p-LAPLACIAN PROBLEMS WITH CRITICAL SOBOLEV EXPONENT

GIAMPIERO PALATUCCI

Abstract. We use variational methods to study the asymptotic behavior of solutions of p-Laplacian problems with nearly subcritical nonlinearity in general, possibly non-smooth, bounded domains.

Contents

1. Introduction

Consider the elliptic Dirichlet problem with nearly subcritical nonlinearity:

$$\begin{cases} -\Delta u_\varepsilon = u_\varepsilon^{2^*-1-\varepsilon}, & u_\varepsilon > 0 \quad \text{in } \Omega, \\ \quad u_\varepsilon = 0 & \qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded set in $\mathbb{R}^N$, $N \geq 3$, $2^* = 2N/(N-2)$ being the critical Sobolev exponent.

It is well known that when ε is positive, problem (1.1) has at least a solution u_ε. On the other hand, when ε is zero, the existence, the multiplicity and the properties of the solutions strongly depend by the shape and the topology of the domain Ω. For instance, in this case $\varepsilon = 0$: Pohozaev in [17] discovered that (1.1) has no solutions if Ω is star-shaped; Bahri and Coron in [4] analyzed wide effects of the topology of the domain, also proving that (1.1) has a solution, when $N = 3$

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and Ω is not contractible; Ding in [8] proved the existence of the solutions when Ω is contractible with a specific geometry.

In view of this qualitative change in the critical case, it is interesting to study the asymptotic behavior of the subcritical solutions u_ε when ε goes to zero.

This problem was widely investigated in the late 80's. Atkinson and Peletier in [2] proposed the first study in this sense, showing that the solutions u_ε of (1.1) in the unit ball in $\mathbb{R}^3$ are such that

$$\lim_{\varepsilon\to 0}\varepsilon u_\varepsilon^2(0)=\frac{32}{\pi}\quad\text{and}\quad\lim_{\varepsilon\to 0}\varepsilon^{-1/2}u_\varepsilon(x)=\frac{\sqrt{\pi}}{4\sqrt{2}}\left(\frac{1}{|x|}-1\right),\ \forall x\neq 0.$$

In [6], Brezis and Peletier returned to this problem in the case of Ω being a spherical domain. They proved the same precise results, along with other interesting statements. The relevance of the results in [6] was that the subcritical solutions concentrate at exactly one point of $\overline{\Omega}$; the authors also conjectured that the same kind of results holds for non spherical domains.

This conjecture was proved in the case of any smooth domains Ω by Han in [12] and Rey in [19]. They showed that the solutions u_ε of (1.1), that are maximizing for the following variational problem

$$S^*_{2,\varepsilon}:=\sup\left\{\int_\Omega |u|^{2^*-\varepsilon}:\int_\Omega|\nabla u|^2dx\leq 1,\ u=0\text{ on }\partial\Omega\right\},\tag{1.2}$$

concentrate at exactly one point x_0 in $\overline{\Omega}$. They also showed that x_0 is a critical point of the Robin function of Ω (the diagonal of the regular part of the Green function), answering to a conjecture by Brezis and Peletier in [6].

All the above results makes effort of some *Standard elliptic regularity* techniques that require to work in smooth domains.

Recently, the author in [15] proved the same concentration result (without identifying the blowing-up) in the case of any bounded domain Ω with no strong regularity assumptions, as well as describing the asymptotic analysis of the Sobolev quotient (1.2) by means of De Giorgi's Γ-convergence.

The inspiration in [15] comes from the paper [1], where Amar and Garroni, following Flucher and Müller [9], used Γ-convergence techniques to study some concentration phenomena related to critical Sobolev exponent. Precisely, Amar and Garroni studied the following problem

$$\sup\left\{\varepsilon^{-2^*}\int_\Omega G(\varepsilon u)\,dx : u\in H^1_0(\Omega),\ ||\nabla u||_{L^2(\Omega)}\leq 1\right\},\tag{1.3}$$

when $\varepsilon \to 0$, where G is a non-negative upper semi-continuous function bounded from above by $c|t|^{2^*}$. They obtained a Γ-convergence result, that implies concentration phenomena arising in critical growth problems.

We note that problem (1.2) can not be reduced to problem (1.3).

We also emphasize on the importance of regularity assumptions on the domain. In fact, in the case of a smooth domain, the Robin function is equal to ∞ on the boundary and has no critical points in a neighborhood of the boundary. In the contrary, in the paper [10], Flucher, Garroni and Müller provided us with an example of a nonsmooth domain $\tilde{\Omega}$ such that its Robin function achieves its infimum on a point $\bar{x}$ on the boundary (see also [11]). As a further matter, Pistoia and Rey showed in [18] that the maximizing solutions of the elliptic Dirichlet problem (with nearly subcritical nonlinearity) in $\tilde{\Omega}$ concentrate at $\bar{x} \in \partial\tilde{\Omega}$.

Following the same variational framework in [1], in this paper we generalize the asymptotic analysis of subcritical solutions of the analogous problem to (1.1) for the p-Laplacian operator:

$$\begin{cases} -\Delta_p u_\varepsilon = u_\varepsilon^{p^*-1-\varepsilon}, \quad u_\varepsilon > 0 & \text{in } \Omega, \\ \qquad u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $p^* = pN/(N-p)$, for any $p \in (1, N)$.

The related variational problem is given by

$$S_\varepsilon^* := \sup \left\{ F_\varepsilon(u) : \int_\Omega |\nabla u|^p dx \leq 1, \, u = 0 \text{ on } \partial\Omega \right\} \tag{1.5}$$

$$\text{with} \quad F_\varepsilon := \int_\Omega |u|^{p^*-\varepsilon} dx, \tag{1.6}$$

Thus, we are interested to the asymptotic behavior of the functional (1.6) in terms of Γ-convergence.

One of the main points of Γ-convergence is the choice of the right topology. Here we need to work with a weak topology that can allow us to recover the desired concentration result. Taking into account the constraint on the p-energy in the variational problem (1.5), we will study every sequence u_ε weakly converging to some function u in the Sobolev space $W_0^{1,p}(\Omega)$ such that its enegy $|\nabla u_\varepsilon|^p$ converges to some measure μ in the sense of measures (see Section 2).

In this setting, the asymptotic behavior of the functional (1.6) is described by the following functional F that depends by the two variables u, belonging to $W_0^{1,p}(\Omega)$, and μ, being a non-negative Borel measure (with its atomic coefficients μ_i):

$$F(u,\mu) := \int_\Omega |u|^{p^*} dx + S^* \sum_{i=0}^{\infty} \mu_i^{\frac{p^*}{p}}, \tag{1.7}$$

where S^* is the best constant for which $\|u\|^{p^*}_{L^{p^*}(\Omega)} \leq S^* \|\nabla u\|^{p^*}_{L^p(\Omega)}$, for every u in $W^{1,p}_0(\Omega)$ (see Theorem 3.1).

As a consequence of a sharp estimation on the limit functional F (see Lemma 3.2), by the Γ-convergence result we can deduce the concentration of the subcritical solutions u_ε of (1.4) that are maximizing for (1.5); i.e., there exists a point $x_0 \in \overline{\Omega}$ such that

$$|\nabla u_\varepsilon|^p \overset{*}{\rightharpoonup} \delta_{x_0} \quad \text{in } \mathcal{M}(\overline{\Omega}),$$

where δ_{x_0} is the Dirac mass at x_0 and $\mathcal{M}(\overline{\Omega})$ denotes the set of non-negative Borel measures on $\overline{\Omega}$ endowed with the weak-$*$ topology (see Theorem 5.1).

It is worth pointing out that every result of this paper holds for any $p \in (1,N)$. In particular, this means that, for $N \geq 3$, we can also recover the "classical" concentration result for the solutions of problem (1.1), while for $N = 2$, we can give a result involving the singular p-Laplacian operator.

A natural question arises: can we localize the blowing up? We think that it will be possible to prove that the maximizing sequences prefer again to concentrate at some particular points (like it happens in the classical $p = 2$ case), being based on the papers [9] and [10], where Flucher, Garroni and Müller showed that the concentration at the critical points of the Robin function is a general phenomenon.

Finally, with Pisante in ([16]), among other results, we study some non-local concentration phenomena in the same spirit of this paper.

Here is the outline of this paper. In the following section, we fix the notation. In Section 3, we discuss the asymptotic behavior of the functional F_ε, giving its characterization. Section 4 is devoted to the proof of the Γ-convergence result for the functional F_ε. Finally, we recover the concentration result in Section 5.

2. Framework

Throughout the paper, the domain Ω will be a general (possible non-smooth) bounded open subset of $\mathbb{R}^N$, with $N > p$, $p \in (1,N)$. $\overline{\Omega}$ denotes the closure of Ω and $\partial\Omega$ is the boundary of Ω.

We use $\mathcal{M}(\overline{\Omega})$ to indicate the set of non-negative bounded total variation Borel measures on $\overline{\Omega}$, endowed with weak-$*$ topology. Let $(\mu_n) \subseteq \mathcal{M}(\overline{\Omega})$, $\mu \in \mathcal{M}(\overline{\Omega})$, we have:

$$\mu_n \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\overline{\Omega}) \Leftrightarrow \int_\Omega \varphi \, d\mu_n \to \int_\Omega \varphi \, d\mu, \ \forall \varphi \in C^0(\overline{\Omega}).$$

Let $f \in L^1(\Omega)$, when no misunderstanding can occur, we will denote by $|f|$ the measure $|f|dx$.

As usual, $W_0^{1,p}(\Omega)$ is the space of functions defined as the closure in L^p-norm of the gradient of the space $C_0^\infty(\Omega)$ of infinitely differentiable functions with compact support in Ω. By Sobolev imbedding, there exists a constant $S^* = S^*(N,p)$ such that

$$\|u\|_{L^{p^*}(\Omega)}^{p^*} \leq S^* \|\nabla u\|_{L^p(\Omega)}^{p^*}, \quad \forall u \in W_0^{1,p}(\Omega), \tag{2.1}$$

where $p^* = pN/(N-p)$ is the critical Sobolev exponent. We note that by a scaling argument one can see that the best Sobolev constant S^* does not depend on the domain Ω.

The value of the best constant in (2.1), together with extremal functions, was found by Rosen [20], Talenti [21] and Aubin [3]. This was based on a symmetrization argument and some optimal one-dimensional bounds discovered by Bliss [5]. We have the equality in (2.1) if and only if

$$u(x) = U_\lambda(x) := \frac{\lambda^{\frac{N-p}{p}}}{\left(1 + c(\lambda|x-x_0|)^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}} \quad \forall x \in \mathbb{R}^N, \tag{2.2}$$

where $x_0 \in \mathbb{R}^N, \lambda > 0$ and c is a constant depending on p and N.

In view of this result, it follows that the best Sobolev constant S^* is not achieved on bounded domain Ω. Otherwise we should be able to find at least one solution v of

$$S^* = \max\left\{\int_{\mathbb{R}^N} |u|^{p^*} dx : u \in W^{1,p}(\mathbb{R}^N), \int_{\mathbb{R}^N} |\nabla u|^p dx \leq 1\right\}$$

with compact support. As a consequence, v would be not of the form (2.2), that is in contrast with the result by Rosen, Talenti and Aubin.

We are now in position to set our problem.

For every $\varepsilon > 0$, we want to study the asymptotic behavior of the following variational problem

$$S_\varepsilon^* = \sup\left\{F_\varepsilon(u)dx : u \in W_0^{1,p}(\Omega), \int_\Omega |\nabla u|^p dx \leq 1\right\},$$

where the functional F_ε is defined by

$$F_\varepsilon(u) = \int_\Omega |u|^{p^*-\varepsilon} dx, \quad \forall u \in W_0^{1,p}(\Omega). \tag{2.3}$$

In order to apply Γ-convergence, we are interested to the asymptotic behavior of the sequence $F_\varepsilon(u_\varepsilon)$ for every sequence u_ε such that $\|\nabla u_\varepsilon\|^p_{L^p(\Omega)} \leq 1$. This constraint on the Dirichlet p-energy of u_ε implies that there exists a nonnegative measure μ in $\mathcal{M}(\overline{\Omega})$ such that $\mu(\overline{\Omega}) \leq 1$ and $|\nabla u_\varepsilon|^p \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\overline{\Omega})$. By Sobolev imbedding, there exists $u \in W_0^{1,p}(\Omega)$ such that (up to subsequences) $u_\varepsilon \rightharpoonup u$ in $L^{p^*}(\Omega)$.

By the lower semicontinuity of the L^p-norm, we deduce $\mu \geq |\nabla u|^p$. Hence, we can decompose μ as follows:

$$\mu = |\nabla u|^p + \tilde{\mu} + \sum_{i=0}^{\infty} \mu_i \delta_{x_i},$$

where $\mu_i \in [0,1]$ and $x_i \in \overline{\Omega}$ is such that $x_i \neq x_j$ if $i \neq j$, δ_{x_i} is the Dirac mass concentrated at x_i; $\tilde{\mu}$ can be view as the "non-atomic part" of the measure $(\mu - |\nabla u|^p)$.

Thus, in analogy with [1], we can declare the setting for the limit functional as the space X defined by

$$X = X(\Omega) := \big\{(u,\mu) \in W_0^{1,p}(\Omega) \times \mathcal{M}(\overline{\Omega}) : \mu \geq |\nabla u|^p,\, \mu(\overline{\Omega}) \leq 1\big\},$$

endowed with the natural product topology τ such that

$$(u_\varepsilon, \mu_\varepsilon) \xrightarrow{\tau} (u,\mu) \overset{\text{def}}{\Leftrightarrow} \begin{cases} u_\varepsilon \rightharpoonup u \text{ in } L^{p^*}(\Omega) \\ \mu_\varepsilon \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\overline{\Omega}). \end{cases}$$

It is worth pointing out that the above topology τ is compact in X. Hence we have that the Γ^+-convergence of functionals in this space implies the convergence of maxima.

Now we need to extend our functional F_ε to the whole space X, keeping the same symbol, in the natural way as follows

$$F_\varepsilon(u,\mu) = \begin{cases} \displaystyle\int_\Omega |u|^{p^*-\varepsilon} dx & \text{if } (u,\mu) \in X : \mu = |\nabla u|^p, \\ \\ 0 & \text{otherwise in } X. \end{cases} \tag{2.4}$$

We conclude this section by recalling the notion of Γ^+-convergence as regards our variational framework (see [7] for further details).

Definition 2.1. *We say that the sequence (F_ε) Γ^+-converges to a functional $F : X \to [0,\infty)$, as $\varepsilon \to 0$, if for every $(u,\mu) \in X$ the following conditions hold:*

(i) *for every sequence $(u_\varepsilon) \subset W_0^{1,p}(\Omega)$ such that $u_\varepsilon \rightharpoonup u$ in $L^{p^*}(\Omega)$ and $|\nabla u_\varepsilon|^p \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\overline{\Omega})$:*

$$F(u,\mu) \geq \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon);$$

(ii) *there exists a sequence $\bar{u}_\varepsilon \subset W_0^{1,p}(\Omega)$ such that $\bar{u}_\varepsilon \rightharpoonup u$ in $L^{p^*}(\Omega)$, $|\nabla \bar{u}_\varepsilon|^p \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\overline{\Omega})$ and*

$$F(u,\mu) \leq \liminf_{\varepsilon\to 0} F_\varepsilon(\bar{u}_\varepsilon).$$

3. The Γ^+-convergence theorem

The characterization of the Γ^+-limit of the sequence (F_ε) is given in the following theorem.

Theorem 3.1. *There exists the Γ^+-limit F of the sequence of functionals (F_ε) defined by (2.4) and F is given by*

$$F(u,\mu) = \int_\Omega |u|^{p^*}\,dx + S^* \sum_{i=0}^{\infty} \mu_i^{\frac{p^*}{p}}, \quad \forall (u,\mu) \in X.$$

Here it is the strategy of the proof.

First of all, we can rewrite the two conditions of Definition 2.1 in terms of the Γ^+-limsup and the Γ^+-liminf functionals, respectly defined for every $(u,\mu) \in X$ by

$$F^+(u,\mu) = \sup\left\{ \limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,\mu_\varepsilon) : (u_\varepsilon,\mu_\varepsilon) \xrightarrow{\tau} (u,\mu) \right\}$$

and

$$F^-(u,\mu) = \sup\left\{ \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon,\mu_\varepsilon) : (u_\varepsilon,\mu_\varepsilon) \xrightarrow{\tau} (u,\mu) \right\}.$$

The Γ^+-limit F exists if and only if $F^- = F^+$ and, in this case, $F = F^- = F^+$. Since $F^- \leq F^+$ always holds, to prove Theorem 3.1 it is enough to show the following inequalities

$$\Gamma^+\text{- lim sup inequality:} \qquad F \ \geq \ F^+ \tag{3.1}$$

$$\Gamma^+\text{- lim inf inequality:} \qquad F \ \leq \ F^-. \tag{3.2}$$

The first inequality follows by Lions' Concentration-Compactness Principle and a precise representation of limit measures of every converging sequence in X.

The proof of the second inequality is more delicate. According to the idea of Amar and Garroni in [1], we will study two separate cases:

$$(u, \mu) = (u, |\nabla u|^p + \tilde{\mu}) \quad \text{and} \quad (u, \mu) = \big(0, \textstyle\sum_i \mu_i \delta_{x_i}\big).$$

This decomposition is the key point in the proof of the Γ^+-lim inf inequality. Indeed, for the pairs of the first type, we can use strong L^{p^*} convergence (see Proposition 4.2); while for the pairs with purely atomic measure part, we can work "locally" on each single Dirac mass (see Proposition 4.3). Finally, we will be able to unify the proved results to obtain the inequality (3.2) for every pair in X (see Lemma 4.5).

We recall that this strategy is similar to the one used by Amar and Garroni in [1]. For the sake of self-containment, we will produce in this paper also the proofs in which minor modifications are required.

Finally, we conclude this section by the following lemma, that provides an optimal upper bound for the limit functional F.

Lemma 3.2. *For every $(u, \mu) \in X$, we have*

$$F(u, \mu) \leq S^* \tag{3.3}$$

and the equality holds if and only if $(u, \mu) = (0, \delta_{x_0})$ for some $x_0 \in \overline{\Omega}$.

Proof. The key point of this proof is the convexity argument in the proof of the concentration-compactness alternative by Lions.

For every $(u, \mu) \in X$, by Sobolev inequality, we have

$$\begin{aligned} F(u, \mu) \ &\equiv \ \int_\Omega |u|^{p^*} dx + S^* \sum_{i=0}^{\infty} \mu_i^{\frac{p^*}{p}} \\ &\leq \ S^* \left(\int_\Omega |\nabla u|^p dx \right)^{\frac{p^*}{p}} + S^* \sum_{i=o}^{\infty} \mu_i^{\frac{p^*}{p}}. \end{aligned}$$

By the convexity of the function $t \mapsto t^{\frac{p^*}{p}}$, for every fixed $p \in (1, N)$, we get

$$\begin{aligned} F(u,\mu) &\leq S^* \left(\int_\Omega |\nabla u|^p dx + \sum_{i=o}^\infty \mu_i \right)^{\frac{p^*}{p}} \\ &\leq S^*, \quad \forall\, (u,\mu) \in X, \end{aligned}$$

where we also used the fact that $\displaystyle\int_\Omega |\nabla u|^p dx + \sum_i \mu_i \leq \mu(\overline{\Omega}) \leq 1$.

It remains to prove that the equality in (3.3) holds if and only if $(u,\mu) = (0, \delta_{x_0})$, for some $x_0 \in \overline{\Omega}$.

For any $x_0 \in \overline{\Omega}$ it is immediately seen that $F(0, \delta_{x_0}) = S^*$. Instead, for all the remaining pairs $(u,\mu) \in X$, we can prove that $F(u,\mu) < S^*$.

Let $(u,\mu) \in X$ with $\mu = |\nabla u|^p + \tilde{\mu} + \sum_i \mu_i \delta_{x_i}$.

If $u \neq 0$, since Ω is bounded the Sobolev inequality is strict and the inequality in (3.3), too. If $\mu = \tilde{\mu}$, the inequality is strict, since $u = 0$ and $F(0, \tilde{\mu}) = 0 < S^*$. Finally, if $\mu \neq \delta_{x_0}$, then there exists at least one coefficient $\mu_i \in (0,1)$, which implies $\mu_i^{p^*/p} < \mu_i$, and again the inequality (3.3) is strict. □

4. Proof of the Γ^+-convergence theorem

4.1. The Γ^+-$\limsup$ inequality.

Proposition 4.1. *For every $(u,\mu) \in X$, we have*

$$F(u,\mu) \geq \limsup_{\varepsilon \to 0} F_\varepsilon(u_\varepsilon, \mu_\varepsilon),$$

for every $(u_\varepsilon, \mu_\varepsilon) \subset X$ such that $(u_\varepsilon, \mu_\varepsilon) \xrightarrow{\tau} (u,\mu)$.

Proof. Let $(u_\varepsilon, \mu_\varepsilon)$ be in X such that $(u_\varepsilon, \mu_\varepsilon) \xrightarrow{\tau} (u,\mu)$; i.e.,

$$u_\varepsilon \rightharpoonup u \text{ in } L^{p^*}(\Omega) \quad \text{and} \quad |\nabla u_\varepsilon|^p \overset{*}{\rightharpoonup} \mu \text{ in } \mathcal{M}(\overline{\Omega}),$$

with $\mu = |\nabla u|^p + \tilde{\mu} + \displaystyle\sum_{i=0}^\infty \mu_i \delta_{x_i}$.

We have

$$|u_\varepsilon|^{p^*} \overset{*}{\rightharpoonup} \nu \text{ in } \mathcal{M}(\overline{\Omega}). \tag{4.1}$$

and ν can always be decomposed as

$$\nu = g + \sum_{i=0}^\infty \nu_i \delta_{x_i}. \tag{4.2}$$

By Lions' Concentration-Compactness Principle [14], we have

$$g = |u|^{p^*}. \tag{4.3}$$

While the atomic coefficients of ν are finite and they can be controlled in a precise sense by the ones of μ. In the following, we will show that

$$\nu_i \leq S^* \mu_i^{\frac{p^*}{p}}. \tag{4.4}$$

For every fixed $x_i \in \overline{\Omega}$, denote by $B_r^i = B_r(x_i) \cap \overline{\Omega}$. Let ξ be a cut-off function in $\mathbb{R}^N$, such that $\xi \equiv 1$ in B_r^i, $\xi \equiv 0$ out B_{2r}^i and $|\nabla \xi| \leq 1/r$.

By Sobolev inequality on the function ξu_ε in Ω, we have

$$\int_\Omega |\xi u_\varepsilon|^{p^*} dx \leq S^* \left(\int_\Omega |\nabla(\xi u_\varepsilon)|^p \right)^{\frac{p*}{p}}. \tag{4.5}$$

By definition of ξ, we get

$$\int_{B_r^i} |u_\varepsilon|^{p^*} dx \leq \int_\Omega |\xi u_\varepsilon|^{p^*}$$

and so (4.5) becomes

$$\int_{B_r^i} |u_\varepsilon|^{p^*} dx \leq S^* \left(\int_{B_r^i} |\nabla u_\varepsilon|^p dx + \int_{B_{2r}^i \setminus B_r^i} |\nabla(\xi u_\varepsilon)|^p dx \right)^{\frac{p^*}{p}}. \tag{4.6}$$

We can estimate the integral on the set $B_{2r}^i \setminus B_r^i$, using the fact that for every $\delta \in (0,1)$ there exists $\alpha(\delta) \to +\infty$ as $\delta \to 0$, such that

$$(A+B)^p \leq (1+\delta)A^p + \alpha(\delta)B^p,$$

for every non negative A, B.

Hence, for every $\delta \in (0,1)$, we have

$$\int_{B_{2r}^i \setminus B_r^i} |\nabla u_\varepsilon|^p dx \leq (1+\delta) \int_{B_{2r}^i \setminus B_r^i} |\xi|^p |\nabla u_\varepsilon|^p dx + \alpha(\delta) \int_{B_{2r}^i \setminus B_r^i} |\nabla \xi|^p |u_\varepsilon|^p dx. \tag{4.7}$$

Using that $|\xi| \leq 1$ in the first term of the right side of the last inequality and then replacing (4.7) in (4.6), we obtain

$$\int_{B_r^i} |u_\varepsilon|^{p^*} dx \leq S^* \left((1+\delta) \int_{B_{2r}^i} |\nabla u_\varepsilon|^p dx + \alpha(\delta) \int_{B_{2r}^i \setminus B_r^i} |\nabla \xi|^p |u_\varepsilon|^p dx \right)^{\frac{p^*}{p}}. \tag{4.8}$$

Since $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ with compact imbedding, passing to a subsequence if necessary, we may assume that u_ε converges strong to u in $L^p(\Omega)$. Thus, letting ε tend to 0 in (4.8), we get

$$\nu(B_r^i) \leq S^* \left((1+\delta)\mu(\overline{B_{2r}^i}) + \alpha(\delta) \int_{B_{2r}^i \setminus B_r^i} |\nabla \xi|^p |u|^p dx \right)^{\frac{p^*}{p}}, \tag{4.9}$$

where we also used the weak-$*$ convergence of the measures $|u_\varepsilon|^{p^*}$ and $|\nabla u_\varepsilon|^p$.

Now, we need an estimation for the second term in the right side of (4.9). By Hölder Inequality, we have

$$\int_{B_{2r}^i \setminus B_r^i} |\nabla \xi|^p |u|^p dx \leq \left(\int_{B_{2r}^i \setminus B_r^i} |\nabla \xi|^{\frac{pp^*}{p^*-p}} \right)^{\frac{p^*-p}{p^*}} \left(\int_{B_{2r}^i \setminus B_r^i} |u|^{p^*} \right)^{\frac{p}{p^*}}. \tag{4.10}$$

By definition of ξ, we have

$$\left(\int_{B_{2r}^i \setminus B_r^i} |\nabla \xi|^{\frac{pp^*}{p^*-p}} \right)^{\frac{p^*-p}{p^*}} \leq \left(\frac{1}{r^N} |B_{2r}^i \setminus B_r^i| \right)^{\frac{p}{N}} = C,$$

where C is a positive constant not depending on r. Combining the above estimation and (4.10), inequality (4.9) becomes

$$\nu(B_r^i) \leq S^* \left((1+\delta)\mu(\overline{B_{2r}^i}) + \alpha(\delta) C \left(\int_{B_{2r}^i \setminus B_r^i} |u|^{p^*} dx \right)^{\frac{p}{p^*}} \right)^{\frac{p^*}{p}}. \tag{4.11}$$

Then, when r goes to 0 in (4.11), we have

$$\nu_i \leq S^* \big((1+\delta)\mu_i \big)^{\frac{p^*}{p}} \quad \text{(for every } \delta \in (0,1)).$$

Finally, taking limit for $\delta \to 0$, we obtain the desired inequality (4.4).

Now, we are in position to prove the Γ^+-$\limsup$ inequality. By Hölder Inequality, we have

$$\begin{aligned} F_\varepsilon(u_\varepsilon, \mu_\varepsilon) &= \int_\Omega |u_\varepsilon|^{p^*-\varepsilon} dx \\ &\leq \left(\int_\Omega |u_\varepsilon|^{p^*} dx \right)^{\frac{p^*-\varepsilon}{p^*}} |\Omega|^{\frac{\varepsilon}{p^*}}. \end{aligned}$$

It follows

$$\begin{aligned}
\limsup_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon,\mu_\varepsilon) &\le \limsup_{\varepsilon\to 0}\left(\left(\int_\Omega |u_\varepsilon|^{p^*}dx\right)^{\frac{p^*-\varepsilon}{p^*}}|\Omega|^{\frac{\varepsilon}{p^*}}\right)\\
&\le \nu(\overline{\Omega})\\
&\le \int_\Omega |u|^{p^*}dx + S^*\sum_{i=0}^{\infty}\mu_i^{\frac{p^*}{p}} \;\equiv\; F(u,\mu),
\end{aligned}$$

where we used (4.1)–(4.4). □

4.2. **The Γ^+-$\liminf$ inequality.** The proof for the pairs $(u,\mu)=(u,|\nabla u|^p+\tilde{\mu})$ follows by strong L^{p^*}-convergence and Lebesgue Convergenge Theorem.

Proposition 4.2. *Let $(u,\mu)\in X$ be such that $\mu=|\nabla u|^p+\tilde{\mu}$. Then*

$$F(u,\mu)\le\liminf_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon,\mu_\varepsilon)$$

for every sequence $(u_\varepsilon,\mu_\varepsilon)\subset X$ such that $(u_\varepsilon,\mu_\varepsilon)\xrightarrow{\tau}(u,\mu)$ as $\varepsilon\to 0$.

Proof. Take $(u_\varepsilon,\mu_\varepsilon)\subset X$ such that $\mu_\varepsilon=|\nabla u_\varepsilon|^p+\tilde{\mu_\varepsilon}$ and $(u_\varepsilon,\mu_\varepsilon)\xrightarrow{\tau}(u,\mu)$ as ε goes to 0. Since the atomic part of μ_ε is zero, u_ε converges strongly to u in $L^{p^*}(\Omega)$, so (up to a subsequence) $u_\varepsilon\to u$ a.e. in Ω as ε goes to 0.

It follows

$$\forall x \text{ a.e. in } \Omega \qquad |u_\varepsilon(x)|^{p^*-\varepsilon}\to|u(x)|^{p^*} \;\text{ as }\; \varepsilon\to 0.$$

Hence, by Lebesgue Convergence Theorem, we have:

$$\lim_{\varepsilon\to 0}\int_\Omega |u_\varepsilon|^{p^*-\varepsilon}dx=\int_\Omega|u|^{p^*}dx\equiv F(u,\mu),$$

that gives the desired inequality. □

In the following two propositions we study the case of pairs with purely atomic measure part.

Proposition 4.3. *For every open set $A\subset\Omega$, for every $x\in\overline{A}$ and for every $(u,\mu)\in X$ such that $(u,\mu)=(0,\delta_x)$, there exists the Γ^+-limit of the sequence (F_ε) restricted to A and the following equality holds:*

$$\left(\Gamma^+\text{-}\lim_{\varepsilon\to 0}F_\varepsilon\right)(0,\delta_x;A)=S^*.$$

Proof. Fix $A \subseteq \Omega$. Let ε_h be such that $\varepsilon_h \to 0$ if $h \to \infty$. By the compactness property of the Γ^+-convergence, it follows that there exist a subsequence (still denoted by ε_h) and a functional $F_A : X(A) \to [0, \infty)$ such that

$$\left(\Gamma^+\text{-}\lim_{h\to\infty} F_{\varepsilon_h}\right)(u, \mu; A) = F_A(u, \mu).$$

We also have

$$S^*_{\varepsilon_h} = \sup_{X(A)} F_{\varepsilon_h} \to \max_{X(A)} F_A \text{ as } h \to \infty. \tag{4.12}$$

Now, we recall that by suitably normalizing the optimal function U_λ defined in (2.2), we can construct a test function v_{ε_h} for $S^*_{\varepsilon_h}$ as follows

$$v_{\varepsilon_h}(x) = \varphi(x) U_{\lambda_{\varepsilon_h}}(x),$$

where φ is a cut-off function and λ_{ε_h} goes to ∞ as h goes to ∞, and we have

$$\int_A |v_{\varepsilon_h}|^{p^*-\varepsilon_h} dx = S^* + o(1).$$

Then it follows

$$\lim_{h\to\infty} S^*_{\varepsilon_h} \geq S^*.$$

Hence, by the above lower bound and (4.12) we obtain

$$\max_{X(A)} F_A \geq S^*. \tag{4.13}$$

Otherwise, by Proposition 4.1 it follows

$$F_A(u, \mu) \leq F^+(u, \mu; A) \leq F(u, \mu; A),$$

and, by Lemma 3.2, we have

$$F(u, \mu; A) < S^*, \ \forall\ (u, \mu) \neq (0, \delta_{\bar{x}}), \ \forall\ \bar{x} \in A.$$

Hence,

$$F_A(u, \mu) < S^*, \ \forall\ (u, \mu) \neq (0, \delta_{\bar{x}}), \ \forall\ \bar{x} \in A. \tag{4.14}$$

Combining (4.13) and (4.14), we recover the existence of $\bar{x} \in \overline{A}$ such that

$$F_A(0, \delta_{\bar{x}}) = S^*.$$

Finally, replacing A by a family of ball $(B_r(q)) \subset A$, centered in $q \in \mathbb{Q}^N \cap A$, with radius $r \in \mathbb{Q}$, we obtain by a density argument that the above equality holds for $(u, \mu) = (0, \delta_x)$, for every $x \in \overline{A}$ (see [1, Proposition 3.7] for a detailed proof).□

Proposition 4.4. *For every* $(u,\mu) \in X$ *such that* $(u,\mu) = \left(0, \sum_{i=0}^{n} \mu_i \delta_{x_i}\right)$, *with* $x_i \in \overline{\Omega}$, *the following equality holds:*

$$F^{-}(u,\mu) = F(u,\mu).$$

Proof. We can assume that $n = 1$; i.e., $\mu = \mu_0\delta_{x_0} + \mu_1\delta_{x_1}$, with $\mu_0, \mu_1 \in (0,1)$ and $\mu_0 + \mu_1 \leq 1$. The general case $n > 1$ can be treated in the same way.

Let us set $A_i := B_{r_i}(x_i) \cap \overline{\Omega}$ for $i = 0,1$, with radii r_0 and r_1 such that $\operatorname{dist}(A_0, A_1) > 0$. By Proposition 4.3 (for $i = 0,1$), we obtain that there exists a sequence $(u^i_\varepsilon, \mu^i_\varepsilon) \subseteq X(A_i)$, with $\mu^i_\varepsilon = |\nabla u^i_\varepsilon|^p$ such that $(u^i_\varepsilon, \mu^i_\varepsilon) \xrightarrow{\tau} (0, \delta_{x_i})$ and

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u^i_\varepsilon, \mu^i_\varepsilon; A_i) \geq F(0, \delta_{x_i}; A_i) = S^*.$$

In particular,

$$\lim_{\varepsilon\to 0} \int_{A_i} |u^i_\varepsilon|^{p^*-\varepsilon} dx = S^*, \quad \text{for } i = 0,1. \tag{4.15}$$

We define $u_\varepsilon = {\mu_0}^{1/p} u^0_\varepsilon + {\mu_1}^{1/p} u^1_\varepsilon$ and $\mu_\varepsilon = |\nabla u_\varepsilon|^p$. We have that the pair $(u_\varepsilon, \mu_\varepsilon)$ belongs to X. In fact,

$$\begin{aligned} \int_\Omega |\nabla u_\varepsilon|^p dx &= \mu_0 \int_{A_0} |\nabla u^0_\varepsilon|^p dx + \mu_1 \int_{A_1} |\nabla u^1_\varepsilon|^p dx \\ &\leq \mu_0 + \mu_1 \ \leq \ 1. \end{aligned}$$

Moreover,

$$F(u_\varepsilon, \mu_\varepsilon) = \int_\Omega |u_\varepsilon|^{p^*-\varepsilon} dx = \mu_0^{\frac{p^*-\varepsilon}{p}} \int_{A_0} |u^0_\varepsilon|^{p^*-\varepsilon} dx + \mu_1^{\frac{p^*-\varepsilon}{p}} \int_{A_1} |u^1_\varepsilon|^{p^*-\varepsilon} dx.$$

Thus, by (4.15), it follows

$$\lim_{\varepsilon\to 0} F(u_\varepsilon, \mu_\varepsilon) = S^* \left(\mu_0^{\frac{p^*}{p}} + \mu_1^{\frac{p^*}{p}} \right) \equiv F(0, \mu_0\delta_{x_0} + \mu_1\delta_{x_1}).$$

□

We are ready to complete the proof of inequality (3.2) for every pair (u,μ) in X, and thus the proof of Theorem 3.1. We need the following technical lemma by Amar and Garroni, applied to our functionals F_ε.

Lemma 4.5. *If* $F^{-}(u,\mu) \geq F(u,\mu)$ *for every* $(u,\mu) \in X$ *such that*

(1) $\mu(\overline{\Omega}) < 1$;

(2) $\mu = |\nabla u|^p + \tilde{\mu} + \sum_{i=0}^{n} \mu_i \delta_{x_i},\ x_i \in \overline{\Omega}$;

(3) $\operatorname{dist}\big(\overline{\operatorname{supp}(|u| + \tilde{\mu})}, \bigcup_{i=0}^{n}\{x_i\}\big) > 0$.

Then

$$F^-(u, \mu) \geq F(u, \mu) \text{ for every } (u, \mu) \in X.$$

Proof. The proof is contained in [1, Lemma 4.1]. Here we indicate the required steps with minor modifications.

Step 1. Consider an arbitrary pair $(u, \mu) \in X$ satisfying (1) and (2), then we are able to construct a sequence (u_ρ, μ_ρ) for every $\rho > 0$ such that it verifies (3) and such that the Γ^+-$\liminf$ inequality holds.

For every $\rho > 0$ and every $i = 0, 1, ..., n$, we define $B^i_\rho = B_\rho(x_i) \cap \overline{\Omega}$ and we consider a cut-off function $\phi_\rho \in C^\infty(\overline{\Omega})$ such that $0 \leq \phi_\rho \leq 1$, $\phi_\rho = 0$ in $\bigcup_{i=0}^{n} B^i_\rho$, $\phi_\rho = 1$ in $\Omega \setminus \bigcup_i B^i_{2\rho}$, $|\nabla \phi_\rho| \leq 1/\rho$. We set

$$(u_\rho, \mu_\rho) = \big(u\phi_\rho, |\nabla(u\phi_\rho)|^p + \mu\phi_\rho + \sum_{i=0}^{n} \mu_i \delta_{x_i}\big).$$

We have $\operatorname{dist}\big(\overline{\operatorname{supp}(|u_\rho| + \tilde{\mu_\rho})}, \bigcup_{i=0}^{n}\{x_i\}\big) \geq \rho > 0$ and $(u_\rho, \mu_\rho) \in X$ is such that $u_\rho \to u$ in $L^{p^*}(\Omega)$ and $\mu_\rho \xrightarrow{*} \mu$ in $\mathcal{M}(\overline{\Omega})$ as ρ goes to 0. Hence

$$F^-(u_\rho, \mu_\rho) \geq F(u_\rho, \mu_\rho) \quad \forall \rho > 0$$

and then, using the strong L^{p^*} convergence of u_ρ to u as ρ goes to zero, we have

$$F^-(u, \mu) \geq F(u, \mu) \;\; \forall (u, \mu) \text{ satisfying (1) and (2).}$$

Step 2. We can assume μ being in the form of (2).

For every $(u, \mu) \in X$, we can consider the sequence (u_n, μ_n) defined by

$$u_n = u \quad \text{and} \quad \mu_n = |\nabla u|^p + \tilde{\mu} + \sum_{i=0}^{n} \mu_i \delta_{x_i}, \quad \forall n \in \mathbb{N}.$$

We have $(u_n, \mu_n) \xrightarrow{\tau} (u, \mu)$ when n goes to infinity and, by taking into account the upper semi-continuity of the Γ^+-limit, it follows

$$F^-(u, \mu) \geq \limsup_{n\to\infty} F^-(u_n, \mu_n) \geq \lim_{n\to\infty} F(u_n, \mu_n) = F(u, \mu).$$

Step 3. We can always assume $\mu(\overline{\Omega}) < 1$.

For every $(u, \mu) \in X$, we can consider the sequence (u_σ, μ_σ) defined by

$$u_\sigma = \frac{u}{1+\sigma} \quad \text{and} \quad \mu_\sigma = \frac{\mu}{(1+\sigma)^p}, \quad \forall \sigma > 0.$$

Since $\mu_\sigma(\overline{\Omega}) < 1$ and $(u_\sigma, \mu_\sigma) \xrightarrow{\tau} (u, \mu)$ as σ goes to 0, by the previous steps, it follows

$$F^-(u_\sigma, \mu_\sigma) \geq F(u_\sigma, \mu_\sigma) = \frac{1}{(1+\sigma)^{p^*}} F(u, \mu),$$

where we used the definition of F. Thus, we have the conclusion letting σ to zero and again taking into account the upper semi-continuity of the Γ^+-liminf. □

Proof of Theorem 3.1

By the previous results (Proposition 4.1, Proposition 4.2 and Proposition 4.4) and in view of Lemma 4.5, it remains only to prove the Γ^+-$\liminf$ inequality

$$F^-(u, \mu) \geq F(u, \mu)$$

for every $(u, \mu) \in X$ such that

$$\mu(\overline{\Omega}) < 1, \quad \mu = |\nabla u|^p + \tilde{\mu} + \sum_{i=0}^{n} \mu_i \delta_{x_i} \text{ and dist}\left(\overline{\operatorname{supp}(|u| + \tilde{\mu})}, \textstyle\bigcup_{i=0}^{n}\{x_i\}\right) > 0.$$

For every (u, μ) set

$$\mu = \mu^A + \mu^B$$

with $\mu^A = \sum_{i=0}^{n} \mu_i \delta_{x_i}$ and $\mu^B = |\nabla u|^p + \tilde{\mu}$.

Let $A, B \subset \Omega$ two open sets such that $\overline{\operatorname{supp}(\mu^A)} \subset \overline{A}$, $\overline{\operatorname{supp}(|u| + \tilde{\mu})} \subset \overline{B}$ and $\overline{A} \cap \overline{B} = \emptyset$.

By Proposition 4.4, it follows that there exists a sequence $(u_\varepsilon^A, \mu_\varepsilon^A)$, with $u_\varepsilon^A \in W_0^{1,p}(A)$, $\mu_\varepsilon^A = |\nabla u_\varepsilon^A|^p$ and $\mu_\varepsilon^A(\overline{\Omega}) < 1$ for ε small enough, such that

$$(u_\varepsilon^A, \mu_\varepsilon^A) \xrightarrow{\tau} (0, \mu^A) \quad \text{as } \varepsilon \to 0$$

and

$$F_\varepsilon(u_\varepsilon^A, \mu_\varepsilon^A) \to F(0, \mu^A) \quad \text{as } \varepsilon \to 0. \tag{4.16}$$

Similarly, by Proposition 4.1 and Proposition 4.2, it follows that there exists a sequence $(u_\varepsilon^B, \mu_\varepsilon^B)$, with $u_\varepsilon^B \in W_0^{1,p}(B)$, $\mu_\varepsilon^B = |\nabla u_\varepsilon^B|^p$ and $\mu_\varepsilon^B(\overline{\Omega}) < 1$ for ε small enough, such that

$$(u_\varepsilon^B, \mu_\varepsilon^B) \xrightarrow{\tau} (u, \mu^B) \quad \text{as } \varepsilon \to 0$$

and

$$F_\varepsilon(u_\varepsilon^B, \mu_\varepsilon^B) \to F(u, \mu^B) \quad \text{as } \varepsilon \to 0. \tag{4.17}$$

Let us set $u_\varepsilon = u_\varepsilon^A + u_\varepsilon^B$ and $\mu_\varepsilon = \mu_\varepsilon^A + \mu_\varepsilon^B$. Since the support of u^A and u^B are disjoint, $(u_\varepsilon, \mu_\varepsilon)$ is such that $u_\varepsilon \in W_0^{1,p}(\Omega)$, $\mu_\varepsilon = |\nabla u_\varepsilon|^p$ and $\mu_\varepsilon(\overline{\Omega}) < 1$ for ε sufficiently small. By (4.16) and (4.17), we obtain

$$\begin{aligned}
\lim_{\varepsilon \to 0} F(u_\varepsilon, \mu_\varepsilon) &= F(u, \mu^B) + F(0, \mu^A) \\
&= \int_\Omega |u|^{p^*} dx + S^* \sum_{i=0}^n \mu_i^{\frac{p^*}{p}} \\
&= F(u, \mu), \quad \forall\ (u, \mu) \text{ as in Lemma 4.5.}
\end{aligned}$$

This concludes the proof of Theorem 3.1. □

5. The concentration result

Thanks to the Γ^+-convergence result, we can deduce that the positive solutions u_ε of (1.4), which are maximizing for the variational problem (1.5), concentrate at one point $x_0 \in \overline{\Omega}$ when ε goes to zero.

Theorem 5.1. *Let u_ε be solution of (1.4) maximizing for S_ε^*. Then u_ε concentrates at some $x_0 \in \overline{\Omega}$, i.e. $(u_\varepsilon, |\nabla u_\varepsilon|^p) \xrightarrow{\tau} (0, \delta_{x_0})$ as ε goes to 0.*

Proof. We are interested in the asymptotic behavior of the subcritical solutions of (1.4) that are maximizing for (1.5).

By Theorem 3.1 and Γ^+-convergence properties, it follows that every maximizing sequence $(u_\varepsilon, |\nabla u_\varepsilon|^p)$ of F_ε must converge to a pair $(u, \mu) \in X$ maximizer for F, i.e.

$$(u_\varepsilon, |\nabla u_\varepsilon|^p) \xrightarrow{\tau} (u, \mu), \quad \text{with } F(u, \mu) = \max_{X(\overline{\Omega})} F.$$

By Lemma 3.2, we have the optimal lower bound for F given by

$$F(u, \mu) \leq S^* \quad \text{for every } (u, \mu) \in X$$

and the equality is achieved if and only if $(u, \mu) = (0, \delta_{x_0})$ with $x_0 \in \overline{\Omega}$, i.e.

$$\max_{X(\Omega)} F = F(0, \delta_{x_0}), \quad \text{with } x_0 \in \overline{\Omega}.$$

Hence, it follows

$$(u_\varepsilon, |\nabla u_\varepsilon|^p) \overset{\tau}{\to} (0, \delta_{x_0}), \quad \text{with } x_0 \in \overline{\Omega},$$

that is the desired concentration result. $\square$

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(Giampiero Palatucci) MIPA, Université de Nîmes, Place Gabriel Péri, 30021 Nîmes, France

E-mail address: giampiero.palatucci@unimes.fr