The passage from non-convex discrete systems to variational problems in Sobolev spaces: the one-dimensional case

Andrea Braides*, Maria Stella Gelli† and Mario Sigalotti‡

December 8, 2004

Introduction

In this note we treat the problem of the description of variational limits of discrete problems in a one-dimensional setting. Given $n \in \mathbb{N}$ and points $x_i^n = i\lambda_n$ ($\lambda_n = L/n$ is the *lattice spacing*) we consider energies of the general form

$$E_n(\{u_i\}) = \sum_{j=1}^{n} \sum_{i=0}^{n-j} \lambda_n \psi_n^j \Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big).$$

If we picture the set $\{x_i^n\}$ as the reference configuration of an array of material points interacting through some forces, and u_i represents the displacement of the i-th point, then ψ_n^j can be thought as the energy density of the interaction of points with distance $j\lambda_n$ (j lattice spacings) in the reference lattice. Note that the only assumption we make is that ψ_n^j depends on $\{u_i\}$ through the differences $u_{i+j} - u_i$, but we find it more convenient to highlight its dependence on 'discrete difference quotients'.

Our goal is to describe the behaviour of problems of the form

$$\min\Big\{E_n(\{u_i\}) - \sum_{i=0}^{n} \lambda_n u_i f_i : \ u_0 = U_0, \ u_n = U_L\Big\}$$

(and similar), and to show that for a quite general class of energies these problems have a limit continuous counterpart. Here $\{f_i\}$ represents the external forces

*Present address: Dipartimento di Matematica, Università di Roma 'Tor Vergata', via della Ricerca Scientifica, 00133 Rome, Italy

†Present address: Dip. di Matematica, via F. Buonarroti 2, 56100 Pisa, Italy

‡SISSA, Via Beirut 4, 34014 Trieste, Italy

and U_0, U_L are the boundary conditions at the endpoints of the interval $(0, L)$. More general statement and different problems can be also obtained. To make this asymptotic analysis precise, we use the notation and methods of De Giorgi's Γ-convergence (see [2], [7]). We will show that, under some growth conditions, upon suitably identifying discrete functions $\{u_i\}$ with their interpolations, the free energies E_n 'Γ-*converge*' to a limit energy F, which is defined on a Sobolev space and takes the form

$$F(u) = \int_0^L \psi(u')\, dt.$$

As a consequence we obtain that minimizers of the problem above are 'very close' to minimizers of a classical problem of the Calculus of Variations

$$\min\Big\{\int_0^L (\psi(u') - fu)\, dt : \ u(0) = U_0,\ u(L) = U_L\Big\}.$$

The energy densities ψ can be explicitly identified by a series of operations on the functions ψ_n^j. The case when only nearest-neighbour interactions are taken into account:

$$E_n(\{u_i\}) = \sum_{i=0}^{n-1} \lambda_n \psi_n\Big(\frac{u_{i+1} - u_i}{\lambda_n}\Big)$$

has been completely solved as a particular case of the results by Braides and Gelli [5]. In this case, the limit energy density is given by the limit of the convex envelopes of the functions $\psi_n(z)$, which exists up to subsequences. The description of the limit energy gets more complex when not only nearest-neighbour interactions come into play. In the case when interactions up to a fixed order K are taken into account:

$$E_n(\{u_i\}) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n \psi_n^j\Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big)$$

(or, equivalently, $\psi_n^j = 0$ if $j > K$), the main idea is to show that (upon some controllable errors) we can find a lattice spacing η_n (possibly much larger than λ_n) such that E_n is 'equivalent' (as Γ-convergence is concerned) to a nearest-neighbour interaction energy on a lattice of step size η_n, of the form

$$\overline{E}_n(\{u_i\}) = \sum_{i=0}^{m-1} \eta_n \overline{\psi}_n\Big(\frac{u_{i+1} - u_i}{\eta_n}\Big),$$

and to which then the recipe above can be applied.

The crucial points are the computation of $\overline{\psi}_n$ and the choice of the scaling η_n. In the case of *next-to-nearest neighbours* this computation is particularly simple, as it consists in choosing $\eta_n = 2\lambda_n$ and in 'integrating out the contribution of first neighbours': in formula,

$$\overline{\psi}_n(z) = \psi_n^2(z) + \frac{1}{2}\min\{\psi_n^1(z_1) + \psi_n^1(z_2) : z_1 + z_2 = 2z\}.$$

In a sense this is a formula of *relaxation type*. If $K > 2$ then the formula giving $\overline{\psi}_n$ resembles more a *homogenization formula*, and we have to choose $\eta_n = K_n \lambda_n$ with K_n large. In this case the reasoning that leads from E_n to $\overline{E}_n$ is that the overall behaviour of a system of interacting point will behave as *clusters* of large arrays of neighbouring points interacting through their 'extremities'. These results are proven in Section 3.

In the description above one crucial point is played by the growth condition on ψ_n^j, which ensure compactness in some Sobolev space of functions with equi-bounded energy. If this condition is not satisfied then the limit F may be defined on some space of functions of bounded variation, and contain both a *bulk* and an *interfacial* energy. The determination of the corresponding energy densities can be obtained from ψ_n^j by an additional *separation of scales* argument (see [5]). Moreover, we may have limits defined on more copies of Sobolev spaces (*multiple density limits*). An example highlighting this phenomenon and the highly-sensitive dependence of the limit process on the chosen topology is presented in Section 4. Another important condition is that we have a sufficiently fast decay of ψ_n^j as $j \to +\infty$. This condition is satisfied if only interactions until a certain order are present. When the number of interaction orders we consider is not bounded the description becomes more complex. In particular, additional *non-local* terms may appear in F (see [1]).

1 Discrete functionals

We will consider the limit of energies defined on one-dimensional discrete systems of n points as n tends to $+\infty$. In order to define a limit energy on a continuum we parameterize these points as a subset of a single interval $(0, L)$. Set

$$\lambda_n = \frac{L}{n}, \qquad x_i^n = \frac{i}{n} L = i\lambda_n, \qquad i = 0, 1, \ldots, n. \tag{1.1}$$

We denote $I_n = \{x_0^n, \ldots, x_n^n\}$ and by $\mathcal{A}_n(0, L)$ the set of functions $u : I_n \to \mathbb{R}$. If n is fixed and $u \in \mathcal{A}_n(0, L)$ we equivalently denote

$$u_i = u(x_i^n).$$

Given $K \in \mathbb{N}$ with $1 \le K \le n$ and functions $f^j : \mathbb{R} \to [0, +\infty]$, with $j = 1, \ldots, K$, we will consider the related functional $E : \mathcal{A}_n(0, L) \to [0, +\infty]$ given by

$$E(u) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} f^j(u_{i+j} - u_i). \tag{1.2}$$

Note that E can be viewed simply as a function $E : \mathbb{R}^n \to [0, +\infty]$.

An interpretation with a physical flavour of the energy E is as the internal interaction energy of a chain of $n + 1$ material points each one interacting with its

K-nearest neighbours, under the assumption that the interaction energy densities depend only on the order j of the interaction and on the distance between the two points $u_{i+j} - u_i$ in the reference configuration. If $K = 1$ then each point interacts with its nearest neighbour only, while if $K = n$ then each pair of points interacts.

Remark 1.1 From elementary calculus we have that E is lower semicontinuous if each f^j is lower semicontinuous, and that E is coercive on bounded sets of $\mathcal{A}_n(0, L)$.

We will describe the limit as $n \to +\infty$ of sequences (E_n) with $E_n : \mathcal{A}_n(0, L) \to [0, +\infty]$ of the general form

$$E_n(u) = \sum_{j=1}^{K_n} \sum_{i=0}^{n-j} f_n^j(u_{i+j} - u_i), \tag{1.3}$$

and show that it provides an energy defined on a Sobolev space. We use standard notation for Sobolev and Lebesgue spaces. The letter c denotes a generic strictly positive constant.

Since each functional E_n is defined on a different space, the first step is to identify each $\mathcal{A}_n(0, L)$ with a subspace of a common space of functions defined on $(0, L)$. In order to identify each discrete function with a continuous counterpart, we extend u by $\tilde{u} : (0, L) \to \mathbb{R}$ as the piecewise-affine function defined by

$$\tilde{u}(s) = u_{i-1} + \frac{u_i - u_{i-1}}{\lambda_n}(s - x_{i-1}) \qquad \text{if } s \in (x_{i-1}, x_i). \tag{1.4}$$

In this case, $\mathcal{A}_n(0, L)$ is identified with those continuous $u \in W^{1,1}(0, L)$ (actually, in $W^{1,\infty}(0, L)$) such that u is affine on each interval (x_{i-1}, x_i). Note moreover that we have

$$\tilde{u}' = \frac{u_i - u_{i-1}}{\lambda_n} \tag{1.5}$$

on (x_{i-1}, x_i). If no confusion is possible, we will simply write u in place of $\tilde{u}$.

As we will treat limit functionals defined on Sobolev spaces, it is convenient to rewrite the dependence of the energy densities in (1.3) with respect to difference quotients rather than the differences $u_{i+j} - u_i$. We then write

$$E_n(u) = \sum_{j=1}^{K_n} \sum_{i=0}^{n-j} \lambda_n \psi_n^j\Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big), \tag{1.6}$$

where

$$\psi_n^j(z) = \frac{1}{\lambda_n} f_n^j(j\lambda_n z).$$

With the identification of u with $\tilde{u}$, E_n may be viewed as an integral functional defined on $W^{1,1}(0,L)$ by the equality

$$E_n(u) = F_n(\tilde{u}), \tag{1.7}$$

where

$$F_n(v) = \begin{cases} \displaystyle\sum_{j=1}^{K_n} \sum_{l=0}^{j-1} \frac{1}{j} \int_{l\lambda_n}^{L-(j-1-l)\lambda_n} \psi_n^j \Big(\frac{1}{j} \sum_{k=-l}^{j-1-l} v'(x + k\lambda_n) \Big) dx & \text{if } v \in \mathcal{A}_n(0,L) \\ +\infty & \text{otherwise.} \end{cases} \tag{1.8}$$

Note that in the particular case $K_n = 1$ we have (set $\psi_n = \psi_n^1$)

$$F_n(v) = \begin{cases} \displaystyle\int_0^L \psi_n(v')dx & \text{if } v \in \mathcal{A}_n(0,L) \\ +\infty & \text{otherwise.} \end{cases} \tag{1.9}$$

Definition 1.2 (Convergence of discrete functions and energies) With the identifications above we will say that u_n *converges to* u (respectively, in L^1, in measure, in $W^{1,1}$, etc.) if $\tilde{u}_n$ converge to u (respectively, in L^1, in measure, weakly in $W^{1,1}$, etc.), and we will say that E_n Γ*-converges to* F (respectively, with respect to the convergence in L^1, in measure, weakly in $W^{1,1}$, etc.) if F_n Γ-converges to F (respectively, with respect to the convergence in L^1, in measure, weakly in $W^{1,1}$, etc.); i.e., if for all u
(i) (*liminf inequality*) $F(u) \le \liminf_n F_n(u_n)$ for all u_n converging to u;
(ii) (*limsup inequality*) there exists u_n converging to u such that $\limsup_n F_n(u) \le F(u)$.

We recall that since our functionals will always be equi-coercive then Γ-convergence entails the desired convergence of minimum problems as described in the Introduction (see [7], [2], [4] for details).

We will often use the following simple Γ-convergence result (see e.g. [7], [5]).

Theorem 1.3 *Let ψ_n be locally equi-bounded convex functions and let $\psi = \lim_n \psi_n$. Then the functionals defined on $W^{1,1}(0,L)$ by*

$$\int_0^L \psi_n(u')\,dt$$

Γ-converge with respect to the weak convergence in $W^{1,1}(0,L)$ to the functional defined on $W^{1,1}(0,L)$ by

$$\int_0^L \psi(u')\,dt.$$

2 Convex energies

First we briefly recall the case when the energies ψ_n^j are convex. We will see that in the case of nearest neighbours, the limit is obtained by simply replacing sums by integrals, while in the case of long-range interactions a superposition principle holds.

2.1 Nearest-neighbour interactions: an identification principle

We start by considering the case $K = 1$, so that the functionals E_n are given by

$$E_n(u) = \sum_{i=0}^{n-1} \lambda_n \psi_n\Big(\frac{u_{i+1} - u_i}{\lambda_n}\Big). \tag{2.1}$$

The integral counterpart of E_n is given by

$$F_n(v) = \begin{cases} \displaystyle\int_0^L \psi_n(v')dx & \text{if } v \in \mathcal{A}_n(0,L) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.2}$$

The following result states that as n approaches ∞ the identification of E_n with its continuous analog is complete.

Theorem 2.1 *Let $\psi_n : \mathbb{R} \to [0, +\infty)$ be convex and locally equi-bounded. Let E_n be given by* (2.1)*and let $\psi = \lim_n \psi_n$ (note that it is not restrictive to suppose that such limit exists upon extraction a subsequence).*

(i) *The Γ-limit of E_n with respect to the weak convergence in $W^{1,1}(0, L)$ is given by F defined by*

$$F(u) = \int_{(0,L)} \psi(u')\, dx. \tag{2.3}$$

(ii) *If*

$$\lim_{|z|\to\infty} \frac{\psi(z)}{|z|} = +\infty \tag{2.4}$$

then the Γ-limit of E_n with respect to the convergence in $L^1(0, L)$ is given by F defined by

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\, dx & \textit{if } u \in W^{1,1}(0,L) \\ +\infty & \textit{otherwise} \end{cases} \tag{2.5}$$

on $L^1(0, L)$.

This theorem can be seen as a particular case of many results. However, since its proof is particularly simple we include it for the reader's convenience.

PROOF. (i) By Theorem 1.3 we have $\Gamma\text{-}\liminf_j F_j(u) \geq F(u)$. Conversely, fixed $u \in W^{1,\infty}(0,L)$ let $u_n \in \mathcal{A}_n(0,L)$ be such that $u_n(x_i^n) = u(x_i^n)$. By convexity we have

$$\int_{x_i^n}^{x_{i+1}^n} \psi(u')\,dt \geq \lambda_n \psi\Big(\frac{1}{\lambda_n}\int_{x_i^n}^{x_{i+1}^n} u'\,dt\Big) = \lambda_n \psi\Big(\frac{u(x_{i+1}^n) - u(x_i^n)}{\lambda_n}\Big);$$

hence, summing up, letting $n \to +\infty$ and using the pointwise convergence of ψ_n to ψ, we get

$$\int_0^L \psi(u')\,dt = \lim_n \int_0^L \psi_n(u')\,dt \geq \limsup_n E_n(u_n).$$

By a density argument we recover the same inequality on the whole $W^{1,1}(0,L)$.

(ii) If (2.4) holds then the sequence (E_n) is equi-coercive on bounded sets of $L^1(0,L)$ with respect to the weak convergence in $W^{1,1}(0,L)$, from which the thesis is easily deduced.

2.2 Long-range interactions: a superposition principle

Let now $K \in N$ be fixed. The energies E_n take the form

$$E_n(u) = \sum_{j=1}^{K}\sum_{i=0}^{n-j} \lambda_n \psi_n^j\Big(\frac{u_{i+j} - u_i}{j\lambda_n}\Big). \tag{2.6}$$

In this case E_n can be seen as the superposition of K nearest-neighbour functionals to which we can apply the result above. The theorem below is a particular case of [6].

Theorem 2.2 *Let $\psi_n^j : \mathbb{R} \to [0, +\infty)$ be convex and locally equi-bounded. Let E_n be given by* (2.6) *and for all j let $\psi^j = \lim_n \psi_n^j$ (note that it is not restrictive to suppose that such limit exists upon extraction a subsequence). Let ψ^1 satisfy*

$$\lim_{|z|\to\infty} \frac{\psi^1(z)}{|z|} = +\infty \tag{2.7}$$

then the Γ-limit of E_n with respect to the convergence in $L^1(0,L)$ is given by F defined by

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dx & \text{if } u \in W^{1,1}(0,L) \\ +\infty & \text{otherwise} \end{cases} \tag{2.8}$$

on $L^1(0,L)$, where

$$\psi = \sum_{j=1}^{K} \psi^j. \tag{2.9}$$

3 Non-convex energies

We now investigate the effects of the lack of convexity.

3.1 Nearest-neighbour interactions: a convexification principle

In the case $K = 1$ the only effect of the passage from the discrete setting to the continuum is a convexification of the integrand. Again the following result can be obtained as a particular case of [5].

Theorem 3.1 *Let $\psi_n : \mathbb{R} \to [0,+\infty)$ be locally equi-bounded Borel functions satisfying* (2.4)*, and suppose that $\psi = \lim_n \psi_n^{**}$. Let E_n be given by* (2.1)*; then the Γ-limit of E_n with respect to the convergence in $L^1(0,L)$ is given by F defined by*

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dx & \text{if } u \in W^{1,1}(0,L) \\ +\infty & \text{otherwise} \end{cases} \tag{3.1}$$

on $L^1(0,L)$. In particular if $\psi_n = \widehat{\psi}$ then $\psi = \widehat{\psi}^{**}$

3.2 Next-to-nearest neighbour interactions: non-convex relaxation

In the non-convex setting, the case $K = 2$ offers an interesting way of describing the two-level interactions between first and second neighbours. Such description is more difficult in the case $K \geq 3$. Essentially, the way the limit continuum theory is obtained is by first integrating-out the contribution due to nearest neighbours by means of an inf-convolution procedure and then by applying the previous results to the resulting functional.

Theorem 3.2 *Let $\psi_n^1, \psi_n^2 : \mathbb{R} \to [0,+\infty)$ be locally equi-bounded Borel functions such that*

$$\lim_{|z|\to\infty} \frac{\psi_n^1(z)}{|z|} = +\infty, \tag{3.2}$$

uniformly in n, and let $E_n(u) : \mathcal{A}_n(0,L) \to [0,+\infty)$ be given by

$$E_n(u) = \sum_{i=0}^{n-1} \lambda_n \psi_n^1\Big(\frac{u_{i+1}-u_i}{\lambda_n}\Big) + \sum_{i=0}^{n-2} \lambda_n \psi_n^2\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big) \tag{3.3}$$

Let $\tilde{\psi}_n : \mathbb{R} \to [0,+\infty)$ be defined by

$$\begin{aligned} \tilde{\psi}_n(z) &= \psi_n^2(z) + \frac{1}{2}\inf\{\psi_n^1(z_1) + \psi_n^1(z_2)) : z_1 + z_2 = 2z\} \\ &= \inf\Big\{\psi_n^2(z) + \frac{1}{2}(\psi_n^1(z_1) + \psi_n^1(z_2)) : z_1 + z_2 = 2z\Big\}, \end{aligned} \tag{3.4}$$

and suppose that

$$\psi = \lim_n \tilde{\psi}_n^{**}. \tag{3.5}$$

Then the Γ*-limit of* E_n *with respect to the convergence in* $L^1(0,L)$ *is given by* F *defined by*

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dx & \textit{if } u \in W^{1,1}(0,L) \\ +\infty & \textit{otherwise} \end{cases} \tag{3.6}$$

on $L^1(0,L)$.

Remark 3.3 (i) The growth conditions on ψ_n^2 can be weakened, by requiring that $\psi_n^2 : \mathbb{R} \to \mathbb{R}$ and

$$-c_1\psi_n^1 \le \psi_n^2 \le c_2(1+\psi_n^1)$$

provided that we still have

$$\lim_{|z|\to\infty} \frac{\psi(z)}{|z|} = +\infty$$

(see [8]).

(ii) If ψ_n^1 is convex then $\tilde{\psi}_n = \psi_n^1 + \psi_n^2$. If also ψ_n^2 is convex then we recover a particular case of Theorem 2.2.

Proof. Let $u \in \mathcal{A}_n(0,L)$. We have, regrouping the terms in the summation,

$$\begin{aligned} E_n(u) &= \sum_{\substack{i=0 \\ i \text{ even}}}^{n-2} \lambda_n \Big(\psi_n^2\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big) + \frac{1}{2}\psi_n^1\Big(\frac{u_{i+2}-u_{i+1}}{\lambda_n}\Big) + \frac{1}{2}\psi_n^1\Big(\frac{u_{i+2}-u_{i+1}}{\lambda_n}\Big)\Big) \\ &\quad + \sum_{\substack{i=0 \\ i \text{ odd}}}^{n-2} \lambda_n \Big(\psi_n^2\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big) + \frac{1}{2}\psi_n^1\Big(\frac{u_{i+2}-u_{i+1}}{\lambda_n}\Big) + \frac{1}{2}\psi_n^1\Big(\frac{u_{i+1}-u_i}{\lambda_n}\Big)\Big) \\ &\quad + \frac{\lambda_n}{2}\psi_n^1\Big(\frac{u_n-u_{n-1}}{\lambda_n}\Big) + \frac{1}{2}\psi_n^1\Big(\frac{u_1-u_0}{\lambda_n}\Big) \\ &\ge \frac{1}{2}\left(\sum_{\substack{i=0 \\ i \text{ even}}}^{n-2} 2\lambda_n\tilde{\psi}_n\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big) + \sum_{\substack{i=0 \\ i \text{ odd}}}^{n-2} 2\lambda_n\tilde{\psi}_n\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)\right) \\ &\ge \frac{1}{2}\left(\sum_{\substack{i=0 \\ i \text{ even}}}^{n-2} 2\lambda_n\tilde{\psi}_n^{**}\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big) + \sum_{\substack{i=0 \\ i \text{ odd}}}^{n-2} 2\lambda_n\tilde{\psi}_n^{**}\Big(\frac{u_{i+2}-u_i}{2\lambda_n}\Big)\right) \\ &= \frac{1}{2}\Big(\int_0^{2\lambda_n[n/2]} \tilde{\psi}_n^{**}(\tilde{u}_1')\,dt + \int_{\lambda_n}^{(1+2[n-1/2])\lambda_n} \tilde{\psi}_n^{**}(\tilde{u}_2')\,dt\Big), \end{aligned} \tag{3.7}$$

where $\tilde{u}_k$, respectively, with $k = 1, 2$, are the continuous piecewise-affine functions such that

$$\tilde{u}_k' = \frac{u_{i+2} - u_i}{2\lambda_n} \qquad \text{on } (x_i^n, x_{i+2}^n) \tag{3.8}$$

for i, respectively, even or odd.

Let now $u_n \to u$ in $L^1(0, L)$ and $\sup_n E_n(u_n) < +\infty$; then $u_n \rightharpoonup u$ in $W^{1,1}(0, L)$. Let $u_{k,n}$ be defined as in (3.8); we then deduce that $u_{k,n} \rightharpoonup u$ as $n \to +\infty$, for $k = 1, 2$. For every fixed $\eta > 0$ by (3.7) we obtain

$$\begin{aligned}\liminf_n E_n(u_n) &\geq \frac{1}{2}\left(\liminf_n \int_\eta^{L-\eta} \tilde{\psi}_n^{**}(u_{1,n}')\,dt + \liminf_n \int_\eta^{L-\eta} \tilde{\psi}_n^{**}(u_{2,n}')\,dt\right) \\ &\geq \int_\eta^{L-\eta} \psi(u')\,dt\end{aligned}$$

by Theorem 1.3, and the liminf inequality follows by the arbitrariness of $\eta > 0$.

Now we prove the limsup inequality. By an easy relaxation argument, it suffices to treat the case when $\tilde{\psi}_n$ is lower semicontinuous, $u(x) = zx$ and $\psi(z) = \lim_n \tilde{\psi}_n(z)$. With fixed $\eta > 0$ let z_1^n, z_2^n be such that $z_1^n + z_2^n = 2z$ and

$$\psi_n^2(z) + \frac{1}{2}(\psi_n^1(z_1^n) + \psi_n^2(z_2^n)) \leq \tilde{\psi}(z) + \eta$$

for all n sufficiently large. We define the recovery sequence u_n as

$$u_n(x_i^n) = \begin{cases} zx_i^n & \text{if } i \text{ is even} \\ z(i-1)\lambda_n + z_1^n\lambda_n & \text{if } i \text{ is odd.}\end{cases}$$

We then have

$$\begin{aligned}E_n(u_n) &= \sum_{i=0}^{n-1} \lambda_n \psi_n^1\Big(\frac{u_n(x_{i+1}^n) - u_n(x_i^n)}{\lambda_n}\Big) + \sum_{i=0}^{n-2} \lambda_n \psi_n^2\Big(\frac{u_n(x_{i+2}^n) - u_n(x_i^n)}{2\lambda_n}\Big) \\ &\leq \frac{L}{2}(\psi_n^1(z_1^n) + \psi_n^1(z_2^n)) + L\psi_n^2(z) \leq L\psi(z) + L\eta = F(u) + L\eta,\end{aligned}$$

and the limsup inequality follows by the arbitrariness of η.

Remark 3.4 (Multiple-scale effects) The formula defining ψ highlights a double-scale effect. The operation of inf-convolution highlights oscillations on the scale λ_n, while the convexification of $\tilde{\psi}$ acts at a much larger scale.

3.3 Long-range interactions: a 'clustering' principle

We consider now the case of a general $K \geq 1$. In this case the effective energy density will be given by a *homogenization* formula. As the statement of the general result (which is postponed to the next section) will be quite complex, we begin by treating the case of energy densities independent of n. We suppose for the sake of simplicity that $\psi^j : \mathbb{R} \to [0, +\infty)$ are lower semicontinuous and there exists $p > 1$ such that

$$\psi^1(z) \geq c_0(|z|^p - 1), \qquad \psi^j(z) \leq c_j(1 + |z|^p). \tag{3.9}$$

for all $j = 1, \dots, K$. Before stating the convergence result we define some energy densities.

Let $N \in \mathbb{N}$. We define $\psi_N : \mathbb{R} \to [0, +\infty)$ as follows:

$$\begin{aligned}\psi_N(z) \quad = \quad & \min\Big\{ \frac{1}{N} \sum_{j=1}^{K} \sum_{i=0}^{N-j} \psi^j \Big(\frac{u(i+j) - u(i)}{j} \Big) \\ & u : \{0, \dots, N\} \to \mathbb{R},\ u(i) = zi \text{ for } i \leq K \text{ or } i \geq N - K \Big\}.\end{aligned} \tag{3.10}$$

Proposition 3.5 *For all $z \in \mathbb{R}$ there exists the limit $\psi(z) = \lim_N \psi_N(z)$.*

Proof. With fixed $z \in \mathbb{R}$, let $N, M \in \mathbb{N}$ with $M > N$, and let u_N be a minimizer for $\psi_N(z)$. We define $u_M : \{0, \dots, M\} \to \mathbb{R}$ as follows:

$$u_M(i) = \begin{cases} u_N(i - lN) + lNz & \text{if } lN \leq i \leq (l+1)N \ (0 \leq l \leq \frac{M}{N} - 1) \\ zi & \text{otherwise.} \end{cases}$$

Then we can estimate

$$\begin{aligned}\psi_M(z) \quad & \leq \quad \frac{1}{M} \sum_{j=1}^{K} \sum_{i=0}^{M-j} \psi^j \Big(\frac{u_M(i+j) - u_M(i)}{j} \Big) \\ & \leq \quad \frac{1}{N} \sum_{j=1}^{K} \sum_{i=0}^{N-j} \psi^j \Big(\frac{u_N(i+j) - u_N(i)}{j} \Big) \\ & \quad + \frac{1}{N} \sum_{j=1}^{K} (2K - j) \psi^j(z) + \sum_{j=1}^{K} \frac{M - [M/N]N + K - j}{M} \psi^j(z) \\ & \leq \quad \psi_N(z) + \frac{2K}{N} \sum_{j=1}^{K} \psi_j(z) + \frac{N+K}{M} \sum_{j=1}^{K} \psi_j(z) \\ & \leq \quad \psi_N(z) + c\Big(\frac{2K}{N} + \frac{N+K}{M} \Big)(1 + |z|^p).\end{aligned} \tag{3.11}$$

Taking first the limsup in M and then the liminf in N we deduce that

$$\limsup_M \psi_M(z) \leq \liminf_N \psi_N(z)$$

as desired

Remark 3.6 (i) $c_0(|z|^p - 1) \le \psi^1(z) \le \psi(z) \le c(1 + |z|^p)$;
(ii) ψ is lower semicontinuous;
(iii) ψ is convex;
(iv) for all $N \in \mathbb{N}$ we have $\psi(z) \le \psi_N(z) + \frac{c}{N}(1 + |z|^p)$.

We can state the convergence theorem.

Theorem 3.7 *Let ψ^j be as above and let E_n be defined by* (2.6)*. Then the Γ-limit of E_n with respect to the convergence in $L^1(0,L)$ is given by F defined by*

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\,dx & \text{if } u \in W^{1,p}(0,L) \\ +\infty & \text{otherwise} \end{cases} \tag{3.12}$$

on $L^1(0,L)$, where ψ is given by Proposition 3.5.

Proof. We begin by establishing the liminf inequality. Let $u_n \to u$ in $L^1(0,L)$ be such that $\sup_n E_n(u_n) < +\infty$. Note that this implies that

$$\sup_n \int_0^L |u_n'|^p\,dt < +\infty,$$

so that indeed $u_n \rightharpoonup u$ weakly in $W^{1,p}(0,L)$ and hence also $u_n \to u$ in $L^\infty(0,L)$.

For all $k \in \{0, \dots, N-1\}$ let

$$\Phi_n(k) = \sum_{l \in \mathbb{N}} \int_{((k+Nl-2K)\lambda_n, (k+Nl+2K)\lambda_n) \cap (0,L)} |u_n'|^p\,dt.$$

We have

$$\sum_{k=0}^{N-1} \Phi_n(k) \le 2K \int_0^L |u_n'|^p\,dt \le c,$$

so that, upon choosing a subsequence if necessary, there exists k such that

$$\Phi_n(k) \le \frac{c}{N}.$$

For the sake of notational simplicity we will suppose that this holds with $k = 0$, and also that $n = MN$ with $M \in \mathbb{N}$, so that the inequality above reads

$$\sum_{l=0}^{M-1} \int_{((Nl-2K)\lambda_n, (Nl+2K)\lambda_n) \cap (0,L)} |u_n'|^p\,dt \le \frac{c}{N}. \tag{3.13}$$

We may always suppose so, upon first reasoning in slightly smaller intervals than $(0,L)$ and then let those intervals invade $(0,L)$.

Let v_n^N be the piecewise-affine function defined on $(0,L)$ such that

$$
\begin{aligned}
&v_n^N(0) = u_n(0)\\
&(v_n^N)' = u_n' \qquad \text{on } (x_i^n, x_{i+1}^n),\ \ Nl+K \le i \le N(l+1)-K-1\\
&(v_n^N)' = \frac{u_n((Nl+N-K)\lambda_n) - u_n((Nl+K)\lambda_n)}{(N-2K)\lambda_n} =: z_{n,l}^N\\
&\qquad\qquad \text{on } (Nl\lambda_n, (Nl+K)\lambda_n) \cup ((N(l+1)-K-1)\lambda_n, N(l+1)\lambda_n).
\end{aligned}
$$

The construction of v_n^N deserves some words of explanation. The function v_n^N is constructed on each interval $(Nl\lambda_n, (N+1)\lambda_n)$ as equal to the function u_n (up to an additive constant) in the middle interval $((Nl+K)\lambda_n, (N(l+1)-K)\lambda_n)$, and as the affine function of slope $z_{n,l}^N$ in the remaining two intervals. Note that the construction implies that the function

$$v_{n,l}^N : \{0,\dots,N\} \to \mathbb{R}$$

defined by

$$v_{n,l}^N(i) = \frac{1}{\lambda_n} v_n^N((lN+i)\lambda_n)$$

is a test function for the minimum problem defining $\psi_N(z_{n,l}^N)$, and that

$$
\begin{aligned}
&\sum_{j=1}^{K} \sum_{i=Nl}^{N(l+1)-j} \lambda_n \psi^j\Big(\frac{v_n^N(x_{i+j}^n) - v_n^N(x_i^n)}{j\lambda_n}\Big)\\
=\ & \sum_{j=1}^{K} \sum_{i=0}^{N-j} \lambda_n \psi^j\Big(\frac{v_{n,l}^N((i+j)) - v_{n,l}^N(i)}{j}\Big) \ge N\lambda_n \psi_N(z_{n,l}^N). \qquad (3.14)
\end{aligned}
$$

Moreover, note that, by Hölder's inequality, we have

$$\int_{(0,L)} |(v_n^N)' - u_n'|\,dt \le \Big(\frac{2K}{N}L\Big)^{1-1/p} \|u_n'\|_{L^p(0,L)} + \frac{2K}{N-2K}\|u_n'\|_{L^1(0,L)},$$

so that, since $u_n(0) = v_n^N(0)$ we have a uniform bound

$$\|v_n^N - u_n\|_{L^\infty(0,L)} \le \frac{c}{N}. \qquad (3.15)$$

We have that

$$
\begin{aligned}
E_n(u_n) \quad &\ge \quad \sum_{l=0}^{M-1} \sum_{j=1}^{K} \sum_{i=Nl+K}^{N(l+1)-K-j} \lambda_n \psi^j\Big(\frac{u_n(x_{i+j}^n) - u_n(x_i^n)}{j\lambda_n}\Big)\\
&= \quad \sum_{l=0}^{M-1} \sum_{j=1}^{K} \sum_{i=Nl+K}^{N(l+1)-K-j} \lambda_n \psi^j\Big(\frac{v_n^N(x_{i+j}^n) - v_n^N(x_i^n)}{j\lambda_n}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{l=0}^{M-1}\sum_{j=1}^{K}\sum_{i=Nl}^{N(l+1)-j} \lambda_n \psi^j\Big(\frac{v_n^N(x_{i+j}^n)-v_n^N(x_i^n)}{j\lambda_n}\Big) \\
&\qquad -\sum_{l=0}^{M-1}\sum_{j=1}^{K}\sum_{i=Nl}^{Nl+K} \lambda_n \psi^j\Big(\frac{v_n^N(x_{i+j}^n)-v_n^N(x_i^n)}{j\lambda_n}\Big) \\
&\qquad -\sum_{l=1}^{M}\sum_{j=1}^{K}\sum_{i=Nl-K-j}^{Nl-j} \lambda_n \psi^j\Big(\frac{v_n^N(x_{i+j}^n)-v_n^N(x_i^n)}{j\lambda_n}\Big) \\
&=: \sum_{l=0}^{M-1}\sum_{j=1}^{K}\sum_{i=Nl}^{N(l+1)-j} \lambda_n \psi^j\Big(\frac{v_n^N(x_{i+j}^n)-v_n^N(x_i^n)}{j\lambda_n}\Big) - I_n^1 - I_n^2 \\
&\geq \sum_{l=0}^{M-1}\sum_{j=1}^{K} N\lambda_n \psi_N(z_{n,l}^N) - I_n^1 - I_n^2, \qquad (3.16)
\end{aligned}
$$

the last estimate being given by (3.14).

We give an estimate of the term I_n^1; the term I_n^2 can be dealt with similarly. Let $i < Nl+K \leq i+j$; by the growth conditions on ψ^j and the convexity of $z \mapsto |z|^p$ we have

$$
\begin{aligned}
&\psi^j\Big(\frac{v_n^N(x_{i+j}^n)-v_n^N(x_i^n)}{j\lambda_n}\Big) \\
&\leq c\Big(1+\Big|\frac{v_n^N(x_{i+j}^n)-v_n^N(x_i^n)}{j\lambda_n}\Big|^p\Big) \\
&\leq c\Big(1+\frac{1}{j}\sum_{k=i}^{i+j-1}\Big|\frac{v_n^N(x_{k+1}^n)-v_n^N(x_k^n)}{\lambda_n}\Big|^p\Big) \\
&\leq c\Big(1+K|z_{n,l}^N|^p+\frac{1}{\lambda_n}\int_{((Nl-2K)\lambda_n,(Nl+2K)\lambda_n)\cap(0,L)}|u_n'|^p\,dt\Big)
\end{aligned}
$$

By (3.13) and the fact that $|z|^p \leq c(1+\psi_N(z))$, we deduce that

$$
\begin{aligned}
I_n^1 &\leq \sum_{l=0}^{M-1}\sum_{j=1}^{K}\sum_{i=Nl}^{Nl+K}\lambda_n c\Big(1+\psi_N(z_{n,l}^N)+\frac{1}{\lambda_n}\int_{((Nl-2K)\lambda_n,(Nl+2K)\lambda_n)}|u_n'|^p\,dt\Big) \\
&\leq \frac{c}{N}+\frac{c}{N}\sum_{l=0}^{M-1}N\lambda_n\psi_N(z_{n,l}^N). \qquad (3.17)
\end{aligned}
$$

Plugging this estimate and the analog for I_n^2 into (3.16) we get

$$
E_n(u_n) \geq \Big(1-\frac{c}{N}\Big)\sum_{l=0}^{M-1}N\lambda_n\psi_N(z_{n,l}^N)-\frac{c}{N}. \qquad (3.18)
$$

By Remark 3.6(iv) we have

$$\psi_N(z) \geq \psi(z) - \frac{c}{N}(1 + |z|^p) \geq \Big(1 - \frac{c}{N}\Big)\psi(z) - \frac{c}{N}.$$

From (3.18) we then have

$$E_n(u_n) \geq \Big(1 - \frac{c}{N}\Big) \sum_{l=0}^{M-1} N\lambda_n \psi(z^N_{n,l}) - \frac{c}{N} \tag{3.19}$$

Now, note that the piecewise-affine functions u^N_n defined by

$$u^N_n(0) = u_n(0) \qquad \text{and} \qquad (u^N_n)' = z^N_{n,l} \text{ on } (Nl\lambda_n, N(l+1)\lambda_n)$$

are weakly precompact in $W^{1,p}(0,L)$, so that we may suppose that $u^N_n \rightharpoonup u^N$. Then by Theorem 2.1 we have

$$\liminf_n \sum_{l=0}^{M-1} N\lambda_n \psi(z^N_{n,l}) = \liminf_n \int_0^L \psi((u^N_n)')\, dt \geq \int_0^L \psi((u^N)')\, dt, \tag{3.20}$$

so that

$$\liminf_n E_n(u_n) \geq \Big(1 - \frac{c}{N}\Big) \int_0^L \psi((u^N)')\, dt - \frac{c}{N} \tag{3.21}$$

By (3.15) and the uniform convergence of u_n to u we have

$$\|u^N - u\|_{L^\infty(0,L)} \leq \frac{c}{N}. \tag{3.22}$$

By letting $N \to +\infty$ we then obtain the thesis by the lower semicontinuity of $\int \psi(u')\, dt$.

To prove the limsup inequality it suffices to deal with the case $u(x) = zx$ since from this construction we easily obtain a recovery sequence for piecewise-affine functions and then reason by density. To exhibit a recovery sequence for such u it suffices to fix $N \in \mathbb{N}$, consider v^N a minimum point for the problem defining $\psi_N(z)$ and define

$$u_n(x^n_i) = v^N(i - Nl)\lambda_n + zNl\lambda_n \qquad \text{if } Nl \leq i \leq N(l+1).$$

We then have

$$\limsup_n E_n(u_n) \leq L\,\psi_N(z) + \frac{c}{N} \sum_{j=1}^{K} \psi^j(z),$$

and the thesis follows by the arbitrariness of N.

3.4 The general convergence theorem

By slightly modifying the proof of Theorem 3.7 we can easily state a general Γ-convergence result, allowing a dependence also on n for the energy densities.

Theorem 3.8 *Let $K \geq 1$. Let $\psi_n^j : \mathbb{R} \to [0, +\infty)$ be lower semicontinuous functions and let $p > 1$ exists such that*

$$\psi_n^1(z) \geq c_0(|z|^p - 1), \qquad \psi_n^j(z) \leq c_j(1 + |z|^p). \tag{3.23}$$

for all $j \in \{1, \dots, K\}$ and $n \in \mathbb{N}$. For all $N, n \in \mathbb{N}$ let $\psi_{N,n} : \mathbb{R} \to [0, +\infty)$ be defined by

$$\begin{aligned}\psi_{N,n}(z) \quad = \quad & \min\Big\{ \frac{1}{N} \sum_{j=1}^{K} \sum_{i=0}^{N-j} \psi_n^j \Big(\frac{u(i+j) - u(i)}{j} \Big) \\ & u : \{0, \dots, N\} \to \mathbb{R},\ u(i) = zi \text{ for } i \leq K \text{ or } i \geq N - K \Big\}\end{aligned} \tag{3.24}$$

Suppose that $\psi : \mathbb{R} \to [0, +\infty)$ exists such that

$$\psi(z) = \lim_N \lim_n \psi_{N,n}^{**}(z) \qquad \text{for all } z \in \mathbb{R} \tag{3.25}$$

(note that this is not restrictive upon passing to a subsequence of n and N). Let E_n be defined on $\mathcal{A}_n(0, L)$ by

$$E_n(u) = \sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n \psi_n^j \Big(\frac{u_{i+j} - u_i}{j \lambda_n} \Big). \tag{3.26}$$

Then the Γ-limit of E_n with respect to the convergence in $L^1(0, L)$ is given by F defined by

$$F(u) = \begin{cases} \displaystyle\int_{(0,L)} \psi(u')\, dx & \text{if } u \in W^{1,p}(0, L) \\ +\infty & \text{otherwise} \end{cases} \tag{3.27}$$

on $L^1(0, L)$.

Proof. Let $u_n \to u$ in $L^1(0, L)$. We can repeat the proof for the liminf inequality for Theorem 3.7, substituting ψ^j by ψ_n^j and ψ_N by $\psi_{N,n}$. We then deduce as in (3.20)–(3.21) that

$$\begin{aligned}\liminf_n E_n(u_n) \quad & \geq \quad \Big(1 - \frac{c}{N}\Big) \liminf_n \int_0^L \psi_{N,n}((u_n^N)')\, dt - \frac{c}{N} \\ & \geq \quad \Big(1 - \frac{c}{N}\Big) \int_0^L \psi_N((u^N)')\, dt - \frac{c}{N},\end{aligned}$$

where $\psi_N = \lim_n \psi^{**}_{N,n}$ and the thesis by letting $N \to +\infty$.

To prove the limsup inequality it suffices to deal with the case $u(x) = zx$ since from this construction we easily obtain a recovery sequence for piecewise-affine functions and then reason by density. To exhibit a recovery sequence for such u it suffices to fix $N \in \mathbb{N}$, consider $z_{1,n}, z_{2,n}$ and $\eta_n \in [0,1]$ such that

$$\psi^{**}_{N,n}(z) = \eta_n \psi_{N,n}(z_{1,n}) + (1-\eta_n)\psi_{N,n}(z_{2,n}), \qquad z = \eta_n z_{1,n} + (1-\eta_n) z_{2,n}.$$

Let $v^N_{1,n}$, $v^N_{2,n}$ be minimum points for the problem defining $\psi_{N,n}(z_{1,n})$, $\psi_{N,n}(z_{2,n})$, respectively. For the sake of simplicity assume that there exists m such that $mN\eta_n \in \mathbb{N}$ for all n. Define

$$u_n(x^n_i) = \begin{cases} v^N_{1,n}(i - Nl)\lambda_n + zmNl\lambda_n & \text{if } mNl \le i \le mNl + mN\eta_n \\[2ex] v^N_{2,n}(i - Nl - mN\eta_n)\lambda_n + zmNl + z_{1,n} mN\eta_n\lambda_n & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } mNl + mN\eta_n \le i \le mN(l+1). \end{cases}$$

By the growth conditions on ψ^j_n it is easily seen that $(z_{k,n})$ are equi bounded and that

$$\sup\{v^N_{k,n}(i) - z_{k,n} i : \ i \in \{0, \dots, N\}, n \in \mathbb{N}\} < +\infty,$$

so that u_n converges to zx uniformly. We then have

$$\limsup_n E_n(u_n) \quad \le \quad L \limsup_n \psi^{**}_{N,n}(z)$$

and the thesis follows by the arbitrariness of N.

3.5 Convergence of minimum problems

From Theorem 3.8 we immediately deduce the following theorem.

Theorem 3.9 *Let E_n and F be given by Theorem 3.8, let $f \in L^1(0,L)$ and $d > 0$. Then the minimum values*

$$m_n = \min\Big\{ E_n(u) + \int_0^L fu\, dt : \ u(0) = 0, \ u(L) = d \Big\} \tag{3.28}$$

converge to

$$m = \min\Big\{ F(u) + \int_0^L fu\, dt : \ u(0) = 0, \ u(L) = d \Big\}, \tag{3.29}$$

and from each sequence of minimizers of (3.28) we can extract a subsequence converging to a minimizer of (3.29).

PROOF. Since the sequence of functionals (E_n) is equi-coercive, it suffices to show that the boundary conditions do not change the form of the Γ-limit; i.e., that for all $u \in W^{1,p}(0,L)$ such that $u(0)=0$ and $u(L)=d$ and for all $\varepsilon>0$ there exists a sequence u_n such that $u_n(0)=0$, $u_n(L)=d$ and $\limsup_n E_n(u_n) \le F(u)+\varepsilon$.

Let $v_n \to u$ in $L^\infty(0,L)$ be such that $\lim_n E_n(v_n)=F(u)$. With fixed $\eta>0$ and $N\in\mathbb{N}$ let $K_n\in\mathbb{N}$ be such that

$$\lim_n K_n\lambda_n = \frac{\eta}{N}.$$

For all $l\in\{1,\dots,N\}$ let $\phi_n^{N,l}:[0,L]\to[0,1]$ be the piecewise-affine function defined by $\phi_n^{N,l}(0)=0$,

$$\phi_n^{N,l\,\prime} = \begin{cases} 1/(K_n\lambda_n) & \text{on } ((l-1)K_n\lambda_n, lK_n\lambda_n) \\ -1/(K_n\lambda_n) & \text{on } ((n-lK_n)\lambda_n, (n-lK_n+K_n)\lambda_n) \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$u_n^{N,l} = \phi_n^{N,l} v_n + (1-\phi_n^{N,l})u.$$

We have

$$\begin{aligned} E_n(u_n^{N,l}) \;\le\;& E_n(u_n) + c\Big(\int_0^{\eta+K\lambda_n}(1+|u'|^p)\,dt + \int_{L-\eta-K\lambda_n}^L (1+|u'|^p)\,dt\Big) \\ &+c\Big(\int_{((l-1)K_n-K)\lambda_n,(lK_n+K)\lambda_n)\cap(0,L)} |u_n'|^p\,dt \\ &+\int_{((n-lK_n-K)\lambda_n,(n-lK_n+K_n+K)\lambda_n)\cap(0,L)} |v_n'|^p\,dt \\ &+\int_0^L \frac{1}{(K_n\lambda_n)^p}|v_n-u|^p\Big) \\ \le\;& E_n(u_n) + c\Big(\int_0^{2\eta}(1+|u'|^p)\,dt + \int_{L-2\eta}^L (1+|u'|^p)\,dt\Big) \\ &+c\Big(\int_{(((l-2)\eta/N,((l+1)\eta/N)\cup(L-(l+1)\eta/N,L-(l-2)\eta/N))\cap(0,L)} |v_n'|^p\,dt\Big) \\ &+c\frac{N^p}{\eta^p}\|v_n-u\|^p_{L^\infty(0,L)} \end{aligned}$$

for n large enough. Since

$$\begin{aligned} &\sum_{l=1}^N \int_{((l-2)\eta/N,((l+1)\eta/N)\cup(L-(l+1)\eta/N,L-(l-2)\eta/N)\cap(0,L)} |u_n'|^p\,dt \\ \le\;& 2\int_0^L (1+|v_n'|^p)\,dt \le c, \end{aligned}$$

for all n there exists $l_n \in \{1, \dots, N\}$ such that

$$\begin{aligned} E_n(u_n^{N,l_n}) \quad &\leq \quad E_n(v_n) + c\Big(\int_0^{2\eta} (1 + |u'|^p)\, dt + \int_{L-2\eta}^{L} (1 + |u'|^p)\, dt\Big) \\ &\qquad + \frac{c}{N} + c\frac{N^p}{\eta^p} \|v_n - u\|^p_{L^\infty(0,L)} \end{aligned}$$

Setting $u_n = u_n^{N,l_n}$ we then have

$$\limsup_n E_n(u_n) \leq F(u) + c\Big(\int_0^{2\eta} (1 + |u'|^p)\, dt + \int_{L-2\eta}^{L} (1 + |u'|^p)\, dt\Big) + \frac{c}{N},$$

and the desired inequality by the arbitrariness of η and N.

4 An example: multiplicity of Γ-limits

The results presented in the previous section give a complete picture of the asymptotic behaviour of discrete systems under superlinear growth conditions. In the case of nearest-neighbour interaction the passage from discrete to continuous variational problems amounts essentially to a convexification of the energy densities. Note that in this case the Γ limit in the weak $W^{1,1}$ topology is the same as the relaxation (i.e., convexification) of the Γ-limit in the strong $W^{1,1}$ topology, so that the two topologies are in this sense equivalent. This is not true anymore if more interactions are present. In order to highlight the importance of the choice of the topology, in this section we give an example of a long-range interaction system with three different Γ-limits when computed in different topologies, and each limit cannot be deduced from those in the stronger topologies. In particular we give an example of a double-density limit.

4.1 Formulation

The energies of discrete functions we will deal with are the fourth-neighbour energies E_n on $\mathcal{A}_n(0, L)$ defined by

$$E_n(u) = \sum_{j=1}^{4} \sum_{i=0}^{n-j} \lambda_n \psi_n^j \left(\frac{u_{i+j} - u_i}{j\lambda_n} \right)$$

where

$$\begin{aligned} \psi_n^1(z) = \lambda_n^2 z^2 \qquad & \psi_n^2(z) = \psi^2(z) = \min\left\{(z-1)^2, (z+1)^2\right\} \\ \psi_n^3(z) = 0 \qquad & \psi_n^4(z) = \psi^4(z) = z^2. \end{aligned}$$

That is, reformulating the previous expression in a more explicit way:

$$E_n(u) = \sum_{i=0}^{n-1} \lambda_n (u_{i+1} - u_i)^2 + \sum_{i=0}^{n-2} \lambda_n \psi^2 \left(\frac{u_{i+2} - u_i}{2\lambda_n} \right) + \sum_{i=0}^{n-4} \lambda_n \left(\frac{u_{i+4} - u_i}{4\lambda_n} \right)^2.$$

We will denote $\mathcal{A}^\infty(0,L)$ the set of sequences $\{u_n\}_{n\in\mathbb{N}}$ of discrete functions such that $u_n \in \mathcal{A}_n(0,L)$ for all $n \in \mathbb{N}$.

If one wishes to analyse the Γ-convergence of E_n with respect to the strong convergence in $L^2(0,L)$ the need of a new strategy occurs. In fact a $L^2(0,L)$-bounded sequence $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{A}^\infty(0,L)$ such that $E_n(u_n)$ is equi-bounded need not be precompact in $L^2(0,L)$. Roughly speaking this happens because we do not get a uniform bound in the $W^{1,2}(0,L)$ norm of $\{u_n\}_{n\in\mathbb{N}}$, as it would have happened if we had an estimate of the form $\psi_n^1(z) \geq cz^2 - d$ as in the previous sections, for some real constants c, d independent of n, $c > 0$. However, such an estimate is available for $\psi_n^2 = \psi^2$, and is thus possible to argue that the functions $u_{n,e} \in \mathcal{A}_{[n/2]}(0,L), u_{n,o} \in \mathcal{A}_{[(n-1)/2]}(0,L)$ (e for even, o for odd) defined by

$$u_{n,e}(i\lambda_{[n/2]}) = u_n(2i\lambda_n), \quad u_{n,o}(i\lambda_{[(n-1)/2]}) = u_n((2i+1)\lambda_n)$$

form two precompact subsequences in $L^2(0,L)$. This makes it reasonable to identify a discrete function u_n in $\mathcal{A}_n(0,L)$ (also) with the pair $(\tilde{u}_{n,e}, \tilde{u}_{n,o}) \in L^2(0,L) \times L^2(0,L)$ and to extend E_n on all $L^2(0,L) \times L^2(0,L)$ by assigning the value $+\infty$ outside $\{(\tilde{u}_{n,e}, \tilde{u}_{n,o}) | u \in \mathcal{A}_n(0,L)\}$. Moreover we say that $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{A}^\infty(0,L)$ converges (in $L^2(0,L)$) to (u^e, u^o) as $n \to +\infty$ if so does $\{(\tilde{u}_{n,e}, \tilde{u}_{n,o})\}_{n\in\mathbb{N}}$. We are now able to state the main result of the present section.

Theorem 4.1 (Γ1) *The Γ-limit of E_n with respect to the strong convergence in $W^{1,2}(0,L)$ is given by $F^{(1)}$, defined by*

$$F^{(1)}(u) = \int_{(0,L)} \psi^2(u')\,dt + \int_{(0,L)} (u')^2\,dt, \quad u \in W^{1,2}(0,L).$$

(Γ2) *The Γ-limit of E_n with respect to the weak convergence in $W^{1,2}(0,L)$ is given by $F^{(2)}$, defined by*

$$F^{(2)}(u) = \int_{(0,L)} \psi^{**}(u')\,dt, \quad u \in W^{1,2}(0,L)$$

where ψ^{**} is the convexification of

$$\psi(z) = z^2 + \min\left\{(z-1)^2, z^2, (z+1)^2\right\}.$$

(Γ3) *The Γ-limit of E_n on $L^2(0,L) \times L^2(0,L)$ with respect to the convergence of u_n to (u^e, u^o) defined above is given by $F^{(3)}$ defined by*

$$F^{(3)}(u^e, u^o) = \begin{cases} \displaystyle\int_{(0,L)} |u^e - u^o|^2\,dt + \int_{(0,L)} \psi^{**}((u^e)')\,dt + \int_{(0,L)} \psi^{**}((u^o)')\,dt & \\ \qquad\qquad \textit{if } u^e, u^o \in W^{1,2}(0,L) & \\ +\infty \qquad\qquad \textit{otherwise in } L^2(0,L) \times L^2(0,L). & \end{cases}$$

Before proving Theorem 4.1 we introduce some additional notation: for each u belonging to $\mathcal{A}_n(0,L)$ and for each $j \in \{1,2,3,4\}$ and $k \in \{0,\ldots,j-1\}$ let $u^{j,k} : (0,L) \to \mathbb{R}$ be defined by

$$
\begin{array}{rcll}
u^{j,k}(k\lambda_n) & = & u_k & \\
(u^{j,k})' & = & \dfrac{u_{jl+k+j} - u_{jl+k}}{j\lambda_n} & \text{in } \lambda_n(jl+k, jl+k+j) \text{ if} \\
 & & & \quad \lambda_n(jl+k, jl+k+j) \subseteq (0,L) \\
(u^{j,k})' & = & 0 & \text{otherwise in } (0,L).
\end{array}
$$

In particular let us remark that $u^{1,0} = \tilde{u}$. Accordingly to this notation:

$$
E_n(u) = \underbrace{\sum_{i=0}^{n-1} \lambda_n (u_{i+1} - u_i)^2}_{=:H_n(u^{2,0},u^{2,1})} + \sum_{i=0}^{1} \int_{(0,L)} \psi^2((u^{2,i})')\, dt + \sum_{i=0}^{3} \int_{(0,L)} ((u^{4,i})')^2 dt.
$$

Lemma 4.2 *Let $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{A}^\infty(0,L)$. For each $j \in \{1,2,3,4\}$ and $k \in \{0,\ldots,j-1\}$ we have*

$$
u_n \rightharpoonup u \quad in \quad W^{1,2}(0,L) \Longrightarrow u_n^{j,k} \rightharpoonup u \quad in \quad W^{1,2}(0,L) \tag{4.1}
$$

$$
u_n \to u \quad in \quad W^{1,2}(0,L) \Longrightarrow u_n^{j,k} \to u \quad in \quad W^{1,2}(0,L). \tag{4.2}
$$

Proof. First, remark that by the definitions just given and the convexity of the squaring function:

$$
\int_{(0,L)} ((u_n^{j,k})')^2 dt \le \int_{(0,L)} (u_n')^2 dt. \tag{4.3}
$$

The implication (4.1) follows by a simple compactness argument. To obtain (4.2) we use (4.1) and show that in addition

$$
\|(u_n^{j,k})'\|_{L^2(0,L)} \to \|u'\|_{L^2(0,L)}.
$$

Indeed

$$
\begin{array}{rclcl}
\int_{(0,L)} (u')^2 dt & \le & \liminf_n \int_{(0,L)} ((u_n^{j,k})')^2 dt & \le & \limsup_n \int_{(0,L)} ((u_n^{j,k})')^2 dt \\
 & \le & \limsup_n \int_{(0,L)} (u_n')^2 dt & = & \int_{(0,L)} (u')^2 dt
\end{array}
$$

where the first inequality is obtained by $L^2(0,L)$-weakly lower semicontinuity of $u \mapsto \int_{(0,L)} u^2$ and the last one by (4.3).

Lemma 4.3 *Let $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{A}^\infty(0,L)$ and suppose, for $k = 0,1$,*

$$
u_n^{2,k} \rightharpoonup u^k \quad in \quad W^{1,2}(0,L).
$$

Then

$$
\lim_n H_n(u_n^{2,0}, u_n^{2,1}) = \int_{(0,L)} |u^0 - u^1|^2 dt.
$$

In particular, as a direct consequence of (4.1)*, if* $\{u_n\}_n$ *is weakly convergent in* $W^{1,2}(0,L)$ *then*

$$\lim_n H_n(u_n^{2,0}, u_n^{2,1}) = 0.$$

The lemma follows by the uniform convergence of $u_n^{2,k}$ to u^k for $k = 0, 1$. From the lemma above we deduce:

Corollary 4.4 *Denote, for each* $n \in \mathbb{N}$ *and* $u \in \mathcal{A}_n(0,L)$*:*

$$\tilde{E}_n(u) = \sum_{i=0}^{n-2} \lambda_n \psi^2 \left(\frac{u_{i+2} - u_i}{2\lambda_n} \right) + \sum_{i=0}^{n-4} \lambda_n \left(\frac{u_{i+4} - u_i}{4\lambda_n} \right)^2.$$

Then the following equalities hold:

$$\begin{aligned}
\Gamma(s\text{-}W^{1,2}(0,L))\text{-}\liminf_n E_n &= \Gamma(s\text{-}W^{1,2}(0,L))\text{-}\liminf_n \tilde{E}_n \\
\Gamma(w\text{-}W^{1,2}(0,L))\text{-}\liminf_n E_n &= \Gamma(w\text{-}W^{1,2}(0,L))\text{-}\liminf_n \tilde{E}_n
\end{aligned}$$

$$\begin{aligned}
&\Gamma(s\text{-}L^2(0,L) \times L^2(0,L))\text{-}\liminf_n E_n(u^e, u^o) = \\
&= \int_{(0,L)} |u^e - u^o|^2 \, dt + \Gamma(s\text{-}L^2(0,L) \times L^2(0,L))\text{-}\liminf_n \tilde{E}_n(u^e, u^o)
\end{aligned}$$

and analogous equalities hold for the Γ*-limsup.*

Note that to obtain the equalities relative to ($\Gamma 3$) one makes use of the coercivity given by

$$\psi^2(z) \geq \frac{1}{2} z^2 - 1.$$

Indeed: let $u_n \to (u^e, u^o)$ in the convergence defined in Section 4.1. Let moreover be $\liminf E_n(u_n) < +\infty$ (equivalent to: $\liminf \tilde{E}_n(u_n) < +\infty$). By the compactness reasoning presented to motivate such definition, we have:

$$u_n^{2,0} \rightharpoonup u^e, \qquad u_n^{2,1} \rightharpoonup u^o \ \text{ in } \ W^{1,2}(0,L)$$

and it now suffies to apply Lemma 4.3.

We give a last auxiliary result, whose proof is just an arithmetic matter:

Lemma 4.5 *The function* ψ *defined in the statement of Theorem* 4.1 *satisfies*

$$\psi(z) = z^2 + \frac{1}{2} \min\{\psi^2(z_1) + \psi^2(z_2) : z_1 + z_2 = 2z\}.$$

Proof of Theorem 4.1 Let us start with the proof of ($\Gamma 1$). It can basically be deduced from (4.2) and Corollary 4.3. Indeed for all sequences $u_n \to u$ in $W^{1,2}(0,L)$ we have, thanks to the strong-$W^{1,2}(0,L)$ continuity of $u \mapsto \int_{(0,L)} \psi^i(u') \, dt$, $i = 2, 4$,

$$\tilde{E}_n(u_n) \to 2 \cdot \frac{1}{2} \int_{(0,L)} \psi^2(u') \, dt + 4 \cdot \frac{1}{4} \int_{(0,L)} (u')^2 dt.$$

To prove the Γ-liminf inequalities relative to (Γ2) and (Γ3), first regroup the terms in the expression of the $\tilde{E}_n$ in order to cause an outing of ψ^{**} (note that: $2\cdot\dfrac{u_{4(l+1)+k}-u_{4l+k}}{4\lambda_n}=\dfrac{u_{4(l+1)+k}-u_{4l+k+2}}{2\lambda_n}+\dfrac{u_{4l+k+2}-u_{4l+k}}{2\lambda_n}$):

$$\begin{aligned}
\tilde{E}_n(u) &= \sum_{k=0}^{3}\sum_{l=0}^{[(n-k)/4]-1}\lambda_n\left[\left(\frac{u_{4(l+1)+k}-u_{4l+k}}{4\lambda_n}\right)^2\right.\\
&\quad\left.+\frac{1}{2}\psi^2\left(\frac{u_{4(l+1)+k}-u_{4l+k+2}}{2\lambda_n}\right)+\frac{1}{2}\psi^2\left(\frac{u_{4l+k+2}-u_{4l+k}}{2\lambda_n}\right)\right]\\
&\quad+\frac{\lambda_n}{2}\left[\psi^2\left(\frac{u_n-u_{n-2}}{2\lambda_n}\right)+\psi^2\left(\frac{u_{n-1}-u_{n-3}}{2\lambda_n}\right)\right.\\
&\quad\left.+\psi^2\left(\frac{u_3-u_1}{2\lambda_n}\right)+\psi^2\left(\frac{u_2-u_0}{2\lambda_n}\right)\right]\\
&\geq \frac{1}{4}\sum_{k=0}^{3}\sum_{l=0}^{[(n-k)/4]-1}4\lambda_n\psi\left(\frac{u_{4(l+1)+k}-u_{4l+k}}{4\lambda_n}\right)\\
&\geq \frac{1}{4}\sum_{k=0}^{3}\int_{k\lambda_n}^{\lambda_n(4[(n-k)/4]+k)}\psi^{**}((u^{4,k})')\,dt.
\end{aligned}$$

Let $\{u_n\}_{n\in\mathbb{N}}\in\mathcal{A}^\infty(0,L)$ converge to (u^e,u^o). We already saw that this implies:

$$u_n^{2,0}\rightharpoonup u^e,\ u_n^{2,1}\rightharpoonup u^o\qquad\text{in}\ \ W^{1,2}(0,L).$$

Moreover, by the same arguments used to prove Lemma 4.2 we get:

$$u_n^{4,k}\rightharpoonup u^e\ \ \text{in}\ \ W^{1,2}(0,L)\quad\text{if } k \text{ is even}$$

$$u_n^{4,k}\rightharpoonup u^o\ \ \text{in}\ \ W^{1,2}(0,L)\quad\text{if } k \text{ is odd}$$

and thus, for all $\eta>0$,

$$\begin{aligned}
\liminf_n\tilde{E}_n(u_n) &\geq \frac{1}{4}\sum_{k=0}^{3}\liminf_n\int_\eta^{L-\eta}\psi^{**}((u_n^{4,k})')\,dt\\
&\geq \frac{1}{2}\left(\int_\eta^{L-\eta}\psi^{**}((u^e)')\,dt+\int_\eta^{L-\eta}\psi^{**}((u^o)')\,dt\right).
\end{aligned}$$

Then, by the arbitrariness of η:

$$\liminf_n\tilde{E}_n(u_n)\geq\frac{1}{2}\left(\int_{(0,L)}\psi^{**}((u^e)')dt+\int_{(0,L)}\psi^{**}((u^o)')\,dt\right).$$

This completes the proof of the Γ-liminf inequalities in (Γ2) and (Γ3).

Let us now prove the Γ-limsup inequalities. To help the comprehension we first consider the ($\Gamma 2$) case alone. Fix $u \in W^{1,2}(0,L)$. By the convexity of $F^{(2)}$ it is possible to find a sequence $\{u_j\}_{j\in\mathbb{N}} \subset W^{1,2}(0,L)$ of piecewise-affine functions such that

$$u_j \rightharpoonup u \text{ in } W^{1,2}(0,L), \quad F^{(2)}(u_j) \to F^{(2)}(u).$$

In this way we reduce to prove the Γ-limsup inequality when u is piecewise-affine, i.e. the continuous collation of a finite number of affine functions on subintervals of $(0,L)$. By the finiteness of such collation, we easily further reduce to the case of u in the form $u(x) = zx$, $z \in \mathbb{R}$. Take $\zeta_1, \zeta_2 \in \mathbb{R}$ and $t \in [0,1]$ such that

$$z = t\zeta_1 + (1-t)\zeta_2, \quad \psi^{**}(\zeta_1) = \psi(\zeta_1), \quad \psi^{**}(\zeta_2) = \psi(\zeta_2)$$

and

$$\psi^{**}(z) = t\psi(\zeta_1) + (1-t)\psi(\zeta_2).$$

Then there exists $\{u_j\}_{j\in\mathbb{N}}$ weakly convergent to u such that $F^{(2)}(u) = \lim_j F^{(2)}(u_j)$ and u_j' takes value in $\{\zeta_1, \zeta_2\}$ for all $j \in \mathbb{N}$.

Thus, without loss of generality, $\psi^{**}(z) = \psi(z)$. Let z_1, z_2 be such that $z_1 + z_2 = 2z$ and $\psi(z) = z^2 + \frac{1}{2}\psi^2(z_1) + \frac{1}{2}\psi^2(z_2)$. Define, for all $n \in \mathbb{N}$, $u_n \in \mathcal{A}_n(0,L)$ by:

$$\begin{aligned} u_{n,4l} &= 4l\lambda_n z & \qquad u_{n,4l+1} &= (4l+1)\lambda_n z \\ u_{n,4l+2} &= 4l\lambda_n z + 2\lambda_n z_1 & \qquad u_{n,4l+3} &= (4l+1)\lambda_n z + 2\lambda_n z_1 \end{aligned}$$

Remark that such definition actually makes $\{u_n\}_{n\in\mathbb{N}}$ an approximating sequence for u. This choice of u_n yields:

$$\left(\frac{u_{n,i+4} - u_{n,i}}{4\lambda_n}\right)^2 + \frac{1}{2}\psi^2\left(\frac{u_{n,i+2} - u_{n,i}}{2\lambda_n}\right) + \frac{1}{2}\psi^2\left(\frac{u_{n,i+4} - u_{n,i+2}}{2\lambda_n}\right) = \psi(z)$$

for each $i = 0, \ldots, n-4$. This completes the proof of ($\Gamma 2$), since it implies (compare the expanded expression of $\tilde{E}_n$ above):

$$\lim_n \tilde{E}_n(u_n) = \int_{(0,L)} \psi^{**}(u')\, dt.$$

In order to prove ($\Gamma 3$) we reduce in an analogous way to seek $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{A}^\infty(0,L)$ such that $u_n^{2,0} \to u^e$, $u_n^{2,1} \to u^o$ in $L^2(0,L)$ and

$$\tilde{E}_n(u_n) \to \int_{(0,L)} \psi^{**}((u^e)')\, dt + \int_{(0,L)} \psi^{**}((u^o)')\, dt$$

where $u^e(x) = z^e x$, $u^o(x) = z^o x, \psi^{**}(z^e) = \psi(z^e), \psi^{**}(z^o) = \psi(z^o)$. A $\{u_n\}_{n\in\mathbb{N}}$ with this properties is obtained by choosing $z_1^e, z_2^e, z_1^o, z_2^o$ which satisfy

$$\begin{aligned} z_1^e + z_2^e &= 2z^e, & \qquad \psi(z^e) &= (z^e)^2 + \frac{1}{2}\psi^2(z_1^e) + \frac{1}{2}\psi^2(z_2^e), \\ z_1^o + z_2^o &= 2z^o, & \qquad \psi(z^o) &= (z^o)^2 + \frac{1}{2}\psi^2(z_1^o) + \frac{1}{2}\psi^2(z_2^o), \end{aligned}$$

and defining each $u_n \in \mathcal{A}_n(0,L)$ by:

$$\begin{aligned} u_{n,4l} &= 4l\lambda_n z^e & u_{n,4l+1} &= (4l+1)\lambda_n z^o \\ u_{n,4l+2} &= 4l\lambda_n z^e + 2\lambda_n z_1^e & u_{n,4l+3} &= (4l+1)\lambda_n z^o + 2\lambda_n z_1^o. \end{aligned}$$

Remark 4.6 All the arguments we used in this note apply to any $\psi^2, \psi^4 : \mathbb{R} \to [0,+\infty)$ provided:

$$c_0 z^2 - c_{-1} \leq \psi^2(z) \leq c_2 z^2$$
$$\psi^4(z) \leq c_4 z^2$$

for suitable $c_{-1}, c_0, c_2, c_4 > 0$.

References

[1] A. Braides, Non-local variational limits of discrete systems, *Comm. Contemporary Math.* **2** (2000), 285–297.

[2] A. Braides, *Γ-convergence for Beginners.* Oxford University Press, Oxford, to appear.

[3] A. Braides, G. Dal Maso and A. Garroni, Variational formulation of softening phenomena in fracture mechanics: the one-dimensional case, *Arch. Rational Mech. Anal.* **146** (1999), 23-58.

[4] A. Braides and A. Defranceschi, *Homogenization of Multiple Integrals*, Oxford University Press, Oxford, 1998.

[5] A. Braides and M.S. Gelli, Continuum limits of discrete systems without convexity hypotheses, *Math. Mech. Solids*, to appear.

[6] A. Braides and M.S. Gelli, Limits of discrete systems with long-range interactions, to appear.

[7] G. Dal Maso, *An Introduction to Γ-convergence*, Birkhäuser, Boston, 1993.

[8] S. Pagano and R. Paroni, A simple model for phase transitions: from the discrete to the continuum problem, Preprint, Oxford, 2000.

[9] G. Puglisi and L. Truskinovsky, Mechanics of a discrete chain with bi-stable elements. *J. Mech. Phys. Solids* **48**(2000), 1-27.

[10] L. Truskinovsky, Fracture as a phase transition, *Contemporary research in the mechanics and mathematics of materials* (R.C. Batra and M.F. Beatty eds.) CIMNE, Barcelona, 1996, 322–332.