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ON THE HOMOGENIZATION OF SOME NON-COERCIVE HAMILTON–JACOBI–ISAACS EQUATIONS

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Abstract. We study the homogenization of Hamilton-Jacobi equations with oscillating initial data and non-coercive Hamiltonian, mostly of the Bellman-Isaacs form arising in optimal control and differential games. We describe classes of equations for which pointwise homogenization fails for some data. We prove locally uniform homogenization for various Hamiltonians with some partial coercivity and some related restrictions on the oscillating variables, mostly motivated by the applications to differential games, in particular of pursuit-evasion type. The effective initial data are computed under some assumptions of asymptotic controllability of the underlying control system with two competing players.

1. **Introduction.**

We are concerned with the homogenization of Hamilton-Jacobi equations with oscillating initial data

$$\begin{cases} \partial_t u^\varepsilon + H(z, \frac{z}{\varepsilon}, D_z u^\varepsilon) = 0 & \text{in } (0,\infty)\times\mathbb{R}^N \\ u^\varepsilon(0,z) = h(z, \frac{z}{\varepsilon}) & \text{in } \mathbb{R}^N \end{cases} \tag{1}$$

with Hamiltonian H and initial data h at least continuous and $\mathbb{Z}^N$-periodic in the second entry where the oscillating variables appear. The goal is finding an effective Hamiltonian $\bar{H} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and effective initial data $\bar{h} : \mathbb{R}^N \to \mathbb{R}$ such that u^ε converges (locally uniformly) as $\varepsilon \to 0$ to the solution of

$$\begin{cases} \partial_t u^\varepsilon + \bar{H}(z, D_z u^\varepsilon) = 0 & \text{in } (0,\infty)\times\mathbb{R}^N \\ u^\varepsilon(0,z) = \bar{h}(z) & \text{in } \mathbb{R}^N. \end{cases} \tag{2}$$

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The problem was studied by several authors for initial data $h = h(z)$ independent of $\frac{z}{\varepsilon}$ and with coercive Hamiltonian, that is, for all $z \in \mathbb{R}^N$,

$$\lim_{|p|\to+\infty} H(z,\zeta,p) = +\infty \quad \text{uniformly in } \zeta \in \mathbb{R}^N. \tag{3}$$

The construction of the effective Hamiltonian $\bar{H}$ by tools of ergodic control and a convergence theorem were first proved in the pioneering paper of Lions, Papanicolaou and Varadhan [31]. A variational proof was given by E [22] for H convex in $p = Du$, and Evans [23] introduced the general approach called Perturbed Test Function Method. Estimates of the rate of convergence were proved in [17] (see also [32]), Hamiltonians with discontinuous dependence on z were studied in [16] and with terms $u^\varepsilon/\varepsilon$ in [29], iterated homogenization in [7]. We refer to [1] for the numerical methods and to [24, 26] for the connections with the weak KAM theory, see also the references therein. Under the coercivity assumption (3) the theory was also extended to oscillations more general than periodic, such as quasi-periodic [15, 30] and stationary ergodic [35, 33, 21, 39], and to domains with periodic holes [28, 2].

Much less is known when the coercivity condition (3) fails and the initial data depend on some oscillating variables, $h = h(z, \frac{z}{\varepsilon})$. Homogenization for non-coercive Hamiltonians with special structures was studied in [8, 14, 36, 27, 4, 5, 11, 18, 20]. For initial data with oscillations and general degenerate parabolic equations, Alvarez and the first author [3] reduced the construction of the effective initial data $\bar{h}$ to a stabilization problem and treated the boundary layer at $t = 0$; applications were made in [6, 5].

The development of a sufficiently general theory of homogenization for non-coercive Hamiltonians faces the obstruction of several counterexamples to pointwise convergence. For the basic equation of front propagation

$$\partial_t u^\varepsilon + g\left(\frac{z}{\varepsilon}\right)|D_z u^\varepsilon| = 0$$

it was observed in [13, 19] that there is no such convergence as soon as g changes sign. In [5] it was shawn that when H depends also on z, besides $\frac{z}{\varepsilon}$, $\bar{H}$ may have a less regular dependence on z that prevents convergence at some points and uniqueness for (2). Simple examples are also the equations

$$\partial_t u^\varepsilon + |u^\varepsilon_x| - |u^\varepsilon_y| = \cos\left(2\pi\frac{x-y}{\varepsilon}\right) \qquad \text{in } (0,\infty)\times\mathbb{R}^2, \tag{4}$$

and

$$\partial_t u^\varepsilon + |u^\varepsilon_x + u^\varepsilon_y| = \cos\left(2\pi\frac{x-y}{\varepsilon}\right) \qquad \text{in } (0,\infty)\times\mathbb{R}^2, \tag{5}$$

that are both solved by $u^\varepsilon(t,x,y) = t\cos(2\pi\frac{x-y}{\varepsilon})$ that does not converge at any point off the line $x = y$. The same two equations with null right hand side do not homogenize the initial data $h(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}) = \cos\left(2\pi\frac{x-y}{\varepsilon}\right)$, because this function is a stationary solution. Note that in (4) the Hamiltonian goes to $+\infty$ in some directions and to $-\infty$ in others, whereas in (5) it is bounded below and fails to be coercive just in one direction. In Section 3 we describe two classes of equations modeled on these examples for which homogenization fails for at least one choice of smooth and periodic forcing term or initial data.

In this paper we focus our attention on the Hamiltonians of Bellman-Isaacs type that arise in deterministic optimal control and in the theory of two-person, zero-sum differential games, namely

$$H(z,\zeta,p)=\min_{b\in B}\max_{a\in A}\{-p\cdot f(z,\zeta,a,b)-l(z,\zeta,a,b)\} \tag{6}$$

under standard boundedness and continuity assumptions on f and l and without condition (3). The game-theoretic interpretation of our problem is presented in Section 2, together with some preliminary results from earlier papers. Our model problem is the convex–concave eikonal-type equation

$$\begin{cases}\partial_t u^\varepsilon + g_1\left(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)|D_x u^\varepsilon| - g_2\left(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)|D_y u^\varepsilon| = l\left(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,+\infty)\times\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}, \\ u^\varepsilon(0,x,y)=h\left(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right), \qquad\qquad\qquad \text{in } \mathbb{R}^{N_1}\times\mathbb{R}^{N_2}.\end{cases} \tag{7}$$

In view of the negative examples mentioned before we look for special forms of oscillations under partial coercivity conditions. Some of our main results concern problems where the oscillating terms involve only $\frac{x-y}{\varepsilon}$, motivated by games of pursuit-evasion type where the costs depend only on the distance between some coordinates of the two players. In other results we make assumptions of decoupling of the vector field f of the form $f(z,\zeta,a,b)=(f_1(z,\zeta,a),f_2(z,\zeta,b))$, that are also common for differential games.

In Section 4 we study problems with oscillations only on a subspace and coercivity in the directions of the subspace. This is related to the work of Barles [11] and the recent paper of Viterbo [38] on symplectic methods in homogenization. We give some applications to control and game problems, among which (7) with

$$N_1=N_2=\frac{N}{2},\ g_1,g_2,l,h \text{ depending only on } \left(x,y,\frac{x-y}{\varepsilon}\right),\ g_1>g_2 \text{ or } g_1<g_2, \tag{8}$$

and this result is sharp if $g_2=\gamma g_1$ for some constant γ.

In Section 5 we study the stabilization property of the Hamiltonian that allows to define the effective initial data $\bar{h}$. This is done by developing a notion of asymptotic controllability of an underlying control system with two players introduced in [9], and studying it by game theoretic ideas combined with PDE-viscosity methods. A byproduct is the homogenization for the model problem (7) when the assumption that $h=h(x,y,\frac{x-y}{\varepsilon})$ in (8) is replaced by the existence of a *saddle* h_s such that

$$h_s(x,y)=\max_{\eta\in\mathbb{R}^{N_2}}\min_{\xi\in\mathbb{R}^{N_1}} h(x,y,\xi,\eta)=\min_{\xi\in\mathbb{R}^{N_1}}\max_{\eta\in\mathbb{R}^{N_2}} h(x,y,\xi,\eta).$$

Note that these two assumptions on h are somehow opposite, because a function of $\xi-\eta$ has a saddle in (ξ,η) only if it is a constant.

In Section 6 we consider equation with decoupled fast variables

$$\partial_t u^\varepsilon + H_1(x,y,\frac{x}{\varepsilon},D_x u^\varepsilon) - H_2(x,y,\frac{y}{\varepsilon},D_y u^\varepsilon)=0 \quad \text{for } t>0,\ (x,y)\in\mathbb{R}^{N_1}\times\mathbb{R}^{N_2}$$

so that the coupling of the fast components takes place only via the initial data. If H_1 and H_2 are coercive we prove homogenization for h that either has a saddle or is of the form $h=h_1(x,y,\frac{x-y}{\varepsilon})+h_2(x,y,\frac{y}{\varepsilon})$.

Finally, in Section 7, for a two-dimensional example we compute explicitly the effective Hamiltonian and find a differential game whose value function is the solution of the effective Cauchy problem. This can be called effective differential game. Here we observe the unexpected phenomenon of a cost independent of the controls in the ε-problem that produces a control-dependent effective cost in the limit.

Some results of this paper appeared in the second author's Ph.D. thesis [37].

2. **Setting and preliminary results.** In this section we describe the problem, list the basic assumptions and recall some known results.

2.1. **The problem and the game model.** We consider the Cauchy problem (1) where $H:\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ is at least continuous, $h:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ is bounded uniformly continuous, and both are $\mathbb{Z}^N$–periodic with respect to the oscillating variable $\zeta=\frac{z}{\varepsilon}$. We are concerned with the limit of the solution u^ε of (1) as $\varepsilon\to 0$. We assume the existence and uniqueness of a viscosity solution u^ε: sufficient conditions for this are well-known, for instance in the case of Hamiltonian H of Isaacs type (6), see [10] and the references therein. Our basic assumptions on Hamilton-Jacobi-Isaacs equations are the following.

$$\begin{cases} A \text{ and } B \text{ are given compact subsets of metric spaces;}\\ \text{the functions } f \text{ and } l \text{ are bounded uniformly continuous}\\ \text{in } \mathbb{R}^N\times\mathbb{R}^N\times A\times B, \text{ with values respectively in } \mathbb{R}^N \text{ and } \mathbb{R};\\ f \text{ is Lipschitz–continuous in } (z,\zeta), \text{ uniformly with respect to } (a,b);\\ \text{the data } f,l,h \text{ are } \mathbb{Z}^N\text{–periodic with respect to the variable } \zeta, \text{ i.e.}\\ \text{for any } z\in\mathbb{R}^N,\ (a,b)\in A\times B,\ \phi=f,l,h \text{ satisfies}\\ \phi(z,\zeta,a,b)=\phi(z,\zeta+k,a,b) \text{ for any } \zeta\in\mathbb{R}^N,\ k\in\mathbb{Z}^N. \end{cases} \tag{9}$$

Lemma 2.1. *Let H be of Isaacs type (6) (9). Then H is coercive if and only if, for any $z,\overline{p}\in\mathbb{R}^N$, there exist $\nu>0$ and $C\geq 0$ such that*

$$H(z,\zeta,p+\overline{p})\geq \nu|p|-C(1+|\overline{p}|),\ \textit{for all } \zeta,p\in\mathbb{R}^N. \tag{10}$$

Proof. Inequality (10) readily implies (3). The viceversa is a consequence of the homogeneity property of H. In fact, for any $z,\zeta\in\mathbb{R}^N$ we have

$$\min_{b\in B}\max_{a\in A}\{-p\cdot f(z,\zeta,a,b)-l(z,\zeta,a,b)\}\leq |p|\min_{b\in B}\max_{a\in A}\left\{-\frac{p}{|p|}\cdot f(z,\zeta,a,b)\right\}+\sup(-l).$$

The coercivity (3) gives

$$\nu(z,q):=\inf_{\zeta\in\mathbb{R}^N}\min_{b\in B}\max_{a\in A}\{-q\cdot f(z,\zeta,a,b)\}>0$$

for any $z,q\in\mathbb{R}^N$, $|q|=1$. By setting $\nu=\nu(z):=\min\{\nu(z,q)\,:\,|q|=1\}>0$, we get

$$\min_{b\in B}\max_{a\in A}\{-(p+\overline{p})\cdot f(z,\zeta,a,b)-l(z,\zeta,a,b)\}\geq \nu|p+\overline{p}|+\inf(-l),\qquad \text{for all } \zeta,p\in\mathbb{R}^N.$$

Thus we obtain (10) with $C:=\max\{\nu,\sup l\}$. ☐

The Cauchy problem (1) with the Isaacs Hamiltonian (6) arises in the theory of differential games [25], as we describe next.

Consider the sets $\mathcal{A}$ and $\mathcal{B}$ of measurable functions defined on $(0,+\infty)$ with values in A and B respectively. In the game theory terminology, such functions are the (open-loop) *controls* associated to two players. The state of the system is governed by the ODE

$$\begin{cases}\dot{z}_t=f(z_t,\frac{z_t}{\varepsilon},a_t,b_t)\\ z_0=z,\end{cases} \tag{11}$$

and depends on $a \in \mathcal{A}$, $b \in \mathcal{B}$. Next define, for a time-horizon $t > 0$, initial state $z \in \mathbb{R}^N$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$, the cost-payoff functional of the game with running cost l and terminal cost h

$$J(t, z, a, b) := \int_0^t l\left(z_s, \frac{z_s}{\epsilon}, a_s, b_s\right) ds + h\left(z_t, \frac{z_t}{\varepsilon}\right).$$

This is the functional the first player wants to minimize and the second wants to maximize.

A *strategy* for the first player (respectively, for the second player) is a map $\alpha : \mathcal{A} \to \mathcal{B}$ (respectively, $\beta : \mathcal{B} \to \mathcal{A}$). A strategy α for the first player it is said *nonanticipating* if for any $t > 0$ and any $b, b' \in \mathcal{B}$, $b_s = b'_s$ for any $0 \le s \le t$ implies $\alpha[b]_s = \alpha[b']_s$ for any $0 \le s \le t$, and the definition for the second player is symmetric. We denote

$$\Gamma := \{\alpha : \mathcal{A} \to \mathcal{B} \text{ nonanticipating strategy for the first player}\}$$

$$\Delta := \{\beta : \mathcal{B} \to \mathcal{A} \text{ nonanticipating strategy for the second player}\}$$

The *lower value function* of the zero-sum differential game just described is

$$u^\varepsilon(t, z) := \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} J(t, z, \alpha[b], b),$$

where z_s obeys (11) with $a = \alpha[b]$. Symmetrically, the *upper value function* is

$$v^\varepsilon(t, z) := \sup_{\beta\in\Delta} \inf_{a\in\mathcal{A}} J(t, z, a, \beta[a]),$$

where z_s obeys (11) with $b = \beta[a]$.

It was proved in [25] that under the assumptions (9) the lower value u^ε is the unique viscosity solution of the Cauchy problem (1) with Isaacs Hamiltonian (6). Symmetrically, the upper value v^ε is the unique viscosity solution of the Cauchy problem (1) with Hamiltonian

$$\tilde{H}(z, \zeta, p) = \max_{a\in A} \min_{b\in B} \{-p \cdot f(z, \zeta, a, b) - l(z, \zeta, a, b)\}.$$

Therefore our results have an interpretation in terms of differential games with fast space-periodic oscillations in the dynamical system and in the cost.

2.2. **Ergodicity, stabilization and the effective Cauchy problem.** Following [31, 8], we consider, for any fixed $\bar{z}$, $\bar{p} \in \mathbb{R}^N$ and $\delta > 0$, the problem

$$\delta w^\delta + H(\bar{z}, \zeta, Dw^\delta + \bar{p}) = 0 \qquad \text{in } \mathbb{R}^N. \tag{12}$$

Assume (6) (9) or, more generally, that for a modulus ω

$$|H(\bar{z}, \zeta, p) - H(\bar{z}, \zeta', p)| \le \omega(|\zeta - \zeta'|(1 + |p|)) \quad \forall \zeta, \zeta', p \in \mathbb{R}^N. \tag{13}$$

Then the problem (12) has a unique viscosity solution $w^\delta(\zeta)$ (see, e.g., [10]).

Definition 2.2. We say that H is *ergodic* if, for all $\bar{z}, \bar{p}$, $\delta w^\delta(\zeta)$ converges to a constant as $\delta \to 0^+$, uniformly with respect to ζ. In this case we set

$$\bar{H}(\bar{z}, \bar{p}) := -\lim_{\delta\to 0^+} \delta w^\delta.$$

Remark 1. If H is ergodic then $\bar{H} : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is continuous, see, e.g., [3]. The value $\bar{H}(\bar{z}, \bar{p})$ can also be characterized as the unique constant $\lambda \in \mathbb{R}$ such that the problem

$$-\lambda + H(\bar{z}, \zeta, D\chi + \bar{p}) = 0, \qquad \text{for } \zeta \in \mathbb{R}^N,$$

also called *(true) cell problem*, admits a solution $\chi(\zeta)$.

If H is of Isaacs type (6) a representation formula for $\bar{H}$ is obtained as soon as w^δ is recognized as the lower value function of a discounted infinite horizon differential game, with dynamics

$$\begin{cases} \dot{\zeta}_t = f(\bar{z}, \zeta_t, a_t, b_t) \\ \zeta_0 = \zeta \in \mathbb{R}^N, \end{cases} \tag{14}$$

and cost functional

$$\int_0^{+\infty} e^{-\delta s} l(\bar{z}, \zeta_t, a_t, b_t)\, ds,$$

that is,

$$w^\delta(\zeta) = \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} \int_0^{+\infty} e^{-\delta s} l(\bar{z}, \zeta_t, \alpha[b]_t, b_t)\, ds,$$

where ζ_s obeys (14) with $a = \alpha[b]$.

Another equivalent characterization of $\bar{H}$ is given through the uniform limit of $-w(t,\zeta)/t$ as $t \to +\infty$, where $w(t,\zeta)$ solves

$$\partial_t w + H(\bar{z}, \zeta, Dw + \bar{p}) = 0, \qquad \text{for } t > 0,\ \zeta \in \mathbb{R}^N,$$

with initial condition $w(0,\zeta) = 0$; see [8, 3, 4]. The latter characterization will not be used in this paper.

Next consider, for any fixed $\bar{z} \in \mathbb{R}^N$, the problem

$$\begin{cases} \partial_t v + H'(\bar{z}, \zeta, Dv) = 0 & \text{in } (0,\infty) \times \mathbb{R}^N \\ v(0,\zeta) = h(\bar{z}, \zeta) & \text{in } \mathbb{R}^N \end{cases} \tag{15}$$

where H' is the *homogeneous part* of H. We refer to [3] for the abstract definition of H'. Here we will use it for H of Isaacs' type (6) and in this case

$$H'(z, \zeta, p) := \min_{b \in B} \max_{a \in A} \{-p \cdot f(z, \zeta, a, b)\}.$$

Definition 2.3. We say that the pair (H, h) is *stabilizing (to a constant)* if, for all $\bar{z}$, $v(t,\zeta)$ converges to a constant as $t \to +\infty$, uniformly with respect to ζ. In this case we set

$$\bar{h}(\bar{z}) := \lim_{t \to \infty} v(t, \zeta).$$

Remark 2. Under the current assumptions (9) $\bar{h}$ is bounded and uniformly continuous in $\mathbb{R}^N$ [3]. The unique solution of (15) is the lower value function of the finite horizon differential game with dynamics (14) and merely terminal cost functional h, that is,

$$v(t,\zeta) = \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} h(\bar{z}, \zeta_t), \tag{16}$$

where ζ_t solves (14) with $a_t = \alpha[b]_t$, see [25]. Then

$$\bar{h}(\bar{z}) = \lim_{t \to +\infty} \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} h(\bar{z}, \zeta_t). \tag{17}$$

For further information, references, and some counterexamples on the long-time behavior of solutions to Hamilton-Jacobi equations, we refer to [12].

Definition 2.4. We say that *there is homogenization* for the problem (1) if for any $z \in \mathbb{R}^N$ and $p \in \mathbb{R}^N$ there exist $\bar{H}(z,p)$ and $\bar{h}(z)$ such that $u^\varepsilon(t,z)$ converges uniformly on compact subsets of $(0,\infty)\times\mathbb{R}^N$, as $\varepsilon\to 0^+$, to the unique solution $u(t,z)$ of

$$\begin{cases}\partial_t u+\bar{H}(z,Du)=0 & \text{in } (0,\infty)\times\mathbb{R}^N\\ u(0,z)=\bar{h}(z) & \text{in } \mathbb{R}^N.\end{cases} \tag{18}$$

In other words, we say there is homogenization if not only u^ε converges but its limit is also characterized as the solution of a well-posed Hamilton-Jacobi equation. The three notions just recalled are linked by the following general result.

Theorem 2.5. *Assume that H is ergodic, the pair (H,h) is stabilizing, and $\bar{H}$ satisfies the comparison principle. Then there is homogenization for (1). Moreover, if $h=h(z)$ only, the convergence is locally uniform on $[0,+\infty)\times\mathbb{R}^N$.*

Proof. See Theorem 1 and Corollary 2 in [3]. □

Saying that $\bar{H}$ satisfies the comparison principle means the following: for any $T>0$, if u and v are bounded and, respectively, an upper semicontinuous subsolution and a lower semicontinuous supersolution in viscosity sense of $\partial_t u+\bar{H}(z,Du)=0$ on $]0,T[\times\mathbb{R}^N$, $u(0,z)\le v(0,z)$ for all $z\in\mathbb{R}^N$, then $u\le v$ everywhere, see, e.g., [10]. The comparison principle holds, for instance, if for a modulus ω

$$|\bar{H}(z,p)-\bar{H}(z',p)|\le\omega(|z-z'|(1+|p|))\quad\forall z,z'\in\mathbb{R}^N. \tag{19}$$

An example of effective Hamiltonian $\bar{H}$ that does not satisfy the comparison principle is in Chapter 8 of [5].

2.3. **The model problem: the convex–concave eikonal equation.** All the results of the paper will be tested on the convex–concave eikonal-type equation (7), where $z\in\mathbb{R}^N$ is written as $z=(x,y)$ with $x\in\mathbb{R}^{N_1}$ and $y\in\mathbb{R}^{N_2}$, $N_1+N_2=N$. The assumptions are

$$\begin{cases} l,h,g_1,g_2:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R} \text{ bounded and uniformly continuous,}\\ g_1 \text{ and } g_2 \text{ Lipschitz-continuous,}\\ l,h,g_1,g_2\ \mathbb{Z}^N\text{–periodic with respect to } (\xi,\eta)=(\frac{x}{\varepsilon},\frac{y}{\varepsilon}).\end{cases}$$

The Hamiltonian in (7) is convex with respect to $D_x u$ if $g_1\ge 0$ and concave with respect to $D_y u$ if $g_2\ge 0$, which motivates the name of the model problem. It is a special case of (6) because

$$g_1(x,y,\xi,\eta)|p_x|=\max_{|a|\le 1}\{-p_x\cdot a g_1(x,y,\xi,\eta)\}$$
$$-g_2(x,y,\xi,\eta)|p_y|=\min_{|b|\le 1}\{-p_y\cdot b g_2(x,y,\xi,\eta)\}.$$

The associated differential game has dynamics

$$\begin{cases}\dot{x}_t=g_1\left(x_t,y_t,\frac{x_t}{\varepsilon},\frac{y_t}{\varepsilon}\right)a_t, & |a_t|\le 1,\\ \dot{y}_t=g_2\left(x_t,y_t,\frac{x_t}{\varepsilon},\frac{y_t}{\varepsilon}\right)b_t, & |b_t|\le 1,\\ x_0=x,\ y_0=y,\end{cases} \tag{20}$$

with cost functional independent of the controls

$$\int_0^t l\left(x_s,y_s,\frac{x_s}{\varepsilon},\frac{y_s}{\varepsilon}\right)ds+h\left(x_t,y_t,\frac{x_t}{\varepsilon},\frac{y_t}{\varepsilon}\right).$$

3. **Failure of pointwise homogenization.** In this section we show that homogenization for non-coercive Hamiltonian may fail to hold, if no assumptions on the operator H and the initial data h are imposed. We will consider the case of convex-concave Hamiltonians, and a class of convex but non-coercive Hamiltonians.

3.1. Convex–concave Hamiltonians.

Example 1. Let $\gamma > 0$ with $\gamma^{-1} \in \mathbb{Z}$. The problem

$$\begin{cases} \partial_t u^\varepsilon + |u^\varepsilon_x| - \gamma |u^\varepsilon_y| = \cos\left(2\pi \frac{x - y\gamma^{-1}}{\varepsilon}\right) & \text{in } (0,\infty)\times\mathbb{R}^2, \\ u^\varepsilon(0,x,y) = 0 & \text{in } \mathbb{R}^2 \end{cases} \tag{21}$$

does not homogenize if $x \neq \frac{y}{\gamma}$. In fact, an explicit solution of (21) is given by

$$u^\varepsilon(t,x,y) = t\cos\left(2\pi \frac{x - y\gamma^{-1}}{\varepsilon}\right)$$

and u^ε has no limit for $\varepsilon \to 0$.

Example 2. Let $\gamma > 0$ with $\gamma^{-1} \in \mathbb{Z}$. The problem

$$\begin{cases} \partial_t u^\varepsilon + |u^\varepsilon_x| - \gamma |u^\varepsilon_y| = 0 & \text{for } (0,\infty)\times\mathbb{R}^2 \\ u^\varepsilon(0,x,y) = \cos\left(2\pi \frac{x - y\gamma^{-1}}{\varepsilon}\right) & \text{in } \mathbb{R}^2 \end{cases} \tag{22}$$

does not homogenize if $x \neq \frac{y}{\gamma}$. In fact, an explicit solution of (22) is given by the steady solution

$$u^\varepsilon(t,x,y) = \cos\left(2\pi \frac{x - y\gamma^{-1}}{\varepsilon}\right)$$

and u^ε does not converge to any function as ε vanishes.

The next Propositions generalize these examples to other convex-concave Hamiltonians in dimension 2.

Proposition 1. *Let u^ε be a solution of*

$$\begin{cases} \partial_t u^\varepsilon + H_1(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon},u^\varepsilon_x) - H_2(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon},u^\varepsilon_y) = l(\frac{x}{\varepsilon},\frac{y}{\varepsilon}), & \text{in } (0,\infty)\times\mathbb{R}^2 \\ u^\varepsilon(0,x,y) = 0, & \text{in } \mathbb{R}^2, \end{cases} \tag{23}$$

and assume that constants $\nu_1, \nu_2, \beta > 0$ and $C_1, C_2 > 0$ exist, such that

$$\left|H_1(x,y,\xi,\eta,p_x) - \nu_1|p_x|^\beta\right| \le C_1, \qquad \left|H_2(x,y,\xi,\eta,p_y) - \nu_2|p_y|^\beta\right| \le C_2. \tag{24}$$

Then, for all $\delta > 0$, there exists $l : \mathbb{R}^2 \to \mathbb{R}$ analytic such that

$$\liminf_{\varepsilon\to 0^+} u^\varepsilon(t,x,y) \le -\delta t, \qquad \limsup_{\varepsilon\to 0^+} u^\varepsilon(t,x,y) \ge \delta t$$

for any $t > 0$ and any x, y such that $x\nu_1^{-\frac{1}{\beta}} \neq y\nu_2^{-\frac{1}{\beta}}$. Moreover l is $\mathbb{Z}^2$-periodic if ${\nu_1}^{-1/\beta}$ and ${\nu_2}^{-1/\beta}$ belong to $\mathbb{Z}$.

Proof. First of all, observe that for any $\lambda, \mu \in \mathbb{R}$, the solution of the problem

$$\begin{cases} \partial_t v^\varepsilon + \nu_1|v^\varepsilon_x|^\beta - \nu_2|v^\varepsilon_y|^\beta = \mu + \lambda\cos\left(\frac{2\pi}{\varepsilon}\left(x\nu_1^{-1/\beta} - y\nu_2^{-1/\beta}\right)\right) & \text{in } (0,\infty)\times\mathbb{R}^2 \\ v^\varepsilon(0,x,y) = 0 & \text{in } \mathbb{R}^2 \end{cases} \tag{25}$$

satisfies

$$\liminf_{\varepsilon\to 0^+} v^\varepsilon(t,x,y) = t(\mu - \lambda), \qquad \limsup_{\varepsilon\to 0^+} v^\varepsilon(t,x,y) = t(\mu + \lambda), \tag{26}$$

for any x, y such that $x\nu_1^{-1/\beta} \neq y\nu_2^{-\frac{1}{\beta}}$. In fact the solution of (25) is explicitly given by

$$v^\varepsilon(t,x,y) = t\left[\mu + \lambda\cos\left(\frac{2\pi}{\varepsilon}\left(x\nu_1^{-1/\beta} - y\nu_2^{-1/\beta}\right)\right)\right].$$

To prove the Proposition we argue by comparison with supersolutions and subsolutions of the auxiliary problem (25). For a fixed $\delta > 0$ define

$$l\,(x,y) := (C_1 + C_2 + \delta)\cos\left(2\pi\left(x\nu_1^{-1/\beta} - y\nu_2^{-1/\beta}\right)\right),$$

with C_1 and C_2 as in (24). We have

$$\begin{aligned}0 &= \partial_t u^\varepsilon + H_1\left(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon},u_x^\varepsilon\right) - H_2\left(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon},u_y^\varepsilon\right) - l\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\\ &\le \partial_t u^\varepsilon + \nu_1|u_x^\varepsilon|^\beta - \nu_2|u_y^\varepsilon|^\beta + C_1 + C_2 - l\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\end{aligned}$$

Put $\mu = -(C_1+C_2)$ and $\lambda = C_1+C_2+\delta$, and let v^ε be the corresponding solution of (25). The previous computation shows that u^ε is a supersolution of (25). Therefore the Comparison Principle gives

$$\limsup_{\varepsilon\to 0^+} u^\varepsilon \ge \limsup_{\varepsilon\to 0^+} v^\varepsilon = t(\mu+\lambda) = \delta t.$$

Analogously, we have

$$\begin{aligned}0 &= \partial_t u^\varepsilon + H_1\left(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon},u_x^\varepsilon\right) - H_2\left(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon},u_y^\varepsilon\right) - l\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\\ &\ge \partial_t u^\varepsilon + \nu_1|u_x^\varepsilon|^\beta - \nu_2|u_y^\varepsilon|^\beta - C_1 - C_2 - l\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)\end{aligned}$$

and, by comparison with the solution $\tilde v^\varepsilon$ of the problem (25) with μ replaced by $\tilde\mu = C_1 + C_2$ and λ as before, we obtain

$$\liminf_{\varepsilon\to 0^+} u^\varepsilon \le \liminf_{\varepsilon\to 0^+} \tilde v^\varepsilon = t(\tilde\mu - \lambda) = -\delta t.$$

Finally, it is straightforward to check that l is $\mathbb{Z}^2$-periodic if and only if ${\nu_1}^{-1/\beta}$ and ${\nu_2}^{-1/\beta}$ belong to $\mathbb{Z}$. □

Remark 3. If the condition ${\nu_1}^{-1/\beta}, {\nu_2}^{-1/\beta} \in \mathbb{Z}$ is weakened to $(\nu_1/\nu_2)^{1/\beta} \in \mathbb{Q}$ the function l constructed in the proof is still T-periodic in both x and y for some $T \in \mathbb{Z}$ possibly larger than 1. Therefore we have a counterexample to homogenization also in this case. The result is sharp for the model problem $H_1(p_x) = \nu_1|p_x|, H_2(p_y) = \nu_2|p_y|$, because a result of Cardaliaguet [18] shows that homogenization holds in the non-resonant case $\nu_1/\nu_2 \notin \mathbb{Q}$.

Proposition 2. *Let u^ε be a solution of*

$$\begin{cases}\partial_t u^\varepsilon + H_1(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon},u_x^\varepsilon) - H_2(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon},u_y^\varepsilon) = 0, & \text{in } (0,\infty)\times\mathbb{R}^2\\ u^\varepsilon(0,x,y) = h(\frac{x}{\varepsilon},\frac{y}{\varepsilon}), & \text{in } \mathbb{R}^2,\end{cases} \tag{27}$$

and assume that (24) holds. Then, for any $\delta > 0$ and any $T > 0$, there exists $h : \mathbb{R}^2 \to \mathbb{R}$ analytic such that that the solution of (27) satisfies

$$\liminf_{\varepsilon\to 0^+} u^\varepsilon(t,x,y) \le -\delta, \qquad \limsup_{\varepsilon\to 0^+} u^\varepsilon(t,x,y) \ge \delta$$

for any $0 \le t \le T$ and any $x, y \in \mathbb{R}$ such that $x\nu_1^{-\frac{1}{\beta}} \neq y\nu_2^{-\frac{1}{\beta}}$. Moreover h is $\mathbb{Z}^2$-periodic if ${\nu_1}^{-1/\beta}$ and ${\nu_2}^{-1/\beta}$ belong to $\mathbb{Z}$.

Proof. For $\lambda, \mu \in \mathbb{R}$ to be chosen consider the problem

$$\begin{cases} \partial_t v^\varepsilon + \nu_1 |v^\varepsilon_x|^\beta - \nu_2 |v^\varepsilon_y|^\beta = \mu, & \text{in } (0,\infty)\times\mathbb{R}^2 \\ v^\varepsilon(0,x,y) = h(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}), & \text{in } \mathbb{R}^2, \end{cases} \tag{28}$$

with

$$h(x,y) := \lambda \cos\left(2\pi\left(x\nu_1^{-\frac{1}{\beta}} - y\nu_2^{-\frac{1}{\beta}}\right)\right).$$

Since $\nu_1|h_x|^\beta = \nu_2|h_y|^\beta$, the solution of (28) is

$$v^\varepsilon(t,x,y) = \mu t + h\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$$

and $\limsup_{\varepsilon\to 0} v^\varepsilon = \lambda + \mu t$ for $x\nu_1^{-\frac{1}{\beta}} \neq y\nu_2^{-\frac{1}{\beta}}$. By (24)

$$0 = \partial_t u^\varepsilon + H_1\left(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon},u^\varepsilon_x\right) - H_2\left(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon},u^\varepsilon_y\right)$$
$$\leq \partial_t u^\varepsilon + \nu_1|u_x|^\beta + C_1 - \nu_2|u_y|^\beta + C_2,$$

so u^ε is a supersolution of (28) if $\mu \leq -C_1 - C_2$. Then the Comparison Principle for (28) gives $u^\varepsilon \geq v^\varepsilon$ and

$$\limsup_{\varepsilon\to 0} u^\varepsilon(t,x,y) \geq \lambda + \mu t \geq \delta, \quad \text{for any } t \in [0,T]$$

if we choose $\lambda \geq \delta - \mu T$. The other inequality is obtained in a symmetric way and the proof of the periodicity of h is straightforward. □

3.2. **Hamiltonians vanishing in a direction.**

Example 3. The problem

$$\begin{cases} \partial_t u^\varepsilon + |u^\varepsilon_x + u^\varepsilon_y| = \cos\left(2\pi\frac{x-y}{\varepsilon}\right) & \text{in } (0,\infty)\times\mathbb{R}^2 \\ u^\varepsilon(0,x,y) = 0 & \text{in } \mathbb{R}^2 \end{cases} \tag{29}$$

does not homogenize. In fact, an explicit solution of (29) is $u^\varepsilon(t,x,y) = t\cos(2\pi\frac{x-y}{\varepsilon})$ which has no limit for $x \neq y$ as $\varepsilon \to 0$.

Example 4. The problem

$$\begin{cases} \partial_t u^\varepsilon + |u^\varepsilon_x + u^\varepsilon_y| = 0 & \text{in } (0,\infty)\times\mathbb{R}^2 \\ u^\varepsilon(0,x,y) = \cos(2\pi\frac{x-y}{\varepsilon}) & \text{in } \mathbb{R}^2 \end{cases} \tag{30}$$

does not homogenize. In fact, an explicit stationary solution of (30) is $u^\varepsilon(t,x,y) \equiv \cos(2\pi\frac{x-y}{\varepsilon})$ which has no limit for $x \neq y$ as $\varepsilon \to 0$.

The following Propositions generalize the previous examples.

Proposition 3. *Consider the problem*

$$\begin{cases} \partial_t u^\varepsilon + H(z, D_z u^\varepsilon) = l(\frac{z}{\varepsilon}), & \text{in } (0,\infty)\times\mathbb{R}^N \\ u^\varepsilon(0,z) = 0, & \text{in } \mathbb{R}^N. \end{cases} \tag{31}$$

Assume that there exists $p_0 \in \mathbb{Z}^N \setminus \{0\}$ such that

$$H(z, \lambda p_0) = 0, \quad \text{for any } \lambda \in \mathbb{R},\ z \in \mathbb{R}^N. \tag{32}$$

Then, for any $\delta > 0$ there exists $l : \mathbb{R}^N \to \mathbb{R}$ analytic and $\mathbb{Z}^N$–periodic, such that

$$\liminf_{\varepsilon\to 0^+} u^\varepsilon(t,z) = -\delta t, \qquad \limsup_{\varepsilon\to 0^+} u^\varepsilon(t,z) = \delta t \tag{33}$$

for any $t > 0$ and any $z \in \mathbb{R}^N$, $z \neq p_0$.

Proof. Fix $\delta > 0$ and set $l(z) = \delta\cos(2\pi p_0 \cdot z)$. Then, the function $u^\varepsilon(t,z) := tl(z/\varepsilon)$ solves the problem (31). In fact, $D_z u^\varepsilon(t,z) = \lambda p_0$, with

$$\lambda = -\frac{2\pi t\delta}{\varepsilon}\sin\left(2\pi p_0 \cdot \frac{z}{\varepsilon}\right).$$

Then u^ε satisfies (33). □

Proposition 4. *Consider the problem*

$$\begin{cases} \partial_t u^\varepsilon + H(z, D_z u^\varepsilon) = 0, & \textit{in } (0,\infty)\times\mathbb{R}^N \\ u^\varepsilon(0,z) = h\left(\frac{z}{\varepsilon}\right), & \textit{in } \mathbb{R}^N. \end{cases} \tag{34}$$

Assume that there exists $p_0 \in \mathbb{Z}^N \setminus \{0\}$ such that (32) holds. Then, for any $\delta > 0$ there exists $h : \mathbb{R}^N \to \mathbb{R}$ analytic and $\mathbb{Z}^N$–periodic, such that

$$\liminf_{\varepsilon\to 0^+} u^\varepsilon(t,z) = -\delta, \qquad \limsup_{\varepsilon\to 0^+} u^\varepsilon(t,z) = \delta \tag{35}$$

for any $t > 0$ and $z \in \mathbb{R}^N$, $z \neq p_0$.

Proof. Fix $\delta > 0$ and set $h(z) = \delta\cos(2\pi p_0 \cdot z)$. Then, the function $u^\varepsilon(t,z) \equiv h(z/\varepsilon)$ is a steady solution of (34) and it satisfies (35). □

Remark 4. In both propositions of this section the condition (32) cannot be weakened by replacing $\lambda \in \mathbb{R}$ with $\lambda > 0$, i.e., the Hamiltoninan vanishing on a half line instead of a whole line: see the examples of homogenization in Chapter 9 of [5].

Also the assumption that $p_0 \in \mathbb{Z}^N$ is sharp for Hamiltonians $H = H(p)$ convex and 1-homogeneous, i.e. of the form $H(p) = \max_{a\in A}\{-p\cdot f(a)\}$. In fact, if this assumption fails, then for all $k \in \mathbb{Z}^N$ $H(k) \neq 0$, and so $k \cdot f(a) \neq 0$ for some $a \in A$. This is the non-resonance condition of Arisawa and Lions [8], which implies the homogenization of (31), see Chapters 7 and 9 of [5]. The same reference also proves that there is homogenization of (34) as well if, in addition, for all $k \in \mathbb{Z}^N$ there exist $a, a' \in A$ such that $k \cdot f(a) \neq k \cdot f(a')$.

4. Homogenization on subspaces.

4.1. The general case.

Let $M < N$ and denote with V an M–dimensional vector subspace in $\mathbb{R}^N$, with θ its generic element, and with z_V the projection of $z \in \mathbb{R}^N$ onto V. We consider $H = H(z,\theta,p) : \mathbb{R}^N \times V \times \mathbb{R}^N \to \mathbb{R}$ $\mathbb{Z}^M$-periodic with respect to θ. The goal of this section is to prove homogenization for the problem

$$\begin{cases} \partial_t u^\varepsilon + H\left(z, \frac{z_V}{\varepsilon}, D_z u^\varepsilon\right) = 0 & \text{in } (0,\infty)\times\mathbb{R}^N \\ u^\varepsilon(0,z) = h\left(z, \frac{z_V}{\varepsilon}\right) & \text{in } \mathbb{R}^N \end{cases} \tag{36}$$

assuming that H satisfy the following coercivity condition restricted to V: for all $z, \bar p \in \mathbb{R}^N$

$$\lim_{\substack{|p|\to+\infty \\ p\in V}} H(z,\theta,p+\bar p) = +\infty, \quad \text{uniformly with respect to } \theta \in V. \tag{37}$$

The basic regularity of the Hamiltonian that we require is (13), that now reads

$$|H(z,\theta,p) - H(z,\theta',p)| \le \omega(|\theta-\theta'|(1+|p|)) \quad \forall\, \theta,\theta' \in V,\ z,p \in \mathbb{R}^N, \tag{38}$$

and the existence of $L \ge 0$ and a modulus ω such that

$$|H(z,\theta,p) - H(z',\theta,p)| \le L|z-z'|(1+|p|) + \omega(|z-z'|) \quad \forall\, z,z',p \in \mathbb{R}^N,\ \theta \in V. \tag{39}$$

For the regularity of the effective Hamiltonian we also use that for some $C_1 \geq 0$

$$|H(z,\theta,p)| \leq C_1(1+|p|) \quad \forall\, z,p \in \mathbb{R}^N, \theta \in V. \tag{40}$$

All the three last conditions are verified if H is of Isaacs type (6) satisfying (9).

Proposition 5. *Assume* (37), (38), *and* (39). *Then H is ergodic. Moreover,*
(i) if $L = 0$ in (39) *then*

$$|\bar{H}(z,p) - \bar{H}(z',p)| \leq \omega(|z-z'|) \quad \forall\, z,z' \in \mathbb{R}^N; \tag{41}$$

(ii) if $L > 0$ and (40) *holds, then there exists $K > 0$ such that*

$$|\bar{H}(z,p) - \bar{H}(z',p)| \leq K|z-z'|(1+|p|) + \omega(|z-z'|) \quad \forall\, z,z',p \in \mathbb{R}^N. \tag{42}$$

Proof. By a suitable choice of the coordinate axes we can assume without loss of generality that V is generated by the first M vectors of the canonical basis of $\mathbb{R}^N$. Thus $z_V = (z_1,\dots,z_M,0,\dots,0)$. We also use the notation

$$\theta_0 := (\theta_1,\dots,\theta_M,0,\dots,0), \quad \theta \in \mathbb{R}^M.$$

For given $z,p \in \mathbb{R}^N$, we will prove that there is a constant λ such that

$$H(z,\zeta,D_\zeta\chi + p) = \lambda, \quad \text{for } \zeta \in \mathbb{R}^N, \tag{43}$$

has a $\mathbb{Z}^M$-periodic viscosity solution $\chi(\zeta)$. This reduces to a problem in $\mathbb{R}^M$ by introducing

$$H_V(\theta,q) = H_V^{z,p}(\theta,q) := H(z,\theta_0,q_0+p), \quad \theta,q \in \mathbb{R}^M,$$

and the cell problem

$$H_V(\theta, D\tilde{\chi}) = \lambda, \quad \text{for } \theta \in \mathbb{R}^M. \tag{44}$$

By (37) H_V satisfies

$$\lim_{|q|\to\infty} H_V(\theta,q) = +\infty \quad \text{uniformly with respect to } \theta \in \mathbb{R}^M. \tag{45}$$

Then there is a unique λ such that (44) has a $\mathbb{Z}^M$-periodic solution $\tilde{\chi}(\theta)$ [31, 23]. We extend $\tilde{\chi}$ to $\mathbb{R}^N$ by setting $\chi(\zeta) := \tilde{\chi}(\zeta_V)$. Since $D\chi(\zeta) = (D\tilde{\chi}(\zeta_V))_0$, χ solves (43) and then $\lambda = \bar{H}(z,p)$.

Next we show the regularity of $\bar{H}$. To this end we recall that

$$\bar{H}(z,p) = -\lim_{\delta\to 0} \delta w^\delta(\theta;z,p),$$

where $w^\delta(\cdot;z,p)$ solves

$$\delta w^\delta + H_V^{z,p}(\theta, Dw^\delta) = 0, \quad \text{for } \theta \in \mathbb{R}^M. \tag{46}$$

If (39) with $L = 0$ holds, then $w^\delta(\cdot;z,p)$ also satisfies in viscosity sense

$$\delta w^\delta + H_V^{z',p}(\theta, Dw^\delta) \leq \omega(|z-z'|) \quad \text{in } \mathbb{R}^M.$$

The Comparison Principle holds for this equation, because $H_V^{z',p}$ is bounded and uniformly continuous on $\mathbb{R}^M \times B_R$ for any ball B_R and the coercivity condition (45) holds. Then

$$w^\delta(\theta;z,p) - w^\delta(\theta;z',p) \leq \omega(|z-z'|) \quad \forall\, \theta \in \mathbb{R}^M,\ z,z',p \in \mathbb{R}^N.$$

By exchanging the roles of z and z' and then letting $\delta \to 0$ we get (41).

In the case $L > 0$ in (39), (40) gives

$$\max_\theta |H_V^{z,p}(\theta,0)| \leq \max_\theta |H(z,\theta_0,p)| \leq C_1(1+|p|), \quad \forall z,p.$$

Then the Comparison Principle for (46) implies

$$\max_\theta |\delta w^\delta(\theta)| \le C_1(1+|p|), \quad \forall z, p.$$

By Lemma 2.1 (with l replaced by $l + f \cdot p$) we have for some $\nu, C_2 > 0$

$$H_V^{z,p}(\theta, q) \ge \nu|q| - C_2(1+|p|).$$

Then

$$|Dw^\delta| \le \frac{C_1 + C_2}{\nu}(1+|p|).$$

By (39) $w^\delta(\cdot; z, p)$ satisfies

$$\delta w^\delta + H_V^{z',p}(\theta, Dw^\delta) \le \omega(|z-z'|) + L|z-z'|(1+|Dw^\delta| + |p|) \quad \text{in } \mathbb{R}^M,$$

so the Comparison Principle for (46) gives

$$w^\delta(\theta; z, p) - w^\delta(\theta; z', p) \le \omega(|z-z'|) + L\frac{C_1 + C_2 + \nu}{\nu}|z-z'|(1+|p|) \qquad \forall\, \theta \in \mathbb{R}^M, z, z', p \in \mathbb{R}^N.$$

As before we let $\delta \to 0$ and get (42). □

Corollary 1. *Under the assumptions of Proposition 5, if the initial condition does not depend on the oscillating variables, i.e., $h = h(z)$, then a solution u^ε of (36) converges as $\varepsilon \to 0^+$ locally uniformly on $[0, +\infty) \times \mathbb{R}^N$ to the unique solution of*

$$\begin{cases} \partial_t u + \bar{H}(z, D_z u) = 0 & \text{for } t > 0,\ z \in \mathbb{R}^N, \\ u(0, z) = h(z) & \text{for } z \in \mathbb{R}^N. \end{cases}$$

Proof. It is enough to put together Proposition 5 and Theorem 2.5, and recall that the condition (42) implies the Comparison Principle for the effective Cauchy problem. □

Theorem 4.1. *Assume that (37) holds and that H is of Isaacs type (6) and satisfies (9). Then, for all $h = h(z, \theta) : \mathbb{R}^N \times V \to \mathbb{R}$, bounded, uniformly continuous and periodic with respect to θ, the pair (H, h) is stabilizing and there exists $\bar{H}(z, p)$ satisfying (41) and such that the solution u^ε of (36) converges, locally uniformly on the compact subsets of $(0, +\infty) \times \mathbb{R}^N$ to u, unique solution of*

$$\begin{cases} \partial_t u + \bar{H}(z, Du) = 0 & \text{for } t > 0,\ z \in \mathbb{R}^N \\ u(0, z) = \min_{\theta \in V} h(z, \theta) & \text{for } z \in \mathbb{R}^N. \end{cases}$$

Proof. As in the proof of Proposition 5 we can assume that V is generated by the first M vectors of the canonical basis of $\mathbb{R}^N$. We also use the same notations, i.e., $z_V = (z_1, \dots, z_M, 0, \dots, 0)$ for $z \in \mathbb{R}^N$, $\theta_0 := (\theta_1, \dots, \theta_M, 0, \dots, 0)$ for $\theta \in \mathbb{R}^M$, and

$$H'_V(\theta, q) := H'(z, \theta_0, q_0), \quad \theta, q \in \mathbb{R}^M,$$

where H' is the homogenous part of H. To show that (H, h) is stabilizing, we fix $\bar{z} \in \mathbb{R}^N$ and consider the problem

$$\begin{cases} \partial_t w + H'(\bar{z}, (\zeta_1, \dots, \zeta_M), D_\zeta w) = 0 & t > 0,\ \zeta \in \mathbb{R}^N, \\ w(0, \zeta) = h(\bar{z}, (\zeta_1, \dots, \zeta_M)), & \zeta \in \mathbb{R}^N. \end{cases} \tag{47}$$

This reduces to the problem in $\mathbb{R}^M$

$$\begin{cases} \partial_t \tilde{w} + H'_V(\bar{z}, \theta, D_\theta \tilde{w}) = 0 & \text{for } t > 0,\ \theta \in \mathbb{R}^M \\ \tilde{w}(0, \theta) = h(\bar{z}, \theta) & \text{for } \theta \in \mathbb{R}^M. \end{cases}$$

Since H is of Isaacs' type and condition (37) holds, Lemma 2.1 implies that for some constant $\nu > 0$

$$H'_V(\bar{z}, \theta, p) \geq \nu |p|, \quad \text{for any } \theta \in \mathbb{R}^M,\ p \in \mathbb{R}^M. \tag{48}$$

Then the comparison principle gives

$$\min_{\mathbb{R}^M} h(\bar{z}, \cdot) \leq \tilde{w}(t, \theta) \leq \hat{w}(t, \theta), \quad \text{for any } t > 0,\ \theta \in \mathbb{R}^M,$$

where $\hat{w}(t, \theta)$ is the solution of

$$\begin{cases} \partial_t \hat{w} + \nu |D_\theta \hat{w}| = 0 & \text{for } t > 0,\ \theta \in \mathbb{R}^M \\ \hat{w}(0, \theta) = h(\bar{z}, \theta) & \text{for } \theta \in \mathbb{R}^M. \end{cases}$$

It is easy to see that $\hat{w}(t, \theta) \equiv \min_{\mathbb{R}^M} h(\bar{z}, \cdot)$ for t large enough. Then

$$\lim_{t \to +\infty} \tilde{w}(t, \theta) = \min_{\mathbb{R}^M} h(\bar{z}, \cdot).$$

Next we observe that $w(t, \zeta) = \tilde{w}(t, \zeta_V)$ by uniqueness of solution of (47) (note that $D_\zeta w(\zeta) = (D_\theta \tilde{w}(\zeta_V))_0$). Hence, the pair (H, h) is stabilizing and $\bar{h}(\bar{z}) = \min_{\mathbb{R}^M} h(\bar{z}, \cdot)$. Now the conclusion follows from Proposition 5 and Theorem 2.5. □

Remark 5. In the statements of Corollary 1 and Theorem 4.1 the assumption of coercivity to $+\infty$ (37) can be replaced by one to $-\infty$, namely,

$$\lim_{\substack{p \in V \\ |p| \to +\infty}} H(z, \theta, p + \bar{p}) = -\infty. \tag{49}$$

The only difference is the formula for the effective initial data $\bar{h}$ in Theorem 4.1 which is now

$$u(0, z) = \max_{\theta \in V} h(z, \theta) \quad \text{for } z \in \mathbb{R}^N.$$

The proofs are essentially the same.

4.2. **Applications and examples. Oscillations of pursuit-evasion type.** We assume that N is even and consider the problem

$$\begin{cases} \partial_t u^\varepsilon + H\left(x, y, \frac{x-y}{\varepsilon}, D_x u^\varepsilon, D_y u^\varepsilon\right) = 0 & \text{for } t > 0,\ x, y \in \mathbb{R}^{N/2}, \\ u^\varepsilon(0, x, y) = h\left(x, y, \frac{x-y}{\varepsilon}\right) & \text{for } x, y \in \mathbb{R}^{N/2}, \end{cases} \tag{50}$$

with H of Isaacs type satisfying (9). The oscillations affect only the difference between the x and the y variables. This fits problems where x and y represent the coordinates of two conflicting players and the cost functional involves the distance of the two opponents. In pursuit-evasion games the cost is an increasing function of such distance that the pursuer wants to minimize and the evader to maximize. Let $H : \mathbb{R}^N \times \mathbb{R}^{N/2} \times \mathbb{R}^N \to \mathbb{R}$ be $\mathbb{Z}^{N/2}$-periodic with respect to $\theta = \frac{x-y}{\varepsilon}$ and assume

$$\text{for all } z \in \mathbb{R}^N, \quad \lim_{\substack{|q| \to +\infty \\ q \in \mathbb{R}^{N/2}}} H(z, \theta, q, -q) = +\infty, \quad \text{uniformly in } \theta \in \mathbb{R}^{N/2}. \tag{51}$$

As a consequence of Theorem 4.1 we have the following.

Corollary 2. *Under the previous assumptions and for $h = h(x, y, \theta)$ bounded uniformly continuous and periodic in θ, the pair (H, h) is stabilizing and there exists $\bar{H}(x, y, p_x, p_y)$, Lipschitz–continuous with respect to (x, y), such that the solution u^ε of (50) converges, locally uniformly on the compact subsets of $(0, +\infty) \times \mathbb{R}^{N/2} \times \mathbb{R}^{N/2}$ to u, unique solution of*

$$\begin{cases} \partial_t u + \bar{H}(x, y, D_x u, D_y u) = 0 & \text{for } t > 0,\ x, y \in \mathbb{R}^{N/2}, \\ u(0, x, y) = \min\limits_{\theta \in \mathbb{R}^{N/2}} h(x, y, \theta) & \text{for } x, y \in \mathbb{R}^{N/2}. \end{cases}$$

Proof. Consider the $N/2$ dimensional subspace of $\mathbb{R}^N$

$$V := \{(q, -q) \,:\, q \in \mathbb{R}^{N/2}\}.$$

An orthogonal basis for V is given by $v^i = e^i - e^{i+N/2}$, $(i = 1, \ldots, N/2)$, where $\{e^j\}_{j=1}^N$ is the canonical basis of $\mathbb{R}^N$. For any $z = (x, y) \in \mathbb{R}^{N/2} \times \mathbb{R}^{N/2}$ we compute:

$$z_V = \sum_{i=1}^{N/2} (z \cdot v^i) v^i = \sum_{i=1}^{N/2} (x_i - y_i) v^i = ((x - y), -(x - y)).$$

Next observe that (9) implies, for some $L \geq 0$, $|H(z, \theta, \bar{p} + p) - H(z, \theta, p)| \leq L|\bar{p}|$. Then we take $p = (p_x, -p_x) \in V$ and use (51) to get

$$H(x, y, \theta, \bar{p} + p) \geq H(x, y, \theta, p_x, -p_x) - L|(\bar{p}_x, \bar{p}_y)| \to +\infty \quad \text{as } |p| = \sqrt{2}|p_x| \to \infty.$$

Therefore (37) holds and Theorem 4.1 gives the desired conlusion. □

Example 5. [Convex-concave eikonal equation] Consider the model problem

$$\begin{cases} \partial_t u^\varepsilon + g_1 |D_x u^\varepsilon| - g_2 |D_y u^\varepsilon| = l & \text{in } (0, +\infty) \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}, \\ u^\varepsilon(0, x, y) = h & \text{in } \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}. \end{cases}$$

Then there is homogenization in the following cases:

1. g_1, g_2, l, h depend only on $(x, y, \frac{x}{\varepsilon})$ and, for some $\nu > 0$, $g_1 \geq \nu\ \forall x, y$;
2. g_1, g_2, l, h depend only on $(x, y, \frac{y}{\varepsilon})$ and, for some $\nu > 0$, $g_2 \geq \nu\ \forall x, y$;
3. $N_1 = N_2 = \frac{N}{2}$, g_1, g_2, l, h depend only on $(x, y, \frac{x-y}{\varepsilon})$ and, for some $\nu > 0$, $(g_1 - g_2)(x, y, \theta) \geq \nu\ \forall x, y, \theta \in \mathbb{R}^{N/2}$.

The three statements follow immediately from Theorem 4.1, Remark 5, and Corollary 2, respectively.

Example 6. Consider the special case of convex-concave eikonal equation with oscillations of pursuit-evasion type and g_1 proportional to g_2, i.e.,

$$\begin{cases} \partial_t u^\varepsilon + g(x, y, \frac{x-y}{\varepsilon}) |D_x u^\varepsilon| - \gamma g(x, y, \frac{x-y}{\varepsilon}) |D_y u^\varepsilon| = l(x, y, \frac{x-y}{\varepsilon}), \\ \hfill \text{in } (0, +\infty) \times \mathbb{R}^N, \\ u^\varepsilon(0, x, y) = h(x, y, \frac{x-y}{\varepsilon}), \hfill \text{in } \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}. \end{cases}$$

with γ constant, $g : \mathbb{R}^N \times \mathbb{R}^{N/2} \to \mathbb{R}$ bounded, $\mathbb{Z}^{N/2}$–periodic in $\theta = \frac{x-y}{\varepsilon}$, Lipschitz–continuous, and such that $g(x, y, \theta) \geq \nu$ for some $\nu > 0$. Then we have the following sharp result:

$$\text{there is homogenization for all data } g, l, h \quad \Longleftrightarrow \quad \gamma \neq 1.$$

In fact, if $\gamma = 1$, Example 1 and Example 2 show that homogenization may not occur even if g and either h or l are constant. On the other hand, if $\gamma < 1$ there is homogenization by case 3 of the previous example, and for $\gamma > 1$ there is homogenization by Remark 5.

Convex non-coercive Hamiltonians. Consider the problem with Hamiltonian convex in the $(D_x u, D_y u)$ variables

$$\begin{cases}\partial_t u^\varepsilon + g(x,y,\frac{x-y}{\varepsilon})|D_x u^\varepsilon + \gamma D_y u^\varepsilon| = l(x,y,\frac{x-y}{\varepsilon}) & \text{for } t>0,\ x,y\in\mathbb{R}^{N/2} \\ u(0,x,y) = h(x,y,\frac{x-y}{\varepsilon}) & \text{for } x,y\in\mathbb{R}^{N/2}\end{cases}$$

under the usual assumptions (9) and with $\gamma \in \mathbb{R}$, $g(x,y,\theta) \geq \nu$ for some $\nu > 0$. Then we have the sharp result:

$$\text{there is homogenization for all data } g,l,h \quad \Longleftrightarrow \quad \gamma \neq 1.$$

In fact, if $\gamma = 1$, Examples 3 and 4 show that homogenization may not occur even if g and either h or l are constant. On the other hand, if $\gamma \neq 1$ (51) holds and then there is homogenization by Corollary 2.

Systems with controlled acceleration. Consider the control system in $\mathbb{R}^M$ with a single player

$$\begin{cases}\ddot{x}_t = f\left(x_t, \dot{x}_t, \frac{\dot{x}_t}{\varepsilon}, a_t\right) \\ x_0 = x,\ \dot{x}_0 = y,\end{cases} \tag{52}$$

with a cost functional

$$\int_0^t l\left(x_s, \dot{x}_s, \frac{\dot{x}_s}{\varepsilon}, a_s\right) ds + h\left(x_t, \dot{x}_t, \frac{\dot{x}_t}{\varepsilon}\right), \quad (t>0).$$

Note that the oscillations affect only the velocity and not the position of the system. By the usual substitution $y_s = \dot{x}_s$ this is rewritten as a standard problem in $\mathbb{R}^N$, $N = 2M$ and we assume the usual conditions (9). The value function $u^\varepsilon(t,x,y)$ satisfies the Hamilton-Jacobi-Bellman equation

$$\begin{cases}\partial_t u^\varepsilon - y\cdot D_x u^\varepsilon + \max_{a\in A}\left\{-D_y u^\varepsilon \cdot f\left(x,y,\frac{y}{\varepsilon},a\right) - l\left(x,y,\frac{y}{\varepsilon},a\right)\right\} = 0, \\ \hfill \text{for } t>0, x,y\in\mathbb{R}^M \\ u^\varepsilon(0,x,y) = h\left(x,y,\frac{y}{\varepsilon}\right) \hfill \text{for } x,y\in\mathbb{R}^M.\end{cases} \tag{53}$$

Assume that for any $(x,y)\in\mathbb{R}^N$ there exists $\nu>0$ such that

$$\overline{\text{co}} f(x,y,\eta,A) \supseteq B(\nu), \quad \text{for any } \eta\in\mathbb{R}^M, \tag{54}$$

where $B(r)$ is the open ball with radius r centered at 0 and $\overline{\text{co}}$ stands for the closed convex hull. Condition (54) is equivalent to the following coercivity condition: for any fixed $(x,y)\in\mathbb{R}^N$ there exist $\nu>0$ and $C\geq 0$ such that

$$\max_{a\in A}\{-p_y\cdot f(x,y,\eta,a) - l(x,y,\eta,a)\} \geq \nu|p_y| - C, \quad \text{for any } p_y,\eta\in\mathbb{R}^M.$$

Thus the Hamiltonian

$$H(x,y,\eta,p_x,p_y) = -y\cdot p_x + \max_a\{-p_y\cdot f(x,y,\eta,a) - l(x,y,\eta,a)\}$$

satisfies (37) with $V = \{(x,y)\in\mathbb{R}^N : x = 0\}$ and therefore there is homogenization for (53) by Theorem 4.1.

The same result holds for differential games where f and l depends also on a second control b_s. The the H-J-Isaacs equation is

$$\partial_t u^\varepsilon - y\cdot D_x u^\varepsilon + \min_{b\in B}\max_{a\in A}\left\{-D_y u^\varepsilon\cdot f\left(x,y,\frac{y}{\varepsilon},a,b\right) - l\left(x,y,\frac{y}{\varepsilon},a,b\right)\right\} = 0,$$

Now the Hamiltonian is coercive in the direction of $D_y u$ if (54) is replaced by

$$\forall b\in B \quad \overline{\text{co}} f(x,y,\eta,A,b) \supseteq B(\nu), \quad \text{for any } \eta\in\mathbb{R}^M,$$

and therefore in this case there is homogenization.

A quadratic Hamiltonian arising in H^∞ control. We present an application where the Hamiltonian is of Isaacs type (6) but does not satisfy the usual assumptions (9) and grows quadratically in some components of Du. Consider a system with controlled acceleration as in the previous example, and assume it is affected by an unbounded noise

$$\begin{cases} \ddot{x}_t = \tilde{f}\left(x_t, \dot{x}_t, \frac{\dot{x}_t}{\varepsilon}, a_t\right) + b_t \\ x_0 = x, \ \dot{x}_0 = y, \end{cases} \tag{55}$$

with $b_t \in \mathbb{R}^M$. As in the theory of robust or H^∞ control we consider a running cost of the form

$$l = \tilde{l}\left(x_s, \dot{x}_s, \frac{\dot{x}_s}{\varepsilon}, a_s\right) - \frac{\gamma^2 |b_s|^2}{2},$$

where the constant $\gamma > 0$ is the disturbance attenuation level. In this example we suppose the terminal cost is $h(x_t, \dot{x}_t)$. We assume that $\tilde{f}, \tilde{l}, A$ satisfy (9). As before, by putting $y_s = \dot{x}_s$ we rewrite the system in $\mathbb{R}^N, N = 2M$. By the theory of Soravia, see Appendix B of [10] and the references therein, $u^\varepsilon(t, x, y)$ is the unique viscosity solution of

$$\begin{cases} \partial_t u^\varepsilon - y \cdot D_x u^\varepsilon - \dfrac{|D_y u^\varepsilon|^2}{2\gamma^2} + \max\limits_{a \in A} \left\{ -D_y u^\varepsilon \cdot \tilde{f}\left(x, y, \dfrac{y}{\varepsilon}, a\right) - \tilde{l}\left(x, y, \dfrac{y}{\varepsilon}, a\right) \right\} = 0, \\ \hfill t > 0, x, y \in \mathbb{R}^M, \\ u^\varepsilon(0, x, y) = h(x, y), \hfill x, y \in \mathbb{R}^M. \end{cases} \tag{56}$$

Note the quadratic term $-\frac{|D_y u^\varepsilon|^2}{2\gamma^2}$ in the Hamiltonian. Since $\tilde{f}$ and $\tilde{l}$ are bounded by (9), the Hamiltonian

$$H(x, y, \eta, p_x, p_y) = -y \cdot p_x - \frac{|p_y|^2}{2\gamma^2} + \max_{a \in A} \left\{ -p_y \cdot \tilde{f}(x, y, \eta, a) - \tilde{l}(x, y, \eta, a) \right\}$$

is coercive to $-\infty$ with respect to p_y, i.e., it satisfies (49) with $V = \{(x, y) \in \mathbb{R}^N : x = 0\}$. Therefore there is homogenization for (56) by Corollary 1 and Remark 5.

5. **Asymptotic controllability and the effective initial data.** This Section is devoted to the study of the stabilizing property of (H, h). Special forms for h are taken into account: in Section 5.1 we suppose that, for any $(x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, $h(x, y, \xi, \eta)$ has a saddle point in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, whereas in Section 5.3 we consider $h(x, y, \xi, \eta)$ of pursuit-evasion type, i.e., depending on the difference $\xi - \eta \in \mathbb{R}^{N/2}$.

We consider Hamilton–Jacobi–Isaacs operators of type (6) with assumption (9). The study of the stabilization of (H, h) is based on the analysis of the associated differential game. For every fixed $z = (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ we look at the control system

$$\begin{cases} \dot{\zeta}_t = f(z, \zeta_t, a_t, b_t) \\ \zeta_0 = \zeta \equiv (\xi, \eta) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \end{cases}$$

where $\zeta_t = (\xi_t, \eta_t)$. As already mentioned in Section 2.2, for H of type (6), (H, h) is stabilizing at (x, y) if the lower value of the game (20) associated to the cost $h(x, y, \cdot, \cdot)$ converges to a constant as $t \to +\infty$, i.e.,

$$\lim_{t \to +\infty} \inf_{\alpha \in \Gamma} \sup_{b \in \mathcal{B}} h(x, y, \xi_t, \eta_t) = \bar{h}(x, y) \quad \text{uniformly in } \zeta,$$

where (ξ_t, η_t) is the solution of (20) with $a = \alpha[b]$. We additionally require that the homogeneous part of H satisfies the Isaacs' condition

$$H'(x, y, \xi, \eta, p_x, p_y) := \min_{b\in B} \max_{a\in A}\{-p\cdot f(x, y, \xi, \eta, a, b)\} \\ = \max_{a\in A} \min_{b\in B}\{-p\cdot f(x, y, \xi, \eta, a, b)\}, \quad (57)$$

for any $(x, y) \in \mathbb{R}^N$, $(\xi, \eta) \in \mathbb{R}^N$, $p \equiv (p_x, p_y) \in \mathbb{R}^N$. It is well known [25] that if (57) holds, the lower value and upper value of the finite horizon game (20) with cost h coincide, that is

$$\sup_{\beta\in\Delta} \inf_{a\in\mathcal{A}} h(x, y, \xi_t, \eta_t) = \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} h(x, y, \xi_t, \eta_t). \quad (58)$$

Remark 6. Condition (57) is satisfied, for example, when H has the form

$$H(z, \zeta, p_x, p_y) = \min_{b\in B} \max_{a\in A}\{-p_x \cdot f_1(z, \zeta, a) - p_y \cdot f_2(z, \zeta, b) - l(z, \zeta, a, b)\}.$$

In particular it holds in the model problem of Section 2.3.

5.1. **Terminal cost with a saddle.** In this subsection we assume that for any $(x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ the function $h(x, y, \cdot, \cdot)$ has a *saddle point* in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, that is, for any $(x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ there exists $h_s(x, y)$ such that

$$h_s(x, y) = \max_{\eta\in\mathbb{R}^{N_2}} \min_{\xi\in\mathbb{R}^{N_1}} h(x, y, \xi, \eta) = \min_{\xi\in\mathbb{R}^{N_1}} \max_{\eta\in\mathbb{R}^{N_2}} h(x, y, \xi, \eta). \quad (59)$$

In what follows we will denote by dist the Euclidean distance in $\mathbb{R}^N$, i.e. for any $x, y \in \mathbb{R}^N$, $\mathrm{dist}(x, y) = |x - y|$. As usual, for any $x \in \mathbb{R}^N$ and any $\mathcal{T} \subset \mathbb{R}^N$, with $\mathrm{dist}(x, \mathcal{T})$ we mean the infimum of $\{\mathrm{dist}(x, y) : y \in \mathcal{T}\}$.

Definition 5.1. Let $z = (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ be fixed and consider the game (20). We say that *the ξ-variables are asymptotically controllable by the first player* if there exists a function κ, with $\kappa(t) \to 0$ as $t \to +\infty$, such that, for any initial state $\zeta \in \mathbb{R}^N$, and any $\bar{\xi} \in \mathbb{R}^{N_1}$, there exists a strategy $\tilde{\alpha} \in \Gamma$ of the first player such that

$$\mathrm{dist}(\xi_t, \bar{\xi}) \le \kappa(t) \quad \text{for any } b \in \mathcal{B}, \text{ any } t > 0$$

where $\zeta_t = (\xi_t, \eta_t)$ is the solution of (20) with $a = \tilde{\alpha}[b]$. The definition of asymptotic controllability of the η-variables by the second player is symmetric.

Proposition 6. *Let $z = (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ be fixed. Suppose that the ξ-variables are asymptotically controllable by the first player, and the η-variables by the second player. Suppose also that (57) and (59) holds. Then the pair (H, h) is stabilizing at (x, y), and $\bar{h}(x, y) = h_s(x, y)$.*

Proof. We denote by $\omega(\cdot)$ a modulus of continuity of the function $h(x, y, \cdot, \cdot)$. Fix an initial state $\zeta \in \mathbb{R}^N$, and a point $\bar{\xi} \in \mathbb{R}^{N_1}$ as in Definition 5.1. Since the ξ-variables are asymptotically controllable by the first player, there exists $\tilde{\alpha} \in \Gamma$ such that the corresponding path $\tilde{\zeta}_t = (\tilde{\xi}_t, \tilde{\eta}_t)$ of (20) with $a = \tilde{\alpha}[b]$ satisfies

$$h(x, y, \tilde{\xi}_t, \tilde{\eta}_t) \le \omega(|\tilde{\xi}_t - \bar{\xi}|) + h(x, y, \bar{\xi}, \tilde{\eta}_t) \le \omega(\kappa(t)) + \max_{\eta} h(x, y, \bar{\xi}, \eta), t > 0.$$

for any $b \in \mathcal{B}$, $t > 0$. Then

$$\inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} h(x, y, \xi_t, \eta_t) \le \sup_{b\in\mathcal{B}} h(x, y, \tilde{\xi}_t, \tilde{\eta}_t) \le \omega(\kappa(t)) + \max_{\eta} h(x, y, \bar{\xi}, \eta)$$

and taking the lim sup for $t \to +\infty$, and the minimum over all $\bar{\xi}$, we get

$$\limsup_{t\to+\infty} \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} h(x, y, \xi_t, \eta_t) \le \min_{\xi} \max_{\eta} h(x, y, \xi, \eta), \qquad \text{uniformly in } \zeta = (\xi_0, \eta_0).$$

A symmetric computation gives:

$$\liminf_{t\to+\infty} \sup_{\beta\in\Delta} \inf_{a\in\mathcal{A}} h(x,y,\xi_t,\eta_t) \geq \max_{\eta} \min_{\xi} h(x,y,\xi,\eta), \qquad \text{uniformly in } \zeta.$$

Therefore (58) and (59) give

$$\lim_{t\to\infty} \inf_{\alpha\in\Gamma} \sup_{b\in\mathcal{B}} h(x,y,\xi_t,\eta_t) = h_s(x,y), \qquad \text{uniformly in } \zeta$$

and then $\bar{h}(x,y) = h_s(x,y)$. □

Example 7. Consider a system (20) of the form

$$\begin{cases} \dot{\xi}_t = f_1(z,\xi_t,\eta_t,a_t) \\ \dot{\eta}_t = f_2(z,\xi_t,\eta_t,b_t) \\ \xi_0 = \xi, \eta_0 = \eta \end{cases} \tag{60}$$

where the first player controls only the first N_1 variables and the second player controls only the last N_2 variables. Assume that for some $\nu_1, \nu_2 > 0$

$$\overline{\text{co}} f_1(z,\xi,\eta,A) \supseteq B(\nu_1), \quad \overline{\text{co}} f_2(z,\xi,\eta,B) \supseteq B(\nu_2), \quad \forall\, z, (\xi,\eta) \in \mathbb{R}^N, \tag{61}$$

where $B(r) := \{q : |q| < r\}$. Then it is well-known that the first player has a strategy that brings the first N_1 components of the system to $\bar{\xi}$ within a time $|\xi - \bar{\xi}|/\nu_1$, no matter what is the control of the second player (see, for instance, [34]). Since the state space $\mathbb{R}^N/\mathbb{Z}^N$ is compact we conclude that the ξ-variables are asymptotically controllable by the first player. Symmetrically, the second player has a strategy that brings the last N_2 components to a given $\bar{\eta}$ within a time $|\eta - \bar{\eta}|/\nu_2$ for any $a \in \mathcal{A}$, so also the η-variables are asymptotically controllable.

5.2. **Homogenization for pursuit-evasion oscillations and initial data with a saddle.** As a first application of the preceding Section 5.1 we consider the problem

$$\begin{cases} \partial_t u^\varepsilon + H_1(x,y,\frac{x-y}{\varepsilon},D_x u^\varepsilon) - H_2(x,y,\frac{x-y}{\varepsilon},D_y u^\varepsilon) = 0, & \text{for } t>0,\ x,y \in \mathbb{R}^{N/2}, \\ u^\varepsilon(0,x,y) = h(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon}) & \text{for } x,y \in \mathbb{R}^{N/2}, \end{cases} \tag{62}$$

with

$$H_1(x,y,\theta,p_x) = \max_{a\in A}\{-p_x \cdot f_1(x,y,\theta,a) - l_1(x,y,\theta,a)\},$$
$$H_2(x,y,\theta,p_y) = -\min_{b\in B}\{-p_y \cdot f_2(x,y,\theta,b) - l_2(x,y,\theta,b)\}.$$

In other words, the underlying control system has the form (60) and the running cost also splits as

$$l(x,y,\theta,a,b) = l_1(x,y,\theta,a) + l_2(x,y,\theta,b).$$

Corollary 3. *Assume that h has a saddle (59), (9) holds, and for any $(x,y) \in \mathbb{R}^N$ there exist constants $\nu_1, \nu_2, \delta > 0$, and C_1, C_2, C_3 such that*

$$H_1(x,y,\theta,p_x) \geq \nu_1|p_x| - C_1, \quad (\nu_1 - \delta)|p_y| + C_3 \geq H_2(x,y,\theta,p_y) \geq \nu_2|p_y| - C_2, \quad \forall\, \theta \in \mathbb{R}^{N/2}.$$

Then there exist $\bar{H}$ such that the solution u^ε of (62) converges locally uniformly on the compact subsets of $(0,+\infty)\times\mathbb{R}^N$ to the unique solution u of

$$\begin{cases}\partial_t u + \bar{H}(x,y,D_x u, D_y u) = 0 & \text{for } t>0,\ x,y\in\mathbb{R}^{N/2},\\ u(0,x,y) = h_s(x,y) & \text{for } x,y\in\mathbb{R}^{N/2}.\end{cases}$$

Proof. Set $H(x,y,\theta,p_x,p_y) := H_1(x,y,\theta,p_x) - H_2(x,y,\theta,p_y)$ and fix $x,y\in\mathbb{R}^{N/2}$. Then

$$H(x,y,\theta,p,-p) \geq \delta|p| - C_1 - C_3, \quad \text{for all } \theta,p\in\mathbb{R}^{N/2}.$$

By Proposition 5 and the proof of Corollary 2, H is ergodic and $\bar{H}$ satisfies the comparison principle. By Lemma 5.2 below, the coercivity of H_1, H_2 implies the properties (61) of the underlying control system. This entails the asymptotic controllability of the ξ and the η variables of (60) by the first player and the second player, respectively, see Example 7. Moreover the Isaacs condition (57) is clearly satisfied. Therefore by Proposition 6 the pair (H,h) is stabilizing and $\bar{h}(x,y) = h_s(x,y)$. The conclusion now follows from Theorem 2.5. □

The next Lemma completes the previous proof.

Lemma 5.2. *Let $K\subset\mathbb{R}^N$ be a closed convex set and $\nu>0$. Then*

$$\max_{v\in K} p\cdot v \geq \nu|p| \quad \text{for any } p\in\mathbb{R}^N \quad \Longleftrightarrow \quad K\supseteq B(\nu). \tag{63}$$

Proof. If $K\supseteq B(\nu)$ then, for any $p\in\mathbb{R}^N$, $\max_{v\in K} p\cdot v \geq \max_{v\in B(\nu)} p\cdot v = \nu|p|$.

To prove the converse implication we observe first that K contains a ball $B(\rho)$, for some $\rho>0$. Otherwise $0\notin \text{int}K$, and $\bar{p}\in\mathbb{R}^N$ with $|\bar{p}|=1$ would exist such that, for any $v\in K$, $\bar{p}\cdot v\leq 0$. Then, by taking the maximum over $v\in K$ we obtain a contradiction with the left hand side of (63).

It remains to prove that we can take $\rho=\nu$. If not, put $\bar{\rho} := \sup\{\rho<\nu : K\supseteq B(\rho)\}$. Thus $B(\bar{\rho})$ is the larger ball contained in K, and there exists $\bar{v}\in\partial K$ with $|\bar{v}|=\bar{\rho}$. For any $v\in K$ we have $(v-\bar{v})\cdot\bar{v}\leq 0$, that is, $v\cdot\bar{v}\leq\bar{\rho}^2$. Then, for any $\lambda>1$,

$$\max_{v\in K} v\cdot(\lambda\bar{v}) \leq \lambda\bar{\rho}^2 = \bar{\rho}|\lambda\bar{v}| < \nu|\lambda\bar{v}|,$$

a contradiction that completes the proof. □

Example 8. In the convex-concave eikonal equation of Example 5, in the hypotheses of Case 3 we can replace $h = h(x,y,\frac{x-y}{\varepsilon})$ with the assumption that h has a saddle (59).

5.3. **Terminal cost of pursuit–evasion type.** In this Subsection we will write $z\in\mathbb{R}^N$ as $z=(x,y)$, with x and y in $\mathbb{R}^{N/2}$. This allows us to consider initial data h of *pursuit–evasion* type plus a perturbation, that is

$$h(x,y,\xi,\eta) = h_1(x,y,\xi-\eta) + h_2(x,y,\eta). \tag{64}$$

Here the perturbation h_2 is independent of ξ, the symmetric case of a perturbation depending on ξ and not on η is treated in Remark 7. Note that h of this form does not have a saddle, i.e., (59) does not hold, unless h_1 is constant with respect ot θ.

Definition 5.3. Let $z=(x,y)\in\mathbb{R}^{N/2}\times\mathbb{R}^{N/2}$ be fixed. We say that (20) is *asymptotically controllable by the first player to a closed target $\mathcal{T}\subset\mathbb{R}^N$* if there exists a function κ, with $\kappa(t)\to 0$ as $t\to+\infty$, such that, for any initial state

$\zeta \in \mathbb{R}^N$ there is a strategy $\tilde\alpha \in \Gamma$ of the first player such that the path ζ_t of (20) with to $a = \tilde\alpha[b]$ satisfies

$$\mathrm{dist}(\zeta_t, \mathcal{T}) \le \kappa(t) \quad \text{for any } b \in \mathcal{B}, \text{ any } t > 0.$$

The definition of asymptotic controllability of the η–variables by the second player to a closed target is symmetric.

Depending on cases we will assume the following:

Assumption 1. (i) for each $z = (x, y) \in \mathbb{R}^{N/2} \times \mathbb{R}^{N/2}$ the system (20) is asymptotically controllable by the first player with respect to

$$\mathcal{T}_1(z) := \{(\xi, \eta) \in \mathbb{R}^N : \xi - \eta \in \arg\min h_1(x, y, \cdot)\};$$

(ii) the η-variables are asymptotically controllable by the second player.

Proposition 7. *Assume that h has the form (64), H' satisfies (57), and Assumption 1 holds. Then the pair (H, h) is stabilizing, and for any $(x, y) \in \mathbb{R}^{N/2} \times \mathbb{R}^{N/2}$,*

$$\bar h(x, y) = \min_\theta h_1(x, y, \theta) + \max_\eta h_2(x, y, \eta). \tag{65}$$

The proof of Proposition 7 uses the following technical Lemma, which holds for h not necessarily satisfying (64).

Lemma 5.4. *Let $z = (x, y) \in \mathbb{R}^{N/2} \times \mathbb{R}^{N/2}$ be fixed, and assume that the game (20) is asymptotically controllable by the first player with respect to the target*

$$\mathcal{G}(z) := \left\{(\xi', \eta') : h(z, \xi', \eta') \le \max_\eta \min_\xi h(z, \xi, \eta)\right\},$$

and that the η-variables of (20) are asymptotically controllable by the second player with respect to the target

$$\mathcal{R}(z) := \arg\max \min_\xi h(z, \xi, \cdot).$$

Finally, assume that (57) is satisfied. Then, the pair (H, h) is stabilizing at $z = (x, y)$, and

$$\bar h(z) = \max_\eta \min_\xi h(z, \xi, \eta).$$

Proof. Fix an initial state $\zeta \in \mathbb{R}^N$ in (20). Since (20) is asymptotically controllable by the first player on $\mathcal{G}(z)$, there exist a function κ_1 independent of ζ and a strategy $\tilde\alpha \in \Gamma$ for the first player such that the solution $\tilde\zeta_t = (\tilde\xi_t, \tilde\eta_t)$ of (20) with $a = \tilde\alpha[b]_t$ satisfies

$$\mathrm{dist}(\tilde\zeta_t, \mathcal{G}(z)) \le \kappa_1(t) \quad \text{for any } b \in \mathcal{B}, \text{ any } t > 0. \tag{66}$$

Let us denote by $\zeta'_t = (\xi'_t, \eta'_t)$ the projection of $\tilde\zeta_t$ on $\mathcal{G}(z)$, so that

$$h(z, \xi'_t, \eta'_t) \le \max_\eta \min_\xi h(z, \xi, \eta) \quad \text{for any } t > 0.$$

Then

$$h(z, \tilde\zeta_t) \le \omega(|\tilde\zeta_t - \zeta'_t|) + \max_\eta \min_\xi h(z, \xi, \eta), \quad \text{for any } t > 0,$$

where $\omega(\cdot)$ is a modulus of continuity of the function $h(z, \cdot, \cdot)$. By (66) we get

$$h(z, \tilde\zeta_t) \le \omega(\kappa_1(t)) + \max_\eta \min_\xi h(z, \xi, \eta), \quad \text{for any } b \in \mathcal{B}, \text{ any } t > 0,$$

and therefore

$$\inf_{\alpha\in\Gamma}\sup_{b\in\mathcal{B}} h(z,\zeta_t) \le \omega(\kappa_1(t)) + \max_\eta \min_\xi h(z,\xi,\eta).$$

Consequently, by taking the $\limsup_{t\to+\infty}$, we obtain

$$\limsup_{t\to+\infty}\inf_{\alpha\in\Gamma}\sup_{b\in\mathcal{B}} h(z,\zeta_t) \le \max_\eta \min_\xi h(z,\xi,\eta), \quad \text{uniformly in } \zeta. \tag{67}$$

Since the η-variables are asymptotically controllable by the second player with respect to $\mathcal{R}(z)$, there exists a function κ_2 independent of ζ and a strategy $\tilde\beta \in \Delta$ of the second player such that

$$\operatorname{dist}(\tilde\eta_t, \mathcal{R}(z)) \le \kappa_2(t), \quad \text{for any } a\in\mathcal{A}, \text{ any } t>0.$$

Now $\tilde\zeta_t = (\tilde\xi_t, \tilde\eta_t)$ stands for a solution of (20) with $b = \tilde\beta[a]$. Let us consider the projection η'_t of $\tilde\eta_t$ on the target $\mathcal{R}(z)$, that is

$$\operatorname{dist}(\tilde\eta_t, \mathcal{R}(z)) = |\eta'_t - \tilde\eta_t|, \qquad \min_\xi h(z,\xi,\eta'_t) = \max_\eta \min_\xi h(z,\xi,\eta).$$

We have, for any $a\in\mathcal{A}$ and any $t>0$,

$$\begin{aligned} h(z,\tilde\xi_t,\tilde\eta_t) \ge -\omega(|\eta'_t - \tilde\eta_t|) + h(z,\tilde\xi_t,\eta'_t) &\ge -\omega(|\eta'_t - \tilde\eta_t|) + \min_\xi h(z,\xi,\eta'_t) \\ &\ge -\omega(\kappa_2(t)) + \max_\eta \min_\xi h(z,\xi,\eta). \end{aligned}$$

Therefore

$$\sup_{\beta\in\Delta}\inf_{a\in\mathcal{A}} h(z,\xi_t,\eta_t) \ge -\omega(\kappa_2(t)) + \max_\eta \min_\xi h(z,\xi,\eta)$$

and, by taking the $\liminf_{t\to+\infty}$,

$$\liminf_{t\to+\infty}\sup_{\beta\in\Delta}\inf_{a\in\mathcal{A}} h(z,\xi_t,\eta_t) \ge \max_\eta \min_\xi h(z,\xi,\eta), \quad \text{uniformly in } \zeta. \tag{68}$$

Now (58), (67), and (68), give

$$\bar h(z) := \lim_{t\to+\infty}\inf_{\alpha\in\Gamma}\sup_{b\in\mathcal{B}} h(z,\xi_t,\eta_t) = \max_\eta \min_\xi h(z,\xi,\eta), \quad \text{uniformly in } \zeta. \qquad \square$$

Proof of Proposition 7. Let $z = (x,y) \in \mathbb{R}^{N/2}\times\mathbb{R}^{N/2}$ be fixed. Observe that, for any $(\xi',\eta') \in \mathcal{T}_1(z)$,

$$h(x,y,\xi',\eta') \le \min_\theta h_1(x,y,\theta) + \max_\eta h_2(z,\eta) = \max_\eta \min_\xi h(z,\xi,\eta).$$

This shows that $\mathcal{T}_1(z) \subseteq \mathcal{G}(z)$. Thus (20) is asymptotically controllable by the first player on $\mathcal{G}(z)$, being controllable on a subset of it. Moreover, since the η-variables are asymptotically controllable by the second player, in particular they are asymptotically controllable on the target $\mathcal{R}(z) = \arg\max\min_\xi h(z,\xi,\cdot)$.Then the conclusion and (65) follow from Lemma 5.4. $\square$

Remark 7. It is easy to adapt the previous proof to handle the case

$$h(x,y,\xi,\eta) = h_1(x,y,\xi-\eta) + h_3(x,y,\xi)$$

if the hypotheses on the two players in Assumption 1 are exchanged. More precisely, we require that (20) is asymptotically controllable by the second player on the target $\mathcal{T}_2(z) = \arg\max h_1(x,y,\cdot)$ and that the ξ-variables are asymptotically controllable by the first player. In this case, instead of (65), we obtain

$$\bar h(x,y) = \max_\theta h_1(x,y,\theta) + \min_\xi h_3(x,y,\xi).$$

Example 9. Consider a system of the form (60) and assume that for some $\nu_1 > \delta > 0$

$$\overline{\text{co}} f_1(z,\xi,\eta,A) \supseteq B(\nu_1), \quad \overline{\text{co}} f_2(z,\xi,\eta,B) \subseteq B(\nu_1-\delta), \quad \forall\, z,(\xi,\eta) \in \mathbb{R}^N. \tag{69}$$

We claim that this condition implies the property *(i)* of Assumption 1. Without loss of generality we assume that $0 \in \arg\min h_1(z,\cdot)$. We begin with the simplified dynamics

$$\begin{cases} \dot{\xi}_t = a_t, & a_t \in B(\nu_1), \\ \dot{\eta}_t = b_t, & b_t \in B(\nu_1-\delta). \end{cases}$$

We define a nonanticipating strategy by setting

$$\alpha[b]_t := b_t - \delta \frac{\xi_t - \eta_t}{|\xi_t - \eta_t|},$$

where the second term on the right hand side is taken to be 0 if $\xi_t - \eta_t = 0$. Then $w_t := |\xi_t - \eta_t|^2$ solves $\dot{w}_t = -2\delta\sqrt{w_t}$ and therefore it reaches 0 at the time $\sqrt{w_0}/\delta$. Therefore the simplified system can be driven by the first player to the target $\mathcal{T}_1(z)$ in a time not larger than $|\xi_0 - \eta_0|/\delta$. In view of (69) the first player can reach the same goal also for system (60) within the time $|\xi_0 - \eta_0|/\delta$. Since the state space $\mathbb{R}^N/\mathbb{Z}^N$ is compact we conclude that (60) is asymptotically controllable by the first player to $\mathcal{T}_1(z)$.

6. **Homogenization with partially decoupled fast variables.**

In this Section we consider the problem

$$\begin{cases} \partial_t u^\varepsilon + H_1(x,y,\frac{x}{\varepsilon},D_x u^\varepsilon) - H_2(x,y,\frac{y}{\varepsilon},D_y u^\varepsilon) = 0 & \text{for } t>0,\ (x,y) \in \mathbb{R}^{N_1}\times\mathbb{R}^{N_2} \\ u^\varepsilon(0,x,y) = h(x,y,\frac{x}{\varepsilon},\frac{y}{\varepsilon}) & \text{for } (x,y) \in \mathbb{R}^{N_1}\times\mathbb{R}^{N_2} \end{cases} \tag{70}$$

The oscillating variables $\xi = \frac{x}{\varepsilon}$ and $\eta = \frac{y}{\varepsilon}$ are separated in two different Hamiltonians but they are coupled in the initial data h. This is why we call these problems *partially decoupled.*

We assume that, for any $(x,y) \in \mathbb{R}^N$,

$$\begin{aligned} &\lim_{|p_x|\to\infty} H_1(x,y,\xi,p_x) = +\infty, \quad \text{uniformly in } \xi \in \mathbb{R}^{N_1} \\ &\lim_{|p_y|\to\infty} H_2(x,y,\eta,p_y) = +\infty, \quad \text{uniformly in } \eta \in \mathbb{R}^{N_2}. \end{aligned} \tag{71}$$

The main results will be in two cases: when $h(x,y,\xi,\eta)$ has a saddle point with respect to ξ and η, and when h has oscillations of pursuit-evasion type. Both results will hold under the further assumption that H_1, H_2 are of Bellman form, i.e.,

$$\begin{aligned} H_1(x,y,\xi,p_x) &= \max_{a\in A}\{-p_x\cdot f_1(x,y,\xi,a) - l_1(x,y,\xi,a)\}, \\ H_2(x,y,\eta,p_y) &= -\min_{b\in B}\{-p_y\cdot f_2(x,y,\eta,b) - l_2(x,y,\eta,b)\}. \end{aligned} \tag{72}$$

6.1. **The effective Hamiltonian.**

Proposition 8. *Assume that the Hamiltonian*

$$H(x,y,\xi,\eta,p_x,p_y) := H_1(x,y,\xi,p_x) - H_2(x,y,\eta,p_y)$$

satisfies (38)*,* (39)*, and* (71) *for all* $(x,y) \in \mathbb{R}^N$*. Then it is ergodic and the effective Hamiltonian can be written in the form*

$$\bar{H}(x,y,p_x,p_y) = \bar{H}_1(x,y,p_x) - \bar{H}_2(x,y,p_y).$$

Moreover, if each H_i verifies either (39) with $L = 0$ or (40), then $\bar{H}$ satisfies the comparison principle. Finally, if H_1 (respectively, H_2) is convex in p_x (respectively, p_y), then $\bar{H}_1$ (respectively, $\bar{H}_2$) is convex in p_x (respectively, p_y).

Proof. Fix $(x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and $(p_x, p_y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and consider the cell problem

$$\delta w^\delta + H(x, y, \xi, \eta, D_\xi w^\delta + p_x, D_\eta w^\delta + p_y) = 0, \qquad \text{in } \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}.$$

We look for a $\mathbb{Z}^N$-periodic solution of the form $w^\delta(\xi, \eta) = w_1^\delta(\xi) - w_2^\delta(\eta)$, where w_1^δ and w_2^δ solve, respectively,

$$\delta w_1^\delta + H_1(x, y, \xi, D w_1^\delta + p_x) = 0, \quad \text{in } \mathbb{R}^{N_1} \tag{73}$$

$$\delta w_2^\delta + H_2(x, y, \eta, D w_2^\delta + p_y) = 0, \quad \text{in } \mathbb{R}^{N_2}. \tag{74}$$

Consider the problem (73). Since H_1 is periodic with respect to ξ, there exists $K_1 > 0$ such that $|H_1(x, y, \xi, p_x)| \leq K_1$ for any ξ. By comparison with K_1/δ and $-K_1/\delta$, we get $|\delta w_1^\delta| \leq K_1$ for any ξ. Thus, by (73), $|H_1(x, y, \xi, D w_1^\delta + p_x)| \leq K_1$ for any ξ. This in turn entails, by (71), $|D w_1^\delta| \leq L_1$ for any ξ and any $\delta > 0$, for some constant $L_1 > 0$ large enough.

Analogously, by (74) and (71) we obtain $|D w_2^\delta| \leq L_2$ for any η and any $\delta > 0$, for some $L_2 > 0$. Then both δw_1^δ and δw_2^δ, converge to some constants, as δ vanishes, uniformly with respect to ξ and η, respectively. Therefore H_1 and H_2 are ergodic, respectively, at (x, y, p_x) and (x, y, p_y). Then

$$\begin{aligned} \bar{H}(x, y, p_x, p_y) &:= - \lim_{\delta \to 0^+} \delta w^\delta(\xi, \eta) \\ &= - \lim_{\delta \to 0^+} \left(\delta w_1^\delta(\xi) - \delta w_2^\delta(\eta) \right) =: \bar{H}_1(x, y, p_x) - \bar{H}_2(x, y, p_y). \end{aligned}$$

The regularity of $\bar{H}$ ensuring the comparison principle is proved as in Proposition 5. Finally, the statement about convexity is well-known in view of the construction of $\bar{H}_1, \bar{H}_2$ by means of (73)(74). □

Remark 8. We immediately derive from Proposition 8 and Theorem 2.5 that under the assumptions of Proposition 8 the solution u^ε of (70) with $h = h(x, y)$ converges as $\varepsilon \to 0^+$ locally uniformly on $[0, +\infty) \times \mathbb{R}^N$ to the unique solution of

$$\begin{cases} \partial_t u + \bar{H}_1(x, y, D_x u) - \bar{H}_2(x, y, D_y u) = 0 & \text{in } (0, \infty) \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \\ u(0, x, y) = h(x, y) & \text{in } \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}. \end{cases}$$

6.2. **Initial data with a saddle.** We recall that the saddle property (59) of h is the existence of h_s such that

$$h_s(x, y) = \max_{\eta \in \mathbb{R}^{N_2}} \min_{\xi \in \mathbb{R}^{N_1}} h(x, y, \xi, \eta) = \min_{\xi \in \mathbb{R}^{N_1}} \max_{\eta \in \mathbb{R}^{N_2}} h(x, y, \xi, \eta).$$

Theorem 6.1. *Assume H_1 and H_2 are given by (72), satisfy (9) and (71) for every $(x, y) \in \mathbb{R}^N$, and h verifies (59) for every $(x, y) \in \mathbb{R}^N$. Then there exist $\bar{H}_1(x, y, p_x), \bar{H}_2(x, y, p_y)$ convex in p_x, p_y, respectively, such that the solution u^ε of (70) converges as $\varepsilon \to 0$ locally uniformly on $(0, +\infty) \times \mathbb{R}^N$ to the unique solution u of*

$$\begin{cases} \partial_t u + \bar{H}_1(x, y, D_x u) - \bar{H}_2(x, y, D_y u) = 0 & \text{for } t > 0,\ (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \\ u(0, x, y) = h_s(x, y) & \text{for } (x, y) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}. \end{cases} \tag{75}$$

Proof. By (72) and Lemma 2.1, (71) implies that there exist $\nu_1, \nu_2 > 0$, C_1, C_2 such that

$$H_1(x, y, \xi, p_x) \geq \nu_1 |p_x| - C_1, \qquad H_2(x, y, \eta, p_y) \geq \nu_2 |p_y| - C_2, \qquad \text{for any } \xi, \eta. \tag{76}$$

Consider the differential game (60). By Lemma 5.2 condition (76) implies the property (61) of the vector fields f_1, f_2, and therefore the asymptotic controllability of the ξ-variables (resp., the η-variables) by the first player (resp., see the second player), see Example 7. The Isaacs condition (57) is clearly satisfied. Then the pair (H, h) is stabilizing by Proposition 6. The conclusion then follows from Proposition 8 and Theorem 2.5. □

Example 10. Under the partial decoupling condition of this section the model problem (7) becomes

$$\begin{cases} \partial_t u^\varepsilon + g_1\left(x, y, \frac{x}{\varepsilon}\right)|D_x u^\varepsilon| - g_2\left(x, y, \frac{y}{\varepsilon}\right)|D_y u^\varepsilon| = l_1\left(x, y, \frac{x}{\varepsilon}\right) + l_2\left(x, y, \frac{y}{\varepsilon}\right) \\ u^\varepsilon(0, x, y) = h\left(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right). \end{cases} \tag{77}$$

If h has the saddle property (59) and $g_1 > 0$, $g_2 > 0$, Theorem 6.1 gives the homogenization for (77) with effective Cauchy problem (75).

6.3. **Initial data of pursuit–evasion type.** Here we consider h with the property (64) that the oscillations are of pursuit–evasion type plus a perturbation term where the oscillations involve only one group of variables, namely,

$$h(x, y, \xi, \eta) = h_1(x, y, \xi - \eta) + h_2(x, y, \eta).$$

Theorem 6.2. *Assume H_1, H_2 are given by (72), satisfy (9), and for any $(x, y) \in \mathbb{R}^N$ there exist constants $\nu_1, \nu_2, \delta > 0$, and C_1, C_2, C_3 such that*

$$H_1(x, y, \theta, p_x) \geq \nu_1 |p_x| - C_1, \quad (\nu_1 - \delta)|p_y| + C_3 \geq H_2(x, y, \theta, p_y) \geq \nu_2 |p_y| - C_2, \quad \forall\, \theta \in \mathbb{R}^{N/2}. \tag{78}$$

Suppose also that h has the form (64). Then there exist $\bar{H}_1(x, y, p_x), \bar{H}_2(x, y, p_y)$ convex in p_x, p_y, respectively, such that the solution u^ε of (70) converges as $\varepsilon \to 0$ locally uniformly on $(0, +\infty) \times \mathbb{R}^N$ to the unique solution u of

$$\begin{cases} \partial_t u + \bar{H}_1(x, y, D_x u) - \bar{H}_2(x, y, D_y u) = 0 & \text{for } t > 0, x, y \in \mathbb{R}^{N/2} \\ u(0, x, y) = \min_\theta h_1(x, y, \theta) + \max_\eta h_2(x, y, \eta) & \text{for } x, y \in \mathbb{R}^{N/2}. \end{cases} \tag{79}$$

Proof. By Lemma 5.2 the bounds on H_1, H_2 and the Bellman structure (72) imply the properties (69) on the vector fields f_1, f_2. Then the differential game (60) has the property *(i)* of Assumption 1, by Example 9. By Lemma 5.2 again, the coercivity of H_2 implies $\overline{\text{co}} f_2(z, \xi, \eta, B) \supseteq B(\nu_2)$ for all $z, (\xi, \eta) \in \mathbb{R}^N$, which gives the property *(ii)* of Assumption 1 by Example 7. The Isaacs condition (57) is clearly satisfied. Then Proposition 7 implies that the pair (H, h) is stabilizing, and

$$\bar{h}(x, y) = \min_\theta h_1(x, y, \theta) + \max_\eta h_2(x, y, \eta).$$

Finally, Proposition 8 and Theorem 2.5. give the conclusion. □

Remark 9. By exchanging the hypotheses on H_1 and H_2 the result is easily adapted to handle initial data of the form

$$u^\varepsilon(0, x, y) = h_1\left(x, y, \frac{x - y}{\varepsilon}\right) + h_3\left(x, y, \frac{x}{\varepsilon}\right).$$

More precisely, if

$$(\nu_2 - \delta)|p_x| + C_3 \geq H_1(x, y, \theta, p_x) \geq \nu_1|p_x| - C_1, \quad H_2(x, y, \theta, p_y) \geq \nu_2|p_y| - C_2, \qquad \forall\, \theta \in \mathbb{R}^{N/2},$$

holds insted of (78), there is homogenization for (70) with initial data as above. In this case the effective initial data turns out to be

$$\bar{h}(x, y) = \max_\theta h_1(x, y, \theta) + \min_\xi h_3(x, y, \xi).$$

Example 11. Consider the model problem (77) with h of the form (64) and

$$g_1(x, y, \xi) - g_2(x, y, \eta) > 0 \quad \text{for all } x, y, \xi, \eta \in \mathbb{R}^{N/2}.$$

Then Theorem 6.2 implies the homogenization with effective Cauchy problem (79).

7. **An explicit example of effective equation and game.** In this section we study in detail the model problem (77) in dimension 2. We further simplify the setting assuming that $g_1 = 1$, $g_2 = \gamma > 0$,

$$\min_{\xi \in [0,1]} l_1(x, y, \xi) = l_1(x, y, \xi_0) = 0 \quad \text{and} \quad \max_{\eta \in [0,1]} l_2(x, y, \eta) = l_2(x, y, \eta_0) = 0, \tag{80}$$

for some $\xi_0, \eta_0 \in [0, 1]$. We deal with the homogenization of

$$\begin{cases} \partial_t u^\varepsilon + |u^\varepsilon_x| - \gamma|u^\varepsilon_y| = l_1(x, y, \frac{x}{\varepsilon}) + l_2(x, y, \frac{y}{\varepsilon}) & t > 0,\ x, y \in \mathbb{R} \\ u^\varepsilon(0, x, y) = h(x, y, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}) & x, y \in \mathbb{R}. \end{cases} \tag{81}$$

Computing the effective Hamiltonian. Fix $x, y, p_x, p_y \in \mathbb{R}$ and consider the (true) cell problem, that is

$$|p_x + \chi_\xi| - \gamma|p_y + \chi_\eta| - l_1(x, y, \xi) - l_2(x, y, \eta) = \lambda \qquad \text{for } \xi, \eta \in \mathbb{R}. \tag{82}$$

Since x and y are frozen we omit them in the next calculations. It is well known that there exists at most one value $\lambda \in \mathbb{R}$ such that the previous equation admits a continuous viscosity solution $\chi(\xi, \eta)$, defined up to additive constants. In the spirit of Proposition 8, we look for a solution of (82) of the form $\chi(\xi, \eta) = \chi_1(\xi) + \chi_2(\eta)$, with χ_1 and χ_2 such that

$$|p_x + \chi_1'(\xi)| - l_1(\xi) = \lambda_1 \qquad \text{for } \xi \in \mathbb{R} \tag{83}$$

$$-\gamma|p_y + \chi_2'(\eta)| - l_2(\eta) = \lambda_2 \qquad \text{for } \eta \in \mathbb{R}. \tag{84}$$

Consider (83). We claim that

$$\lambda_1 := (|p_x| - \langle l_1 \rangle)^+ \tag{85}$$

is the unique constant such that the problem (83) has a solution $\chi_1(\xi)$. Here $(\cdot)^+$ stands for the positive part, and $\langle l_1 \rangle$ denotes the average of l_1 on a period, that is,

$$\langle l_1 \rangle := \langle l_1 \rangle(x, y) := \int_0^1 l_1(x, y, \xi) d\xi.$$

To prove the claim we construct an explicit solution of (83) under the position (85). Suppose first that $|p_x| \leq \langle l_1 \rangle$, and put

$$\chi_1(\xi) := \begin{cases} \int_{\xi_0}^{\xi} (l_1(s) - p_x) ds & \text{if } \xi_0 \leq \xi \leq \bar{\xi} \\ \int_{\xi}^{\xi_0+1} (l_1(s) + p_x) ds & \text{if } \bar{\xi} \leq \xi \leq \xi_0 + 1 \end{cases}$$

where ξ_0 is defined in (80) and $\bar{\xi} \in [\xi_0, \xi_0 + 1]$ is such that

$$\int_{\xi_0}^{\bar{\xi}} (l_1(s) - p_x) ds = \int_{\bar{\xi}}^{\xi_0+1} (l_1(s) + p_x) ds.$$

Thus χ_1 can be extended to a continuous periodic function in $\mathbb{R}$. It is easy to check that χ_1 solves (83) with $\lambda_1 = 0$ at all points of differentiability. Moreover

$$\lim_{\xi \to \xi_0+} \chi_1'(\xi) = l_1(\xi_0) - p_x = \lim_{\xi \to \xi_0+1-} \chi_1'(\xi)$$

by (80), so χ_1 is differentiable at $\xi_0 + n$ for all integer n. Finally,

$$\lim_{\xi \to \bar{\xi}-} \chi_1'(\xi) = l_1(\bar{\xi}) \geq -l_1(\bar{\xi}) = \lim_{\xi \to \bar{\xi}+} \chi_1'(\xi),$$

so χ_1 is a viscosity solution of (83) in all $\mathbb{R}$.

In the case $|p_x| \geq \langle l_1 \rangle$ we define

$$\chi_1(\xi) := (\text{sign } p_x) \int_0^{\xi} (l_1(s) - \langle l_1 \rangle) ds, \qquad \xi \in \mathbb{R},$$

and observe it is periodic and C^1. Moreover, for all $\xi \in \mathbb{R}$,

$$|p_x + \chi_1'(\xi)| = ||p_x| + l_1(\xi) - \langle l_1 \rangle| = l_1(\xi) + \lambda_1, \qquad \lambda_1 = |p_x| - \langle l_1 \rangle.$$

In a similar way we find that

$$\lambda_2 := -(\gamma |p_y| + \langle l_2 \rangle)^+$$

is the unique value such that (84) admits a viscosity solution $\chi_2(\eta)$. Thus, $\lambda = \lambda_1 + \lambda_2$ is the unique value such that (82) has a solution and therefore

$$\bar{H}(x, y, p_x, p_y) = (|p_x| - \langle l_1 \rangle(x, y))^+ - (\gamma |p_y| + \langle l_2 \rangle(x, y))^+.$$

Remark 10. It is easy to adapt the previous computation to the general case, in which $\min l_1 \neq 0$ and $\max l_2 \neq 0$. In this case we find

$$\lambda_1(p_x) = (|p_x| - \langle l_1 \rangle)^+ + \min_{[0,1]} l_1,$$
$$\lambda_2(p_y) = -(\gamma |p_y| + \langle l_2 \rangle)^+ - \max_{[0,1]} l_2.$$

Homogenization for h with a saddle. Suppose $h(x, y, \xi, \eta)$ satisfies (59), that is, it has a saddle value $h_s(x, y)$ when it is minimized with respect to ξ and maximized with respect to η. Under the assumptions of this Section, Theorem 6.1 states that the solution u^ε of (81) converges as $\varepsilon \to 0^+$ to the unique solution of

$$\begin{cases} \partial_t u + (|u_x| - \langle l_1 \rangle(x, y))^+ - (\gamma |u_y| + \langle l_2 \rangle(x, y))^+ = 0 & \text{for } t > 0,\ x, y \in \mathbb{R}, \\ u(0, x, y) = h_s(x, y) & \text{for } x, y \in \mathbb{R}. \end{cases} \tag{86}$$

Homogenization for h of pursuit–evasion type. Take as initial condition in (81)

$$u^\varepsilon(0, x, y) = h_1\left(x, y, \frac{x - y}{\varepsilon}\right) + h_2\left(x, y, \frac{y}{\varepsilon}\right)$$

and assume $\gamma < 1$. Then (see Example 11) u^ε converges as $\varepsilon \to 0^+$ to the unique solution of

$$\begin{cases} \partial_t u + (|u_x| - \langle l_1 \rangle(x, y))^+ - (\gamma |u_y| + \langle l_2 \rangle(x, y))^+ = 0 & \text{for } t > 0,\ x, y \in \mathbb{R}, \\ u(0, x, y) = \min_\theta h_1(x, y, \theta) + \max_\eta h_2(x, y, \eta) & \text{for } x, y \in \mathbb{R}. \end{cases} \tag{87}$$

The effective differential game. By means of Legendre transforms we can rewrite the effective Hamiltonian as an Isaacs' one

$$\bar{H}(x,y,p_x,p_y) = \max_{|a|\le 1}\{ap_x - |a|\langle l_1\rangle(x,y)\} + \min_{|b|\le 1}\{b\gamma p_y - |b|\langle l_2\rangle(x,y)\}.$$

Then we can write a representation formula for the solution of (86) or (87) as the value function of a differential games. We consider the control system

$$\dot{x}_s = a_s, \quad |a_s| \le 1; \qquad \dot{y}_s = \gamma b_s, \quad |b_s| \le 1;$$

the running cost

$$\bar{l}(x,y,a,b) := |a|\langle l_1\rangle(x,y) + |b|\langle l_2\rangle(x,y),$$

and the payoff functional with time horizon t and initial position of the system $x_0 = x$, $y_0 = y$

$$\bar{J}(t,x,y,a,b) := \int_0^t \bar{l}(x_s,y_s,a_s,b_s)\,ds + \bar{h}(x_t,y_t),$$

where the terminal cost $\bar{h}$ is the effective initial condition of the Hamilton-Jacobi equation, namely, $\bar{h}(x,y) = h_s(x,y)$ in (86) and

$$\bar{h}(x,y) = \min_{\theta} h_1(x,y,\theta) + \max_{\eta} h_2(x,y,\eta)$$

in (87). By the results of [25] $u(t,x,y) = \lim_{\varepsilon\to 0} u^{\varepsilon}(t,x,y)$ is the lower value function of this game, that is,

$$u(t,x,y) = \inf_{\alpha\in\Gamma}\sup_{b\in\mathcal{B}} \bar{J}(t,x,y,\alpha[b],b).$$

Note that the effective cost $\bar{l}$ of the limit game depends on the controls, although the cost $l = l_1 + l_2$ does not.

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