K)$ is finite $\gamma-$a.e. in E.

Lemma 5 *Let K be a Borel set in E such that $\gamma(K) > 0$. Then $d_{\mathcal{H}}(\cdot, K)$ is finite $\gamma-$a.e. in E. In addition, $\nabla_{\mathcal{H}} d_{\mathcal{H}}(\cdot, K) = 0$ γ-a.e. on K.*

Proof. We have to prove that for $\gamma-$a.e. $x \in E$ there exists $h \in \mathcal{H}$ such that $x + h \in K$. Let $\rho = \frac{\chi_K}{\gamma(K)} \in L^1(\gamma)$. Observe that $\int_E \rho\, d\gamma = 1$ and that

$$\int_E \rho \ln \rho\, d\gamma = \frac{1}{\gamma(K)} \int_E \chi_K \ln \frac{\chi_K}{\gamma(K)} d\gamma = \ln \frac{1}{\gamma(K)} < \infty.$$

Now, applying the estimation (2) we know that there exists $\beta \in \mathcal{P}(E \times E)$ having γ as first marginal and $\rho\gamma$ as second marginal such that $\int c\, d\beta$ is finite. We have $y - x \in \mathcal{H}$ $\beta-$a.e., by the finiteness of the integral, and $y \in K$ $\beta-$a.e., since the second marginal of β is $\rho\gamma$. It follows that $d_{\mathcal{H}}(\cdot, K)$ is finite $\beta-$a.e., and using the fact that γ is the first marginal we obtain the finiteness $\gamma-$a.e. of $d_{\mathcal{H}}(\cdot, K)$. □

In the sequel we denote by $j : E^* \to E$ the map

$$j(e^*) := \int_E \langle e^*, x\rangle x\, d\gamma(x).$$

It is well-known that this map takes its values in $\mathcal{H}$ and its image is a dense subspace of $\mathcal{H}$.

Now, we have all the ingredients to prove Theorem 4.

Proof of Theorem 4. The strategy of the proof is the following. First, we show that for a fixed direction $h \in j(E^*)$, f is differentiable along h $\gamma-$a.e. in $S(f)$. Second, we fix a sequence $(e_i^*) \subset E^*$ and consider the $\mathbb{Q}$-subspaces $\mathcal{H}_n'$ spanned on $\mathbb{Q}$ by $j(e_1^*), \ldots, j(e_n^*)$, and also $\mathcal{H}_n$ the corresponding $\mathbb{R}$-subspaces; obviously we can choose (e_i^*) in such a way that $(j(e_i^*))$ are an orthonormal basis of $\mathcal{H}$. We prove in Step 2 that for $\gamma-$a.e. x the derivative along directions in $\mathcal{H}_n$ exists and is $\mathbb{Q}$-linear. Once this is done, we construct in Step 3 a continuous linear functional $\nabla f(x) : \cup_n \mathcal{H}_n' \to \mathbb{R}$ for $\gamma-$a.e. $x \in S(f)$. Finally, we show in Step 4 that ∇f is the $\mathcal{H}-$Gateâux derivative of f at $\gamma-$a.e. point of $S(f)$.

Step 1. We show that, for a fixed direction $h \in j(E^*)$, f is differentiable along h $\gamma-$a.e. in $S(f)$. We have to prove that the set

$$N_h := \big\{x \in S(f) : f \text{ is not differentiable along } h \text{ at } x\big\},$$

is $\gamma-$negligible. Observe first that the set N_h is $\gamma-$measurable. Indeed, for each $t \in \mathbb{R}$, define the function

$$F_t(x) = \frac{f(x + th) - f(x)}{t}.$$

Following the proof of Proposition 2, one can check that F_t is $\gamma-$measurable, and therefore the functions

$$x \mapsto \liminf_{t\to 0} F_t(x) \quad \text{and} \quad x \mapsto \limsup_{t\to 0} F_t(x)$$

are also $\gamma-$measurable. To finish, notice that

$$N_h = \{x \in S(f) : \limsup_{t\to 0} F_t(x) > \liminf_{t\to 0} F_t(x)\}.$$

Now, let $x \in E$ and set $D_x = \{t \in \mathbb{R} : x + th \in S(f)\}$. The function $f_x : t \to f(x + th)$ satisfies $\operatorname{Lip} f_x(t) \le \operatorname{Lip}_{\mathcal{H}} f(x + th) < \infty$ in D_x and so, applying the 1-dimensional Stepanov's theorem ([F, 3.1.8]) we obtain that f_x is differentiable at $\mathscr{L}^1-$a.e. $t \in D_x$, that is, $\mathscr{L}^1(\{t \in \mathbb{R} : x + th \in N_h\}) = 0$ for all $x \in E$. By [AD, Lem. 1] (or (3) with $B = N_h$ in Step 2 below), we obtain that $\gamma(N_h) = 0$.

Step 2. We fix an integer $n \ge 1$ and prove that the set

$$S^n := \left\{x \in S(f) : \partial_h f(x) \text{ exists for all } h \in \mathcal{H}'_n \text{ and is } \mathbb{Q}\text{-linear in } \mathcal{H}'_n\right\}$$

has full measure in $S(f)$, i.e. $\gamma(S(f) \setminus S^n) = 0$. Notice that, by Step 1, it is easily seen that S^n is γ-measurable and that for γ-a.e. $x \in S(f)$, $\partial_h f(x)$ exists for all $h \in \mathcal{H}'_n$. Therefore, we need only to check the $\mathbb{Q}$-linearity property, that is, that for γ-a.e. $x \in S(f)$ the map $\mathcal{H}'_n \to \mathbb{R}$, $h \mapsto \partial_h f(x)$ is linear over the field of rationals.

We denote by $\pi : E \to \mathcal{H}_n$ the canonical projection map

$$\pi(x) := \sum_{i=1}^{n} \langle e_i^*, x\rangle j(e_i^*)$$

and by E_n the kernel of π, so that $E = E_n \bigoplus \mathcal{H}_n$ algebraically. Using the process of decomposing a gaussian measure (see for example [B, 3.10.2]), for all $\gamma-$measurable sets B one has

$$\gamma(B) = \int_{E_n} \mathscr{L}^n(B_y)\, d\nu(y), \tag{3}$$

where ν is the image of γ under π and

$$B_y := \left\{ z \in \mathbb{R}^n : \ y + \sum_{i=1}^{n} z_i\, j(e_i^*) \in B \right\}.$$

Fix $y \in E_n$. Let us observe that $f_{n,y}(z) := f(y + \sum_i z_i\, j(e_i^*))$ is pointwise Lipschitz on $(S(f))_y$, since $\operatorname{Lip} f_{n,y}(z) \le \operatorname{Lip}_{\mathcal{H}} f(y + \sum_i z_i j(e_i^*))$. Applying the n-dimensional Stepanov's theorem ([F, 3.1.8]) we obtain that $f_{n,y}$ is differentiable $\mathscr{L}^n-$a.e. in $(S(f))_y$, so that f is Gateâux differentiable along directions in $\mathcal{H}_n$ at $y + \sum_i z_i\, j(e_i^*)$ for $\mathscr{L}^n$-a.e. $z \in (S(f))_y$. Since y is arbitrary we conclude from (3) with $B = S(f) \setminus S^n$ the claimed property of S^n.

Step 3. We construct for $x \in \cap_n S^n$ a continuous linear functional $\nabla f(x) : \mathcal{H} \to \mathbb{R}$ with norm less than $\operatorname{Lip} f(x)$ such that $\langle \nabla f(x), h\rangle = \partial_h f(x)$. Indeed, Step 2 provides us with a continuous $\mathbb{Q}$-linear functional $\nabla f(x) : \cup_n \mathcal{H}'_n \to \mathbb{R}$ with norm less than $\operatorname{Lip} f(x)$, and by density we have a unique continuous linear extension.

Step 4. We prove that $\nabla f(x)$ is $\gamma-$a.e. the $\mathcal{H}-$Gateâux derivative of f. Now, given $\varepsilon > 0$, fix $m \in \mathbb{N}$ sufficiently large, so that $\gamma(S(f) \setminus A_m) < \varepsilon$ and set $K = A_m$. By the definition of A_m we have

$$\|f(x+h) - f(x)\| \leq m\|h\|_{\mathcal{H}} \quad \forall x \in K\,, h \in \mathcal{H} \text{ with } \|h\|_{\mathcal{H}} < \frac{1}{m}. \tag{4}$$

Now we consider a point $x_0 \in \cap_n S^n$ where $d_{\mathcal{H}}(\cdot, K)$ has null Gateâux $\mathcal{H}$-derivative and prove that f is Gateâux $\mathcal{H}$-differentiable at x_0. Since (by Step 2) $\gamma-$a.e. $x_0 \in K$ has this property and $\gamma(S(f) \setminus K) < \varepsilon$, this will conclude the proof.

Fix $h \in \mathcal{H}$, $\delta > 0$ and find $h_k \in \cup_n \mathcal{H}'_n$ with $\|h - h_k\|_{\mathcal{H}} < \delta$. We can find $\bar{t} > 0$ (depending only on h and δ) such that $\delta \bar{t} < 1/(2m)$ and

$$d_{\mathcal{H}}(x_0 + th, K) < \delta |t|, \qquad \forall t \in (-\bar{t}, \bar{t}). \tag{5}$$

This means that for $t \in (-\bar{t}, \bar{t})$ we can write $x_0 + th = x + u$ with $x \in K$ and $\|u\|_{\mathcal{H}} < \delta|t|$; we can apply (4) to obtain

$$|f(x_0 + th) - f(x)| = |f(x+u) - f(x)| \leq m\delta|t|$$

and analogously

$$|f(x_0 + th_k) - f(x)| = |f(x + u + t(h_k - h)) - f(x)| \leq m(\delta + \|h - h_k\|_{\mathcal{H}})|t| \leq 2m\delta|t|.$$

It follows that $|f(x_0 + th) - f(x_0 + th_k)| \leq 3m\delta|t|$ for $|t| < \bar{t}$, hence

$$\limsup_{t \to 0} \left\| \frac{f(x_0 + th) - f(x_0)}{t} - \nabla f(x_0)(h) \right\|$$

can be estimated from above with

$$\limsup_{t \to 0} \left\| \frac{f(x_0 + th_k) - f(x_0)}{t} - \nabla f(x_0)(h_k) \right\| + (3m + \operatorname{Lip}_{\mathcal{H}} f(x_0))\delta.$$

Since $x_0 \in \cap_n S^n$, the $\limsup$ above is null. Since δ is arbitrary we conclude. □

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