Thin Notched Beams

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Abstract. The three-dimensional elasticity energy for a notched rod under axial loads is shown to converge in a variational sense to a class of one-dimensional models with a localized compliance when the beam becomes very slender and when the ratio between the depth and the width of each notch vanishes in a suitable way. Also the case of notched thermal conductors is considered, in a parallel treatment, because it falls in a similar but simpler framework.

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1. Introduction

Structural behavior of slender elastic beams is altered by geometrical discontinuities resulting from notches. The main hindrance in describing the effect of such discontinuities in one-dimensional models is due to the fact that classical Saint-Venant's principle, starting-point for most of beam theories, cannot possibly be true in these cases, see Toupin [15]. So, several mechanical models have been proposed in the technical literature to include the effect of a notch in one-dimensional systems, and an account of them can be found in Petroski [13]. A recurring assumption in modelling a notch is that its effects essentially have a local character, inducing a beam stiffness reduction in a region close to the discontinuity, and rapidly decaying far from the notch, [7]. Usually, physical experiments are carried out in order to characterize quantitatively the stiffness perturbation produced by real notches of various widths and depths, [14].

A different approach to notch modelization in linear elastic straight beams subject to longitudinal loads is presented here. The basic idea comes from the papers by Acerbi, Buttazzo and Percivale [1], Anzellotti, Baldo and Percivale [3] and Percivale [12], where it is shown that classical one-dimensional models of slender beams with smooth longitudinal profile can be obtained as variational limit of the corresponding three-dimensional elasticity problems when the radius of the cross-section goes to zero. By adopting a flow of ideas which is typical of the Γ-convergence of functionals defined on measures, in this paper we extend some of the above results to include the occurrence of abrupt geometrical changes on the longitudinal profile of the beam such as those induced by a notch.

It turns out that the three-dimensional elasticity problem for a notched rod under axial loads converges to a class of one-dimensional models with a localized compliance when the beam becomes very slender and when the ratio between the depth and the width of each notch vanishes in a suitable way. Roughly speaking, in the limit family the classical one-dimensional model for axial deformations holds for the two segments of beam adjacent to the notch, and the notch is represented by means of a translational elastic spring located at the damaged cross-section. This macroscopic description of localized discontinuities in beams is well known in the technical field and it is usually assumed by engineers for studying damage detection problems via dynamic methods, [2], [10]. As a matter of fact, experiments show that the analytical model with a localized compliance describes accurately the behavior of a notched beam under axial loads as well as the classical model does for a homogeneous beam, [11].

A similar dimension reduction problem can be considered also for thin thermal conductors with abrupt changes in the cross section. In fact, the scalar character of the thermal energy functional allows to simplify proofs and considerably reduces technicalities while, on the other hand, the same basic ideas can be used as well as in the elastic case. This is the main motivation for our parallel presentation of the two cases.

From the mathematical point of view, Continuum Mechanics is one of the most fruitful field of application of Γ-convergence in Calculus of Variations. Since this kind of convergence of functionals preserves in general minima and minimizers, it can be viewed as a reasonable setting to study the asymptotic behaviour of structures with spatial rapidly oscillating material properties, such as microstructures, and many other situations, like optimal design or optimal shape design for conducting or elastic media, when the material properties of the structure are unknown.

This paper is devoted to some progress of recent ideas in Γ-convergence with varying domains for dimensional reduction problems. More precisely, we consider, as a sequence of functionals, the stored energy of a cylindrical isotropic linear elastic rod, whose cross section has a small radius ε. Making this parameter go to zero, we need to recall and to use a suitable notion of Γ-convergence when the functionals are defined on a corresponding sequence of vector spaces, each related to a different material domain for the admissible fields of displacement. Once the Γ-limit has been identified, this functional can be viewed as the elastic energy of the bar after being reduced to a one-dimensional structure. Its domain is a "limit space" of possible generalized functions which are defined on the line interval representing the axis itself.

Asymptotic behaviour of energy functionals in dimension reduction problems has been studied within this setting by Anzellotti, Baldo and Percivale in [3] for beams with circular cross section having constant radius, and by Percivale in [12] for beams with arbitrary but constant cross section. However, we extend here the analysis to the more general situation when the cross section varies along the axis, both for the scalar case, in conductivity, and for the vector one of elasticity. Referring to the latter, in this brief presentation, for its more relevant interest, we let some cylindrical portion of the beam to shrink, either radially or axially, in accordance with powers of ε of higher order with respect to the complement. We then expect that the limit model should provide discontinuous displacements, whose jumps are due to the arising of pointwise concentrations of elastic rigidities, such as springs located in the thinner regions.

2. Notation

Let us denote by $C_0(a,b)$ the space of continuous functions which vanish at a and b while the elements of $C_0[a,b)$ are the continuous functions which vanish at b. Their dual $\mathcal{M}(a,b) = C_0(a,b)^*$ and $\mathcal{M}[a,b) = C_0[a,b)^*$ are the spaces of bounded Borel measures on the interval (a,b) and $[a,b)$, respectively. They can be endowed by the weak* topology induced by the duality; let's recall that a sequence μ_n weakly* converges to μ in $\mathcal{M}(I)$ (where I denotes the interval (a,b) or $[a,b)$) if and only if

$$\int_I \varphi \, d\mu_n \to \int_I \varphi \, d\mu$$

for every $\varphi \in C_0(I)$. If $f \in L^1(a,b)$ then it can be identified with a measure in $\mathcal{M}[a,b)$ (hence in $\mathcal{M}(a,b)$) which will be denoted by $f(x_1)dx_1$; in particular dx_1 is the Lebesgue measure on $[a,b)$. As usual, given two measures λ and m, $d\lambda/dm$ will denote the Radon-Nikodym derivative, that is the density, of λ with respect to m. Finally, $BV(a,b)$ will denote the space of functions with bounded variation on the open interval (a,b), that is all $L^1(a,b)$ functions with first distributional derivative belonging to $\mathcal{M}(a,b)$; a sequence v_n is said to weakly* converge to a function v in $BV(a,b)$ if and only if it converges in the norm of $L^1(a,b)$ while the derivatives v_n' weakly* converge to v' in $\mathcal{M}(a,b)$.

3. Main theorems

3.1. Thin thermal conductors

In this section we consider the problem of reducing the thermal energy functional from three dimensions to one. Let $a, b \in \mathbb{R}$, $a < b$ and $\Sigma = [a,b)$; we shall take the subdomains of $\mathbb{R}^3$

$$\Sigma_\varepsilon = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ x_1 \in \Sigma,\ (x_2, x_3) \in S_\varepsilon(x_1)\}$$

where the cross sections $S_\varepsilon(x_1)$ are two dimensional open balls with Lipschitz continuous radii satisfying $C\varepsilon \geq r_\varepsilon(x_1) \geq c_\varepsilon > 0$ for suitable contants C and c_ε. Our aim is to prove that, if r_ε converges to 0 when $\varepsilon \to 0^+$, then the minimizers of the energies of Σ_ε converge to the minimizer of the energy of a possibly inhomogeneous one dimensional limit medium, whose conductivity depends on the rate of convergence of the radii. But, of course, the minimum value of the energies tends to zero as $\varepsilon \to 0^+$. Anyway, the desired convergence of such minima can be recovered by considering for every $\varepsilon > 0$ the functional $\mathcal{F}_\varepsilon - \mathcal{L}_\varepsilon$ describing the "scaled" total energy of a thermal homogeneous conductive medium occupying a region of space Σ_ε under given distributed heat sources and boundary conditions. The functionals $\mathcal{F}_\varepsilon$ are defined on $W^{1,2}(\Sigma_\varepsilon)$ by

$$\mathcal{F}_\varepsilon(u) = \frac{1}{2\varepsilon^2} \int_{\Sigma_\varepsilon} |\nabla u(x)|^2 \, dx + \chi_a(u)$$

where the function $\chi_a(u)$ takes the value 0 if $u = 0$ on $S_\varepsilon(a)$ and $+\infty$ otherwise. On the other hand, the functionals $\mathcal{L}_\varepsilon$ are the scaled potential energies of the heat sources and are given by

$$\mathcal{L}_\varepsilon(u) = \frac{1}{\varepsilon^2} \int_{\Sigma_\varepsilon} g(x) u(x) \, dx$$

where, for the sake of simplicity, the given bounded function g is taken to depend on x_1 only, although more general cases can be considered as well. The stored energy of the one-dimensional limit structure will be found to be of the following form

$$\mathcal{F}(v) = \begin{cases} \dfrac{\pi}{2}\Big(\displaystyle\int_a^b \Big|\frac{dv'}{dm}\Big|^2 dm + \frac{|v(a^+)|^2}{m(\{a\})}\Big) & \text{if } v \in BV(a,b),\ v' \ll m \\ +\infty & \text{elsewhere in } L^1(a,b) \end{cases} \tag{3.1}$$

where m denotes the measure on $[a,b)$ defined by (3.5) and where we have set by convention that $\dfrac{|v(a^+)|^2}{m(\{a\})} = 0$ if $m(\{a\}) = 0$. Finally, the limit potential energy is given by

$$\mathcal{L}(v) = \int_a^b g(x_1)v(x_1)\, dx_1. \tag{3.2}$$

Our result can be more easily stated by firstly performing a suitable change of variables which trasforms the ε-depending domain Σ_ε into the fixed domain $\Omega = [a,b) \times S_1$. Afterwards, setting

$$v(x_1, x_2, x_3) = u\big(x_1, r_\varepsilon(x_1)x_2, r_\varepsilon(x_1)x_3\big),$$

then v is defined on Ω and

$$\mathcal{F}_\varepsilon(u) = \mathcal{F}_\varepsilon^\Omega(v), \qquad \mathcal{L}_\varepsilon(u) = \mathcal{L}_\varepsilon^\Omega(v),$$

where $\mathcal{F}_\varepsilon^\Omega, \mathcal{L}_\varepsilon^\Omega$ are the functionals defined on $W^{1,2}(\Omega)$ by

$$\begin{aligned} \mathcal{F}_\varepsilon^\Omega(v) &= \frac{1}{2\varepsilon^2}\int_\Omega [(r_\varepsilon v_{x_1} - \dot{r}_\varepsilon x_2 v_{x_2} - \dot{r}_\varepsilon x_3 v_{x_3})^2 + v_{x_2}^2 + v_{x_3}^2]\, dx + \chi_a(v), \\ \mathcal{L}_\varepsilon^\Omega(v) &= \frac{1}{\varepsilon^2}\int_\Omega r_\varepsilon^2 g v\, dx. \end{aligned} \tag{3.3}$$

Our main goal, in the thermal case, consists in proving the following theorem where the convergence of minimizers will be expressed in terms of convergence in $L^1(a,b)$ of their averages on the cross section, $q : W^{1,2}(\Omega) \to W^{1,2}(a,b)$, defined by

$$q(v)(x_1) = \frac{1}{\pi}\int_{S_1} v(x_1, x_2, x_3)\, dx_2 dx_3. \tag{3.4}$$

THEOREM 3.1. *Let us assume that*

$$\frac{\varepsilon^2}{r_\varepsilon^2(x_1)}\, dx_1 \to m \quad \textit{weakly* in } \mathcal{M}[a,b), \tag{3.5}$$

$$\frac{\varepsilon^2}{r_\varepsilon^2(x_1)}\, \dot{r}_\varepsilon^2(x_1)\, dx_1 \to 0 \quad \textit{weakly* in } \mathcal{M}[a,b). \tag{3.6}$$

If v_ε minimizes $\mathcal{F}_\varepsilon^\Omega - \mathcal{L}_\varepsilon^\Omega$ then *(i) if* (ε_n) *is a sequence such that* $\lim_{n\to\infty} \varepsilon_n = 0$ *and if* $q(v_{\varepsilon_n})$ *converges to some* v *in* $L^1(a,b)$*, then* v *is a minimizer of* $\mathcal{F} - \mathcal{L}$ *and* $\lim_{n\to\infty} [\mathcal{F}^\Omega_{\varepsilon_n}(v_{\varepsilon_n}) - \mathcal{L}^\Omega_{\varepsilon_n}(v_{\varepsilon_n})] = \mathcal{F}(v) - \mathcal{L}(v)$*;*

(ii) there is a sequence ε_n *such that* $\lim_{n\to\infty} \varepsilon_n = 0$ *and a minimizer* v *of* $\mathcal{F} - \mathcal{L}$ *such that* $q(v_{\varepsilon_n})$ *converges to* v *weakly* in* $BV(a,b)$*, hence strongly in* $L^1(a,b)$*.*

REMARK 3.1. By the uniform coercivity with respect to the transverse derivatives, the limit v of minimizing sequences v_ε cannot depend on x_2 and x_3. This fact makes the use of the averaging operator q ininfluent but, on the other hand, it improves readability particularly in view of its application to the elastic case where, on the contrary, it plays an important role.

EXAMPLE 3.1. Let's consider now the case of a conductor shaped as in the picture, the main peculiarity being that the radius of the cross section vanishes, within a small interval, with a higher order than outside it. In this case we take $\Sigma = [-1,1)$ and we assume that the radius r_ε be given by

$$r_\varepsilon(x_1) = \begin{cases} \varepsilon & \text{if } |x_1| > \dfrac{\varepsilon^{\alpha-2}+\varepsilon^\beta}{2k} \\ \dfrac{\varepsilon - \varepsilon^{\alpha/2}}{\varepsilon^\beta}(2k|x_1| - \varepsilon^{\alpha-2}) + \varepsilon^{\alpha/2} & \text{if } \dfrac{\varepsilon^{\alpha-2}}{2k} < |x_1| \le \dfrac{\varepsilon^{\alpha-2}+\varepsilon^\beta}{2k} \\ \varepsilon^{\alpha/2} & \text{if } |x_1| \le \dfrac{\varepsilon^{\alpha-2}}{2k} \end{cases}$$

with $\alpha > 2$ and $k > 0$. In order that (3.5) and (3.6) hold we have to assume respectively that α and β satisfy the following inequalities

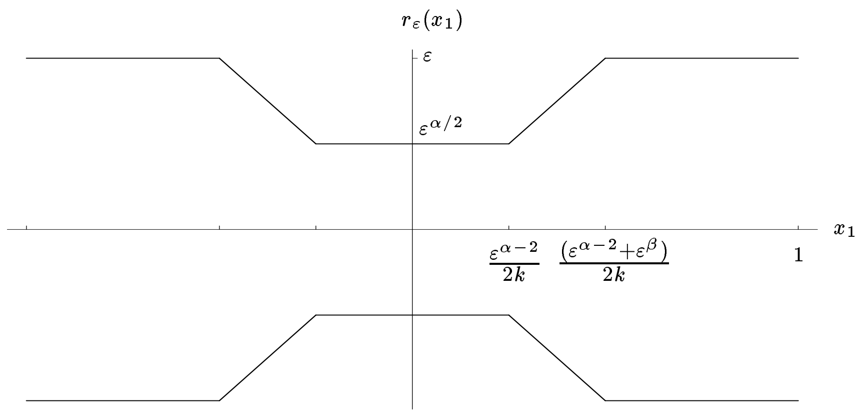
$$\frac{\alpha}{2} - 1 \le \beta < 3 - \frac{\alpha}{2}. \tag{3.7}$$

Then it is easy to see that the limit measure is $m = dx_1 + \dfrac{1}{k}\delta_0$, hence the limit energy turns out to be the functional

$$\mathcal{F}(v) = \begin{cases} \dfrac{\pi}{2}\displaystyle\int_{-1}^{1} |v'(x_1)|^2 dx_1 + \dfrac{k\pi}{2}[v]_0^2 & \text{if } v \in W^{1,2}(\Sigma \setminus \{0\}),\, v(-1) = 0 \\ +\infty & \text{elsewhere in } L^1(\Sigma), \end{cases}$$

where $[v]_0 = |v(0^+) - v(0^-)|$ is the jump of v at 0. Let's remark that if the second inequality in (3.7) is not satisfied (that is for instance if the slope of the linear joint exceeds a certain critical value) then (3.6) may be not satisfied letting the integrals $\int_{-1}^{1} \frac{\varepsilon^2}{r_\varepsilon^2(x_1)} \dot{r}_\varepsilon^2(x_1)\, dx_1$ (and therefore also $\int_{-1}^{1} \frac{1}{a_\varepsilon(x_1)}\, dx_1$) to be unbounded, and the family of minimizers may lose compactness (see Section 6 below).

Concerning the physical meaning of the limit functional, we can notice that the first term is just the classical stored thermal energy distributed on the one dimensional limit structure. The remaining term expresses, actually, a concentration of conductivity k at the point 0 which allows the minimizer to jump.



3.2. Thin elastic beams

Let, in this section, $\mathcal{F}_\varepsilon$ be the functionals defined on $W^{1,2}(\Sigma_\varepsilon; I\!R^3)$ by

$$\mathcal{F}_\varepsilon(u) = \int_{\Sigma_\varepsilon} f(\boldsymbol{e}(u))\, dx + \chi_a(u) \tag{3.8}$$

where

$$\boldsymbol{e}(u) = (e_{ij}(u))_{i,j=1,2,3}\,, \quad e_{ij} = \frac{u_{i,j} + u_{j,i}}{2}$$

is the linearized strain tensor associated to the displacement vector u and $f(\boldsymbol{e})$ is the deformation energy density. Moreover we shall make the assumption that the body is made of linear elastic, isotropic and homogeneous material, so that

$$f(\boldsymbol{e}) = \mu|\boldsymbol{e}|^2 + \frac{\lambda}{2}|\mathrm{tr}(\boldsymbol{e})|^2$$

where $\lambda \geq 0$, $\mu > 0$ are the Lamé moduli.

We assume that the structure be loaded only by axial forces which are expressed by a vector field g of the form $(g(x_1), 0, 0)$, so that their scaled potential energies are given by

$$\mathcal{L}_\varepsilon(u) = \frac{1}{\varepsilon^2} \int_{\Sigma_\varepsilon} g(x_1) u_1(x)\, dx. \tag{3.9}$$

The stored energy of the one-dimensional limit structure will be the functional

$$\mathcal{F}(v) = \begin{cases} \dfrac{\pi E}{2} \Big(\displaystyle\int_a^b \left| \frac{dv'}{dm} \right|^2 dm + \frac{|v(a^+)|^2}{m(\{a\})} \Big) & \text{if } v \in BV(a,b),\ v' \ll m \\ +\infty & \text{elsewhere in } L^1(a,b), \end{cases} \tag{3.10}$$

where $E = \mu \dfrac{3\lambda + 2\mu}{\lambda + \mu}$ is the Young's modulus of the material. Finally, the limit potential energy is given by

$$\mathcal{L}(v) = \int_a^b g(x_1) v_1(x_1)\, dx_1. \tag{3.11}$$

By performing the same change of variables as in the thermal case and denoting by

$$v_i(x_1, x_2, x_3) = u_i\big(x_1, r_\varepsilon(x_1) x_2, r_\varepsilon(x_1) x_3\big), \quad i = 1, 2, 3$$

the transformed functional $\mathcal{F}_\varepsilon^\Omega : W^{1,2}(\Omega; I\!R^3) \to \overline{I\!R}$ is given by

$$\begin{aligned} \mathcal{F}_\varepsilon^\Omega(v) = {} & \frac{\mu}{\varepsilon^2} \int_\Omega \Big[|r_\varepsilon v_{1,1} - \dot r_\varepsilon x_2 v_{1,2} - \dot r_\varepsilon x_3 v_{1,3}|^2 + |v_{2,2}|^2 + |v_{3,3}|^2 + \\ & + \frac{1}{2} |v_{1,2} + r_\varepsilon v_{2,1} - \dot r_\varepsilon x_2 v_{2,2} - \dot r_\varepsilon x_3 v_{2,3}|^2 + \\ & + \frac{1}{2} |v_{1,3} + r_\varepsilon v_{3,1} - \dot r_\varepsilon x_2 v_{3,2} - \dot r_\varepsilon x_3 v_{3,3}|^2 + \frac{1}{2} |v_{2,3} + v_{3,2}|^2 \Big] dx + \\ & + \frac{\lambda}{2\varepsilon^2} \int_\Omega \Big[|r_\varepsilon v_{1,1} - \dot r_\varepsilon x_2 v_{1,2} - \dot r_\varepsilon x_3 v_{1,3}|^2 + |v_{2,2}|^2 + |v_{3,3}|^2 + \\ & + (r_\varepsilon v_{1,1} - \dot r_\varepsilon x_2 v_{1,2} - \dot r_\varepsilon x_3 v_{1,3})(v_{2,2} + v_{3,3}) + 2 v_{2,2} v_{3,3} \Big]\, dx + \chi_a(u). \end{aligned} \tag{3.12}$$

while

$$\mathcal{L}_\varepsilon^\Omega(v) = \frac{1}{\varepsilon^2} \int_\Omega r_\varepsilon^2(x_1) g(x_1) v_1(x)\, dx. \tag{3.13}$$

Let q denote the same averaging map introduced in the thermal case.

THEOREM 3.2. *Let us assume that*

$$\frac{\varepsilon^2}{r_\varepsilon^2(x_1)}\, dx_1 \to m \quad \textit{weakly* in } \mathcal{M}[a,b), \tag{3.14}$$

$$\frac{\varepsilon^2}{r_\varepsilon^2(x_1)}\, \dot r_\varepsilon^2(x_1)\, dx_1 \to 0 \quad \textit{weakly* in } \mathcal{M}[a,b). \tag{3.15}$$

If v^ε minimizes $\mathcal{F}_\varepsilon^\Omega - \mathcal{L}_\varepsilon^\Omega$ then

(i) if (ε_n) is a sequence such that $\lim_{n\to\infty} \varepsilon_n = 0$ and if $q(v_1^{\varepsilon_n})$ converges to some v in $L^1(a,b)$ then v is a minimizer of $\mathcal{F} - \mathcal{L}$ and $\lim_{n\to\infty} [\mathcal{F}_{\varepsilon_n}^\Omega(v^{\varepsilon_n}) - \mathcal{L}_{\varepsilon_n}^\Omega(v^{\varepsilon_n})] = \mathcal{F}(v) - \mathcal{L}(v)$;

(ii) there is a sequence ε_n such that $\lim_{n\to\infty} \varepsilon_n = 0$ and a minimizer v of $\mathcal{F} - \mathcal{L}$ such that $q(v_1^{\varepsilon_n})$ converges to v weakly in $BV(a,b)$, hence strongly in $L^1(a,b)$.*

4. Sequential Γ-convergence

For the purposes of the paper we need of a kind of variational convergence which allows to treat families of functionals $F_\varepsilon : X \to \overline{I\!R}$ defined on a space which may be different from the domain of the limiting functional. It is a variant of De Giorgi's Γ-convergence, and has been introduced by Anzellotti, Baldo and Percivale in [3] in order to study dimension reduction problems in mechanics.

Let X be a set, let (Y,τ) be a topological space and let $q : X \to Y$. Given a sequence of functionals $F_n : X \to \overline{I\!R}$ and a point $v \in Y$, let us denote by

$$\Gamma(q,\tau Y^-)\liminf_{n\to\infty} F_n(v) := \inf\{\liminf_{n\to\infty} F_n(x_n) \ : \ q(x_n) \xrightarrow{\tau} v\},$$

$$\Gamma(q,\tau Y^-)\limsup_{n\to\infty} F_n(v) := \inf\{\limsup_{n\to\infty} F_n(x_n) \ : \ q(x_n) \xrightarrow{\tau} v\},$$

which are called, respectively, the Γ^--lower and the Γ^--upper limit at the point v.

DEFINITION 4.1. *Given a sequence (ε_n) such that $\lim_{n\to\infty} \varepsilon_n = 0$, we say that a sequence $F_{\varepsilon_n} : X \to \overline{I\!R}$ (sequentially) $\Gamma(q,\tau Y^-)$-converges to a functional $F : Y \to I\!R$ at a point $v \in Y$, and we write*

$$\Gamma(q,\tau Y^-)\lim_{n\to\infty} F_{\varepsilon_n}(v) = F(v)$$

if

$$\Gamma(q,\tau Y^-)\liminf_{n\to\infty} F_{\varepsilon_n}(v) = \Gamma(q,\tau Y^-)\limsup_{n\to\infty} F_{\varepsilon_n}(v) = F(v). \qquad (4.1)$$

We say that the family of functionals $F_\varepsilon : X \to \overline{I\!R}$ (sequentially) $\Gamma(q,\tau Y^-)$-converges to a functional $F : Y \to I\!R$ at a point $v \in Y$, and we write

$$\Gamma(q,\tau Y^-)\lim_{\varepsilon\to 0^+} F_\varepsilon(v) = F(v) \qquad (4.2)$$

if for any sequence (ε_n) such that $\lim_{n\to\infty}\varepsilon_n = 0$ we have that

$$\Gamma(q,\tau Y^-)\lim_{n\to\infty} F_{\varepsilon_n}(v) = F(v)$$

We say that a family of functionals $\Gamma(q,\tau Y^-)$-converges on a set if it $\Gamma(q,\tau Y^-)$-converges at every point of the set.

REMARK 4.1. Let us remark that (4.1) holds if and only if the following two conditions are satisfied

1. for every sequence $x_n \in X$ such that $q(x_n) \xrightarrow{\tau} v$ one has

$$\liminf_{n\to\infty} F_{\varepsilon_n}(x_n) \geq F(v);$$

2. there exists a sequence $\overline{x}_n \in X$ such that $q(\overline{x}_n) \xrightarrow{\tau} v$ and

$$\lim_{n\to\infty} F_{\varepsilon_n}(\overline{x}_n) = F(v).$$

If $X = Y$ and q is the identity map, then the Γ-limits defined above are the classical De Giorgi's sequential Γ-limits which will be denoted by the classical notation

$$\Gamma(\tau X^-)\lim_{\varepsilon\to 0^+} F_\varepsilon = F.$$

Moreover, it is easy to see that, setting for every $n \in I\!N$

$$I_\varepsilon(v) = \inf\{F_\varepsilon(x) \ : \ q(x) = v\}$$

then (4.2) holds true if and only if

$$\Gamma(\tau Y^-)\lim_{\varepsilon\to 0^+} I_\varepsilon(v) = F(v).$$

Besides the notion of sequential Γ-convergence, also a more general concept of Γ-convergence, associated with a topology on X, can be introduced (see for instance Dal Maso [8]) and the two kinds of Γ-convergence coincide on first countable topological spaces. Hence when the topological space (Y,τ) is first countable, and the family F_ε is equi-coercive then the $\Gamma(q,\tau Y^-)$-convergence has the variational character which is typical of Γ-convergence, that is, it ensures lower semicontinuity to the limit which, moreover, turns out to be coercive when uniform coercivity assumptions are made on the family (F_ε). Moreover, roughly speaking, it preserves convergence of minima and of minimizers. This is summarized by the following theorem.

PROPOSITION 4.1. *Let us assume that* $\Gamma(q,\tau Y^-)\lim\limits_{\varepsilon\to 0^+} F_\varepsilon = F$ *on* Y *and that the family* F_ε *be equi-coercive in the sense that for every real number* M *there exists a* τ*-compact and* τ*-closed subset* K_M *of* Y *such that* $\{q(x)\ :\ F_\varepsilon(x)\le M\}\subseteq K_M$ *for any* $\varepsilon>0$. *Then we have:*

(i) F *is* τ*-lower semicontinuos;*

(ii) F *is* τ*-coercive;*

(iii) if $x_\varepsilon\in X$ *satisfy* $\liminf\limits_{\varepsilon\to 0^+} F_\varepsilon(x_\varepsilon) = \liminf\limits_{\varepsilon\to 0^+}\inf F_\varepsilon$ *(e.g. if* x_ε *minimizes* F_ε*) then*

- *(a) if* (ε_n) *is a sequence such that* $\lim\limits_{n\to\infty}\varepsilon_n=0$ *and if* $q(x_{\varepsilon_n})\xrightarrow{\tau} v$ *then* v *is a minimizer of* F *on* Y *and* $\lim\limits_{n\to\infty} F_{\varepsilon_n}(x_{\varepsilon_n}) = F(v)$*;*
- *(b) there is a sequence* ε_n *such that* $\lim\limits_{n\to\infty}\varepsilon_n=0$ *and a minimizer* v *of* F *on* Y *such that* $q(x_{\varepsilon_n})\xrightarrow{\tau} v$.

REMARK 4.2. We have to observe, because it will be subsequently used, that the semicontinuity property does hold not only for the Γ-lower but also for the Γ-upper limit of a sequence or a family of functionals (see for instance Dal Maso [8], Proposition 6.8).

Γ-limits of sums behave differently from usual limits because, generally, they are not additive, except in particular cases like that considered in the following proposition (see Dal Maso [8], Proposition 6.20).

PROPOSITION 4.2. *Let* $F_\varepsilon : X\to\overline{I\!R}$ *and* $G_\varepsilon : X\to I\!R$ *be two families of functionals. Let us assume that*

1. *there exists the* Γ*-limit* $\Gamma(q,\tau Y^-)\lim\limits_{\varepsilon\to 0^+} F_\varepsilon = F$ *on* Y*;*

2. G_ε converges continuously to a real valued limit functional G defined on Y, that is

$$q(x_\varepsilon) \xrightarrow{\tau} v \quad \Longrightarrow \quad G_\varepsilon(x_\varepsilon) \to G(v).$$

Then $\Gamma(q, \tau Y^-) \lim\limits_{\varepsilon \to 0^+} (F_\varepsilon + G_\varepsilon) = F + G.$

5. Γ-limits of functionals defined on measures

Let us consider the sequence of functionals defined on $L^1(a,b)$ by

$$F_n(u) = \int_a^b a_n(x)|u(x)|^2 \, dx$$

where $a_n : (a,b) \to [0, +\infty)$ are given functions. It is well known, and a simple fact, that if the coefficients a_n satisfy the equi-uniform ellipticity condition

$$a_n(x) \geq \alpha > 0 \quad \text{for every } n \in I\!N \text{ and every } x \in (a,b) \tag{5.1}$$

and $1/a_n \to 1/a$ in the weak* topology of $L^\infty(a,b)$, then the sequence (F_n) Γ-converges in the weak topology of $L^1(a,b)$ to the limit functional

$$F(u) = \int_a^b a(x)|u(x)|^2 \, dx.$$

On the contrary, if condition (5.1) is not satisfied, then the existence of Γ-converging subsequences of (F_n) in the weak topology of L^1 may fail. Nevertheless, under the weaker condition

$$\int_a^b \frac{1}{a_n(x)} \, dx \leq C \quad \text{for every } n \in I\!N \tag{5.2}$$

compactness can be recovered, for instance, in the larger space $\mathcal{M}[a,b)$ for the sequence of extended functionals

$$F_n(u) = \begin{cases} \displaystyle\int_a^b a_n(x)|u(x)|^2 dx & \text{if } u \in L^1(a,b) \\ \quad +\infty & \text{elsewhere in } \mathcal{M}[a,b) \end{cases}$$

and there exists a subsequence which Γ-converges in the weak* sense of $\mathcal{M}[a,b)$. The choice of including the left extreme into the interval $[a,b)$ allows us to take into account a boundary datum which will be eventually prescribed on a; also the extreme b must be included if one wants to prescribe a datum also at this point.

The Γ-limit has been completely identified by Bouchitté [4] (see also Buttazzo and Freddi [6]). This result becomes more expressive by writing F_n in the following equivalent form

$$F_n(\lambda) = \begin{cases} \displaystyle\int_a^b a_n(x)\Big|\frac{d\lambda}{dx}(x)\Big|^2 dx & \text{if } \lambda \ll dx \\ +\infty & \text{elsewhere in } \mathcal{M}[a,b) \end{cases}$$

where dx is the Lebesgue measure on $[a,b)$ and d/dx denotes the Radon-Nikodym derivative with respect to dx (that is the absolutely continuous part of λ with respect to the Lebesgue measure).

THEOREM 5.1. (Buttazzo and Freddi [6], Theorem 2.2) *Let us assume that the measures $(a_n(x))^{-1}dx$ weakly* converge to a measure m in the space $\mathcal{M}[a,b)$. Then, for every $\lambda \in \mathcal{M}[a,b)$ we have*

$$\Gamma(w^*\mathcal{M}[a,b)^-)\lim_{n\to\infty} F_n(\lambda) = F(\lambda)$$

where

$$F(\lambda) = \begin{cases} \displaystyle\int_{[a,b)} \Big|\frac{d\lambda}{dm}\Big|^2 dm & \textit{if } \lambda \ll m \\ +\infty & \textit{elsewhere in } \mathcal{M}[a,b). \end{cases}$$

5.1. Application

Theorem 5.1 can be used to characterize the asymptotic behaviour of a sequence of functionals defined on $W^{1,2}(a,b)$ by

$$F_n(u) = \int_a^b a_n(x)|u'(x)|^2\, dx,$$

where $a_n : (a,b) \to [0,+\infty)$ are given functions, under the weak ellipticity condition (5.2). The sequence of extended functionals

$$F_n(u) = \begin{cases} \displaystyle\int_a^b a_n(x)|u'(x)|^2 dx & \text{if } u \in W^{1,1}(a,b) \\ +\infty & \text{elsewhere in } L^1(a,b) \end{cases}$$

turns out to be indeed equi-coercive in the norm topology of $L^1(a,b)$. The following theorem provides a complete description of the Γ-limit.

THEOREM 5.2. (Buttazzo and Freddi [6], Theorem 3.3) *Let us assume that the measures $(a_n(x))^{-1}dx$ weakly* converge to a measure m in the space $\mathcal{M}[a,b)$. Then, for every $u \in L^1(a,b)$ we have*

$$\Gamma(L^1(a,b)^-)\lim_{n\to\infty}[F_n + \chi_a](u) = F(u)$$

where

$$F(u) = \begin{cases} \displaystyle\int_{(a,b)} \left|\frac{du'}{dm}\right|^2 dm + \frac{|u(a^+)|^2}{m(\{a\})} & \textit{if } u \in BV(a,b),\ u' \ll m \\ +\infty & \textit{elsewhere in } L^1(a,b). \end{cases}$$

6. Reducing the thermal energy

In this section we consider the problem, already stated in Section 3.1 of reducing the thermal energy functional from three dimensions to one, and we give the proof of Theorem 3.1. We shall prove, in fact, the Γ-convergence of the family of the energies of the loaded conductors in the strongest topology under which the family of minimizers turns out to be compact. Theorem 3.1 follows then by the variational property of Γ-convergence stated in Proposition 4.1.

6.1. Coercivity of energies and compactness of minimizers

Let us write the functional $\mathcal{F}_\varepsilon^\Omega$ defined in (3.3) in the form

$$\mathcal{F}_\varepsilon^\Omega(v) = \frac{1}{2\varepsilon^2}\int_\Omega \langle A_\varepsilon(x_1,x_2,x_3)\nabla v, \nabla v\rangle\, dx + \chi_a(v)$$

where the symmetric matrix $A_\varepsilon(x_1,x_2,x_3)$ is given by

$$\begin{pmatrix} r_\varepsilon^2(x_1) & -r_\varepsilon(x_1)\dot r_\varepsilon(x_1)x_2 & -r_\varepsilon(x_1)\dot r_\varepsilon(x_1)x_3 \\ -r_\varepsilon(x_1)\dot r_\varepsilon(x_1)x_2 & 1+\dot r_\varepsilon^2(x_1)x_2^2 & \dot r_\varepsilon^2(x_1)x_2x_3 \\ -r_\varepsilon(x_1)\dot r_\varepsilon(x_1)x_3 & \dot r_\varepsilon^2(x_1)x_2x_3 & 1+\dot r_\varepsilon^2(x_1)x_3^2 \end{pmatrix}$$

The eigenvalues of A_ε turn out easily to be

$$\begin{aligned} \lambda_1 &= \frac{1+r_\varepsilon^2(x_1)+\dot r_\varepsilon^2(x_1)(x_2^2+x_3^2)}{2} \\ &\quad - \frac{\sqrt{\left[1+r_\varepsilon^2(x_1)+\dot r_\varepsilon^2(x_1)(x_2^2+x_3^2)\right]^2 - 4r_\varepsilon^2}}{2}, \\ \lambda_2 &= \frac{1+r_\varepsilon^2(x_1)+\dot r_\varepsilon^2(x_1)(x_2^2+x_3^2)}{2} \\ &\quad + \frac{\sqrt{\left[1+r_\varepsilon^2(x_1)+\dot r_\varepsilon^2(x_1)(x_2^2+x_3^2)\right]^2 - 4r_\varepsilon^2}}{2}, \\ \lambda_3 &= 1, \end{aligned}$$

and therefore, recalling also that $x_2^2 + x_3^2 \le 1$, we have

$$\lambda_i \ge \frac{r_\varepsilon^2}{1 + r_\varepsilon^2 + \dot{r}_\varepsilon^2}, \quad i = 1, 2, 3. \tag{6.1}$$

Then, setting

$$a_\varepsilon(x_1) = \frac{r_\varepsilon^2(x_1)}{\varepsilon^2 \left[1 + r_\varepsilon^2(x_1) + \dot{r}_\varepsilon^2(x_1)\right]} \tag{6.2}$$

we can prove the following theorem.

THEOREM 6.1. *If there exists a constant $K > 0$ such that*

$$\int_a^b \frac{1}{a_\varepsilon(x_1)}\, dx_1 \le K \tag{6.3}$$

then

(i) the family of the energy functionals $\{\mathcal{F}_\varepsilon^\Omega - \mathcal{L}_\varepsilon^\Omega\}_\varepsilon$ is equi-coercive in $L^1(a,b)$, in the sense that, for every $M \in I\!R$ there is a compact subset K_M of $L^1(a,b)$ such that $\{q(v) : \mathcal{F}_\varepsilon^\Omega(v) - \mathcal{L}_\varepsilon^\Omega(v) \le M\} \subseteq K_M$ for every $\varepsilon > 0$;

(ii) the family $\{q(v^\varepsilon)\}_\varepsilon$, where v^ε are minimizers of $\mathcal{F}_\varepsilon^\Omega - \mathcal{L}_\varepsilon^\Omega$, is weakly relatively compact in $BV(a,b)$.*

REMARK 6.1. Condition (6.3) is satisfied under assumptions (3.5) and (3.6).

Proof. Let's prove *(i)*. Let v be a function in $W^{1,2}(\Omega)$ such that $\mathcal{F}_\varepsilon^\Omega(v) - \mathcal{L}_\varepsilon^\Omega(v) \le M$ for every $\varepsilon > 0$; then

$$\mathcal{F}_\varepsilon^\Omega(v) \le M + |\mathcal{L}_\varepsilon^\Omega(v)| \le M + \pi \|g\|_\infty \|q(v)\|_1 \quad \forall\, \varepsilon > 0. \tag{6.4}$$

By definition of $\mathcal{F}_\varepsilon^\Omega$, (6.4) implies that $q(v)(a) = 0$, and the Poincaré inequality gives

$$\begin{aligned} \int_a^b |q(v)|\, dx_1 &\le C_p \int_a^b |q(v)'|\, dx_1 \\ &= \frac{C_p}{\pi} \int_a^b \left| \int_{S_1} v_{x_1}^\varepsilon \, dx_2 dx_3 \right| dx_1 \\ &\le \frac{C_p}{\pi} \int_\Omega |\nabla v|\, dx. \end{aligned} \tag{6.5}$$

On the other hand, by Hölder's inequality

$$\begin{aligned}\int_\Omega |\nabla v|\,dx &\le \int_\Omega |\nabla v|\sqrt{a_\varepsilon}\frac{1}{\sqrt{a_\varepsilon}}\,dx\\ &\le \Big(\int_\Omega a_\varepsilon(x_1)|\nabla v|^2 dx\Big)^{1/2}\Big(\int_\Omega \frac{1}{a_\varepsilon(x_1)}\,dx\Big)^{1/2}\end{aligned}\tag{6.6}$$

for every $\varepsilon > 0$, while, from (6.1) and (6.2) we get

$$\int_\Omega a_\varepsilon(x_1)|\nabla v|^2\,dx \le 2\mathcal{F}_\varepsilon^\Omega(v).\tag{6.7}$$

By (6.4) and using the fact that $r_\varepsilon(x_1) \le C\varepsilon$, together with (6.6), (6.7) and assumption (6.3), we have

$$\|\nabla v\|_{1,\Omega}^2 \le K\pi(M + \pi\|g\|_\infty\|q(v)\|_1)$$

and, by (6.5), there are two positive constants C and D such that $\|\nabla v\|_{1,\Omega}^2 \le C + D\|\nabla v\|_{1,\Omega}$. Hence $\|\nabla v\|_{1,\Omega} \le \widetilde{C}$ which can be used in (6.5) to conclude that there exists a constant $R > 0$, depending only on M, such that

$$\mathcal{F}_\varepsilon^\Omega(v) - \mathcal{L}_\varepsilon^\Omega(v) \le M\ \forall\,\varepsilon > 0 \quad\Longrightarrow\quad \|q(v)\|_1 \le C_p\|q(v)'\|_1 \le R.$$

Then, for every $\varepsilon > 0$ the sets $\{q(v)\ :\ \mathcal{F}_\varepsilon^\Omega(v) - \mathcal{L}_\varepsilon^\Omega(v) \le M\}$ are all contained into a same closed ball of $W^{1,1}(a,b)$ which is a weakly* compact subset of $BV(a,b)$, hence strongly compact in $L^1(a,b)$, and the family of the total energy functionals turns out to be equi-coercive.

(ii) follows by *(i)* and by the fact that, if v^ε minimizes $\mathcal{F}_\varepsilon^\Omega - \mathcal{L}_\varepsilon^\Omega$, then $\mathcal{F}_\varepsilon^\Omega(v^\varepsilon) - \mathcal{L}_\varepsilon^\Omega(v^\varepsilon) \le 0$. Let's remark that the weak* convergence in $BV(a,b)$ stated in *(ii)* is only apparently stronger then the norm convergence in $L^1(a,b)$ because, as pointed out at the end of the proof of point *(i)*, any family $q(v_\varepsilon)$ such that the corresponding total energies are uniformly bounded from above is weakly* relatively compact in $BV(a,b)$. $\square$

In the sequel, it will be convenient to deal with the extension of the functional $\mathcal{F}_\varepsilon^\Omega$ to $L^1(\Omega)$ by putting $\mathcal{F}_\varepsilon^\Omega(v) = +\infty$ if $v \in L^1(\Omega)\backslash W^{1,2}(\Omega)$. It is clear that Theorem 6.1 still holds for the extended functionals.

6.2. Γ-CONVERGENCE OF THE THERMAL ENERGY

THEOREM 6.2. *Let the convergence assumptions* (3.5) *and* (3.6) *of Theorem* 3.1 *hold true. Then, for every* $v \in L^1(a,b)$

$$\Gamma(q, L^1(a,b)^-) \lim_{\varepsilon\to 0^+} \mathcal{F}^\Omega_\varepsilon(v) = \mathcal{F}(v)$$

where $\mathcal{F}$ *is the functional defined in* (3.1).

Proof. Let us observe, first of all, that assumptions (3.5) and (3.6) imply that (see (6.2))

$$\frac{1}{a_\varepsilon(x_1)} dx_1 \to m \quad \text{weakly* in } \mathcal{M}[a,b]. \tag{6.8}$$

Let $v \in L^1(a,b)$ and let (ε_n) be a sequence such that $\lim_{n\to\infty} \varepsilon_n = 0$; according to 1. and 2. of Remark 4.1, we have to prove the following two statements:

1. for any sequence (v^n) of functions in $L^1(\Omega)$ such that $q(v^n) \to v$ in $L^1(a,b)$ we have

$$\liminf_{n\to\infty} \mathcal{F}^\Omega_{\varepsilon_n}(v^n) \geq \mathcal{F}(v), \tag{6.9}$$

2. there exists a sequence of functions $\bar{v}^n \in L^1(\Omega)$ such that $q(\bar{v}^n) \to v$ in $L^1(a,b)$ and

$$\lim_{n\to\infty} \mathcal{F}^\Omega_{\varepsilon_n}(\bar{v}^n) = \mathcal{F}(v).$$

Let's prove *1.* To avoid trivialities we may assume that the left hand side of (6.9) be finite and that the lim inf be a limit, which implies that $v^n \in W^{1,2}(\Omega)$ for every n large enough and there exists a positive constant C such that

$$\sup_{n\in\mathbb{N}} \mathcal{F}^\Omega_{\varepsilon_n}(v^n) \leq C. \tag{6.10}$$

From inequality (6.7) we have

$$\mathcal{F}^\Omega_{\varepsilon_n}(v^n) \geq \frac{1}{2}\int_\Omega a_{\varepsilon_n}(x_1)|v^n_{x_1}(x_1,x_2,x_3)|^2\, dx \tag{6.11}$$

and therefore, by Jensen's inequality, we get

$$\begin{aligned}
\liminf_{n\to\infty} \mathcal{F}^\Omega_{\varepsilon_n}(v^n) &\geq \frac{1}{2}\liminf_{n\to\infty}\int_\Omega a_{\varepsilon_n}(x_1)|v^n_{x_1}(x_1,x_2,x_3)|^2 dx \\
&= \frac{\pi}{2}\liminf_{n\to\infty}\int_a^b \frac{a_{\varepsilon_n}(x_1)}{\pi}\Big(\int_{S_1}|v^n_{x_1}(x_1,x_2,x_3)|^2 dx_2 dx_3\Big) dx_1 \\
&\geq \frac{\pi}{2}\liminf_{n\to\infty}\int_a^b a_{\varepsilon_n}(x_1)|q(v^n)'(x_1)|^2 dx_1.
\end{aligned}$$

By metrizability of the weak* topology on bounded sets, we have that $q(v^n)'$ weakly* converges in $\mathcal{M}[a,b)$ to v' so that, by Theorem 5.1, we get

$$\liminf_{n\to\infty} \int_a^b a_{\varepsilon_n}(x_1)|q(v^n)'(x_1)|^2 dx_1 \geq \begin{cases} \int_{[a,b)} \left|\frac{dv'}{dm}\right|^2 dm \text{ if } v' \ll m \\ +\infty \text{ elsewhere in } L^1(a,b). \end{cases}$$

and, finally,

$$\liminf_{n\to\infty} \mathcal{F}^\Omega_{\varepsilon_n}(v^n) \geq \mathcal{F}(v)$$

which proves *1.*

It remains now to prove *2.* By Theorem 5.2, there exists a sequence $w^n \in W^{1,2}(a,b)$, with $w^n(a) = 0$, such that

$$w^n \to v \quad \text{in } L^1(a,b)$$

and

$$\int_a^b \frac{r^2_{\varepsilon_n}(x_1)}{\varepsilon_n^2} |w^{n\prime}|^2 \, dx_1 \to 2\frac{\mathcal{F}(v)}{\pi}.$$

Let's define

$$\bar{v}^n(x_1, x_2, x_3) = w^n(x_1)$$

for every $n \in I\!N$. Then $\bar{v}^n$ satisfy the boundary condition, converge to v in $L^1(\Omega)$, and

$$\mathcal{F}^\Omega_{\varepsilon_n}(\bar{v}^n) = \frac{\pi}{2\varepsilon_n^2} \int_a^b r^2_{\varepsilon_n}(x_1)|w^{n\prime}|^2 \, dx_1$$

which tends, of course, to $\mathcal{F}(v)$. □

COROLLARY 6.1. *Let the convergence assumptions* (3.5) *and* (3.6) *of Theorem* 3.1 *hold true. Then, for every* $v \in L^1(a,b)$

$$\Gamma(q, L^1(a,b)^-) \lim_{\varepsilon\to 0^+} [\mathcal{F}^\Omega_\varepsilon - \mathcal{L}^\Omega_\varepsilon](v) = \mathcal{F}(v) - \mathcal{L}(v)$$

where $\mathcal{F}$ *and* $\mathcal{L}$ *are the functionals defined in* (3.1) *and* (3.2).

Proof. It's enough to observe that the real valued functionals $\mathcal{L}^\Omega_\varepsilon$ continuously converge to $\mathcal{L}$ and to apply Proposition 4.2. □

Proof of Theorem 3.1 By the Γ-convergence Theorem 6.2 and the compactness Theorem 6.1, Proposition 4.1 applies and *(i)* and *(ii)* of Theorem 3.1 follows immediately by *(a)* and *(b)* in Proposition 4.1. □

7. Reducing the elastic energy

In this section we consider the problem, already stated in Section 3.2, of dimensional reduction of the functionals $\mathcal{F}_\varepsilon - \mathcal{L}_\varepsilon$ defined in (3.8) and (3.9) which, after having changed variables take the forms $\mathcal{F}_\varepsilon^\Omega - \mathcal{L}_\varepsilon^\Omega$ displayed in (3.12) and (3.13).

7.1. Coercivity of energies and compactness of minimizers

The averaging map q, introduced in (3.4), is now considered as defined on the space $W^{1,2}(\Omega;\mathbb{R}^3)$ instead than $W^{1,2}(\Omega)$, but acting only on the first component v_1 of $v \in W^{1,2}(\Omega;\mathbb{R}^3)$; for this reason we will not distiguish, in the sequel, between $q(v)$ and $q(v_1)$.

Let a_ε be the function of x_1 defined in (6.2). We can prove the following theorem.

THEOREM 7.1. *If there exists a constant $K > 0$ such that*

$$\int_a^b \frac{1}{a_\varepsilon(x_1)}\,dx_1 \le K \tag{7.1}$$

then

(i) *the family of the energy functionals $\{\mathcal{F}_\varepsilon^\Omega - \mathcal{L}_\varepsilon^\Omega\}_\varepsilon$ is equi-coercive in $L^1(a,b)$, in the sense that, for every $M \in \mathbb{R}$ there is a compact subset K_M of $L^1(a,b)$ such that $\{q(v_1) : \mathcal{F}_\varepsilon^\Omega(v) - \mathcal{L}_\varepsilon^\Omega(v) \le M\} \subseteq K_M$ for every $\varepsilon > 0$;*

(ii) *the family $\{q(v_1^\varepsilon)\}_\varepsilon$, where v^ε are minimizers of $\mathcal{F}_\varepsilon^\Omega - \mathcal{L}_\varepsilon^\Omega$, is weakly* relatively compact in $BV(a,b)$.*

REMARK 7.1. Condition (7.1) is satisfied under assumptions (3.14) and (3.15).

Proof. Let's prove *(i)*. Let $v \in W^{1,2}(\Omega;\mathbb{R}^3)$ be such that $\mathcal{F}_\varepsilon^\Omega(v) - \mathcal{L}_\varepsilon^\Omega(v) \le M$ for every $\varepsilon > 0$; then

$$\mathcal{F}_\varepsilon^\Omega(v) \le M + |\mathcal{L}_\varepsilon^\Omega(v)| \le M + \pi\|g\|_\infty\|q(v)\|_1 \quad \forall\,\varepsilon > 0. \tag{7.2}$$

Let us write the functional $\mathcal{F}_\varepsilon^\Omega$ into the form

$$\mathcal{F}_\varepsilon^\Omega(v) = \frac{1}{\varepsilon^2}\int_\Omega \langle A_\varepsilon(x)\nabla v, \nabla v\rangle\,dx + \chi_a(v)$$

with $\nabla v = (v_{1,1}, v_{1,2}, v_{1,3}, v_{2,1}, v_{2,2}, v_{2,3}, v_{3,1}, v_{3,2}, v_{3,3})^T$, $A_\varepsilon = \mu S_\varepsilon + \frac{\lambda}{2}T_\varepsilon$ and S_ε and T_ε are the 9×9 positive semi-definite symmetric matrices associated to the corresponding quadratic forms. We remark that, since $f(\boldsymbol{e}(u)) = 0$ if and only if u is a rigid displacement of the body Σ_ε, that is if and only if $u = a + b \wedge x$; then

$$\langle A_\varepsilon(x)\nabla v, \nabla v\rangle = 0$$

if and only if

$$v = a + (b_2 r_\varepsilon x_3 - b_3 r_\varepsilon x_2,\ b_3 x_1 - b_1 r_\varepsilon x_3,\ b_1 r_\varepsilon x_2 - b_2 x_1)$$

that is if and only if $\nabla v(x)$ belongs to the space E_ε spanned by

$$\begin{array}{l} n_1^\varepsilon = (\ \ 0, \quad 0, \quad 0, \quad -\dot r_\varepsilon x_3,\ 0,\ -r_\varepsilon,\ \dot r_\varepsilon x_2,\ r_\varepsilon,\ 0), \\ n_2^\varepsilon = (-\dot r_\varepsilon x_3, \quad 0, \quad -r_\varepsilon, \quad 0, \quad 0, \quad 0, \quad 1, \quad 0,\ 0), \\ n_3^\varepsilon = (-\dot r_\varepsilon x_2,\ -r_\varepsilon, \quad 0, \quad 1, \quad 0, \quad 0, \quad 0, \quad 0,\ 0). \end{array}$$

Now, the gradient ∇v can be written as sum of a term, $\nabla_{E_\varepsilon} v$, belonging to the null eigenspace E_ε and another term, $\nabla_{E_\varepsilon^\perp} v$, belonging to the ortogonal $E_\varepsilon^\perp$

$$\nabla v = \nabla_{E_\varepsilon} v + \nabla_{E_\varepsilon^\perp} v.$$

Since $A_\varepsilon \nabla_{E_\varepsilon} v = 0$, we have

$$\int_\Omega \langle A_\varepsilon \nabla v, \nabla v\rangle\, dx = \int_\Omega \langle A_\varepsilon \nabla_{E_\varepsilon^\perp} v, \nabla_{E_\varepsilon^\perp} v\rangle\, dx.$$

The eigenvalues of the matrix S_ε can be explicitly computed and the least positive one among them is

$$s_\varepsilon(x_1, x_2, x_3) = \frac{1 + r_\varepsilon^2(x_1) + \dot r_\varepsilon^2(x_1)(x_2^2 + x_3^2)}{2} - \frac{\sqrt{\left[1 + r_\varepsilon^2(x_1) + \dot r_\varepsilon^2(x_1)(x_2^2 + x_3^2)\right]^2 - 4r_\varepsilon^2}}{2}$$

and, as in the thermal case, it can be estimated from below by $\varepsilon^2 a_\varepsilon$ (see (6.2)), hence

$$\int_\Omega a_\varepsilon(x_1)|\nabla_{E_\varepsilon^\perp} v|^2\, dx \le \mathcal{F}_\varepsilon^\Omega(v). \tag{7.3}$$

Concerning the component of the gradient along E_ε, since we have $v_{1,1|E_\varepsilon} = b_2^\varepsilon \dot r_\varepsilon x_3 - b_3^\varepsilon \dot r_\varepsilon x_2$ for suitable constants b_2^ε and b_3^ε, then

$$q(v_{1,1|E_\varepsilon}^\varepsilon) = \frac{1}{\pi}\int_{S_1} v_{1,1|E_\varepsilon}\, dx_2 dx_3 = 0 \tag{7.4}$$

so that

$$q(v_1)' = q(v_{1,1}) = q(v_{1,1|E_\varepsilon}) + q(v_{1,1|E_\varepsilon^\perp}) = q(v_{1,1|E_\varepsilon^\perp}) \tag{7.5}$$

and therefore, by the Poincaré inequality,

$$\int_a^b |q(v_1)|dx_1 \le C_p \int_a^b |q(v_1)'|dx_1 = \\ = C_p \int_a^b |q(v_{1,1|E_\varepsilon^\perp})|dx_1 \le \frac{C_p}{\pi} \int_\Omega |\nabla_{E_\varepsilon^\perp} v|\, dx. \tag{7.6}$$

On the other hand, by the Holder's inequality,

$$\int_\Omega |\nabla_{E_\varepsilon^\perp} v| dx \le \left(\int_\Omega a_\varepsilon(x_1) |\nabla v_{E_\varepsilon^\perp}|^2 dx \right)^{1/2} \left(\int_\Omega \frac{1}{a_\varepsilon(x_1)}\, dx \right)^{1/2}. \tag{7.7}$$

By (7.2) and using the fact that $r_\varepsilon(x_1) \le C\varepsilon$, together with (7.3), (7.7) and assumption (7.1), we have

$$\|\nabla_{E_\varepsilon^\perp} v\|_{1,\Omega}^2 \le K\pi(M + \pi\|g\|_\infty \|q(v_1)\|_1)$$

and, by (7.6), there are two positive constants C and D such that $\|\nabla_{E_\varepsilon^\perp} v\|_{1,\Omega}^2 \le C + D\|\nabla_{E_\varepsilon^\perp} v\|_{1,\Omega}$. Hence $\|\nabla_{E_\varepsilon^\perp} v\|_{1,\Omega} \le \widetilde{C}$ which can be used in (7.6), as in the thermal case, to conclude the proof.

(ii) follows by the fact that, if v^ε is a minimizer of $\mathcal{F}_\varepsilon^\Omega - \mathcal{L}_\varepsilon^\Omega$, then $\mathcal{F}_\varepsilon^\Omega(v^\varepsilon) - \mathcal{L}_\varepsilon^\Omega(v^\varepsilon) \le 0$. □

7.2. Γ-convergence of the elastic energy

THEOREM 7.2. *Let the convergence assumptions* (3.14) *and* (3.15) *of Theorem 3.2 hold true. Then, for every* $v \in L^1(a,b)$

$$\Gamma(q, L^1(a,b)^-) \lim_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon^\Omega(v) = \mathcal{F}(v)$$

where $\mathcal{F}(v)$ *is the functional defined in* (3.10).

Proof. Let (ε_n) be a sequence such that $\lim_{n\to\infty} \varepsilon_n = 0$. The claim will be attained by proving the following two statements:
1. for any sequence (v^n) of functions in $L^1(\Omega; I\!R^3)$ such that $q(v_1^n) \to v$ in $L^1(a,b)$ we have

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}^\Omega(v^n) \ge \mathcal{F}(v); \tag{7.8}$$

2. for every $v \in L^1(a,b)$

$$\mathcal{F}(v) \ge \Gamma(L^1(a,b)^-) \limsup_{n\to\infty} \mathcal{F}_{\varepsilon_n}^\Omega(v). \tag{7.9}$$

Let's prove *1.* Avoiding trivialities we can assume that the left hand side of (7.8) be finite and that the lower limit be a limit, which implies that $v^n \in W^{1,2}(\Omega; \mathbb{R}^3)$ for every n large enough and that there exists a positive constant C such that

$$\sup_{n\in\mathbb{N}} \mathcal{F}^{\Omega}_{\varepsilon_n}(v^n) \leq C. \tag{7.10}$$

We have

$$\mathcal{F}^{\Omega}_{\varepsilon_n}(v^n) = \frac{1}{\varepsilon_n^2}\int_\Omega \langle A_{\varepsilon_n}\nabla_{E^{\perp}_{\varepsilon_n}} v^n, \nabla_{E^{\perp}_{\varepsilon_n}} v^n\rangle \, dx$$

where $A_\varepsilon = \mu S_\varepsilon + \dfrac{\lambda}{2}T_\varepsilon$. The only non zero eigenvalue of the rank one matrix T_ε is

$$t_\varepsilon(x_1, x_2, x_3) = 2 + r^2_\varepsilon(x_1) + \dot{r}^2_\varepsilon(x_1)(x_2^2 + x_3^2)$$

and it is easy to see that

$$\mu s_\varepsilon + \frac{\lambda}{2}t_\varepsilon \geq \frac{E}{2}\varepsilon^2 a_\varepsilon$$

for any $\varepsilon > 0$ small enough, uniformly with respect to x_1, x_2, x_3. Then, by Jensen's inequality, we have

$$\begin{aligned}
\mathcal{F}^{\Omega}_{\varepsilon_n}(v^n) &\geq \frac{E}{2}\int_\Omega a_{\varepsilon_n}(x_1)|\nabla_{E^{\perp}_{\varepsilon_n}} v^n|^2 \, dx \\
&\geq \frac{E}{2}\int_\Omega a_{\varepsilon_n}(x_1)|v^n_{1,1|E^{\perp}_{\varepsilon_n}}|^2 \, dx \\
&\geq \pi\frac{E}{2}\int_a^b a_{\varepsilon_n}(x_1)\Big(\frac{1}{\pi}\int_{S_1} v^n_{1,1|E^{\perp}_{\varepsilon_n}} \, dx_2 dx_3\Big)^2 dx_1 \\
&= \pi\frac{E}{2}\int_a^b a_{\varepsilon_n}(x_1)|q(v^n_{1,1|E_{\varepsilon_n^{\perp}}})|^2 dx_1 \\
&= \pi\frac{E}{2}\int_a^b a_{\varepsilon_n}(x_1)|q(v^n_1)'|^2 dx_1
\end{aligned} \tag{7.11}$$

where in the last equality we used the fact that $q(v^n_{1,1|E_{\varepsilon_n^{\perp}}}) = q(v^n_1)'$ (see (7.5)). Since

$$q(v^n_1)' \to v' \quad \text{weakly* in } \mathcal{M}[a,b),$$

by Theorem 5.1 we have

$$\liminf_{n\to\infty}\int_a^b a_{\varepsilon_n}(x_1)|q(v^n_1)'(x_1)|^2 dx_1 \geq \begin{cases} \displaystyle\int_{[a,b)}\Big|\frac{dv'}{dm}\Big|^2 dm \text{ if } v' \ll m \\ +\infty \text{ elsewhere in } L^1(a,b). \end{cases}$$

and therefore we get

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}^{\Omega}(v^n) \geq \mathcal{F}(v)$$

which proves *1.*

Let's prove *2.* Let $v \in L^1(a,b)$ be such that $v' \ll m$, and moreover that the function $\theta(x_1) = \dfrac{dv'}{dm}(x_1)$ belong to $C_0^1[a,b)$. Let

$$w_\varepsilon(x_1, x_2, x_3) = \varepsilon^2 \int_a^{x_1} \frac{\theta(t)}{r_\varepsilon^2(t)}\, dt.$$

Then $w_\varepsilon \in W^{1,2}(\Omega)$ and, by assumptions (3.14) and (3.15),

$$q(w_\varepsilon) \to v \quad \text{in } L^1(a,b)$$

$$q(w_\varepsilon)' \to v' \quad \text{weakly* in } \mathcal{M}[a,b)$$

and moreover

$$\lim_{\varepsilon\to 0^+} \int_a^b \frac{r_\varepsilon^2}{\varepsilon^2} |q(w_\varepsilon)'|^2 dx_1 = \int_{[a,b)} \left|\frac{dv'}{dm}\right|^2 dm.$$

Let now be

$$v^\varepsilon = (w_\varepsilon, -\frac{\lambda}{2(\lambda+\mu)} x_2 r_\varepsilon q(w_\varepsilon)', -\frac{\lambda}{2(\lambda+\mu)} x_3 r_\varepsilon q(w_\varepsilon)').$$

Then

$$\mathcal{F}_\varepsilon^{\Omega}(v^\varepsilon) = \pi \frac{E}{2} \int_a^b \frac{r_\varepsilon^2}{\varepsilon^2} |q(w_\varepsilon)'|^2 dx_1 + \frac{\mu\lambda^2}{8(\lambda+\mu)^2} \int_a^b \frac{r_\varepsilon^4}{\varepsilon^2} |q(w_\varepsilon)''|^2 dx_1.$$

Since

$$q(w_\varepsilon)'' = \theta' \frac{\varepsilon^2}{r_\varepsilon^2} - 2\theta \frac{\varepsilon^2}{r_\varepsilon^2} \dot{r}_\varepsilon^2$$

we have

$$\lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon^{\Omega}(v^\varepsilon) = \pi \frac{E}{2} \int_{[a,b)} \left|\frac{dv'}{dm}\right|^2 dm = \mathcal{F}(v).$$

Then, denoted by

$$\mathcal{F}^+ = \Gamma\big(L^1(a,b)^-\big) \limsup_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon^{\Omega}$$

we can say that

$$\mathcal{F}^+(v) \leq \mathcal{F}(v) \quad \forall v \in L^1(a,b) \ : \ \frac{dv'}{dm} \in C_0^1[a,b).$$

In order to prove (7.9) we can assume $\mathcal{F}(v) < +\infty$ which implies $v \in L^1(a,b)$ and the function $\theta(x_1) = \dfrac{dv'}{dm}(x_1)$ belongs to $L^2([a,b),m)$. Then there exists a sequence $\theta_h \in C_0^1[a,b)$ such that $\theta_h \to \theta$ in the space $L^2([a,b),m)$ and setting

$$v_h(x_1) = \int_{[a,x_1)} \theta_h(t)dm(t)$$

we have $v_h \to v$ in $L^1(a,b)$ and, by the lower semicontinuity of $\mathcal{F}^+$ we get

$$\begin{aligned}\mathcal{F}^+(v) &\le \liminf_{h\to\infty} \mathcal{F}^+(v_h) \le \liminf_{h\to\infty} \mathcal{F}(v_h)\\ &= \liminf_{h\to\infty} \pi\frac{E}{2}\int_{[a,b)} |\theta_h(t)|^2 dm(t) = \pi\frac{E}{2}\int_{[a,b)} |\theta(t)|^2 dm(t)\\ &= \pi\frac{E}{2}\int_{[a,b)} \left|\frac{dv'}{dm}\right|^2 dm = \mathcal{F}(v)\end{aligned}$$

which concludes the proof. □

COROLLARY 7.1. *Let the convergence assumptions* (3.14) *and* (3.15) *of Theorem* 3.2 *hold true. Then, for every* $v \in L^1(a,b)$

$$\Gamma(q, L^1(a,b)^-)\lim_{\varepsilon\to 0^+} [\mathcal{F}_\varepsilon^\Omega - \mathcal{L}_\varepsilon^\Omega](v) = \mathcal{F}(v) - \mathcal{L}(v)$$

where $\mathcal{F}$ *and* $\mathcal{L}$ *are the functionals defined in* (3.10) *and* (3.11).

Proof. It's enough to observe that the real valued functionals $\mathcal{L}_\varepsilon^\Omega$ continuosly converge to $\mathcal{L}$ and to apply Proposition 4.2. □

REMARK 7.2. The proof of Theorem 3.2 is analogous to that of Theorem 3.1, already given at the end of Section 6.

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