FROM SPECIAL BOUNDED DEFORMATION TO SPECIAL BOUNDED HESSIAN : THE ELASTIC-PLASTIC BEAM

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Abstract - Slender beams with small cracks described by Γ *limits: a description of an elastic–perfectly plastic beam or rod is obtained as a variational limit of 2D or 3D bodies with damage at small scale satisfying the Kirchhoff cinematic restriction on the deformations.*

0. Introduction

In this paper we describe the asymptotic behaviour as $\varepsilon \to 0_+$ of minimizers of the functionals

$$\mathcal{F}^\varepsilon(\mathbf{v}) \;=\; \varepsilon^{-\delta}\int_{\Omega^\varepsilon} Q\left(\mathcal{E}(\mathbf{v})\right)\,d\mathbf{x} \;+\; \varepsilon^2\,\alpha\,\mathcal{H}^{n-1}(J_\mathbf{v}) \;+\; \varepsilon\,\beta\int_{J_\mathbf{v}} |[\mathbf{v}]\odot\nu_\mathbf{v}|d\mathcal{H}^{n-1} \;-\; \int_{\Omega^\varepsilon}\mathbf{g}^\varepsilon\cdot\mathbf{v}\,dx \quad (0.1)$$

where the open set $\Omega^\varepsilon \subset \mathbf{R}^n$ is a tubular neighborhood of a C^3 regular simple arc representing the un-stressed configuration of a beam of finite length L , $\mathbf{v}$ is a vector field with special bounded deformation ([ACD]) in Ω^ε, $n = 2, 3,\ \alpha > 0,\ \beta > 0,\ \delta \geq 0$, $\mathcal{E}(\mathbf{v})$ is the absolutely continuous part of the linear strain tensor $\mathrm{e}(\mathbf{v}) \;=\; \frac{1}{2}\left(D\mathbf{v} + (D\mathbf{v})^T\right)$ and

$$Q\left(\mathcal{E}(\mathbf{v})\right) \;\;=\;\; \mu\,|\mathcal{E}(\mathbf{v})|^2 \;+\; \frac{\lambda}{2}\,|\ \mathrm{Tr}\ \mathcal{E}(\mathbf{v})\,|^2$$

where λ, μ are the Lamè coefficients. $J_\mathbf{v}$ is the jump set of $\mathbf{v}$, $\nu_\mathbf{v}$ is the normal to $J_\mathbf{v}$ and $\mathcal{H}^{n-1}$ denotes the $(n-1)$ dimensional Hausdorff measure.
The functional $\mathcal{F}^\varepsilon$ represents the mechanical energy for a linear elastic body, with natural reference Ω^ε , subject to transverse dead load $\mathbf{g}^\varepsilon$, with free small cracks whose geometry (the set $J_\mathbf{v}$) is not "a priori" prescribed (see [PT1,2],[BDG]). The first term in (0.1) represents the elastic energy in undamaged regions, the second one is a surface energy (area of material surfaces where damage occurs ([G])), the third one describes a weak resistance of the material to compression or crack opening. The sum of the second and third term is an interface energy which is suitable to describe the damage process according to the Barenblatt cohesive model of crack ([Ba]). Non trivial loads in the last term of (0.1) are consistent with our analysis even without artificial confinement of the body.
In the paper we do not assume any condition on $\mathbf{v}$ at the boundary $\partial\Omega$: hence the extremals of (0.1) formally satisfy a Neumann boundary condition; this choice is done just for simplicity; in [PT1] we performed the analysis for the cantilever.
The functional (0.1) achieves a minimum over the displacements $\mathbf{v}$ in *SBD*(Ω^ε), vector fields with special bounded deformation ([ACD],[BCD]), provided $\mathbf{g}^\varepsilon$ is small and has vanishing resultant and angular momentum ([CT]).

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By denoting with $\mathbf{v}$ the displacement and with $\mathbb{e}(\mathbf{v})$ the linear strain tensor, we assume the following cinematic restriction on the deformations

$$\mathbb{e}(\mathbf{v}) \cdot \mathcal{N} \;=\; \mathbf{0} \tag{0.2}$$

for every continuous vector field $\mathcal{N}(\mathbf{x})$ normal to the beam at the projection $\mathbf{p}(\mathbf{x})$ of $\mathbf{x}$ on the arc itself: such projection is well defined for small ε (see (2.9),(3.11) for a precise statement of (0.2), respectively in the 2D and 3D approximation).
We notice explicitly that the internal constraint (0.2) requires that any normal to the central strand lies in the kernel of $\mathbb{e}(\mathbf{v})$, which is the usual Kirchhoff cinematic restriction on the deformations in 2D, say *"the material fibers orthogonal to the middle arc before loading remain approximately orthogonal to it after loading and suffer negligible stretching"* (see ([K],[PG]), while in 3D (0.2) is a point-wise linearization of the Bernoulli-Navier cinematic restriction, say "*the intrinsic frame stays approximately orthonormal after deformation,* " (see [V] pag.70]). More explicitly, if $\psi(\mathbf{x}) = \mathbf{x} + d\,\mathbf{v}(\mathbf{x})$ and $E(\mathbf{v}) = (\nabla\psi)^T\,\nabla\psi - I$ denote deformation and strain, then assumption (0.2) is the formal linearization of the internal constraint, for small values of d, say, if $\mathbf{N}, \mathbf{B}$ are the intrinsic normal and bi-normal, then $E(\psi)\cdot\mathbf{N} = E(\psi)\cdot\mathbf{B} = \mathbf{0}$ and both $\mathbb{e}(\mathbf{v})\cdot\mathbf{N}$, $\mathbb{e}(\mathbf{v})\cdot\mathbf{B}$ tends to 0 as d tend to 0_+ .

The nonconvex energy (0.1) achieves a finite minimum (at a possibly not unique minimizer) among the vector fields $\mathbf{v}$ in $SBD(\Omega^\varepsilon)$ satisfying constraint (0.2), provided $\mathbf{g}^\varepsilon$ has vanishing resultant and moments (compatibility condition) and $\mathbf{g}^\varepsilon$ is small (safe load condition) compared to β and with respect to the geometry of Ω^ε (Theorems 2.9 and 3.9).

By assuming (0.2) and $\mathbf{v}^\varepsilon \in \operatorname{argmin} \mathcal{F}^\varepsilon$, we show that the averages $\mathbf{u}^\varepsilon$ of $\mathbf{v}^\varepsilon$ over cross-sections of Ω^ε (as functions of arc-length and up to sub-sequences) converges to a minimizer of the functionals listed below as $\varepsilon \to 0_+$ (u', $\dot{u}$ denote respectively distributional derivative and its absolutely continuous part).
(*Theorem 2.11*) If $n=2$, $\delta = 0$, $\mathbf{g}^\varepsilon$ and f are defined by (2.7) and the force field $\mathbf{g}^\varepsilon$ is traction-free, then $\mathbf{u}^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ converges to a minimizer $\mathbf{u} = (u_1, u_2)$ of the functional $\mathcal{F}_2$ defined on $SBH((0,L), \mathbf{R}^2)$ by

$$\mathcal{F}_2(\mathbf{u}) \;=\; \begin{cases} \dfrac{2}{3}\left(\mu + \dfrac{\lambda}{2}\right)\displaystyle\int_0^L (|\ddot{u}_2 + \kappa\dot{u}_1 + \dot{\kappa}u_1|^2 - fu_2)ds + 2\alpha\mathcal{H}^0(S_{\dot{u}_2}) + \beta\sum_{s\in S_{\dot{u}_2}} |[\dot{u}_2]| & \text{if } u_1' = \kappa u_2 \\ +\infty & \text{otherwise}\,. \end{cases} \tag{0.3}$$

(*Theorem 4.1*) If $\delta > 0$, $n = 2$ and $\mathbf{g}^\varepsilon$ is traction-free (defined by (2.7)), $\mathbf{u}^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$, then $\mathcal{F}^\varepsilon$ converges to a minimizer $\mathbf{u} = (u_1, u_2)$ of the functional Λ defined on $SBH((0,L), \mathbf{R}^2)$ by

$$\Lambda(\mathbf{v}) \;=\; \begin{cases} -\displaystyle\int_0^L fu_2 ds \;+\; 2\alpha\mathcal{H}^0(S_{\dot{u}_2}) \;+\; \beta\sum_{S_{\dot{u}_2}} |[\dot{u}_2]| & \text{if} \quad u_1' = \kappa u_2\,, \text{ and } \; u_2'' + \kappa u_1' + \dot{\kappa}u_1 = 0 \\ +\infty & \text{otherwise .} \end{cases} \tag{0.4}$$

(*Theorem 3.11*) If $n = 3$, $\delta = 0$, $\mathbf{g}^\varepsilon$ and $\mathbf{f}$ defined by (3.8), $\mathbf{g}^\varepsilon$ is traction-free, u_1^ε, u_2^ε, u_3^ε are the averaged intrinsic components of $\mathbf{v}^\varepsilon$ over the cross-sections of Ω^ε and $u_4^\varepsilon(s)$ denotes the mean value of resultant moment of $\mathbf{v}^\varepsilon$ over the cross-section of Ω^ε through $\boldsymbol{\gamma}(s)$, expressed as functions of arc-length s, $\mathbf{u}^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon, u_4^\varepsilon)$, then $\mathbf{u}^\varepsilon$ converges to a minimizer $\mathbf{u} = (u_1, u_2, u_3, u_4)$ of the functional $\mathcal{F}_3$ defined on $SBH((0,L), \mathbf{R}^4)$ by

$$\mathcal{F}_3(\mathbf{u}) \;=\; \begin{cases} \Phi(\mathbf{u}) & \text{if } {u_1}' = Ku_2\,,\; u_4' = -K(u_3' + \tau u_2)\,, \\ +\infty & \text{otherwise} \end{cases} \tag{0.5}$$

where

$$\Phi(\mathbf{u}) \;=\; \frac{\pi}{4}\left(\mu+\frac{\lambda}{2}\right)\int_0^L \left|A_1\mathbf{u}\right|^2 + \left|A_2\mathbf{u}\right|^2 ds - \pi\int_0^L \mathbf{f}\cdot\mathbf{u}\, ds + \pi\alpha\mathcal{H}^0(S_{\dot u_2}\cup S_{\dot u_3}) + \frac{4}{3}\,\beta \sum_{S_{\dot u_2}\cup S_{\dot u_3}} \sqrt{[\dot u_2]^2+[\dot u_3]^2}\,,$$

$$A_1\mathbf{u} = \ddot u_2 - \dot\tau u_3 - 2\tau\dot u_3 + K\dot u_1 + \dot K u_1 - \tau^2 u_2\,, \qquad A_2\mathbf{u} = \ddot u_3 + \dot\tau u_2 + 2\tau\dot u_2 + \tau K u_1 - K u_4 - \tau^2 u_3\,.$$

In each one of the previous cases we prove also the convergence of the naturally scaled energies of minimizers: $\varepsilon^{-3}\mathcal{F}_2^\varepsilon(\mathbf{v}^\varepsilon)$ to $\mathcal{F}_2(\mathbf{u})$ (to $\Lambda(\mathbf{u})$ if $\delta > 0$) and $\varepsilon^{-4}\mathcal{F}_3^\varepsilon(\mathbf{v}^\varepsilon)$ to $\mathcal{F}_3(\mathbf{u})$, as $\varepsilon \to 0_+$, where we denote $\mathcal{F}_n^\varepsilon$ instead of $\mathcal{F}^\varepsilon$ since a dimension labelling is needed).
Hence, in a different perspective, one can say that any minimizer $\mathbf{v}^\varepsilon$ of $\mathcal{F}_n^\varepsilon$ can be approximatively recovered, for positive ε, respectively by a minimizer of $\mathcal{F}_n$, $n = 2, 3$, due to

$$\min \mathcal{F}_n^\varepsilon \;=\; \varepsilon^{n+1}\min\mathcal{F}_n \;+\; o(\varepsilon^{n+1})\;, \qquad \text{when } \delta = 0\;.$$

The analogous statement holds for Λ when $\delta > 0$.
Functional $\mathcal{F}^\varepsilon$ describes the stored energy of the (2D or 3D) thick elastic body with damage at meso-scale undergoing small deformations.
Functional $\mathcal{F}_2$ is a model energy of a planar (possibly not straight) linear elastic plastic beam, whose natural configuration is the middle arc of the ribbon Ω^ε.
Functional $\mathcal{F}_3$ is a model energy of a (possibly not planar) linear elastic plastic beam, whose natural configuration is the central strand of the tube Ω^ε.
Functional Λ is a model energy of a (possibly not straight) rigid plastic beam, whose natural configuration is the middle arc of the ribbon Ω^ε.
The simple cases of a straight beam or a circular ring can be dealt with in the same way for both 2D or 3D approximation framework(see Examples 2.12, 2.13, 3.14 Remark 3.13 and Section 4), and that of a cylindrical helix in the 3D approximation (see Ex.3.15).
It is worth noticing that in the $3D$ approximation of the beam, the admissible vector fields , with values in $\mathbf{R}^3$, are completely and simply described by four scalar functions $\mathbf{u} = \mathcal{U}_3(\mathbf{v})$ of one variable, each one with a physical interpretation (see Lemmas 3.2, 3.4 and Remark 3.8). While in the 2D-approximation the admissible vector fields are described by two scalar functions of one variable ($\mathbf{u} = \mathcal{U}_2(\mathbf{v})$) (see Lemmas 2.2, 2.4 and Remark 2.8). This holds true also in the limit where the functionals $\mathcal{F}_3, \mathcal{F}_2$ depend on the 4D and 2D vector-valued arguments (we refer to [Pe2] for an analogous property in linear elasticity).
The above results are the natural extensions of those in [PT1] [PT2] to neither straight nor flat beams, but the proofs cannot be obtained by a straightforward application of the approach in [PT1] or [PT2] since it is not available an estimate on the behavior of the Korn-Poincaré inequality for this general context (this kind of estimate was proved in a flat geometry: Thm 4.1 of [PT1] and Thm 2.1 of [PT2]).
Here we estimate the asymptotic behavior (as $\varepsilon \to 0$) of the Korn-Poincaré inequality constant in $BD(\Omega)$ for the 2D approximation under the Kirchhoff cinematic restriction (Lemma 2.6, Rmk 2.7), and for the 3D approximation under the Bernoulli-Navier cinematic restriction (Lemma 3.6, Rmk.3.7) of a curved beam with bounded curvature and torsion (respectively (2.5),(3.5)).
The key points in the proofs are Lemmas 2.2, 3.2 which exploit the cinematic constraint (0.2) to express all the relevant physical quantities in term of intrinsic coordinates.

The outline of the paper is the following.

1. Functional framework.
2. Two–dimensional approximation of a linear elastic–plastic curved beam ($\mathbf{LCB_2}$).
3. Three–dimensional approximation of a linear elastic–plastic curved beam ($\mathbf{LCB_3}$).
4. Three–dimensional approximation of a linear elastic-plastic straight beam ($\mathbf{LB_3}$).
5. Approximation of a rigid-plastic beam (**RB**).
6. References

1. Functional framework

We denote by U an open bounded subset of $\mathbf{R}^n$ with Lipschitz boundary, and by $\mathcal{L}^n$ the n–dimensional Lebesgue measure . $\{\mathbf{e}_i\}$ denote the canonical basis of $\mathbf{R}^n$. For a given set $Q \subset \mathbf{R}^n$ we denote by ∂Q its topological boundary, by $\mathcal{H}^m(Q)$ its m–dimensional Hausdorff measure and by $|Q|$ its Lebesgue outer measure; $p' = p/(p-1)$ denotes the conjugate exponent of any $p \in [1, +\infty]$. We denote by $B_\rho(\mathbf{x})$ the open ball $\{\mathbf{y} \in \mathbf{R}^n; |\mathbf{y}-\mathbf{x}| < \rho\}$, and we set $B_\rho = B_\rho(\mathbf{0})$. Moreover $s \wedge t = \min\{s,t\}$, $s \vee t = \max\{s,t\}$ for every $s, t \in \mathbf{R}$. spt denote the support of a distribution.

$$\fint_Q v\,dx \;=\; |Q|^{-1}\int_Q v\,dx \quad \forall \mathcal{L} \text{ measurable set } Q,\ \text{and } \mathcal{L} \text{ integrable function } v \text{ in } Q.$$

$M_{k,n}$ denotes the $k \times n$ matrices and I_k the identity matrix in $M_{k,k}$; given any two vectors $\mathbf{a} = \{\mathbf{a}_i\}$, $\mathbf{b} = \{\mathbf{b}_i\}$, and matrices $A = \{A_{ij}\}$, $B = \{B_{ij}\}$, we set $\mathbf{a}\cdot\mathbf{b} = \sum_i \mathbf{a}_i\mathbf{b}_i$, $(\mathbf{a}\otimes\mathbf{b})_{ij} = \mathbf{a}_i\mathbf{b}_j$, $(\mathbf{a}\odot\mathbf{b})_{ij} = 1/2\,(\mathbf{a}_i\mathbf{b}_j + \mathbf{a}_j\mathbf{b}_i)$, $(A\cdot\mathbf{b})_i = \sum_j A_{ij}\mathbf{b}_j$, $|\mathbf{a}|^2 = \mathbf{a}\cdot\mathbf{a}$, $(\mathbf{b}\cdot A)_j = \sum_i A_{ij}\mathbf{b}_i$, $(AB)_{ij} = \sum_k A_{ik}B_{kj}$, $A : B = \sum_{ij} A_{ij}B_{ij}$, the euclidian norm is denoted by $|A|$ where $|A|^2 = A : A \;=\; \sup\{\sum_{ij} A_{ij}B_{ij} \;:\; \sum_{ij} B_{ij}^2 = 1\}$.

We say that a subset E of $\mathbf{R}^n$ is *countably* $(\mathcal{H}^{n-1}, n-1)$ *rectifiable* if (up to a set of vanishing $\mathcal{H}^{n-1}$ measure) is the countable union of C^1 images of bounded subsets of $\mathbf{R}^{n-1}$; if in addition $\mathcal{H}^{n-1}(E) < +\infty$ then we say that E is $(\mathcal{H}^{n-1}, n-1)$ *rectifiable*.

For $p \in [1, +\infty]$, and Y is a finite dimensional space, we denote by $L^p(U,Y)$ and by $W^{1,p}(U,Y)$ the Lebesgue and Sobolev spaces of functions with values in Y, endowed with the usual norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{1,p}}$ respectively. $\mathcal{M}(U,Y)$ denotes the space of the bounded measures on Ω with values in Y .

We write shortly $L^p(U), W^{1,p}(U), \mathcal{M}(U)$ when $Y = \mathbf{R}$.

$\|\cdot\|_{\mathcal{M}}$ denotes the total variation of a measure in $\mathcal{M}(U,Y)$, i.e.

$$\|\mu\|_{\mathcal{M}}(U) \;=\; \int_U |\mu| \;=\; \sup\left\{\int_U \sum_{ij} \varphi_{ij}\,d\mu_{ij} \;:\; \varphi_{ij} \in C_0^0(U),\ \sum_{ij}\varphi_{ij}^2 \le 1 \text{ in } U\right\}.$$

$\mu^a = \frac{d\mu}{d\mathcal{L}^m}$ is the Radon–Nikodym derivative with respect to $\mathcal{L}^m$ of μ, and $\mu^s = \mu - \mu^a$.

If $O \subset U$ is any open set then $|\mu|_{\mathcal{M}(O)}$ is defined in the same way with $\varphi_{ij} \in C_0^0(O)$ and we define a Borel measure $|\mu|$ by setting $|\mu|(B) = \inf\left\{\int_O |\mu|;\ B \subset O,\ O \text{ open}\right\}$ for every Borel set $B \subset U$.

If I is an interval, we denote by $S\mathcal{M}(I)$ the subspace of $\mathcal{M}(I)$ of *special bounded measures* or measures without Cantor part, say

$$\mathcal{SM}(I) \;=\; \left\{\mu \in \mathcal{M}(I) :\ \exists\, x_j \in I\,,\ \text{and } c_j \ \text{ s.t.} \quad \mu^s \;=\; \textstyle\sum_j c_j\delta(x - x_j)\right\}$$

which is closed in the strong BV norm, and dense in BV with the weak and intermediate topology.

Let $\mathbf{v} : U \to \mathbf{R}^k$ be a Borel function, (we write v in the scalar case, $k = 1$) ; for $\mathbf{x} \in U$ and $\mathbf{z} \in \widetilde{\mathbf{R}^k} = \mathbf{R}^k \cup \{\infty\}$ (the one point compactification of $\mathbf{R}^k$) we say that $\mathbf{z}$ is the approximate limit of $\mathbf{v}$ at $\mathbf{x}$, and we write

$$\mathbf{z} = \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} \mathbf{v}(\mathbf{y}), \qquad \text{if, for every } g \in C^0(\widetilde{\mathbf{R^k}}), \qquad g(\mathbf{z}) = \lim_{\rho\to 0}\frac{\int_{B_\rho(\mathbf{x})} g(\mathbf{v}(\mathbf{y}))d\mathbf{y}}{|B_\rho|} \quad .$$

The *singular set* $S_{\mathbf{v}} := \{\mathbf{x} \in U : \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}}\mathbf{v}(\mathbf{y}) \text{ does not exist}\}$ is a Borel set; moreover by $\widetilde{\mathbf{v}} : \Omega \setminus S_{\mathbf{v}} \to \widetilde{\mathbf{R}}^k$ we denote the function $\widetilde{\mathbf{v}}(\mathbf{x}) = \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}}\mathbf{v}(\mathbf{y})$.

Let $\mathbf{x} \in U \setminus S_{\mathbf{v}}$ s.t. $\widetilde{\mathbf{v}}(\mathbf{x}) \in \mathbf{R}^k$: we say that $\mathbf{v}$ is approximately differentiable at $\mathbf{x}$ iff there is a $k \times n$ matrix $\nabla\mathbf{v}(\mathbf{x})$ s.t.

$$\operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}}\frac{|\mathbf{v}(\mathbf{y}) - \widetilde{\mathbf{v}}(\mathbf{x}) - \nabla\mathbf{v}(\mathbf{x})(\mathbf{y}-\mathbf{x})|}{|\mathbf{y}-\mathbf{x}|} = 0.$$

If $\mathbf{v}$ is a smooth function then $\nabla\mathbf{v}$ coincides with the classical gradient.
We recall the definition of the space of functions with bounded variation in U with values in $\mathbf{R}^k$:

$$BV(U,\mathbf{R}^k) = \{\mathbf{v}\in L^1(\Omega,\mathbf{R}^k):\ D\mathbf{v}\in\mathcal{M}(U,M_{k,n})\}$$

$$\|\mathbf{v}\|_{BV(U)} \ = \ \|\mathbf{v}\|_{L^1(U)} + \int_U |D\mathbf{v}|$$

where $D\mathbf{v} = \{D_j\mathbf{v}_i\}_{\substack{i=1,..k\\ j=1,..m}}$ denotes the derivatives of $\mathbf{v}$ in the sense of distributions.
In the one dimensional case ($n=1$) we shall use the notation $\dot{\mathbf{v}}$ in place of $\nabla\mathbf{v}$ and $\mathbf{v}'$ instead of Dv.
To simplify notations we set, for any $n\geq 1$,

$$\mathbf{v}_{,i} \ := \ \frac{\partial\mathbf{v}}{\partial x_i} \ = \ D_i\mathbf{v} \ = \ D\mathbf{v}\cdot\mathbf{e}_i \qquad\qquad \nabla_i\mathbf{v} \ := \ (\mathbf{e_i}\cdot\nabla)\,\mathbf{v} \qquad i=1,\cdots,n\ .$$

For every $\mathbf{v}\in BV(U\mathbf{R}^k)$ the following properties hold:

BV1) $\widetilde{\mathbf{v}}(\mathbf{x})\in\mathbf{R}^k$ for $\mathcal{H}^{n-1}$–almost all $\mathbf{x}\in U\setminus S_{\mathbf{v}}$ (see [Z], 5.9.6);

BV2) $S_{\mathbf{v}}$ has null Lebesgue measure and is countably $(\mathcal{H}^{n-1};n-1)$ rectifiable (see [Z], 5.9.6);

BV3) $\nabla\mathbf{v}$ exists a.e. in U and coincides with the Radon–Nikodym derivative of $D\mathbf{v}$ with respect to the Lebesgue measure (see [FE], 4.5.9(26));

BV4) for $\mathcal{H}^{n-1}$ almost all $\mathbf{x}\in S_{\mathbf{v}}$ there exist $\nu=\nu_{\mathbf{v}}(\mathbf{x})\in\partial B_1$, $\mathbf{v}^+(\mathbf{x}),\mathbf{v}^-(\mathbf{x})\in\mathbf{R}^k$ (outer and inner trace, respectively, of $\mathbf{v}$ at $\mathbf{x}$ in the direction ν) such that (see [Z], 5.14.3 and [F], 4.5.9(15))

$$\lim_{\varrho\to 0^+}\varrho^{-n}\fint_{\{\mathbf{y}\in B_\varrho(\mathbf{x});(\mathbf{y}-\mathbf{x})\cdot\nu>0\}}|\mathbf{v}(\mathbf{y})-\mathbf{v}^+(\mathbf{x})|\,d\mathbf{y}=0, \tag{1.1}$$

$$\lim_{\varrho\to 0^+}\varrho^{-n}\fint_{\{\mathbf{y}\in B_\varrho(\mathbf{x});(\mathbf{y}-\mathbf{x})\cdot\nu<0\}}|\mathbf{v}(\mathbf{y})-\mathbf{v}^-(\mathbf{x})|\,d\mathbf{y}=0, \tag{1.2}$$

$$\|Dv\|_{\mathcal{M}(U)} \ \geq \ \int_U\|\nabla v(\mathbf{x})\|\,d\mathbf{x}+\int_{S_v}|\mathbf{v}^+(\mathbf{x})-\mathbf{v}^-(\mathbf{x})|\,d\mathcal{H}^{n-1}(\mathbf{x})\ ; \tag{1.3}$$

BV5) by setting $j_{\mathbf{v}} = (\mathbf{v}^+-\mathbf{v}^-)\otimes\nu_{\mathbf{v}}\,d\mathcal{H}^{n-1}\llcorner S_{\mathbf{v}}$, $C_{\mathbf{v}}=(Dv)^s-j_{\mathbf{v}}$, we have the decomposition

$$D\mathbf{v} \ = \ \nabla\mathbf{v}\,d\mathbf{x} \ + \ j_{\mathbf{v}} \ + \ C_{\mathbf{v}}\ .$$

The space of vector fields with bounded deformation has been introduced to deal with variational problems in perfect plasticity (see [TS],[Te]):

$$BD(U) = \left\{\mathbf{v}\in L^1(\Omega,\mathbf{R}^n):\ \mathbf{e}(\mathbf{v}) := \frac{1}{2}\left(D\mathbf{v}+(D\mathbf{v})^T\right)\ \in\ \mathcal{M}(U,M_{n,n})\right\}$$

$$\|\mathbf{v}\|_{BD(U)} \ = \ \|\mathbf{v}\|_{L^1(U)} + \int_U|\mathbf{e}(\mathbf{v})|\ .$$

$BD(U)$ is the dual of a separable Banach space. For any $\mathbf{v}\in BD(U)$ we define

$$J_{\mathbf{v}} \ := \ \{\mathbf{x}\in U:\ \exists\ \mathbf{v}^+(\mathbf{x}),\mathbf{v}^-(\mathbf{x}),\,\nu_{\mathbf{v}}(\mathbf{x})\in\partial B_1(0),\ \text{s.t. } (1.1),(1.2) \text{ hold with } k=n\}$$

which is the subset of $S_{\mathbf{v}}$ where that $\mathbf{v}$ has one-sided approximate limits with respect to a suitable direction $\nu_{\mathbf{v}}$ "normal" to $J_{\mathbf{v}}$. $J_{\mathbf{v}}$ is called the *jump set* of $\mathbf{v}$ and plays a role analogous to the singular set $S_{\mathbf{v}}$ in the theory of BV functions (see [ACD],[BCD]).
We notice that for $\mathbf{v}\in BV(U,\mathbf{R}^k)$, the set $S_{\mathbf{v}}\setminus J_{\mathbf{v}}$ is $\mathcal{H}^{n-1}$ negligible, while it is not known whether the same property holds in $BD(U)$. Moreover, for every $\mathbf{v}\in BD$

BD1) the linear strain tensor $\mathrm{e}(\mathbf{v})$ has the following decomposition

$$\mathrm{e}(\mathbf{v}) \;=\; \mathrm{e}^a(\mathbf{v}) + \mathrm{e}^s(\mathbf{v}) \;=\; \mathcal{E}(\mathbf{v})\,d\mathbf{x} + \mathrm{e}^j(\mathbf{v}) + \mathrm{e}^c(\mathbf{v})$$

where $\mathrm{e}^a(\mathbf{v}) = \mathcal{E}(\mathbf{v})d\mathbf{x}$ and $\mathrm{e}^s(\mathbf{v})$ are respectively the absolutely continuous and the singular part of $\mathrm{e}(\mathbf{v})$ with respect to $\mathcal{L}^n$; $\mathrm{e}^j(\mathbf{v})$, $\mathrm{e}^c(\mathbf{v})$ are respectively the restriction of $\mathrm{e}^s(\mathbf{v})$ to $J_\mathbf{v}$ and the restriction of e^s to its complement (say the *jump* and *Cantor part* of $\mathrm{e}(\mathbf{v})$).
All along the paper we set $\operatorname{div}\mathbf{v} = \operatorname{Tr}\mathcal{E}(\mathbf{v})$.

BD2) $J_\mathbf{v}$ is a Borel set with null Lebesgue measure and is countably $(\mathcal{H}^{n-1}, n-1)$ rectifiable (see [ACD], Prop.3.5), and there are $\nu_\mathbf{v} = \nu_\mathbf{v}(x) \in \partial B_1$, $\mathbf{v}^+(x), \mathbf{v}^-(x)$ (respectively geometric measure theory normal, outer and inner trace in the ν direction) $\mathcal{H}^{n-1}$ *a.e.* in $J_\mathbf{v}$, s.t.

$$\mathrm{e}^j(\mathbf{v}) \;=\; (\mathbf{v}^+ - \mathbf{v}^-) \odot \nu_\mathbf{v}\, \mathcal{H}^{n-1} \llcorner J_\mathbf{v} \ ,$$

and the jump part $\mathrm{e}^j(\mathbf{v})$ can be represented on every Borel set B by the formula

$$\mathrm{e}^j(\mathbf{v})(B) \;=\; \int_{B\cap J_\mathbf{v}} [\mathbf{v}] \odot \nu_\mathbf{v}\, d\mathcal{H}^{n-1} \qquad\qquad \text{where } [\mathbf{v}] := \mathbf{v}^+ - \mathbf{v}^-$$

BD3) If $\mathcal{R}$ denotes the set of *rigid displacements* (the affine maps of type $A\cdot x + b$ where $A \in M_{n,n}$ is skew-symmetric and $b \in \mathbf{R}^n$), then ([Te] Prop.2.2,2.3 p.155 and Theorem 3.1 [ACD]) for every bounded connected open set U with Lipschitz boundary, and every continuous linear map $R : BD(U) \to \mathcal{R}$ which leaves fixed the elements of $\mathcal{R}$, there is a constant $c_1 = c_1(U,R)$ such that

$$\|\mathbf{v} - R(\mathbf{v})\|_{L^{n/(n-1)}(U)} \;\leq\; c_1(U,R)\ |\mathrm{e}(\mathbf{v})|(U) \qquad \forall \mathbf{v} \in BD(U)\ .$$

The constant c_1 is invariant by dilations (εU , $\varepsilon > 0$) of the open set.

BD4) If ψ is a continuous semi-norm on $BD(U)$ and a norm on $\mathcal{R}$, then $\psi(\mathbf{v}) + \int_U |\mathrm{e}(\mathbf{v})|$ is a norm on $BD(U)$ equivalent to $\|\cdot\|_{BD(U)}$.

BD5) $\quad BD(U) \subset L^s(U) \quad \forall s \in [1, n/(n-1)]$ with compact embedding if $s < n/(n-1)$.

The spaces $BH(U)$ (bounded hessian functions) and $SBH(U)$ (special bounded hessian functions) were introduced and studied in the analysis of elastic–perfectly plastic beams and plates (see [De],[Te], [CLT1], [SaT]):

$$\|v\|_{BH(U)} \;=\; \|v\|_{L^1(U)} + \|Dv\|_{L^1(U)} + \|D^2 v\|_{\mathcal{M}(U)}\ ,$$

$BH(U)$ endowed with this norm is the dual of a Banach space.

When $I \subset \mathbf{R}$ is a bounded interval then the $\mathbf{R}^m$ vector-valued functions with special bounded second derivative are denoted by $BH(I,\mathbf{R}^m)$ (or shortly by $(BH(I)^m)$)

$$BH(I,\mathbf{R}^m) \;=\; \{v \in W^{1,\infty}(I,\mathbf{R}^m) :\ v'' \in \mathcal{M}(I,\mathbf{R}^m)\} = \{v \in L^1(\Omega) :\ Dv \in BV(I,\mathbf{R}^m)\}.$$

Now we recall the definition and main properties of the following spaces: functions with special bounded variation (see [DGA]), vector fields with special bounded deformation (see [ACD]) and functions with special bounded hessian (see [CLT1,2],[SaT]); then we point out some of their properties. These spaces are characterized by the property that some combinations of distributional derivatives are De Giorgi special measures ([DG2]). We set

$$SBV(U,\mathbf{R}^k) = \big\{\mathbf{v} \in BV(U,\mathbf{R}^k) :\ C_\mathbf{v} \equiv 0\big\}$$

$$SBD(U) = \big\{\mathbf{v} \in BD(U) :\ \mathrm{e}^c(\mathbf{v}) \equiv 0\ \big\}$$

$$SBH(U) = \big\{w \in W^{1,1}(U) :\ Dw \in SBV(U,\mathbf{R}^n)\big\}$$

We notice that $\mathbf{v} \in SBV(U, \mathbf{R}^k)$ if and only if

$$\mathbf{v} \in BV(U, \mathbf{R}^k) \quad \text{and} \quad D\mathbf{v} \;=\; \nabla \mathbf{v}\, d\mathbf{x} \;+\; \big(\mathbf{v}^+ - \mathbf{v}^-\big) \otimes \nu_{\mathbf{v}}\, d\mathcal{H}^{\,n-1} \llcorner S_{\mathbf{v}},$$

where $\mathcal{H}^{\,n-1} \llcorner S_{\mathbf{v}}(B) = \mathcal{H}^{\,n-1}(B \cap S_{\mathbf{v}})$ for any Borel set B (see [A]). Moreover (by [BCD] App.)

$$SBD(U) \cap BV(U, \mathbf{R}^n) \;=\; SBV(U, \mathbf{R}^n)$$
$$SBV(U, \mathbf{R}^n) \underset{\neq}{\subset} SBD(U) \underset{\neq}{\subset} BD(U) \;.$$

We remark that $Dw = \nabla w$ in $SBH(U)$ and in $BH(U)$. Moreover we set

$$S_{Dw} \;=\; S_{\nabla w} \;, \qquad \Delta^a w \;=\; \nabla \cdot Dw \qquad\qquad \forall w \in SBH(U)\,.$$

By definition $SBH(U)$ is a closed subspace of $BH(U)$ with respect to the strong norm, while it is not closed with respect to the w^*–$BH(U)$ topology. Moreover we have (see [CLT1,2]):

SBH1) $\displaystyle\int_U |D^2 w| \;=\; \int_U |\nabla^2 w|\, d\mathbf{x} \;+\; \int_{S_{Dw}} |[Dw]|\, d\mathcal{H}^{\,n-1} \qquad \text{where } [Dv] = (Dv)^+ - (Dv)^- \;.$

SBH2) If $I \subset \mathbf{R}$ is a bounded open interval, then $BH(I) \subset W^{1,\infty}(I)$ and the embedding in $W^{1,p}$ is compact for $1 \le p < +\infty$.

2. Two–dimensional approximation of a linear elastic–plastic curved beam ($\mathbf{LCB_2}$)

We describe the geometry of the un-stressed curved beam by an arc-length parametric path: let

$$\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2) \quad \text{be a} \quad C^3([0, L]; \mathbf{R}^2) \quad \text{regular simple arc s.t.} \quad |\dot{\boldsymbol{\gamma}}(s)| = 1 \quad \forall\, s \in [0, L] \tag{2.1}$$

$$\Sigma \;=\; \{\boldsymbol{\gamma}(s) : s \in [0, L]\}. \tag{2.2}$$

We denote by $\mathbf{t}(s) = \dot{\boldsymbol{\gamma}}(s)$ and $\mathbf{n}(s) = (-\dot{\boldsymbol{\gamma}}_2(s), \dot{\boldsymbol{\gamma}}_1(s))$, respectively the unit tangent vector to Σ at $\boldsymbol{\gamma}(s)$ and (our choice of) the unit normal vector. Moreover we denote

$$\kappa \;=\; \kappa(s) \;=\; \dot{\mathbf{t}} \cdot \mathbf{n} \qquad \text{the signed curvature of } \boldsymbol{\gamma}. \tag{2.3}$$

Hence $\ddot{\boldsymbol{\gamma}} \;=\; \dot{\mathbf{t}} \;=\; \kappa \mathbf{n} \;=\; K\mathbf{N}$ where $\mathbf{N}$ is the intrinsic normal and $K = |\kappa|$ the (absolute) scalar curvature (see Fig.1).
The reference configuration of the two dimensional thick rod is the open set

$$\Sigma^\varepsilon \;=\; \{\boldsymbol{\gamma}(s) + \xi \mathbf{n}(s) \;:\; |\xi| < \varepsilon\,,\; 0 < s < L\} \;. \tag{2.4}$$

We do not exclude closed simple arcs in (2.1): in this case $\boldsymbol{\gamma}(0) = \boldsymbol{\gamma}(L)$ and (2.4) is substituted by $\Sigma^\varepsilon = \{\boldsymbol{\gamma}(s) + \xi \mathbf{n}(s) : |\xi| < \varepsilon\,, 0 \le s \le L\}$. All along this chapter we assume

$$0 \;<\; \varepsilon \;\le\; \varepsilon_0 \;\;=\;\; \min \big\{ \big(2\,|\kappa(s)|\big)^{-1} \;:\; s \in [0, L] \,\big\} \;. \tag{2.5}$$

The choice of ε_0 in (2.5) is such that every $\mathbf{x} \in \Sigma^\varepsilon$ has a unique orthogonal projection $\mathbf{p}(\mathbf{x})$ on Σ. Hence the functions $\mathbf{p}(\mathbf{x})$, $s(\mathbf{x}) = \boldsymbol{\gamma}^{-1}(\mathbf{p}(\mathbf{x}))$, $\mathbf{t}(\mathbf{x}) = \mathbf{t}(s(\mathbf{x}))$, $\mathbf{n}(\mathbf{x}) = \mathbf{n}(s(\mathbf{x}))$, $\xi(\mathbf{x}) = \big(\mathbf{x} - \mathbf{p}(\mathbf{x})\big) \cdot \mathbf{n}(\mathbf{x})$ are well defined and smooth on Σ^ε. Since we are interested in the asymptotic behaviour as $\varepsilon \to 0_+$, condition (2.5) is not restrictive, actually $\boldsymbol{\gamma} \in C^k$ for $k \ge 2$ entails the existence of $\varepsilon > 0$ s.t. the distance from Σ is a C^k function on Σ^ε (see [DFN], App.14.6).

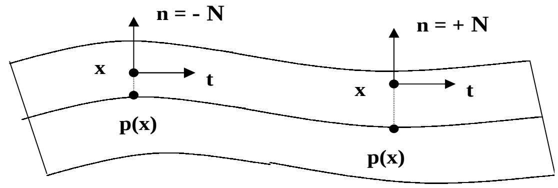

Fig.1 -Open set Σ^ε (n is the $\frac{\pi}{2}$ counterclockwise rotation of t, N is the intrinsic normal)

Assume that Σ^ε is the natural reference of a two dimensional elastic body with damage at meso-scale and that internal strain energy of the body is given, for every displacement field $\mathbf{v} \in SBD(\Sigma^\varepsilon)$, by the following functional

$$\begin{aligned}\mathcal{G}_2^\varepsilon(\mathbf{v}) \;&=\; \int_{\Sigma^\varepsilon} W\left(\mathcal{E}(\mathbf{v})\right)\, dx \;+\; \varepsilon^2 \,\alpha\, \mathcal{H}^1(J_\mathbf{v}) \;+\; \varepsilon\,\beta \int_{J_\mathbf{v}} |[\mathbf{v}] \odot \nu_\mathbf{v}| d\mathcal{H}^1 \\ \text{with} \quad W(A) \;&=\; \mu\,|A|^2 \,+\, \frac{\lambda}{2}\,(\mathrm{Tr}\,A)^2 \;\geq\; C_2\,(\mu,\lambda)\,|A|^2 \qquad \forall A \in M_{2,2}\ , \\ &\alpha,\, \beta,\, \mu \;>\; 0 \;\;,\;\; \mu+\lambda > 0, \qquad C_2(\mu,\lambda) = \min\{\mu, \mu+\lambda\} \;>\; 0\ .\end{aligned} \tag{2.6}$$

Suppose that on Σ^ε acts a transverse dead load $\mathbf{g}^\varepsilon$ such that

$$\mathbf{g}^\varepsilon(\mathbf{x}) \;=\; \varepsilon^2\, f(s(\mathbf{x}))\,\mathbf{n} \qquad\qquad f \in L^p(0,L) \qquad\qquad p \geq 2 \tag{2.7}$$

and so $\mathbf{g}^\varepsilon \in L^p(\Omega^\varepsilon, \mathbf{R}^3)$ the (total) energy functional is

$$\mathcal{F}_2^\varepsilon(\mathbf{v}) \;=\; \mathcal{G}_2^\varepsilon(\mathbf{v}) - \int_{\Sigma^\varepsilon} \mathbf{g}^\varepsilon \cdot \mathbf{v}\, d\mathbf{x} \tag{2.8}$$

The subscript 2 (introduced to distinguish from the 3D approximation of chapter 3) will be dropped whenever this does not create any risk of confusion, and is always implicit in chapter 2.
As in the case of an elastic plastic straight rod (see [PT1]) we assume that the displacement field $\mathbf{v}$ satisfies the cinematic constraint $\mathrm{e}(\mathbf{v}) \cdot \mathbf{n} = \mathbf{0}$ in the sense of measures ([K],[PG]), say

$$\int_{\Sigma^\varepsilon} \mathrm{e}(\mathbf{v}) \cdot \mathbf{n}\ \ \varphi \;\;=\;\; \mathbf{0} \qquad \forall \varphi \in C_0^0(\Sigma^\varepsilon) \tag{2.9}$$

so that we are led to study the following minimization problem

$$(\mathbf{LCB}_2^\varepsilon) \qquad\qquad \inf\,\{\, \mathcal{F}_2^\varepsilon(\mathbf{v}) :\ \mathbf{v} \in SBD(\Sigma^\varepsilon),\ \mathrm{e}(\mathbf{v}) \cdot \mathbf{n} = \mathbf{0} \text{ in } \Sigma^\varepsilon \}$$

Our goal is to study the existence and asymptotic behaviour of minimizers of $(\mathbf{PLB}_2^\varepsilon)$. The first step is to fix the relationship between physical and intrinsic coordinates in Σ^ε.

Lemma 2.1 - *The map* $\Psi\ :\ (s,\xi) \to \mathbf{x} = \boldsymbol{\gamma}(s) + \xi\mathbf{n}(s)$ *is one-to-one, moreover* $\det D\Psi = 1 - \kappa(s)\xi$, $d\mathbf{x} \;=\; \big(1-\kappa(s)\xi\big)\, ds\, d\xi$ and $|\Sigma^\varepsilon| = 2L\varepsilon$.

Proof - $\Psi \in C^1$ is one-to-one by (2.5). By direct computation $D\Psi = \begin{bmatrix} \dot{\boldsymbol{\gamma}}_1 + \xi\dot{\mathbf{n}}_1 & \mathbf{n}_1 \\ \dot{\boldsymbol{\gamma}}_2 + \xi\dot{\mathbf{n}}_2 & \mathbf{n}_2 \end{bmatrix}$, hence the thesis.

The next step is to exploit the cinematic constraint and show, in particular, that that admissible vector fields are completely described by two scalar functions of arc-length and have rank-1 strain tensor.

Lemma 2.2 - *Assume (2.1)-2.5) and fix $\varepsilon \in (0,\varepsilon_0)$. Assume $\mathbf{v} \in SBD(\Sigma^\varepsilon)$ and $\mathrm{e}(\mathbf{v})\cdot\mathbf{n} = \mathbf{0}$ in the sense of measures. Then there are $u_1 \in SBV(0,L)$, $u_2 \in SBH(0,L)$ such that , by labeling $v_{\mathbf{t}} = \mathbf{v}\cdot\mathbf{t}$, $v_{\mathbf{n}} = \mathbf{v}\cdot\mathbf{n}$ the tangential and normal components of $\mathbf{v}$, we have*

i) $\mathbf{v} \;=\; v_t\,\mathbf{t} \,+\, v_n\,\mathbf{n} \;=\; \Big(\big(1-\kappa(s)\xi\big)\,u_1(s) \,-\, \xi\dot u_2(s)\Big)\mathbf{t} \,+\, u_2(s)\,\mathbf{n}$

ii) $$\mathrm{e}(\mathbf{v}) \;=\; \big(1-\kappa\xi\big)^{-1}\,(v_{\mathbf{t},s} - \kappa v_{\mathbf{n}})\,(\mathbf{t}\otimes\mathbf{t}) \;=$$ $$=\; \big(1-\kappa(s)\xi\big)^{-1}\,\Big\{\,\big(u_1'(s)-\kappa(s)u_2(s)\big) \,-\, \xi\,\big(u_2''(s)+\dot\kappa(s)u_1(s)+\kappa(s)u_1'(s)\big)\,\Big\}\;(\mathbf{t}\otimes\mathbf{t})$$

iii) $J_{\mathbf{v}} \;\subset\; (J_{u_1}\cup J_{\dot u_2+\kappa u_1})\otimes(-\varepsilon,\varepsilon) = (\mathbf{S_u})\otimes(-\varepsilon,\varepsilon)$ *and*

$$\mathcal{H}^{\,1}(J_{\mathbf{v}}\setminus(J_{u_1}\cup J_{\dot u_2+\kappa u_1})\times(-\varepsilon,\varepsilon)) = 0\,;$$

$$d\mathcal{H}^1\llcorner J_{\mathbf{v}} \;=\; \sum_{\mathbf{S_u}}(1-\kappa(s)\xi)\,d\xi$$

where $\mathbf{S_u} = S_{u_1}\cup S_{\dot u_2+\kappa u_1}$

iv) $\mathrm{e}(\mathbf{v}) \,=\, \mathbf{0}$ *if and only if* $\dot u_1 - \kappa\,u_2 \;=\; \ddot u_2 \,+\, \dot\kappa\,u_1 \,+\, \kappa\,\dot u_1 \;=\; j_{u_1} \;=\; j_{\dot u_2} \;\equiv\; 0$

v) $$\mathcal{G}^\varepsilon(\mathbf{v}) \;=\; (1+o(1))\left(\mu+\frac{\lambda}{2}\right)\int_0^L\left\{2\,\varepsilon\,|\dot u_1-\kappa\,u_2|^2 \,+\, \frac{2}{3}\,\varepsilon^3\,|\ddot u_2 \,+\, \dot\kappa\,u_1 \,+\, \kappa\,\dot u_1|^2\right\}ds \,+$$ $$+\,(1+o(1))\left(2\,\varepsilon^3\alpha\,\mathcal{H}^0(S_{u_1}\cup S_{\dot u_2+\kappa u_1}) \,+\, \varepsilon\,\beta\sum_{s\,\in\,S_{u_1}\cup S_{\dot u_2+\kappa u_1}}\int_{-\varepsilon}^{\varepsilon}\big|\,[u_1]-[\,\dot u_2+\kappa u_1]\,\xi\,\big|\,d\xi\right)$$

where $o(1)$ tends to 0 as $\varepsilon\to 0_+$.

On the other hand, for every given pair $\mathbf{u} = (u_1,u_2)$ in $SBV(0,L)\times SBH(0,L)$ the vector field $\mathbf{v}$ defined by i) belongs to $SBD(\Sigma^\varepsilon)$ and fulfills (2.9), ii), iii), iv), v).
Moreover the linear map $\mathcal{U}_2^\varepsilon\,:\;\mathbf{v}\to\mathbf{u}$ defined by (i) (transforming orthogonal components into intrinsic components) satisfies

vi) $$u_2 \;=\; \mathbf{v}\cdot\mathbf{n} \;=\; \fint_\varepsilon^\varepsilon \mathbf{v}\cdot\mathbf{n}\,d\xi \quad,\qquad u_1 \;=\; (1-\kappa\xi)^{-1}\,(\mathbf{v}\cdot\mathbf{t}+\xi\dot u_2) \;=\; \fint_\varepsilon^\varepsilon \mathbf{v}\cdot\mathbf{t}\,d\xi$$

and $\mathcal{U}_2^\varepsilon$ is one-to-one and bi-continuous in the strong topologies from the closed subspace of $SBD(\Sigma^\varepsilon)$ satisfying (2.9) to $SBV(0,L)\times SBH(0,L)$.

Proof - The summation convention over repeated indexes is used in the proof. For $\mathbf{x}\in\Sigma^\varepsilon$ we have $\mathbf{x} = \boldsymbol{\gamma}(s(\mathbf{x}))+\xi(\mathbf{x})\mathbf{n}(\boldsymbol{\gamma}(s(\mathbf{x})))$ and $\xi = \big(\mathbf{x}-\boldsymbol{\gamma}(s(\mathbf{x}))\big)\cdot\mathbf{n}$, hence $D_j\xi \;=\; \mathbf{n}_j - \mathbf{t}\cdot\mathbf{n}\,s_{\mathbf{x}_j} + (\mathbf{x}-\boldsymbol{\gamma}(s(\mathbf{x})))\cdot\dot{\mathbf{n}}s_{\mathbf{x}_j}$ say $D_j\xi = \mathbf{n}_j$, moreover $D_\xi\mathbf{x}_j = \mathbf{n}_j$.
By differentiating $\mathbf{x} = \boldsymbol{\gamma}(s(\mathbf{x}))+\xi\mathbf{n}(\boldsymbol{\gamma}(s(\mathbf{x})))$ with respect to $\mathbf{x}_j$, and denoting by $\delta_{i,j}$ the Kronecker symbol, we evaluate :

$$D_j\boldsymbol{\gamma}_i + \mathbf{n}_iD_j\xi + \xi D_j\mathbf{n}_i = \delta_{ij} \tag{2.10}$$

Since $D_j = s_{\mathbf{x}_j}\cdot D_s$, $\mathbf{t} = \dot{\boldsymbol{\gamma}}$ and $\mathbf{t}\cdot\mathbf{n} = 0$, by multiplying (2.10) times $\mathbf{t}_j$ and $\mathbf{n}_j$ we get respectively

$$\mathbf{t}_i = \dot{\boldsymbol{\gamma}}_is_{\mathbf{x}_j}\mathbf{t}_j + \xi\mathbf{t}_jD_j\mathbf{n}_i + \mathbf{n}_i\mathbf{t}_jD_j\xi$$

$$\mathbf{n}_i = \dot{\boldsymbol{\gamma}}_is_{\mathbf{x}_j}\mathbf{n}_j + \mathbf{n}_i\mathbf{n}_jD_j\xi + \xi\mathbf{n}_jD_j\mathbf{n}_i$$

By multiplying the above equations times $\mathbf{t}_i$, recalling that $\dot{\mathbf{n}} = -\kappa\mathbf{t}$, and summing up

$$1 = \mathbf{t}_js_{\mathbf{x}_j} + \xi\mathbf{t}_i\mathbf{t}_jD_j\mathbf{n}_i = (1-\kappa(s)\xi)\mathbf{t}_js_{\mathbf{x}_j} \tag{2.11}$$

$$\mathbf{n}_j s_{\mathbf{x}_j} + \xi \mathbf{t}_i \mathbf{n}_j D_j \mathbf{n}_i = 0 \tag{2.12}$$

By (2.11) we get $s_{\mathbf{x}_j}\mathbf{t}_j = (1-\kappa\xi)^{-1}$. Moreover, since (2.12) must hold for every $\xi \in (-\varepsilon, \varepsilon)$, we get $\mathbf{t}_i\mathbf{n}_j\, D_j\mathbf{n}_i \;=\; 0$ and $\operatorname{div}\mathbf{t} = \mathbf{N}\cdot\nabla_{\mathbf{x}} s = (\operatorname{sign}\kappa)\mathbf{n}_j s_{\mathbf{x}_j} = 0$ hence $\operatorname{div}\mathbf{t} = s_{\mathbf{x}_j}\mathbf{n}_j = 0$.
Now, the relationship $\mathcal{E}(\mathbf{v})\cdot\mathbf{n} = \mathbf{0}$ in the sense of measures is equivalent to the following ones

$$\mathbf{n}_i\mathbf{n}_j\, D_j\mathbf{v}_i = 0 \tag{2.13}$$

$$\mathbf{n}_i\mathbf{t}_j\, D_j\mathbf{v}_i + \mathbf{t}_i\mathbf{n}_j D_j\mathbf{v}_i = 0 \tag{2.14}$$

Equality (2.13) entails $D_\xi v_{\mathbf{n}} = 0$ and then $v_{\mathbf{n}} = u_2(s)$ for a suitable $u_2 \in L^1(0,L)$.
By $\mathbf{n}_j s_{x_j} = 0$, $D_j = s_{x_j} D_s$, $t_j s_{x_j} = (1-\kappa\xi)^{-1}$, $D_\xi\mathbf{t} = \mathbf{0}$, $D_\xi\mathbf{n} = \mathbf{0}$, $D_j\xi = \mathbf{n}_j$, $v_n(\mathbf{x}) = u_2(s(\mathbf{x}))$, if $\mathbf{v}$ is smooth we get

$$\mathbf{t}_i\mathbf{n}_j\, D_j\mathbf{v}_i \;=\; D_\xi v_{\mathbf{t}} \tag{2.15}$$

$$\mathbf{n}_i\mathbf{t}_j\, D_j\mathbf{v}_i \;=\; \mathbf{t}_j D_j\mathbf{v}_n - \mathbf{t}_j\mathbf{v}_i D_j\mathbf{n}_i \;=\; \mathbf{t}_j s_{\mathbf{x}_j} D_s v_{\mathbf{n}} - \mathbf{t}_j\mathbf{v}_i D_s\mathbf{n}_i s_{\mathbf{x}_j} = (1-\kappa\xi)^{-1}\left(D_s v_{\mathbf{n}} + \kappa v_{\mathbf{t}}\right) \tag{2.16}$$

then (2.14) implies the following differential equation in $\mathcal{D}'\left((0,L)\times(-\varepsilon,\varepsilon)\right)$ say in $\mathcal{D}'\left((-\varepsilon,\varepsilon)\right)$

$$D_\xi v_{\mathbf{t}} + (\dot u_2 + \kappa v_{\mathbf{t}})(1-\kappa\xi)^{-1} = 0. \tag{2.17}$$

and then there exists $u_1 = u_1(s)$ such that $u_1 \in L^1(0,L)$ and i) holds in the sense of distributions.
If $\mathbf{v}$ is not a Lipschitz function we cannot use the chain rule in the derivation of (2.16) to evaluate $\mathbf{t}_j D_j\mathbf{v}_{\mathbf{n}}$; on the other hand if $\mathbf{v}$ is not a Sobolev function we cannot perform the change of variables when proving (2.15),(2.16). Nevertheless we may prove in a weak sense (2.15), (2.16) as follows (hence, the differential equation (2.17) holds true (in the sense of distributions shown by (2.17') below, in term of intrinsic coordinates) for $a.e.\, s \in (0,L)$) for every $\mathbf{v} \in \mathit{SBD}(\Sigma^\varepsilon)$ satisfying (2.9).

Since $\operatorname{div}\mathbf{n} = D_j\mathbf{n}_j = s_{\mathbf{x}_j}\dot{\mathbf{n}}_j = -\kappa s_{\mathbf{x}_j}\mathbf{t}_j = -\kappa(1-\kappa\xi)^{-1}$, then for every $\varphi \in C^1_0(\Sigma^\varepsilon)$ and $\mathbf{v} \in \mathit{SBD}(\Sigma^\varepsilon)$, by setting $\widetilde z(s,\xi) = z(\mathbf{x}(s,\xi))$ for every function $z = z(\mathbf{x})$,

$$\begin{aligned}
&\int_{\Sigma^\varepsilon} (\mathbf{t}_i\mathbf{n}_j D_j\mathbf{v}_i)\varphi\, d\mathbf{x} = \int_{\Sigma^\varepsilon} (\mathbf{n}_j D_j v_t)\varphi\, d\mathbf{x} - \int_{\Sigma^\varepsilon} v_i(\mathbf{n}_j D_j\mathbf{t}_i)\varphi\, d\mathbf{x} = \\
&= \int_{\Sigma^\varepsilon} (\mathbf{n}_j D_j v_t)\varphi\, d\mathbf{x} - \int_{\Sigma^\varepsilon} (v_i D_\xi\mathbf{t}_i)\varphi\, d\mathbf{x} = \int_{\Sigma^\varepsilon} (\mathbf{n}_j D_j v_{\mathbf{t}})\varphi\, d\mathbf{x} = \\
&= -\int_{\Sigma^\varepsilon} (\mathbf{n}_j v_t) D_j\varphi\, d\mathbf{x} - \int_{\Sigma^\varepsilon} (v_{\mathbf{t}}\operatorname{div}\mathbf{n})\varphi\, d\mathbf{x} = \\
&= -\int_{-\varepsilon}^{\varepsilon}\int_0^L (1-\kappa\xi)\,\widetilde v_{\mathbf{t}} D_\xi\,\widetilde\varphi\, ds\, d\xi - \int_{-\varepsilon}^{\varepsilon}\int_0^L (1-\kappa\xi)\,\widetilde v_{\mathbf{t}}\,\widetilde{\operatorname{div}\mathbf{n}}\,\widetilde\varphi\, ds\, d\xi = \\
&= -\int_{-\varepsilon}^{\varepsilon}\int_0^L (1-\kappa\xi)\,\widetilde v_{\mathbf{t}}\, D_\xi\widetilde\varphi\, ds\, d\xi \;+ \int_{-\varepsilon}^{\varepsilon}\int_0^L \kappa\,\widetilde v_{\mathbf{t}}\,\widetilde\varphi\; ds\, d\xi \;= \\
&= \;-\int_{-\varepsilon}^{\varepsilon}\int_0^L \widetilde v_{\mathbf{t}}\, D_\xi\{(1-\kappa\xi)\widetilde\varphi\}\, ds\, d\xi \;= \int_{-\varepsilon}^{\varepsilon}\int_0^L (D_\xi\widetilde v_{\mathbf{t}})\;\widetilde\varphi\,(1-\kappa\xi)\, ds\, d\xi
\end{aligned} \tag{2.15'}$$

say $= \int_{\Sigma^\varepsilon}(D_\xi v_{\mathbf{t}})\;\varphi\, d\mathbf{x}$ if change of variables is performed.
Since $\operatorname{div}\mathbf{t} = \mathbf{n}_j s_{x_j} = 0$ for every $\varphi \in C^1_0(\Sigma^\varepsilon)$ we have

$$\begin{aligned}
\int_{\Sigma^\varepsilon} (\mathbf{t}_j D_j v_n)\varphi\, d\mathbf{x} \;&= -\int_{\Sigma^\varepsilon} (v_n\operatorname{div}\mathbf{t})\varphi\, d\mathbf{x} \,-\, \int_{\Sigma^\varepsilon} (\mathbf{t}_j v_n) D_j\varphi\, d\mathbf{x} \;= \\
&= \;-\int_{\Sigma^\varepsilon} (\mathbf{t}_j v_n) D_j\varphi\, d\mathbf{x} \;=\; -\int_{-\varepsilon}^{\varepsilon}\int_0^L \widetilde v_n (1-\kappa\xi)^{-1} D_s\widetilde\varphi\,(1-\kappa\xi)\, ds\, d\xi \;= \\
&= \;-\int_{-\varepsilon}^{\varepsilon}\int_0^L \widetilde v_n D_s\widetilde\varphi\, ds\, d\xi \;=\; \int_{-\varepsilon}^{\varepsilon}\int_0^L (D_s\widetilde v_n)\;\widetilde\varphi\, ds\, d\xi
\end{aligned}$$

say $= \int_{\Sigma^\varepsilon} (1-\kappa\xi)^{-1}\, (D_s v_n)\, \varphi\, d\mathbf{x}$ if the change of variables is performed.
And, since $\mathbf{n}_i \mathbf{t}_j D_j \mathbf{v}_i \;=\; \mathbf{t}_j D_j \mathbf{v_n} - \mathbf{t}_j \mathbf{v}_i D_j \mathbf{n}_i \;=\; \mathbf{t}_j D_j \mathbf{v_n} + \kappa(1-\xi\kappa)^{-1}\mathbf{v_t}$, we get

$$\int_{\Sigma^\varepsilon} (\mathbf{n}_i \mathbf{t}_j D_j \mathbf{v}_i)\; \varphi\, d\mathbf{x} \;=\; \int_0^L \int_{-\varepsilon}^{\varepsilon} (D_s \widetilde{v}_\mathbf{n} + \kappa \widetilde{v}_\mathbf{t})\; \widetilde{\varphi}\, ds\, d\xi\ . \tag{2.16$'$}$$

By summarizing

$$\int_0^L \int_{-\varepsilon}^{\varepsilon} (\, (1-\kappa\xi) D_\xi \widetilde{\mathbf{v}}_\mathbf{t} \,+\, (\dot{u}_2 + \kappa \widetilde{\mathbf{v}}_\mathbf{t})\,)\, \widetilde{\varphi}\, ds\, d\xi \;=\; 0 \qquad \forall \widetilde{\varphi} \in C_0^1\big((0,L)\times(-\varepsilon,\varepsilon)\big)\,. \tag{2.17$'$}$$

Once established $i)$ we proceed in the proof. Since $\mathrm{e}(\mathbf{v})\cdot\mathbf{n} = \mathbf{0}$ we have

$$\mathrm{e}(\mathbf{v}) \;=\; (\mathbf{t}_i \mathbf{t}_j D_i \mathbf{v}_j)\,(\mathbf{t}\otimes\mathbf{t})$$

hence, by substitution of $i)$, we get $ii)$. Therefore $D_s v_t \;=\; (1-\kappa\xi)^{-1}(\mathrm{e}(\mathbf{v})\cdot\mathbf{t})\cdot\mathbf{t} \,+\, \kappa v_n$ belongs to $\mathcal{M}((0,L)\times(-\varepsilon,\varepsilon))$ and is a measure without Cantor part; then, by a standard elimination technique, ${u_1}' - \kappa u_2 \in \mathcal{SM}(0,L)$, ${u_2}'' + \kappa' u_1 + \kappa {u_1}' \in \mathcal{SM}(0,L)$, hence $u_1 \in SBV(0,L)$, $u_2 \in SBH(0,L)$ and $iii)$ holds true. Then we get , for every Lipschitz $\mathbf{v}$,

$$\mathbf{t}_i\mathbf{t}_j D_i \mathbf{v}_j = \mathbf{t}_i D_i v_\mathbf{t} - \mathbf{t}_i \mathbf{v}_j D_i \mathbf{t}_j = \mathbf{t}_i s_{x_i} D_s v_t - \mathbf{t}_i \mathbf{v}_j s_{x_i} \dot{\mathbf{t}}_j = (1-\kappa\xi)^{-1}(D_s v_\mathbf{t} - \kappa v_\mathbf{n})\ , \tag{2.18}$$

and (by arguing as in the proof of (2.15),(2.16),(2.15'),(2.16')), for every $\mathbf{v} \in SBD(\Sigma^\varepsilon)$ and $\varphi \in C_0^1(\Sigma^\varepsilon)$, we find

$$\begin{aligned}
&\int_{\Sigma^\varepsilon} (\mathbf{t}_i D_i v_\mathbf{t})\; \varphi\, d\mathbf{x} \;= \\
&\quad= -\int_{\Sigma^\varepsilon} (\mathbf{t}_i v_\mathbf{t})\; D_i\varphi\, d\mathbf{x} - \int_{\Sigma^\varepsilon} (v_\mathbf{t}\,\mathrm{div}\,\mathbf{t})\; \varphi\, d\mathbf{x} \;=\; \int_{\Sigma^\varepsilon} D_i\,(\mathbf{t}_i v_\mathbf{t})\; \varphi\, d\mathbf{x} - \int_{\Sigma^\varepsilon} (v_\mathbf{t}\,\mathrm{div}\,\mathbf{t})\; \varphi\, d\mathbf{x} \;= \\
&\quad= \int_{\Sigma^\varepsilon} (s_{x_i}\mathbf{t}_i D_s v_\mathbf{t})\; \varphi\, d\mathbf{x} \;=\; \int_{\Sigma^\varepsilon} (1-\kappa\xi)^{-1} D_s v_\mathbf{t}\, \varphi\, d\mathbf{x} \;=\; \int_0^L \int_{-\varepsilon}^{\varepsilon} D_s \widetilde{\mathbf{v}}_\mathbf{t}\, \widetilde{\varphi}\, ds\, d\xi
\end{aligned}$$

$$\int_{\Sigma^\varepsilon} (\mathbf{t}_i\mathbf{t}_j D_i \widetilde{\mathbf{v}}_j)\; \varphi\, d\mathbf{x} \;=\; \int_0^L \int_{-\varepsilon}^{\varepsilon} (\,(D_s \widetilde{v}_\mathbf{t} - \kappa \widetilde{v}_\mathbf{n})\,)\; \widetilde{\varphi}\, ds\, d\xi\ , \tag{2.18$'$}$$

now we know that $D_s v_\mathbf{t} - \kappa v_\mathbf{n}$ is a measure, hence its image through the map Ψ (by Lemma 2.1) gives (2.18) for $\mathbf{v} \in SBD(\Sigma^\varepsilon)$.
Recalling that $\mathrm{e}(\mathbf{v})$ is a measure without Cantor part we may deduce that, for almost every $\xi \in (-\varepsilon,\varepsilon)$, $\mathbf{v_t} \in SBV(0,L)$. Hence, $iii), iv), v)$ follow by $ii)$.
The definition and injectivity of the map $\mathcal{U}_2^\varepsilon$ are trivial. The continuity of $(\mathcal{U}_2^\varepsilon)^{-1}$ follows from $i), ii)$. Then the continuity of $\mathcal{U}_2^\varepsilon$ follows from surjectivity and the Open Mapping Theorem. ■

We state some technical lemmas that will be useful tools in the proof of the main result.

Lemma 2.3 - *For every $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$ the following inequality holds*

$$\int_{-\varepsilon}^{\varepsilon} |\,\mathbf{a} + \mathbf{b}z\,|\; dz \quad\geq\quad \theta\,|\mathbf{b}|\,\varepsilon^2 \;+\; 2(1-\theta)\,|\mathbf{a}|\,\varepsilon \qquad \forall\varepsilon \geq 0\ , \forall\theta \in [0,1]\ . \tag{2.19}$$

Moreover, if $\mathbf{b} = \mathbf{0}$, then equality in (2.19) holds iff $\theta = 0$; if $\mathbf{b} \neq \mathbf{0}$, then equality in (2.19) holds iff $\mathbf{a} = \mathbf{0}$ and $\theta = 1$.

Proof - See Lemma 2.2 of [PT2].

We introduce the Hilbert space $\mathcal{W}$ of pairs $\mathbf{u} = (u_1, u_2)$, related via Lemma 2.1 to vector fields $\mathbf{v}$ (admissible for $\mathbf{LCB}_2^\varepsilon$ in the sense (2.9)), which is a closed sub-space of $H^1(0,L) \times H^2(0,L)$.

$$\mathcal{W} = \{\mathbf{w} \in H^1(0,L) \times H^2(0,L) : \dot{w}_1 - \kappa w_2 = \ddot{w}_2 + \dot{\kappa} w_1 + \kappa \dot{w}_1 = 0\} \quad \text{where } \kappa \text{ is defined by (2.3)}$$

$\mathcal{W}$ describes the space $\mathcal{R}$ of infinitesimal rigid displacements in terms of intrinsic coordinates. Notice that $j_{w_1} = j_{\dot{w}_2} = 0$ for any $w \in \mathcal{W}$. Moreover, due to Lemma 2.2 ii),vi) , $w \in \mathcal{W}$ is as regular as $\mathbf{n}$ and κ are. The next Lemma investigates additional properties of $\mathcal{W}$

Lemma 2.4 - *The linear space $\mathcal{W}$ is closed in the $L^2(0,L) \times H^1(0,L)$ topology. Moreover For every $\mathbf{u} = (u_1, u_2) \in SBV(0,L) \times SBH(0,L)$ the following minimum is achieved*

$$\min \left\{ \int_0^L |\mathbf{u} - \mathbf{w}|^2 + |\dot{u}_2 - \dot{w}_2|^2 ds : \mathbf{w} \in \mathcal{W} \right\},$$

so that we can introduce the following notation: the unique minimizer $\mathbf{w} = \mathbf{w}(\mathbf{u})$ of the above problem will be denoted by $\overline{\mathbf{u}} \stackrel{\text{def}}{=} \mathcal{P}(\mathbf{u})$. Then for every $\mathbf{v} \in SBD(\Sigma^\varepsilon)$, $\overline{\mathbf{v}} \in \mathcal{R}$ will denote the vector field $\overline{\mathbf{v}} = (\mathcal{U}_2^\varepsilon)^{-1} \mathcal{P}(\mathbf{u})$ where $\mathbf{u}$ is related to $\mathbf{v}$ through Lemma 2.2, that is $\mathbf{u} = \mathcal{U}_2^\varepsilon(\mathbf{v})$ and $\overline{\mathbf{u}} = \mathcal{U}_2^\varepsilon(\overline{\mathbf{v}})$:

$$\overline{\mathbf{v}} \ = \ \big((1 - \kappa(s)\xi)\, \overline{u}_1(s) \ - \ \xi \dot{\overline{u}}_2(s) \big) \mathbf{t} + \overline{u}_2(s) \mathbf{n} \ . \tag{2.20}$$

The map $\mathcal{P} :\ \mathbf{u} \to \overline{\mathbf{u}}$ is linear and continuous from $SBV(0,1) \times SBH(0,1))$ to $\mathcal{W}$ with norm $L^2 \times H^1$. The map $(\mathcal{U}_2^\varepsilon)^{-1} \mathcal{P} \mathcal{U}_2^\varepsilon :\ \mathbf{v} \to \overline{\mathbf{v}}$ is linear and continuous from $SBD(\Sigma^\varepsilon)$ to $\mathcal{R}$. (see Fig.2)

Proof. The differential equations $\dot{w}_1 = \kappa w_2$ and $\ddot{w}_2 = -\dot{\kappa} w_1 - \kappa \dot{w}_1$ together with $\kappa \in C^2$ entail the equivalence on $\mathcal{W}$ of $L^2 \times H^1$ and $H^1 \times H^2$ topologies. Hence the $L^2 \times H^1$ closedness of $\mathcal{W}$ is proved. The minimum problem in the statement defines the projection $\mathcal{P}$ onto the closed subspace $\mathcal{W}$ with respect to the $L^2 \times H^1$ topology. This proves (for every $\mathbf{u} \in SBV \times SBH$) the existence of the (unique by strict convexity) minimizer $\mathbf{w} = \mathcal{P}(\mathbf{u})$ which will be denoted with $\overline{\mathbf{u}}$.
The continuity of $\mathbf{v} \to \overline{\mathbf{v}}$ follows from $\overline{\mathbf{v}} \ = \ (\mathcal{U}^\varepsilon)^{-1} \mathcal{P} \mathcal{U}^\varepsilon \, \mathbf{v}$ and the continuity of the maps $\mathcal{U}_2^\varepsilon$ from $SBD(\Sigma^\varepsilon)$ to $SBV(0,L) \times SBH(0,L)$, $\mathcal{P}$ from $SBV(0,L) \times SBH(0,L)$ to $H^1(0,L) \times H^2(0,L)$, and $(\mathcal{U}^\varepsilon)^{-1}$ from $SBV(0,L) \times SBH(0,L)$ to $SBD(\Sigma^\varepsilon)$ (and from $H^1(0,L) \times H^2(0,L)$ to $H^1(\Sigma^\varepsilon, \mathbf{R}^3)$ too).

Lemma 2.5 - *For every $r \in [1,2]$ there is $\mathcal{C}_r = \mathcal{C}(\boldsymbol{\gamma}, r) > 0$ such that for every $\mathbf{u} = (u_1, u_2)$ in $SBV(0,L) \times SBH(0,L)$*

$$\left(\int_0^L |\mathbf{u} - \overline{\mathbf{u}}|^r + |\dot{u}_2 - (\dot{\overline{\mathbf{u}}})_2|^r \right)^{1/r} \ \le \ \mathcal{C}_r \int_0^L |\dot{u}_1 - \kappa u_2| + |\ddot{u}_2 + \dot{\kappa} u_1 + \kappa \dot{u}_1| + \mathcal{C}_r \sum_{S_{u_1} \cup S_{\dot{u}_2 + \kappa u_1}} |[u_1]| + |[\dot{u}_2 + \kappa u_1]|$$

Proof - Indeed by Hölder inequality and arguing by contradiction we suppose that there exists a sequence $\mathbf{u}_h \in SBV(0,L) \times SBH(0,L)$ such that, by setting $\mathbf{z}_h = \mathbf{u}_h - \overline{\mathbf{u}}_h$,

$$\int_0^L (\, |\mathbf{z}_h|^r + |(\dot{z}_2)_h|^r \,) \ = \ 1$$

and
$$(z_1)_h{}' - \kappa (z_2)_h \ \to \ 0 \ , \qquad (z_h)_2{}'' + \kappa' (z_1)_h + \kappa (z_1)_h{}' \ \to \ 0$$

both strongly in $\mathcal{M}(0,L)$. Hence the sequences $(z_1)_h$ and $(z_2)_h$ are sequentially weakly compact in $SBV(0,L)$ and in $SBH(0,L)$ respectively, and we may suppose that, up to sub-sequences, $(z_1)_h{}' \to$

$\kappa z_2 = (z_1)'$ and $(z_2)_h{}'' \to (z_2)'' = -\kappa(z_1)' - \kappa' z_1$ both strongly in $\mathcal{M}(0,L)$. Therefore $\mathbf{z}_h \to \mathbf{z}$ strongly in $SBV(0,L) \times SBH(0,L)$, hence $\mathbf{z}$ belongs to $\mathcal{W}$, and by Lemma 2.4 and the linearity of $\mathcal{P}$, we have $\mathbf{0} = \overline{\mathbf{z}_h} \to \overline{\mathbf{z}} = \mathbf{0}$ which implies (since $\mathcal{P}$ is the identity on $\mathcal{W}$) $\mathbf{z} = \mathbf{0}$ and both $\mathbf{z}_h, (\dot{z}_2)_h \to 0$ in $L^r(0,L)$ for every $r \in [1,2]$, which is a contradiction. ■

The previous Lemma entails an estimate of the blow-up for the Korn-Poincaré inequality constant in $SBD(\Sigma^\varepsilon)$, as $\varepsilon \to 0_+$ (see Rmk 2.7).

Lemma 2.6 - *Let $\mathcal{C}_2$ be the constant provided by Lemma 2.5. Then for every $\mathbf{v} \in SBD(\Sigma^\varepsilon)$ satisfying (2.9), if $\overline{\mathbf{v}} \in SBD(\Sigma^\varepsilon)$ denotes the vector field defined in the statement of Lemma 2.4, for all $\varepsilon < \min(\varepsilon_0, 1)$, we have:*

$$\left(\int_{\Sigma^\varepsilon} |\mathbf{v} - \overline{\mathbf{v}}|^2 \, dx \right)^{\frac{1}{2}} \le 8\, \mathcal{C}\, \varepsilon^{-\frac{3}{2}} \int_{\Sigma^\varepsilon} |\mathrm{e}(\mathbf{v})|$$

Proof. Let $\mathbf{v} \in SBD(\Sigma^\varepsilon)$ such that $\mathrm{e}(\mathbf{v}) \cdot \mathbf{n} = 0$ then, by Lemmas 2.1, 2.2 , 2.4 and 2.5 we get

$$\int_{\Sigma^\varepsilon} |\mathbf{v} - \overline{\mathbf{v}}|^2 \, dx = \int_{-\varepsilon}^{\varepsilon} \int_0^L \left| \{(1-\kappa\xi)(u_1 - \overline{u}_1) - \xi(\dot{u}_2 - \dot{\overline{u}}_2)\}\mathbf{t} + (u_2 - \overline{u}_2)\mathbf{n} \right|^2 (1-\kappa\xi)\, ds\, d\xi \le$$
$$\le 4\varepsilon \int_0^L |\mathbf{u} - \overline{\mathbf{u}}|^2 \, ds + 2\varepsilon^3 \int_0^L |\dot{u}_2 - \dot{\overline{u}}_2|^2 \, ds \le$$
$$\le 4{\mathcal{C}_2}^2 \varepsilon \left\{ \int_0^L |\dot{u}_1 - \kappa u_2| + |\ddot{u}_2 + \kappa \dot{u}_1 + \dot{\kappa} u_1| \, ds + \sum_{S_{u_1} \cup S_{\dot{u}_2 + \kappa u_1}} |[u_1]| + |[\dot{u}_2 + \kappa u_1]| \right\}^2 \le$$
$$\le 16{\mathcal{C}_2}^2 \varepsilon^{-3} \left\{ \int_0^L \varepsilon |\dot{u}_1 - \kappa u_2| + \frac{\varepsilon^2}{2} |\ddot{u}_2 + \kappa \dot{u}_1 + \dot{\kappa} u_1| ds + \sum_{S_{u_1} \cup S_{\dot{u}_2 + \kappa u_1}} \varepsilon |[u_1]| + \frac{\varepsilon^2}{2} |[\dot{u}_2 + \kappa u_1]| \right\}^2 .$$

Since (2.5) entails $|1 - \kappa\xi|^{-1} \le 2$, by applying Lemma 2.2ii and 2.3 twice, for $n = 1,\ \theta = 1/2,$, at first with $\mathbf{a} = \dot{u}_1 - \kappa u_2,\ \mathbf{b} = \ddot{u}_2 + \dot{\kappa} u_1 + \kappa \dot{u}_1$ and then with $\mathbf{a} = [u_1]$ $\mathbf{b} = [\dot{u}_2 + \kappa u_1]$, we deduce

$$16\, \mathcal{C}_2^2 \varepsilon^{-3} \left\{ \int_0^L \varepsilon |\dot{u}_1 - \kappa u_2| + \frac{\varepsilon^2}{2} |\ddot{u}_2 + \kappa \dot{u}_1 + \dot{k} u_1| ds + \sum_{S_{u_1} \cup S_{\dot{u}_2 + \kappa u_1}} \varepsilon |[u_1]| + \frac{\varepsilon^2}{2} |[\dot{u}_2 + \kappa u_1]| \right\}^2 \le$$
$$\le 16\, \mathcal{C}_2^2 \varepsilon^{-3} \left\{ \int_0^L ds \int_{-\varepsilon}^{\varepsilon} |\dot{u}_1 - k u_2 - \xi(\ddot{u}_2 + \kappa \dot{u}_1 + \dot{\kappa} u_1)| \, d\xi + \int_{-\varepsilon}^{\varepsilon} \sum_{S_{u_1} \cup S_{\dot{u}_2 + \kappa u_1}} |\, [u_1] - \xi [\dot{u}_2 + \kappa u_1] \,| \, d\xi \right\}^2$$
$$\le 16\, \mathcal{C}_2^2 \varepsilon^{-3} \left\{ \int_0^L ds \int_{-\varepsilon}^{\varepsilon} (1 - \kappa\xi) |\mathcal{E}(\mathbf{v})| \, d\xi + 2 \int_{\varepsilon}^{\varepsilon} \sum_{\mathbf{S_u}} (1 - \kappa\xi) [\mathbf{v_t}] \, d\xi \right\}^2 \le$$
$$\le 64\, \mathcal{C}^2 \varepsilon^{-3} \left(\int_{\Sigma^\varepsilon} |\mathrm{e}(\mathbf{v})| \right)^2 .$$

An obvious consequence, via Hölder inequality, of the previous Lemma is the next statement.

Remark 2.7 - Let $\mathcal{C} = \mathcal{C}_2$ the constant of Lemma 2.5. Then for every $\mathbf{v} \in SBD(\Sigma^\varepsilon)$ such that $\mathrm{e}(\mathbf{v}) \cdot \mathbf{n} = 0$ and for every $q \in [1,2]$ we have

$$\left(\int_{\Sigma^\varepsilon} |\mathbf{v} - \overline{\mathbf{v}}|^q \, dx \right)^{\frac{1}{q}} \le 2^{\frac{1}{q} - \frac{1}{2}}\, 8\, \mathcal{C}\, L^{\frac{2-q}{2q}}\, \varepsilon^{-\frac{3}{2} + \frac{2-q}{2q}} \, \|\mathrm{e}(\mathbf{v})\|_{\mathcal{M}(\Sigma^\varepsilon)} \tag{2.21}$$

Remark 2.8 - We emphasize that any displacement field $\mathbf{v} \in SBD(\Sigma)$ such that $\mathrm{e}(\mathbf{v}) \cdot \mathbf{n} = \mathbf{0}$ can be identified with the pair $\mathbf{u} = (u_1, u_2) \in SBV(0,L) \times SBH(0,L)$. Such $\mathbf{u}$ describes $\mathbf{v}$ in the sense of Lemma 2.1 and, from now on, we make use of this relationship $\mathbf{u} = \mathbf{u}(\mathbf{v}) = (u_1, u_2)$, where u_2 and $\big((1-\kappa\xi)u_1 - \xi\dot{u}_2\big)$ are respectively the normal and the tangential displacement. In the same spirit, for every $\mathbf{u} = \mathbf{u}(\mathbf{v}) \in SBV(0,L) \times SBH(0,L)$ such that $\mathrm{e}(\mathbf{v}) \cdot \mathbf{n} = \mathbf{0}$, Lemma 2.4 produces an infinitesimal rigid displacement $\overline{\mathbf{v}}$ associated to $\overline{\mathbf{u}} = \mathcal{P}\mathcal{U}_2^\varepsilon(\mathbf{v}) \in \mathcal{W}$. Notice that in general $\overline{\mathbf{u}} \neq \mathbf{u}$ $\overline{\mathbf{v}} \neq \mathbf{v}$. Nevertheless, if $\mathbf{v} \in \mathcal{R}$ and $\mathbf{u} = \mathcal{U}_2^\varepsilon(\mathbf{v})$, then $\overline{\mathbf{u}} = \mathbf{u}$ and $\overline{\mathbf{v}} = \mathbf{v} = \big((1-\kappa\xi)u_1 - \xi\dot{u}_2\big)\mathbf{t} + u_2\mathbf{n}$. The map $\mathcal{U}_2^\varepsilon$ is bijective from $SBD(\Sigma^\varepsilon)$ with constraint (2.9) to $BV(0,L) \times BH(0,L)$. The norm of the linear operator $\mathcal{U}_2^\varepsilon$ depends on ε and is not bounded uniformly in ε, while its inverse $(\mathcal{U}_2^\varepsilon)^{-1}$ has norm uniformly bounded in ε by Lemma 2.2 i) . The framework is summarized in Fig.2 below.

$$\begin{array}{ccc}
\mathbf{v} \in SBD(\Sigma^\varepsilon) \text{ s.t. (2.9) holds} & & \overline{\mathbf{v}} \in \mathcal{R} \subset SBD(\Sigma^\varepsilon) \\
\Big\downarrow \mathcal{U}_2^\varepsilon & & \Big\uparrow \left(\mathcal{U}_2^\varepsilon\right)^{-1} \\
\mathbf{u} \in SBV \times SBH(0,L) & \xrightarrow{\mathcal{P}} & \overline{\mathbf{u}} \in \mathcal{W} \subset L^2 \times H^1(0,L)
\end{array}$$

Fig.2 - Mappings diagram (2D approximation)

Theorem 2.9 - *Assume (2.1)-(2.9), $\mathcal{C} = \mathcal{C}_2$ is the constant defined in Lemma 2.5 and*

$$\int_{\Sigma^\varepsilon} f\,\mathbf{n}\cdot\mathbf{w} \;=\; 0 \quad \forall \mathbf{w} \in \mathcal{R}(\Sigma^\varepsilon) \;\; (\textbf{compatibility})\,; \quad \|\, f \,\|_{L^p(0,L)} < \frac{\beta}{32\,\mathcal{C}\,L^{\frac{p-2}{2p}}} \;\; (\textbf{safe load}). \tag{2.22}$$

Then the minimum problem

$$(\mathbf{LCB_2^\varepsilon}) \qquad \min\,\{\, \mathcal{F}_2^\varepsilon(\mathbf{v}) : \mathbf{v} \in SBD(\Sigma^\varepsilon),\; \mathrm{e}(\mathbf{v}) \cdot \mathbf{n} = 0 \textit{ in } \Sigma^\varepsilon \,\}$$

admits a (not necessarily unique) solution.

Obviously $\mathbf{LCB_2^\varepsilon}$ has solution for small loads, but the smallness condition could depend on the thickness 2ε : actually Theorem 2.9 proves a safe load condition independent of ε . Notice that a smallness condition on the linear term, like the safe load condition (2.22) or (3.22) in the 3D approximation, is a common feature of variational functionals with linear growth (see [BBGT]).

Proof of Theorem 2.9 - Let $\mathbf{v}_h$ be a minimizing sequence. c will denotes various constants independent of ε, the superscript ε in the functions $\mathbf{v}_h^\varepsilon$ will be dropped in the proof. Then, by the compatibility assumption,

$$\mathcal{G}_2^\varepsilon(\mathbf{v}_h) - \int_{\Sigma^\varepsilon} \mathbf{g}^\varepsilon \cdot (\mathbf{v}_h - \overline{\mathbf{v}}_h)\, d\mathbf{x} \;=\; \mathcal{F}_2^\varepsilon(\mathbf{v}_h - \overline{\mathbf{v}}_h) \;=\; \mathcal{F}_2^\varepsilon(\mathbf{v}_h) \;\leq\; \mathcal{F}_2^\varepsilon(\mathbf{0}) \;\leq\; 0 \tag{2.23}$$

hence, by exploiting the safe load condition, (2.6),(2.7), Remark 2.7 with $q = p'$, and Young inequality,

$$\begin{aligned}
&\varepsilon\beta \int_{\Sigma^\varepsilon} |\mathrm{e}(\mathbf{v}_h)| \;-\; \varepsilon^3 \frac{\beta^2 L}{C_2(\mu,\lambda)} \;\leq \\
&\quad \leq C_2(\mu,\lambda) \int_{\Sigma^\varepsilon} ||\mathcal{E}(\mathbf{v}_h)||^2\, d\mathbf{x} \;+\; \varepsilon\,\beta \int_{J_{\mathbf{v}_h}} |[\mathbf{v}_h] \odot \nu_{\mathbf{v}_h}| d\mathcal{H}^1 \;\leq\; +\varepsilon^2 \int_{\Sigma^\varepsilon} f\,\mathbf{n} \cdot (\mathbf{v}_h - \overline{\mathbf{v}}_h)\, dx \;\leq \\
&\quad \leq 2^{1/p} \varepsilon^{2+1/p} \|f\|_{L^p(0,L)} \|\mathbf{v}_h - \overline{\mathbf{v}}_h\|_{L^{p'}} \;\leq\; 2^{\frac{1}{p}+\frac{1}{p'}-\frac{1}{2}}\, 8\, \mathcal{C}_2 L^{\frac{p-2}{2p}}\, \varepsilon \|f\|_{L^p(0,L)} \int_{\Sigma^\varepsilon} |\mathrm{e}(\mathbf{v}_h)| < \\
&\quad < \frac{1}{2} \varepsilon\beta \int_{\Sigma^\varepsilon} |\mathrm{e}(\mathbf{v}_h)|
\end{aligned}$$

say

$$\int_{\Sigma^\varepsilon} |\mathrm{e}(\mathbf{v}_h)| \quad \text{and hence} \int_{\Sigma^\varepsilon} |\mathbf{v}_h - \overline{\mathbf{v}}_h| \quad \text{are uniformly bounded in } h \text{ and } \varepsilon\ . \tag{2.24}$$

From Theorem 1.1 and Corollary 1.2 of [BCD] and Theorem 2.1 of [PT1] we know both the sequential compactness of the sequence $\mathbf{v}_h - \overline{\mathbf{v}}_h$ in w^* $SBD(\Sigma^\varepsilon)$ and the lower semi-continuity of $\mathcal{F}$ with respect to the same topology; taking into account that the constraint $\mathrm{e}(\mathbf{v})\cdot\mathbf{n} = \mathbf{0}$ is closed in w^* $SBD(\Sigma^\varepsilon)$, we get $\mathbf{v}_h - \overline{\mathbf{v}}_h \ \rightarrow^{w^*}\ \mathbf{v}^\varepsilon \ \in$ argmin $\mathbf{LCB}^\varepsilon_\mathbf{2}$ up to sub-sequences, and the thesis follows. ■

Now we show some estimates for minimizers of ($\mathbf{LCB}^\varepsilon_\mathbf{2}$) by exploiting the intrinsic coordinates formulation.

Lemma 2.10 - *Assume (2.1)-(2.9),(2.22). Then there are a constant $c > 0$ independent of ε and $\mathbf{v}^\varepsilon \in \operatorname{argmin} \mathbf{LCB}^\varepsilon_\mathbf{2}$, $0 < \varepsilon \le \varepsilon_0$, such that, for $\mathbf{u}^\varepsilon = \mathcal{U}^\varepsilon_2(\mathbf{v}^\varepsilon)$,*

$$\begin{cases} \displaystyle\int_{S_{u_1^\varepsilon}} |[u_1^\varepsilon]| \le c\,\varepsilon\ ; \qquad \int_{S_{\dot u_2^\varepsilon+\kappa u_1}} |[\dot u_2^\varepsilon + \kappa u_1]| \le c \\ \mathcal{H}^0(S_{u_1^\varepsilon}) \le c\ ; \qquad \mathcal{H}^0(\dot u_2^\varepsilon) \le c, \\ \displaystyle\int_0^L |\dot u_1^\varepsilon - \kappa u_2^\varepsilon|^2 \le c\,\varepsilon^2\ , \qquad \int_0^L |\ddot u_2^\varepsilon + \dot\kappa u_1^\varepsilon + \kappa \dot u_1|^2 \le c\ , \\ \displaystyle \|u_2^\varepsilon\|_{BH(0,L)} \le c\ ; \qquad \int_0^L |\dot u_1^\varepsilon| \le c\ . \end{cases} \tag{2.25}$$

Proof - We drop the superscript ε of various functions along the proof.
Due to compatibility, when $\mathbf{v}$ is a minimizer of $\mathcal{F}^\varepsilon_2$ then $\mathbf{v} - \overline{\mathbf{v}}$ is a minimizer too. Hence we may assume $\overline{\mathbf{v}} = \mathbf{0}$ without loss of generality. We set $\mathbf{u} = \mathcal{U}(\mathbf{v})$. Then by using Lemma 2.3 with $\theta = 1/2$ we get, for every $s \in S_{u_1} \cup S_{\dot u_2+\kappa u_1}$,

$$\varepsilon^2\beta|[u_1](s)| + \frac{1}{2}\varepsilon^3\beta|[\dot u_2 + \kappa u_1](s)| \ \le\ \varepsilon\beta\int_{-\varepsilon}^{\varepsilon} |(1-\kappa\xi)[u_1](s) - \xi[\dot u_2](s)|\, d\xi \tag{2.26}$$

and, by arguing as in the proof of Th. 2.9, exploiting the intrinsic variables , by (2.22),(2.23), Lemma 2.2 $i)v)$, Lemma 2.5 ($r = p'$), and the fact that $\mathcal{C}_{p'} \le L^{\frac{p-2}{2p}}\mathcal{C}_2$ by Hölder inequality,we get, for every $\delta > 0$,

$$\begin{aligned} &\big(1+o(1)\big)\left(\mu+\frac{\lambda}{2}\right)\int_0^L \left\{2\,\varepsilon\,|\dot u_1 - \kappa u_2|^2 + \frac{2}{3}\,\varepsilon^3\,|\ddot u_2 + \dot\kappa u_1 + \kappa\dot u_1|^2\right\} + \\ &\qquad + 2\,\varepsilon^3\alpha\,\mathcal{H}^0(S_{u_1}\cup S_{\dot u_2+\kappa u_1}) + \varepsilon\beta\sum_{s\in S_{u_1}\cup S_{\dot u_2+\kappa u_1}}\int_{-\varepsilon}^{\varepsilon} |[u_1] - [\dot u_2 + \kappa u_1]\xi|\,d\xi \ = \\ &= \ \mathcal{G}^\varepsilon(\mathbf{v}) \ \le\ \varepsilon^2\int_{\Sigma^\varepsilon} f\,\mathbf{n}\cdot\mathbf{v} \ =\ \varepsilon^2\int_{\Sigma^\varepsilon} f\,u_2 \ \le\ 2\varepsilon^3\|f\|_{L^p(0,L)}\ \|u_2\|_{L^{p'}(0,L)} \ \le \\ &\le \frac{\beta\,\varepsilon^3}{16C_2L^{\frac{p-2}{2p}}}||u_2||_{L^{p'}(0,L)} \le \frac{\beta\,\mathcal{C}_{p'}\,\varepsilon^3}{16C_2\,L^{\frac{p-2}{2p}}}\left(\int_0^L |\dot u_1 - \kappa u_2| + |\ddot u_2 + \dot\kappa u_1 + \kappa\dot u_1| + \sum_{\mathbf{S_u}} |[u_1] + [\dot u_2 + \kappa u_1]|\right) \\ &\le \frac{\beta\,\mathcal{C}_{p'}\,\varepsilon^3}{16C_2\,L^{\frac{p-2}{2p}}}\left(\int_0^L 2\delta + \delta^{-1}(|\dot u_1 - \kappa u_2|^2 + |\ddot u_2 + \dot\kappa u_1 + \kappa\dot u_1|^2)\right) + \frac{1}{8}\beta\varepsilon^3\sum_{\mathbf{S_u}} |[u_1] + [\dot u_2 + \kappa u_1]|\,. \end{aligned} \tag{2.27}$$

Gathering the inequalities (2.26)(2.27), by a suitable choice of δ we get the first six estimates in (2.25). Hence, by applying Lemma 2.5 again

$$\int_0^L (|u_1|^2 + |u_2|^2 + |\dot u_2|^2)\, ds \;\le$$
$$\mathcal{C}_2^2 \left(\int_0^L |\dot u_1 - \kappa u_2| + |\ddot u_2 + \kappa \dot u_1 + \dot\kappa u_1|\, ds + \sum_{\mathbf{S_u}} |[u_1]| + |[\dot u_2 + \kappa u_1]| \right)^2 \le c\ .$$

$$\int_0^L (|\dot u_1|^2 + |\ddot u_2|^2) \;\le 2 \int_0^L |\kappa u_2|^2 + |\dot u_1 - \kappa u_2|^2 + |\dot\kappa u_1 + \kappa \dot u_1|^2 + |\ddot u_2 + \kappa \dot u_1 + \dot\kappa u_1|^2 \le c\ . \blacksquare$$

We are now in position to state and prove our main theorem.

Theorem 2.11 - *Assume (2.1)-(2.9),(2.22). Then there exists* $\mathbf{v}^\varepsilon \in \operatorname{argmin} \mathbf{LCB}_2^\varepsilon$, *such that, up to sub-sequences,* $u_1^\varepsilon \overset{*}{\rightharpoonup} u_1$ *in* $SBV(0,L)$, $u_2^\varepsilon \overset{*}{\rightharpoonup} u_2$ *in* $SBH(0,L)$, *where* $\mathbf{u}^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon) = \mathcal{U}_2^\varepsilon(\mathbf{v}^\varepsilon)$ *and* $\mathbf{u} = (u_1, u_2)$ *is a solution of*

$$(\mathbf{LCB}_2) \qquad \min\{\mathcal{F}_2(\mathbf{u}) : \mathbf{u} = (u_1, u_2) \in SBH((0,L), \mathbf{R}^2)\}$$

where

$$\mathcal{F}_2(\mathbf{u}) \;=\; \begin{cases} \displaystyle\int_0^L \left(\frac{2}{3}\left(\mu + \frac{\lambda}{2}\right) |\ddot u_2 + \kappa \dot u_1 + \dot\kappa u_1|^2 - f u_2 \right) \; + 2\alpha \mathcal{H}^0(S_{\dot u_2}) + \beta \int_{S_{\dot u_2}} |[\dot u_2]| & \text{if } u_1' = \kappa u_2 \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover the re-scaled energies converge, that is

$$\lim_{\varepsilon \to 0} \varepsilon^{-3} \mathcal{F}_2^\varepsilon(\mathbf{v}^\varepsilon) = \mathcal{F}_2(\mathbf{u}).$$

Notice that the relationship $u_1' = \kappa u_2$ expresses non extensibility of the limit beam.

Proof - The problem $\mathbf{LCB_2}$ achieves a finite minimum for every $\varepsilon \in [0, \varepsilon_0]$ ([CLT2,3],[CT]). Let $\mathbf{v}^\varepsilon$ be a minimizer then $\mathcal{F}_2^\varepsilon(\mathbf{v}^\varepsilon) \le 0$. Without loss of generality we assume that $\overline{\mathbf{v}^\varepsilon} = 0$, and we set $\mathbf{u}^\varepsilon = \mathcal{U}_2^\varepsilon(\mathbf{v}^\varepsilon)$. From Theorem 2.10 we get: $u_1^\varepsilon \in SBV(0,L)$, $u_2^\varepsilon \in SBH(0,L)$ and

$$\| u_1^\varepsilon \|_{BV(0,L)} \le c\ , \qquad \| u_2^\varepsilon \|_{BH(0,L)} \le c\ , \qquad \int_{S_{\dot u_2^\varepsilon}} |[\dot u_2^\varepsilon]| \le c$$

$$\dot u_1^\varepsilon - \kappa u_2^\varepsilon \to 0 \ \text{ in } \ L^2(0,L)\ , \qquad \int_{S_{u_1^\varepsilon}} |\,[u_1^\varepsilon]\,| \;\to\; 0\ ,$$

then, up to sub-sequences,

$$u_2^\varepsilon \overset{*}{\rightharpoonup} u_2 \in SBH, \qquad \dot u_1^\varepsilon \;\to\; \dot u_1 \;=\; \kappa u_2 \ \text{ strongly in } \ L^2(0,L), \qquad \text{hence } u_1 \in SBH.$$

Moreover

$$\int_0^L |u_1'| \;\le\; \liminf_{\varepsilon\to 0} \left\{ \int_0^L |\dot u_1^\varepsilon|\, ds \;+\; \int_{S_{u_1^\varepsilon}} |[u_1^\varepsilon]|\, d\mathcal{H}^0(s) \right\} \;=\; \liminf_{\varepsilon\to 0} \int_0^L |\dot u_1^\varepsilon|\, ds \;=\; \int_0^L |\dot u_1| ds \;\le\; \int_0^L |u_1'|$$

then

$$u_1' = \dot u_1 = \kappa u_2 \in L^2(0,L) \ \text{ and } \ u_1 \in H^1(0,L). \tag{2.28}$$

The compact embedding $BH(0,L) \subset W^{1,1}(0,L)$ entails that $u_2^\varepsilon \to u_2$ in $W^{1,1}(0,L)$ and, by Lemmas 2.2 v) and 2.3 with $\theta = 1$,

$$\begin{aligned} \varepsilon^{-3}\mathcal{F}_2^\varepsilon(\mathbf{v}^\varepsilon) \ &\geq \ (1+o(1)\,)\,\frac{2}{3}\left(\mu+\frac{\lambda}{2}\right)\int_0^L |\ddot{u}_2^\varepsilon+\kappa\dot{u}_1^\varepsilon+\dot{\kappa}u_1^\varepsilon|^2\,ds - \int_0^L f u_2^\varepsilon\,ds + \\ &\quad + \ 2\alpha\,\mathcal{H}^0(S_{\dot{u}_2^\varepsilon+\kappa u_1^\varepsilon}) \ + \ \beta\int_{S_{\dot{u}_2^\varepsilon+\kappa u_1^\varepsilon}} |[\dot{u}_2^\varepsilon+\kappa u_1^\varepsilon]| \end{aligned} \tag{2.29}$$

By the argument of the proof of Theorem 3.10, $\mathcal{F}_2^\varepsilon(\mathbf{v}^\varepsilon) \leq c\varepsilon^3$. Then (2.29) entails $\dot{u}_2^\varepsilon + \kappa u_1^\varepsilon$ belongs to $SBV(0,L)$ and is bounded in $BV(0,L)$, moreover, by (2.24),(2.25), (2.28) and lower semi-continuity,

$$\begin{aligned} &\liminf_{\varepsilon\to 0}\varepsilon^{-3}\mathcal{F}_2^\varepsilon(\mathbf{v}^\varepsilon) \ \geq \\ &\geq \ \frac{2}{3}\left(\mu+\frac{\lambda}{2}\right)\int_0^L (|\ddot{u}_2+\kappa\dot{u}_1+\dot{\kappa}u_1|^2 - fu_2) + 2\alpha\mathcal{H}^0(S_{\dot{u}_2+\kappa u_1}) + 2\beta\int_{S_{\dot{u}_2+\kappa u_1}} |[\dot{u}_2+\kappa u_1]| \ = \\ &= \ \frac{2}{3}\left(\mu+\frac{\lambda}{2}\right)\int_0^L (|\ddot{u}_2+\kappa\dot{u}_1+\dot{\kappa}u_1|^2 - fu_2) \ + \ 2\alpha\,\mathcal{H}^0(S_{\dot{u}_2}) \ + \ \beta\int_{S_{\dot{u}_2}} |[\dot{u}_2]| \ . \end{aligned} \tag{2.30}$$

By setting $\mathbf{u} = (u_1, u_2)$ and, for $\mathbf{x} \in \Sigma^\varepsilon,\ s \in [0,L],\ \xi \in (-\varepsilon, \varepsilon)$,

$$\mathbf{z}^\varepsilon(\mathbf{x}) \ = \ (\mathcal{U}_2^\varepsilon)^{-1}\,\mathbf{u} \ = \ (u_1(s)(1-\kappa\xi) - \xi\dot{u}_2(s))\mathbf{t} + u_2(s)\mathbf{n}$$

we have

$$\begin{aligned} &\varepsilon^{-3}\mathcal{F}_2^\varepsilon(\mathbf{v}^\varepsilon) \ \leq \ \varepsilon^{-3}\mathcal{F}_2^\varepsilon(\mathbf{z}^\varepsilon) \ = \ \mathcal{F}_2(\mathbf{u}) + o(1) \ = \\ &\qquad = \ \frac{2}{3}\left(\mu+\frac{\lambda}{2}\right)\int_0^L |\ddot{u}_2+\kappa\dot{u}_1+\dot{\kappa}u_1|^2 ds \ - \ \int_0^L f u_2 ds + \ 2\,\alpha\,\mathcal{H}^0(S_{\dot{u}_2}) \ + \ \beta\int_{S_{\dot{u}_2}} |[\dot{u}_2]| + o(1) \ . \end{aligned}$$

This proves that

$$\varepsilon^{-3}\mathcal{F}_2^\varepsilon(\mathbf{v}^\varepsilon) \to \mathcal{F}_2(\mathbf{u}) \tag{2.31}$$

and therefore it remains only to prove that $\mathbf{u}$ is a minimizer of $\mathcal{F}$. To this aim take any other $\mathbf{w} \in SBV(0,L) \times SBH(0,L)$: then either $\mathcal{F}(\mathbf{w}) = +\infty$ or $\dot{w}_1 = \kappa w_2$. Only the second case has to be examined: let us set

$$\mathbf{W}^\varepsilon(\mathbf{x}) \ = \ (\mathcal{U}_2^\varepsilon)^{-1}\,\mathbf{w} \ = \ (\,(1-\kappa\xi)w_1(s) - \xi\dot{w}_2(s)\,)\,\mathbf{t} + w_2(s)\mathbf{n}$$

we have

$$\mathcal{F}_2(\mathbf{w}) \ = \ \varepsilon^{-3}\mathcal{F}_2^\varepsilon(\mathbf{W}^\varepsilon) + o(1) \ \geq \ \varepsilon^{-3}\mathcal{F}_2^\varepsilon(\mathbf{v}^\varepsilon) + o(1)$$

and by passing to the limit as $\varepsilon \to 0$ from (2.31) we get

$$\mathcal{F}_2(\mathbf{u}) \leq \mathcal{F}_2(\mathbf{w}) \ .$$

Then the proof is complete, since $\mathcal{F}_2(\mathbf{u}) < +\infty$ entails $u_1' = \kappa u_2$ hence also $u_1 \in SBH$ and $\mathbf{u}$ belongs to $SBH((0,L),\mathbf{R}^2)$. ■

Actually the proof above says more: $(2.1)-(2.9), (2.22)$ and $\mathbf{v}^\varepsilon \in \operatorname{argmin}\mathbf{LCB}_2^\varepsilon$ entail :
$\exists\mathbf{u} \in SBH((0,L),\mathbf{R}^2)$ and a subsequence of $\mathbf{v}^\varepsilon$ s.t., without relabeling,

$$\mathbf{u}^\varepsilon - \overline{\mathbf{u}^\varepsilon} \ = \ \mathcal{U}_2^\varepsilon(\mathbf{v}^\varepsilon - \overline{\mathbf{v}^\varepsilon}) \quad \overset{w^*BH}{\rightharpoonup} \quad \mathbf{u} \qquad \text{and} \qquad \varepsilon^{-3}\mathcal{F}_2^\varepsilon(\mathbf{v}^\varepsilon) \ \to \ \mathcal{F}_2(\mathbf{u}) \ .$$

Example 2.12 - *Straight rod:* in the simplest case we recover the case of straight linear elastic plastic beam (**LB**) of [PT1], that is $\boldsymbol{\gamma}(s) = (s, 0)$, $\mathbf{t}(s) = (1, 0)$, $\mathbf{n}(s) = (0, 1)$, $\kappa = \dot{\kappa} = 0$. In this case analysis of lemma 2.5 can be refined and the constant $\mathcal{C}_r$ is $L^{1/r}(L+1)$ by:

$$\begin{aligned} ||u_1||_{L^r(0,L)} + ||u_2||_{L^r(0,L)} + ||u_2'||_{L^r(0,L)} \ &\leq \ L^{1/r}\left(||u_1||_{L^\infty(0,L)} + ||u_2||_{L^\infty(0,L)} + ||u_2'||_{L^\infty(0,L)}\right) \leq \\ \leq \ L^{1/r}\left(\int |u_1'| + |u_2'| + |u_2''|\right) \ &\leq \ L^{1/r}(L+1)\left(\int |u_1'| + |u_2''|\right) \end{aligned} .$$

Hence the compatibility and safe load conditions (2.22) read

$$\int_0^L f(s)\, ds = \int_0^L s f(s)\, ds = 0 \ , \qquad\qquad \| f \|_{L^p(0,L)} < \frac{\beta}{32\ L^{\frac{p-1}{p}}(L+1)} \ ,$$

and the limit functional becomes

$$\mathcal{F}_2(\mathbf{u}) \ = \ \begin{cases} \dfrac{2}{3}\left(\mu + \dfrac{\lambda}{2}\right)\displaystyle\int_0^L (|\ddot{u}_2|^2 - f u_2) + 2\alpha\mathcal{H}^0(S_{\dot{u}_2}) + \beta\int_{S_{\dot{u}_2}} |[\dot{u}_2]| & \text{if } u_1 \equiv \text{const}, u_2 \in SBH(0,L) \\ +\infty & \text{otherwise.} \end{cases}$$

Actually here we deal with Neumann conditions, while in [PT1] the study of cantilever was detailed as an example of straight rod: say the one-sided Dirichlet condition.

Example 2.13 - *Circular rod:* we consider here the case in which Σ is an arc of circumference with radius $R > 0$ and length L, that is $\gamma(s) = (R\cos(s/R), R\sin(s/R)$, $\kappa = R^{-1}$, $\dot{\kappa} = 0$, $s \in [0, L]$. Due to Lemma 2.2 i)ii), $\ddot{u}_2 + R^{-2}u_2 = 0$ for every infinitesimal rigid displacement $\mathbf{v}$, and the compatibility condition in (2.22) reads $\int_0^L f w\, ds = 0\ \forall w\ :\ \ddot{w} + R^{-2}w = 0$, that is

$$\int_0^L f(s)\cos(s/R)\, ds = \int_0^L f(s)\sin(s/R)\, ds = 0 \ ,$$

the safe load conditions in (2.22) reads $\| f \|_{L^p(0,L)} < \frac{\beta}{32\,\mathcal{C}_2\, L^{\frac{p-2}{2p}}}$ and the limit functional

$$\mathcal{F}_2(\mathbf{u}) = \begin{cases} \dfrac{2}{3}\left(\mu + \dfrac{\lambda}{2}\right)\displaystyle\int_0^L (|\ddot{u}_2 + R^{-2}u_2|^2 - f u_2) + 2\alpha\mathcal{H}^0(S_{\dot{u}_2}) + \beta\int_{S_{\dot{u}_2}} |[\dot{u}_2]| & \text{if } u_2 \in SBH, u_1' = R^{-1}u_2, \\ +\infty & \text{otherwise .} \end{cases}$$

3. Three–dimensional approximation of a linear elastic–plastic curved beam (LCB$_3$)

We describe the geometry of the un-stressed beam by an arc-length parametric path: let

$$\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3) \in C^3([0, L]; \mathbf{R}^3) \text{ a simple arc s.t. } \quad |\dot{\boldsymbol{\gamma}}(s)| = 1 \ , \quad |\ddot{\boldsymbol{\gamma}}(s)| > 0 \quad \text{a.e.} s \in [0, L] \ , \tag{3.1}$$

$$T = \{\boldsymbol{\gamma}(s) : s \in [0, L]\} \ , \tag{3.2}$$

we denote by $\mathbf{T}(s) = \dot{\boldsymbol{\gamma}}(s)$ and $\mathbf{N}(s) = \frac{\ddot{\boldsymbol{\gamma}}(s)}{\|\ddot{\boldsymbol{\gamma}}(s)\|}$ respectively the unit tangent vector and the intrinsic unit normal vector to T at $\boldsymbol{\gamma}(s)$. Moreover set

$$\mathbf{B} \ = \ \mathbf{T} \wedge \mathbf{N} \ , \qquad K \ = \ \dot{\mathbf{T}} \cdot \mathbf{N} = \| \ddot{\boldsymbol{\gamma}}(s) \| \ , \qquad \tau \ = \ -\dot{\mathbf{B}} \cdot \mathbf{N} \tag{3.3}$$

respectively the bi-normal vector, the (absolute) curvature and the torsion of $\boldsymbol{\gamma}$. We consider a thick beam whose reference configuration is the open set

$$T^\varepsilon \;=\; \{\, \boldsymbol{\gamma}(s) + \xi \mathbf{N} + \zeta \mathbf{B} :\; 0 < s < L \,,\; \xi^2 + \zeta^2 < \varepsilon^2 \,\} \tag{3.4}$$

We do not exclude closed simple arcs in (3.1) : in this case (3.4) is substituted by

$$T^\varepsilon = \{\boldsymbol{\gamma}(s) + \xi \mathbf{N}(s) + \zeta \mathbf{B} :\; 0 \le s \le L \,,\, \xi^2 + \zeta^2 < \varepsilon^2\}. \tag{3.4$'$}$$

All along this chapter we assume

$$0 < \varepsilon \le \varepsilon_0 \;, \qquad\qquad \varepsilon_0 = \frac{1}{2} \min\, \big\{ K(s)^{-1} \;:\; s \in [0, L] \big\} \;, \tag{3.5}$$

here ε_0 is chosen such that ([DNF]) every $\mathbf{x} \in T^\varepsilon$ has a unique orthogonal projection $\mathbf{P}(\mathbf{x})$ on T . Hence the functions $\mathbf{P}(\mathbf{x})$, $s(\mathbf{x}) = \boldsymbol{\gamma}^{-1}(\mathbf{P}(\mathbf{x}))$, $\mathbf{T}(\mathbf{x}) = \mathbf{T}(s(\mathbf{x}))$, $\mathbf{N}(\mathbf{x}) = \mathbf{N}(s(\mathbf{x}))$, $\mathbf{B}(\mathbf{x}) = \mathbf{B}(s(\mathbf{x}))$, $\xi(\mathbf{x}) = \big(\mathbf{x} - \mathbf{P}(\mathbf{x}))\big) \cdot \mathbf{N}(\mathbf{x})$, $\zeta(\mathbf{x}) = \big(\mathbf{x} - \mathbf{P}(\mathbf{x}))\big) \cdot \mathbf{B}(\mathbf{x})$ are well defined and smooth on Σ^ε. Since we are interested in the asymptotic behaviour as $\varepsilon \to 0_+$, condition (3.5) is not restrictive. Actually $\boldsymbol{\gamma} \in C^k$ for $k \ge 2$ entails the existence of $\varepsilon > 0$ s.t. the distance from T is a C^k function on T^ε (see [DFN], App.14.6).

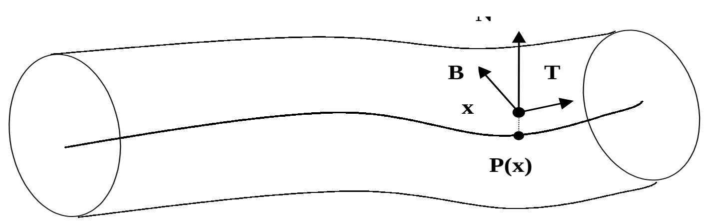

Fig.3 - The open set T^ε .

In contrast to section 2, the requirement $|\ddot{\boldsymbol{\gamma}}| > 0$ is necessary here in order to define $\mathbf{N}, \mathbf{B}$. Nevertheless whenever a smoothly varying intrinsic cartesian coordinate frame is available (e.g. when T is a straight beam, see Sect.4) this requirement may be dropped, and the analysis can be done in the same way.
The set of infinitesimal rigid displacements in T^ε is denoted by $\mathcal{R}(T^\varepsilon)$. The region T^ε is the natural reference of a three-dimensional linear elastic body with free damage at mesoscopic scale, whose internal strain energy is given, for every displacement field $\mathbf{v} \in SBD(T^\varepsilon)$, by the following functional

$$\mathcal{G}_3^\varepsilon(\mathbf{v}) \;=\; \int_{T^\varepsilon} W\left(\mathcal{E}(\mathbf{v})\right) d\mathbf{x} \;+\; \varepsilon^2\, \alpha\, \mathcal{H}^2(J_\mathbf{v}) \;+\; \varepsilon\, \beta \int_{J_\mathbf{v}} |[\mathbf{v}] \odot \nu_\mathbf{v}| d\mathcal{H}^2 \tag{3.6}$$

where

$$\begin{aligned} &W(A) \;=\; \mu\,|A|^2 + \frac{\lambda}{2}\,(\mathrm{Tr}\,A)^2 \;\ge\; C_3(\mu,\lambda)\,|A|^2 \qquad \forall A \in M_{3,3}\,, \\ &\alpha\,,\beta\,,\mu \;>0\;,\quad 2\mu + 3\lambda \;>\; 0 \qquad\quad C_3(\mu,\lambda) = \min\{\mu, \mu + \frac{3}{2}\lambda\} > 0 \end{aligned} \tag{3.7}$$

The beam T^ε is subject to a transverse body force field $\mathbf{g}^\varepsilon$ of the following kind

$$\begin{aligned} \mathbf{g}^\varepsilon(x) &= \varepsilon^2 f_2(s)\mathbf{N} + \varepsilon^2 f_3(s)\mathbf{B} + 2f_4(s)(\xi\mathbf{B} - \zeta\mathbf{N}) \\ \mathbf{f} &= (0, f_2, f_3, f_4) \in L^p((0,L), \mathbf{R}^4) \qquad p \geq 3\,. \end{aligned} \tag{3.8}$$

The term $\varepsilon^2\,(f_2(s)\mathbf{N} + f_3(s)\mathbf{B})$ is the bending force, while $-2f_4(s)(\zeta\mathbf{N} - \xi\mathbf{B})$ is the twisting force, here we assume that there is no tangential component of the force, say $f_1 = \mathbf{g}^\varepsilon \cdot \mathbf{T} = 0$.
We define

$$\mathcal{L}^\varepsilon(\mathbf{v}) = \int_{T^\varepsilon} \mathbf{g}^\varepsilon \cdot \mathbf{v}\, d\mathbf{x} \tag{3.9}$$

and so the (total) energy functional is

$$\mathcal{F}_3^\varepsilon(\mathbf{v}) = \mathcal{G}_3^\varepsilon(\mathbf{v}) - \mathcal{L}^\varepsilon(\mathbf{v})\ . \tag{3.10}$$

We assume the displacement field $\mathbf{v}$ satisfies the Bernoulli-Navier cinematic constraint $\mathrm{e}(\mathbf{v})\cdot\mathcal{N} = \mathbf{0}$ in the sense of measures for all continuous vector fields $\mathcal{N}$ which are normal to the central strand ([K],[PG],[V]), say

$$\int_{T^\varepsilon} \mathrm{e}(\mathbf{v})\cdot\mathbf{N}\ \varphi \;=\; \int_{T^\varepsilon} \mathrm{e}(\mathbf{v})\cdot\mathbf{B}\ \varphi \;=\; \mathbf{0} \qquad \forall\varphi \in C_0^0(T^\varepsilon) \tag{3.11}$$

so that we are led to study the following minimization problem

$$(\mathbf{LCB_3^\varepsilon}) \qquad \min\,\{\mathcal{F}_3^\varepsilon(\mathbf{v}) \,:\, \mathbf{v} \in \mathit{SBD}(T^\varepsilon),\, \mathrm{e}(\mathbf{v})\cdot\mathbf{N} = \mathbf{0} = \mathrm{e}(\mathbf{v})\cdot\mathbf{B} \text{ in } T^\varepsilon\,\}\ .$$

The goal of this section is to study the existence and asymptotic behaviour of minimizers of $(\mathbf{LCB_3^\varepsilon})$; the first step is to exploit the geometric assumption (3.5).

Lemma 3.1 - *The map* $\Psi_3 \,:\, (s,\xi,\zeta) \to \mathbf{x} = \boldsymbol{\gamma}(s) + \xi\mathbf{N}(s) + \zeta\mathbf{B}(s)$ *is one-to-one, moreover* $\det D\Psi_3 = 1 - K\xi$, $d\mathbf{x} = \big(1 - K(s)\xi\big)\, ds\, d\xi\, d\zeta$ and $|T^\varepsilon| = \pi L\varepsilon^2$.

Proof - Ψ is a C^1 map and, by (3.5), is one-to-one. By direct computation

$$D\Psi_3 = \begin{bmatrix} \dot{\boldsymbol{\gamma}}_1 + \xi\dot{\mathbf{N}}_1 + \zeta\dot{\mathbf{B}}_1 & \mathbf{N}_1 & \mathbf{B}_1 \\ \dot{\boldsymbol{\gamma}}_2 + \xi\dot{\mathbf{N}}_2 + \zeta\dot{\mathbf{B}}_3 & \mathbf{N}_2 & \mathbf{B}_2 \\ \dot{\boldsymbol{\gamma}}_3 + \xi\dot{\mathbf{N}}_3 + \zeta\dot{\mathbf{B}}_3 & \mathbf{N}_3 & \mathbf{B}_3 \end{bmatrix}$$

hence $\dot{\boldsymbol{\gamma}} = \mathbf{T}$ and the Frenet-Serret formulae ($\dot{\mathbf{T}} = K\mathbf{N}$, $\dot{\mathbf{N}} = -K\mathbf{T} + \tau\mathbf{B}$, $\dot{\mathbf{B}} = -\tau\mathbf{N}$) give the thesis. ■

The second step is to exploit the cinematic constraint (3.11) to show that admissible vector fields have rank-1 strain tensor, and are completely described by four scalar functions of arc-length s: the averages of intrinsic components over the cross sections and the rotation angle.

Lemma 3.2 - *Fix* $\varepsilon \in (0,\varepsilon_0)$ *. Suppose that* $\mathbf{v} \in \mathit{SBD}(T^\varepsilon)$ *and* $\mathrm{e}(\mathbf{v})\cdot\mathbf{N} = \mathbf{0} = \mathrm{e}(\mathbf{v})\cdot\mathbf{B}$ *in the sense of measures. Then, by labeling* $v_\mathbf{T} = \mathbf{v}\cdot\mathbf{T}$, $v_\mathbf{N} = \mathbf{v}\cdot\mathbf{N}$, $v_\mathbf{B} = \mathbf{v}\cdot\mathbf{B}$,

i) there exist unique $u_1 \in \mathit{SBV}(0,L), u_2, u_3, u_4 \in \mathit{SBH}(0,L)$ *such that*

$$\mathbf{v} = v_\mathbf{T}\,\mathbf{T} + v_\mathbf{N}\,\mathbf{N} + v_\mathbf{B}\,\mathbf{B} = (u_1 - \xi(Ku_1 + u_2' - \tau u_3) - \zeta(u_3' + \tau u_2))\mathbf{T} + (u_2 - \zeta u_4)\mathbf{N} + (u_3 + \xi u_4)\mathbf{B}\,;$$

ii)
$$\begin{aligned} \mathrm{e}(\mathbf{v}) &= (1-K\xi)^{-1}(D_s v_\mathbf{T} + \tau\zeta D_\xi v_\mathbf{T} - \tau\xi D_\zeta v_\mathbf{T} - K v_\mathbf{N})(\mathbf{T}\otimes\mathbf{T}) = \\ &= (1-K\xi)^{-1}\,\{\,\mathcal{A}_0\mathbf{u}\,(s) - \xi\,\mathcal{A}_1\mathbf{u}\,(s) - \zeta\,\mathcal{A}_2\mathbf{u}\,(s)\,\}\,(\mathbf{T}\otimes\mathbf{T}) \end{aligned}$$

with $\mathbf{u} = (u_1, u_2, u_3, u_4)$ *defined by i) and* $\mathcal{A}_j$ $j = 0, 1, 2$, *and their absolutely continuous part* A_j *by*

$$\mathcal{A}_0\mathbf{u} = u_1' - Ku_2\,, \qquad\qquad A_0\mathbf{u} = \dot{u}_1 - Ku_2\,,$$

$$\mathcal{A}_1\mathbf{u} = u_2'' - \dot{\tau}u_3 - 2\tau\dot{u}_3 + Ku_1' + \dot{K}u_1 - \tau^2 u_2\,, \;\; A_1\mathbf{u} = \ddot{u}_2 - \dot{\tau}u_3 - 2\tau\dot{u}_3 + K\dot{u}_1 + \dot{K}u_1 - \tau^2 u_2\,,$$

$$\mathcal{A}_2\mathbf{u} = u_3'' + \dot{\tau}u_2 + 2\tau\dot{u}_2 + \tau K u_1 - Ku_4 - \tau^2 u_3\,, \; A_2\mathbf{u} = \ddot{u}_3 + \dot{\tau}u_2 + 2\tau\dot{u}_2 + \tau K u_1 - Ku_4 - \tau^2 u_3;$$

iii) $J_{\mathbf{v}} \subset (J_{u_1} \cup J_{\dot{u}_2 + Ku_1} \cup J_{\dot{u}_3}) \otimes B_\varepsilon(\mathbf{0}) = (S_{u_1} \cup S_{\dot{u}_2 + Ku_1} \cup S_{\dot{u}_3}) \otimes B_\varepsilon(\mathbf{0}) = \mathbf{S_u} \otimes B_\varepsilon(\mathbf{0})$ *and*

$$\mathcal{H}^2(J_{\mathbf{v}} \setminus (J_{u_1} \cup J_{\dot{u}_2 + Ku_1} \cup J_{\dot{u}_3}) \times B(0, \varepsilon)) = 0\,;$$

$$d\mathcal{H}^2 \llcorner J_{\mathbf{v}} \;=\; \sum_{\mathbf{S_u}} (1 - K(s)\xi)\, d\xi\, d\zeta$$

where $\mathbf{S_u} \;=\; S_{u_1} \cup S_{\dot{u}_2 + Ku_1} \cup S_{\dot{u}_3}$

iv) $\mathrm{e}(\mathbf{v}) = \mathbf{0}$ *iff* $\mathcal{A}_j(\mathbf{u}) = 0\,,\; j = 0, 1, 2$ *(e.g.* $A_j(\mathbf{u}) = 0,\; j = 0, 1, 2\,,$ *and* $j_{u_1} = j_{\dot{u}_2} = j_{\dot{u}_3} = 0)\,;$

v)
$$\mathcal{G}^\varepsilon(\mathbf{v}) \;=\; (1 + o(1))\left(\mu + \frac{\lambda}{2}\right)\pi \int_0^L \left\{\varepsilon^2 |A_0\mathbf{u}|^2 + \frac{\varepsilon^4}{4}(|A_1\mathbf{u}|^2 + |A_2\mathbf{u}|^2)\right\} ds \;+$$
$$+\;(1 + o(1))\left(\pi\varepsilon^4\alpha\,\mathcal{H}^0(\mathbf{S_u}) \;+\; \varepsilon\beta \sum_{s\,\in\,\mathbf{S_u}} \int_{B_\varepsilon(\mathbf{0})} |[u_1] - \xi[\dot{u}_2 + Ku_1] - \zeta[\dot{u}_3]|\, d\xi d\zeta\right)$$

where $o(1)$ *tends to 0 as* $\varepsilon \to 0_+$.

On the other hand, for every $\mathbf{u} = (u_1, u_2, u_3, u_4) \in SBV(0, L) \times SBH\left((0, L), \mathbf{R}^3)\right)$ *the vector field* $\mathbf{v}$ *defined by i) belongs to* $SBD(T^\varepsilon)$ *and fulfills (3.11) and* $ii), iii), iv), v)$.

vi) The linear map $\mathcal{U}_3^\varepsilon : \mathbf{v} \to \mathbf{u}$ *(from orthogonal to intrinsic components) defined by i) satisfies*

$$u_1 = \fint_{B_\varepsilon(\mathbf{0})} \mathbf{v} \cdot \mathbf{T}\, d\xi\, d\zeta, \qquad u_2 = \fint_{B_\varepsilon(\mathbf{0})} \mathbf{v} \cdot \mathbf{N}\, d\xi\, d\zeta, \qquad u_3 = \fint_{B_\varepsilon(\mathbf{0})} \mathbf{v} \cdot \mathbf{B}\, d\xi\, d\zeta,$$

$u_4 = \dfrac{1}{I_0} \displaystyle\int_{B_\varepsilon(\mathbf{0})} \mathbf{v} \cdot (\xi\mathbf{B} - \zeta\mathbf{N})\; d\xi\, d\zeta\,,$ $\quad I_0 = \pi\varepsilon^4/2$ *polar moment of inertia over circular cross sections*

and $\mathcal{U}_3^\varepsilon$ *is one-to-one and bi-continuous in the strong topologies from* $\{\mathbf{v} \in SBD(\Sigma^\varepsilon) : \mathrm{e}(\mathbf{v}) \cdot \mathbf{N} = \mathrm{e}(\mathbf{v}) \cdot \mathbf{B} = 0\}$ *to the closed subspace of* $SBV(0, L) \times SBH((0, L), \mathbf{R}^3)$ *spanned by the solutions of* $u_4' = -K(u_3' + \tau u_2)$ *. We notice that* $(\,u_1(s), u_2(s), u_3(s)\,)$ *is the resultant of* $\mathbf{v}$ *and* u_4 *is the twisting moment of* $\mathbf{v}$ *over the circular cross section of* T_ε *through* $\boldsymbol{\gamma}(s)$ *.*

vii) we emphasize that u_4 *is dependent on* u_3, u_2, *namely:* $u_4' = -K(u_3' + \tau u_2)$ *.*

Proof. We recall that, for every $\mathbf{x} \in \Sigma^\varepsilon$, $\mathbf{P}(\mathbf{x})$ denotes the unique projection on T and hence there are (uniquely defined) $(\xi(\mathbf{x}), \zeta(\mathbf{x})) \in B(0, \varepsilon)$ and $s(\mathbf{x}) \in [0, L]$ s.t. $\mathbf{P}(\mathbf{x}) = \boldsymbol{\gamma}(s(\mathbf{x}))$ and $\mathbf{x} = \boldsymbol{\gamma}(s(\mathbf{x})) + \xi(\mathbf{x})\mathbf{N} + \zeta(\mathbf{x})\mathbf{B}$. The last equality together with Frenet-Serret formulae yield

$$\begin{aligned} D_{\mathbf{x}} s &= (1 - K\xi)^{-1}\,\mathbf{T} \\ D_{\mathbf{x}} \xi &= \mathbf{N} + \tau\zeta(1 - K\xi)^{-1}\,\mathbf{T} \\ D_{\mathbf{x}} \zeta &= \mathbf{B} - \xi\tau(1 - K\xi)^{-1}\,\mathbf{T} \end{aligned} \tag{3.12}$$

then, by denoting D_s, D_ξ, D_ζ the distribution derivatives with respect to intrinsic coordinates and assuming the summation convention,

$$\begin{aligned}
\mathbf{N}_i D_i &= D_\xi \\
\mathbf{B}_i D_i &= D_\zeta \\
\mathbf{T}_i D_i &= (1-K\xi)^{-1}(D_s + \tau\zeta D_\xi - \tau\xi D_\zeta)
\end{aligned} \tag{3.13}$$

then by using the Frenet-Serret formulae again

$$\begin{aligned}
D_\mathbf{x}\mathbf{B} &= -\tau(1-K\xi)^{-1}\mathbf{N}\otimes\mathbf{T} \\
D_\mathbf{x}\mathbf{N} &= -K(1-K\xi)^{-1}\mathbf{T}\otimes\mathbf{T} + \tau(1-K\xi)^{-1}\mathbf{B}\otimes\mathbf{T} \\
D_\mathbf{x}\mathbf{T} &= K(1-K\xi)^{-1}\mathbf{N}\otimes\mathbf{T}
\end{aligned} \tag{3.14}$$

By using (3.12),(3.13),(3.14) and $D_\xi\mathbf{T} = D_\zeta\mathbf{T} = D_\xi\mathbf{B} = D_\zeta\mathbf{B} = \mathbf{0}$ we get

$$\begin{aligned}
2\,\mathrm{e}(\mathbf{v}) : \mathbf{N}\otimes\mathbf{B} &= \big(B_j(\mathbf{N}\cdot D_\mathbf{x})v_j + \mathbf{N}_j(\mathbf{B}\cdot D_\mathbf{x})v_j\big) = \\
&= (\mathbf{N}\cdot D_\mathbf{x})v_\mathbf{B} - v_j(\mathbf{N}\cdot D_\mathbf{x})B_j + (\mathbf{B}\cdot D_\mathbf{x})v_\mathbf{N} - v_j(\mathbf{B}\cdot D_\mathbf{x})\mathbf{N}_j = \\
&= v_{\mathbf{B},\xi} + v_{\mathbf{N},\zeta} \\
\mathrm{e}(\mathbf{v}) : \mathbf{N}\otimes\mathbf{N} &= \frac{1}{2}\big(N_j(\mathbf{N}\cdot D_\mathbf{x})v_j + \mathbf{N}_j(\mathbf{N}\cdot D_\mathbf{x})v_j\big) = v_{\mathbf{N},\xi} \\
\mathrm{e}(\mathbf{v}) : \mathbf{B}\otimes\mathbf{B} &= \frac{1}{2}\big(B_j(\mathbf{B}\cdot D_\mathbf{x})v_j + \mathbf{B}_j(\mathbf{B}\cdot D_\mathbf{x})v_j\big) = v_{\mathbf{B},\zeta}
\end{aligned}$$

$$\begin{aligned}
2\,\mathrm{e}(\mathbf{v}) : \mathbf{T}\otimes\mathbf{N} &= \mathbf{T}_i\mathbf{N}_j D_i\mathbf{v}_j + \mathbf{T}_i\mathbf{N}_j D_j\mathbf{v}_i = \big(T_j(\mathbf{N}\cdot D_\mathbf{x})v_j + \mathbf{N}_j(\mathbf{T}\cdot D_\mathbf{x})v_j\big) = \\
&= \{(\mathbf{T}\cdot D_\mathbf{x})v_\mathbf{N} - v_j(\mathbf{T}\cdot D_\mathbf{x})N_j + (\mathbf{N}\cdot D_\mathbf{x})v_\mathbf{T} - v_j(\mathbf{N}\cdot D_\mathbf{x})\mathbf{T}_j\} = \\
&= \{(\mathbf{T}\cdot D_\mathbf{x})v_\mathbf{N} - v_j(\mathbf{T}\cdot D_\mathbf{x})N_j + (\mathbf{N}\cdot D_\mathbf{x})v_\mathbf{T} = \\
&= (\mathbf{T}\cdot D_\mathbf{x})v_\mathbf{N} + v_{\mathbf{T},\xi} + (1-K\xi)^{-1}(Kv_\mathbf{T} - \tau v_\mathbf{B})
\end{aligned}$$

$$2\,\mathrm{e}(\mathbf{v}) : \mathbf{T}\otimes\mathbf{B} = (\mathbf{T}\cdot D_\mathbf{x})v_\mathbf{B} + v_{\mathbf{T},\zeta} + (1-K\xi)^{-1}\tau v_\mathbf{N}$$

$$\mathrm{e}(\mathbf{v}) : \mathbf{T}\otimes\mathbf{T} = (1-K\xi)^{-1}(D_s v_\mathbf{T} + \tau\zeta D_\xi v_\mathbf{T} - \tau\xi D_\zeta v_\mathbf{T} - Kv_\mathbf{N})\ .$$

Then conditions $\mathrm{e}(\mathbf{v})\cdot\mathbf{N} = \mathbf{0} = \mathrm{e}(\mathbf{v})\cdot\mathbf{B}$ lead to

$$\mathrm{e}(\mathbf{v}) = ((\mathrm{e}(\mathbf{v}) : \mathbf{T}\otimes\mathbf{T})\ \mathbf{T}\otimes\mathbf{T}$$

and the following equalities in $\mathcal{D}'((0,L)\times B(0,\varepsilon))$

$$\begin{aligned}
& v_{\mathbf{B},\xi} + v_{\mathbf{N},\zeta} = 0 \\
& v_{\mathbf{B},\zeta} = v_{\mathbf{N},\xi} = 0 \\
& (\mathbf{T}\cdot D_\mathbf{x})v_\mathbf{B} + v_{\mathbf{T},\zeta} + (1-K\xi)^{-1}\tau v_\mathbf{N} = 0 \\
& (\mathbf{T}\cdot D_\mathbf{x})v_\mathbf{N} + v_{\mathbf{T},\xi} + (1-K\xi)^{-1}(Kv_\mathbf{T} - \tau v_\mathbf{B}) = 0
\end{aligned} \tag{3.15}$$

and then

$$\begin{aligned}
v_\mathbf{B} &= u_3(s) + \xi u_4(s) \\
v_\mathbf{N} &= u_2(s) - \zeta u_4(s) \\
v_{\mathbf{T},\zeta} &+ (1-K\xi)^{-1}(u_3' + u_4'\xi + \tau u_2) = 0 \\
v_{\mathbf{T},\xi} &+ (1-K\xi)^{-1}Kv_\mathbf{T} + (1-K\xi)^{-1}(u_2' - \zeta u_4' - \tau u_3) = 0\ .
\end{aligned} \tag{3.16}$$

By suitable averaging the first two equalities in (3.16) over the cross section, we obtain the last three equalities in *vi)*. By integrating the third equality in (3.16) there is a function $\phi = \phi(s,\xi)$ such that

$$v_{\mathbf{T}} \;=\; -(1-K\xi)^{-1}(u_3' + \tau u_2 + \xi u_4')\zeta + \phi(s,\xi)$$

and, by substituting in the fourth of (3.16),

$$-K(1-K\xi)^{-2}(u_3'+\tau u_2+\xi u_4')\zeta - (1-K\xi)^{-1}u_4'\zeta + \phi_\xi(s,\xi) - \zeta(1-K\xi)^{-2}K(u_3'+\tau u_2+\xi u_4') + \\ + K(1-K\xi)^{-1}\phi(s,\xi) + (1-K\xi)^{-1}(u_2' - \zeta u_4' - \tau u_3) = 0$$

then, by grouping the terms dependent of ζ, we have for every $\xi,\zeta \in B_\varepsilon(0)$

$$\phi_\xi + (1-K\xi)^{-1}K\phi + (1-K\xi)^{-1}(u_2'-\tau u_3) \;-\; 2(1-K\xi)^{-2}\big(K(u_3'+\tau u_2)+u_4'\big)\zeta \;=\; 0\;,$$

that is a pair of decoupled equations:

$$\phi_\xi + (1-K\xi)^{-1}(K\phi + u_2' - \tau u_3) = 0 \tag{3.17}$$

$$u_4' + K(u_3'+\tau u_2) = 0\;.$$

Hence $vii)$.
The general solution of (3.17) is $\phi(s,\xi) \;=\; u_1(s) + B(s)\xi$ with $B(s) = -Ku_1(s) - u_2' + \tau u_3$; then

$$v_{\mathbf{T}} \;=\; -(u_3'+\tau u_2)\zeta \;+\; u_1(s) - (Ku_1 \;+\; u_2' - \tau u_3)\xi \tag{3.18}$$

hence $-u_3'+\tau u_2, Ku_1+u_2'-\tau u_3, u_1 \in L^1(0,L)$ and then $u_3', u_2', u_4' \in L^1(0,L)$, and by averaging (3.18) over a cross section we get the first equality in *vi)*.
Actually the formal computations in deduction of (3.15)-(3.18) by use of (3.12)-(3.14)are correct if $\mathbf{v}$ is Lipschitz. Anyway they can be made rigorous in the general case, by the same method used when proving (2.15)'-(2.17)' as follows: the following equations hold

$$\int_0^L\int_{B_\varepsilon(0)} D_\zeta \widetilde{v}_{\mathbf{T}}\;\widetilde{\varphi}\;ds\,d\xi\,d\zeta \;=\; -\int_0^L\int_{B_\varepsilon(0)}(1-K\xi)^{-1}(u_3'+u_4'\xi+\tau u_2)\;\widetilde{\varphi}(s,\xi,\zeta)\;ds\,d\xi\,d\zeta \tag{3.16$'$}$$

$$\int_0^L\int_{B_\varepsilon(0)} \widetilde{v}_{\mathbf{T}}\;\widetilde{\varphi}\;ds\,d\xi\,d\zeta \;=\; \int_0^L\int_{B_\varepsilon(0)}\big(-(u_3'+\tau u_2)\,\zeta \;+\; u_1 \;-\; (Ku_1+u_2'-\tau u_3)\,\xi\,\big)\;\widetilde{\varphi}(s,\xi,\zeta)\;ds\,d\xi\,d\zeta \tag{3.18$'$}$$

$$\forall\varphi\in C_0^1(T^\varepsilon)\,,\quad \text{where } \widetilde{\mathbf{v}} \;=\; \widetilde{\mathbf{v}}(s,\xi,\zeta) \;\stackrel{\text{def}}{=}\; \mathbf{v}(\mathbf{x}(s,\xi,\zeta))\,,\quad \widetilde{\varphi} \;=\; \widetilde{\varphi}(s,\xi,\zeta) \;\stackrel{\text{def}}{=}\; \varphi(\mathbf{x}(s,\xi,\zeta))\,.$$

From (3.16) we get that both $J_{v_{\mathbf{B}}}$ and $J_{v_{\mathbf{N}}}$ are empty and therefore $J_{\mathbf{v}} = J_{v_{\mathbf{T}}}$ and (3.18) yields $\nu_{\mathbf{v}} = \mathbf{T}$
We have now

$$\mathrm{e}(\mathbf{v}) \;=\; (\mathrm{e}(\mathbf{v}) : \mathbf{T}\otimes\mathbf{T})\;\mathbf{T}\otimes\mathbf{T}$$

and by previous computations we may write

$$\mathrm{e}(\mathbf{v}) \;=\; (1-K\xi)^{-1}\big(D_s v_{\mathbf{T}} + \tau\zeta D_\xi v_{\mathbf{T}} - \tau\xi D_\zeta v_{\mathbf{T}} - Kv_{\mathbf{N}}\big)\,\mathbf{T}\otimes\mathbf{T}$$

and by (i) $\nu_{\mathbf{v}} = \mathbf{T}$ and that $J_{\mathbf{v}} = J_{v_{\mathbf{T}}}$,

$$\mathrm{e}(\mathbf{v}) : \mathbf{T}\otimes\mathbf{T} = \mathcal{E}(\mathbf{v}) : \mathbf{T}\otimes\mathbf{T} + [v_{\mathbf{T}}]\,d\mathcal{H}^2 \llcorner J_{v_{\mathbf{T}}}.$$

By (3.18) we have that $D_\xi v_{\mathbf{T}}, D_\zeta v_{\mathbf{T}} \in L^1(0,L)$ and therefore $Dv_{\mathbf{T}} = (D_s v_{\mathbf{T}}, D_\zeta v_{\mathbf{T}}, D_\xi v_{\mathbf{T}})$ is a vector measure having vanishing Cantor part in $(0,L)\times B_\varepsilon(0)$, that is $v_{\mathbf{T}} \in SBV((0,L)\times B(0,\varepsilon)$ and this implies that

$$u_3'+\tau u_2,\quad u_1,\quad \kappa u_1 + u_2' - \tau u_3 \in SBV(0,L),$$

hence recalling that $u_3', u_2', u_4' \in L^1(0,L)$ it follows that $u_1 \in SBV(0,L), u_2, u_3, u_4 \in SBH(0,L)$. $\mathcal{U}_3^{-1}$ is continuous by $i), ii)$. Then $\mathcal{U}_3$ is continuous due to injectivity and Open Mapping Theorem. ■

Lemma 3.3 - *For every* $\mathbf{a},\mathbf{b},\mathbf{c} \in \mathbf{R}^n$ *the following inequality holds*

$$\int_{B_\varepsilon(0)} |\,\mathbf{a} + \mathbf{b}\xi + \mathbf{c}\zeta\,|\, d\xi\, d\zeta \;\geq\; \frac{4}{3}\varepsilon^3\,(\theta\,|\mathbf{b}| + \omega|\mathbf{c}|\,) + \pi(1-\theta-\omega)\,|\mathbf{a}|\,\varepsilon^2 \qquad \forall\varepsilon \geq 0\,, \forall\,\theta,\omega \in [0,1]\,:\,\theta+\omega \leq 1\,. \tag{3.19}$$

Moreover, the equality in (3.19) may hold iff at most one among **a, b, c** *is different from* **0** *and* $\theta = \omega = 0$ *when* $\mathbf{a} \neq \mathbf{0}$, $\theta = 1$ *when* $\mathbf{b} \neq 0$, $\omega = 1$ *when* $\mathbf{c} = 0$.

Proof - By denoting **sign** the $\mathbf{R}^n$ valued sign function ($\mathbf{sign}(\mathbf{y}) = \mathbf{y}/|\mathbf{y}|$ if $\mathbf{y} \neq \mathbf{0}$, $\mathbf{sign}(\mathbf{0}) = \mathbf{0}$), the convexity of the euclidean norm entails

$$\begin{aligned}
|\mathbf{a}+\mathbf{b}\xi+\mathbf{c}\zeta| &\geq |\mathbf{a}| + \mathbf{sign}(\mathbf{a})\cdot(\mathbf{b}\xi+\mathbf{c}\zeta)\,,\\
|\mathbf{a}+\mathbf{b}\xi+\mathbf{c}\zeta| &\geq |\mathbf{b}\xi| + \mathbf{sign}(\mathbf{b}\xi)\cdot(\mathbf{a}+\mathbf{c}\zeta)\,,\\
|\mathbf{a}+\mathbf{b}\xi+\mathbf{c}\zeta| &\geq |\mathbf{c}\zeta| + \mathbf{sign}(\mathbf{c}\zeta)\cdot(\mathbf{a}+\mathbf{b}\xi)\,.
\end{aligned}$$

At least one of the above inequalities holds strictly in a subset of $B_\varepsilon(0)$ of non-vanishing measure, unless **a**, **b**, **c** span a 1–dimensional space. By integration over $B_\varepsilon(0)$ we get

$$\int_{B_\varepsilon(0)} |\mathbf{a}+\mathbf{b}\xi+\mathbf{c}\zeta|\,d\xi\,d\zeta \;\geq\; \int_{B_\varepsilon(0)} |\mathbf{a}| + \int_{B_\varepsilon(0)} \mathbf{sign}(\mathbf{a})\cdot\big(\mathbf{b}\xi+\mathbf{c}\zeta\big)\,d\xi\,d\zeta \;=\; \pi|\mathbf{a}|\,\varepsilon^2$$

$$\int_{B_\varepsilon(0)} |\mathbf{a}+\mathbf{b}\xi+\mathbf{c}\zeta|\,d\xi\,d\zeta \;\geq\; \int_{B_\varepsilon(0)} |\mathbf{b}\xi| + \int_{B_\varepsilon(0)} \mathbf{sign}(\mathbf{b}\xi)\cdot\big(\mathbf{a}+\mathbf{c}\zeta\big)\,d\xi\,d\zeta \;=\; \frac{4}{3}|\mathbf{b}|\,\varepsilon^3$$

$$\int_{B_\varepsilon(0)} |\mathbf{a}+\mathbf{b}\xi+\mathbf{c}\zeta|\,d\xi\,d\zeta \;\geq\; \int_{B_\varepsilon(0)} |\mathbf{c}\zeta| + \int_{B_\varepsilon(0)} \mathbf{sign}(\mathbf{c}\zeta)\cdot\big(\mathbf{a}+\mathbf{b}\xi\big)\,d\xi\,d\zeta \;=\; \frac{4}{3}|\mathbf{c}|\,\varepsilon^3$$

A convex combination of above inequalities gives (3.19).
The statement about equality is trivial if $\mathbf{a} = \mathbf{b} = \mathbf{c} = \mathbf{0}$.

From now on the equality in (3.19) is assumed and **a, b, c** span a 1dimensional space.
At first we assume that $\mathbf{c} \neq \mathbf{0}$, hence $\mathbf{a} = \eta\mathbf{c}$, $\mathbf{b} = \sigma\mathbf{c}$, and

$$|\mathbf{c}|\int_{B_\varepsilon(0)} |\eta+\sigma\xi+\zeta|\,d\xi d\zeta = \int_{B_\varepsilon(0)} |\mathbf{a}+\mathbf{b}\xi+\mathbf{c}\zeta|\,d\xi d\zeta = \Big(\frac{4}{3}\varepsilon^3\,(\theta\,|\sigma| + \omega) + \pi(1-\theta-\omega)\,|\eta|\,\varepsilon^2\Big)|\mathbf{c}|\;.$$

If $\eta \neq 0$ the set$A(\eta,\sigma) = \{(\xi,\zeta) \in B_\varepsilon(0):\; (\sigma\xi+\zeta)(\eta+\sigma\xi+\zeta) < 0\}$ has positive measure: then for every $(\xi,\zeta) \in A(\eta,\sigma)$ we get$|\eta+\sigma\xi+\zeta| > |\sigma\xi+\zeta| + \eta\,\mathrm{sign}(\sigma\xi+\zeta)$ and by integration

$$(\theta+\omega)|\mathbf{c}|\int_{B_\varepsilon(0)}|\eta+\sigma\xi+\zeta|d\xi d\zeta > (\theta+\omega)|\mathbf{c}|\int_{B_\varepsilon(0)}|\sigma\xi+\zeta|\,d\xi d\zeta = (\theta+\omega)|\mathbf{c}|\frac{4}{3}\,\varepsilon^3\sqrt{1+\sigma^2} \geq \frac{4}{3}\,\varepsilon^3|\mathbf{c}|(\theta|\sigma|+\omega)$$

$$(1-\theta-\omega)|\mathbf{c}|\int_{B_\varepsilon(0)} |\eta+\sigma\xi+\zeta|d\xi d\zeta \;\geq (1-\theta-\omega)\pi|\eta||\mathbf{c}|\varepsilon^2\;.$$

for every $\theta,\omega \geq 0$ such that $\theta+\omega \leq 1$. By summing up the previous inequalities we get a contradiction. Hence $\eta = 0$ and therefore $\mathbf{a} = \mathbf{0}$ and, if we suppose that $\sigma \neq 0$ then

$$\int_{B_\varepsilon(0)} |\mathbf{a}+\mathbf{b}\xi+\mathbf{c}\zeta|\,d\xi d\zeta = |\mathbf{c}|\int_{B_\varepsilon(0)} |\sigma\xi+\zeta|\,d\xi d\zeta \;=\; \frac{4}{3}\,\varepsilon^3\,|\mathbf{c}|\,\sqrt{1+\sigma^2} \;>\; \frac{4}{3}\;|\mathbf{c}|\varepsilon^3$$

which is a contradiction too. Then $\sigma = |\mathbf{b}| = 0$.
The case $\mathbf{b} \neq \mathbf{0}$ leads to the conclusion $\mathbf{a} = \mathbf{c} = \mathbf{0}$ by the same argument.
The case $\mathbf{a} \neq \mathbf{0}$ together with equality in (3.19) leads to $\mathbf{b} = \eta\mathbf{a}, \mathbf{c} = \sigma\mathbf{a}$ for suitable real constants η, σ.

$$|\mathbf{a}| \int_{B_\varepsilon(0)} |1 + \eta\xi + \sigma\zeta| \, d\xi d\zeta = \int_{B_\varepsilon(0)} |\mathbf{a} + \mathbf{b}\xi + \mathbf{c}\zeta| \, d\xi d\zeta = \big(\frac{4}{3}\varepsilon^3 \left(\theta\, |\eta| + \omega|\sigma|\right) \; + \; \pi(1 - \theta - \omega)\, \varepsilon^2\big)|\mathbf{a}| \, .$$

The set $U(\eta, \sigma) = \{(\xi, \zeta) \in B_\varepsilon(0) : \; (\eta\xi + \sigma\zeta)(1 + \eta\xi + \sigma\zeta) < 0\}$ has positive measure if $\eta^2 + \sigma^2 \neq 0$: then for every $(\xi, \zeta) \in U(\eta, \sigma)$ we get $|1 + \eta\xi + \sigma\zeta| > |\eta\xi + \sigma\zeta| + \text{sign}(\eta\xi + \sigma\zeta)$ and by integration

$$(\theta + \omega)|\mathbf{a}|\int_{B_\varepsilon(0)} |1 + \eta\xi + \sigma\zeta| d\xi d\zeta > (\theta + \omega)|\mathbf{a}| \int_{B_\varepsilon(0)} |\eta\xi + \sigma\zeta| \, d\xi d\zeta = (\theta + \omega)|\mathbf{a}| \frac{4}{3}\, \varepsilon^3 \sqrt{\eta^2 + \sigma^2} \geq$$
$$\geq \frac{4}{3}\, \varepsilon^3 |\mathbf{a}|(\theta|\eta| + \omega|\sigma|)$$

$$(1 - \theta - \omega)|\mathbf{a}| \int_{B_\varepsilon(0)} |1 + \eta\xi + \sigma\zeta| d\xi d\zeta \;\; \geq (1 - \theta - \omega)\pi|\mathbf{a}|\varepsilon^2 \; ,$$

by summing up we get a contradiction. Then $\mathbf{a} \neq 0$ entails $\eta = \sigma = |\mathbf{b}| = |\mathbf{c}| = 0$. ■

Due to Lemma 3.2 the set $\mathcal{R}$ of rigid displacements on T^ε can be identified with the set of deformations $\mathbf{v} \in SBD(T^\varepsilon)$ such that $\mathbf{u} = \mathcal{U}_3^\varepsilon(\mathbf{v})$ belongs to the space $\mathcal{V}$ defined as follows

$$\mathcal{V} = \big\{ \mathbf{u} \in SBV(0, L) \times SBH^3(0, L) : \; u_4' = -K(u_3' + \tau u_2); \; A_l(\mathbf{u}) \equiv 0, \; l = 0, 1, 2, \; j_{u_1} = j_{\dot{u}_2} = j_{\dot{u}_3} = 0 \big\}$$

where K, τ are defined by (3.3) and $\mathcal{U}_3$ is defined by Lemma 3.2. It is worth noticing that if $\mathbf{u} \in \mathcal{V}$ then $\mathbf{u} \in H^1(0, L) \times (H^2(0, L))^3$.

In order to achieve suitable asymptotic estimates of type Korn inequality (stated in Remark 3.7) for the displacement $\mathbf{v}$ in cartesian coordinates, we show and exploit the behaviour of Poincaré inequality for $\mathbf{u} = \mathcal{U}_3^\varepsilon(\mathbf{v})$ in intrinsic coordinates (see Lemmas 3.4-3.6).

Lemma 3.4 - *The linear space $\mathcal{V}$ is closed in the $L^{\frac{3}{2}}(0, L) \times W^{1,3/2}\big((0, L), \mathbf{R}^3\big)$ norm. Moreover For every $\mathbf{u} \in SBV(0, L) \times SBH\big((0, L), \mathbf{R}^3\big)$ the following*

$$\min \left\{ \int_0^L (|\mathbf{u} - \mathbf{w}|^{\frac{3}{2}} + |\dot{u}_2 - \dot{w}_2|^{\frac{3}{2}} + |\dot{u}_3 - \dot{w}_3|^{\frac{3}{2}}) \, ds : \mathbf{w} \in \mathcal{V} \right\}$$

is achieved, its minimizer $\mathbf{w} = \mathbf{w}(\mathbf{u})$ is unique and will be denoted by $\overline{\mathbf{u}} \stackrel{def}{=} \mathcal{Q}\mathbf{u}$.
The map $\mathcal{Q} \; : \; \mathbf{u} \rightarrow \overline{\mathbf{u}}$ is continuous from $SBV(0, L) \times SBH\big((0, L), \mathbf{R}^3\big)$ to $\mathcal{V}$, though $\mathcal{Q}$ is not linear. Nevertheless the identity $\mathcal{Q}(\mathbf{u} - \mathcal{Q}(\mathbf{u})) = \mathbf{0}$, say $\overline{\mathbf{u} - \overline{\mathbf{u}}} = \mathbf{0}$, holds true.
Hence, to every $\mathbf{v} \in SBD(T^\varepsilon)$ such that $\mathrm{e}(\mathbf{v}) \cdot \mathbf{N} = \mathrm{e}(\mathbf{v}) \cdot \mathbf{B} = 0$ we may associate an infinitesimal rigid displacement $\overline{\mathbf{v}} = (\mathcal{U}_3^\varepsilon)^{-1} \mathcal{Q}(\mathbf{u})$ where $\mathbf{u}$ is related to $\mathbf{v}$ through Lemma 3.2, that is $\mathbf{u} = \mathcal{U}_3^\varepsilon(\mathbf{v})$ and $\overline{\mathbf{u}} = \mathcal{U}_3^\varepsilon(\overline{\mathbf{v}})$:

$$\overline{\mathbf{v}} = (\overline{u}_1 - \xi(K\overline{u}_1 + \dot{\overline{u}}_2 - \tau\overline{u}_3) - \zeta(\dot{\overline{u}}_3 + \tau\overline{u}_2))\mathbf{T} + (-\overline{u}_4\zeta + \overline{u}_2)\mathbf{N} + (\overline{u}_4\xi + \overline{u}_3)\mathbf{B} \tag{3.20}$$

with $\dot{\overline{u}}_4 = -K(\dot{\overline{u}}_3 + \tau\overline{u}_2)$.

Proof - In contrast to the analogous statement of Lemma 2.4 $\mathcal{V}$ is not a Hilbert space. Anyway the constraint entails equivalence on $\mathcal{V}$ of $L^{3/2}(0, L) \times W^{1,3/2}((0, L), \mathbf{R}^3)$ and $W^{1/3/2}(0, L) \times W^{2,3/2}((0, L), \mathbf{R}^3)$ topology . Moreover the constraint are w^* closed in this topology ([A]).

Fix $\mathbf{u} \in SBV(0,L)\times SBH((0,L,\mathbf{R}^3)$ and let $\mathbf{w}^h$ be a minimizing sequence. Then w^h, $\dot w_2^h$, $\dot w_3^h$ are equi-bounded in $L^{3/2}$ and by ${w_4^h}' = -K({w_3^h}'+\tau w_2^h)$, $\mathcal{A}_l\mathbf{w}^h \equiv 0$, $l=1,2,3$, $j_{w_1}=j_{\dot w_2}=j_{\dot w_3}=j_{\dot w_4}\equiv 0$, we deduce that $\mathbf{w}^h$ is equi-bounded in $W^{1,3/2}(0,L)\times W^{2,3/2}((0,L),\mathbf{R}^3)$. The functional to be minimized is l.s.c. and strictly convex. ■

Lemma 3.5 - *For every $r\in[1,3/2]$ there exists $M=M(\boldsymbol{\gamma},r)$ s.t. for every $\mathbf{u}\in SBV(0,L)\times SBH\big((0,L),\mathbf{R}^3\big)$ s.t. $u_4'=-K(u_3'+\tau u_2)$ we have*

$$\begin{aligned}\|u_4-\overline{u}_4\|_{L^r(0,L)} &\le L^{\frac1r-\frac23}\Big(\|\mathbf{u}-\overline{\mathbf{u}}\|_{L^{\frac32}(0,L)}+\|\dot u_2-\dot{\overline{u_2}}\|_{L^{\frac32}(0,L)}+\|\dot u_3-\dot{\overline{u_3}}\|_{L^{\frac32}(0,L)}\Big)\le\\ &\le M\left(\int_0^L\sum_{j=0}^{2}|A_j\,\mathbf{u})|\,ds+\sum_{J_{u_1}\cup J_{\dot u_2}\cup J_{\dot u_3}}|\,[u_1]\,|+|\,[\dot u_2+Ku_1]\,|+|\,[\dot u_3]\,|\right)\end{aligned}$$

Proof - The first inequality is trivial. The second one can be proved as like as in lemma 2.5. ■

The previous Lemma entails an estimate of the blow-up for the Korn-Poincaré inequality constant in $SBD(T^\varepsilon)$, as $\varepsilon\to 0_+$.

Lemma 3.6 - *Let $M=M(\boldsymbol{\gamma},3/2)$ be the constant provided by Lemma 3.5. Then for every $\mathbf{v}\in SBD(T^\varepsilon)$ satisfying (3.11), and for all $\varepsilon<(\pi^{1/3}-1)\,\min\big\{\|K\|_{L^\infty}^{-1},(1+\|\tau\|_{L^\infty})^{-1}\big\}<\min(\varepsilon_0,1)$, we have:*

$$\left(\int_{T^\varepsilon}|\mathbf{v}-\overline{\mathbf{v}}|^{\frac32}\,dx\right)^{\frac23}\le\frac94\pi M\,\varepsilon^{-\frac53}\int_{T^\varepsilon}|\mathrm{e}(\mathbf{v})|$$

$$\|\mathbf{u}-\overline{\mathbf{u}}\|_{L^{3/2}(0,L)}\le\frac94 M\varepsilon^{-3}\int_{T^\varepsilon}|\mathrm{e}(\mathbf{v})|$$

where $\overline{\mathbf{v}}=(\mathcal{U}_3^\varepsilon)^{-1}Q\mathcal{U}_3^\varepsilon\mathbf{v}\in SBD(T^\varepsilon)$ and $\overline{\mathbf{u}}$ denote the vector fields defined in Lemma 3.4.

Proof - The proof relies on the same idea of Lemma 2.6: by using Minkowski inequality,(3.5) and Lemmas 3.1-3.5 with $r=3/2$, $\theta=\omega=1/3$, we get

$$\begin{aligned}&\left(\int_{T^\varepsilon}|\mathbf{v}-\overline{\mathbf{v}}|^{\frac32}\,dx\right)^{\frac23}\le\\ &\le\ \pi\,\varepsilon^{\frac43}\Big(\|\mathbf{u}-\overline{\mathbf{u}}\|_{L^{\frac32}(0,L)}+\|\dot u_2-\dot{\overline{u_2}}\|_{L^{\frac32}(0,L)}+\|\dot u_3-\dot{\overline{u_3}}\|_{L^{\frac32}(0,L)}\Big)\le\\ &\le\ \pi\,M(\boldsymbol{\gamma},3/2)\,\varepsilon^{-5/3}\left(\varepsilon^3\int_0^L\sum_{j=0}^{2}|A_j\,\mathbf{u})|\,ds\ +\ \varepsilon^3\sum_{J_{u_1}\cup J_{\dot u_2}\cup J_{\dot u_3}}|\,[u_1]\,|+|\,[\dot u_2+Ku_1]\,|+|\,[\dot u_3]\,|\right)\le\\ &\le\frac94\,\pi\,M(\boldsymbol{\gamma},3/2)\,\varepsilon^{-5/3}\int_0^L\int_{B_\varepsilon(0)}|\mathcal{A}_0\mathbf{u}-\xi\,\mathcal{A}_1\mathbf{u}-\zeta\,\mathcal{A}_2\mathbf{u}|=\\ &=\frac94\,\pi\,M(\boldsymbol{\gamma},3/2)\,\varepsilon^{-5/3}\int_0^L\int_{B_\varepsilon(0)}|(1-K\xi)^{-1}\big(\mathcal{A}_0\mathbf{u}-\xi\,\mathcal{A}_1\mathbf{u}-\zeta\,\mathcal{A}_2\mathbf{u}\big)|\,(1-K\xi)\,ds\,d\xi\,d\zeta=\\ &\le\frac94\,\pi\,M(\boldsymbol{\gamma},3/2)\,\varepsilon^{-5/3}\int_{T^\varepsilon}|\mathrm{e}(\mathbf{v})|\,d\mathbf{x}.\qquad\blacksquare\end{aligned}$$

The following statement is a straightforward consequence, via Hölder inequality, of the Lemma 3.6 .

Remark 3.7 - *There exists a constant $c=c(\boldsymbol{\gamma},r)=\frac94\pi(\pi L)^{\frac{3-2r}{3r}}M(\boldsymbol{\gamma},3/2)$ independent of ε such that for every $\mathbf{v}\in SBD(T^\varepsilon)$ with $\mathrm{e}(\mathbf{v})\circ\mathbf{N}=\mathrm{e}(\mathbf{v})\circ\mathbf{B}=0$ and*

$$\left(\int_{T^\varepsilon}|\mathbf{v}-\overline{\mathbf{v}}|^r\,d\mathbf{x}\right)^{\frac1r}\le\ c(\boldsymbol{\gamma},r)\,\varepsilon^{\frac{2-3r}{r}}\int_{T^\varepsilon}|\mathrm{e}(\mathbf{v})|\qquad\forall\ r\in[\,1,3/2\,]\qquad(3.21)$$

$$\| \mathbf{u} - \overline{\mathbf{u}} \|_{L^{p'}(0,L)} \;\leq\; \frac{1}{2}\, \pi^{\frac{2p'-3}{3p'}}\; c(\boldsymbol{\gamma}, p')\, \varepsilon^{-3} \int_{T^\varepsilon} |\mathrm{e}(\mathbf{v})| \qquad\qquad \forall p' \in [\,1, 3/2\,] \qquad (3.21')$$

Remark 3.8 - Analogously to the previous section, besides the more complicate geometry, we observe that by Lemma 3.2 to any displacement field $\mathbf{v} \in SBD(\Sigma^\varepsilon)$ such that $\mathrm{e}(\mathbf{v}) \cdot \mathbf{N} = \mathrm{e}(\mathbf{v}) \cdot \mathbf{B} = 0$ we may associate a quadruple $\mathbf{u} = \mathcal{U}_3^\varepsilon(\mathbf{v}) = (u_1, u_2, u_3, u_4) \in SBV(0,L) \times SBH\big((0,L), \mathbf{R}^3\big)$ with $\dot u_4 = -K(\dot u_3 + \tau u_2)$. In the same spirit, for every $\mathbf{u} = \mathcal{U}_3^\varepsilon(\mathbf{v}) \in SBV(0,L) \times SBH\big((0,L), \mathbf{R}^3\big)$ such that $\mathrm{e}(\mathbf{v}) \cdot \mathbf{N} = \mathrm{e}(\mathbf{v}) \cdot \mathbf{B} = \mathbf{0}$, Lemma 3.4 produces an infinitesimal rigid displacement $\overline{\mathbf{v}} = (\mathcal{U}_3^\varepsilon)^{-1}(\mathbf{u})$ associated to $\overline{\mathbf{u}} = \mathcal{Q}\mathcal{U}_3^\varepsilon(\mathbf{v}) \in \mathcal{V}$. Notice that, if $\mathbf{v} \in \mathcal{R}$, then $\overline{\mathbf{u}} = \mathbf{u}$ and $\overline{\mathbf{v}} = \mathbf{v}$, while, in general, $\overline{\mathbf{u}} = \mathcal{Q}(\mathbf{u}) \neq \mathbf{u}$ and $\overline{\mathbf{v}} \neq \mathbf{v}$ (see Fig.4). The map $\mathcal{U}_3 = \mathcal{U}_3^\varepsilon$, bijective from $SBD(T^\varepsilon)$ with constraint (3.11) to the subspace of $SBV \times SBH^3$ where $u_4' = -K(u_3' + \tau u_2)$, $\mathcal{U}_3^\varepsilon$ depends on ε and its norm, as linear operator, is not bounded uniformly in ε, while its inverse $(\mathcal{U}_3^\varepsilon)^{-1}$ has norm uniformly bounded in ε, thanks to Lemma 3.2 i).

$$\begin{array}{ccc}
\mathbf{v} \in SBD(T^\varepsilon) \text{ s.t. (3.11) holds} & & \overline{\mathbf{v}} \in \mathcal{R} \subset SBD(T^\varepsilon) \\
\Big\downarrow \mathcal{U}_3^\varepsilon & & \Big\uparrow (\mathcal{U}_3^\varepsilon)^{-1} \\
\mathbf{u} \in SBV(0,L) \times SBH(0,L)^3 & \xrightarrow{\mathcal{Q}} & \overline{\mathbf{u}} \in \mathcal{W} \subset H^1(0,L) \times H^2(0,L)^3
\end{array}$$

Fig.4 - Mappings diagram (3d approximation)

Theorem 3.9 - *Assume (3.1)-(3.11), $c = c(\boldsymbol{\gamma}, r)$ is the constant defined in Remark 3.7, and*

$$\int_{T^\varepsilon} \mathbf{g}^\varepsilon \cdot \mathbf{w} = 0 \quad \forall \mathbf{w} \in \mathcal{R} \quad (\textbf{compatibility})\,, \quad \| \,\mathbf{f}\, \|_{L^p(0,L)} < \frac{2\,\beta}{\pi^{\frac{1}{p} + \frac{2}{3}}\, c(\boldsymbol{\gamma}, p')} \quad (\textbf{safe load})\,. \quad (3.22)$$

Then there is a (not necessarily unique) solution of the following minimization problem.

$$(\mathbf{LCB_3^\varepsilon}) \qquad \min \left\{ \mathcal{F}^\varepsilon(\mathbf{v}) : \mathbf{v} \in SBD(T^\varepsilon),\ \mathrm{e}(\mathbf{v}) \cdot \mathbf{N} = \mathrm{e}(\mathbf{v}) \cdot \mathbf{B} = \mathbf{0} \text{ in } T^\varepsilon \right\}\,.$$

Obviously $\mathbf{LCB_3^\varepsilon}$ has solution for small loads, but the smallness condition could depend on the radius ε : actually Theorem 3.9 proves a safe load condition independent of ε.

Proof - Let $\mathbf{v}_h^\varepsilon$ be a minimizing sequence, $\mathbf{u}^\varepsilon = \mathcal{U}_3^\varepsilon \mathbf{v}_h^\varepsilon$, $c(\boldsymbol{\gamma}, r)$ is the constant appearing in Remark 3.7, $C_3(\mu, \lambda)$ is the constant appearing in (3.7). Then, by the compatibility assumption in (3.22),

$$\mathcal{G}_3^\varepsilon(\mathbf{v}_h^\varepsilon) - \int_{T^\varepsilon} \mathbf{g}^\varepsilon \cdot (\mathbf{v}_h^\varepsilon - \overline{\mathbf{v}_h^\varepsilon})\, d\mathbf{x} \;=\; \mathcal{F}_3^\varepsilon(\mathbf{v}_h^\varepsilon - \overline{\mathbf{v}_h^\varepsilon}) \;=\; \mathcal{F}_3^\varepsilon(\mathbf{v}_h^\varepsilon) \;\leq\; \mathcal{F}_3^\varepsilon(\mathbf{0}) \;=\; 0 \qquad (3.23)$$

hence, by exploiting Young inequality, (3.6),(3.7),(3.8), the safe load condition in (3.22), Remark 3.7

with $r = p'$, (3.21) and cancellation of integrals over $B_\varepsilon(\mathbf{0})$ for terms with odd dependance on $\xi,\ \zeta$

$$
\begin{aligned}
&2\varepsilon\beta \int_{T^\varepsilon} |\mathrm{e}(\mathbf{v}_h^\varepsilon)| - \frac{\pi\beta^2 L}{C_3(\mu,\lambda)}\varepsilon^4 \ \le \\
&\quad\le\ C_3(\mu,\lambda)\int_{T^\varepsilon} |\mathcal{E}(\mathbf{v}_h^\varepsilon)|^2\, d\mathbf{x} \ +\ \varepsilon\,\beta \int_{J_{\mathbf{v}_h^\varepsilon}} |[\mathbf{v}_h^\varepsilon]\odot \nu_{\mathbf{v}_h^\varepsilon}|\, d\mathcal{H}^2 \ +\ \varepsilon^2\alpha\,\mathcal{H}^2(J_{\mathbf{v}_h^\varepsilon}) \le \\
&\quad\le \int_{T^\varepsilon} \mathbf{g}^\varepsilon \cdot (\mathbf{v}_h^\varepsilon - \overline{\mathbf{v}_h^\varepsilon})\, dx \ = \\
&\quad= \int_0^L\int_{B_\varepsilon} (\varepsilon^2\{f_2(s)(u_2^\varepsilon - \overline{u_2^\varepsilon}) + f_3(s)(u_3^\varepsilon - \overline{u_3^\varepsilon})\} + 2f_4(s)(\xi^2+\zeta^2)(u_4^\varepsilon - \overline{u_4^\varepsilon}))(1-K(s)\xi)\, ds d\xi d\zeta = \\
&\quad= \int_0^L\int_{B_\varepsilon} \left(\varepsilon^2 f_2(u_2^\varepsilon - \overline{u_2^\varepsilon}) + \varepsilon^2 f_3(u_3^\varepsilon - \overline{u_3^\varepsilon}) + 2f_4(\xi^2+\zeta^2)(u_4^\varepsilon - \overline{u_4^\varepsilon})\right)\, ds d\xi d\zeta = \\
&\quad=\ \pi\,\varepsilon^4 \int_0^L \mathbf{f}\cdot(\mathbf{u}_h^\varepsilon - \overline{\mathbf{u}_h^\varepsilon})\, ds \ \ \le\ \pi\,\varepsilon^4 \|\mathbf{f}\|_{L^p}\|\mathbf{u}^\varepsilon - \overline{\mathbf{u}^\varepsilon}\|_{L^{p'}} \ < \\
&\quad<\ \pi\,\varepsilon^4\, \frac{2\,\beta}{\pi^{\frac{1}{p}+\frac{2}{3}}\, c(\boldsymbol{\gamma},p')}\, \frac{1}{2}\pi^{\frac{2p'-3}{3p'}}\ c(\boldsymbol{\gamma},p')\,\varepsilon^{-3}\int_{T^\varepsilon} |\mathrm{e}(\mathbf{v}_h^\varepsilon)| \ =\ \beta\,\varepsilon \int_{T^\varepsilon} |\mathrm{e}(\mathbf{v}_h^\varepsilon)|
\end{aligned}
$$

then $\int_{T^\varepsilon} |\mathrm{e}(\mathbf{v}_h^\varepsilon)|$ and , $\int_{T^\varepsilon} |\mathbf{v}_h^\varepsilon - \overline{\mathbf{v}_h^\varepsilon}|$ are bounded uniformly in $h\,,\varepsilon$ since, by Remark 3.7,

$$
\varepsilon \int_{T^\varepsilon} |\mathbf{v}_h^\varepsilon - \overline{\mathbf{v}_h^\varepsilon}|\, ds\, d\xi\, d\zeta \ \ \le\ \ c(\boldsymbol{\gamma},p')\int_{T^\varepsilon} |\mathrm{e}(\mathbf{v}_h^\varepsilon)| \ \ \le\ \ \frac{c(\boldsymbol{\gamma},p')\,\pi\,\beta}{C_3(\mu,\lambda)}\,\varepsilon^3\ .
$$

Hence, by summarizing the information contained in the previous chain of inequalities and taking into account Lemma 2.3 of [PT2]

$$
\begin{cases}
\displaystyle\int_{T^\varepsilon} |\mathbf{v}_h^\varepsilon - \overline{\mathbf{v}}_h^\varepsilon|\, ds\, d\xi\, d\zeta \ \le\ \frac{\pi\,\beta\,L\,c(\boldsymbol{\gamma},p')}{C_3(\mu,\lambda)}\,\varepsilon^2 \\
\displaystyle\int_{T^\varepsilon} |\mathrm{e}(\mathbf{v}_h^\varepsilon)| \ \le\ \frac{\pi\,\beta\,L}{C_3(\mu,\lambda)}\,\varepsilon^3 \\
\displaystyle\int_{T^\varepsilon} |\mathcal{E}(\mathbf{v}_h^\varepsilon)|^2\, d\mathbf{x} \ \le\ \frac{\pi\,\beta^2\,L}{(C_3(\mu,\lambda))^2}\,\varepsilon^4 \\
\displaystyle\int_{J_{\mathbf{v}_h^\varepsilon}} |[\mathbf{v}_h^\varepsilon]|\, d\mathcal{H}^2 \ \le\ \sqrt{2}\int_{J_{\mathbf{v}_h^\varepsilon}} |[\mathbf{v}_h^\varepsilon]\odot\nu_{\mathbf{v}_h^\varepsilon}|\, d\mathcal{H}^2 \ \le\ \frac{\sqrt{2}\,\pi\,\beta}{C_3(\mu,\lambda)}\,\varepsilon^3 \\
\displaystyle\mathcal{H}^2(J_{\mathbf{v}_h^\varepsilon}) \ \le\ \frac{\pi\,\beta^2\,L}{\alpha\,C_3(\mu,\lambda)}\,\varepsilon^2
\end{cases}
\tag{3.24}
$$

By Theorem 1.1 and Corollary 1.2 of [BCD] and Theorem 2.1 of [PT1] we know both the sequential compactness of sequence $\mathbf{v}_h^\varepsilon - \overline{\mathbf{v}_h^\varepsilon}$ in w^* *SBD*(T^ε) and the lower semi-continuity of $\mathcal{F}$ with respect to the same topology; taking into account that the constraint $\mathrm{e}(\mathbf{v})\cdot\mathbf{N} = \ \mathrm{e}(\mathbf{v})\cdot\mathbf{B} = \mathbf{0}$ is w^* closed in *SBD*(T^ε), we get $\mathbf{v}_h^\varepsilon - \overline{\mathbf{v}_h^\varepsilon} \overset{w^*}{\rightharpoonup} \mathbf{v}^\varepsilon\ \in$ argmin $\mathbf{LCB_3^\varepsilon}$ up to sub-sequences, as $h\to\infty\,,\ \forall\varepsilon>0$, and the thesis follows. ∎

We notice that we need an explicit (ε-uniform) estimate for minimizers of $\mathbf{LCB_3^\varepsilon}$ expressed in terms of explicit coordinates: this will be done in Theorem 3.10 below.

Theorem 3.10 - *Assume (3.1)-(3.11) and the compatibility and safe load conditions (3.22) hold:*

Then there are $\mathbf{v}^\varepsilon \in \operatorname{argmin} \mathbf{LCB_3^\varepsilon}$ *such that* $\mathbf{u}^\varepsilon = \mathcal{U}_3^\varepsilon(\mathbf{v}^\varepsilon)$ *fulfills*

$$\int_0^L |A_0(\mathbf{u}^\varepsilon)|^2 \le \frac{2\,\beta^2\,L}{(C_3(\mu,\lambda))^2}\,\varepsilon^2 \;\;,\;\; \int_0^L |A_1(\mathbf{u}^\varepsilon)|^2 \le \frac{8\,\beta^2\,L}{(C_3(\mu,\lambda))^2} \;\;,\;\; \int_0^L |A_2(\mathbf{u}^\varepsilon)|^2 \le \frac{8\,\beta^2\,L}{(C_3(\mu,\lambda))^2} \;\;,$$

$$\mathcal{H}^0(S_{u_1^\varepsilon}) \;+\; \mathcal{H}^0(S_{\dot u_2^\varepsilon + K u_1^\varepsilon}) \;+\; \mathcal{H}^0(S_{\dot u_3^\varepsilon}) \;\le\; \frac{\beta^2\,L}{\alpha\,(C_3(\mu,\lambda))} \;\;,$$

$$\sum_{\mathbf{S}_{\mathbf{u}^\varepsilon}} |[\dot u_2^\varepsilon + K u_1^\varepsilon]| \le \frac{9\,\pi\,\beta\,L}{\sqrt{2}\,C_3(\mu,\lambda)} \;\;,\;\; \sum_{\mathbf{S}_{\mathbf{u}^\varepsilon}} |[\dot u_3^\varepsilon]| \le \frac{9\,\pi\,\beta\;L}{\sqrt{2}\,C_3(\mu,\lambda)} \;\;,\;\; \sum_{\mathbf{S}_{\mathbf{u}^\varepsilon}} |[u_1^\varepsilon]| \le \frac{3\,\sqrt{2}\,\beta^2}{C_3(\mu,\lambda)}\,\varepsilon \;\;,$$

hence, by Lemma 3.5, there is $C>0$ *independent of* ε *such that*

$$\int_0^L |\dot u_j^\varepsilon| \le C\,,\;\; j=1,2,3,4, \text{ and } \int_0^L |\ddot u_k^\varepsilon| \le C\,,\;\; \|u_k^\varepsilon\|_{BH(0,L)} \le C,\;\; k=2,3,4,\;\; \|u_1^\varepsilon\|_{BV(0,L)} \le C\,; \quad (3.25)$$

Proof - We drop the superscript ε everywhere in the proof. Let $\mathbf{v}$ be a minimizer of $\mathcal{F}_3$. By performing the computations in intrinsic coordinates as in the proof of theorem 3.9, we notice that $\mathbf{v} - \overline{\mathbf{v}}$ is a minimizer too, hence we may assume without loss of generality, $\overline{\mathbf{v}} = \mathbf{0}$, $\overline{\mathbf{u}} = \mathbf{0}$ and set $\mathbf{u} = \mathcal{U}(\mathbf{v})$. By taking into account (3.5), $d\mathbf{x} = (1-K\xi)\,d\xi\,d\zeta\,ds$, inequality (3.24), and Lemma 3.2 ii),iii) we get

$$\frac{1}{2}\int_0^L \left(\pi\varepsilon^2|A_0(\mathbf{u})|^2 + \pi\frac{\varepsilon^4}{4}(|A_1(\mathbf{u})|^2 + |A_2(\mathbf{u})|^2)\right) ds \;\le$$
$$\le \int_0^L\int_{B^\varepsilon(0)} (1-K\xi)^{-1}|A_0(\mathbf{u}) - \xi A_1(\mathbf{u}) - \zeta A_2(\mathbf{u})|^2\,ds\,d\xi\,d\zeta \;=\; \int_{T^\varepsilon} |\mathcal{E}(\mathbf{v})|^2\,d\mathbf{x} \;\le\; \frac{\pi\,\beta^2\,L}{(C_3(\mu,\lambda))^2}\,\varepsilon^4$$

$$\pi\,\varepsilon^2\,\left(\mathcal{H}^0(S_{u_1^\varepsilon}) + \mathcal{H}^0(S_{\dot u_2^\varepsilon + K u_1^\varepsilon}) + \mathcal{H}^0(S_{\dot u_3^\varepsilon})\right) \;=\; \mathcal{H}^2(J_{\mathbf{v}}) \;\le\; \frac{\pi\,\beta^2\,L}{\alpha\,(C_3(\mu,\lambda))}\,\varepsilon^2$$

By $(1-K\xi)>0$, $|(1-K\xi)^{-1}|<2$, Lemma 3.2 v) and Lemma 3.3 with $\theta=\omega=1/3$, we obtain

$$\begin{aligned}
\beta\sum_{\mathbf{S}_\mathbf{u}}\left(\frac{\pi}{3}\,\varepsilon^3\,|\,[u_1]\,| \;+\; \frac{4}{9}\,\varepsilon^4\;\;[\dot u_2 + K u_1]\,| \;+\; \frac{4}{9}\,\varepsilon^4\,|\,[\dot u_3]\,|\right) \;\le& \\
\le\;\; \varepsilon\,\beta\sum_{s\,\in\,\mathbf{S}_\mathbf{u}}\int_{B_\varepsilon(0)} |\,[u_1] - \xi[\dot u_2 + K u_1] - \zeta[\dot u_3]\,|\;d\xi\,d\zeta =& \\
=\;\; \varepsilon\beta\sum_{s\in\mathbf{S}_\mathbf{u}}\int_{B_\varepsilon(\mathbf{0})} |[v_\mathbf{T}]|\,(1-K\xi)\,(1-K\xi)^{-1}\,d\xi\,d\zeta \;\;\le& \qquad (3.26)\\
\le\;\; 2\,\varepsilon\beta\sum_{\mathbf{S}_\mathbf{u}}\int_{B_\varepsilon(\mathbf{0})} |[v_\mathbf{T}]|\,(1-K\xi)\,d\xi\,d\zeta \;\;=& \\
=\;\; 2\varepsilon\beta\int_{J_\mathbf{v}} |[v_\mathbf{T}]|\,d\mathcal{H}^2 \;\;=\;\; 2\,\varepsilon\,\beta\int_{J_\mathbf{v}} |[\mathbf{v}]|\,d\mathcal{H}^2 \;\;\le\;\; \frac{2\,\sqrt{2}\,\pi\,\beta^2\,L}{C_3(\mu,\lambda)}\,\varepsilon^4&
\end{aligned}$$

And gathering together we get the thesis. ■

The next theorem is the main result of this section

Theorem 3.11 - *Assume (3.1)-(3.11), (3.22). Then there exists* $\mathbf{v}^\varepsilon \in \operatorname{argmin} \mathbf{LCB}_3^\varepsilon$ *, with* $\overline{\mathbf{v}^\varepsilon} = \mathbf{0}$ *, and a displacements expressed in intrinsic coordinates* $\mathbf{u} = (u_1, u_2, u_3, u_4) \in SBV \times SBH^3$ *such that, by setting* $\mathbf{u}^\varepsilon = \mathcal{U}_3^\varepsilon(\mathbf{v}^\varepsilon)$*,* $\overline{\mathbf{u}^\varepsilon} = \mathbf{0}$ *and up to sub-sequences,*

$$u_1^\varepsilon \;=\; ⨍_{B_\varepsilon(0)} \mathbf{v}^\varepsilon \cdot \mathbf{T} \;\overset{*}{\rightharpoonup}\; u_1 \qquad \text{in } SBV(0,L) \;\text{ as } \varepsilon \to 0_+$$

$$u_j^\varepsilon \;\overset{*}{\rightharpoonup}\; u_j \quad \text{in } SBH(0,L) \qquad \text{for } j = 2,3,4 \;\text{ as } \varepsilon \to 0_+$$

where $u_2^\varepsilon = ⨍_{B_\varepsilon(\mathbf{0})} \mathbf{v}^\varepsilon \cdot \mathbf{N}\,,\quad u_3^\varepsilon = ⨍_{B_\varepsilon(\mathbf{0})} \mathbf{v}^\varepsilon \cdot \mathbf{B}\,,\quad u_4^\varepsilon = \dfrac{1}{I_0}\displaystyle\int_{B_\varepsilon(\mathbf{0})} \mathbf{v}^\varepsilon \cdot (\xi \mathbf{B} - \zeta \mathbf{N})\;.$
Moreover this $\mathbf{u} = (u_1, u_2, u_3, u_4)$ *is a solution of*

$$(\mathbf{LCB}_3) \qquad\qquad \min\{\mathcal{F}_3(\mathbf{u}) :\; \mathbf{u} \in SBH\big((0,L), \mathbf{R}^4\big)\,\}$$

where

$$\mathcal{F}_3(\mathbf{u}) \;=\; \begin{cases} \Phi(\mathbf{u}) & \text{if} \quad u_1' = \dot u_1 = K u_2\,, \text{ and } \dot u_4 = -K(\dot u_3 + \tau u_2) \\ +\infty & \text{otherwise,} \end{cases}$$

and (referring to Lemma 3.2ii) for the definition of A_1*,* A_2*)*

$$\Phi(\mathbf{u}) \;=\; \frac{\pi}{4}\left(\mu + \frac{\lambda}{2}\right) \int_0^L \big(|A_1(\mathbf{u})|^2 + |A_2(\mathbf{u})|^2\big)\, ds \;-\; \pi \int_0^L \sum_{j=2}^4 f_j u_j \, ds \;+$$
$$+\; \pi\alpha\, \mathcal{H}^0(S_{\dot u_2} \cup S_{\dot u_3}) \;+\; \frac{4}{3}\,\beta \sum_{s \in S_{\dot u_2} \cup S_{\dot u_3}} \sqrt{[\dot u_2]^2 + [\dot u_3]^2}\;.$$

Moreover $\varepsilon^{-4} \min \mathbf{LCB_3^\varepsilon} \;\le\; \min \mathbf{LCB_3} \; + o(1)$ *and the re-scaled energies converge, that is*

$$\lim_{\varepsilon \to 0_+} \varepsilon^{-4} \mathcal{F}_3^\varepsilon(\mathbf{v}^\varepsilon) \;=\; \mathcal{F}_3(\mathbf{u}).$$

Proof - Problem $\mathbf{LCB}_3^\varepsilon$ achieves a finite minimum for every $\varepsilon > 0$ by Theorem 3.9 (see also [CLT2,3],[CT]). Let $\mathbf{v}^\varepsilon$ be a minimizer then, by (3.23), $\mathcal{F}^\varepsilon(\mathbf{v}^\varepsilon) \le 0$. Without loss of generality we can assume that $\overline{\mathbf{v}}^\varepsilon \equiv 0$, and we set $\mathbf{u}^\varepsilon = \mathcal{U}_3^\varepsilon(\mathbf{v}^\varepsilon)$, from Theorem 3.10 there is $\mathbf{u} = (u_1, u_2, u_3, u_4) \in SBV \times (SBH)^3$, such that, up to sub-sequences, $u_1^\varepsilon \overset{*}{\rightharpoonup} u_1$ in $SBV(0,L)$, $u_j^\varepsilon \overset{*}{\rightharpoonup} u_j$ in $SBH(0,L)$ for $j = 2,3,4$. Then by using lower semi-continuity of total variation an Theorem 3.10

$$\int_0^L |\mathcal{A}_0 \mathbf{u}| \;\le\; \liminf_{\varepsilon \to 0} \int_0^L |\mathcal{A}_0 \mathbf{u}^\varepsilon|\, ds \;\le\; \liminf_{\varepsilon \to 0} \left\{ \sqrt{L} \left(\int_0^L |A_0 \mathbf{u}^\varepsilon|^2 \right)^{\frac{1}{2}} + \sum_{S_{u_1^\varepsilon}} |[u_1^\varepsilon(s)]| \right\} \;\le 0$$

then,

$$u_1' = \dot u_1 = K u_2 \in L^2(0,L) \;\text{ and }\; u_1 \in H^1(0,L) \cap SBH(0,L)$$

say $\mathcal{A}_0 \mathbf{u} = 0$ and

$$\dot u_4 = -K(\dot u_3 + \tau u_2)\;.$$

By the convexity of the euclidian norm, since $\int_{B^\varepsilon(\mathbf{0})} |a\xi + b\zeta|\, d\xi d\zeta \;=\; \frac{4}{3}\,\varepsilon^3\,\sqrt{a^2+b^2}$ for every $a, b \in \mathbf{R}$,

$$\begin{aligned}
&\int_{B_\varepsilon(\mathbf{0})} |\,[u_1^\varepsilon] - \xi[\dot u_2^\varepsilon + K u_1^\varepsilon] - \zeta[\dot u_3^\varepsilon]|\, d\xi\, d\zeta \;\geq \\
\geq\; &\int_{B_\varepsilon(\mathbf{0})} |\zeta[\dot u_3^\varepsilon] + \xi[\dot u_2^\varepsilon + K u_1^\varepsilon]\,|\, d\xi\, d\zeta \;-\; \int_{B_\varepsilon(\mathbf{0})} [u_1^\varepsilon]\;\; \mathbf{sign}(\xi[\dot u_2^\varepsilon + K u_1^\varepsilon] - \zeta[\dot u_3^\varepsilon])\;\, d\xi\, d\zeta \;= \\
=\; &\frac{4}{3}\,\varepsilon^3\,\sqrt{[\dot u_3^\varepsilon]^2 + [\dot u_2^\varepsilon + K u_1^\varepsilon]^2}
\end{aligned}$$

By using now Lemma 3.2 i), Theorem 3.10, and vanishing of integrals with odd dependance on ξ, ζ, we obtain

$$\begin{aligned}
\mathcal{F}_3^\varepsilon(\mathbf{v}^\varepsilon) \;=\; &(1+o(1))\,\pi\,\left(\mu + \frac{\lambda}{2}\right)\int_0^L \left\{\varepsilon^2|A_0\mathbf{u}^\varepsilon|^2 + \frac{\varepsilon^4}{4}(|A_1\mathbf{u}^\varepsilon|^2 + A_2\mathbf{u}^\varepsilon|^2)\right\} ds \;+ \\
&+\; \pi\varepsilon^4\alpha\,\mathcal{H}^0(\mathbf{S}_{\mathbf{u}^\varepsilon}) \;+\; \varepsilon\beta \sum_{s\,\in\,\mathbf{S}_{\mathbf{u}_\varepsilon}} \int_{B_\varepsilon(0)} |[u_1^\varepsilon] - \xi[\dot u_2^\varepsilon + K u_1^\varepsilon] - \zeta[\dot u_3^\varepsilon]|\, d\xi\, d\zeta \;+ \\
&-\int_0^L ds \int_{B_\varepsilon(0)} \left(\varepsilon^2(f_2\mathbf{v_N} + f_3\mathbf{v_B}) + 2f_4(\xi\mathbf{v_B} - \zeta\mathbf{v_N})\right)(1-K\xi)\, d\xi\, d\zeta \;\geq \\
\geq\;\; &\frac{\varepsilon^4}{4}\left(1+o(1)\right)\pi\,\left(\mu + \frac{\lambda}{2}\right)\int_0^L \left(|A_1\mathbf{u}^\varepsilon|^2 + |A_2\mathbf{u}^\varepsilon|^2\right)\, ds + \pi\,\varepsilon^4\,\alpha\,\mathcal{H}^0(\mathbf{S}_{\mathbf{u}^\varepsilon}) \;+ \\
&+\;\; \frac{4}{3}\,\beta\,\varepsilon^4 \sum_{\mathbf{S}_{\mathbf{u}^\varepsilon}} \sqrt{[\dot u_2^\varepsilon + K u_1^\varepsilon]^2 + [\dot u_3^\varepsilon]^2} \;-\; \pi\,\varepsilon^4 \int_0^L \sum_{j=2}^4 f_j u_j\, ds \geq \\
\geq\;\; &\frac{\varepsilon^4}{4}\left(1+o(1)\right)\pi\,\left(\mu + \frac{\lambda}{2}\right)\int_0^L \left(|A_1\mathbf{u}^\varepsilon|^2 + |A_2\mathbf{u}^\varepsilon|^2\right)\, ds + \pi\,\varepsilon^4\,\alpha\,\mathcal{H}^0(S_{\dot u_2^\varepsilon + K u_1^\varepsilon} \cup S_{\dot u_3^\varepsilon}) \;+ \\
&+\;\; \frac{4}{3}\,\beta\,\varepsilon^4 \sum_{\mathbf{S}_{\mathbf{u}^\varepsilon}} \sqrt{[\dot u_2^\varepsilon + K u_1^\varepsilon]^2 + [\dot u_3^\varepsilon]^2} \;-\; \pi\,\varepsilon^4 \int_0^L \sum_{j=2}^4 f_j u_j\, ds
\end{aligned}$$

where $\mathbf{S}_{\mathbf{u}^\varepsilon} \;=\; S_{u_1^\varepsilon} \cup S_{\dot u_2^\varepsilon + K u_1^\varepsilon} \cup S_{\dot u_3^\varepsilon}$.
By Theorem 3.10 there is $C > 0$ independent of ε such that

$$\int_0^L |\dot u_j^\varepsilon| \leq C\,,\;\; j = 1,2,3,4, \text{ and } \int_0^L |\ddot u_k^\varepsilon| \leq C\,,\;\; \|u_k^\varepsilon\|_{BH(0,L)} \leq C,\;\; k = 2,3,4,\;\; \|u_1^\varepsilon\|_{BV(0,L)} \leq C$$

and

$$\sum_{\mathbf{S}_{\mathbf{u}^\varepsilon}} |[\dot u_2^\varepsilon + K u_1^\varepsilon]| \leq \frac{9\,\pi\,\beta\,L}{\sqrt{2}\,C_3(\mu,\lambda)} \quad , \qquad \sum_{\mathbf{S}_{\mathbf{u}^\varepsilon}} |[\dot u_3^\varepsilon]| \leq \frac{9\,\pi\,\beta\;L}{\sqrt{2}\,C_3(\mu,\lambda)} \quad , \qquad \sum_{\mathbf{S}_{\mathbf{u}^\varepsilon}} |[u_1^\varepsilon]| \leq \frac{3\,\sqrt{2}\,\beta^2}{C_3(\mu,\lambda)}\,\varepsilon$$

and therefore there exists a compact set $\mathbf{E} \subset \mathbf{R}^2$ such that $(\dot u_2^\varepsilon + K u_1^\varepsilon, \dot u_3^\varepsilon) \in \mathbf{E}$ for a.e. $s \in (0,L)$ and $\varepsilon \in (0, \varepsilon_0)$.
By applying Theorem 5.22 of [AFP] with $\phi(i,j,p) \;=\; \left(\pi\,\alpha \;+\; \frac{4}{3}\,\beta\,\left|i-j\right|\right)\,\left|\,p\,\right|$ (and [AFP] Ex.5.23) we get

$$\begin{aligned}
\liminf_{\varepsilon\to 0}\;\; &\pi\alpha\,\mathcal{H}^0(S_{\dot u_2^\varepsilon + K u_1^\varepsilon} \cup S_{\dot u_3^\varepsilon}) \;+\; \frac{4}{3}\,\beta \sum_{\mathbf{S}_{\mathbf{u}_\varepsilon}} \sqrt{[\dot u_2^\varepsilon + K u_1^\varepsilon]^2 + [\dot u_3^\varepsilon]^2} \;\;\geq \\
&\geq\;\; \pi\alpha\,\mathcal{H}^0(S_{\dot u_2 + K u_1} \cup S_{\dot u_3}) \;+\; \frac{4}{3}\,\beta \sum_{\mathbf{S}_{\mathbf{u}}} \sqrt{[\dot u_2 + K u_1]^2 + [\dot u_3]^2}.
\end{aligned}$$

Hence by Ioffe Theorem (see [AFP] Th. 5.8)

$$\liminf_{\varepsilon\to 0} \quad \varepsilon^{-4}\,\mathcal{F}_3^{\varepsilon}(\mathbf{v}^{\varepsilon}) \;\geq\; \mathcal{F}_3(\mathbf{u})\;.$$

By defining the recovery sequence $\mathbf{z}^{\varepsilon}$, for $\mathbf{x}\in T^{\varepsilon},\ s\in[0,L],\ (\xi,\,\zeta)\in B_{\varepsilon}(0)\,,$

$$\mathbf{z}^{\varepsilon}(\mathbf{x}) \;=\; (\mathcal{U}_3^{\varepsilon})^{-1}\mathbf{u} \;=\; (\,u_1-\xi(Ku_1+u_2'-\tau u_3)-\zeta(u_3'+\tau u_2)\,)\,\mathbf{T} \,+\, (u_2-\zeta u_4)\,\mathbf{N} \,+\, (u_3+\xi u_4)\,\mathbf{B}$$

where $\mathbf{u}=\lim\mathbf{u}^{\varepsilon}=$ belongs to $SBH(0,L)$, and applying Lemma 3.2, we get

$$\varepsilon^{-4}\mathcal{F}_3^{\varepsilon}(\mathbf{v}^{\varepsilon}) \;\leq\; \varepsilon^{-4}\,\mathcal{F}_3^{\varepsilon}(\mathbf{z}^{\varepsilon}) \;=\; \mathcal{F}_3(\mathbf{u})+o(1) \qquad\qquad \varepsilon\in[0,\varepsilon_0].$$

This proves that

$$\varepsilon^{-4}\mathcal{F}_3^{\varepsilon}(\mathbf{v}^{\varepsilon})\to\mathcal{F}_3(\mathbf{u})$$

then $\mathbf{u}$ is a minimizer of $\mathcal{F}_3(\mathbf{u})$ in $SBH\big((0,L),\mathbf{R}^4\big)$. ■

Actually the proof above says more: if $(3.1)-(3.11),(3.22)$ and $\mathbf{v}^{\varepsilon}\in\operatorname{argmin}\mathbf{LCB}_3^{\varepsilon}$ then $\exists\mathbf{u}\in SBV\times SBH\big((0,L),\mathbf{R}^3\big)$ and a subsequence of $\mathbf{v}^{\varepsilon}$ s.t., without re-labeling

$$\mathbf{u}^{\varepsilon}-\overline{\mathbf{u}}^{\varepsilon} \;=\; \mathcal{U}_3^{\varepsilon}(\mathbf{v}^{\varepsilon}-\overline{\mathbf{v}}^{\varepsilon}) \;\rightharpoonup^{*}\; \mathbf{u} \qquad \text{and} \qquad \varepsilon^{-4}\mathcal{F}_3^{\varepsilon}(\mathbf{v}^{\varepsilon}) \;\to\; \mathcal{F}_2(\mathbf{u})$$

Remark 3.12 - It is worth noticing that the compatibility condition $\int_{\Sigma^{\varepsilon}}\mathbf{g}^{\varepsilon}\cdot\mathbf{w}=\mathbf{0}\ \ \forall\,\mathbf{w}\in\mathcal{R}(T^{\varepsilon})$ can be rewritten as $\int_0^L\mathbf{f}\cdot\mathbf{u}\,ds=0$ for every $\mathbf{u}\in\mathcal{V}$.

Remark 3.13 - Although the case of a straight beam doesn't fall within the assumptions of the present section, since the curvature K is not strictly positive , still all the results hold true when restated by formal substitution of $K=\tau=0$ as shown in Section 4. In the following examples we give explicit form to general formulation of Theorem 3.11 in some case of simple geometries of the beam.

Example 3.14 (*Circular ring*) - Assume T is a circle of radius R , say:
$\boldsymbol{\gamma}(s)=\Big(R\,\cos\dfrac{s}{R},\,R\,\sin\dfrac{s}{R},\,0\Big),\ \ R>0,$, and $s\in[0,L]$. Then $K=R^{-1}>0\,,\ \dot K\equiv 0\,,\ \tau\equiv 0\,.$
The compatibility condition in (3.22) reads as follows (by (3.8) and Lemma 3.2i))

$$\int_0^L f_i u_i\,ds=0 \qquad i\ =2,3,4$$

for every quadruple (u_1,u_2,u_3,u_4) of functions solving the system of differential equations (which characterizes rigid displacements in terms of intrinsic coordinates, in the ring geometry):

$$\left\{\begin{array}{l}\dot u_1=R^{-1}u_2\\ \ddot u_2+R^{-1}\dot u_1=0\\ \ddot u_3-R^{-1}u_4=0\\ \dot u_4=-R^{-1}\dot u_3\end{array}\right. \quad \text{i.e} \quad \left\{\begin{array}{l}\ddot u_2+R^{-2}u_2=0\\ \dot u_1=R^{-1}u_2\\ \ddot u_4+R^{-2}u_2=0\\ \dot u_3=R\dot u_4\end{array}\right. \quad \text{i.e} \quad \left\{\begin{array}{l}u_1(s)=A_2\cos\left(\frac{s-s_0}{R}\right)+A_1\\ u_2(s)=A_2\sin\left(\frac{s-s_0}{R}\right)\\ u_3(s)=RA_4\sin\left(\frac{s-s_4}{R}\right)+A_3\\ u_4(s)=A_4\sin\left(\frac{s-s_4}{R}\right)\ .\end{array}\right.$$

The limit functional Φ is

$$\Phi(\mathbf{u}) \;=\; \frac{\pi}{4}\Big(\mu+\frac{\lambda}{2}\Big)\int_0^L\Big(|\ddot u_2+R^{-2}u_2|^2+|\ddot u_3-R^{-1}u_4|^2-\pi\sum_{j=2}^{4}f_ju_j\Big)\,ds\;+$$
$$+\;\pi\alpha\,\mathcal{H}^0(S_{\dot u_2}\cup S_{\dot u_3})\;+\;\frac{4}{3}\,\beta\sum_{s\in\,S_{\dot u_2}\cup\,S_{\dot u_3}}\sqrt{[\dot u_2]^2+[\dot u_3]^2}\;.$$

if $u_1' = \dot u_1 = R^{-1} u_2$ and $u_4 = constant$, while $\Phi(\mathbf{u}) = +\infty$ else.
It is interesting to compare this Φ with the one related to the 2D approximation in example 2.13.

Example 3.15 (*Cylindrical helix*) - We consider here the case in which T is a cylindrical helix, e.g. the image of the map $\boldsymbol{\gamma}$, with $\boldsymbol{\gamma}(s) = \left(R\cos\frac{s}{A}\,,\, R\sin\frac{s}{A}\,,\, \frac{p}{2\pi}\,s\right)$, $R > 0,\ p > 0\,,\ \ A = \sqrt{R^2 + p^2/(4\pi^2)}\,,$ and $s \in [0, L]$.
Then $K = |\ddot{\boldsymbol{\gamma}}| = R/A^2 = \left(R + p^2/(4\pi^2 R)\right)^{-1}$, $\dot K \equiv 0$, $\tau = |\tau| = (4\pi^2 R^2 p)/A^3$, $\dot\tau \equiv 0$.
The compatibility condition in (3.22) (thanks to (3.8), Lemma 3.2i)) reads as follows

$$\int_0^L f_i u_i \, ds = 0 \qquad i \ = 2, 3, 4$$

for every quadruple (u_1, u_2, u_3, u_4) of functions solving the constant coefficients system of differential equations (which characterizes rigid displacements in terms of intrinsic coordinates, in the helix geometry):

$$\begin{cases} \dot u_1 = K u_2 \\ \ddot u_2 - 2\tau \dot u_3 + K \dot u_1 - \tau^2 u_2 = 0 \\ \ddot u_3 + 2\tau \dot u_2 + \tau K u_1 - K u_4 - \tau^2 u_3 = 0 \\ \dot u_4 = -K(\dot u_3 + \tau u_2) \end{cases}$$

(hence he compressed helicoidal spring is not compatible with the above condition on the load). ■

4. Three–dimensional approximation of a linear elastic–plastic straight beam($\mathbf{LB_3}$)

As a remarkable example we explicit the analysis of the 3D approximation of an elastic-plastic straight beam with Neumann boundary conditions (for the 2D approximation of the cantilever we refer to [PT1]).
We emphasize that the statement of Theorem 4.3 at the end of this section corresponds exactly to the formal substitution of $K(s) = \tau(s) \equiv 0$ in Theorem 3.11, but here the assumption (3.5) fails. Hence the proof has to be modified at certain steps (and actually is much simpler). In addition the simple geometric structure allows to evaluate the constant M in Lemma 3.5 and the constant in the safe load condition.
We study a straight beam whose un-stressed configuration is the segment $[0, L]$ parameterized by arc-length x

$$\boldsymbol{\gamma} = (\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \boldsymbol{\gamma}_3) \ = \ (x, 0, 0) \ = \ x\,\mathbf{e}_1 \quad x \in [0, L] \ , \tag{4.1}$$

In $\mathbf{R}^3$ we denote the cartesian coordinates by x, y, z and we consider a thick beam whose reference configuration is the open set

$$C^\varepsilon \ = \ \{\, x\,\mathbf{e}_1 \ + \ y\,\mathbf{e}_2 \ + \ z\,\mathbf{e}_3 \ : \ 0 < x < L\,,\ y^2 + z^2 < \varepsilon^2 \,\} \tag{4.2}$$

The set of infinitesimal rigid displacements in C^ε is denoted by $\mathcal{R}(C^\varepsilon)$. The region C^ε is the natural reference of a three-dimensional linear elastic body with free damage at mesoscopic scale, whose internal strain energy is given, for every displacement field $\mathbf{v} \in SBD(T^\varepsilon)$, by the following functional

$$\mathcal{G}_3^\varepsilon(\mathbf{v}) \ = \ \int_{T^\varepsilon} W\left(\mathcal{E}(\mathbf{v})\right) d\mathbf{x} \ + \ \varepsilon^2\,\alpha\,\mathcal{H}^2(J_\mathbf{v}) \ + \ \varepsilon\,\beta \int_{J_\mathbf{v}} |[\mathbf{v}] \odot \nu_\mathbf{v}| d\mathcal{H}^2 \tag{4.3}$$

where

$$\begin{aligned} & W(A) \ = \ \mu\,|A|^2 + \frac{\lambda}{2}(\mathrm{Tr}\,A)^2 \geq C_3(\mu,\lambda)|A|^2 \qquad \forall A \in M_{3,3}\,, \\ & \alpha\,,\beta\,,\mu \ > 0\ , \quad 2\mu + 3\lambda \ > \ 0 \qquad\qquad C_3(\mu,\lambda) = \min\{\mu, \mu + \frac{3}{2}\lambda\} > 0 \end{aligned} \tag{4.4}$$

The beam T^ε is subject to a transverse body force field $\mathbf{g}^\varepsilon$ of the following kind

$$\begin{aligned} \mathbf{g}^\varepsilon(x) \;&=\; \varepsilon^2 f_2(s)\,\mathbf{e}_2 \,+\, \varepsilon^2 f_3(s)\,\mathbf{e}_3 \,+\, 2\, f_4(s)\,(\,y\,\mathbf{e}_3 \,-\, z\,\mathbf{e}_2\,) \\ \mathbf{f} \;&=\; (0, f_2, f_3, f_4) \in L^p((0,L),\mathbf{R}^4) \qquad p \geq 3\,. \end{aligned} \tag{4.4}$$

The term $\varepsilon^2\,(f_2(s)\mathbf{e}_1 + f_3(s)\mathbf{e}_2)$ is the bending force, while $-2f_4(s)(z\mathbf{e}_2 - y\mathbf{e}_3)$ is the twisting force, here we assume that there is no tangential component of the force, say $f_1 = \mathbf{g}^\varepsilon \cdot \mathbf{e}_1 = 0$.
We then define

$$\mathcal{L}^\varepsilon(\mathbf{v}) \,=\, \int_{C^\varepsilon} \mathbf{g}^\varepsilon \cdot \mathbf{v}\, d\mathbf{x} \tag{4.6}$$

and so the (total) energy functional will be

$$\mathcal{F}_3^\varepsilon(\mathbf{v}) \,=\, \mathcal{G}_3^\varepsilon(\mathbf{v}) \,-\, \mathcal{L}^\varepsilon(\mathbf{v})\ . \tag{4.7}$$

We assume that the displacement field $\mathbf{v}$ satisfies the Bernoulli-Navier cinematic constraint $\mathrm{e}(\mathbf{v})\cdot\mathcal{N} = \mathbf{0}$ in the sense of measures for all continuous vector fields $\mathcal{N}$ which are normal to the central strand ([K],[PG],[V]), say

$$\mathrm{e}(\mathbf{v})\cdot\mathbf{e}_2 \;=\; \mathrm{e}(\mathbf{v})\cdot\mathbf{e}_3 \;=\; \mathbf{0} \qquad \text{in } \mathcal{D}'(C^\varepsilon) \tag{4.8}$$

so that we are led to study the following minimization problem

$(\mathbf{LB_3^\varepsilon})$ $\qquad \min\,\{\mathcal{F}_3^\varepsilon(\mathbf{v})\,:\,\mathbf{v} \in SBD(T^\varepsilon),\, \mathrm{e}(\mathbf{v})\cdot\mathbf{e}_2 = \mathbf{0} = \mathrm{e}(\mathbf{v})\cdot\ e_3 \text{ in } T^\varepsilon\,\}\ .$

The analysis of existence and asymptotic behavior for minimizers of $(\mathbf{LB_3^\varepsilon})$ goes as like as in section 3, except for the proof of lemmas 3.2 , 3.5 and Theorem 3.11, which we restate and prove as follows (see respectively Lemmas 4.1, 4.2 and theorem 4.3).

We exploit the cinematic constraint (4.8) by showing that admissible vector fields have rank-1 strain tensor and are completely described by four scalar functions of arc-length s: the averages of cartesian components of the displacements over the cross sections and the rotation angle (which turns out to be constant for the straight beam).

Lemma 4.1 - *Fix $\varepsilon > 0$. Suppose that $\mathbf{v} \in SBD(T^\varepsilon)$ and $\mathrm{e}(\mathbf{v})\cdot\mathbf{e}_2 = \mathbf{0} = \mathrm{e}(\mathbf{v})\cdot\mathbf{e}_3$ in the sense of vector-valued measures. Then, by labelling $v_1 = \mathbf{v}\cdot\mathbf{e}_1, \quad v_2 = \mathbf{v}\cdot\mathbf{e}_2, \quad v_3 = \mathbf{v}\cdot\mathbf{e}_3$,*

i) there exist unique $u_1 \in SBV(0,L), u_2, u_3, \in SBH(0,L)$ and a constant function $u_4 = c$ such that

$$\mathbf{v} \;=\; v_1\,\mathbf{e}_1 \,+\, v_2\,\mathbf{e}_2 \,+\, v_3\,\mathbf{e}_3 \;=\; (u_1 - yu_2' - zu_3')\,\mathbf{e}_1 + (u_2 - cz)\,\mathbf{e}_2 + (u_3 + cy)\,\mathbf{e}_3\,;$$

ii) $\mathrm{e}(\mathbf{v}) \;=\; (\,u_1' \,-\, y\,u_2'' \,-\, z\,u_3''\,)\,\mathbf{e}_1 \otimes \mathbf{e}_1$

here $\mathbf{u} = (u_1,u_2,u_3,u_4)$ is defined by i) (and, just for comparison with section 3, we notice that $\mathcal{A}_0\mathbf{u} = u_1',\ A_0\mathbf{u} = \dot{u}_1\ ,\ \mathcal{A}_1\mathbf{u} = u_2''\ ,\ A_1\mathbf{u} = \ddot{u}_2\,, \qquad \mathcal{A}_2\mathbf{u} = u_3''\ ,\ A_2\mathbf{u} = \ddot{u}_3$ *) .*

iii) $J_\mathbf{v} \subset (J_{u_1} \cup J_{\dot{u}_2} \cup J_{\dot{u}_3}) \otimes B_\varepsilon(\mathbf{0}) = (S_{u_1} \cup S_{\dot{u}_2} \cup S_{\dot{u}_3}) \otimes B_\varepsilon(\mathbf{0}) = \mathbf{S_u} \otimes B_\varepsilon(\mathbf{0})$ *and*

$$\mathcal{H}^2(J_\mathbf{v} \setminus (J_{u_1} \cup J_{\dot{u}_2} \cup J_{\dot{u}_3}) \times B(0,\varepsilon)) = 0\,;$$

$$d\mathcal{H}^2 \llcorner J_\mathbf{v} \;=\; \sum_{\mathbf{S_u}} (1 - K(s)\xi)\, d\xi\, d\zeta$$

iv) $\mathrm{e}(\mathbf{v}) = \mathbf{0}$ *iff* $u_1' = u_2'' = u_3'' = 0$ *(e.g.* $\dot{u}_1 = \ddot{u}_2 = \ddot{u}_3 = 0$ *, and* $j_{u_1} = j_{\dot{u}_2} = j_{\dot{u}_3} = 0)$*;*

$v)\qquad \mathcal{G}^{\varepsilon}(\mathbf{v}) \;=\; (1+o(1))\left(\mu+\dfrac{\lambda}{2}\right)\pi\displaystyle\int_0^L \left\{\varepsilon^2|\dot u_1|^2+\dfrac{\varepsilon^4}{4}(|\ddot u_2|^2+\ddot u_3|^2)\right\}ds \;+$

$$+\;\pi\varepsilon^4\alpha\,\mathcal{H}^0(\mathbf{S_u})\;+\;\varepsilon\beta\sum_{s\,\in\,\mathbf{S_u}}\int_{B_\varepsilon(\mathbf{0})}|[u_1]-\xi[\dot u_2]-\zeta[\dot u_3]|\,d\xi d\zeta$$

where $\mathbf{S_u}\;=\;S_{u_1}\cup S_{\dot u_2}\cup S_{\dot u_3}$ *and* $o(1)$ *tends to 0 as* $\varepsilon\to 0_+$.

On the other hand, for every $\mathbf{u}=(u_1,u_2,u_3,u_4)\in SBV(0,L)\times SBH\left((0,L),\mathbf{R}^3)\right)$ *the vector field* $\mathbf{v}$ *defined by i) belongs to* $SBD(T^\varepsilon)$ *and fulfills (3.11) and* $ii),iii),iv),v)$.

vi) The linear map $\mathcal{U}_3^\varepsilon:\mathbf{v}\to\mathbf{u}$ *(from orthogonal to intrinsic coordinates) defined by i) satisfies*

$$u_1=\fint_{B_\varepsilon(\mathbf{0})}\mathbf{v}\cdot\mathbf{e}_1\,d\xi\,d\zeta,\qquad u_2=\fint_{B_\varepsilon(\mathbf{0})}\mathbf{v}\cdot\mathbf{e_2}\,d\xi\,d\zeta,\qquad u_3=\fint_{B_\varepsilon(\mathbf{0})}\mathbf{v}\cdot\mathbf{e}_3\,d\xi\,d\zeta,$$

$u_4=\dfrac{1}{I_0}\displaystyle\int_{B_\varepsilon(\mathbf{0})}\mathbf{v}\cdot(y\,\mathbf{e_3}\,-\,z\,\mathbf{e}_2)\,d\xi\,d\zeta,\quad I_0=\pi\varepsilon^4/2$ *polar moment of inertia over circular cross sections*

and $\mathcal{U}_3^\varepsilon$ *is one-to-one and bi-continuous in the strong topologies from* $\{\mathbf{v}\in SBD(\Sigma^\varepsilon):\,\mathrm{e}(\mathbf{v})\cdot\mathbf{e}_2=\mathrm{e}(\mathbf{v})\cdot\mathbf{e}_3=\mathbf{0}\}$ *to the closed subspace of* $SBV(0,L)\times SBH((0,L),\mathbf{R}^3)$ *spanned by the solutions of* $u_4'=0$. *We notice that* $(\,u_1(s),u_2(s),u_3(s)\,)$ *is the resultant of* $\mathbf{v}$ *and* u_4 *is the twisting moment of* $\mathbf{v}$ *over the circular cross section of* C_ε *through* $\boldsymbol{\gamma}(x)$.

vii) we emphasize that u_4 *is constant, namely:* $u_4'=0$.

Proof - We exploit

$$\mathbf{v}_{2,y}\;=\;\mathbf{v}_{3,z}\;=\;\mathbf{v}_{1,y}+\mathbf{v}_{2,x}\;=\;\mathbf{v}_{1,z}+\mathbf{v}_{3,x}\;=\;\mathbf{v}_{2,z}\,+\,\mathbf{v}_{3,y}\;=\;0\qquad\mathcal{D}'(C^\varepsilon)$$

$$\mathbf{v}_{2,z}\,+\,\mathbf{v}_{3,y}\;=\;0\qquad\Rightarrow\qquad\begin{cases}\mathbf{v}_{2,zz}\;=\;-\mathbf{v}_{3,yz}\;=\;0\\ \mathbf{v}_{3,yy}\;=\;-\mathbf{v}_{2,yz}\;=\;0\end{cases}\qquad\overset{v_{2,y}=v_{3,z}=0}{\Rightarrow}\qquad\begin{cases}\mathbf{v}_2\;=\;u_2(x)\,+\,z\,\varphi(x)\\ \mathbf{v}_3\;=\;u_3(x)\,-\,y\,\psi(x)\end{cases}$$

Hence $\mathbf{v}_{2,z}\,+\,\mathbf{v}_{3,y}\;=\;0$ entails $\psi(x)=\varphi(x)$, say $\begin{cases}\mathbf{v}_2\;=\;u_2(x)\,+\,z\,\varphi(x)\\ \mathbf{v}_3\;=\;u_3(x)\,-\,y\,\varphi(x)\,.\end{cases}$

$$\begin{cases}\mathbf{v}_{1,y}\;=\;-\mathbf{v}_{2,x}\;=\;-u_2'(x)\,-\,z\,\varphi'(x)\\ \mathbf{v}_{1,z}\;=\;-\mathbf{v}_{3,x}\;=\;-u_3'(x)\,+\,y\,\varphi'(x)\end{cases}$$

Then there is $\tau=\tau(x,z)$, such that

$$\mathbf{v}_1(x,y,z)=-\,y\,u_2'(x)\,-\,y\,z\,\varphi'(x)\,-\tau(x,z)$$

and since $\mathbf{v}_{1,z}=-\mathbf{v}_{3,x}$ we get

$$\tau_z(x,z)\,-\,u_3'(x)\,+\,2y\,\varphi'(x)\;=\;0\qquad\text{in }\mathcal{D}'(C^\varepsilon)$$

hence $\tau_z\,-\,u_3'\;=\;\varphi'\;=\;0$.
Therefore $\tau(x,z)\;=\;-zu_3'(x)\,+\,u_1(x)$, $\varphi(x)=u_4(x)=c$ constant.
Regularity properties of u_1,u_2,u_3 can be proven by arguing as in Lemma 3.2 and i) follows. Then ii)-vi) are easily deduced by substitution. Then i) is proved; ii)-vi) are easily deduced by substitution.

Lemma 4.2 - *For every $r \in [1, 3/2]$ for every $\mathbf{u} \in SBV(0,L) \times SBH(0L)^3$ s.t. $u_4' = 0$ we have*

$$\|u_4 - \overline{u}_4\|_{L^r(0,L)} \;\le\; L^{\frac{1}{r}-\frac{2}{3}} \left(\|\mathbf{u} - \overline{\mathbf{u}}\|_{L^{\frac{3}{2}}(0,L)} + \|\dot u_2 - \overline{\dot u_2}\|_{L^{\frac{3}{2}}(0,L)} + \|\dot u_3 - \overline{\dot u_3}\|_{L^{\frac{3}{2}}(0,L)} \right) \;\le$$

$$\le\; L^{1/r}(1 + L^{1/3}) \left(\int_0^L (|\dot u_1| + |\ddot u_2| + |\ddot u_3|)\, ds + \sum_{J_{u_1} \cup J_{\dot u_2} \cup J_{\dot u_3}} |\,[u_1]\,| + |\,[\dot u_2 + K u_1]\,| + |\,[\dot u_3]\,| \right)$$

say when $\boldsymbol{\gamma}$ is a straight line of length L the constant $M = M(\boldsymbol{\gamma}, r)$ of lemma 3.5 can be chosen as $M = L^{1/r}(1 + L^{1/3})$.

Proof - The first inequality is trivial. The second one can be proved by using

$$\left(\int_0^L |w - \overline{w}|^r\, ds \right)^{1/r} \le L^{1/r} \int_0^L |w'| \qquad \text{for every } w \in BV(0,L)$$

as follows: by assuming, without loss of generality, that $\overline{\mathbf{u}} = \mathbf{0}$ we have

$$\|\mathbf{u}\|_{L^{\frac{3}{2}}(0,L)} + \|\dot u_2\|_{L^{\frac{3}{2}}(0,L)} + \|\dot u_3\|_{L^{\frac{3}{2}}(0,L)} \;\le$$

$$\le L^{2/3} \int_0^L \sum_{j=1}^4 |u_j'| + L^{2/3} \int_0^L (|u_2''| + |u_3''|) \le L^{2/3} \int_0^L |u_1'| + (L^{2/3} + L) \int_0^L (|u_2''| + |u_3''|) \le$$

$$L^{2/3}(1 + L^{1/3}) \left(\int_0^L (|\dot u_1| + |\ddot u_2| + |\ddot u_3|)\, ds + \sum_{J_{u_1} \cup J_{\dot u_2} \cup J_{\dot u_3}} |\,[u_1]\,| + |\,[\dot u_2 + K u_1]\,| + |\,[\dot u_3]\,| \right)$$

hence the constant in Lemma 3.5 may be chosen as $M(\boldsymbol{\gamma}, r) = L^{1/r}(1 + L^{1/3})$. ■
Since $c = c(\boldsymbol{\gamma}, r) = \frac{9}{4}\,\pi\,(\pi L)^{\frac{3-2r}{3r}} M(\boldsymbol{\gamma}, 3/2) = \frac{9}{4}\,\pi^{\frac{r+3}{3r}} L^{1/r}(1 + L^{1/3})$ the right-hand side in (3.22) reads explicitly

$$\frac{2\,\beta}{\pi^{\frac{1}{p}+\frac{2}{3}}\, c(\boldsymbol{\gamma}, p')} = \frac{8\,\beta}{9\,\pi^2\,(1 + L^{\frac{1}{3}})\, L^{\frac{1}{p'}}}\ ,$$

hence, in the straight beam geometry ((4.1)-(4.8)), the safe load condition becomes

$$\|\,\mathbf{f}\,\|_{L^p(0,L)} < \frac{8\,\beta}{9\pi^2\,(1 + L^{\frac{1}{3}})\, L^{\frac{1}{p'}}}\ . \tag{4.9}$$

Moreover in the straight beam geometry, the compatibility condition reads

$$\int_0^L f_j = 0 \quad j = 2,3,4 \qquad\qquad \int_0^L s f_j(s) = 0 \quad j = 2,3 \tag{4.10}$$

since, by Lemma 4.1 iv, if $\mathbf{v}$ is a rigid displacement and $\mathbf{u} = \mathcal{U}_3^\varepsilon \mathbf{v}$, then $u_1 = c_1$, $u_4 = c_4$, and $u_j = c_j + d_j s\ \ j = 2,3$.

The conclusion is summarized in the following statement

Theorem 4.3 - (*Straight beam under flexural force and torque*)
Assume (4.1)-(4.10). Then there exists $\mathbf{v}^\varepsilon \in \operatorname{argmin} \mathbf{LB}_3^\varepsilon$ with $\overline{\mathbf{v}^\varepsilon} = \mathbf{0}$, and a displacement expressed in intrinsic coordinates $\mathbf{u} = (u_1, u_2, u_3, u_4) \in SBV \times SBH^3$ such that, referring to Lemma 4.1 vi), by setting $\mathbf{u}^\varepsilon = \mathcal{U}_3^\varepsilon(\mathbf{v}^\varepsilon)$, we have $\overline{\mathbf{u}^\varepsilon} = \mathbf{0}$ and up to sub-sequences,

$$u_1^\varepsilon \;=\; ⨍_{B_\varepsilon(0)} \mathbf{v}^\varepsilon \cdot \mathbf{e}_1 \;\stackrel{*}{\rightharpoonup}\; u_1 \qquad \text{in } SBV(0,L) \ \text{ as } \varepsilon \to 0_+$$

$$u_j^\varepsilon \quad \overset{*}{\rightharpoonup} \quad u_j \quad \textit{in } SBH(0,L) \qquad \textit{for } j=2,3,4 \quad \textit{as } \varepsilon \to 0_+$$

where $u_2^\varepsilon = \displaystyle\fint_{B_\varepsilon(\mathbf{0})} \mathbf{v}^\varepsilon \cdot \mathbf{e}_2\,, \quad u_3^\varepsilon = \fint_{B_\varepsilon(\mathbf{0})} \mathbf{v}^\varepsilon \cdot \mathbf{e}_3\,, \quad u_4^\varepsilon = \frac{1}{I_0}\int_{B_\varepsilon(\mathbf{0})} \mathbf{v}^\varepsilon \cdot (\xi \mathbf{e}_3 - \zeta \mathbf{e}_2)\ .$
Moreover this $\mathbf{u} = (u_1, u_2, u_3, u_4)$ *is a solution of*

$$(\mathbf{LB}_3) \qquad\qquad \min\{\mathcal{F}_3(\mathbf{u}) :\ \mathbf{u} \in SBH\big((0,L), \mathbf{R}^4\big)\ \}$$

where

$$\mathcal{F}_3(\mathbf{u}) \ = \ \begin{cases} \Phi(\mathbf{u}) & \textit{if} \quad u_1' = \dot u_1 = \dot u_4 = 0 \\ \\ +\infty & \textit{otherwise,} \end{cases}$$

and

$$\begin{aligned} \Phi(\mathbf{u}) \ &= \ \frac{\pi}{4}\left(\mu + \frac{\lambda}{2}\right)\int_0^L (|\ddot u_2|^2 + |\ddot u_3|^2)\, ds \ - \ \pi \int_0^L \sum_{j=2}^4 f_j u_j\, ds \, + \\ &\quad + \ \pi\alpha\, \mathcal{H}^0(S_{\dot u_2} \cup S_{\dot u_3}) + \frac{4}{3}\,\beta \sum_{S_{\dot u_2}\cup S_{\dot u_3}} \sqrt{[\dot u_2]^2 + [\dot u_3]^2}\ . \end{aligned} \tag{4.11}$$

Moreover $\varepsilon^{-4}\, \min \mathbf{LB_3^\varepsilon} \ \le\ \min \mathbf{LB_3} \ + o(1)$ *and the re-scaled energies converge, that is*

$$\lim_{\varepsilon\to 0_+} \ \varepsilon^{-4}\mathcal{F}_3^\varepsilon(\mathbf{v}^\varepsilon) \quad = \quad \mathcal{F}_3(\mathbf{u}).$$

It is worth noticing that the constraint under which $\mathcal{F}(\mathbf{u})$ is finite imply, as in the elastic case, that the limit beam is inextensible and that the rotation angle u_4 is constant.

Proof - The only difference with respect to Th.3.11 consists in the choice $\mathbf{T} = \mathbf{e}_1$, $\mathbf{N} = \mathbf{e}_2$, $\mathbf{B} = \mathbf{e}_3$, hence $\dot u_1 = \ddot u_2 = \ddot u_3 = \ddot u_4 = 0$ characterize rigid displacements

Following step by step the proof of Theorem 3.11 it is not difficult to show that an analogous result holds even in this case.

5. Approximation of a rigid-plastic beam (RB)

In this section we study the effects of a divergent weight (which blows-up as ε tends to 0) in the volume integrals of (2.6) and (3.6): as a consequence we find a more stiff structure in the limit. As like as in sections 2 and 3 we can consider the 2D and 3D approximations: referring to (2.1)-(2.4) and (3.1)-3.4) we set

$$\begin{cases} \mathcal{T}_n^\varepsilon(\mathbf{v}) \ = \ \varepsilon^{-\delta}\displaystyle\int_{\Omega^\varepsilon} W\left(\mathcal{E}(\mathbf{v})\right)\, dx \ + \ \varepsilon^2\, \alpha\, \mathcal{H}^{n-1}(J_\mathbf{v}) \ + \ \varepsilon\, \beta \int_{J_\mathbf{v}} |[\mathbf{v}] \odot \nu_\mathbf{v}|\, d\mathcal{H}^{n-1} \\ W(A) \ = \ \mu\, |A|^2 \, + \, \dfrac{\lambda}{2}\,(\mathrm{Tr}\, A)^2 \quad \forall A \in M_{n,n}\ , \qquad \alpha,\, \beta,\, \delta,\, \mu \ > \ 0\ , \\ n = 2,3, \qquad \Omega^\varepsilon = \Sigma^\varepsilon \text{ if } n=2, \quad \Omega^\varepsilon = T^\varepsilon \text{ if } n = 3, \qquad (n-1)\mu + n\lambda > 0\ , \\ \mathbf{g}^\varepsilon \text{ given by (2.7) if } n=2 \text{ , and by (3.8) if } n=3, \end{cases} \tag{5.1}$$

$$\Lambda_2^\varepsilon(\mathbf{v}) = \ \mathcal{T}_2^\varepsilon(\mathbf{v}) \ - \int_{\Sigma^\varepsilon} \mathbf{g}^\varepsilon \cdot \mathbf{v} \qquad\qquad n=2$$

$$\Lambda_3^\varepsilon(\mathbf{v}) = \ \mathcal{T}_3^\varepsilon(\mathbf{v}) \ - \int_{T^\varepsilon} \mathbf{g}^\varepsilon \cdot \mathbf{v} \qquad\qquad n=3$$

We will consider the following problems

$(\mathbf{RCB}_2^\varepsilon)$ $$\min\{\Lambda_2^\varepsilon(\mathbf{v}) \ : \mathbf{v}\in SBD(\Sigma^\varepsilon),\ \mathrm{e}(\mathbf{v})\cdot\mathbf{n}=\mathbf{0} \text{ in } \Sigma^\varepsilon\}\ ,$$

$(\mathbf{RCB}_3^\varepsilon)$ $$\min\{\Lambda_3^\varepsilon(\mathbf{v}) \ : \mathbf{v}\in SBD(T^\varepsilon),\ \mathrm{e}(\mathbf{v})\cdot\mathbf{N}=\mathbf{0}=\mathrm{e}(\mathbf{v})\cdot\mathbf{B} \quad \text{in } T^\varepsilon\}\ .$$

Then following statements can be proved by the same methods used in the previous sections.
We recall that (by Lemma 2.2) if $n=2$, then under the Kirchhoff cinematic restriction, in intrinsic coordinates $u_1'=\dot u_1=\kappa u_2$ is the *tangential non extensibility*, while , $\ddot u_2+\kappa\dot u_1+\dot\kappa u_1=0$ is the *flexural rigidity*.

Theorem 5.1 - *Assume (5.1), $n=2$, and (2.1)-(2.9)-(2.22). Then there exists $\mathbf{v}^\varepsilon\in\operatorname{argmin}\mathbf{RCB}_2^\varepsilon$, such that, up to sub-sequences, $u_1^\varepsilon\overset{*}{\rightharpoonup}u_1$ in $SBV(0,L)$, $u_2^\varepsilon\overset{*}{\rightharpoonup}u_2$ in $SBH(0,L)$, where $\mathbf{u}^\varepsilon=(u_1^\varepsilon,u_2^\varepsilon)=\mathcal{U}^\varepsilon(\mathbf{v}^\varepsilon)$ and $\mathbf{u}=(u_1,u_2)$ is a solution of*

$(\mathbf{RCB_2})$ $$\min\{\Lambda_2(\mathbf{u}):\ \mathbf{u}\in SBH\left((0,L),\mathbf{R}^2\right)\}$$

where

$$\Lambda_2(\mathbf{u}) \ = \ \begin{cases} -\displaystyle\int_0^L f u_2\,ds \ +2\alpha\mathcal{H}^0(S_{\dot u_2})+\beta\int_{S_{\dot u_2}}|[\dot u_2]| & \text{if } u_1'=\dot u_1=\kappa u_2,\ \ddot u_2+\kappa\dot u_1+\dot\kappa u_1=0\\ \\ +\infty & \text{otherwise}\end{cases}$$

Moreover $\varepsilon^{-3}\min\mathbf{RCB_2^\varepsilon}\le\min\mathbf{RCB_2}+o(1)$ *and the re-scaled energies converge, that is*

$$\lim_{\varepsilon\to0}\varepsilon^{-3}\Lambda_2(\mathbf{v}^\varepsilon)=\Lambda_2(\mathbf{u}).\ \blacksquare$$

We recall that, by Lemma 3.2, if $n=3$ then the Bernoulli–Navier cinematic restriction on the deformations entails $u_4{}'=-K(u_3{}'+\tau u_2)$ moreover , in intrinsic coordinates,
$u_1'=\dot u_1=\kappa u_2$ describes the *tangential non extensibility*, while $A_1(\mathbf{u})=A_2(\mathbf{u})=0$ describes the *flexural rigidity*.

Theorem 5.2 - *Assume (5.1), $n=3$, (3.1)-(3.11) (3.22) hold; then there exists $\mathbf{v}^\varepsilon\in\operatorname{argmin}\mathbf{RCB}_3^\varepsilon$ and $\mathbf{u}=(u_1,u_2,u_3,u_4)\in SBV\times SBH^3$ such that, by setting $\mathbf{u}^\varepsilon=\mathcal{U}_3^\varepsilon(\mathbf{v}^\varepsilon)$, up to sub-sequences, $u_1^\varepsilon\overset{*}{\rightharpoonup}u_1$ in $SBV(0,L)$, $u_j^\varepsilon\overset{*}{\rightharpoonup}u_j$ in $SBH(0,L)$ for $j=2,3,4$ and $\mathbf{u}=(u_1,u_2,u_3,u_4)$ is a solution of*

$(\mathbf{RCB}_3)$ $$\min\{\Lambda_3(\mathbf{u}):\mathbf{u}\in SBH((0,L),\mathbf{R}^4)\}$$

where

$$\Lambda_3(\mathbf{u})=\begin{cases}\Phi(\mathbf{u}) & \text{if } u_1'=\dot u_1=Ku_2,\ A_1\,\mathbf{u}=A_2\,\mathbf{v}=0,\dot u_4=-K(\dot u_3+\tau u_2)\\ \\ +\infty & \text{otherwise,}\end{cases}$$

and

$$\Phi(\mathbf{u}) \ = \ \pi\int_0^L\sum_{j=2}^4 f_j u_j\,ds+\pi\alpha\mathcal{H}^0(S_{\dot u_2}\cup S_{\dot u_3})+\frac{4}{3}\beta\sum_{s\in S_{\dot u_2}\cup S_{\dot u_3}}\sqrt{[\dot u_2]^2+[\dot u_3]^2}$$

Moreover $\varepsilon^{-4}\min \mathbf{RCB}_3^\varepsilon \le \min \mathbf{RCB_3} + o(1)$ *and the re-scaled energies converge, that is*

$$\lim_{\varepsilon\to 0_+} \varepsilon^{-4}\Lambda^\varepsilon(\mathbf{v}^\varepsilon) = \Lambda_3(\mathbf{u}).$$

Theorems 5.1, 5.2 clarify how stiff thin structures undergoing small deformations can be described by assuming that the elastic deformation is irrelevant if compared to the plastic flow occurring along "a priori" unknown plastic yield points. Hence it is natural to couple rigid deformations with plastic hinges positioned at an unknown pattern of points: on this lines the deformation is still continuous but the gradients may undergo jump discontinuities of rank 1 (see [SM],[CLT4]).

We recall that, for an elastic beam, the integrals $\int_0^L |A_0\mathbf{u}|^2 ds$ and $\int_0^L (|A_1\mathbf{u}|^2 + |A_2\mathbf{u}|^2)\, ds$ denotes respectively the resistance to traction and the resistance to flexion.
So that here the constraint $\mathcal{A}_0\mathbf{u} = A_0\mathbf{u} \equiv 0$ (i.e. $u_1' = \dot u_1 = K u_2$) corresponds to tangential rigidity, and the constraint $A_1\mathbf{u} = A_2\mathbf{u} \equiv 0$ corresponds to (piece-wise) flexural rigidity.
Hence the whole set of constraint $\mathcal{A}_0\mathbf{u} = A_1\mathbf{u} = A_2\mathbf{u} \equiv 0$ describe the rigid-plastic beam.

Remark 5.3 - The 2D and 3D approximation of the rigid-plastic straight beam are obtained, respectively, by formal substitution of $\kappa = 0$ in Theorem 5.1, and $K = \tau \equiv 0$ in theorem 5.2, and arguing as like as in section 4.

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