QUASISTATIC EVOLUTION IN NON-ASSOCIATIVE PLASTICITY – THE CAP MODEL

JEAN-FRANÇOIS BABADJIAN, GILLES A. FRANCFORT, AND MARIA GIOVANNA MORA

Abstract. Non-associative elasto-plasticity is the working model of plasticity for soil and rock mechanics. Yet, it is usually viewed as non-variational. In this work, we prove *a contrario* the existence of a variational evolution for such a model under a natural capping assumption on the hydrostatic stresses and a less natural mollification of the stress admissibility constraint. The obtained elasto-plastic evolution is expressed for times that are conveniently rescaled.

Contents

In spite of its mechanical success, small strain elasto-plasticity has been relatively seldom broached in the applied mathematics literature, at least as far as elasto-plastic evolution is concerned. The reader is referred to the work of P.-M. Suquet [33] for the first "modern" treatment of that evolution; see also the treatise [34]. After those works, small strain elasto-plastic evolution was not systematically revisited until [6] which reformulated the problem within the framework of the new variational theory of rate independent evolutions.

Traditionally, small strain elasto-plasticity is formally modeled as follows for a homogeneous elasto-plastic material occupying a volume $\Omega \subset \mathbb{R}^n$, with Hooke's law (elasticity tensor) A. Assume

Date: October 7, 2011.
Key words and phrases. Elasto-plasticity, Functions of bounded deformation, Calculus of variations, Quasistatic evolution, Radon measures, Duality, Convex analysis.

that the body is subjected to a time-dependent loading process with, say, $f(t)$ as body loads, $g(t)$ as surface loads on a part Γ_s of $\partial\Omega$, and $w(t)$ as displacement loads (hard device) on the complementary part Γ_d of $\partial\Omega$. Let $Eu(t)$ denote the infinitesimal strain at t, that is, the symmetric part of the spatial gradient of the displacement field $u(t)$ at t. Let $\sigma(t)$ be the Cauchy stress tensor at time t, and let $e(t)$ and $p(t)$ (a deviatoric symmetric matrix) be the elastic and plastic strain at t. The classical formulation of the quasistatic evolution problem in a time interval $[0, T]$ for the functions $u(t)$, $e(t)$, $p(t)$, $\sigma(t)$ reads formally as

- Kinematic compatibility: $Eu(t) = e(t) + p(t)$ in Ω and $u(t) = w(t)$ on Γ_d;
- Equilibrium: $\text{div}\sigma(t) + f(t) = 0$ in Ω and $\sigma(t)\nu = g(t)$ on Γ_s, where ν denotes the outer unit normal to $\partial\Omega$;
- Constitutive law: $\sigma(t) = Ae(t)$ in Ω;
- Stress constraint: $\sigma_D(t) \in K$, where σ_D is the deviatoric part of the Cauchy stress σ, and K is the admissible set of stresses (a convex and compact subset of deviatoric $n \times n$ matrices);
- Flow rule: $\dot{p}(t) = 0$ if $\sigma_D(t) \in \text{int } K$, while $\dot{p}(t)$ belongs to the normal cone to K at $\sigma_D(t)$ if $\sigma_D(t) \in \partial K$.

The corresponding variational evolution, as discussed in [6], formally consists in the following four-pronged formulation, for $t \in [0, T]$,

- Kinematic compatibility: $Eu(t) = e(t) + p(t)$ in Ω and $u(t) = w(t)$ on Γ_d;
- Global stability: The triplet $(u(t), e(t), p(t))$ globally minimizes
$$\frac{1}{2}\int_\Omega A\eta : \eta \, dx + \int_\Omega H(q - p(t)) \, dx$$
among all admissible triplets (v, η, q), where $H(p)$ is the support function of K, *i.e.*, $H(p) := \sup\{\sigma_D : p : \sigma_D \in K\}$;
- Constitutive law: $\sigma(t) = Ae(t)$ in Ω;
- Energy balance:
$$\frac{d\mathcal{E}}{dt}(t) = \int_{\Gamma_d} (\sigma(t)\nu) \cdot \dot{w}(t) \, d\mathcal{H}^{n-1} - \int_\Omega \dot{f}(t) \cdot u(t) \, dx - \int_{\Gamma_s} \dot{g}(t) \cdot u(t) \, d\mathcal{H}^{n-1},$$
where
$$\mathcal{E}(t) := \frac{1}{2}\int_\Omega Ae(t) : e(t) \, dx + \int_0^t \int_\Omega H(\dot{p}(s)) \, dx \, ds - \int_\Omega f(t) \cdot u(t) \, dx - \int_{\Gamma_s} g(t) \cdot u(t) \, d\mathcal{H}^{n-1}.$$

The existence of a variational evolution and the extent to which that evolution is "equivalent" to the original formulation is the main focus of [6]. This has been further extended to various elasto-plastic settings in subsequent works [7, 8]. The most relevant from our standpoint is the investigation of Cam-Clay elasto-plastic evolutions performed in [9], and in its sequel [10]. In those papers a model of elasto-plastic clay which exhibits hardening and/or softening is being studied. The most striking feature of that model is that it exhibits a certain degree of non-associativity, that is, that the time derivative of the pair $(p(t), \zeta(t))$ – $\zeta(t)$ denotes the additional internal variable that describes the hardening/softening behavior – does not follow a standard flow rule. That model is usually referred to as one with a non-associative hardening rule.

This is but a specific instance of a generic response for both soils and rocks. Indeed, although definitely exhibiting elasto-plastic behavior, those do not fall within the traditional purview of elasto-plasticity because, on the one hand, granularity or defaults drive volume changes under hydrostatic pressure (see, *e.g.*, [4] where a mathematical model of plastic dilatancy is discussed within the framework of associative plasticity), and, on the other hand, their dilatancy is much smaller than that predicted by an associative model (one for which the flow rule applies); see, *e.g.*, [25, Section 6.1]. This could be due to the impact of kinematic hardening and/or softening as in

Cam-Clay, but, more commonly especially in soils, it is a byproduct of inter-granular friction, and this without any kind of hardening/softening phenomenon.

In any case, non-associative elasto-plasticity is a widely used model in both soil and rock mechanics and it has long been thought not to be tractable from a variational standpoint (even as a static problem).

In this paper, we show that this view is "misguided" and develop a variational formulation for a rather generic model of non-associative elasto-plasticity. The starting point of our analysis is a remarkable observation by P. Laborde [22, 23] which has been – rather auspiciously in our view – completely ignored up to now. That observation is briefly described in Formulation 1.1 in the next section. It imparts a variational structure upon non-associative elasto-plasticity and serves as a fundational building block for our analysis.

Classical rock and soil models allow for infinitely large hydrostatic compressions. This is obviously mechanically unsound because it gives rise to unbounded sets of admissible stresses. Indeed, for geomaterials like porous rocks, pressure-dependent models, such as Drucker-Prager or Mohr-Coulomb models, overestimate the yield stress, and inadequately predict the volumetric strain response. Following an original idea of [17], a cap model was proposed in [16]. It consists in closing the yield surface with a cap that crosses the hydrostatic stress axis. The possible hydrostatic tractions are thereby bounded and this gives rise to what is referred to as a non-associative cap model (see, *e.g.*, [28, 29, 24, 13]). We adopt this restriction in the present work.

The obtained variational formulation cannot proceed, however, without a further "artificial" assumption. Because Laborde's formulation results in a set of admissible stresses that depends on the actual stress (see (1.3)), it becomes impossible to carry the mathematical analysis through without assuming a spatially continuous stress. But, at present, there are no stress regularity results that extend to the closure of the whole domain, even in the simplest case of classical elasto-plasticity. To our knowledge, the only result in that direction is [2] (revisited in [15]) where the H^1_{loc}-regularity of the stress is established. Our only way out seems to be a mollification of the dependence of the set of admissible stresses upon the actual stress. That very same issue was also a stumbling block in [9] and resulted in a similar use of a mollification of the dependence of the set of admissible stresses upon the actual variable $\zeta(t)$.

We make every effort to postpone the mollifying process for as long as feasible and produce a non-associative visco-plastic evolution for the cap model which is not mollified in Section 2. But we have to surrender when letting the viscosity become vanishingly small in Section 3.

It would seem plausible to address the evolution through a time-incremental process because that method has proved successful in the handling of classical elasto-plasticity in [6], as well as in many other investigations of rate independent evolutions. Unfortunately, that approach soon grinds to a halt because a lack of Lipschitz estimate in time on the plastic strain seems to impede the proof of the lower inequality in the energy balance. The idea, borrowed from [26, 27], but whose germ is in [18], is then to recover a Lipschitz estimate on the plastic strain through an adequate re-parameterization of time. Unfortunately, once again the analysis grinds to a halt because the re-parameterization involves both piecewise constant and piecewise affine interpolations of the incremental plastic strains. But we are unable, in passing to the limit in the time discretization, to prove that the limits of both interpolations coincide. Our only remaining avenue is then to use viscous regularization in lieu of a time-incremental process, and then to perform a time rescaling of the resulting visco-plastic evolution in the spirit of [18].

The paper is organized as follows. A first section is devoted to the formulation of the non-associative cap model in a framework that will be palatable to the subsequent analysis; as mentioned before, it makes critical use of Laborde's formulation. The last subsection, Subsection 1.4, of that section addresses the precise mathematical framework needed for the analysis, paying heed to the issues of duality that are pivotal in elasto-plasticity. Section 2 addresses the existence of a visco-plastic evolution through an incremental process. Section 3 implements the rescaling and proves the main result of the paper, namely Theorem 3.1 which states an existence theorem that sits in between a classical formulation and a variational evolution as detailed earlier. The main technical hurdle (among many) lies in proving the differentiable character of the rescaled limit elastic strain. This is the object of Subsection 3.6, the last subsection of Section 3 and it is a direct consequence of the energy balance which is also proved in that subsection.

In all fairness, our analysis has been heavily influenced by that of the Cam-Clay model in [9, 10] and parts of this paper (especially Subsection 3.6) should not be construed as much more than an adaptation of results obtained in those references to a different context.

We also expect that one should be able to recover an evolution in real time because, in contrast with the setting of [9] (see [5, 12]), the solution in the spatially homogeneous case has been shown in [22, Theorem 5] to be at minimum continuous in time, at least for particular classes of sets of admissible stresses.

Finally, we decided to limit the loading process to a hard device, that is to a displacement $w(t)$ acting on the entirety of the boundary $\partial\Omega$ of our domain. This is certainly a simplifying assumption because it alleviates the need for safe load conditions on the loads $f(t)$ and $g(t)$. Those can become at times a thorny issue in plasticity. We are confident that the analysis remains the same if general loads were to be incorporated into the evolution. However, the added complexity would render the paper less readable than it is now. Of course, a malicious reader might object that the paper is already barely readable as is!

1. Description of the model

In this section, we provide an overview of the proposed model with minimal concern for the functional properties of the fields involved in that description.

1.1. The original model revisited.

The context is that of small strains. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set occupied by a homogeneous elasto-plastic material. We denote by $u : \Omega \to \mathbb{R}^n$ the displacement field and by $Eu := (Du + Du^T)/2$ the strain tensor. As is usual in small deformations plasticity, the strain tensor is *additively* decomposed as

$$Eu = e + p,$$

where e and p respectively stand for the elastic and plastic strains. This is part of what will be referred to as *kinematic compatibility.* The constitutive equation which relates the (Cauchy) stress tensor σ to the elastic part e of the linearized strain is also assumed to be linear, *i.e.*,

$$\sigma = Ae$$

where A is the Hooke tensor. At equilibrium, and if no volume forces are applied to the sample, the stress satisfies

$$\mathrm{div}\sigma = 0 \quad \text{in } \Omega.$$

It is also constrained to remain in a closed convex set K of the set $\mathbb{M}^{n\times n}_{sym}$ of $n \times n$ symmetric matrices:

$$\sigma \in K := \{\tau \in \mathbb{M}^{n\times n}_{sym} : f(\tau) \le 0\},$$

where $f : \mathbb{M}^{n\times n}_{sym} \to \mathbb{R}$ is the *yield function.* We assume that f is continuous and convex, which implies that K is closed and convex, and that $f(0) < 0$, so that $0 \in \mathrm{int}\, K$.

The behavior of the plastic strain is governed by a non-associative flow rule. Specifically, denoting by $\dot p$ the time derivative of p,

$$\dot p \in \mathcal{A}\sigma,$$

where, according to [25], $\mathcal{A}\sigma = \{0\}$ if $\sigma \in \operatorname{int} K$, while $\mathcal{A}\sigma$ is the normal cone at σ to the level set $\{\tau \in \mathbb{M}^{n\times n}_{sym} : g(\tau) \le g(\sigma)\}$ of g if $\sigma \in \partial K$. Note that if $\sigma \in \partial K$ is not a minimum point of g, then we have from [30, Corollary 23.7.1]

$$\mathcal{A}\sigma = \{\lambda\xi : \lambda \ge 0 \text{ and } \xi \in \partial g(\sigma)\}.$$

In the previous expression, $g : \mathbb{M}^{n\times n}_{sym} \to \mathbb{R}$ is the *plastic potential*, a continuous and convex function and $\partial g(\sigma)$ is the subdifferential of g at σ. Finally, as announced in the introduction, the material is subject to a hard loading device; in other words, a Dirichlet boundary condition $u(x,t) = w(x,t)$ is imposed on $\partial\Omega$.

Our goal is to obtain a triplet $(u(x,t), e(x,t), p(x,t))$ such that

$$\begin{cases} Eu(x,t) = e(x,t) + p(x,t), \\ \sigma(x,t) = Ae(x,t), \\ \operatorname{div}\sigma(x,t) = 0, \\ \sigma(x,t) \in K, \\ \dot p(x,t) \in \mathcal{A}\sigma(x,t), \end{cases}$$

together with the Dirichlet boundary condition. Of course, we know from prior works on plasticity that we cannot expect the boundary condition to be satisfied because plastic strains may develop at the boundary, so that, as seen later, we will have to replace that condition by

$$p(x,t) = (w-u)(x,t) \odot \nu \text{ on } \partial\Omega.$$

Throughout, the symbol $\odot$ stands for the symmetrized tensor product, while ν denotes the outer unit normal to $\partial\Omega$.

When $f = g$ the model is *associative*, and the plastic strain rate obeys the usual normality flow rule (see [25])

$$\dot p \in N_K(\sigma),$$

where $N_K(\sigma)$ is the normal cone to K at $\sigma \in K$. Elementary convex analysis points to the well-known formal equivalence between flow rule and the principle of maximum dissipation, namely

$$\sigma : \dot p = \max_{\tau\in K} \tau : \dot p.$$

From a thermodynamical standpoint, the associated materials are examples of standard generalized materials. The associative theory seems to be vindicated, as far as the principle of maximal dissipation is concerned, by those materials for which plasticity does not promote volumetric changes, that is whenever the yield criterion is independent of the average pressure (see [31]). This is for instance the case for ductile metals and for most alloys.

However, whenever significant volume variations accompany the plastic deformation, the principle of maximum plastic work is no longer valid and thus the associative flow should be abandoned in favor of a non-associative model with $f \neq g$.

With an eye on applications, we are specifically interested in the following type of functions f and g :

$$\begin{aligned} f(\sigma) &= \kappa(\sigma^D) + \sigma^m \sin\varphi - 2c\cos\varphi, \\ g(\sigma) &= \kappa(\sigma^D) + \sigma^m \sin\psi - 2c\cos\psi, \end{aligned} \tag{1.1}$$

where the parameters φ, ψ, and c satisfy $0 < \psi < \varphi < \frac{\pi}{2}$, $c > 0$, and $\kappa : \mathbb{M}^{n\times n}_D \to [0, +\infty)$ is a convex, positively 1-homogeneous function on $\mathbb{M}^{n\times n}_D$ such that $\kappa(0) = 0$.

Here, $\mathbb{M}^{n\times n}_D = \{\sigma \in \mathbb{M}^{n\times n}_{sym} : \operatorname{tr}\sigma = 0\}$, $\sigma^D = \sigma - \operatorname{tr}\sigma/n\, \mathcal{I}d \in \mathbb{M}^{n\times n}_D$ and $\sigma^m = \operatorname{tr}\sigma \in \mathbb{R}$ are respectively the deviatoric and spherical part of σ so that $\sigma = \sigma^D + \sigma^m/n\, \mathcal{I}d$. In particular, for $n = 3$, the Drucker-Prager model corresponds to

$$\kappa(\sigma^D) = \sqrt{\frac{1}{6}\sum_{i<j}(\sigma^D_i - \sigma^D_j)^2} = \frac{1}{\sqrt{2}}\,|\sigma^D|,$$

while the Mohr-Coulomb model[1] corresponds to

$$\kappa(\sigma^D) = \max_{i,j}\{\sigma^D_i - \sigma^D_j\},$$

where σ^D_i, $i = 1, 2, 3$, are the ordered eigenvalues of σ^D. The parameters c, φ and ψ are the cohesion, the angle of internal friction, and the angle of dilatancy, respectively.

We now reformulate this problem variationally in the footstep of Laborde [22, 23].

Formulation 1.1. *(***Laborde's formulation***) Consider a continuous function $h : \mathbb{M}^{n\times n}_{sym} \to \mathbb{R}$ (to be explicited below in (1.7)) satisfying*

$$\begin{cases} h(\tau) = g(\tau) & \text{if} \quad f(\tau) = 0, \\ h(\tau) > g(\tau) & \text{if} \quad f(\tau) < 0, \\ h(\tau) < g(\tau) & \text{if} \quad f(\tau) > 0. \end{cases} \tag{1.2}$$

For every $\sigma \in \mathbb{M}^{n\times n}_{sym}$, we further define the closed and convex set

$$K(\sigma) := \{\tau \in \mathbb{M}^{n\times n}_{sym} : g(\tau) \le h(\sigma)\}.$$

From [23, Proposition 4], $\sigma \in K$ if and only if $\sigma \in K(\sigma)$, and, in this case, $\dot p \in \mathcal{A}\sigma$ if and only if $\dot p : (\tau - \sigma) \le 0$ for any $\tau \in K(\sigma)$. Since $K(\sigma)$ is a closed and convex set in $\mathbb{M}^{n\times n}_{sym}$, that last property can be expressed as $\dot p \in \partial I_{K(\sigma)}(\sigma)$.

We next define the dissipation potential $H : \mathbb{M}^{n\times n}_{sym} \times \mathbb{M}^{n\times n}_{sym} \to \mathbb{R}$ as the support function of $K(\sigma)$, that is,

$$H(\sigma, p) = \max_{\tau\in K(\sigma)} \tau : p,$$

and note that, for a fixed σ, it is convex, sub-additive and positively 1-homogeneous in p. By standard convex analysis, the property $\dot p \in \partial I_{K(\sigma)}(\sigma)$ is equivalent to $\sigma \in \partial_2 H(\sigma, \dot p)$, where $\partial_2 H(\sigma, \dot p)$ denotes the subdifferential of $H(\sigma, \cdot)$ at $\dot p$. Thus, an equivalent formulation of the problem is

$$\begin{cases} Eu(x,t) = e(x,t) + p(x,t), \\ p(x,t) = (w-u)(x,t) \odot \nu(x) \text{ on } \partial\Omega, \\ \sigma(x,t) = Ae(x,t), \\ \operatorname{div}\sigma(x,t) = 0, \\ \sigma(x,t) \in K(\sigma(x,t)), \\ \sigma(x,t) \in \partial_2 H(\sigma(x,t), \dot p(x,t)). \end{cases} \tag{1.3}$$

It is by now fairly classical that the starting point of the variational method for studying an evolution such as (1.3) consists in discretising time and in solving, at the discrete times t_i, the following minimization problem:

Find (u_i, e_i, p_i), with $\sigma_i := Ae_i$, that minimizes

$$\frac{1}{2}\int_\Omega A\eta : \eta \, dx + \int_{\overline\Omega} H(\sigma_i, q - p_{i-1})\, dx$$

among every triplet (v, η, q) such that $Ev = \eta + q$ and $q = (w(t_i) - v) \odot \nu$ on $\partial\Omega$.

Unfortunately, when attempting to carry the analysis of this model through, we grind to a halt because of a lack of a priori bound on the admissible stresses. Indeed, the set K of all admissible stresses is not bounded in the direction of hydrostatic stresses. This together with our choice of the function h imply that the set $K(\sigma)$ is not contained in a uniform ball with respect to $\sigma \in \mathbb{M}^{n\times n}_{sym}$, and consequently the potential H may not be finite along some directions in the plastic strain

[1]According to [25] the Mohr-Coulomb model actually corresponds to $f(\sigma) = \sigma_{max} - \sigma_{min} + (\sigma_{max} + \sigma_{min})\sin\varphi - 2c\cos\varphi$, and similarly for g. But one can rewrite f as $f(\sigma) = \sigma_{max} - \sigma_{min} + 1/3[(\sigma_{max} - \sigma_{int}) - (\sigma_{int} - \sigma_{min})]\sin\varphi + 2/3\,\sigma^m \sin\varphi - 2c\cos\varphi$, where σ_{int} is the intermediate eigenvalue, so that, in that case, $\kappa(\sigma^D) = 3/2\{(\sigma^D_3 - \sigma^D_1) + \sin\varphi/3(\sigma^D_3 - \sigma^D_2) - \sin\varphi/3\;(\sigma^D_2 - \sigma^D_1)\}$.

space. Thus, mathematics, as well as mechanics as was explained in the introduction prompt a modified version of the previous model where the set K of admissible stresses is replaced by

$$\widetilde{K} := K \cap \{(\sigma^D, \sigma^m) \in \mathbb{M}^{n\times n}_D \times \mathbb{R} : \ \sigma^m \geq -R\} \tag{1.4}$$

with $R > 0$ sufficiently large. That model is described in the next subsection.

1.2. **The cap model.** In the engineering literature the terminology "cap model" usually refers to a modification of the original model, where the yield surface is closed with a cap, the non-associative flow rule is kept on the portion of the yield surface that coincides with ∂K, while an associative flow rule is considered on the remaining part. This is, for instance, the case of the so-called Drucker-Prager cap model and Mohr-Coulomb cap model (see, *e.g.*, [28, 29]). In the non-associative setting, the construction of the adequate cap (usually an ellipsoid) is guided by an attempt at keeping the direction of the flow, and this up to the intersection between the cap and the original yield surface. This is so because the envisioned models usually include strain hardening in the direction of the flow. But, here, we do not consider strain hardening – which would, by the way, simplify the mathematical analysis – so that the impact of the choice of a "correct" cap model that would keep the same flow directions everywhere on the original yield surface is minimal in the worst case scenario. Since in turn the mathematical impact of the shape of the cap is nil, we feel entitled to use as working model a variant of the above models, where the cap is chosen as in (1.4) and the flow rule coincides with that described above, except on a portion of the yield surface close to the cap.

Remark 1.2. Let f and g be the yield function and the plastic potential introduced in (1.1). Set

$$G := \{\sigma \in \mathbb{M}^{n\times n}_{sym} : g(\sigma) \leq 0\}, \quad \hat{g}(\sigma) := \mathrm{dist}(\sigma, \partial G),$$

where $\mathrm{dist}(\cdot, \partial G)$ denotes the signed distance from ∂G in the (σ^D, σ^m)-plane (which is equivalent to the usual distance in $\mathbb{M}^{n\times n}_{sym}$, because of the orthogonality of the deviatoric and spherical components). Then $g = \sqrt{1 + \sin^2\psi}\, \hat{g}$ on a neighborhood of K, since, in the (σ^D, σ^m)-plane, the set K is a cone with vertex $(0, 2c\cot\varphi)$ and aperture $\arctan(\sin\varphi)$, while the 0-level set of g is a cone with vertex $(0, 2c\cot\psi)$ and aperture $\arctan(\sin\psi)$, and further $2c\cot\psi > 2c\cot\varphi$ by the assumptions on φ and ψ. Therefore,

$$\bigcup_{\lambda\geq 0} \lambda\, \partial g(\sigma) = \bigcup_{\lambda \geq 0} \lambda\, \partial\hat{g}(\sigma)$$

for every $\sigma \in \partial K$ and we can replace g by $\hat{g}$ without changing the problem. Analogously, if we introduce

$$\hat{f}(\sigma) = \mathrm{dist}(\sigma, \partial K),$$

clearly we have that K coincides with the set $\{\sigma \in \mathbb{M}^{n\times n}_{sym} : \hat{f}(\sigma) \leq 0\}$, so that we can replace f by $\hat{f}$ without changing the problem. Also note that the functions $\hat{f}$ and $\hat{g}$ are still convex, since the signed distance from the boundary of a convex set is a convex function (see, *e.g.*, [14, Chapter 7, Theorem 10.1]). From now on we will work with $\hat{f}$ and $\hat{g}$, and for simplicity will denote them f and g. ¶

By virtue of Remark 1.2, our variant of the cap model can be concisely described in terms of distance functions in the following way. We introduce the sets

$$\begin{aligned} \widetilde{K} &:= K \cap \{(\sigma^D, \sigma^m) \in \mathbb{M}^{n\times n}_D \times \mathbb{R} : \ \sigma^m \geq -R\}, \\ \widetilde{G} &:= G \cap \{(\sigma^D, \sigma^m) \in \mathbb{M}^{n\times n}_D \times \mathbb{R} : \ \sigma^m \geq -R\}, \end{aligned}$$

and the functions

$$\tilde{f} := \mathrm{dist}(\cdot, \partial\widetilde{K}), \quad \tilde{g} := \mathrm{dist}(\cdot, \partial\widetilde{G}).$$

We define as cap model the plastic problem with yield surface given by $\tilde f$ and flow rule described by $\tilde g$. Clearly, the set $\{\sigma \in \mathbb{M}^{n\times n}_{sym} : \tilde f(\sigma) \le 0\}$ coincides with $\widetilde K$. Moreover, $\partial \tilde g$ coincides with ∂g on $\partial \widetilde K \cap \partial K$, except for a small region close to the cap, while is pointing in the same directions of $\partial \tilde f$ on $\partial \widetilde K \cap \partial \tilde G \cap \{\sigma^m = -R\}$.

The above considerations prompt us to propose an "abstract" setting for non-associative cap plasticity. This is detailed in the next subsection.

1.3. **Abstract setting.** We denote by K_D a compact convex subset of $\mathbb{M}^{n\times n}_D$ containing 0 as an interior point. Given $\alpha > \beta > 0$ and $\bar\sigma^m_G > \bar\sigma^m_K > 0$, we consider the subsets of $\mathbb{M}^{n\times n}_{sym}$ given by

$$\widehat K = \{(\sigma^D, \sigma^m) \in \mathbb{M}^{n\times n}_D \times [-R, \bar\sigma^m_K] : \ \sigma^D \in \alpha(\bar\sigma^m_K - \sigma^m) K_D\},$$
$$\widehat G = \{(\sigma^D, \sigma^m) \in \mathbb{M}^{n\times n}_D \times [-R, \bar\sigma^m_G] : \ \sigma^D \in \beta(\bar\sigma^m_G - \sigma^m) K_D\}.$$

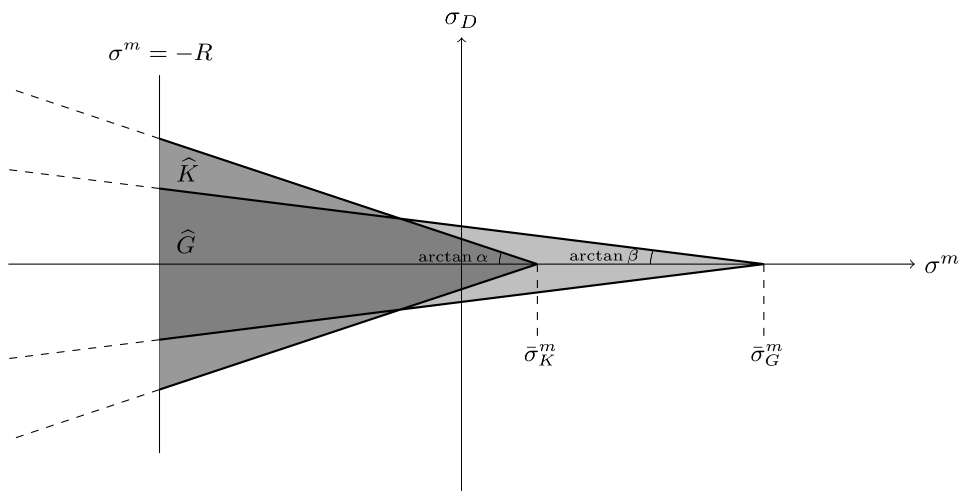

FIGURE 1. The sets $\widehat K$ and $\widehat G$.

If we take $K_D = \{\sigma \in \mathbb{M}^{n\times n}_D : \kappa(\sigma) \le 1\}$, $\alpha = \sin\varphi$, $\beta = \sin\psi$, $\bar\sigma^m_K = 2c\cot\varphi$, $\bar\sigma^m_G = 2c\cot\psi$ and use the 1-homogeneous character of κ, then, for R sufficiently large,

$$\widehat K = K \cap Q_R, \quad \widehat G = G \cap Q_R,$$

which is precisely the cap model as detailed above (see Figure 1). Therefore, from now on

$$f := \mathrm{dist}(\cdot, \partial \widehat K), \quad g := \mathrm{dist}(\cdot, \partial \widehat G). \tag{1.5}$$

We observe that for $a \in [\min g, 0]$ the sublevel set

$$G_a := \{\sigma \in \mathbb{M}^{n\times n}_{sym} : \ g(\sigma) \le a\}$$

is given in the (σ^D, σ^m)-plane by a cone with vertex on the σ^m-axis and aperture $\arctan\beta$, cut by the plane $\sigma^m = -R - a$. In particular, for R sufficiently large we have that the set

$$G_{g(0)} = \{(\sigma^D, \sigma^m) \in \mathbb{M}^{n\times n}_D \times [-R - g(0), 0] : \ \sigma^D \in -\beta\sigma^m K_D\}$$

and is therefore contained in the interior of $\widehat K$. Hence, by continuity of g there exists $\delta > 0$ small enough so that, setting $\lambda := g(0) + 2\delta$, we have

$$G_\lambda \subset \mathrm{int}\, \widehat K. \tag{1.6}$$

We now use Laborde's formulation and we define

$$h(\sigma) := \min \big\{ \max\{g(\sigma), \lambda\}, \mathcal{G} \big\} - \min\{f(\sigma), \delta\} \tag{1.7}$$

for

$$\mathcal{G} := \max_{\widehat{K}} g + 1. \tag{1.8}$$

Since $\widehat{K}$ is closed and bounded, the maximum of g on $\widehat{K}$ exists and, using (1.6), h satisfies (1.2). Set

$$K(\sigma) := \{\tau \in \mathbb{M}^{n\times n}_{sym} : g(\tau) \leq h(\sigma)\}, \quad H(\sigma, p) := \sup_{\tau\in K(\sigma)} \tau : p.$$

As already noted, the map $p \mapsto H(\sigma, p)$ is convex, continuous, positively 1-homogeneous and sub-additive (as the support function of a closed and convex set). We now state other properties of the function H.

Lemma 1.3. *The map H is continuous over $\mathbb{M}^{n\times n}_{sym} \times \mathbb{M}^{n\times n}_{sym}$.*

Proof. Let $(\sigma_k, p_k) \to (\sigma, p)$. We first start by proving upper semi-continuity. Since $K(\sigma_k)$ is compact, for each k there exists $\tau_k \in K(\sigma_k)$ such that $H(\sigma_k, p_k) = \tau_k : p_k$. By the continuity of h, there exists a constant $M > 0$ such that $h(\sigma_k) \leq M$, $k \in \mathbb{N}$. But g is the distance to a compact set, so that, for some $r > 0$ depending only on M,

$$K(\sigma_k) = \{\tau \in \mathbb{M}^{n\times n}_{sym} : g(\tau) \leq h(\sigma_k)\} \subset \{\tau \in \mathbb{M}^{n\times n}_{sym} : g(\tau) \leq M\} \subset Q_r.$$

We can thus extract a subsequence – still denoted $\{\tau_k\}$ – such that $\tau_k \to \tau$ for some symmetric matrix τ. As

$$g(\tau) = \lim_{k\to+\infty} g(\tau_k) \leq \lim_{k\to+\infty} h(\sigma_k) = h(\sigma),$$

we deduce that $\tau \in K(\sigma)$, and thus

$$\limsup_{k\to+\infty} H(\sigma_k, p_k) = \limsup_{k\to+\infty} \tau_k : p_k = \tau : p \leq H(\sigma, p).$$

We now show the lower semi-continuity of H. We first observe that

$$H(\sigma, p) := \sup_{\tau\in \operatorname{int} K(\sigma)} \tau : p,$$

where $\operatorname{int} K(\sigma) := \{\tau \in \mathbb{M}^{n\times n}_{sym} : g(\tau) < h(\sigma)\}$ denotes the interior of $K(\sigma)$. Assume that $\tau \in \operatorname{int} K(\sigma)$, then $g(\tau) < h(\sigma)$, and thus $g(\tau) \leq h(\sigma_k)$ for k large enough. Consequently, one has $\tau \in K(\sigma_k)$ for k large enough, hence

$$\liminf_{k\to+\infty} H(\sigma_k, p_k) \geq \liminf_{k\to+\infty} \tau : p_k = \tau : p.$$

Taking the supremum over all $\tau \in \operatorname{int} K(\sigma)$ leads to

$$\liminf_{k\to+\infty} H(\sigma_k, p_k) \geq H(\sigma, p),$$

which completes the proof of the lemma. □

The following property will be instrumental for the forthcoming analysis.

Lemma 1.4. *There exist $0 < \alpha_H < \beta_H < +\infty$ such that*

$$B(0, \alpha_H) \subset K(\sigma) \subset B(0, \beta_H), \tag{1.9}$$

or still

$$\alpha_H|p| \leq H(\sigma, p) \leq \beta_H|p| \tag{1.10}$$

for every σ and $p \in \mathbb{M}^{n\times n}_{sym}$.

Proof. Lower bound: To prove the lower bound we will show that for every $\sigma \in \mathbb{M}^{n\times n}_{sym}$ the set $K(\sigma)$ contains a ball centered at 0 with a uniform radius with respect to σ. In particular, by continuity of g it is enough to show that

$$g(0) < \inf_{\sigma \in \mathbb{M}^{n\times n}_{sym}} h(\sigma). \tag{1.11}$$

This follows immediately from the definition of h. Indeed, by (1.6) and (1.8) we have that $\mathcal{G} > \lambda$, so that

$$h(\sigma) \geq \lambda - \delta > g(0)$$

for every $\sigma \in \mathbb{M}^{n\times n}_{sym}$.

Upper bound. By construction (see (1.7)-(1.8)),

$$h(\sigma) \leq \mathcal{G} - \min f =: \mathcal{G}_0 \quad \text{for every } \sigma \in \mathbb{M}^{n\times n}_{sym}.$$

Therefore, for every $\sigma \in \mathbb{M}^{n\times n}_{sym}$ we have

$$K(\sigma) \subset \{\tau \in \mathbb{M}^{n\times n}_{sym} : \ g(\tau) \leq \mathcal{G}_0\}.$$

The function g being a distance function to a compact set, the set above is bounded. This implies the upper bound on H. □

Finally we show that H is Lipschitz with respect to its first variable.

Lemma 1.5. *There exists a constant $C_H > 0$, which only depends on f and g, such that*

$$|H(\sigma_1, p) - H(\sigma_2, p)| \leq C_H |p||\sigma_1 - \sigma_2|$$

for any σ_1, σ_2 and $p \in \mathbb{M}^{n\times n}_{sym}$.

Proof. We assume without loss of generality that $h(\sigma_2) \leq h(\sigma_1)$, so that $K(\sigma_2) \subset K(\sigma_1)$ and $H(\sigma_2, p) \leq H(\sigma_1, p)$. Since $K(\sigma_1)$ is compact, there exists $\tau_1 \in K(\sigma_1)$ such that $H(\sigma_1, p) = \tau_1 : p$.

Note that, in view of the conical form of the level sets of g, the Hausdorff distance

$$d_H(\partial K(\sigma_1), \partial K(\sigma_2)) \leq c|h(\sigma_1) - h(\sigma_2)|,$$

for some constant $c > 1$ depending only on g.

We claim that if τ_1 is given as before, then there exists $\tau_2 \in K(\sigma_2)$ such that $|\tau_1 - \tau_2| \leq c|h(\sigma_1) - h(\sigma_2)|$. Indeed if $\tau_1 \in K(\sigma_2)$, then it suffices to take $\tau_2 = \tau_1$ and the property is trivial. On the other hand, if $\tau_1 \in K(\sigma_1) \setminus K(\sigma_2)$, then we simply take τ_2 as the minimal distance projection of τ_1 onto the convex set $K(\sigma_2)$. In this case, it follows that $|\tau_1 - \tau_2| \leq d_H(\partial K(\sigma_1), \partial K(\sigma_2)) \leq c|h(\sigma_1) - h(\sigma_2)|$, and the claim is proved.

By definition of H,

$$H(\sigma_1, p) - H(\sigma_2, p) \leq (\tau_1 - \tau_2) : p \leq |\tau_1 - \tau_2||p| \leq c|p||h(\sigma_1) - h(\sigma_2)|.$$

Since f and g are both 1-Lipschitz (see their definition (1.5)), h defined in (1.7) is 2-Lipschitz as well. It follows that there exists a constant $C_H > 2$ such that

$$H(\sigma_1, p) - H(\sigma_2, p) \leq C_H |p||\sigma_1 - \sigma_2|.$$

The other inequality is obtained by inverting the roles of σ_1 and σ_2. □

Remark 1.6. Since it is easily seen that, if $h(\sigma_2) \leq h(\sigma_1)$, then, for any $\tau \in \mathbb{M}^{n\times n}_{sym}$, $P_{K(\sigma_2)}(\tau) = P_{K(\sigma_2)}(P_{K(\sigma_1)}(\tau))$, we actually proved in Lemma 1.5 that if σ_1, σ_2 and $\tau \in \mathbb{M}^{n\times n}_{sym}$, then

$$|P_{K(\sigma_1)}(\tau) - P_{K(\sigma_2)}(\tau)| \leq C_H |\sigma_1 - \sigma_2|. \tag{1.12}$$

¶

We introduce the perturbed dissipation potential $H_\varepsilon : \mathbb{M}^{n\times n}_{sym} \times \mathbb{M}^{n\times n}_{sym} \to [0, +\infty)$ defined, for each $\varepsilon > 0$, as

$$H_\varepsilon(\sigma, p) := H(\sigma, p) + \frac{\varepsilon}{2}|p|^2. \tag{1.13}$$

The convex conjugate $H^*_\varepsilon : \mathbb{M}^{n\times n}_{sym} \times \mathbb{M}^{n\times n}_{sym} \to [0, +\infty)$ of H_ε with respect to the second variable is defined by

$$H^*_\varepsilon(\sigma, \tau) := \sup_{p\in\mathbb{M}^{n\times n}_{sym}} \{\tau : p - H_\varepsilon(\sigma, p)\}.$$

Using standard convex analysis, see [30, Theorem 16.4],

$$H^*_\varepsilon(\sigma, \tau) = \frac{|\tau - P_{K(\sigma)}(\tau)|^2}{2\varepsilon},$$

where $P_{K(\sigma)}$ denotes the minimal distance projection onto the convex set $K(\sigma)$. In particular H^*_ε is differentiable in the second variable, and its partial derivative is given by

$$N_\varepsilon(\sigma, \tau) = \partial_2 H^*_\varepsilon(\sigma, \tau) = \frac{\tau - P_{K(\sigma)}(\tau)}{\varepsilon}. \tag{1.14}$$

Note that, since $0 \in K(\sigma)$ (see (1.9)),

$$|N_\varepsilon(\sigma, \tau)| \le \frac{1}{\varepsilon}|\tau|, \tag{1.15}$$

and it follows that, for any σ, τ_1 and $\tau_2 \in \mathbb{M}^{n\times n}_{sym}$,

$$|H^*_\varepsilon(\sigma, \tau_1) - H^*_\varepsilon(\sigma, \tau_2)| \le \frac{1}{\varepsilon}(|\tau_1| + |\tau_2|)|\tau_1 - \tau_2|.$$

Actually, N_ε is Lipschitz. Indeed, we have the following result.

Lemma 1.7. *Let C_H be the constant in Lemma 1.5, then*

$$|N_\varepsilon(\sigma_1, \tau_1) - N_\varepsilon(\sigma_2, \tau_2)| \le \frac{C_H}{\varepsilon}\left(|\sigma_1 - \sigma_2| + |\tau_1 - \tau_2|\right)$$

for any σ_1, σ_2, τ_1 and $\tau_2 \in \mathbb{M}^{n\times n}_{sym}$.

Proof. By definition of N_ε and since the projection is 1-Lipschitz,

$$|N_\varepsilon(\sigma, \tau_1) - N_\varepsilon(\sigma, \tau_2)| \le \frac{2}{\varepsilon}|\tau_1 - \tau_2|.$$

On the other hand, by Remark 1.6 we have

$$|N_\varepsilon(\sigma_1, \tau) - N_\varepsilon(\sigma_2, \tau)| \le \frac{C_H}{\varepsilon}|\sigma_1 - \sigma_2|.$$

Observing that $C_H > 2$ by construction, we obtain the thesis. □

As a final note, given $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, define the set

$$\mathcal{K}(\sigma) := \{\tau \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) : \tau(x) \in K(\sigma(x)) \text{ for a.e. } x \in \Omega\}.$$

Then, if $\tau \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$,

$$\|N_\varepsilon(\sigma, \tau)\|_2 = \frac{\operatorname{dist}_2(\tau, \mathcal{K}(\sigma))}{\varepsilon}, \tag{1.16}$$

where, for any closed set $\mathcal{C} \subset L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, $\operatorname{dist}_2(\tau, \mathcal{C})$ is the L^2-distance from τ to $\mathcal{C}$.

1.4. **Mathematical setting.** Throughout the paper, Ω is a bounded connected open set in $\mathbb{R}^n$ with Lipschitz boundary. The Lebesgue measure in $\mathbb{R}^n$ and the $(n-1)$-dimensional Hausdorff measure are respectively denoted by $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$.

We use standard notation for Lebesgue and Sobolev spaces. In particular, for $1 \le p \le \infty$, the L^p-norms of the various quantities are denoted by $\|\cdot\|_p$. The space $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ is that of all $\mathbb{M}^{n\times n}_{sym}$-valued bounded Radon measures on $\overline{\Omega}$, and the norm in that space is denoted by $\|\cdot\|_1$. By the Riesz representation Theorem, $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ can be identified with the dual of $\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$. Finally, $BD(\Omega)$ stands for the space of functions with bounded deformations on Ω, *i.e.*, $u \in BD(\Omega)$ if $u \in L^1(\Omega;\mathbb{R}^n)$ and $Eu \in \mathcal{M}(\Omega;\mathbb{M}^{n\times n}_{sym})$, where $Eu := (Du + Du^T)/2$ and Du is the distributional derivative of u. We refer to [34] for general properties of that space.

Let $u \in BD(\Omega)$, $w \in H^1(\Omega;\mathbb{R}^n)$, $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $p \in \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ be such that

$$Eu = e + p \text{ in } \Omega,$$
$$p = (w-u)\odot \nu\mathcal{H}^{n-1} \text{ on } \partial\Omega.$$

If $\sigma \in L^\infty(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $\mathrm{div}\sigma \in L^n(\Omega;\mathbb{R}^n)$, it is possible to define the "scalar product" of σ and p as the distribution $[\sigma : p]$ on $\mathbb{R}^n$ by setting

$$[\sigma : p](\varphi) := -\int_\Omega \varphi(u-w)\cdot \mathrm{div}\sigma\, dx - \int_\Omega \sigma : (u-w)\odot \nabla\varphi\, dx - \int_\Omega \sigma : \varphi(e - Ew)\, dx,$$

for every $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$. It turns out that $[\sigma : p]$ is independent of u, w and e, and that it defines a bounded Radon measure on $\overline{\Omega}$. We also define the global duality pairing $\langle \sigma, p\rangle$ by setting

$$\langle \sigma, p\rangle := [\sigma : p](1) = \int_\Omega (w-u)\cdot \mathrm{div}\sigma\, dx - \int_\Omega \sigma : (e - Ew)\, dx. \tag{1.17}$$

It can be proved (see [20, Section 6]) that

$$|\langle \sigma, p\rangle| \le \|\sigma\|_\infty \|p\|_1.$$

Moreover, if σ further belongs to $\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$, then

$$\langle \sigma, p\rangle = \int_{\overline{\Omega}} \sigma(x) : \frac{dp}{d|p|}(x)\, d|p|(x) \tag{1.18}$$

is the usual duality pairing between $\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ and $\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$. In the previous formula we have denoted by $|p|$ the variation measure of p. If instead p further belongs to $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, then the duality pairing $\langle \sigma, p\rangle$ coincides with the L^2 product.

The space $L^1(0,T;\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}))$ is the space of all strongly measurable maps $t \mapsto f(t) \in \mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ such that

$$\int_0^T \|f(t)\|_\infty\, dt < +\infty.$$

From [19, Theorem 2.112], since $\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ is separable, the dual of the space $L^1(0,T;\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}))$ can be identified to the space $L^\infty_w(0,T;\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}))$ of all weakly* measurable maps $t \mapsto \lambda(t) \in \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})$ such that

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|\lambda(t)\|_1 < +\infty,$$

through the duality pairing

$$\langle \lambda, f\rangle = \int_0^T \langle \lambda(t), f(t)\rangle_{\mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}),\mathcal{C}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym})}\, dt.$$

Let A be a fourth order Hooke tensor satisfying the usual symmetry properties $A_{ijkh} = A_{jikh} = A_{khij}$ for every i, j, k, $h \in \{1, \dots, n\}$, and

$$\alpha_A |\xi|^2 \le A\xi : \xi \le \beta_A |\xi|^2, \tag{1.19}$$

for some $0 < \alpha_A \le \beta_A < +\infty$ and every $\xi \in \mathbb{M}^{n\times n}_{sym}$. Then define, for any $e \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, the elastic energy as

$$\mathcal{Q}(e) := \frac{1}{2}\int_\Omega Ae : e\, dx.$$

If $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $p \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ we define the functionals

$$\mathcal{H}(\sigma, p) := \int_\Omega H(\sigma, p)\, dx, \quad \mathcal{H}_\varepsilon(\sigma, p) := \int_\Omega H_\varepsilon(\sigma, p)\, dx,$$

while, if $\sigma \in \mathcal{C}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ and $p \in \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ the first functional is defined as

$$\mathcal{H}(\sigma, p) := \int_{\overline{\Omega}} H\left(\sigma, \frac{dp}{d|p|}\right) d|p|.$$

Remark 1.8. The following (lower semi-)continuity results hold:

1. If $\{\sigma_k\}$, $\{p_k\} \subset L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, $\sigma_k \to \sigma$ strongly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, and $p_k \rightharpoonup p$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, then
$$\mathcal{H}(\sigma, p) \le \liminf_{k\to+\infty} \mathcal{H}(\sigma_k, p_k).$$
Moreover if $p_k \to p$ strongly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, then
$$\mathcal{H}(\sigma, p) = \lim_{k\to+\infty} \mathcal{H}(\sigma_k, p_k).$$
2. If $\{\sigma_k\} \subset \mathcal{C}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$, $\{p_k\} \subset \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$, $\sigma_k \to \sigma$ uniformly in $\overline{\Omega}$, and $p_k \xrightharpoonup{*} p$ weakly* in $\mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$, then
$$\mathcal{H}(\sigma, p) \le \liminf_{k\to+\infty} \mathcal{H}(\sigma_k, p_k).$$

Indeed, in the first case, Lemma 1.5 implies that

$$|\mathcal{H}(\sigma, p_k) - \mathcal{H}(\sigma_k, p_k)| \le C_H \|\sigma - \sigma_k\|_2 \|p_k\|_2 \to 0,$$

since $\{p_k\}$ is bounded in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. Hence

$$\liminf_{k\to+\infty} \mathcal{H}(\sigma_k, p_k) = \liminf_{k\to+\infty} \mathcal{H}(\sigma, p_k).$$

Now since H is convex in its second variable, we infer that

$$\liminf_{k\to+\infty} \mathcal{H}(\sigma, p_k) \ge \mathcal{H}(\sigma, p),$$

while an easy application of dominated convergence yields the convergence result in the case where $\{p_k\}$ strongly converges in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$.

In the second case, that same lemma implies that

$$|\mathcal{H}(\sigma, p_k) - \mathcal{H}(\sigma_k, p_k)| \le C_H \|\sigma - \sigma_k\|_\infty \|p_k\|_1 \to 0,$$

since $\{p_k\}$ is bounded in $\mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$. Hence

$$\liminf_{k\to+\infty} \mathcal{H}(\sigma_k, p_k) = \liminf_{k\to+\infty} \mathcal{H}(\sigma, p_k).$$

Since $(x, \xi) \mapsto H(\sigma(x), \xi)$ is (lower semi-)continuous, while $H(\sigma(x), \cdot)$ is convex and positively 1-homogeneous, we infer from Reshetnyak's lower semi-continuity Theorem (see, *e.g.*, [1, Theorem 2.38]) that $\liminf_k \mathcal{H}(\sigma, p_k) \ge \mathcal{H}(\sigma, p)$. ¶

When dealing with the visco-plastic approximation of the elasto-plastic problem, we will obtain the first type of convergence on our approximating sequences, while, when letting the viscosity parameter tend to 0, we will only obtain weak convergence in L^2 of the approximating σ-sequence, and convergence in the space of measures of the approximating p-sequence.

Unfortunately, Reshetnyak lower semi-continuity Theorem is false when H fails to be (lower semi)-continuous, so that we are pretty much forced to restrict our analysis to continuous stresses; but continuity is not preserved under L^2-weak convergence, which is the best we can hope for the various sequences of stresses that will enter the formulation. Consequently, the analysis will soon grind to a halt for lack of lower semi-continuity of H. This is why we will propose, in the spirit of [9], to introduce a regularization of σ in the definition of $\mathcal{K}(\sigma)$. This will be done by introducing a convolution kernel ρ and replacing $\mathcal{K}(\sigma)$ by $\mathcal{K}(\sigma * \rho)$ defined below.

We fix $\rho \in \mathcal{C}^1_c(\mathbb{R}^n)$ and set, for $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$,

$$x \in \overline{\Omega} \mapsto \sigma * \rho(x) := \int_\Omega \rho(x-y)\sigma(y)\, dy.$$

The convolution $\sigma * \rho$ defines an element in $\mathcal{C}^1(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$.

Note that, with our definition of the convolution, if $\sigma_k \rightharpoonup \sigma$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, then, in particular,

$$\sigma_k * \rho \to \sigma * \rho \text{ uniformly on } \overline{\Omega}. \tag{1.20}$$

For now, we address in the next section the visco-plastic regularization.

2. The visco-plastic model

We propose to establish existence of the solution to the visco-plastic regularization through a time incremental process.

Consider a boundary displacement $\hat{w} \in H^1(\Omega; \mathbb{R}^n)$. We set

$$A_{\rm reg}(\hat{w}) := \Big\{ (v, \eta, q) \in H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) : \\ Ev = \eta + q \text{ a.e. in } \Omega,\ v = \hat{w}\ \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega \Big\}. \tag{2.1}$$

The main result of this section is the following existence result for the non-associative visco-plastic evolution.

Theorem 2.1. *Let $w \in H^1(0,T; H^1(\Omega, \mathbb{R}^n))$, let $(u_0, e_0, p_0) \in A_{\rm reg}(w(0))$ be such that ${\rm div}\sigma_0 = 0$ a.e. in Ω, where $\sigma_0 := Ae_0$, and let $\varepsilon > 0$. Then, there exists a unique triplet $(u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t)) \in A_{\rm reg}(w(t))$ with*

$$u_\varepsilon \in H^1(0,T; H^1(\Omega; \mathbb{R}^n))$$
$$e_\varepsilon \in H^1(0,T; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$$
$$p_\varepsilon \in W^{1,\infty}(0,T; L^2(\Omega; \mathbb{M}^{n\times n}_{sym})),$$

such that, setting $\sigma_\varepsilon(t) := Ae_\varepsilon(t)$, the following conditions are satisfied:

Initial condition: $(u_\varepsilon(0), e_\varepsilon(0), p_\varepsilon(0)) = (u_0, e_0, p_0)$;

Kinematic compatibility: *For every $t \geq 0$,*

$$Eu_\varepsilon(t) = e_\varepsilon(t) + p_\varepsilon(t) \text{ a.e. in } \Omega,$$
$$u_\varepsilon(t) = w(t)\ \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega;$$

Equilibrium condition: *For every $t \geq 0$,*

$$\operatorname{div} \sigma_\varepsilon(t) = 0 \text{ a.e. in } \Omega;$$

Regularized non-associative flow rule: *For a.e.* $t \in [0,T]$,

$$\dot{p}_\varepsilon(t) = N_\varepsilon(\sigma_\varepsilon(t), \sigma_\varepsilon(t)) \text{ for a.e. } x \in \Omega,$$

or equivalently,

$$\sigma_\varepsilon(t) - \varepsilon \dot{p}_\varepsilon(t) \in \partial_2 H(\sigma_\varepsilon(t), \dot{p}_\varepsilon(t)) \text{ for a.e. } x \in \Omega.$$

In particular, $\varepsilon \|\dot{p}_\varepsilon(t)\|_2 = \mathrm{dist}_2(\sigma_\varepsilon(t), \mathcal{K}(\sigma_\varepsilon(t)))$.

We call such a triplet a non-associative visco-plastic solution.

2.1. **Preliminaries.** We first prove a few preliminary results that will help us in deriving a meaningful incremental process.

Proposition 2.2. *Let* $\varepsilon > 0$ *and* $\delta > 0$ *be such that* $\frac{C_H \beta_A^2 \delta}{\alpha_A \varepsilon} < \frac{1}{3}$. *If* $\hat{w} \in H^1(\Omega;\mathbb{R}^n)$ *and* $\hat{p} \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ *then there exists a triplet* $(u,e,p) \in A_{\mathrm{reg}}(\hat{w})$*, satisfying*

$$\mathcal{Q}(e) + \mathcal{H}(\sigma, p - \hat{p}) + \frac{\varepsilon}{2\delta}\|p - \hat{p}\|_2^2 \le \mathcal{Q}(\eta) + \mathcal{H}(\sigma, q - \hat{p}) + \frac{\varepsilon}{2\delta}\|q - \hat{p}\|_2^2 \tag{2.2}$$

for any $(v,\eta,q) \in A_{\mathrm{reg}}(\hat{w})$*, with* $\sigma := Ae$.

Proof. Let us define $(u_0,e_0,p_0) := (\hat{w}, E\hat{w}, 0)$, and for any $k \ge 1$, consider the minimization problem

$$\min_{(v,\eta,q)\in A_{\mathrm{reg}}(\hat{w})} \left\{ \mathcal{Q}(\eta) + \mathcal{H}(\sigma_{k-1}, q - \hat{p}) + \frac{\varepsilon}{2\delta}\|q - \hat{p}\|_2^2 \right\}, \tag{2.3}$$

where $\sigma_{k-1} := Ae_{k-1}$. Let $(v_j,\eta_j,q_j) \in A_{\mathrm{reg}}(\hat{w})$ be a minimizing sequence, then by (1.10), (1.19) and the Poincaré-Korn's inequality, one has

$$\sup_{j\in\mathbb{N}} \{\|\eta_j\|_2 + \|q_j\|_2 + \|v_j\|_{H^1(\Omega;\mathbb{R}^n)}\} < +\infty.$$

Hence, up to a subsequence, $v_j \rightharpoonup u_k$ in $H^1(\Omega;\mathbb{R}^n)$, $\eta_j \rightharpoonup e_k$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and $q_j \rightharpoonup p_k$ in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ as $j \to +\infty$. Since $v_j = \hat{w}$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, $u_k = \hat{w}$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$ and thus $(u_k,e_k,p_k) \in A_{\mathrm{reg}}(\hat{w})$. Then, by convexity of the functional, we get that

$$\mathcal{Q}(e_k) + \mathcal{H}(\sigma_{k-1}, p_k - \hat{p}) + \frac{\varepsilon}{2\delta}\|p_k - \hat{p}\|_2^2 \le \liminf_{j\to+\infty} \left\{ \mathcal{Q}(\eta_j) + \mathcal{H}(\sigma_{k-1}, q_j - \hat{p}) + \frac{\varepsilon}{2\delta}\|q_j - \hat{p}\|_2^2 \right\},$$

from which we conclude that (u_k,e_k,p_k) is a solution of the minimization problem (2.3). By strict convexity, we infer that this solution is actually unique.

We now prove that $\{e_k\}$ is a Cauchy sequence in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. Indeed, writing the Euler-Lagrange equations of the minimization problem (2.3) (see, *e.g.*, Proposition 2.3 below), we get that

$$\operatorname{div} \sigma_k = 0 \quad \text{a.e. in } \Omega,$$

and also, by (1.13), and the homogeneity of degree 0 of the multifunction $p \mapsto \partial_2 H(\sigma, p)$, that

$$\sigma_k \in \partial_2 H_\varepsilon \left(\sigma_{k-1}, \frac{p_k - \hat{p}}{\delta} \right) \quad \text{a.e. in } \Omega.$$

Now, convex duality and (1.14) imply that the latter is equivalent to $p_k - \hat{p} = \delta N_\varepsilon(\sigma_{k-1}, \sigma_k)$ a.e. in Ω. Since $Eu_k = e_k + p_k$,

$$e_k - e_{k-1} = Eu_k - Eu_{k-1} - \delta \left(N_\varepsilon(\sigma_{k-1}, \sigma_k) - N_\varepsilon(\sigma_{k-2}, \sigma_{k-1}) \right).$$

Taking the L^2-scalar product with $\sigma_k - \sigma_{k-1}$, using (1.19) and the fact that, by Lemma 1.7, N_ε is Lipschitz continuous (with a Lipschitz constant of order $1/\varepsilon$), we deduce that

$$\begin{aligned}\alpha_A \|e_k - e_{k-1}\|_2^2 \leq \int_\Omega (\sigma_k - \sigma_{k-1}) : (e_k - e_{k-1})\, dx \\ \leq \int_\Omega (\sigma_k - \sigma_{k-1}) : (Eu_k - Eu_{k-1})\, dx \\ + \frac{C_H \delta}{\varepsilon} \|\sigma_k - \sigma_{k-1}\|_2 \left(\|\sigma_k - \sigma_{k-1}\|_2 + \|\sigma_{k-1} - \sigma_{k-2}\|_2\right).\end{aligned}$$

But since $\operatorname{div} \sigma_k = \operatorname{div} \sigma_{k-1} = 0$ a.e. in Ω and $u_k - u_{k-1} \in H^1_0(\Omega;\mathbb{R}^n)$, we deduce that the integral in the right hand side of the previous inequality vanishes, hence

$$\|e_k - e_{k-1}\|_2 \leq \frac{C_H \beta_A^2 \delta}{\alpha_A \varepsilon} \left(\|e_k - e_{k-1}\|_2 + \|e_{k-1} - e_{k-2}\|_2\right).$$

Hence, if $\frac{C_H \beta_A^2 \delta}{\alpha_A \varepsilon} < \frac{1}{3}$, then

$$\|e_k - e_{k-1}\|_2 \leq \frac{1}{2} \|e_{k-1} - e_{k-2}\|_2 \leq \frac{1}{2^{k-1}} \|e_1 - e_0\|_2,$$

which shows that $\{e_k\}$ is a Cauchy sequence in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. As a consequence, there exists $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ such that $e_k \to e$ strongly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, and in particular $\sigma_k \to \sigma := Ae$ strongly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. Moreover, as above,

$$\sup_{k\in\mathbb{N}} \{\|p_k\|_2 + \|u_k\|_{H^1(\Omega;\mathbb{R}^n)}\} < +\infty,$$

and thus, up to the possible extraction of a subsequence, $u_k \rightharpoonup u$ weakly in $H^1(\Omega;\mathbb{R}^n)$ and $p_k \rightharpoonup p$ weakly in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, where $(u,e,p) \in A_{\rm reg}(\hat w)$. We next use the first item in Remark 1.8 to obtain that

$$\mathcal{Q}(e) + \mathcal{H}(\sigma, p - \hat p) + \frac{\varepsilon}{2\delta}\|p - \hat p\|_2^2 \leq \liminf_{k\to+\infty} \left\{ \mathcal{Q}(e_k) + \mathcal{H}(\sigma_{k-1}, p_k - \hat p) + \frac{\varepsilon}{2\delta}\|p_k - \hat p\|_2^2 \right\}.$$

Thanks to the dominated convergence Theorem, for any $(v,\eta,q) \in A_{\rm reg}(\hat w)$, we have

$$\lim_{k\to+\infty} \left\{ \mathcal{Q}(\eta) + \mathcal{H}(\sigma_{k-1}, q - \hat p) + \frac{\varepsilon}{2\delta}\|q - \hat p\|_2^2 \right\} = \mathcal{Q}(\eta) + \mathcal{H}(\sigma, q - \hat p) + \frac{\varepsilon}{2\delta}\|q - \hat p\|_2^2.$$

The proof is complete. □

We now derive the Euler-Lagrange equation satisfied by the solution of the minimization problem (2.2).

Proposition 2.3. *Let $\varepsilon > 0$ and $\delta > 0$ be such that $\frac{C_H \beta_A^2 \delta}{\alpha_A \varepsilon} < \frac{1}{3}$. Let $\hat w \in H^1(\Omega;\mathbb{R}^n)$ and $\hat p \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. The following statements are equivalent:*

- (f_1) *$(u,e,p) \in A_{\rm reg}(\hat w)$ is a solution of (2.2) with $\sigma := Ae$;*
- (f_2) *for any $(v,\eta,q) \in A_{\rm reg}(0)$,*

$$\mathcal{H}(\sigma, q + p - \hat p) - \mathcal{H}(\sigma, p - \hat p) \geq -\int_\Omega \sigma : \eta\, dx - \frac{\varepsilon}{\delta} \int_\Omega (p - \hat p) : q\, dx;$$

- (f_3) *$\operatorname{div} \sigma = 0$ and $\sigma - \frac{\varepsilon}{\delta}(p - \hat p) \in \partial_2 H(\sigma, p - \hat p)$ a.e. in Ω.*

Proof. We first prove that (f_1) implies (f_2). Let $(v,\eta,q) \in A_{\rm reg}(0)$, then $(u+sv, e+s\eta, p+sq) \in A_{\rm reg}(\hat w)$ is an admissible triplet for (2.2) for every $0<s<1$. Hence

$$\begin{aligned}\mathcal{Q}(e) + \mathcal{H}(\sigma, p-\hat p) + \frac{\varepsilon}{2\delta}\|p-\hat p\|_2^2 \\ &\le \mathcal{Q}(e+s\eta) + \mathcal{H}(\sigma, p+sq-\hat p) + \frac{\varepsilon}{2\delta}\|p+sq-\hat p\|_2^2 \\ &= \mathcal{Q}(e) + s\int_\Omega \sigma : \eta\,dx + s^2\mathcal{Q}(\eta) + \mathcal{H}(\sigma, p+sq-\hat p) \\ &\qquad + \frac{\varepsilon}{2\delta}\|p-\hat p\|_2^2 + s\frac{\varepsilon}{\delta}\int_\Omega (p-\hat p):q\,dx + s^2\frac{\varepsilon}{2\delta}\|q\|_2^2.\end{aligned}$$

Using the convexity of $\mathcal{H}$, we deduce that $\psi(s) := \mathcal{H}(\sigma, p+sq-\hat p)$ is convex as well, and thus $\psi(s)-\psi(0) \le s(\psi(1)-\psi(0))$. Dividing the previous inequality by s, and letting s tend to zero implies that

$$\int_\Omega \sigma : \eta\,dx + \mathcal{H}(\sigma, q+p-\hat p) - \mathcal{H}(\sigma, p-\hat p) + \frac{\varepsilon}{\delta}\int_\Omega (p-\hat p) : q\,dx \ge 0,$$

which is (f_2).

We now deduce (f_3) from (f_2). Taking first $\pm(\varphi, E\varphi, 0) \in A_{\rm reg}(0)$, for any $\varphi \in \mathcal{C}^\infty_c(\Omega;\mathbb{R}^n)$, as test in (f_2), we get that $\int_\Omega \sigma : E\varphi\,dx = 0$ and thus $\operatorname{div}\sigma = 0$ a.e. in Ω. Then choosing $(0, -q+p-\hat p, q-p+\hat p) \in A_{\rm reg}(0)$, for any $q \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, as test in (f_2), we deduce that

$$\mathcal{H}(\sigma, q) \ge \mathcal{H}(\sigma, p-\hat p) + \int_\Omega \Big(\sigma - \frac{\varepsilon}{\delta}(p-\hat p)\Big) : (q-(p-\hat p))\,dx,$$

which, by definition of the subdifferential, implies that $\sigma - \frac{\varepsilon}{\delta}(p-\hat p) \in \partial_2\mathcal{H}(\sigma, p-\hat p)$. By a standard result on convex integrands this is equivalent to (f_3).

Finally, we show that (f_3) implies (f_1). Indeed, consider $(v,\eta,q) \in A_{\rm reg}(\hat w)$, then from the subdifferential condition in (f_3), we get that

$$\begin{aligned}&\mathcal{H}(\sigma, q-\hat p) \ge \mathcal{H}(\sigma, p-\hat p) + \int_\Omega \Big(\sigma - \frac{\varepsilon}{\delta}(p-\hat p)\Big) : ((q-\hat p)-(p-\hat p))\,dx \\ &= \mathcal{H}(\sigma, p-\hat p) + \int_\Omega \sigma : (Ev-Eu)\,dx - \int_\Omega \sigma : (\eta-e)\,dx - \frac{\varepsilon}{\delta}\int_\Omega (p-\hat p) : ((q-\hat p)-(p-\hat p))\,dx.\end{aligned}$$

Since $\operatorname{div}\sigma = 0$ a.e. in Ω and $u-v \in H^1_0(\Omega;\mathbb{R}^n)$,

$$\int_\Omega \sigma : (Ev-Eu)\,dx = 0.$$

On the other hand,

$$-\int_\Omega \sigma : (\eta-e)\,dx = \mathcal{Q}(e) + \mathcal{Q}(e-\eta) - \mathcal{Q}(\eta),$$

and similarly

$$-\int_\Omega (p-\hat p) : ((q-\hat p)-(p-\hat p))\,dx = \frac{1}{2}(\|p-\hat p\|_2^2 + \|p-q\|_2^2 - \|q-\hat p\|_2^2).$$

Gathering everything yields

$$\begin{aligned}\mathcal{Q}(\eta) + \mathcal{H}(\sigma, q-\hat p) + \frac{\varepsilon}{2\delta}\|q-\hat p\|_2^2 \\ &\ge \mathcal{Q}(e) + \mathcal{H}(\sigma, p-\hat p) + \frac{\varepsilon}{2\delta}\|p-\hat p\|_2^2 + \mathcal{Q}(e-\eta) + \frac{\varepsilon}{2\delta}\|p-q\|_2^2 \\ &\ge \mathcal{Q}(e) + \mathcal{H}(\sigma, p-\hat p) + \frac{\varepsilon}{2\delta}\|p-\hat p\|_2^2,\end{aligned}$$

hence (f_1). □

We now detail the incremental problem. In all that follows, the boundary datum w is the trace on $\partial\Omega$ of a function, still denoted by w, which lies in $H^1(0,T;H^1(\Omega;\mathbb{R}^n))$. The initial conditions on the triplet (u,e,p) are $(u_0,e_0,p_0)\in A_{\rm reg}(w(0))$ (see (2.1)) such that $\mathrm{div}\sigma_0=0$ a.e. in Ω, where $\sigma_0:=Ae_0$.

2.2. **The incremental problem.** Let $T>0$, and consider a sequence of nested subdivisions $(t_k^i)_{0\le i\le N(k)}$ of the time interval $[0,T]$, with the following properties:

$$\begin{gathered}0=t_k^0<t_k^1<\ldots<t_k^{N(k)-1}<t_k^{N(k)}=T\\ t_k^i-t_k^{i-1}=\delta_k\searrow 0 \text{ with } k\nearrow\infty\\ N(k)\delta_k=T\\ \{t_k^i: i=1,\ldots,N(k)\}\subset\{t_l^i: i=1,\ldots,N(l)\},\ k\le l.\end{gathered}$$

Let $\varepsilon>0$. We assume that k is large enough so that

$$\frac{C_H\beta_A^2\delta_k}{\alpha_A\varepsilon}<\frac{1}{3}. \tag{2.4}$$

For every $i\in\{1,\ldots,N(k)\}$, we set $w_k^i:=w(t_k^i)$, and we define (u_k^i,e_k^i,p_k^i) by induction. We first set $(u_k^0,e_k^0,p_k^0):=(u_0,e_0,p_0)$ the initial datum which belongs by assumption to $A_{\rm reg}(w(0))$. For $i\in\{1,\ldots,N(k)\}$, we define $(u_k^i,e_k^i,p_k^i)\in A_{\rm reg}(w_k^i)$ as a solution to the minimization problem

$$\mathcal{Q}(e_k^i)+\mathcal{H}(\sigma_k^i,p_k^i-p_k^{i-1})+\frac{\varepsilon}{2\delta_k}\|p_k^i-p_k^{i-1}\|_2^2\le\mathcal{Q}(\eta)+\mathcal{H}(\sigma_k^i,q-p_k^{i-1})+\frac{\varepsilon}{2\delta_k}\|q-p_k^{i-1}\|_2^2, \tag{2.5}$$

for any $(v,\eta,q)\in A_{\rm reg}(w_k^i)$, where $\sigma_k^i=Ae_k^i$. By (2.4) and Proposition 2.2 such a solution exists.
Also, as a consequence of Proposition 2.3,

$$\operatorname{div}\sigma_k^i=0 \text{ a.e. in } \Omega \tag{2.6}$$

$$\sigma_k^i-\tfrac{\varepsilon}{\delta_k}(p_k^i-p_k^{i-1})\in\partial_2\mathcal{H}(\sigma_k^i,p_k^i-p_k^{i-1}). \tag{2.7}$$

Now, by (1.13) and the homogeneity of degree 0 of $\partial_2\mathcal{H}$ in the second variable, (2.7) also reads as

$$\sigma_k^i\in\partial_2\mathcal{H}_\varepsilon\left(\sigma_k^i,\frac{p_k^i-p_k^{i-1}}{\delta_k}\right),$$

and, by convex duality and (1.14), this is equivalent to

$$\frac{p_k^i-p_k^{i-1}}{\delta_k}=N_\varepsilon(\sigma_k^i,\sigma_k^i) \text{ a.e. in } \Omega. \tag{2.8}$$

We next define, for $t\in[t_k^i,t_k^{i+1})$, the piecewise constant interpolations

$$u_k(t):=u_k^i,\quad e_k(t):=e_k^i,\quad \sigma_k(t):=Ae_k^i,\quad p_k(t):=p_k^i,\quad w_k(t):=w_k^i,$$

and the piecewise affine interpolations

$$\hat e_k(t):=e_k^i+\frac{t-t_k^i}{\delta_k}(e_k^{i+1}-e_k^i),\quad \hat\sigma_k(t):=A\hat e_k(t),\quad \hat p_k(t):=p_k^i+\frac{t-t_k^i}{\delta_k}(p_k^{i+1}-p_k^i),$$

and derive a discrete energy inequality between two arbitrary times.

Lemma 2.4. *There exists a sequence $\omega_k \to 0^+$ such that for every $k \in \mathbb{N}$ and every $0 \le t_1 \le t_2 \le T$ with $t_1 \in [t_k^{j_1}, t_k^{j_1+1})$ and $t_2 \in [t_k^{j_2}, t_k^{j_2+1})$,*

$$\mathcal{Q}(e_k(t_2)) + \int_{t_k^{j_1}}^{t_k^{j_2}} \mathcal{H}(\sigma_k(s), \dot{\hat{p}}_k(s))\, ds + \frac{\varepsilon}{2}\int_{t_k^{j_1}}^{t_k^{j_2}} \|\dot{\hat{p}}_k(s)\|_2^2\, ds \le \mathcal{Q}(e_k(t_1)) + \int_{t_k^{j_1}}^{t_k^{j_2}} \int_\Omega \sigma_k(s) : E\dot{w}(s)\, dx\, ds + \omega_k. \quad (2.9)$$

Proof. In (2.5), we take as competitor $(u_k^{i-1} + w_k^i - w_k^{i-1}, e_k^{i-1} + Ew_k^i - Ew_k^{i-1}, p_k^{i-1}) \in A_{\rm reg}(w_k^i)$. Then

$$\mathcal{Q}(e_k^i) + \mathcal{H}(\sigma_k^i, p_k^i - p_k^{i-1}) + \frac{\varepsilon}{2\delta_k}\|p_k^i - p_k^{i-1}\|_2^2 \le \mathcal{Q}(e_k^{i-1} + Ew_k^i - Ew_k^{i-1}) = \mathcal{Q}(e_k^{i-1}) + \mathcal{Q}(Ew_k^i - Ew_k^{i-1}) + \int_\Omega \sigma_k^{i-1} : (Ew_k^i - Ew_k^{i-1})\, dx. \quad (2.10)$$

Since Ew is absolutely continuous in time with values in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, then

$$Ew_k^i - Ew_k^{i-1} = \int_{t_k^{i-1}}^{t_k^i} E\dot{w}(s)\, ds.$$

By (1.19),

$$\mathcal{Q}(Ew_k^i - Ew_k^{i-1}) \le \frac{\beta_A}{2}\left\|\int_{t_k^{i-1}}^{t_k^i} E\dot{w}(s)\, ds\right\|_2^2 \le \frac{\beta_A}{2}\left(\int_{t_k^{i-1}}^{t_k^i} \|E\dot{w}(s)\|_2\, ds\right)^2 \le \frac{\beta_A}{2}\omega(\delta_k)\int_{t_k^{i-1}}^{t_k^i} \|E\dot{w}(s)\|_2\, ds, \quad (2.11)$$

where $\omega : [0,+\infty) \to [0,+\infty)$ is an infinitesimal function in 0. In view of (2.10) and (2.11),

$$\mathcal{Q}(e_k^i) + \mathcal{H}(\sigma_k^i, p_k^i - p_k^{i-1}) + \frac{\varepsilon}{2\delta_k}\|p_k^i - p_k^{i-1}\|_2^2 \le \mathcal{Q}(e_k^{i-1}) + \frac{\beta_A}{2}\omega(\delta_k)\int_{t_k^{i-1}}^{t_k^i} \|E\dot{w}(s)\|_2\, ds + \int_{t_k^{i-1}}^{t_k^i}\int_\Omega \sigma_k(s) : E\dot{w}(s)\, dx\, ds.$$

Let $0 \le t_1 \le t_2 \le T$, then there exist unique j_1, $j_2 \in \{1, \dots, N(k)\}$ such that $t_1 \in [t_k^{j_1}, t_k^{j_1+1})$ and $t_2 \in [t_k^{j_2}, t_k^{j_2+1})$. Summing up for $i = j_1 + 1$ to j_2, and using the 1-homogeneity of $\mathcal{H}$ in its second variable leads to

$$\mathcal{Q}(e_k(t_2)) + \sum_{i=j_1+1}^{j_2} \delta_k \mathcal{H}\left(\sigma_k^i, \frac{p_k^i - p_k^{i-1}}{\delta_k}\right) + \frac{\varepsilon}{2}\sum_{i=j_1+1}^{j_2} \delta_k \left\|\frac{p_k^i - p_k^{i-1}}{\delta_k}\right\|_2^2 \le \mathcal{Q}(e_k(t_1)) + \frac{\beta_A}{2}\omega(\delta_k)\int_0^T \|E\dot{w}(s)\|_2\, ds + \int_{t_k^{j_1}}^{t_k^{j_2}}\int_\Omega \sigma_k(s) : E\dot{w}(s)\, dx\, ds,$$

which implies (2.9) with $\omega_k := \frac{\beta_A}{2}\omega(\delta_k)\int_0^T \|E\dot{w}(s)\|_2\, ds$. □

With the help of the previously derived energy inequality, we deduce bounds on the triplet $(u_k(t), e_k(t), p_k(t))$ in suitable energy spaces, namely, that there exists a constant $C_T > 0$ (independent of k and ε) such that

$$\sup_{t\in[0,T]} \|e_k(t)\|_2 \le C_T, \tag{2.12}$$

$$\int_0^T \|\dot{e}_k(s)\|_2^2\, ds \le \frac{C_T}{\varepsilon}, \tag{2.13}$$

and

$$\int_0^T \|\dot{p}_k(s)\|_2^2\, ds \le \frac{C_T}{\varepsilon}. \tag{2.14}$$

Indeed, apply Lemma 2.4 with $t_1 = 0$ and $t_2 = t$. Coercivity and boundedness of A (see (1.19)) implies

$$\alpha_A \sup_{t\in[0,T]} \|e_k(t)\|_2^2 \le \beta_A \|e_0\|_2^2 + 2\beta_A \sup_{t\in[0,T]} \|e_k(t)\|_2 \int_0^T \|E\dot{w}(s)\|_2\, ds + 2\omega_k,$$

hence that

$$\sup_{t\in[0,T]} \|e_k(t)\|_2 \le C_T,$$

for some constant $C_T > 0$ independent of k and ε. Thanks again to Lemma 2.4 with $t_1 = 0$ and $t_2 = T$,

$$\int_0^T \|\dot{p}_k(s)\|_2^2\, ds \le \frac{C_T}{\varepsilon}.$$

By kinematic compatibility,

$$\frac{e_k^{i+1} - e_k^i}{\delta_k} = \frac{Eu_k^{i+1} - Eu_k^i}{\delta_k} - \frac{p_k^{i+1} - p_k^i}{\delta_k}.$$

Taking the L^2-scalar product with $\sigma_k^{i+1} - \sigma_k^i$, and using the fact that $\mathrm{div}\sigma_k^{i+1} = \mathrm{div}\sigma_k^i = 0$ a.e. in Ω and $u_k^{i+1} - u_k^i = w_k^{i+1} - w_k^i$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$,

$$\int_\Omega \frac{e_k^{i+1} - e_k^i}{\delta_k} : (\sigma_k^{i+1} - \sigma_k^i)\, dx = \int_\Omega \left(\frac{Ew_k^{i+1} - Ew_k^i}{\delta_k} - \frac{p_k^{i+1} - p_k^i}{\delta_k} \right) : (\sigma_k^{i+1} - \sigma_k^i)\, dx.$$

In view of (1.19), this yields

$$\alpha_A \int_{t_k^i}^{t_k^{i+1}} \|\dot{e}_k(s)\|_2^2\, ds \le \int_{t_k^i}^{t_k^{i+1}} \int_\Omega (E\dot{w}(s) - \dot{p}_k(s)) : \dot{\sigma}_k(s)\, dx\, ds,$$

and thanks to the Cauchy-Schwartz and the triangle inequality, we then get, using (1.19) again,

$$\int_{t_k^i}^{t_k^{i+1}} \int_\Omega (E\dot{w}(s) - \dot{p}_k(s)) : \dot{\sigma}_k(s)\, dx\, ds$$
$$\le \beta_A \left(\int_{t_k^i}^{t_k^{i+1}} (\|E\dot{w}(s)\|_2 + \|\dot{p}_k(s)\|_2)^2\, ds \right)^{1/2} \left(\int_{t_k^i}^{t_k^{i+1}} \|\dot{e}_k(s)\|_2^2\, ds \right)^{1/2}.$$

Using both previous relations, summing from $i = 0$ to $i = N(k) - 1$ and applying Cauchy-Schwartz inequality once more, we finally obtain,

$$\int_0^T \|\dot{e}_k(s)\|_2^2\, ds \le \frac{\beta_A^2}{\alpha_A^2} \int_0^T (\|E\dot{w}(s)\|_2^2 + \|\dot{p}_k(s)\|_2^2)\, ds.$$

The bound (2.13) is then a direct consequence of (2.14).

Remark 2.5. Note that, in lieu of the constant C_T, the bounds (2.12), (2.13) and (2.14) can be restated in terms of an expression of the form $a\big(\int_0^T \|E\dot w(s)\|_2\,ds + T\omega(\delta_k)\big)$, with $a \ge 0$ continuous and non decreasing. In particular, that will imply that the bounds in Remark 2.9 below are given in terms of a constant C_T whose dependence on T is of the form $a(\int_0^T \|E\dot w(s)\|_2^2\,ds)$. ¶

In order to pass to the limit in the discrete flow rule (2.7) we need to get strong compactness on the sequence of stresses $\{\sigma_k\}$. To that aim, we next prove that $\{e_k\}$ satisfies the Cauchy condition in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$.

Lemma 2.6. *The sequences $\{e_k\}$ and $\{p_k\}$ satisfy the Cauchy condition in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, and $\{u_k\}$ is a Cauchy sequence in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^n))$.*

Proof. From (2.8),

$$p_k^i - p_k^{i-1} = \delta_k N_\varepsilon(\sigma_k^i,\sigma_k^i),$$

and since $Eu_k^i = e_k^i + p_k^i$, and $Eu_k^{i-1} = e_k^{i-1} + p_k^{i-1}$,

$$e_k^i - e_k^{i-1} = Eu_k^i - Eu_k^{i-1} - \delta_k N_\varepsilon(\sigma_k^i,\sigma_k^i).$$

Summation for $i=1$ to j leads to

$$e_k^j - e_0 = Eu_k^j - Eu_0 - \sum_{i=1}^j \delta_k N_\varepsilon(\sigma_k^i,\sigma_k^i).$$

Consequently, if $t\in[t_k^j,t_k^{j+1})$, then, in view of (1.15), (1.19), (2.12), and the fact that $|t-t_k^{j+1}|\le\delta_k$ and $|t_k^1|\le\delta_k$,

$$e_k(t) - e_0 = Eu_k(t) - Eu_0 - \int_0^t N_\varepsilon(\sigma_k(s),\sigma_k(s))\,ds + R_k(t)$$

with

$$\sup_{t\in[0,T]} \|R_k(t)\|_2 \le \frac{4}{\varepsilon}\beta_A\delta_k C_T. \tag{2.15}$$

Then, by difference,

$$e_k(t) - e_l(t) = Eu_k(t) - Eu_l(t) - \int_0^t \left(N_\varepsilon(\sigma_k(s),\sigma_k(s)) - N_\varepsilon(\sigma_l(s),\sigma_l(s))\right)\,ds + R_k(t) - R_l(t).$$

Taking the scalar product in L^2 with $\sigma_k(t)-\sigma_l(t)$, Cauchy-Schwartz inequality, (2.15) and Lemma 1.7, imply that

$$\begin{aligned}\alpha_A\|e_k(t)-e_l(t)\|_2^2 &\le \int_\Omega (\sigma_k(t)-\sigma_l(t)) : (e_k(t)-e_l(t))\,dx\\ &\le \int_\Omega (\sigma_k(t)-\sigma_l(t)) : (Eu_k(t)-Eu_l(t))\,dx\\ &\quad + \beta_A\|e_k(t)-e_l(t)\|_2\left(\frac{4\beta_A C_T}{\varepsilon}(\delta_k+\delta_l) + \frac{2\beta_A C_H}{\varepsilon}\int_0^t \|e_k(s)-e_l(s)\|_2\,ds\right).\end{aligned}$$

But, since $\operatorname{div}\sigma_k(t) = \operatorname{div}\sigma_l(t) = 0$ a.e. in Ω and $u_k(t)-u_l(t) = w_k(t)-w_l(t)$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$,

$$\int_\Omega (\sigma_k(t)-\sigma_l(t)) : (Eu_k(t)-Eu_l(t))\,dx = \int_\Omega (\sigma_k(t)-\sigma_l(t)) : (Ew_k(t)-Ew_l(t))\,dx,$$

so that the Cauchy-Schwartz inequality leads to

$$\|e_k(t)-e_l(t)\|_2 \le C_\varepsilon\left(\delta_k+\delta_l+\|Ew_k(t)-Ew_l(t)\|_2 + \int_0^t \|e_k(s)-e_l(s)\|_2\,ds\right),$$

for some constant C_ε independent of k, l and t.

Since $Ew \in H^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, then it is Hölder continuous with values in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. Hence $\{Ew_k\}$ is a Cauchy sequence in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, thus $\|Ew_k(t)-Ew_l(t)\|_2 \le \omega(k,l) \searrow 0$ as $k,\, l \nearrow +\infty$. Application of Gronwall's inequality yields in turn

$$\|e_k(t)-e_l(t)\|_2 \le C_\varepsilon\,(\delta_k+\delta_l+\omega(k,l))\,e^{C_\varepsilon T}.$$

and this completes the proof that $\{e_k\}$, hence $\{\sigma_k\}$, are Cauchy sequences in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$.

Now, similarly we would get

$$p_k(t)-p_l(t)=\int_0^t (N_\varepsilon(\sigma_k(s),\sigma_k(s))-N_\varepsilon(\sigma_l(s),\sigma_l(s)))\,ds+R_k(t)-R_l(t),$$

from which we would deduce that $\{p_k\}$ is also a Cauchy sequence in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$. From the kinematic compatibility, it follows that $\{Eu_k\}$ is Cauchy in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$ as well. Thanks to the Poincaré-Korn's inequality we deduce that $\{u_k\}$ is a Cauchy sequence in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^n))$. ☐

We are now in a position to show the existence of a non-associative visco-plastic evolution.

2.3. **Proof of Theorem 2.1.** At the possible expense of extracting a subsequence, we conclude, thanks to (2.13), (2.14), Lemma 2.6 and because $u_k(0)=u_0$, $\hat p_k(0)=p_0$, to the existence of a quintuplet $(u_\varepsilon,e_\varepsilon,\hat e_\varepsilon,p_\varepsilon,\hat p_\varepsilon)$ such that

$$\begin{aligned} u_k &\to u_\varepsilon \text{ strongly in } L^\infty(0,T;H^1(\Omega;\mathbb{R}^n)),\\ e_k &\to e_\varepsilon \text{ strongly in } L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})),\\ p_k &\to p_\varepsilon \text{ strongly in } L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})),\\ \hat p_k &\rightharpoonup \hat p_\varepsilon \text{ weakly in } H^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})),\\ \hat e_k &\rightharpoonup \hat e_\varepsilon \text{ weakly in } H^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})). \end{aligned} \tag{2.16}$$

Consider the quintuplet $(u_\varepsilon(t),e_\varepsilon(t),\hat e_\varepsilon(t),p_\varepsilon(t),\hat p_\varepsilon(t))$ obtained through the convergences in (2.16) and remark first that the kinematic compatibility relation passes to the limit, *i.e.*, $Eu_\varepsilon(t)=e_\varepsilon(t)+p_\varepsilon(t)$ a.e. in Ω and $u_\varepsilon(t)=w(t)$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$. Now, since

$$\hat e_k(t)=e_k(t)+\int_{t_k^i}^t \dot{\hat e}_k(s)\,ds,$$

if $t\in[t_k^i,t_k^{i+1})$, then, from bound (2.13),

$$\sup_{t\in[0,T]}\|\hat e_k(t)-e_k(t)\|_2\le C_\varepsilon\sqrt{\delta}_k\to 0,$$

as $k\to+\infty$, which implies that $\hat e_\varepsilon(t)=e_\varepsilon(t)$. The same argument shows that $\hat p_\varepsilon(t)=p_\varepsilon(t)$ and thus, e_ε and $p_\varepsilon\in H^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$. Moreover, the Poincaré-Korn's inequality yields $u_\varepsilon\in H^1(0,T;H^1(\Omega;\mathbb{R}^n))$.

By (2.6), $\operatorname{div}\sigma_k(t)=0$ a.e. in Ω, so that $\operatorname{div}\sigma_\varepsilon(t)=0$ a.e. in Ω.

In view of (2.8), we have, for $t\in[t_k^i,t_k^{i+1})$,

$$\begin{aligned}\dot{\hat p}_k(t) &= N_\varepsilon(\sigma_k(t)+\sigma_k^{i+1}-\sigma_k^i,\sigma_k(t)+\sigma_k^{i+1}-\sigma_k^i)\\ &= N_\varepsilon(\sigma_k(t),\sigma_k(t))+\tilde R_k(t),\end{aligned}$$

where, thanks to Lemma 1.7, (1.19) and (2.13), $\|\tilde R_k(t)\|_2\le C_\varepsilon\sqrt{\delta_k}$. As a consequence, $\tilde R_k\to 0$ strongly in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$. On the other hand, since, by (2.16), $\sigma_k=Ae_k\to\sigma_\varepsilon:=Ae_\varepsilon$ strongly in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$, we conclude that $N_\varepsilon(\sigma_k,\sigma_k)\to N_\varepsilon(\sigma_\varepsilon,\sigma_\varepsilon)$ strongly in $L^\infty(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$. This implies that, in particular,

$$\dot p_\varepsilon(t)=N_\varepsilon(\sigma_\varepsilon(t),\sigma_\varepsilon(t))$$

for a.e. $t \in [0,T]$ and a.e. $x \in \Omega$, and that actually $p_\varepsilon \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$. The remaining relations are obtained by convex duality and thanks to (1.16). The proof of the existence of the announced triplet is complete.

Now, let us prove uniqueness. Let $(u_1(t), e_1(t), p_1(t))$ and $(u_2(t), e_2(t), p_2(t))$ be two non-associative visco-plastic solutions with boundary datum w and initial condition (u_0, e_0, p_0). For $i = 1, 2$, set $\sigma_i(t) := Ae_i(t)$. Thanks to the regularized flow rule, $\dot p_i(t) = N_\varepsilon(\sigma_i(t), \sigma_i(t))$ for a.e. $t \in [0,T]$ and a.e. $x \in \Omega$, and, using kinematic compatibility, we get $\dot e_i(t) = E\dot u_i(t) - N_\varepsilon(\sigma_i(t), \sigma_i(t))$. Hence

$$\dot e_2(t) - \dot e_1(t) = E\dot u_2(t) - E\dot u_1(t) - (N_\varepsilon(\sigma_2(t), \sigma_2(t)) - N_\varepsilon(\sigma_1(t), \sigma_1(t))).$$

Taking the L^2-scalar product with $\sigma_2(t) - \sigma_1(t)$, using the fact that $\operatorname{div}\sigma_1(t) = \operatorname{div}\sigma_2(t) = 0$ a.e. in Ω and $\dot u_2(t) - \dot u_1(t) \in H^1_0(\Omega;\mathbb{R}^n)$, we get that

$$\frac{d}{dt}\mathcal{Q}(e_2(t) - e_1(t)) \le C_\varepsilon \|e_2(t) - e_1(t)\|_2^2 \le C'_\varepsilon \mathcal{Q}(e_2(t) - e_1(t)),$$

where we used the Lipschitz continuous character of N_ε (see Lemma 1.7) and (1.19). By the initial condition, we have $e_2(0) - e_1(0) = 0$, hence from Gronwall's Lemma, we deduce that $\mathcal{Q}(e_2(t) - e_1(t)) = 0$, and by (1.19) that $e_1(t) = e_2(t)$. Consequently, $\sigma_1(t) = \sigma_2(t)$ and the regularized flow rule, together with the fact that $p_1(0) = p_2(0) = p_0$, implies that $p_1(t) = p_2(t)$. Finally, from kinematic compatibility, $u_2(t) - u_1(t) \in H^1_0(\Omega;\mathbb{R}^n)$ and $Eu_1(t) = Eu_2(t)$, and consequently, Korn's inequality leads to $u_1(t) = u_2(t)$.

The proof of Theorem 2.1 is now complete.

Remark 2.7. Note that the same theorem applies with $N_\varepsilon(\sigma_\varepsilon(t), \sigma_\varepsilon(t))$ replaced by $N_\varepsilon(\sigma_\varepsilon(t) * \rho, \sigma_\varepsilon(t))$ (and $\partial_2 H(\sigma_\varepsilon(t), \dot p_\varepsilon(t))$ replaced by $\partial_2 H(\sigma_\varepsilon(t) * \rho, \dot p_\varepsilon(t))$), where ρ is a convolution kernel. We then call a solution triplet a *non-associative ρ-visco-plastic solution*, and in that case,

$$\varepsilon \|\dot p_\varepsilon(t)\|_2 = \operatorname{dist}_2(\sigma_\varepsilon(t), \mathcal{K}(\sigma_\varepsilon(t) * \rho)).$$

Remark 2.8 below also applies to that case. ¶

We end this subsection with a remark whose proof is very close to that of [9, Theorem 3.4]; we omit the proof here.

Remark 2.8. The regularized non-associative flow rule in Theorem 2.1 can be equivalently replaced by

1. Modified Stress Constraint: $\sigma_\varepsilon(t) - \varepsilon\dot p_\varepsilon(t) \in \mathcal{K}(\sigma_\varepsilon(t))$, for a.e. $t \in [0,T]$, or equivalently, since, as is classical in convex analysis, $K(\sigma) = \partial_2 H(\sigma, 0)$,

$$\sigma_\varepsilon(t) - \varepsilon\dot p_\varepsilon(t) \in \partial_2 H(\sigma_\varepsilon(t), 0) \text{ for a.e. } (x,t) \in \Omega \times [0,T];$$

2. Energy equality: $(u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t))$ satisfies the following energy equality, for every $t \in [0,T]$:

$$\mathcal{Q}(e_\varepsilon(t)) + \int_0^t \mathcal{H}(\sigma_\varepsilon(s), \dot p_\varepsilon(s))\,ds + \varepsilon\int_0^t \|\dot p_\varepsilon(s)\|_2^2\,ds = \mathcal{Q}(e_0) + \int_0^t\int_\Omega \sigma_\varepsilon(s) : E\dot w(s)\,dx\,ds,$$

or still

$$\mathcal{Q}(e_\varepsilon(t)) + \int_0^t \mathcal{H}(\sigma_\varepsilon(s), \dot p_\varepsilon(s))\,ds + \int_0^t \|\dot p_\varepsilon(s)\|_2 \operatorname{dist}_2(\sigma_\varepsilon(s), \mathcal{K}(\sigma_\varepsilon(s)))\,ds$$
$$= \mathcal{Q}(e_0) + \int_0^t\int_\Omega \sigma_\varepsilon(s) : E\dot w(s)\,dx\,ds.$$

The same applies to the non-associative ρ-visco-plastic evolution defined in Remark 2.7 above. ¶

Remark 2.9. In view of the energy equality in Remark 2.8, of (1.19), (1.10), and of bound (2.12), which is independent of ε, we have

$$\|e_\varepsilon(t)\|_2 \le C_T, \quad \|\sigma_\varepsilon(t)\|_2 \le C_T, \quad \int_0^T \|\dot p_\varepsilon(s)\|_1 ds \le C_T,$$

where C_T is an ε-independent constant. Actually, in view of Remark 2.5, the dependence of C_T in T is of the form $a\big(\int_0^T \|E\dot w(s)\|_2^2\, ds\big)$, with $a \ge 0$ continuous and non decreasing.

Once again, the same applies to the non-associative ρ-visco-plastic evolution defined in Remark 2.7 above. ¶

3. Time rescaling

In this section, we propose a rescaling of time which will permit to pass to the vanishing viscosity limit in the non-associative ρ-visco-plastic evolution. Such an idea was first put forth in [18], then, in a setting close to the present one, that of Cam-Clay plasticity, in [9].

Because of the bounds established in Remark 2.9, we can only guarantee that the limit plastic strain will be a function of bounded variation with respect to time when passing to the 0-viscosity limit. But, because of the dependence of the dissipation potential upon the (mollified) stress, this will prevent us from proving the lower semi-continuity of the dissipated energy, a must if one is to hope for some kind of energy balance. This is remedied through a rescaling of the time variable which results in a limit plastic strain with Lipschitz regularity in (rescaled time). In turn, this allows us to derive a limiting energy equality, that includes the limit of the viscous dissipation (see (3.28)–(3.43) below) and can be written in an equivalent differential form (the maximum plastic work condition in Theorem 3.1).

Jumps in the original time correspond to intervals where the mapping from the rescaled time to the original one remains constant. In those intervals the stress constraint may not be satisfied (see the partial stress constraint condition in Theorem 3.1 below). This is also reflected in the expression for the maximum plastic work.

By default, each result stated in this section will be assumed to apply solely to the non-associative ρ-visco-plastic evolution, where $\rho \in \mathcal{C}^1_c(\mathbb{R}^n)$, unless otherwise stated.

Because of the bounds in Remark 2.9, we cannot expect to keep the L^2-regularity of the fields Eu and p when passing to the 0-viscosity limit and we thus have to redefine $A_{\rm reg}(\hat w)$ from (2.1) as

$$A(\hat w) := \Big\{(v,\eta,q) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}) : \\ Ev = \eta + q \text{ in } \Omega,\ q = (\hat w - v)\odot \nu \mathcal{H}^{n-1} \text{ on } \partial\Omega\Big\}$$

with $\hat w \in H^1(\Omega;\mathbb{R}^n)$. The interpretation of the boundary condition is that, if the displacement u does not match the prescribed boundary displacement w, then the loaded boundary can experience plastic slips.

We still keep a boundary datum $w \in H^1(0,T;H^1(\Omega;\mathbb{R}^n))$. Without loss of generality, we extend w by $w(T)$ for $t \ge T$.

The main result of the paper is the following existence result for a rescaled quasistatic evolution model in non-associative plasticity.

Theorem 3.1. *Let $w \in H^1(0,T;H^1(\Omega;\mathbb{R}^n))$ and let $(u_0,e_0,p_0) \in A(w(0))$ be such that*

$$\text{div}\sigma_0 = 0 \text{ a.e. in } \Omega \text{ and } \sigma_0 \in \mathcal{K}(\sigma_0 * \rho),$$

where $\sigma_0 := Ae_0$. *Then, there exist* $\overline{T}$ *and a mapping* $[0,\overline{T}] \ni s \mapsto (u^\circ(s), e^\circ(s), p^\circ(s), t^\circ(s))$ *such that*

$$u^\circ : [0,\overline{T}] \to BD(\Omega) \text{ is strongly continuous and a.e. weakly* differentiable};$$
$$e^\circ : [0,\overline{T}] \to L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \text{ is strongly continuous and a.e. differentiable};$$
$$p^\circ : [0,\overline{T}] \to \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}) \text{ is 1-Lipschitz};$$
$$t^\circ : [0,\overline{T}] \to [0,+\infty) \text{ is nondecreasing and 1-Lipschitz, with } t^\circ(\overline{T}) \geq T.$$

Further, setting $\sigma^\circ := Ae^\circ$, *the following properties are satisfied:*

Initial condition: $(u^\circ(0), e^\circ(0), p^\circ(0), t^\circ(0)) = (u_0, e_0, p_0, 0)$;
Kinematic compatibility: *For every* $s \in [0,\overline{T}]$,

$$Eu^\circ(s) = e^\circ(s) + p^\circ(s) \text{ in } \Omega,$$
$$p^\circ(s) = (w(t^\circ(s)) - u^\circ(s)) \odot \nu\, \mathcal{H}^{n-1} \text{ on } \partial\Omega;$$

Equilibrium condition: *For every* $s \in [0,\overline{T}]$,

$$\operatorname{div} \sigma^\circ(s) = 0 \text{ a.e. in } \Omega;$$

Partial stress constraint: *For every* $s \in [0,\overline{T}] \setminus U^\circ$,

$$\sigma^\circ(s) \in \mathcal{K}(\sigma^\circ(s) * \rho),$$

where $U^\circ := \{s \in (0,\overline{T}] : t^\circ \text{ is constant in a neighborhood of } s\}$;
Maximum plastic work: *For a.e.* $s \in [0,\overline{T}]$ *with* $\sigma^\circ(s) \notin \mathcal{K}(\sigma^\circ(s) * \rho)$ *we have* $\dot{p}^\circ(s) \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ *and for a.e.* $s \in [0,\overline{T}]$,

$$\langle \sigma^\circ(s) - P_{\mathcal{K}(\sigma^\circ(s)*\rho)}(\sigma^\circ(s)), \dot{p}^\circ(s)\rangle + \mathcal{H}(\sigma^\circ(s) * \rho, \dot{p}^\circ(s)) = \langle \sigma^\circ(s), \dot{p}^\circ(s)\rangle.$$

The proof is given in Subsections 3.1 to 3.6.

Remark 3.2. The extent to which one could recover a (rescaled) flow rule from Theorem 3.1 will not be detailed here. The interested reader is directed to [10, Theorems 3.8, 3.10] which would equally apply in the current framework. It would then remain to interpret the evolution in terms of the original time variable, a task which is performed in the setting of Cam-Clay plasticity in [10, Section 5], but that will not be undertaken here. ¶

The next proposition guarantees the existence of initial data such as in Theorem 3.1.

Proposition 3.3. *There exists a triplet* $(u_0, e_0, p_0) \in A(w(0))$ *such that, setting* $\sigma_0 := Ae_0$, *then*

$$\operatorname{div}\sigma_0 = 0 \text{ a.e. in } \Omega \text{ and } \sigma_0 \in \mathcal{K}(\sigma_0 * \rho).$$

Proof. Define $K_0 := \bigcap_{\tau \in \mathbb{M}^{n\times n}_{sym}} K(\tau)$. We proved in Lemma 1.4 that each $K(\tau)$ contain a uniform ball with respect to τ. Consequently, K_0 is a non empty closed and convex subset of $\mathbb{M}^{n\times n}_{sym}$. Let us denote by

$$H_0(q) := \sup_{\tau \in K_0} \tau : q, \quad q \in \mathbb{M}^{n\times n}_{sym},$$

its support function, and

$$\mathcal{H}_0(p) := \int_{\overline{\Omega}} H_0\left(\frac{dp}{d|p|}\right) d|p|, \quad p \in \mathcal{M}(\overline{\Omega};\mathbb{M}^{n\times n}_{sym}).$$

By Lemma 1.4, H_0 satisfies the following growth and coercivity properties:

$$\alpha_H |q| \leq H_0(q) \leq \beta_H |q| \quad \text{for all } q \in \mathbb{M}^{n\times n}_{sym}. \tag{3.1}$$

Let us define $(v_0, \eta_0, q_0) := (w(0), Ew(0), 0)$, and by induction, for any $k \geq 1$, let $(v_k, \eta_k, q_k) \in A(w(0))$ be a solution of

$$\min_{(v,\eta,q)\in A(w(0))} \{\mathcal{Q}(\eta) + \mathcal{H}_0(q - q_{k-1})\}.$$

The direct method in the calculus of variations, the Poincaré-Korn inequality in $BD(\Omega)$ – see [34, Chapter II, Remark 2.5 (ii)]; referred to henceforth as Poincaré-Korn's inequality – and the convexity of $\mathcal{Q}$ and $\mathcal{H}_0$ ensure the existence of such a solution. Taking $(v, \eta, q) = (v_{k-1}, \eta_{k-1}, q_{k-1})$ as test function in the previous minimization problem, and summing up leads to

$$\|\eta_k\|_2 + \|q_k\|_1 \leq \|\eta_k\|_2 + \sum_{l=1}^{k} \|q_l - q_{l-1}\|_1 \leq C,$$

for some constant $C > 0$ independent of k, where we used (1.19) and (3.1). Using again the Poincaré-Korn inequality in $BD(\Omega)$, we can assume that, up to a subsequence, $v_k \rightharpoonup u_0$ weakly* in $BD(\Omega)$, $\eta_k \rightharpoonup e_0$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, and $q_k \rightharpoonup p_0$ weakly* in $\mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$, with $(u_0, e_0, p_0) \in A(w(0))$ (see [6, Lemma 2.1]).

Using the triangle inequality satisfied by H_0, we deduce that

$$\mathcal{Q}(\eta_k) \leq \mathcal{Q}(\eta) + \mathcal{H}_0(q - q_k) \tag{3.2}$$

for any $(v, \eta, q) \in A(w(0))$. Taking $(v, \eta, q) := (v_k + t\varphi, \eta_k + tE\varphi, q_k)$ as test function in (3.2), where $t \in \mathbb{R}$ and $\varphi \in \mathcal{C}^\infty_c(\Omega; \mathbb{M}^{n\times n}_{sym})$, and letting $t \to 0^\pm$ yields $\mathrm{div}(A\eta_k) = 0$ a.e. in Ω. Passing to the limit as $k \to +\infty$ in the distributional sense leads to $\mathrm{div}\sigma_0 = 0$ a.e. in Ω, where $\sigma_0 := Ae_0$.

Finally, taking $(v, \eta, q) = (v_k, \eta_k - t\eta, q_k + t\eta)$ as test function in (3.2), where $t > 0$ and $\eta \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, and letting again $t \to 0^+$ leads to $A\eta_k \in \partial\mathcal{H}_0(0)$. Since $\partial\mathcal{H}_0(0)$ is a sequentially weakly closed subset of $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, it follows by passing to the limit as $k \to +\infty$ that $\sigma_0 = Ae_0 \in \partial\mathcal{H}_0(0)$. By a standard localization argument, we obtain that $\sigma_0(x) \in \partial H_0(0) = K_0$ for a.e. $x \in \Omega$. In particular, we get that $\sigma_0(x) \in K([\sigma_0 * \rho](x))$ which completes the proof of the proposition. □

We are now in a position to prove Theorem 3.1. For the reader's convenience the proof is split into the following six subsections.

3.1. **The rescaled non-associative visco-plastic evolution.**

Let $w \in H^1(0, T; H^1(\Omega; \mathbb{R}^n))$ (extended by $w(T)$ for $t \geq T$) and let $(u_0, e_0, p_0) \in A(w(0))$ be such that

$$\mathrm{div}\sigma_0 = 0 \text{ a.e. in } \Omega \text{ and } \sigma_0 \in \mathcal{K}(\sigma_0 * \rho),$$

where $\sigma_0 := Ae_0$. According to [9, Lemma 5.1], there exists a sequence $\{u^\varepsilon_0\} \subset H^1(\Omega; \mathbb{R}^n)$ such that $u^\varepsilon_0 = w(0)$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, $u^\varepsilon_0 \to u_0$ strongly in $L^1(\Omega; \mathbb{R}^n)$ and $Eu^\varepsilon_0 \rightharpoonup Eu_0$ weakly* in $\mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$. Setting $p^\varepsilon_0 := Eu^\varepsilon_0 - e_0$, we get that $(u^\varepsilon_0, e_0, p^\varepsilon_0) \in A_{\mathrm{reg}}(w(0))$ satisfies

$$\begin{cases} u^\varepsilon_0 \rightharpoonup u_0 \text{ weakly* in } BD(\Omega), \\ p^\varepsilon_0 \rightharpoonup p_0 \text{ weakly* in } \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}). \end{cases} \tag{3.3}$$

Theorem 2.1 and Remark 2.7 then provide, for every $\varepsilon > 0$, a unique non-associative ρ-visco-plastic (or visco-plastic) solution $(u_\varepsilon(t), e_\varepsilon(t), p_\varepsilon(t))$, for $0 \leq t < +\infty$, with $(u^\varepsilon_0, e_0, p^\varepsilon_0)$ as initial condition.

We rescale time as follows:

$$s^\circ_\varepsilon(t) := \int_0^t (\|\dot{p}_\varepsilon(s)\|_1 + \|E\dot{w}(s)\|_2 + 1)\, ds,$$

so that $s \mapsto t^\circ_\varepsilon(s) := (s^\circ_\varepsilon)^{-1}(s)$ is strictly monotonically increasing and equi-Lipschitz on $[0, +\infty)$, *i.e.*,

$$|t^\circ_\varepsilon(s_1) - t^\circ_\varepsilon(s_2)| \leq |s_1 - s_2|$$

for every s_1 and $s_2 \geq 0$. Note that $t^\circ_\varepsilon(0) = 0$ and that, by virtue of Remark 2.9,

$$\overline{T} := a\left(\int_0^T \|E\dot{w}(s)\|_2^2\,ds\right) + T \geq s^\circ_\varepsilon(T),$$

so that $t^\circ_\varepsilon(\overline{T}) \geq T$, for each $\varepsilon > 0$.

Define, on $[0, \overline{T}]$,

$$w^\circ_\varepsilon(s) := w(t^\circ_\varepsilon(s)), \quad u^\circ_\varepsilon(s) := u_\varepsilon(t^\circ_\varepsilon(s)), \quad e^\circ_\varepsilon(s) := e_\varepsilon(t^\circ_\varepsilon(s)),$$
$$\sigma^\circ_\varepsilon(s) := \sigma_\varepsilon(t^\circ_\varepsilon(s)), \quad p^\circ_\varepsilon(s) := p_\varepsilon(t^\circ_\varepsilon(s)).$$

3.2. **Compactness.** In this subsection we deduce some compactness properties for the sequences defined above and show that the limit mappings satisfy the intial condition, the kinematic compatibility, and the equilibrium condition of Theorem 3.1. First, an application of the chain rule shows that Ew°_ε and p°_ε are 1-Lipschitz in s with values in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $\mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ respectively, *i.e.*, for every s_1 and $s_2 \in [0, \overline{T}]$,

$$\|Ew^\circ_\varepsilon(s_1) - Ew^\circ_\varepsilon(s_2)\|_2 + \|p^\circ_\varepsilon(s_1) - p^\circ_\varepsilon(s_2)\|_1 \leq |s_1 - s_2|,$$

and in particular, from (3.3)

$$\|p^\circ_\varepsilon(s)\|_1 \leq \|p^\varepsilon_0\|_1 + \overline{T} \leq M, \tag{3.4}$$

for every $s \in [0, \overline{T}]$ and for some contant $M > 0$ which is independent of ε and s. Hence, upon application of Ascoli's theorem (bounded sets in $\mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ are relatively compact and metrizable for the weak* topology), a subsequence (still indexed by ε) of p°_ε is such that

$$p^\circ_\varepsilon(s) \rightharpoonup p^\circ(s) \text{ weakly* in } \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}), \tag{3.5}$$

uniformly in $[0, \overline{T}]$ with $p^\circ \in \mathrm{Lip}([0, \overline{T}]; \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}))$. In particular $p^\circ(0) = p_0$. Because of the already mentioned properties of the space $\mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$, p° is weakly* differentiable for a.e. $s \in (0, \overline{T})$ (see, *e.g.*, [6, Theorem 7.1]) and

$$\|\dot{p}^\circ(s)\|_1 \leq 1 \quad \text{for a.e. } s \in (0, \overline{T}). \tag{3.6}$$

Also, since $s \mapsto \dot{p}^\circ(s)$ is weakly* measurable (with values in $\mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$), we conclude that

$$\dot{p}^\circ \in L^\infty_w(0, \overline{T}; \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})). \tag{3.7}$$

Finally, in view of the last bound in Remark 2.9 together with (3.5), we easily conclude that

$$\dot{p}^\circ_\varepsilon \rightharpoonup \dot{p}^\circ \text{ weakly* in } \mathcal{M}([0, T] \times \overline{\Omega})) \text{ and in } L^\infty_w(0, \overline{T}; \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})). \tag{3.8}$$

Because of the equi-Lipschitz character of t_ε, Ascoli's theorem implies the existence of a Lipschitz and nondecreasing function $t^\circ : [0, \overline{T}] \to [0, +\infty)$ such that (a subsequence of) t°_ε – still indexed by ε – converges uniformly to t° as $\varepsilon \searrow 0$. Hence, $w^\circ_\varepsilon(s) \to w^\circ(s) := w(t^\circ(s))$ strongly in $H^1(\Omega; \mathbb{R}^n)$, where $w^\circ \in H^1(0, \overline{T}; H^1(\Omega; \mathbb{R}^n))$.

We next derive some compactness properties for the sequences $\{u^\circ_\varepsilon\}$ and $\{e^\circ_\varepsilon\}$.

Lemma 3.4. *In both the non-associative ρ-visco-plastic evolution setting, and the non-associative visco-plastic evolution setting, for every $s \in [0, \overline{T}]$, there exists a triplet*

$$(u^\circ(s), e^\circ(s), p^\circ(s)) \in A(w^\circ(s))$$

such that, with $\sigma^\circ(s) := Ae^\circ(s)$, $\operatorname{div}\sigma^\circ(s) = 0$ a.e. in Ω and, for any sequence $s_\varepsilon \to s$,

$$\begin{aligned} &u^\circ_\varepsilon(s_\varepsilon) \rightharpoonup u^\circ(s) \textit{ weakly* in } BD(\Omega),\\ &e^\circ_\varepsilon(s_\varepsilon) \rightharpoonup e^\circ(s) \textit{ weakly in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym}),\\ &\sigma^\circ_\varepsilon(s_\varepsilon) \rightharpoonup \sigma^\circ(s) \textit{ weakly in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym}). \end{aligned} \tag{3.9}$$

Moreover, $(u^\circ(0), e^\circ(0), p^\circ(0)) = (u_0, e_0, p_0)$ *and finally,* $s \mapsto u^\circ(s)$ *is weakly* continuous in* $BD(\Omega)$*, and* $s \mapsto e^\circ(s)$ *and* $s \mapsto \sigma^\circ(s)$ *are weakly continuous in* $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$.

Proof. Consider an arbitrary sequence $s_\varepsilon \to s$. From Remark 2.9, we have

$$\sup_{s\in[0,\overline{T}]} \|e^\circ_\varepsilon(s)\|_2 \le C_{\overline{T}},$$

so that a (possibly s-dependent) subsequence $\{e^\circ_{\varepsilon_j}(s_{\varepsilon_j})\}$ of $\{e^\circ_\varepsilon(s_\varepsilon)\}$ is such that

$$e^\circ_{\varepsilon_j}(s_{\varepsilon_j}) \rightharpoonup e^\circ(s) \text{ weakly in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym}). \tag{3.10}$$

In view of (3.4), (3.10), and since $u^\circ_\varepsilon(s_\varepsilon) = w^\circ_\varepsilon(s_\varepsilon)$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, the Poincaré-Korn inequality in $BD(\Omega)$ implies that $\{u^\circ_\varepsilon(s_\varepsilon)\}$ is bounded in $BD(\Omega)$, hence, a (possibly s-dependent) subsequence $\{u^\circ_{\varepsilon_j}(s_{\varepsilon_j})\}$ of $\{u^\circ_\varepsilon(s_\varepsilon)\}$ is such that

$$u^\circ_{\varepsilon_j}(s_{\varepsilon_j}) \rightharpoonup u^\circ(s) \text{ weakly* in } BD(\Omega). \tag{3.11}$$

Since p°_ε is 1-Lipschitz with values in $\mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$, we deduce that $p^\circ_{\varepsilon_j}(s_{\varepsilon_j}) \rightharpoonup p^\circ(s)$ weakly* in $\mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$, where p° is defined in (3.5). Further,

$$(u^\circ_{\varepsilon_j}(s_{\varepsilon_j}), e^\circ_{\varepsilon_j}(s_{\varepsilon_j}), p^\circ_{\varepsilon_j}(s_{\varepsilon_j})) \in A(w^\circ_{\varepsilon_j}(s_{\varepsilon_j}))$$

and $w^\circ_{\varepsilon_j}(s_{\varepsilon_j}) \to w^\circ(s)$ strongly in $H^1(\Omega; \mathbb{R}^n)$, so that, through an extension argument (see [6, Lemma 2.1]),

$$(u^\circ(s), e^\circ(s), p^\circ(s)) \in A(w^\circ(s)).$$

From (3.10)

$$\sigma^\circ_{\varepsilon_j}(s_{\varepsilon_j}) \rightharpoonup \sigma^\circ(s) := Ae^\circ(s) \text{ weakly in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \tag{3.12}$$

with

$$\operatorname{div} \sigma^\circ(s) = 0 \text{ a.e. in } \Omega.$$

In particular, for any $\varphi \in H^1_0(\Omega; \mathbb{R}^n)$ we infer that

$$\mathcal{Q}(e^\circ(s)) \le \mathcal{Q}(e^\circ(s) + E\varphi). \tag{3.13}$$

We now prove that the limits u°, e° and σ° are independent of the choice of the sequence $\{s_\varepsilon\}$. To this end, consider another sequence $s'_\varepsilon \to s$. Arguing as before, we can assume, at the expense of extracting a further subsequence still denoted $\{\varepsilon_j\}$, that $u^\circ_{\varepsilon_j}(s'_{\varepsilon_j}) \rightharpoonup \hat{u}(s)$ weakly* in $BD(\Omega)$, and $e^\circ_{\varepsilon_j}(s'_{\varepsilon_j}) \rightharpoonup \hat{e}(s)$ and $\sigma^\circ_{\varepsilon_j}(s_{\varepsilon_{j_k}}) \rightharpoonup \hat{\sigma}(s) = A\hat{e}(s)$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, with $(\hat{u}(s), \hat{e}(s), p^\circ(s)) \in A(w^\circ(s))$. Hence $E(u^\circ(s) - \hat{u}(s)) = e^\circ(s) - \hat{e}(s) \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $-(u^\circ(s) - \hat{u}(s)) \odot \nu = p^\circ(s) - p^\circ(s) = 0$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$. Consequently, $\hat{u}(s) - u^\circ(s) \in H^1_0(\Omega; \mathbb{R}^n)$. Taking $\varphi = \hat{u}(s) - u^\circ(s)$ in (3.13) we deduce that

$$\mathcal{Q}(e^\circ(s)) \le \mathcal{Q}(\hat{e}(s)),$$

and inverting the roles of $e^\circ(s)$ and $\hat{e}(s)$, we get that this inequality is actually an equality. By strict convexity of $\mathcal{Q}$ we conclude that $\hat{e}(s) = e^\circ(s)$, and thus $\hat{\sigma}(s) = \sigma^\circ(s)$ and $\hat{u}(s) = u^\circ(s)$. Hence the limits are independent of the choice of the sequence $\{s_\varepsilon\}$, and by uniqueness, we infer that there is no need to extract a subsequence to get the convergences (3.10), (3.11) and (3.12). Moreover, thanks to (3.3), we clearly have that $(u^\circ(0), e^\circ(0), p^\circ(0)) = (u_0, e_0, p_0)$.

We now show that these maps are weakly continuous. Let $s_k \to s$, then by taking $s_\varepsilon = s_k$, we have $e^\circ_\varepsilon(s_k) \rightharpoonup e^\circ(s_k)$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ as $\varepsilon \to 0$. Let

$$M := \sup_{\varepsilon>0, k\in\mathbb{N}} \|e^\circ_\varepsilon(s_k)\|_2.$$

Since the weak topology of $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ is metrizable on bounded sets, we deduce that there exist a distance, denoted by d_M, in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ such that the weak topology on $\{\varphi \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) :$

$\|\varphi\|_2 \leq M\}$ is compatible with the topology induced by the distance d_M. Hence there exists $\varepsilon_k \to 0$ as $k \to +\infty$ such that $d_M(e^\circ_{\varepsilon_k}(s_k), e^\circ(s_k)) < 1/k$, and by (3.9), we also have $\lim_k d_M(e^\circ_{\varepsilon_k}(s_k), e^\circ(s)) = 0$. We conclude that $e^\circ(s_k) \rightharpoonup e^\circ(s)$ weakly in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ which completes the proof of the weak continuity of e° and also of σ°. Finally $s \mapsto Eu^\circ(s)$ is weakly* continuous in $\mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ and by the Poincaré-Korn's inequality, we get the desired weak* continuity of $s \mapsto u^\circ(s)$ in $BD(\Omega)$. □

Remark 3.5. Note that the previous result implies in particular that $s \mapsto e^\circ(s)$ and $s \mapsto \sigma^\circ(s)$ are weakly measurable (with values in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$), hence strongly measurable, so that, in view of Remark 2.9, e°, σ° both belong to $L^\infty(0, \bar{T}; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$. ¶

3.3. **Stress constraint.** We next establish the partial stress constraint of Theorem 3.1. This property only applies to the rescaled limit of the ρ-visco-plastic evolutions. The presence of the regularization kernel ρ is indeed crucial in improving the convergence of the viscous stresses and allowing the computation of the 0-viscosity limit.

We focus on the modified stress constraint in the unscaled time, stated in the first item of Remark 2.8. Passing to the 0-viscosity limit in the rescaled modified stress constraint is not convenient because the chain rule introduces a term of the form $\varepsilon\dot{p}^\circ_\varepsilon/\dot{t}^\circ_\varepsilon$ which we do not control.

Let us introduce the left-continuous (resp. right-continuous) inverse of t° defined by $s^\circ_-(t) := \sup\{s : t^\circ(s) < t\}$ (resp. $s^\circ_+(t) := \inf\{s : t^\circ(s) > t\}$) and $S^\circ := \{t \in (0,T) : s^\circ_-(t) < s^\circ_+(t)\}$. Here we use the convention $\sup\emptyset = 0$, so that $s^\circ_-(0) = 0$. Observe that $t^\circ(s^\circ_-(t)) = t^\circ(s^\circ_+(t)) = t$ and that the set S° is at most countable. By [9, Lemma 5.2], we know that for each $t \notin S^\circ$, $s^\circ_\varepsilon(t) \to s^\circ_-(t) = s^\circ_+(t)$. Hence, owing to convergences (3.5), (3.9) and the fact that p°_ε is 1-Lipschitz, we have

$$\begin{cases} u_\varepsilon(t) \rightharpoonup u^\circ(s^\circ_-(t)) \text{ weakly* in } BD(\Omega), \\ e_\varepsilon(t) \rightharpoonup e^\circ(s^\circ_-(t)) \text{ weakly in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym}), \\ \sigma_\varepsilon(t) \rightharpoonup \sigma^\circ(s^\circ_-(t)) \text{ weakly in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym}), \\ p_\varepsilon(t) \rightharpoonup p^\circ(s^\circ_-(t)) \text{ weakly* in } \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}), \end{cases} \tag{3.14}$$

for all $t \notin S^\circ$. Define

$$U^\circ := \{s \in (0, \overline{T}] : t^\circ \text{ is constant in a neighborhood of } s\},$$

and note that

$$U^\circ = \bigcup_{t\in S^\circ} (s^\circ_-(t), s^\circ_+(t)),$$

hence that it is open.

We are now in a position to prove the partial stress constraint property of Theorem 3.1.

Lemma 3.6. *For every $s \notin U^\circ$, one has*

$$\sigma^\circ(s) \in \mathcal{K}(\sigma^\circ(s) * \rho). \tag{3.15}$$

Proof. Thanks to the energy equality in the second item of Remark 2.8 and of the already established bounds on the various quantities in the right hand-side of that equality (see Remark 2.9),

$$\varepsilon\dot{p}_\varepsilon(t) \to 0 \text{ strongly in } L^2(0,T; L^2(\Omega; \mathbb{M}^{n\times n}_{sym})), \tag{3.16}$$

and also a.e. in $\Omega \times (0,T)$. Recall the modified stress constraint from that same remark, namely

$$\sigma_\varepsilon(t,x) - \varepsilon\dot{p}_\varepsilon(t,x) \in \partial_2 H([\sigma_\varepsilon(t) * \rho](x), 0) \text{ for a.e. } (x,t) \in \Omega \times [0,T].$$

Then, for any $q \in \mathbb{M}^{n\times n}_{sym}$,

$$H([\sigma_\varepsilon(t) * \rho](x), q) \geq (\sigma_\varepsilon(t,x) - \varepsilon\dot{p}_\varepsilon(t,x)) : q \text{ for a.e. } (x,t) \in \Omega \times [0,T].$$

Because of the third convergence in (3.14) and of Remark 2.9, $\sigma_\varepsilon(t) * \rho \to \sigma^\circ(s^\circ_-(t)) * \rho$ a.e. in Ω and in $L^p(\Omega; \mathbb{M}^{n\times n}_{sym})$, $p < \infty$. Consider a measurable subset $E \subset \Omega$. Integrating the relation above over E, recalling Lemma 1.5 and convergence (3.16), we may pass to the limit in that relation and we obtain, for a.e. $t \in [0, T]$,

$$H([\sigma^\circ(s^\circ_-(t)) * \rho](x), q) \geq \sigma^\circ(s^\circ_-(t), x) : q \text{ for a.e. } x \in \Omega,$$

or, equivalently,

$$\sigma^\circ(s^\circ_-(t)) \in \mathcal{K}(\sigma^\circ(s^\circ_-(t)) * \rho).$$

By the left continuity of s°_- and the weak continuity in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ of σ°, we conclude that the previous relation actually holds for every $t \in [0, T]$. A similar argument would lead to

$$\sigma^\circ(s^\circ_+(t)) \in \mathcal{K}(\sigma^\circ(s^\circ_+(t)) * \rho).$$

Since $s^\circ_-(t) < s^\circ_+(t)$ if, and only if $t^\circ(s)$ is constant over the interval $[s_0^-(t), s_0^+(t)]$, we finally obtain (3.15). □

3.4. **Some qualitative properties at rescaled jump times.** In view of Lemma 3.6, it is not clear that the stress constraint should be satisfied for all rescaled times. This suggests the introduction of the following sets:

$$A^\circ := \{s \in [0, \overline{T}] : \text{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s) * \rho)) > 0\}, \quad B^\circ := [0, \overline{T}] \setminus A^\circ.$$

By Lemma 3.6, the inclusion $A^\circ \subset U^\circ$ holds and, in view of (1.9), $\sigma^\circ(s) \in L^\infty(\Omega; \mathbb{M}^{n\times n}_{sym})$ for all $s \in B^\circ$. However, the function $\sigma^\circ(s)$ may fail to belong to $L^\infty(\Omega; \mathbb{M}^{n\times n}_{sym})$ for $s \in A^\circ$ and it will be an obstacle to a rigorous definition of the product of the stress by the plastic strain for such times (see Subsection 1.4). In fact, this lack of regularity on the stress will be compensated by a higher regularity on the plastic strain.

Lemma 3.7. *The set A° is relatively open in $[0, \overline{T}]$, and for every $S \in [0, \overline{T}]$,*

$$\int_{A^\circ \cap [0,S]} \|\dot{p}^\circ(s)\|_2 \, \text{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s) * \rho)) \, ds \leq \liminf_{\varepsilon \to 0} \int_{A^\circ \cap [0,S]} \|\dot{p}^\circ_\varepsilon(s)\|_2 \, \text{dist}_2(\sigma^\circ_\varepsilon(s), \mathcal{K}(\sigma^\circ_\varepsilon(s) * \rho)) \, ds. \quad (3.17)$$

Moreover, $p^\circ \in W^{1,1}_{\text{loc}}(A^\circ; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$.

Proof. We first show that A° is open. For any $\tau \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, the set $\mathcal{K}(\tau)$ is convex, and then the map

$$L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \ni \sigma \mapsto \text{dist}_2(\sigma, \mathcal{K}(\tau))$$

is convex as well and consequently weakly lower semicontinuous in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. On the other hand, from (1.12) we have that

$$|\text{dist}_2(\sigma, \mathcal{K}(\tau_1)) - \text{dist}_2(\sigma, \mathcal{K}(\tau_2))| \leq C_H \|\tau_1 - \tau_2\|_2,$$

for every σ, τ_1 and $\tau_2 \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, from which we deduce that $(\sigma, \tau) \to \text{dist}_2(\sigma, \mathcal{K}(\tau))$ is lower semicontinuous for the product of the weak $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$-convergence in σ with the strong $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$-convergence in τ. As a consequence, by the weak continuity in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ of $s \mapsto \sigma^\circ(s)$ established in Lemma 3.4 which implies in particular the strong continuity of $s \mapsto \sigma^\circ(s) * \rho$ in $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, we deduce that $s \mapsto \text{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s) * \rho))$ is lower semicontinuous, and thus B° is closed.

The proof of the lower semicontinuity (3.17) is verbatim that in [9, Lemma 6.4].

By the second form of the energy equality in Remark 2.8, the first bound in Remark 2.9, the Cauchy-Schwartz inequality and (1.19), we get after performing the change of variables $s = s_\varepsilon^\circ(t)$,

$$\int_0^{\overline{T}} \|\dot{p}_\varepsilon^\circ(s)\|_2 \operatorname{dist}_2(\sigma_\varepsilon^\circ(s), \mathcal{K}(\sigma_\varepsilon^\circ(s) * \rho) \, ds \leq C, \tag{3.18}$$

for some constant $C > 0$ independent of ε. Since A° is open, let us consider a connected component (a, b) of A°. In the light of the already established lower semicontinuity of $s \mapsto \operatorname{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s)* \rho)$, for any $[a', b'] \subset (a, b)$, there exists $\delta > 0$ such that

$$\operatorname{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s) * \rho) > \delta$$

for every $s \in [a', b']$. Hence from the lower semicontinuity (3.17) and the uniform bound (3.18), we get that

$$\int_{a'}^{b'} \|\dot{p}^\circ(s)\|_2 \, ds \leq \frac{C}{\delta}. \tag{3.19}$$

We know that $\dot{p}^\circ \in L^\infty_w(0, \overline{T}; \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}))$, in other words, the function $s \mapsto \langle \dot{p}^\circ(s), \varphi\rangle$ is measurable for every $\varphi \in \mathcal{C}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$. From (3.19), $\|\dot{p}^\circ(s)\|_2 < +\infty$ for a.e. $s \in (a, b)$. Hence by density the function $(a, b) \ni s \mapsto \langle \dot{p}^\circ(s), \varphi\rangle$ is measurable for every $\varphi \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, which shows that $\dot{p}^\circ : (a, b) \to L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ is weakly measurable, and thus strongly measurable, owing to Pettis Theorem. Consequently, by (3.19) we have that $\dot{p}^\circ \in L^1_{\text{loc}}(a, b; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$, and thus $p^\circ \in W^{1,1}_{\text{loc}}(a, b; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$. □

We are now in a position to state that e° and u° are absolutely continuous in time at those points where the stress constraint is not satisfied.

Lemma 3.8. *We have $e^\circ \in W^{1,1}_{\text{loc}}(A^\circ; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$ and $u^\circ \in W^{1,1}_{\text{loc}}(A^\circ; H^1(\Omega; \mathbb{R}^n))$. Moreover, if (a, b) is any connected component of A°, then $u^\circ(s_1) - u^\circ(s_2) \in H^1_0(\Omega; \mathbb{R}^n)$ for any $a < s_1 \leq s_2 < b$.*

Proof. Let (a, b) be a connected component of A°, and let $a < s_1 < s_2 < b$. Since $A^\circ \subset U^\circ$, t° is constant in (a, b) and thus,

$$w^\circ(s_1) = w^\circ(s_2). \tag{3.20}$$

By Lemma 3.7, $p^\circ(s_1) - p^\circ(s_2) \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, so that it does not charge $\partial\Omega$. Thus, $Eu^\circ(s_1) - Eu^\circ(s_2) = e^\circ(s_1) - e^\circ(s_2) + p^\circ(s_1) - p^\circ(s_2) \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ and $(u^\circ(s_1) - u^\circ(s_2)) \odot \nu = 0$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$. As a consequence, we get that $u^\circ(s_1) - u^\circ(s_2) \in H^1_0(\Omega; \mathbb{R}^n)$.

Since $\operatorname{div}\sigma^\circ(s_1) = \operatorname{div}\sigma^\circ(s_2) = 0$ a.e. in Ω,

$$\langle \sigma^\circ(s_2) - \sigma^\circ(s_1), e^\circ(s_2) - e^\circ(s_1)\rangle = -\langle \sigma^\circ(s_2) - \sigma^\circ(s_1), p^\circ(s_2) - p^\circ(s_1)\rangle.$$

Then using (1.19), we infer that

$$\|e^\circ(s_2) - e^\circ(s_1)\|_2 \leq \frac{\beta_A}{\alpha_A} \|p^\circ(s_2) - p^\circ(s_1)\|_2, \tag{3.21}$$

hence $e^\circ \in W^{1,1}_{\text{loc}}(A^\circ; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$.

Moreover, according to the Poincaré inequality and (3.21), there exists a constant $C > 0$ (depending only on Ω, α_A and β_A) such that

$$\|u^\circ(s_2) - u^\circ(s_1)\|_{H^1(\Omega;\mathbb{R}^n)} \leq C\|p^\circ(s_2) - p^\circ(s_1)\|_2,$$

hence $u^\circ \in W^{1,1}_{\text{loc}}(A^\circ; H^1(\Omega; \mathbb{R}^n))$. □

Remark 3.9. Since $A^\circ \ni s \mapsto (u^\circ(s), e^\circ(s), p^\circ(s))$ is absolutely continuous with values in (the reflexive space) $H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ we deduce, from [3, Appendix], that the derivative $(\dot{u}^\circ(s), \dot{e}^\circ(s), \dot{p}^\circ(s))$ exists for a.e. $s \in A^\circ$ for the strong topology of $H^1(\Omega; \mathbb{R}^n) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. Moreover, from [6, Lemma 2.1], we get that for a.e. $s \in A^\circ$,

$(\dot u^\circ(s), \dot e^\circ(s), \dot p^\circ(s)) \in A(0)$ since $s \mapsto w^\circ(s)$ is constant in each connected components of A°. Finally $\operatorname{div} \dot\sigma^\circ(s) = 0$ a.e. in Ω. ¶

Let us show that the dissipated energy is lower semicontinuous.

Lemma 3.10. *For every $S \in [0, \overline{T}]$, we have*

$$\liminf_{\varepsilon\to 0} \int_0^S \mathcal{H}(\sigma^\circ_\varepsilon(s) * \rho, \dot p^\circ_\varepsilon(s))\, ds \geq \int_0^S \mathcal{H}(\sigma^\circ(s) * \rho, \dot p^\circ(s))\, ds.$$

Proof. By virtue of Lemma 1.5, together with (3.6), for a.e. $s \in [0, \overline{T}]$,

$$\begin{aligned} |\mathcal{H}(\sigma^\circ_\varepsilon(s) * \rho, \dot p^\circ_\varepsilon(s)) - \mathcal{H}(\sigma^\circ(s) * \rho, \dot p^\circ_\varepsilon(s))| &\leq C_H \|(\sigma^\circ_\varepsilon(s) - \sigma^\circ(s)) * \rho\|_\infty \|\dot p^\circ_\varepsilon(s)\|_1 \\ &\leq C_H \|(\sigma^\circ_\varepsilon(s) - \sigma^\circ(s)) * \rho\|_\infty, \end{aligned}$$

while, by (3.9) and (1.20), $\sigma^\circ_\varepsilon(s) * \rho \to \sigma^\circ(s) * \rho$ uniformly on $\overline{\Omega}$. Recalling Remarks 2.9 and 3.5, dominated convergence yields

$$\liminf_{\varepsilon\to 0} \int_0^S \mathcal{H}(\sigma^\circ_\varepsilon(s) * \rho, \dot p^\circ_\varepsilon(s))\, ds = \liminf_{\varepsilon\to 0} \int_0^S \mathcal{H}(\sigma^\circ(s) * \rho, \dot p^\circ_\varepsilon(s))\, ds.$$

But the weak L^2-continuity in time of $s \mapsto \sigma^\circ(s)$ established in Lemma 3.4 implies that $\sigma^\circ * \rho \in \mathcal{C}([0, \overline{T}] \times \overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$, so that $H([\sigma^\circ(s) * \rho](x), \xi)$ is continuous in (x, s, ξ) and convex, one-homogeneous in ξ. In view of (3.8), Reshetnyak's Theorem (see, *e.g.*, [1, Theorem 2.38]) yields

$$\liminf_{\varepsilon\to 0} \int_0^S \mathcal{H}(\sigma^\circ(s) * \rho, \dot p^\circ_\varepsilon(s))\, ds \geq \int_{[0,S]\times\overline{\Omega}} H\left([\sigma^\circ(s) * \rho](x), \frac{d\dot p^\circ}{d|\dot p^\circ|}(s,x)\right) d|\dot p^\circ|(s,x).$$

But, by virtue of (3.7), the measure $\dot p^\circ$ disintegrates as

$$\dot p^\circ = |\dot p^\circ(s)|(\Omega) \mathcal{L}^1_s \otimes \frac{\dot p^\circ(s)}{|\dot p^\circ(s)|(\Omega)},$$

so that, by [1, Corollary 2.29],

$$|\dot p^\circ| = |\dot p^\circ(s)|(\Omega) \mathcal{L}^1_s \otimes \frac{|\dot p^\circ(s)|}{|\dot p^\circ(s)|(\Omega)}.$$

Then, application of [1, Theorem 2.28] implies that

$$s \mapsto \int_{\overline{\Omega}} H\left([\sigma^\circ(s) * \rho](x), \frac{d\dot p^\circ}{d|\dot p^\circ|}(s,x)\right) \frac{1}{|\dot p^\circ(s)|(\Omega)} d|\dot p^\circ(s)|(x) \in L^1(0, S; |\dot p^\circ(s)|(\Omega)\mathcal{L}^1_s),$$

that is that $s \mapsto \mathcal{H}(\sigma^\circ(s) * \rho, \dot p^\circ(s))$ is $\mathcal{L}^1$-measurable and that

$$\int_{[0,S]\times\overline{\Omega}} H\left([\sigma^\circ(s) * \rho](x), \frac{d\dot p^\circ}{d|\dot p^\circ|}(s,x)\right) d|\dot p^\circ|(s,x) = \int_0^S \mathcal{H}(\sigma^\circ(s) * \rho, \dot p^\circ(s))\, ds,$$

which completes the proof of the lemma. □

3.5. **Rescaled principle of maximum plastic work.** It is common mechanical knowledge that, in classical elasto-plasticity, the flow rule is equivalent to what is usually referred to as Hill's principle of maximum plastic work. In our (rescaled) context, this would read as

$$\begin{aligned} &\langle \tau, \dot p^\circ(s)\rangle \leq \langle \sigma^\circ(s), \dot p^\circ(s)\rangle \text{ for a.e. } s \in (0, \overline{T}), \text{ and every } \tau \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \\ &\qquad\qquad \text{with } \operatorname{div}\tau = 0 \text{ and } \tau(x) \in K([\sigma^\circ(s) * \rho](x)) \text{ for a.e. } x \in \Omega. \end{aligned}$$

Further, since H is a support function, this is also formally equivalent to

$$\mathcal{H}(\sigma^\circ(s) * \rho, \dot p^\circ(s)) = \langle \sigma^\circ(s), \dot p^\circ(s)\rangle \text{ for a.e. } s \in (0, \overline{T}).$$

The maximum plastic work identity of Theorem 3.1 can be viewed as a variant of the previous equality, accounting for the fact that the stress constraint could not be met at all times, but only at those that live in B^0.

As mentioned in the introduction, the extent to which this relation will imply a more classical flow rule has been analyzed in great details in [10] in a different context. The results obtained there would apply verbatim to the situation at hand.

The proof will be based on the following derivability result, the proof of which is postponed to the next subsection.

Theorem 3.11. *The map $s \mapsto e^\circ(s)$ is differentiable for a.e. $s \in [0, \overline{T}]$ for the strong $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$-topology. Moreover, for every $0 \le s_1 \le s_2 \le \overline{T}$,*

$$\mathcal{Q}(e^\circ(s_2)) - \mathcal{Q}(e^\circ(s_1)) = \int_{s_1}^{s_2} \int_\Omega \sigma^\circ(s,x) : \dot{e}^\circ(s,x)\, dx\, ds. \tag{3.22}$$

Finally, the map $s \mapsto (u^\circ(s), e^\circ(s))$ is strongly continuous in $BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$.

Remark 3.12. In view of the Lipschitz character of $p^\circ(s)$, of Theorem 3.11, and upon appealing to Poincaré-Korn's inequality in $BD(\Omega)$, we conclude that the map $s \mapsto u^\circ(s)$ is weakly* differentiable in $BD(\Omega)$ for a.e. $s \in (0, \overline{T})$, and that the triplet $(\dot{u}^\circ(s), \dot{e}^\circ(s), \dot{p}^\circ(s))$ belongs to $A(\dot{w}^0(s))$ for those s's. ¶

We also need the following remark.

Remark 3.13. Let $\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$. It can be checked that, since the function $(x,q) \mapsto H([\sigma * \rho](x), q)$ is continuous with respect to $(x,q) \in \Omega \times \mathbb{M}^{n\times n}_{sym}$ by virtue of Lemma 1.3, the multi-valued map $x \mapsto K([\sigma * \rho](x))$ is continuous on Ω in the sense that

$$\forall \varepsilon > 0, \forall x \in \Omega,\ \exists r_x > 0 \text{ such that } \forall y \in B(x, r_x):\ d_H(K([\sigma * \rho](x)), K([\sigma * \rho](y))) < \varepsilon,$$

where d_H denotes the Hausdorff distance. Consider now $\tau \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \cap \mathcal{K}(\sigma * \rho)$ with $\operatorname{div} \tau = 0$, and let $p \in \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ be such that $(u,e,p) \in A(w)$ for some $(u,e,w) \in BD(\Omega) \times L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times H^1(\Omega; \mathbb{R}^n)$. Using a partition of unity, together with a translation and a convolution process, it is easily shown, in the spirit of [32, Lemma 3.1], that there exists a sequence $\{\tau_j\} \subset \mathcal{C}^\infty(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$ such that

$$\tau_j \to \tau \text{ strongly in } L^2(\Omega; \mathbb{M}^{n\times n}_{sym}),\ \operatorname{div} \tau_j = 0,$$

and, for each $\varepsilon > 0$, there exists $j_\varepsilon \in \mathbb{N}$ such that

$$\operatorname{dist}\big(\tau_j(x), K([\sigma * \rho](x))\big) < \varepsilon, \text{ for all } x \in \overline{\Omega}, \text{ and all } j \ge j_\varepsilon.$$

See also [20, Proposition 3.9] for a proof that only uses local approximations of τ.

Then, using the definition of H,

$$\varepsilon + H\left([\sigma * \rho](x), \frac{dp}{d|p|}(x)\right) \ge \tau_j(x) : \frac{dp}{d|p|}(x) \quad \text{ for } |p|\text{-a.e. } x \in \overline{\Omega}.$$

Hence, integrating over $\overline{\Omega}$ with respect to $|p|$ and using (1.18), we infer that

$$\begin{aligned}\varepsilon|p|(\overline{\Omega}) + \mathcal{H}(\sigma * \rho, p) &= \varepsilon|p|(\overline{\Omega}) + \int_{\overline{\Omega}} H\left([\sigma * \rho](x), \frac{dp}{d|p|}(x)\right) d|p|(x) \\ &\ge \int_{\overline{\Omega}} \tau_j(x) : \frac{dp}{d|p|}(x) d|p|(x) = \langle \tau_j, p\rangle.\end{aligned}$$

From the integration by parts formula (1.17) together with the established convergences of the sequence $\{\tau_j\}$, we get that $\langle \tau_j, p\rangle \to \langle \tau, p\rangle$. Consequently, passing to the limit first as $j \to +\infty$ and then as $\varepsilon \to 0$ in the previous relation yields

$$\mathcal{H}(\sigma * \rho, p) \ge \langle \tau, p\rangle.$$

¶

We are now in a position to show the maximum plastic work identity for the vanishing viscosity limit.

Theorem 3.14. *For a.e.* $s \in A^\circ$*, one has*

$$\int_\Omega \big(\sigma^\circ(s) - P_{\mathcal{K}(\sigma^\circ(s)*\rho)}(\sigma^\circ(s))\big) : \dot p^\circ(s)\,dx + \mathcal{H}(\sigma^\circ(s)*\rho, \dot p^\circ(s)) = \int_\Omega \sigma^\circ(s) : \dot p^\circ(s)\,dx, \quad (3.23)$$

and for a.e. $s \in B^\circ$*,*

$$\mathcal{H}(\sigma^\circ(s)*\rho, \dot p^\circ(s)) = \langle \sigma^\circ(s), \dot p^\circ(s)\rangle. \quad (3.24)$$

Proof. Recall the flow rule for the regularized non-associative visco-plastic solution in Theorem 2.1, namely, for a.e. $t \in [0,T]$,

$$\sigma_\varepsilon(t) - \varepsilon \dot p_\varepsilon(t) \in \partial_2 H(\sigma_\varepsilon(t)*\rho, \dot p_\varepsilon(t)) \text{ a.e. in } \Omega.$$

Consider a null test function, and integrate the previous relation over $\Omega\times(0, t^\circ_\varepsilon(S))$, for $S \in [0, \overline{T}]$. We get

$$\int_0^{t^\circ_\varepsilon(S)} \int_\Omega (\sigma_\varepsilon(s) - \varepsilon \dot p_\varepsilon(s)) : \dot p_\varepsilon(s)\; dx\,ds \geq \int_0^{t^\circ_\varepsilon(S)} \mathcal{H}(\sigma_\varepsilon(s)*\rho, \dot p_\varepsilon(s))\,ds.$$

Rescaling time with the map t°_ε, it also reads, thanks to the chain rule, to (1.16), and to the 1-homogeneous character of $p \mapsto H(\sigma, p)$, as

$$\liminf_{\varepsilon\to 0}\left\{\int_0^S \mathcal{H}(\sigma^\circ_\varepsilon(s)*\rho, \dot p^\circ_\varepsilon(s))\,ds + \int_0^S \|\dot p^\circ_\varepsilon(s)\|_2 \operatorname{dist}_2(\sigma^\circ_\varepsilon(s), \mathcal{K}(\sigma^\circ_\varepsilon(s)*\rho))\,ds\right\}$$
$$\leq \limsup_{\varepsilon\to 0} \int_0^S \int_\Omega \sigma^\circ_\varepsilon(s) : \dot p^\circ_\varepsilon(s)\; dx\,ds.$$

Then, according to Lemmas 3.7 and 3.10, we get in particular that

$$\int_0^S \mathcal{H}(\sigma^\circ(s)*\rho, \dot p^\circ(s))\,ds + \int_0^S \|\dot p^\circ(s)\|_2 \operatorname{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s)*\rho))\,ds$$
$$\leq \limsup_{\varepsilon\to 0} \int_0^S \int_\Omega \sigma^\circ_\varepsilon(s) : \dot p^\circ_\varepsilon(s)\; dx\,ds. \quad (3.25)$$

We now investigate the limit in the right hand-side of the above inequality. First, kinematic compatibility implies that

$$\int_0^S \int_\Omega \sigma^\circ_\varepsilon(s) : \dot p^\circ_\varepsilon(s)\; dx\,ds = \int_0^S \int_\Omega \sigma^\circ_\varepsilon(s) : E\dot u^\circ_\varepsilon(s)\; dx\,ds - \int_0^S \int_\Omega Ae^\circ_\varepsilon(s) : \dot e^\circ_\varepsilon(s)\; dx\,ds$$

and, since $\operatorname{div}\sigma^\circ_\varepsilon(s) = 0$ a.e. in Ω and $u^\circ_\varepsilon(s) = w^\circ_\varepsilon(s)$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$, then this yields in turn

$$\int_0^S \int_\Omega \sigma^\circ_\varepsilon(s) : \dot p^\circ_\varepsilon(s)\; dx\,ds = \int_0^S \int_\Omega \sigma^\circ_\varepsilon(s) : E\dot w^\circ_\varepsilon(s)\; dx\,ds - \int_0^S \int_\Omega Ae^\circ_\varepsilon(s) : \dot e^\circ_\varepsilon(s)\; dx\,ds$$
$$= \int_0^S \int_\Omega \sigma^\circ_\varepsilon(s) : E\dot w^\circ_\varepsilon(s)\; dx\,ds - \frac{1}{2}\int_\Omega Ae^\circ_\varepsilon(S) : e^\circ_\varepsilon(S)\; dx + \frac{1}{2}\int_\Omega Ae_0 : e_0\; dx.$$

But the first integral in the last term in the string of equalities above also reads as

$$\int_0^{t^\circ_\varepsilon(S)} \int_\Omega \sigma_\varepsilon(t) : E\dot w(t)\; dx\,dt,$$

and, thanks to the third convergence in (3.14), to the uniform convergence of t°_ε to t°, to Remark 2.9, and to the dominated convergence theorem, it converges to

$$\int_0^{t^\circ(S)} \int_\Omega \sigma^\circ(s_0^-(t)) : E\dot w(t)\,dx\,dt = \int_0^S \int_\Omega \sigma^\circ(s_0^-(t^\circ(s))) : E\dot w^\circ(s)\,dx\,ds,$$

where we used the change of variable $t = t^\circ(s)$. But since $E\dot w^\circ(s) = 0$ for all $s \in U^\circ$ and $s_0^-(t^\circ(s)) = s$ for a.e. $s \notin U^\circ$, we get that

$$\int_0^S \int_\Omega \sigma^\circ_\varepsilon(s) : E\dot w^\circ_\varepsilon(s)\,dx\,ds \to \int_0^S \int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\,dx\,ds.$$

Further, (3.10) immediately implies, by weak lower semicontinuity that

$$\liminf_{\varepsilon\to 0} \int_\Omega Ae^\circ_\varepsilon(S) : e^\circ_\varepsilon(S)\,dx \geq \int_\Omega Ae^\circ(S) : e^\circ(S)\,dx$$

and thus by Theorem 3.11, we deduce that

$$\begin{aligned}\limsup_{\varepsilon\to 0} &\int_0^S \int_\Omega \sigma^\circ_\varepsilon(s) : \dot p^\circ_\varepsilon(s)\,dx\,ds \\ &\leq \frac{1}{2}\int_\Omega Ae_0 : e_0\,dx - \frac{1}{2}\int_\Omega Ae^\circ(S) : e^\circ(S)\,dx + \int_0^S \int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\,dx\,ds \\ &\qquad = -\int_0^S \int_\Omega \sigma^\circ(s) : \dot e^\circ(s)\,dx\,ds + \int_0^S \int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\,dx\,ds.\end{aligned}$$

In view of Remark 3.12, for a.e. $s \in [0, \overline{T}]$ the triplet $(\dot u^\circ(s), \dot e^\circ(s), \dot p^\circ(s))$ belongs to $A(\dot w^\circ(s))$ and $\operatorname{div} \dot\sigma^\circ(s) = 0$ a.e. in Ω. Hence, appealing to the duality formula (1.17),

$$-\int_0^S \int_\Omega \sigma^\circ(s) : \dot e^\circ(s)\,dx\,ds + \int_0^S \int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\,dx\,ds = \int_0^S \langle \sigma^\circ(s), \dot p^\circ(s)\rangle\,ds,$$

and thus

$$\limsup_{\varepsilon\to 0} \int_0^S \int_\Omega \sigma^\circ_\varepsilon(s) : \dot p^\circ_\varepsilon(s)\,dx\,ds \leq \int_0^S \langle \sigma^\circ(s), \dot p^\circ(s)\rangle\,ds. \tag{3.26}$$

We collect (3.25) and (3.26), and obtain

$$\begin{aligned}\int_0^S \mathcal{H}(\sigma^\circ(s) * \rho, \dot p^\circ(s))\,ds + \int_0^S \|\dot p^\circ(s)\|_2 \operatorname{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s) * \rho))\,ds& \\ \leq \int_0^S \langle \sigma^\circ(s), \dot p^\circ(s)\rangle\,ds.&\end{aligned} \tag{3.27}$$

Next, appealing to Remark 3.13, for a.e. $s \in B^\circ$ we have

$$\mathcal{H}(\sigma^\circ(s) * \rho, \dot p^\circ(s)) \geq \langle \sigma^\circ(s), \dot p^\circ(s)\rangle.$$

On the other hand, since $\dot p^\circ(s) \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ for a.e. $s \in A^\circ$, the duality pairing $\langle \sigma^\circ(s), \dot p^\circ(s)\rangle$ coincides with the L^2 product for a.e. $s \in A^\circ$, so that

$$\langle \sigma^\circ(s) - P_{\mathcal{K}(\sigma^\circ(s)*\rho)}(\sigma^\circ(s)), \dot p^\circ(s)\rangle + \langle P_{\mathcal{K}(\sigma^\circ(s)*\rho)}(\sigma^\circ(s)), \dot p^\circ(s)\rangle = \langle \sigma^\circ(s), \dot p^\circ(s)\rangle$$

for a.e. $s \in A^\circ$. From the definition of H it immediately follows that

$$\mathcal{H}(\sigma^\circ(s) * \rho, \dot p^\circ(s)) \geq \langle P_{\mathcal{K}(\sigma^\circ(s)*\rho)}(\sigma^\circ(s)), \dot p^\circ(s)\rangle,$$

so that

$$\|\dot p^\circ(s)\|_2 \operatorname{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s) * \rho)) + \mathcal{H}(\sigma^\circ(s) * \rho, \dot p^\circ(s)) \geq \langle \sigma^\circ(s), \dot p^\circ(s)\rangle$$

for a.e. $s \in A^\circ$. Hence we conclude from (3.27) that for a.e. $s \in [0, \overline{T}]$,

$$\|\dot{p}^\circ(s)\|_2 \operatorname{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s) * \rho)) + \mathcal{H}(\sigma^\circ(s) * \rho, \dot{p}^\circ(s)) = \langle \sigma^\circ(s), \dot{p}^\circ(s) \rangle.$$

The proof of (3.23) and (3.24) is complete. □

3.6. **Energy equality.** The aim of this section is to prove Theorem 3.11. Unfortunately, we were not able to bypass the energy equality in establishing that result; therefore, our proof follows closely that of a related result in [9, Sections 7, 8] and the reader familiar with that work can skip most of the following and proceed directly to the proof of Theorem 3.11 which concludes the section.

We first address the proof of the energy inequality which will be achieved through the lower semicontinuity of the different terms involved in the total energy.

Proposition 3.15. *For every $S \in [0, \overline{T}]$,*

$$\mathcal{Q}(e^\circ(S)) + \int_0^S \mathcal{H}(\sigma^\circ(s) * \rho, \dot{p}^\circ(s))\, ds + \int_0^S \|\dot{p}^\circ(s)\|_2 \operatorname{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s) * \rho)\, ds \\ \leq \mathcal{Q}(e^\circ(0)) + \int_0^S \int_\Omega \sigma^\circ(s) : E\dot{w}^\circ(s)\, dx\, ds. \quad (3.28)$$

Proof. According to Remarks 2.7 and 2.8, together with a change of variable in time, we infer that

$$\mathcal{Q}(e^\circ_\varepsilon(S)) + \int_0^S \mathcal{H}(\sigma^\circ_\varepsilon(s) * \rho, \dot{p}^\circ_\varepsilon(s))\, ds + \int_0^S \|\dot{p}_\varepsilon(s)\|_2 \operatorname{dist}_2(\sigma^\circ_\varepsilon(s), \mathcal{K}(\sigma^\circ_\varepsilon(s) * \rho)\, ds \\ = \mathcal{Q}(e^\circ_\varepsilon(0)) + \int_0^S \int_\Omega \sigma^\circ_\varepsilon(s) : E\dot{w}^\circ_\varepsilon(s)\, dx\, ds.$$

By Lemma 3.4 and the convexity of $\mathcal{Q}$, we clearly have

$$\mathcal{Q}(e^\circ(S)) \leq \liminf_{\varepsilon \to 0} \mathcal{Q}(e^\circ_\varepsilon(S)).$$

The conclusion then follows from the lower semicontinuity results obtained in Lemmas 3.7 and 3.10. □

We now prove that the energy inequality is actually an equality. The idea consists in subdividing the interval $[0, \overline{T}]$ into a suitable partition, and using fine approximation results of the Bochner integral by suitable Riemann sums. In particular (see, *e.g.*, [11, Lemma 4.12]), there exists a sequence of subdivision $\{s^i_k\}_{1 \leq i \leq i_k}$ of the interval $[0, S]$ satisfying

$$0 = s^0_k \leq s^1_k \leq \ldots \leq s^{i_k}_k = S, \quad \eta_k := \max_{1 \leq i \leq i_k} (s^i_k - s^{i-1}_k) \to 0, \quad (3.29)$$

and

$$\lim_{k \to +\infty} \sum_{i=1}^{i_k} \int_{s^{i-1}_k}^{s^i_k} \|\sigma^\circ(s) - \sigma^\circ(s^{i-1}_k)\|_2\, ds = 0 = \lim_{k \to +\infty} \sum_{i=1}^{i_k} \int_{s^{i-1}_k}^{s^i_k} \|\sigma^\circ(s) - \sigma^\circ(s^i_k)\|_2\, ds, \quad (3.30)$$

and

$$\lim_{k \to +\infty} \sum_{i=1}^{i_k} \int_{s^{i-1}_k}^{s^i_k} |\chi_{B^\circ}(s) - \chi_{B^\circ}(s^{i-1}_k)|\, ds = 0 = \lim_{k \to +\infty} \sum_{i=1}^{i_k} \int_{s^{i-1}_k}^{s^i_k} |\chi_{B^\circ}(s) - \chi_{B^\circ}(s^i_k)|\, ds. \quad (3.31)$$

On the other hand, from [9, Lemma 7.3], we get that, for any infinitesimal subdivision satisfying (3.29),

$$\lim_{k\to+\infty}\sum_{i=1}^{i_k}\left|\mathcal{H}(\sigma^\circ(s_k^i)*\rho, p(s_k^i)-p(s_k^{i-1}))-\int_{s_k^{i-1}}^{s_k^i}\mathcal{H}(\sigma^\circ(s)*\rho,\dot p^\circ(s))\,ds\right|$$
$$=0=\lim_{k\to+\infty}\sum_{i=1}^{i_k}\left|\mathcal{H}(\sigma^\circ(s_k^{i-1})*\rho, p(s_k^i)-p(s_k^{i-1}))-\int_{s_k^{i-1}}^{s_k^i}\mathcal{H}(\sigma^\circ(s)*\rho,\dot p^\circ(s))\,ds\right|. \quad (3.32)$$

Since the regularity of the triplet $(u^\circ(s), e^\circ(s), p^\circ(s))$ depends on whether $s \in A^\circ$ or $s \in B^\circ$, it proves convenient to distinguish between those indices i for which s_k^i belongs to B° and those for which s_k^i belongs to A°. To this effect, we define the following sets of indices:

$$I_k := \{i \in \{1,\dots,i_k\} : s_k^{i-1} \in B^\circ\cap[0,S], \text{ and } s_k^i \in B^\circ\cap[0,S]\},$$
$$J_k := \{i \in \{1,\dots,i_k\} : s_k^{i-1} \notin B^\circ\cap[0,S], \text{ or } s_k^i \notin B^\circ\cap[0,S]\}.$$

Then, in view of (3.20) and (3.30), it is nearly immediate (see [9, Lemma 8.8]) that

$$\lim_{k\to+\infty}\sum_{i\in I_k}\int_\Omega \sigma^\circ(s_k^{i-1}) : (Ew^\circ(s_k^i)-Ew^\circ(s_k^{i-1}))\,dx = \int_0^S\int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\,dx\,ds$$
$$=\lim_{k\to+\infty}\sum_{i\in I_k}\int_\Omega \sigma^\circ(s_k^{i}) : (Ew^\circ(s_k^i)-Ew^\circ(s_k^{i-1}))\,dx. \quad (3.33)$$

Let us start with the contribution of the set B° to the total energy .

Lemma 3.16. *For every $S \in [0,\overline{T}]$, we have*

$$\int_0^S\int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\,dx\,ds$$
$$\le \liminf_{k\to+\infty}\sum_{i\in I_k}\left\{\mathcal{Q}(e^\circ(s_k^i))-\mathcal{Q}(e^\circ(s_k^{i-1}))+\int_{s_k^{i-1}}^{s_k^i}\mathcal{H}(\sigma^\circ(s)*\rho,\dot p^\circ(s))\,ds\right\}. \quad (3.34)$$

Proof. Let s_1 and $s_2 \in B^\circ$. Using the integration by parts formula (1.17) together with the fact that $\operatorname{div}\sigma^\circ(s_1) = \operatorname{div}\sigma^\circ(s_2) = 0$ a.e. in Ω, we deduce that

$$\int_\Omega(\sigma^\circ(s_1)+\sigma^\circ(s_2)) : (Ew^\circ(s_2)-Ew^\circ(s_1))\,dx =$$
$$\int_\Omega(\sigma^\circ(s_1)+\sigma^\circ(s_2)) : (e^\circ(s_2)-e^\circ(s_1))\,dx + \langle\sigma^\circ(s_1)+\sigma^\circ(s_2), p^\circ(s_2)-p^\circ(s_1)\rangle.$$

Then, since $\sigma^\circ(s_1) \in \mathcal{K}(\sigma^\circ(s_1)*\rho)$ and $\sigma^\circ(s_2) \in \mathcal{K}(\sigma^\circ(s_2)*\rho)$, Remark 3.13 implies that

$$\frac{1}{2}\int_\Omega(\sigma^\circ(s_1)+\sigma^\circ(s_2)) : (Ew^\circ(s_2)-Ew^\circ(s_1))\,dx \le \mathcal{Q}(e^\circ(s_2))-\mathcal{Q}(e^\circ(s_1))$$
$$+\frac{1}{2}\mathcal{H}(\sigma^\circ(s_1)*\rho, p^\circ(s_2)-p^\circ(s_1))+\frac{1}{2}\mathcal{H}(\sigma^\circ(s_2)*\rho, p^\circ(s_2)-p^\circ(s_1)). \quad (3.35)$$

Now let $i \in I_k$ so that s_k^{i-1} and $s_k^i \in B^\circ \cap [0,S]$. Then writing the previous inequality with $s_1 = s_k^{i-1}$ and $s_2 = s_k^i$, and summing up for all $i \in I_k$ leads to

$$\frac{1}{2}\sum_{i\in I_k}\int_\Omega (\sigma^\circ(s_k^{i-1})+\sigma^\circ(s_k^i)):(Ew^\circ(s_k^i)-Ew^\circ(s_k^{i-1}))\,dx$$
$$\leq \sum_{i\in I_k}\Big\{\mathcal{Q}(e^\circ(s_k^i))-\mathcal{Q}(e^\circ(s_k^{i-1}))\Big)+\frac{1}{2}\mathcal{H}(\sigma^\circ(s_k^{i-1})*\rho,p^\circ(s_k^i)-p^\circ(s_k^{i-1}))$$
$$+\frac{1}{2}\mathcal{H}(\sigma^\circ(s_k^i)*\rho,p^\circ(s_k^i)-p^\circ(s_k^{i-1}))\Big\}.$$

Then according to (3.32) and (3.33), we obtain the desired relation (3.34) by letting $k\to+\infty$. □

We next examine the contribution of the energy on A°.

Lemma 3.17. *For every* $S\in[0,\overline{T}]$,

$$0\leq \liminf_{k\to+\infty}\sum_{i\in J_k}\left\{\mathcal{Q}(e^\circ(s_k^i))-\mathcal{Q}(e^\circ(s_k^{i-1}))+\int_{s_k^{i-1}}^{s_k^i}\mathcal{H}(\sigma^\circ(s)*\rho,\dot p^\circ(s))\,ds\right\}$$
$$+\int_0^S\|\dot p^\circ(s)\|_2\,\mathrm{dist}_2(\sigma^\circ(s),\mathcal{K}(\sigma^\circ(s)*\rho))\,ds. \quad (3.36)$$

Proof. Let s_1 and $s_2\in(a,b)$ be such that $s_1<s_2$, where (a,b) is a connected component of A°. Then, setting $\tau^\circ(s):=P_{\mathcal{K}(\sigma^\circ(s)*\rho)}(\sigma^\circ(s))\in\mathcal{K}(\sigma^\circ(s)*\rho)$ for a.e. $s\in(s_1,s_2)$, we obtain

$$\mathcal{H}(\sigma^\circ(s)*\rho,\dot p^\circ(s))\geq\int_\Omega\tau^\circ(s):\dot p^\circ(s)\,dx=\int_\Omega\sigma^\circ(s):\dot p^\circ(s)\,dx-\int_\Omega(\sigma^\circ(s)-\tau^\circ(s)):\dot p^\circ(s)\,dx.$$

From Remark 3.9, $\dot p^\circ(s)=E\dot u^\circ(s)-\dot e^\circ(s)$ a.e. in Ω and for a.e. $s\in(s_1,s_2)$, while $\dot u^\circ(s)=0$ $\mathcal{H}^{n-1}$-a.e. on $\partial\Omega$. Hence, integrating over (s_1,s_2) and using integrations by parts in space and time and the fact that $\mathrm{div}\sigma^\circ(s)=0$, we obtain

$$\mathcal{Q}(e^\circ(s_2))\geq\mathcal{Q}(e^\circ(s_1))-\int_{s_1}^{s_2}\mathcal{H}(\sigma^\circ(s)*\rho,\dot p^\circ(s))\,ds$$
$$-\int_{s_1}^{s_2}\|\dot p^\circ(s)\|_2\,\mathrm{dist}_2(\sigma^\circ(s),\mathcal{K}(\sigma^\circ(s)*\rho)\,ds. \quad (3.37)$$

Since the mapping $s\mapsto e^\circ(s)$ is weakly continuous in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, we deduce by lower semicontinuity of the right hand-side of (3.37) with respect to s_1, that relation (3.37) holds as well for s_1 possibly equal to a. Moreover, application of [9, Lemma 8.5] establishes the existence of a sequence $\{s_k\}\nearrow b$ such that $\lim_k\|e^\circ(s_k)-e^\circ(b)\|_2=0$. Consequently, (3.37) can be further extended to any $a\leq s_1<s_2\leq b$.

Now let $i\in J_k$. Then either $(s_k^{i-1},s_k^i)\subset A^\circ$ or $(s_k^{i-1},s_k^i)\cap B^\circ\neq\emptyset$ so that $J_k=\hat J_k\cup\check J_k$, where

$$\hat J_k:=\{i\in J_k:(s_k^{i-1},s_k^i)\subset A^\circ\},\quad \check J_k:=\{i\in J_k:(s_k^{i-1},s_k^i)\cap B^\circ\neq\emptyset\}.$$

Assume first that $i \in \hat{J}_k$ so that $(s_k^{i-1}, s_k^i) \subset A^\circ$. Then applying (3.37) with $s_1 = s_k^{i-1}$ and $s_2 = s_k^i$, and summing up for all $i \in \hat{J}_k$ leads to

$$0 \leq \sum_{i \in \hat{J}_k} \left\{ \mathcal{Q}(e^\circ(s_k^i)) - \mathcal{Q}(e^\circ(s_k^{i-1})) + \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\sigma^\circ(s) * \rho, \dot{p}^\circ(s))\, ds \right.$$
$$\left. + \int_{s_k^{i-1}}^{s_k^i} \|\dot{p}^\circ(s)\|_2 \operatorname{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s) * \rho)\, ds \right\}. \quad (3.38)$$

On the other hand, if $i \in \check{J}_k$, then $(s_k^{i-1}, s_k^i) \cap B^\circ \neq \emptyset$. Let us define

$$s_k^{i-\frac{2}{3}} := \inf\{s \in B^\circ \cap (s_k^{i-1}, s_k^i)\},$$
$$s_k^{i-\frac{1}{3}} := \sup\{s \in B^\circ \cap (s_k^{i-1}, s_k^i)\},$$

and observe that since B° is closed, then $s_k^{i-\frac{2}{3}}$ and $s_k^{i-\frac{1}{3}} \in B^\circ$. Then the open intervals $(s_k^{i-1}, s_k^{i-\frac{2}{3}})$ and $(s_k^{i-\frac{1}{3}}, s_k^i)$ are contained in A°, and it follows again from (3.37) that

$$\mathcal{Q}(e^\circ(s_k^{i-\frac{2}{3}})) \geq \mathcal{Q}(e^\circ(s_k^{i-1})) - \int_{s_k^{i-1}}^{s_k^{i-\frac{2}{3}}} \mathcal{H}(\sigma^\circ(s) * \rho, \dot{p}^\circ(s))\, ds$$
$$- \int_{s_k^{i-1}}^{s_k^{i-\frac{2}{3}}} \|\dot{p}^\circ(s)\|_2 \operatorname{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s) * \rho)\, ds, \quad (3.39)$$

and

$$\mathcal{Q}(e^\circ(s_k^i)) \geq \mathcal{Q}(e^\circ(s_k^{i-\frac{1}{3}})) - \int_{s_k^{i-\frac{1}{3}}}^{s_k^i} \mathcal{H}(\sigma^\circ(s) * \rho, \dot{p}^\circ(s))\, ds$$
$$- \int_{s_k^{i-\frac{1}{3}}}^{s_k^i} \|\dot{p}^\circ(s)\|_2 \operatorname{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s) * \rho)\, ds. \quad (3.40)$$

Hence, adding (3.39) and (3.40), and summing up for all $i \in \check{J}_k$, we deduce that

$$0 \leq \sum_{i \in \check{J}_k} \left\{ \mathcal{Q}(e^\circ(s_k^i)) - \mathcal{Q}(e^\circ(s_k^{i-1})) + \int_{s_k^{i-1}}^{s_k^i} \mathcal{H}(\sigma^\circ(s) * \rho, \dot{p}^\circ(s))\, ds \right.$$
$$\left. + \int_{s_k^{i-1}}^{s_k^i} \|\dot{p}^\circ(s)\|_2 \operatorname{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s) * \rho)\, ds \right\} + \delta_k, \quad (3.41)$$

where δ_k is the intermediate term defined by

$$\delta_k := \sum_{i \in \check{J}_k} \left\{ \mathcal{Q}(e^\circ(s_k^{i-\frac{2}{3}})) - \mathcal{Q}(e^\circ(s_k^{i-\frac{1}{3}})) - \int_{s_k^{i-\frac{2}{3}}}^{s_k^{i-\frac{1}{3}}} \mathcal{H}(\sigma^\circ(s) * \rho, \dot{p}^\circ(s))\, ds \right.$$
$$\left. - \int_{s_k^{i-\frac{2}{3}}}^{s_k^{i-\frac{1}{3}}} \|\dot{p}^\circ(s)\|_2 \operatorname{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s) * \rho)\, ds \right\}.$$

We claim that

$$\limsup_{k \to +\infty} \delta_k \leq 0. \quad (3.42)$$

If so, adding (3.38) and (3.41) leads to the desired inequality (3.36).

We conclude the proof of the lemma by showing that δ_k indeed satisfies (3.42). Since $s_k^{i-\frac{1}{3}}$ and $s_k^{i-\frac{2}{3}} \in B^\circ$, it follows from (3.35) that

$$\delta_k \le \sum_{i\in \check{J}_k} \left\{ \frac{1}{2}\mathcal{H}(\sigma^\circ(s_k^{i-\frac{2}{3}})*\rho, p^\circ(s_k^{i-\frac{1}{3}}) - p^\circ(s_k^{i-\frac{2}{3}})) + \frac{1}{2}\mathcal{H}(\sigma^\circ(s_k^{i-\frac{1}{3}})*\rho, p^\circ(s_k^{i-\frac{1}{3}}) - p^\circ(s_k^{i-\frac{2}{3}})) - \int_{s_k^{i-\frac{2}{3}}}^{s_k^{i-\frac{1}{3}}} \mathcal{H}(\sigma^\circ(s)*\rho, \dot p^\circ(s))\,ds - \frac{1}{2}\int_\Omega (\sigma^\circ(s_k^{i-\frac{2}{3}}) + \sigma^\circ(s_k^{i-\frac{1}{3}})) : (Ew^\circ(s_k^{i-\frac{1}{3}}) - Ew^\circ(s_k^{i-\frac{2}{3}}))\,dx \right\}.$$

Using the fact that $\sigma^\circ \in L^\infty(\Omega\times B^\circ; \mathbb{M}^{n\times n}_{sym})$, that $s\mapsto Ew^\circ(s)$ is constant on each connected component of A° and 1-Lipschitz (with values in $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$) on B°, we deduce that

$$\delta_k \le C\sum_{i\in \check{J}_k} \int_{s_k^{i-1}}^{s_k^i} \chi_{B^\circ}(s)\,ds + \frac{1}{2}\sum_{i\in \check{J}_k}\left(\mathcal{H}(\sigma^\circ(s_k^{i-\frac{2}{3}})*\rho, p^\circ(s_k^{i-\frac{1}{3}}) - p^\circ(s_k^{i-\frac{2}{3}})) - \int_{s_k^{i-\frac{2}{3}}}^{s_k^{i-\frac{1}{3}}} \mathcal{H}(\sigma^\circ(s)*\rho, \dot p^\circ(s))\,ds\right) + \frac{1}{2}\sum_{i\in \check{J}_k}\left(\mathcal{H}(\sigma^\circ(s_k^{i-\frac{1}{3}})*\rho, p^\circ(s_k^{i-\frac{1}{3}}) - p^\circ(s_k^{i-\frac{2}{3}})) - \int_{s_k^{i-\frac{2}{3}}}^{s_k^{i-\frac{1}{3}}} \mathcal{H}(\sigma^\circ(s)*\rho, \dot p^\circ(s))\,ds\right)$$

for some constant $C>0$ independent of k. Then according to (3.32) (which holds for any infinitesimal subdivision) and because (3.31) allows us to replace (in the limit) each term $\int_{s_k^{i-1}}^{s_k^i} \chi_{B^\circ}(s)\,ds$ by either $(s_k^i - s_k^{i-1})\chi_{B^\circ}(s_k^i) = 0$ if $s_k^i \in A_0$, or by $(s_k^i - s_k^{i-1})\chi_{B^\circ}(s_k^{i-1}) = 0$ if $s_k^{i-1}\in A_0$, we conclude that δ_k satisfies (3.42) as claimed. □

As a consequence of (3.34) and (3.36) we deduce the second energy inequality:

Proposition 3.18. *For every* $S\in[0,\overline{T}]$,

$$\mathcal{Q}(e^\circ(S)) + \int_0^S \mathcal{H}(\sigma^\circ(s)*\rho, \dot p^\circ(s))\,ds + \int_0^S \|\dot p^\circ(s)\|_2 \operatorname{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s)*\rho)\,ds \ge \mathcal{Q}(e^\circ(0)) + \int_0^S\int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\,dx\,ds. \tag{3.43}$$

Propositions 3.15 and 3.18 yield in turn the following energy equality:

$$\mathcal{Q}(e^\circ(S)) + \int_0^S \mathcal{H}(\sigma^\circ(s)*\rho, \dot p^\circ(s))\,ds + \int_0^S \|\dot p^\circ(s)\|_2 \operatorname{dist}_2(\sigma^\circ(s), \mathcal{K}(\sigma^\circ(s)*\rho)\,ds = \mathcal{Q}(e^\circ(0)) + \int_0^S\int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\,dx\,ds,$$

and in particular, for every $0\le s_1\le s_2\le \overline{T}$, we get

$$\mathcal{Q}(e^\circ(s_2)) + \int_{s_1}^{s_2} \mathcal{H}(\sigma^\circ(s)*\rho, \dot p^\circ(s))\,ds \le \mathcal{Q}(e^\circ(s_1)) + \int_{s_1}^{s_2}\int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\,dx\,ds. \tag{3.44}$$

Thanks to this relation, we are in a position to prove the almost everywhere differentiability in time of $s \mapsto e^\circ(s)$.

Finally, the energy equality immediately implies that

$$Q(e^\circ(s)) \in W^{1,1}(0,\overline{T}). \tag{3.45}$$

Proof of Theorem 3.11. According to Lemma 3.8, we already know that $s \mapsto e^\circ(s)$ is absolutely continuous in A° with values in the reflexive space $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$. Hence from [3, Appendix], we conclude that it is differentiable almost everywhere in A° for the strong $L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$ topology. It suffices to prove the almost everywhere differentiability of e° in B°.

Let $0 \leq s_1 \leq s_2 \leq \overline{T}$, and assume that $s_1 \in B^\circ$. Thanks to the Lipschitz continuity of H in its first variable (see Lemma 1.5) and (3.6), for a.e. $s \in (s_1, s_2)$,

$$\begin{aligned}\mathcal{H}(\sigma^\circ(s_1)*\rho,\dot p^\circ(s)) &\leq \mathcal{H}(\sigma^\circ(s)*\rho,\dot p^\circ(s)) + C_H\|\dot p^\circ(s)\|_1\|(\sigma^\circ(s)-\sigma^\circ(s_1))*\rho\|_\infty\\ &\leq \mathcal{H}(\sigma^\circ(s)*\rho,\dot p^\circ(s)) + C\|e^\circ(s)-e^\circ(s_1)\|_2,\end{aligned}$$

for some constant $C > 0$ independent of s and s_1. Next, using (3.44) between s_1 and s_2, we infer that

$$\begin{aligned}&\mathcal{Q}(e^\circ(s_2)) + \int_{s_1}^{s_2}\mathcal{H}(\sigma^\circ(s_1)*\rho,\dot p^\circ(s))\,ds\\ &\qquad\leq \mathcal{Q}(e^\circ(s_1)) + \int_{s_1}^{s_2}\int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\,dx\,ds + C\int_{s_1}^{s_2}\|e^\circ(s)-e^\circ(s_1)\|_2\,ds.\end{aligned} \tag{3.46}$$

Now, since $\sigma^\circ(s_1)*\rho$ is continuous, it is uniformly continuous on $\overline{\Omega}$. Thus for each $\varepsilon > 0$, there exists $\delta > 0$ such that if x and $y \in \overline{\Omega}$ are such that $|x-y| < \delta$, then $|[\sigma^\circ(s_1)*\rho](x) - [\sigma^\circ(s_1)*\rho](y)| < \varepsilon$. Let us split $\overline{\Omega}$ into a finite family of pairwise disjoint sets $\{Q_i\}_{1\leq i\leq m_\varepsilon}$ such that

$$\begin{cases}\overline{\Omega} = \bigcup_{i=1}^{m_\varepsilon}\overline{Q_i},\\ \operatorname{diam}(Q_i) < \delta,\\ \int_{s_1}^{s_2}|\dot p^\circ(s)|(\Omega\cap\partial Q_i)\,ds = |p^\circ(s_2)-p^\circ(s_1)|(\Omega\cap\partial Q_i) = 0\end{cases}$$

for all $i \in \{1,\dots,m_\varepsilon\}$. Fix a point $x_i \in Q_i$. Then appealing to Lemma 1.5, for a.e. $s \in (s_1, s_2)$,

$$\begin{aligned}\int_{\overline{Q_i}}\left|H\left([\sigma^\circ(s_1)*\rho](x),\frac{d\dot p^\circ(s)}{d|\dot p^\circ(s)|}(x)\right) - H\left([\sigma^\circ(s_1)*\rho](x_i),\frac{d\dot p^\circ(s)}{d|\dot p^\circ(s)|}(x)\right)\right| d|\dot p^\circ(s)|(x)\\ \leq C_H\varepsilon|\dot p^\circ(s)|(Q_i).\end{aligned}$$

Hence, using (3.6), we get that

$$\begin{aligned}&\int_{s_1}^{s_2}\mathcal{H}(\sigma^\circ(s_1)*\rho,\dot p^\circ(s))\,ds\\ &\qquad\geq \sum_{i=1}^{m}\int_{s_1}^{s_2}\int_{\overline{Q_i}} H\left([\sigma^\circ(s_1)*\rho](x_i),\frac{d\dot p^\circ(s)}{d|\dot p^\circ(s)|}(x)\right) d|\dot p^\circ(s)|(x)\,ds\\ &\qquad\qquad - C_H\varepsilon(s_2-s_1).\end{aligned} \tag{3.47}$$

By virtue of [6, Theorem 7.1] applied to $\mathcal{H}([\sigma^\circ(s_1) * \rho](x_i), \cdot)$, we get, for each $i \in \{1, \dots, m_\varepsilon\}$,

$$\begin{aligned}\int_{s_1}^{s_2} \int_{\overline{Q_i}} H\left([\sigma^\circ(s_1) * \rho](x_i), \frac{d\dot p^\circ(s)}{d|\dot p^\circ(s)|}(x)\right) d|\dot p^\circ(s)|(x)\, ds \\ \geq \int_{\overline{Q_i}} H\left([\sigma^\circ(s_1) * \rho](x_i), \frac{d[p^\circ(s_2) - p^\circ(s_1)]}{d|p^\circ(s_2) - p^\circ(s_1)|}\right) d|p^\circ(s_2) - p^\circ(s_1)|(x) \\ \geq \int_{\overline{Q_i}} H\left([\sigma^\circ(s_1) * \rho](x), \frac{d[p^\circ(s_2) - p^\circ(s_1)]}{d|p^\circ(s_2) - p^\circ(s_1)|}\right) d|p^\circ(s_2) - p^\circ(s_1)|(x) \\ - C_H \varepsilon |p^\circ(s_2) - p^\circ(s_1)|(Q_i),\end{aligned}$$

where we used again Lemma 1.5. Finally, since $p^\circ \in \mathrm{Lip}(0, \overline{T}; \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym}))$ with Lipschitz constant less than 1, we conclude from (3.47) that

$$\int_{s_1}^{s_2} \mathcal{H}(\sigma^\circ(s_1) * \rho, \dot p^\circ(s))\, ds \geq \mathcal{H}(\sigma^\circ(s_1) * \rho, p^\circ(s_2) - p^\circ(s_1)) - 2C_H \varepsilon (s_2 - s_1),$$

and letting $\varepsilon \searrow 0$ that

$$\int_{s_1}^{s_2} \mathcal{H}(\sigma^\circ(s_1) * \rho, \dot p^\circ(s))\, ds \geq \mathcal{H}(\sigma^\circ(s_1) * \rho, p^\circ(s_2) - p^\circ(s_1)).$$

Thus, (3.46) yields

$$\begin{aligned}\mathcal{Q}(e^\circ(s_2)) + \mathcal{H}(\sigma^\circ(s_1) * \rho, p^\circ(s_2) - p^\circ(s_1)) \\ \leq \mathcal{Q}(e^\circ(s_1)) + \int_{s_1}^{s_2} \int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\, dx\, ds + C \int_{s_1}^{s_2} \|e^\circ(s) - e^\circ(s_1)\|_2\, ds.\end{aligned}$$

Since $s_1 \in B^\circ$, then $\sigma^\circ(s_1) \in \mathcal{K}(\sigma^\circ(s_1) * \rho)$ and thus, by Remark 3.13, $\langle \sigma^\circ(s_1), p^\circ(s_2) - p^\circ(s_1)\rangle \leq \mathcal{H}(\sigma^\circ(s_1) * \rho, p^\circ(s_2) - p^\circ(s_1))$. Hence,

$$\begin{aligned}\mathcal{Q}(e^\circ(s_2)) + \langle \sigma^\circ(s_1), p^\circ(s_2) - p^\circ(s_1)\rangle \\ \leq \mathcal{Q}(e^\circ(s_1)) + \int_{s_1}^{s_2} \int_\Omega \sigma^\circ(s) : E\dot w^\circ(s)\, dx\, ds + C \int_{s_1}^{s_2} \|e^\circ(s) - e^\circ(s_1)\|_2\, ds.\end{aligned}$$

Since $\mathcal{Q}(e^\circ(s_2)) - \mathcal{Q}(e^\circ(s_1)) = \mathcal{Q}(e^\circ(s_2) - e^\circ(s_1)) + \int_\Omega \sigma^\circ(s_1) : (e^\circ(s_2) - e^\circ(s_1))\, dx$,

$$\mathcal{Q}(e^\circ(s_2) - e^\circ(s_1)) \leq \int_{s_1}^{s_2} \int_\Omega (\sigma^\circ(s) - \sigma^\circ(s_1)) : E\dot w^\circ(s)\, dx\, ds + C \int_{s_1}^{s_2} \|e^\circ(s) - e^\circ(s_1)\|_2\, ds.$$

In deriving the inequality above, we have also made use of kinematic compatibility, and of the duality (1.17), together with the fact that $\sigma^\circ(s_1)$ is divergence free.

Owing to the coercivity (1.19) of $\mathcal{Q}$, the Cauchy-Schwartz inequality and the fact that $\|E\dot w^\circ(s)\|_2 \leq 1$ for a.e. s, we obtain that

$$\|e^\circ(s_2) - e^\circ(s_1)\|_2^2 \leq C \int_{s_1}^{s_2} \|e^\circ(s) - e^\circ(s_1)\|_2\, ds,$$

for some constant $C > 0$ independent of s_1 and s_2. Hence Gronwall's Lemma implies that

$$\|e^\circ(s_2) - e^\circ(s_1)\|_2 \leq L(s_2 - s_1), \tag{3.48}$$

for some constant $L > 0$ (independent of s_1 and s_2) for every $0 \leq s_1 \leq s_2 \leq \overline{T}$ with $s_1 \in B^\circ$. Thus, by [10, Theorem 3.1], $s \mapsto e^\circ(s)$ is differentiable almost everywhere in B° for the strong $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$ topology.

Then, we deduce that

$$\frac{d}{ds}\mathcal{Q}(e^\circ(s)) = \int_\Omega \sigma^\circ(s) : \dot e^\circ(s)\, dx$$

for a.e. $s \in [0, \overline{T}]$, and thus relation (3.22) follows, since, thanks to (3.45), $\int_\Omega \sigma^\circ(s) : \dot{e}^\circ(s)\, dx \in L^1(0, \overline{T})$.

Finally, by (3.21), (3.48), we conclude that $s \mapsto e^\circ(s)$ is actually strongly continuous into $L^2(\Omega; \mathbb{M}^{n\times n}_{sym})$, and recalling that $p^\circ(s)$ is Lipschitz into $\mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_{sym})$, together with Poincaré-Korn's inequality and the H^1-regularity of w°, that $s \mapsto u^\circ(s)$ is strongly continuous into $BD(\Omega)$. □

Acknowledgements

The authors are very pleased to express their gratitude to Gianni Dal Maso, Jean-Jacques Marigo and Francesco Solombrino for their expert advice on the multiple facets of this topic and for many fruitful suggestions. The research of JFB has been supported by the Agence Nationale de la Recherche under Grant No. ANR 10-JCJC 0106. He also thanks the Functional Analysis Sector of SISSA for the hospitality and support during the preparation of this paper. MGM wishes to acknowledge the hospitality of the LPMTM at Université Paris-Nord and of the Courant Institute of Mathematical Sciences where part of this work was carried out. GAF wishes to acknowledge the hospitality of the Courant Institute of Mathematical Sciences, where a large part of his contribution to this work was carried out.

References

[1] L. Ambrosio, N. Fusco & D. Pallara: *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York (2000).
[2] A. Bensoussan & J. Frehse: Asymptotic behaviour of the time dependent Norton-Hoff law in plasticity theory and H^1 regularity, *Comment. Math. Univ. Carolin.* **37** (1996) 285–304.
[3] H. Brézis: *Operateurs maximaux monotones et semi groupes de contractions dans les espaces de Hilbert*, North-Holland, Amsterdam-London; American Elsevier, New York (1973).
[4] G. Dal Maso, A. Demyanov & A. De Simone: Quasistatic evolution problems for pressure sensitive plastic materials, *Milan J. Maths.* **75** (2007) 114–134.
[5] G. Dal Maso & A. De Simone: Quasistatic evolution problems for Cam-Clay plasticity: examples of spatially homogeneous solutions, *Math. Models Methods Appl. Sci.* **19** (2009) 1–69.
[6] G. Dal Maso, A. De Simone & M. G. Mora: Quasistatic evolution problems for linearly elastic–perfectly plastic materials, *Arch. Rational Mech. Anal.* **180** (2006) 237–291.
[7] G. Dal Maso, A. De Simone, M. G. Mora & M. Morini: A vanishing viscosity vpproach to quasistatic evolution in plasticity with softening, *Arch. Rational Mech. Anal.* **189** (2008) 469–544.
[8] G. Dal Maso, A. De Simone, M. G. Mora & M. Morini: Globally stable quasistatic evolution in plasticity with softening, *Netw. Heterog. Media* **3** (2008) 567–614.
[9] G. Dal Maso, A. De Simone & F. Solombrino: Quasistatic evolution for Cam-Clay plasticity: a weak formulation via viscoplastic regularization and time rescaling, *Calc. Var. PDEs*, **40** (2011) 125–181.
[10] G. Dal Maso, A. De Simone & F. Solombrino: Quasistatic evolution for Cam-Clay plasticity: properties of the viscosity solutions, to appear in *Calc. Var. PDEs.*
[11] G. Dal Maso, G.A. Francfort & R. Toader: Quasistatic crack growth in nonlinear elasticity, *Arch. Rational Mech. Anal.* **176** (2005) 165–225.
[12] G. Dal Maso & F. Solombrino: Quasistatic evolution for Cam-Clay plasticity: the spatially homogeneous case, *Netw. Heterog. Media* **5** (2010) 97–132.
[13] R. de Borst & A.E. Groen: Computational strategies for standard soil plasticity models, in *Models in Geomechanics*, M. Zaman, J. Booker & G. Gioda, eds., John Wiley, New York, 2000, 23–49.
[14] M.C. Delfour & J.-P. Zolésio: *Shapes and geometries. Analysis, differential calculus, and optimization.* Advances in Design and Control, 4. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2001.
[15] A. Demyanov: Regularity of stresses in Prandtl-Reuss perfect plasticity, *Calc. Var. PDEs* **34** (2009) 23–72.
[16] F. L. DiMaggio & I. S. Sandler: Material model for granular soils, *J. Engrg. Mech.* **97** (1971) 935–950.
[17] D. C. Drucker & W. Prager: Soil mechanics and plastic analysis of rock and concrete, *Quarterly Appl. Math.* **10** (1952) 157–175.
[18] M. Efendiev & A. Mielke: On the rate-independent limit of systems with dry friction and small viscosity, *J. Convex Anal.* **13** (2006) 151–167.

[19] I. Fonseca & G. Leoni: *Modern methods in the calculus of variations: L^p spaces*, Springer Monographs in Mathematics. Springer, New York, 2007.
[20] G.A. Francfort & A. Giacomini: Small strain heterogeneous elasto-plasticity revisited, to appear in *Comm. Pure Appl. Math.*
[21] R.V. Kohn & R. Temam: Dual spaces of stresses and strains, with applications to Hencky plasticity, *Appl. Math. Optim.* **10** (1983) 1–35.
[22] P. Laborde: A nonmonotone differential inclusion, *Nonlin. Anal. Theory, Math. & Appl.* **11** (1986) 757–767.
[23] P. Laborde: Analysis of the strain-stress relation in plasticity with non-associative laws, *Int. J. Engrg Sci.* **25** (1987) 655–666.
[24] V. Lubarda, S. Mastilovic & J. Knap: Brittle-ductile transition in porous rocks by cap model , *J. Engrg Mech.* **122** (1996) 633–642.
[25] J. Lubliner: *Plasticity Theory*, Macmillan Publishing Company, New York, 1990.
[26] A. Mielke, R. Rossi & G. Savaré: Modeling solutions with jumps for rate-independent systems on metric spaces, *Discrete Contin. Dyn. Syst.* **25** (2009) 585–615.
[27] A. Mielke, R. Rossi & G. Savaré: BV solutions and viscosity approximations of rate-independent systems, to appear in *ESAIM Control Optim. Calc. Var.*
[28] L. Resende & J.B. Martin: Formulation of Drucker-Prager Cap Model, *J. Engrg. Mech.* **111** (1985) 855–881.
[29] L. Resende & J.B. Martin: Closure of "Formulation of Drucker-Prager Cap Model", *J. Engrg. Mech.* **113** (1987) 1257–1259.
[30] R.T. Rockafellar: *Convex Analysis*, Princeton University Press, Princeton, 1970.
[31] J. Salençon: *Application of the theory of plasticity to soil mechanics*, Wiley series in geotechnical engineering (1977).
[32] F. Solombrino: Quasistatic evolution problems for nonhomogeneous elastic plastic materials, *J. Convex Anal.* **16** (2009) 89–119.
[33] P. Suquet: Sur les équations de la plasticité: existence et régularité des solutions, *J. Mécanique* **20** (1981) 3–39.
[34] R. Temam: *Mathematical problems in plasticity*, Gauthier-Villars, Paris (1985). Translation of *Problèmes mathématiques en plasticité*, Gauthier-Villars, Paris (1983).

(J.-F. Babadjian) Université Pierre et Marie Curie – Paris 6, CNRS, UMR 7598 Laboratoire Jacques-Louis Lions, Paris, F-75005, France
E-mail address: babadjian@ann.jussieu.fr

(G.A. Francfort) Université Paris-Nord, LAGA, Avenue J.-B. Clément, 93430 - Villetaneuse, France & Institut Universitaire de France
E-mail address: gilles.francfort@univ-paris13.fr

(M.G. Mora) Scuola Internazionale Superiore di Studi Avanzati, Via Bonomea 265, 34136 Trieste (Italy)
E-mail address: mora@sissa.it