The passage from discrete to continuous variational problems: a nonlinear homogenization process. Continuum limits with bulk and surface energies

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1 Introduction

In the past years a number of researches have been devoted to the study of discrete systems with a large number of interactions viewed as a variational limit of energies indexed by the number of nodes of the system. In this framework the setting in which we have a fairly complete set of results is that of *central interactions* for *lattice systems*; i.e., systems where the reference positions of the interacting points lie on a prescribed lattice, whose parameters change as the number of points increases. In more precise terms, we consider an open set $\Omega \subset \mathbb{R}^n$ and take as reference lattice $Z_\varepsilon = \Omega \cap \varepsilon\mathbb{Z}^n$. The general form of a pair-potential energy is then

$$E_\varepsilon(u) = \sum_{i,j \in Z_\varepsilon} f^\varepsilon_{ij}(u(i), u(j)), \tag{1.1}$$

where $u : Z_\varepsilon \to \mathbb{R}^m$. The analysis of energies of the form (1.1) has been performed under various hypotheses on f_{ij}. The first simplifying assumption is that F is invariant under translations (in the target space); that is,

$$f^\varepsilon_{ij}(u, v) = g^\varepsilon_{ij}(u - v). \tag{1.2}$$

Furthermore, an important class is that of homogeneous interactions (i.e., invariant under translations in the reference space); this condition translates into

$$f^{\varepsilon}_{ij}(u,v) = g^{\varepsilon}_{(i-j)/\varepsilon}(u,v). \tag{1.3}$$

If both conditions are satisfied, we may rewrite the energies E_ε above as

$$E_\varepsilon(u) = \sum_{k\in Z^n} \sum_{i,j\in Z_\varepsilon, i-j=\varepsilon k} \varepsilon^n \psi^{\varepsilon}_k\Big(\frac{u(i)-u(j)}{\varepsilon}\Big), \tag{1.4}$$

where $\psi^{\varepsilon}_k(\xi) = \varepsilon^{-n} g^{\varepsilon}_k(\varepsilon\xi)$. In this new form the interactions appear through the (discrete) difference quotients of the function u. Upon identifying each function u with its piecewise-constant interpolation (extending the definition of u arbitrarily outside Ω), we can consider E_ε as defined on (a subset of) $L^1(\Omega;\mathbb{R}^m)$, and hence consider the Γ-limit with respect to the $L^1_{\rm loc}$-topology. Under some coerciveness conditions the computation of the Γ-limit will give a continuous approximate description of the behaviour of minimum problems involving the energies E_ε for ε small (see further for definitions, and [11] for a quick introduction to the subject).

1.1 Some convergence results. Continuum energies on Sobolev spaces

Growth conditions on energy densities ψ^{ε}_k imply correspondingly boundedness conditions on gradient norms of piecewise-affine interpolations of functions with equi-bounded energy. The simplest type of growth condition that we encounter is on *nearest neighbours*; i.e., for $|k| = 1$. If $p > 1$ exists such that

$$c_1|z|^p - c_2 \le \psi^{\varepsilon}_k(z) \le c_2(1+|z|^p) \tag{1.5}$$

($c_1, c_2 > 0$ for $|k| = 1$), and if $\psi^{\varepsilon}_k \ge 0$ for all k then the energies are *equi-coercive*: if (u_ε) is a bounded sequence in $L^1(\Omega;\mathbb{R}^m)$ and $\sup_\varepsilon E_\varepsilon(u_\varepsilon) < +\infty$, then from every sequence (u_{ε_j}) we can extract a subsequence converging to a function $u \in W^{1,p}(\Omega;\mathbb{R}^m)$. In this section we will consider energies satisfying this assumption. Hence, their Γ-limits are defined in the Sobolev space $W^{1,p}(\Omega;\mathbb{R}^m)$.

First, we remark that the energies E_ε can also be seen as an integration with respect to measures concentrated on Dirac deltas at the points of $Z_\varepsilon \times Z_\varepsilon$. If each ψ^{ε}_k satisfies a growth condition $\psi^{\varepsilon}_k(z) \le c^{\varepsilon}_k(1+|z|^p)$, then we have

$$E_\varepsilon(u) \le \int_{\Omega\times\Omega} (1+|u(x)-u(y)|^p)\,d\mu_\varepsilon,$$

where

$$\mu_\varepsilon = \sum_{k=1}^{\infty} \sum_{i-j=\varepsilon k,\ i,j\in Z_\varepsilon} c^{\varepsilon}_k \frac{1}{\varepsilon^p}\delta_{(i,j)}.$$

A natural condition for the finiteness of the limit of E_ε is the equi-boundedness of these measures (as $\varepsilon \to 0$), regardless to the set Ω; namely,

$$\limsup_{\varepsilon\to 0} \sum_{|k|\le K/\varepsilon} \frac{(K-\varepsilon k)^{n-1}}{\varepsilon^{p+n-1}} c_k^\varepsilon < +\infty$$

for each fixed $K > 0$ (for fixed Ω this condition is applied with $K = \operatorname{diam}\Omega$). However, under such assumption, example can be shown exhibiting a non-local Γ-limit, of the form

$$F(u) = \int_\Omega f(\nabla u(x))\,dx + \int_{\Omega\times\Omega} \psi(u(x)-u(y))d\mu(x,y),$$

where μ is the weak*-limit of the measures μ_ε outside the 'diagonal' of $\mathbb{R}^n \times \mathbb{R}^n$ (see [10] for a detailed example). Under some decay conditions, such long-range behaviour may be ruled out. The following compactness result proved by Alicandro and Cicalese [1] shows that a wide class of discrete systems possesses a 'local' continuous limit. We state it in a general 'space-dependent' case.

Theorem 1.1 (compactness) *Let ψ_k^ε satisfy*

$$c_1|z|^p - c_2 \le \psi_{e_i}^\varepsilon(x,z) \tag{1.6}$$

for all $(x,z) \in \Omega \times \mathbb{R}^m$ and $i \in \{1,\dots,n\}$,

$$\psi_\varepsilon^k(x,z) \le c_k^\varepsilon(1+|z|^p) \tag{1.7}$$

for all $(x,z) \in \Omega \times \mathbb{R}^m$, and $k \in \mathbb{Z}^n$, where $c_1 > 0$ and c_k^ε satisfy

(H1): $\displaystyle\limsup_{\varepsilon\to 0^+} \sum_{\xi\in\mathbb{Z}^n} c_\xi^\varepsilon < +\infty;$

(H2): *for all $\delta > 0$ $M_\delta > 0$ exists such that* $\displaystyle\limsup_{\varepsilon\to 0^+} \sum_{|\xi|>M_\delta} c_k^\varepsilon < \delta.$

Let E_ε be defined by

$$E_\varepsilon(u) = \sum_{k\in\mathbb{Z}^n} \sum_{i\in R_\varepsilon^k} \varepsilon^n f_k^\varepsilon\left(i, \frac{u(i+\varepsilon k)-u(\imath)}{\varepsilon|k|}\right),$$

where $R_\varepsilon^k := \{i \in Z_\varepsilon :\ i+\varepsilon k \in Z_\varepsilon\}$. Then for every sequence (ε_j) of positive real numbers converging to 0, there exists a subsequence (ε_{j_k}) and a Carathéodory function $f : \Omega \times \mathbb{R}^{d\times N}$ satisfying

$$c(\|M\|^p - 1) \le f(x,M) \le C(\|M\|^p + 1),$$

with $0 < c < C$, such that $(E_{\varepsilon_{j_k}}(\cdot))$ Γ-converges with respect to the $L^p(\Omega)$-topology to the functional $F : L^p(\Omega) \to [0, +\infty]$ defined as

$$F(u) = \begin{cases} \displaystyle\int_\Omega f(x, \nabla u)\, dx & \text{if } u \in W^{1,p}(\Omega; \mathbb{R}^m) \\ +\infty & \text{otherwise.} \end{cases} \tag{1.8}$$

Moreover, for any $u \in W^{1,p}(\Omega)$ and $A \in \mathcal{A}(\Omega)$,

$$\Gamma\text{-}\lim_k F_{\varepsilon_{j_k}}(u, A) = \int_A f(x, \nabla u)\, dx.$$

In the case of energies defined by a scaling process; i.e., when

$$\psi_k^\varepsilon(x, z) = \psi_k\Big(\frac{z}{\varepsilon}\Big),$$

then the limit φ is independent of the subsequence, and is characterized by the *asymptotic homogenization formula*

$$\varphi(M) = \lim_{T \to +\infty} \frac{1}{T^N} \min \{\mathcal{F}_T(u),\ u|_{\partial Q_T} = Mi\} \tag{1.9}$$

where

$$\mathcal{F}_T(u) = \sum_{k \in \mathbb{Z}^n} \sum_{i \in R_1^k(Q_T)} \psi_k \left(\frac{u(i+k) - u(i)}{|k|} \right)$$

and $u|_{\partial Q_T} = Mi$ means that "near" the boundary of Q_T the function u is the discrete interpolation of the affine function Mx (see [1] for further details). This formula can be compared with that giving the 'pointwise' limit treated by Blanc, Le Bris and Lions [8], and is analogous to that used by Friesecke and Theil to check the validity of the Cauchy-Born assumption [26].

It is worth examining this formula in some special cases. First, if only *nearest-neighbour interactions* are present then it reduces to

$$\varphi(M) = \sum_{i=1}^n \psi_i^{**}(Me_i),$$

where $\psi_i = \psi_{e_i}$ and f^{**} denotes the lower semicontinuous and convex envelope of f. Note that convexity is not a necessary condition for lower semicontinuity at the discrete level: this convexification operation should be interpreted as an effect due to oscillations at a 'mesoscopic scale' (i.e., much larger than the 'microscopic scale' ε but still vanishing as $\varepsilon \to 0$).

If not only nearest neighbours are taken into account then the mesoscopic oscillations must be coupled with microscopic ones (see [17, 30]). A simple example is given by the one-dimensional case with next-to-nearest neighbours; i.e. when only f_1 and f_2 are non zero. In this case $\varphi = \psi^{**}$, where

$$\psi(\xi) = f_2(2\xi) + \frac{1}{2}\min\{f_1(\xi_1) + f_1(\xi_2) : \xi_1 + \xi_2 = 2\xi\}.$$

The second term obtained by minimization is due to oscillations at the microscopic level: nearest neighbours rearrange so as to minimize their interaction coupled with that between second neighbours (see [11] for a simple treatment of these one-dimensional problems). A similar optimization argument is used in [26] to show the non-validity of the Cauchy-Born rule for some types of very simple lattice interactions in dimension two.

Finally, it must be noted that formula (1.9) does not simplify even in the simplest case of three levels of interactions in dimension one, thus showing that this effect, typical of nonlinear homogenization, is really due to the lattice interactions and not restricted to vector-valued functions as in the case of homogenization on the continuum (see [14]).

1.2 Continuous energies on functions of bounded variation. Some new results contained in this paper

In this paper we extend the homogenization formula as above for systems whose continuous counterpart is naturally defined on some set of functions of bounded variation (see below for definitions). In particular we will consider energies defined on the Ambrosio-De Giorgi SBV spaces (see [6, 23, 4, 9]. In the one-dimensional case we can picture a function $u \in SBV(a,b)$ as a piecewise-Sobolev functions. If denoting by $S(u)$ its set of discontinuity points, then (local, homogeneous and translation-invariant) energies on $SBV(a,b)$ are of the form

$$\mathcal{E}(u) = \int_a^b f(u'(x))\,dx + \sum_{t\in S(u)} g([u](t)), \tag{1.10}$$

where $u^{\pm}(t)$ are the right and left-hand side limits of u at t.

The pioneering example for this case is due to Chambolle [18], who treated the limit of some finite-difference schemes in Computer Vision (see [7]), producing as the continuum counterpart the one-dimensional version of the Mumford-Shah functional ([29]). That is however a 'limit' case when the potential is a truncated quadratic function and the decoupling process between bulk and surface parts is obtained by considering the effect of the quadratic and constant parts separately. A different approach is initiated in a subsequent paper by Braides, Dal Maso and Garroni [13], who consider potentials of convex-concave type and express the limit in terms of different scalings of the two parts (the same type of interactions are analyzed by Truskinovsky [33]).

The general case of nearest-neighbour interactions has been treated by Braides and Gelli in [16], where the following result is proved (for missing definitions see the sections below).

Theorem 1.2 (Nearest-neighbour interactions) *Let $\Omega = (a,b)$ and let $\psi_\varepsilon : \mathbb{R} \to [0,+\infty]$ satisfy*

$$\psi_\varepsilon(z) \geq c_1|z|^p - c_2 \qquad \text{for all } z < 0 \tag{1.11}$$

for some $p > 1$ and $c_1 > 0$, and let E_ε be given by

$$E_\varepsilon(u) = \sum_{i,i+1\in Z_\varepsilon} \varepsilon\psi_\varepsilon\Big(\frac{u(i+1)-u(u_i)}{\varepsilon}\Big).$$

Let $T_\varepsilon \in \mathbb{R}$ be an arbitrary sequence satisfying

$$\lim_{\varepsilon\to 0^+} T_\varepsilon = +\infty, \qquad \lim_{\varepsilon\to 0^+} \varepsilon T_\varepsilon = 0, \tag{1.12}$$

and let $F_\varepsilon, G_\varepsilon : \mathbb{R} \to [0,+\infty]$ be defined by

$$F_\varepsilon(z) = \begin{cases} \psi_\varepsilon(z) & z \leq T_\varepsilon \\ +\infty & z > T_\varepsilon \end{cases} \tag{1.13}$$

$$G_\varepsilon(z) = \begin{cases} \varepsilon\psi_\varepsilon\left(\dfrac{z}{\varepsilon}\right) & \text{if } z > \varepsilon T_\varepsilon \\ +\infty & \text{otherwise.} \end{cases} \tag{1.14}$$

Assume that there exist $F, G : \mathbb{R} \to [0,+\infty]$ such that (note that this assumption is always satisfied, upon extracting a subsequence)

$$\Gamma\text{-}\lim_{\varepsilon\to 0^+} F_\varepsilon^{**} = F \text{ on } \mathbb{R}, \tag{1.15}$$

$$\Gamma\text{-}\lim_{\varepsilon\to 0^+} \mathrm{sub}^- G_\varepsilon = G \text{ on } \mathbb{R}\setminus\{0\}, \tag{1.16}$$

where $\mathrm{sub}^- g$ is the lower semicontinuous and subadditive envelope of g. Then, $(E_\varepsilon)_\varepsilon$ Γ-converges to E with respect to the convergence in $L^1_{\mathrm{loc}}(0,L)$ and the convergence in measure, where

$$E(u) = \begin{cases} \displaystyle\int_a^b \overline{F}(\dot u)\,dx + \sum_{S(u)} \overline{G}([u]) + \sigma D u_c^+(0,L) \\ \qquad\qquad \text{if } u \in BV_{\mathrm{loc}}(0,L)\ D_c u^- = 0 \text{ and } [u] > 0 \text{ on } S(u) \\ +\infty \qquad \text{otherwise in } L^1(0,L), \end{cases}$$

where $\overline{F}$ and $\overline{G}$ are defined by (for notation convenience we set $G(0)=0$)

$$\overline{F}(z) := \inf\{F(z_1) + G^0(z_2) : z_1 + z_2 = z\},$$

$$\overline{G}(z) := \inf\{F^\infty(z_1) + G(z_2) : z_1 + z_2 = z\},$$

and $\sigma := \overline{F}^\infty(1)$, where $F^\infty(z) = \lim_{z\to+\infty} \frac{F(z)}{z}$ and $G^0(z) = \lim_{z\to 0^+} \frac{G(z)}{z}$.

The fundamental issue here is the separation of scale effect highlighted by equations (1.13)–(1.14), that allows to derive the bulk and surface energy densities of the limit from the discrete interactions. In this paper, always remaining in the one-dimensional framework, we show how in the case of long-range interactions this scale separation can be coupled with the nonlinear homogenization process described in formula (1.9) (see Theorem 3.2 further on). The general n-dimensional vector-valued case can be dealt with by using the localization and integral representation methods of Γ-convergence ([2]). Particular cases when a simpler description of the limit energy densities is possible are treated in [19, 20, 15]. Some special cases when also non-central interactions are considered are dealt with in [3, 25]. Higher-order effects are considered in [21, 12].

2 Notation and preliminaries

For a set A of $\mathbb{R}$ we denote $\operatorname{int} A$ the interior of A. We write $\operatorname{sgn} t$ and $[t]$ to denote the sign of t and the integer part of t, respectively. We write $\mathcal{L}_1(A)$ or $|A|$ to denote the Lebesgue measure of $A \subset \mathbb{R}$, $\#$ for the counting measure and δ_t for the Dirac mass at t. We use standard notation for Sobolev and Lebesgue spaces. If ϕ is a measurable function then $\fint_B \phi\, dx$ is its mean value on the set B. If μ is a (signed) Borel measure then μ^+ and μ^- denote its positive and negative parts, respectively, and $|\mu|$ its total variation; if B is a Borel set, then the measure $\mu \llcorner B$ is defined as $\mu \llcorner B(A) := \mu(A \cap B)$. The letter c will denote a strictly positive constant whose value may vary from line to line.

2.1 Functions of bounded variation

We recall that the space $BV(a,b)$ of *functions of bounded variation* on (a,b) is defined as the space of functions $u \in L^1(a,b)$ whose *distributional derivative* Du is a signed Borel measure. For each such u there exists $f \in L^1(a,b)$, a (at most countable) set $S(u) \subset (a,b)$, a sequence of real numbers $(a_t)_{t\in S(u)}$ with $\sum_t |a_t| < +\infty$ and a non-atomic measure $D_c u$ singular with respect to the Lebesgue measure such that the equality of measures $Du = f\,\mathcal{L}_1 + \sum_{t\in S(u)} a_t\delta_t + D_c u$ holds. It can be easily seen that for such functions the left hand-side and right hand-side *approximate limits* $u^-(t)$, $u^+(t)$ exist at every point, and that $S(u) = \{t \in \mathbb{R} :\ u^-(t) \neq u^+(t)\}$ and $a_t = u^+(t) - u^-(t) =: [u](t)$. We will write $u' = f$, which is an approximate gradient of u. $D_c u$ is called the *Cantor part* of Du. A sequence u_j converges weakly to u in $BV(a,b)$ if $u_j \to u$ in $L^1(a,b)$ and $\sup_j |Du_j|(a,b) < +\infty$.

The space $SBV(a,b)$ of *special functions of bounded variation* is defined as the space of functions $u \in BV(a,b)$ such that $D_c u = 0$; i.e., whose distributional derivative Du can be written as $Du = u'\,\mathcal{L}_1 + \sum_{t\in S(u)}(u^+(t) - u^-(t))\delta_t$. This notation describes a particular case of a SBV-functions space as introduced by De Giorgi and Ambrosio [23]. We will mainly deal with functionals whose natural domain is that of piecewise-$W^{1,p}$ functions, which is a particular subclass of $SBV(a,b)$ corresponding to the conditions $\dot{u} \in L^p(a,b)$ and $\#(S(u)) < +\infty$, but we nevertheless use the more general SBV notation for future reference and for further generalization to higher dimensions (see [5]). For an introduction to BV and SBV functions we refer to the book by Ambrosio, Fusco and Pallara [6], while approximation methods for free-discontinuity problems are discussed by Braides [9].

A class of energies on $SBV(a,b)$ are those of the form (1.10) with $f, g : \mathbb{R} \to [0,+\infty]$. Lower semicontinuity conditions on $\mathcal{E}$ are equivalent to requiring that f is lower semicontinuous and convex and g is lower semicontinuous and subadditive; i.e., $g(x+y) \le g(x) + g(y)$. The latter can be interpreted as a condition penalizing fracture fragmentation, whereas convexity penalizes oscillations. If φ is not lower semicontinuous and convex (respectively, subadditive) then we may consider its *lower semicontinuous and convex* (respectively, *subadditive*) *envelope*; i.e., the greatest lower semicontinuous and convex (respectively, subadditive) function not greater than φ, that we denote by φ^{**} (respectively, $\mathrm{sub}^-\varphi$). For a discussion on the role of these conditions for the lower semicontinuity of $\mathcal{E}$ we refer to [9] Section 2.2 or [11]. Energies in BV must satisfy further compatibility conditions between f and g.

2.2 Γ-convergence

We recall the definition of De Giorgi's Γ-*convergence* in a metric space space (X,d): given a family of functionals $F_n : X \to [0,+\infty]$, $n \in \mathbb{N}$, for $u \in X$ we define the Γ-*lower limit* and Γ-*upper limit* of (F_n) as

$$F'(u) = \Gamma(d)\text{-}\liminf_n F_n(u) := \inf\Big\{\liminf_n F_n(u_n) \ : \ \lim_n d(u_n,u) = 0\Big\}, \quad (2.1)$$

and

$$F''(u) = \Gamma(d)\text{-}\limsup_n F_n(u) := \inf\Big\{\limsup_n F_n(u_n) \ : \ \lim_n d(u_n,u) = 0\Big\}. \quad (2.2)$$

Note that the functions F' and F'' are lower semicontinuous. If these two quantities coincide then their common value is called the Γ-*limit* of the sequence (F_n) at u, and is denoted by $\Gamma\text{-}\lim_n F_n(u)$ or $\Gamma(d)\text{-}\lim_n F_n(u)$. Equivalently, $F(u) = \Gamma\text{-}\lim_n F_n(u)$ if and only if the two following conditions are satisfied:

(i) (*lower semicontinuity inequality*) for all sequences (u_n) converging to u in X we have $F(u) \le \liminf_n F_n(u_n)$;

(ii) (*existence of a recovery sequence*) there exists a sequence (u_n) converging to u in X such that $F(u) \ge \limsup_n F_n(u_n)$.

We will use as d the L^1-metric or a metric giving convergence in measure.

For a comprehensive study of Γ-convergence we refer to the books of Dal Maso [22] and Braides [11] (see also [14] Part II). The reason for the introduction of this notion is explained by the following fundamental theorem.

Theorem 2.1 *Let $F = \Gamma\text{-}\lim_n F_n$, and let a compact set $K \subset X$ exist such that $\inf_X F_n = \inf_K F_n$ for all n. Then*

$$\exists \min_X F = \lim_n \inf_X F_n.$$

Moreover, if (u_n) is a converging sequence with $\lim_n F_n(u_n) = \lim_n \inf_X F_n$ then its limit is a minimum point for F.

3 Statement of the main result

With fixed $L > 0$, consider an open interval $(0, L)$ of $\mathbb{R}$ and for $n \in \mathbb{N}$ let $\lambda_n = L/n$. This parameter will play the role played by ε in the Introduction. We use the notation $x_n^i = i\lambda_n$. We define $\mathcal{A}_n(0, L)$ as the set of discrete functions $u : \lambda_n\mathbb{Z} \cap [0, L] \to \mathbb{R}$. This set will be identified as the subset of $L^1(0, L)$ of functions constant almost everywhere on each interval $\big(x_n^i - \frac{\lambda_n}{2}, x_n^i + \frac{\lambda_n}{2}\big)$, $i \in \{1, \dots, n\}$.

Let $K \in \mathbb{N}$ be fixed and for $n \in \mathbb{N}$ and $j \in \{1, \dots, K\}$ let $\psi_n^j : \mathbb{R} \to (-\infty, +\infty]$ be given Borel functions bounded below. Define $E_n : L^1(0, L) \to [0, +\infty]$ as

$$E_n(u) = \begin{cases} \displaystyle\sum_{j=1}^{K} \sum_{i=0}^{n-j} \lambda_n \psi_n^j \left(\frac{u(x_n^{i+j}) - u(x_n^i)}{j\lambda_n} \right) & x \in \mathcal{A}_n(0, L) \\ +\infty & \text{otherwise in } L^1(0, L). \end{cases} \tag{3.1}$$

We will describe the asymptotic behaviour of E_n as $n \to +\infty$ when the energy densities are potentials of Lennard-Jones type. More precisely, we will make the following assumption:

(*growth conditions: superlinearity at $-\infty$, mixed type at $+\infty$*) there exists a convex function $\Psi : \mathbb{R} \to [0, +\infty]$ such that

$$\lim_{z \to -\infty} \frac{\Psi(z)}{|z|} = +\infty$$

and there exist constants c_j^1, $c_j^2 > 0$ such that

$$c_j^1(\Psi(z) - 1) \le \psi_n^j(z) \le c_j^2 \max\{\Psi(z), |z|\} \quad \text{for all } z \in \mathbb{R}. \tag{3.2}$$

Under this hypothesis it will be possible to describe explicitly the behaviour of the energies E_n by means of their Γ-limit. The exact statement of the result will be given at the end of this section.

Remark 3.1 Hypothesis (3.2) is designed to cover the case of Lennard-Jones potential (and potential of the same shape), where $\psi_n^j = \psi$ is equal for all j and n, and

$$\psi(z) = \frac{k_1}{z^{12}} - \frac{k_2}{z^6}$$

for some $k_1, k_2 > 0$. In this case, we can take

$$\Psi(z) = \begin{cases} \dfrac{1}{z^{12}} & \text{if } z > 0 \\ +\infty & \text{otherwise.} \end{cases}$$

Another case included in hypothesis (3.2) is when all functions satisfy a uniform growth condition of order $p > 1$; i.e.,

$$c_j(|z|^p - 1) \leq \psi_n^j(z) \leq C\, c_j(|z|^p + 1)$$

for all j and n.

Note that in the case $K = 1$ (nearest-neighbour interaction) the right hand-side of (3.2) can be dropped (see Theorem 1.2).

Before stating our main result, we have to introduce the counterpart of the energy densities F_n and G_n in Theorem 1.2 for the case $K > 1$. Following the idea already performed in [17], we consider clusters of N subsequent points (N large) and define an average discrete energy for each of those clusters, so that the energy E_n may be approximately regarded as a 'nearest neighbour interaction energy' acting between such clusters, to which the above description applies.

Actually, we fix a sequence (N_n) of natural numbers with the property

$$\lim_n N_n = +\infty, \qquad \lim_n \frac{N_n}{n} = 0, \tag{3.3}$$

and we define

$$\psi_n(z) = \min\Big\{ \frac{1}{N_n} \sum_{j=1}^{K} \sum_{i=0}^{N_n - j} \psi_n^j\Big(\frac{u(i+j) - u(i)}{j}\Big) : \ u : \{0, \dots, N_n\} \to \mathbb{R},$$
$$u(x) = zx \text{ if } x = 0, \dots, K, N_n - K, \dots, N_n \Big\}. \tag{3.4}$$

By using the energies ψ_n we will regard a system of N_n neighbouring points as a single interaction between the two extremal ones, up to a little error which is negligible as $N_n \to +\infty$. We can now state our convergence result, whose thesis is exactly the same as that of Theorem 1.2 with $\varepsilon_n := N_n \lambda_n$.

Theorem 3.2 *Let ψ_n^j satisfy (3.2) and let (E_n) be given by (3.1). Let ψ_n be given by (3.4) and let $\varepsilon_n = N_n \lambda_n$. For all $n \in \mathbb{N}$ let $T_n \in \mathbb{R}$ be defined as in*

(1.12), *and let* $F_n, G_n : \mathbb{R} \to [0, +\infty]$ *be defined by*

$$F_n(z) = \begin{cases} \psi_n(z) & z \le T_n \\ +\infty & z > T_n \end{cases} \tag{3.5}$$

$$G_n(z) = \begin{cases} \varepsilon_n \psi_n\left(\dfrac{z}{\varepsilon_n}\right) & \text{if } z > \varepsilon_n T_n \\ +\infty & \text{otherwise.} \end{cases} \tag{3.6}$$

Assume that there exist $F, G : \mathbb{R} \to [0, +\infty]$ *such that*

$$\Gamma\text{-}\lim_n F_n^{**} = F \text{ on } \mathbb{R}, \tag{3.7}$$

$$\Gamma\text{-}\lim_n \operatorname{sub}^- G_n = G \text{ on } \mathbb{R} \setminus \{0\}. \tag{3.8}$$

Note that this assumption is always satisfied, upon extracting a subsequence. Then, $(E_n)_n$ Γ*-converges to* E *with respect to the convergence in* $L^1_{\mathrm{loc}}(0, L)$ *and the convergence in measure, where*

$$E(u) = \begin{cases} \displaystyle\int_0^L \overline{F}(\dot{u})\, dx + \sum_{S(u)} \overline{G}([u]) + \sigma D u_c^+(0, L) \\ \qquad\qquad \text{if } u \in BV_{\mathrm{loc}}(0, L)\ D_c u^- = 0\ ,\ \text{and } [u] > 0 \text{ on } S(u) \\ +\infty \qquad\qquad\qquad \text{otherwise in } L^1(0, L), \end{cases}$$

where $\overline{F}$ *and* $\overline{G}$ *are defined by (for notation convenience we set* $G(0) = 0$*)*

$$\overline{F}(z) := \inf\{F(z_1) + G^0(z_2) : z_1 + z_2 = z\},$$

$$\overline{G}(z) := \inf\{F^\infty(z_1) + G(z_2) : z_1 + z_2 = z\},$$

and $\sigma := \overline{F}^\infty(1)$.

Proof. Upon adding a fixed positive constant we may assume that all ψ_n^j are non negative. We begin by proving the liminf inequality. We thus fix $u_n, u \in L^1_{\mathrm{loc}}(0, L)$ such that $u_n \to u$ in measure and $\sup_n E_n(u_n) < +\infty$. Upon extracting a subsequence we may suppose that $u_n \to u$ a.e. and that the limit $\lim_n E_n(u_n)$ exists. By using (3.2) and reasoning as in the proof of Theorem 1.2 we get that $u \in BV_{\mathrm{loc}}(0, L)$ and $u_n \to u$ weakly in $BV_{\mathrm{loc}}(0, L)$ (see [16] for details). It will be enough to show that for all $0 < a < b < L$ fixed we have

$$\int_a^b \overline{F}(\dot{u}(x))\, dx + \sum_{t \in S(u) \cap (a,b)} \overline{G}([u](t)) + \sigma D u_c^+(a, b) \le \lim_n E_n(u_n). \tag{3.9}$$

As already mentioned, for all $n \in \mathbb{N}$ we will estimate the energy $E_n(u_n)$ with a ‘nearest neighbour interaction’ one, with energy density ψ_n and discretization step $\varepsilon_n = \lambda_n N_n$. Thus, with fixed N_n, we will choose $w \in \{1, \dots, N_n\}$ in such a way that we can find a suitable piecewise affine interpolation of u_n on the lattice $(\lambda_n w + \varepsilon_n \mathbb{Z}) \cap (a,b)$, call it v_n, so that v_n still converges to u in measure and $E_n(u_n) = E_n^1(v_n) + o(1)$.

For all $w \in \{1, \dots, N_n\}$ let

$$Z_n(w) = \{\ell \in w + N_n\mathbb{Z} : \ \lambda_n \ell \in (a,b)\},$$

$$\begin{aligned}\Phi_n(w) \quad &= \quad \sum_{\ell \in Z_n(w)} \Big(\sum_{j=1}^{K} \sum_{i=\ell-2K}^{\ell+2K-1} \lambda_n \psi_n^j\Big(\frac{u_n(\lambda_n(i+j)) - u_n(\lambda_n i)}{j\lambda_n}\Big)\Big) \\ &\quad \sum_{s=\ell-2K}^{\ell+2K-1} |u_n(\lambda_n(s+1)) - u_n(\lambda_n s)| \qquad (3.10)\end{aligned}$$

Note that $\sum_{s=\ell-2K}^{\ell+2K-1} |u_n(\lambda_n(s+1)) - u_n(\lambda_n s)| = |Du_n|([(\ell-2K)\lambda_n, (\ell+2K)\lambda_n])$, where we consider the discrete function u_n identified with the L^1 function piecewise constant. Since each interval $[\lambda_n s, \lambda_n(s+1)]$ belongs at most to $4K$ interval of the type $[\lambda_n(\ell - 2K), \lambda_n(\ell + 2K)]$ we have that

$$\sum_{w=1}^{N_n} \sum_{\ell \in Z_n(w)} \sum_{s=\ell-2K}^{\ell+2K-1} |u_n(\lambda_n(s+1)) - u_n(\lambda_n s)| \le 4K|Du_n|(a,b).$$

Moreover, we also have

$$\begin{aligned}&\sum_{w=1}^{N_n} \sum_{\ell \in Z_n(w)} \sum_{j=1}^{K} \sum_{i=\ell-2K}^{\ell+2K-1} \lambda_n \psi_n^j\Big(\frac{u_n(\lambda_n(i+j)) - u_n(\lambda_n i)}{j\lambda_n}\Big) \\ &= \sum_{i'=0}^{4K} \sum_{w=1}^{N_n} \sum_{\ell \in Z_n(w)} \lambda_n \psi_n^j\Big(\frac{u_n(\lambda_n(\ell - 2K + j + i')) - u_n(\lambda_n(\ell - 2K + i')}{j\lambda_n}\Big) \\ &\le 5KE_n(u_n). \qquad (3.11)\end{aligned}$$

Hence,

$$\sum_{w=1}^{N_n} \Phi_n(w) \le 5KE_n(u_n) + 4K|Du_n|(a,b) \le c$$

and we can find $w_n \in \{1, \dots, N_n\}$ such that

$$\Phi_n(w_n) \le \frac{c}{N_n}.$$

For all $\ell \in Z_n(w_n)$ we define

$$z_n^\ell = \frac{u_n(\lambda_n(\ell + N_n - K)) - u_n(\lambda_n(\ell + K))}{\lambda_n(N_n - 2K)}$$

and $\tilde u_n^\ell$ on $\{0,\dots,N_n\}$ by

$$\tilde u_n^\ell(i)=\begin{cases}\dfrac{1}{\lambda_n}(u_n(\lambda_n(\ell+i))-u_n(\lambda_n(\ell+K)))+z_n^\ell K & \\ & \text{if } i=K,\dots,N_n-K\\ z_n^\ell i & \text{otherwise.}\end{cases}$$

Finally, we define v_n as the continuous piecewise-affine interpolate function such that

$$v_n(a)=u(a)$$

and

$$v_n'=z_n^\ell \text{ on } (\lambda_n\ell,\lambda_n(\ell+N_n)),\qquad \ell\in Z_n(w_n).$$

Note that $v_n\to u$ in measure and $\tilde u_n^\ell$ is a test function for the minimum problem defining $\psi_n(z_n^\ell)$. We then have

$$\begin{aligned}E_n(u_n)\quad &\geq\quad \sum_{\ell\in Z_n(w_n)}\sum_{j=1}^{K}\sum_{i=\ell+K}^{\ell+N_n-K-j}\lambda_n\psi_n^j\Big(\frac{u_n(\lambda_n(i+j))-u_n(\lambda_n i)}{j\lambda_n}\Big)\\ &\geq\quad \sum_{\ell\in Z_n(w_n)}\sum_{j=1}^{K}\sum_{i=0}^{N_n-j}\lambda_n\psi_n^j\Big(\frac{\tilde u_n^\ell(i+j)-\tilde u_n^\ell(i)}{j}\Big)\\ &\quad-\sum_{\ell\in Z_n(w_n)}\sum_{j=1}^{K}\sum_{i=0}^{K-1}\lambda_n\psi_n^j\Big(\frac{\tilde u_n^\ell(i+j)-\tilde u_n^\ell(i)}{j}\Big)\\ &\quad-\sum_{\ell\in Z_n(w_n)}\sum_{j=1}^{K}\sum_{i=N_n-K-j+1}^{N_n-j}\lambda_n\psi_n^j\Big(\frac{\tilde u_n^\ell(i+j)-\tilde u_n^\ell(i)}{j}\Big).\end{aligned}\tag{3.12}$$

As for the first term in (3.12), for any $\ell\in Z_n(w_n)$, we have

$$\sum_{j=1}^{K}\sum_{i=0}^{N_n-j}\psi_n^j\Big(\frac{\tilde u_n^\ell(i+j)-\tilde u_n^\ell(i)}{j}\Big)\geq N_n\psi_n(z_n^\ell),$$

so that, denoting $Z_n=\{i\in\mathbb{Z}:\lambda_n w_n+\varepsilon_n i\in Z_n(w_n)\}$,

$$\begin{aligned}&\sum_{\ell\in Z_n(w_n)}\sum_{j=1}^{K}\sum_{i=0}^{N_n-j}\lambda_n\psi_n^j\Big(\frac{\tilde u_n^\ell(i+j)-\tilde u_n^\ell(i)}{j}\Big)\geq\\ &\qquad\sum_{i\in Z_n}\varepsilon_n\psi_n\Big(\frac{v_n(\varepsilon_n(i+1))-v_n(\varepsilon_n i)}{\varepsilon_n}\Big).\end{aligned}\tag{3.13}$$

By plugging estimate (3.13) in (3.12) we get

$$
\begin{aligned}
E_n(u_n) \quad \geq \quad & \sum_{i\in Z_n} \varepsilon_n \psi_n\Big(\frac{v_n(\varepsilon_n(i+1))-v_n(\varepsilon_n i)}{\varepsilon_n}\Big) \\
& - \sum_{\ell\in Z_n(w_n)} \sum_{j=1}^{K} \sum_{i=0}^{K-1} \lambda_n \psi_n^j \Big(\frac{\tilde{u}_n^\ell(i+j)-\tilde{u}_n^\ell(i)}{j}\Big) \\
& - \sum_{\ell\in Z_n(w_n)} \sum_{j=1}^{K} \sum_{i=N_n-K-j+1}^{N_n-j} \lambda_n \psi_n^j \Big(\frac{\tilde{u}_n^\ell(i+j)-\tilde{u}_n^\ell(i)}{j}\Big) \\
=: \quad & \sum_{i\in Z_n} \varepsilon_n \psi_n\Big(\frac{v_n(\varepsilon_n(i+1))-v_n(\varepsilon_n i)}{\varepsilon_n}\Big) - I_n^1 - I_n^2. \qquad (3.14)
\end{aligned}
$$

It remains to give an estimate of the terms I_n^1, I_n^2. We will confine ourselves to prove that $I_n^1 = o(1)$, the proof of the same estimate for I_n^2 being analogous. Indeed, by using hypothesis (3.2), we have

$$
\begin{aligned}
I_n^1 \quad \leq \quad c_j^2 & \sum_{\ell\in Z_n(w_n)} \sum_{j=1}^{K} \sum_{i=0}^{K-1} \Big(\lambda_n \Psi\Big(\frac{u_n(\lambda_n(\ell+i+j))-u_n(\lambda_n(\ell+i))}{j\lambda_n}\Big) \\
& + \Big|\frac{u_n(\lambda_n(\ell+i+j))-u_n(\lambda_n(\ell+i))}{j}\Big|\Big)
\end{aligned}
\qquad (3.15)
$$

and, by convexity, we have also

$$
\begin{aligned}
& I_n^1 \\
\leq \quad & \sum_{\ell\in Z_n(w_n)} \sum_{j=1}^{K} c_j^2 \sum_{i=0}^{K-1} \sum_{s=i}^{i+j-1} \frac{1}{j}\Big(\lambda_n \Psi\Big(\frac{u_n(\lambda_n(\ell+s+1))-u_n(\lambda_n(\ell+s))}{j\lambda_n}\Big) \\
& + |u_n(\lambda_n(\ell+s+1))-u_n(\lambda_n(\ell+s))|\Big) \\
\leq \quad & c \sum_{\ell\in Z_n(w_n)} \sum_{j=1}^{K} \sum_{i>K-j}^{2K-j} \sum_{s=i}^{i+j-1} \frac{1}{j}\lambda_n \psi_n^1\Big(\frac{u_n(\lambda_n(\ell+s+1))-u_n(\lambda_n(\ell+s))}{\lambda_n}\Big) \\
& + \frac{c}{N_n} + \Phi_n(w_n). \qquad (3.16)
\end{aligned}
$$

By taking into account our choice of w_n, it follows

$$
I_n^1 \leq \frac{c}{N_n}\Big(1 + \sum_{i\in Z_n} \varepsilon_n \psi_n\Big(\frac{v_n(\varepsilon_n(i+1))-v_n(\varepsilon_n i)}{\varepsilon_n}\Big)\Big). \qquad (3.17)
$$

Eventually we get

$$
E_n(u_n) \geq (1-\frac{c}{N_n}) \sum_{i\in Z_n} \varepsilon_n \psi_n\Big(\frac{v_n(\varepsilon_n(i+1))-v_n(\varepsilon_n i)}{\varepsilon_n}\Big) - \frac{c}{N_n}
$$

and it suffices to pass to the liminf and use Theorem 1.2 on the interval (a,b) to have the desired inequality.

To prove the limsup inequality we will first prove that

$$\Gamma\text{-}\limsup_n E_n(u) \leq \int_0^L F(\dot{u}(x))\,dx + \sum_{t\in S(u)} G([u](t))$$

on the functions $u \in SBV(0,L)$ with $\#S(u) < +\infty$ and then we will use a relaxation argument. For the sake of simplicity as a first step we provide a recovery sequence for functions of the form $u(x) = \xi x + z\chi_{(x_0,L)}$. Thus, let us consider the case of $u(x) = \xi x$. By proceeding as in the proof of Theorem 1.2 (for details see [16]) we can find $\xi_n^1, \xi_n^2 \in \mathbb{R}$ and a set of indices $J_n \subset \{0, \dots, \left[\frac{L}{\varepsilon_n}\right]\}$ such that, set $\beta_n := \#(J_n)$ and $M_n := \left[L/\varepsilon_n\right]$, there holds

$$\begin{aligned}
&\varepsilon_n\beta_n\xi_n^1 + \varepsilon_n(M_n - \beta_n)\xi_n^2 = L\xi + o(1)\\
&\varepsilon_n\beta_n\psi_n(\xi_n^1) + \varepsilon_n(M_n - \beta_n)\psi_n(\xi_n^2) = LF(\xi) + o(1)
\end{aligned} \tag{3.18}$$

For $s = 1,2$, let v_n^s be a minimum point for the problem defining $\psi_n(\xi_n^i)$, and define u_n as

$$u_n(i\lambda_n) := \lambda_n v_n^s(i-\ell) + \xi_n^2\ell\lambda_n + \sum_{j=1}^{\ell}(\xi_n^1 - \xi_n^2)\chi_{J_n}(j)N_n\lambda_n$$

if $i \in [\ell, \ell+N_n)$. It can be proved that $u_n \to u$ in $L^1(0,L)$ and

$$\begin{aligned}
E_n(u_n) &\leq \varepsilon_n\beta_n\psi_n(\xi_n^1) + \varepsilon_n(M_n-\beta_n)\psi_n(\xi_n^2)\\
&+ \sum_{\ell\in Z_n(0)}\sum_{j=1}^{K}\sum_{i=0}^{K-1}\lambda_n\psi_n^j\Big(\frac{u_n((i+j+\ell)\lambda_n) - u_n((i+\ell)\lambda_n)}{j\lambda_n}\Big)\\
&+ \sum_{i=N_n-K-j+1}^{N_n-j}\lambda_n\psi_n^j\Big(\frac{u_n((i+j+\ell)\lambda_n) - u_n((i+\ell)\lambda_n)}{j\lambda_n}\Big).
\end{aligned}$$

Since by (3.18) $\lim_n \varepsilon_n\beta_n\psi_n(\xi_n^1) + \varepsilon_n(M_n-\beta_n)\psi_n(\xi_n^2) = LF(\xi)$, it remains to prove that the second terms in (3.19) tend to 0. To do this we will proceed exactly as in the estimate of the terms I_n^1, I_n^2 in the liminf inequality, by taking into account hypothesys (3.2) and using the convexity of $\Psi(z) + |z|$ to consider only interactions of order one. Thus, it holds

$$\begin{aligned}
E_n(u_n) \quad &\leq \quad LF(\xi) + o(1)\\
&+c\sum_{\ell\in Z_n(0)}\sum_{j=1}^{K}\sum_{i=0}^{K-1}\sum_{s=0}^{j-1}\frac{1}{j}\lambda_n\psi_n^1\Big(\tfrac{u_n(\lambda_n(\ell+i+s+1)) - u_n(\lambda_n(\ell+i+s))}{\lambda_n}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+c\sum_{\ell\in Z_n(0)}\sum_{j=1}^{K}\sum_{i=N_n-K-j}^{N_n-K}\sum_{s=0}^{j-1}\frac{1}{j}\lambda_n\psi_n^1\Big(\tfrac{u_n(\lambda_n(\ell+i+s+1))-u_n(\lambda_n(\ell+i+s))}{\lambda_n}\Big)\\
&+\frac{c}{N_n}(1+\varepsilon_n(\beta_n|\xi_n^1|+(M_n-\beta_n)|\xi_n^2|))\\
\leq\ &\frac{c}{N_n}(F(\xi)+1+\varepsilon_n(\beta_n|\xi_n^1|+(M_n-\beta_n)|\xi_n^2|)). \qquad (3.19)
\end{aligned}
$$

By the growth condition on Ψ (and then on ψ_n, F) the terms $\varepsilon_n(\beta_n|\xi_n^1|+(M_n-\beta_n)|\xi_n^2|)$ are equibounded. Then it suffices to take the limsup as n goes to $+\infty$.

Let us come now to the case $u(x)=\xi x+z\chi_{(x_o,L)}(x)$ with $z\neq 0$ and $G(z)<+\infty$. Analogously to the proof of Theorem 1.2 we can find natural numbers $L_n\in\mathbb{N}$ and real numbers $z_n^s\in[0,+\infty)$ for $s=\{1,\dots,L_n\}$ with the following properties

$$
\lim_n L_n\varepsilon_n=0,\qquad \lim_n\sum_{s=1}^{L_n}z_n^s=z,\qquad G(z)=\lim_n\sum_{s=1}^{L_n}G_n(z_n^s). \qquad (3.20)
$$

Let $\ell_o\in Z_n(0)$ be such that $x_o\in[\ell_o\lambda_n,(\ell_o+N_n)\lambda_n)$. If v_n^s is a minimum point for the problem defining $\psi_n\Big(z/\varepsilon_n\Big)$ for $s=1,\dots,L_n$ and u_n is the recovery sequence for ξx previously defined, we set for $i\in[\ell\lambda_n,(\ell+N_n)\lambda_n)$ with $\ell\in Z_n(0)$

$$
w_n(i\lambda_n)=\begin{cases}
u_n(i\lambda_n) & \text{if } \ell<\ell_o\\[2ex]
\lambda_n v_n^s(i-\ell)+u_n(\ell_0\lambda_n)+\sum_{s'=1}^{s}z_n^s & \text{if } \ell\in[\ell_o,\ell_o+L_nN_n)\\[2ex]
u_n(i\lambda_n+(\ell_o+L_nN_n)\lambda_n)+\sum_{s=1}^{L_n}z_n^s & \text{if } \ell\geq\ell_o+L_nN_n.
\end{cases}
$$

By using (3.20) it can be easily checked that w_n converges to u in $L^1(0,L)$ and, by proceeding as above,

$$
E_n(u_n)=LF(\xi)+G(z)+o(1).
$$

This procedure can be applied, up to slight modifications, to any function in $SBV(0,L)$, piecewise affine and with a finite number of jumps. So the thesis follows by usual density and relaxation arguments (see e.g. [11]) . □

Remark 3.3 (next-to-nearest neighbour interactions) In the case that $\psi_n^j=0$ for all $j>2$ then we can equivalently take in place of ψ_n the function $\tilde{\psi}_n$ defined as in the Sobolev case, by

$$
\tilde{\psi}_n(z)=\psi_n^2(z)+\frac{1}{2}\min\{\psi_n^1(z_1)+\psi_n^1(z_2):z_1+z_2=2z\}.
$$

With the previous remark in mind, we can examine the behaviour of next-to-nearest neighbour systems for Lennard-Jones potentials.

Example 3.4 (next-to-nearest neighbour Lennard Jones interactions) Let ψ be as in Remark 3.1, for any n let $\psi_n^1(z) = \psi(z)$, $\psi_n^2(z) = \psi(2z)$ and $\psi_n^j = 0$ for $j \geq 3$. By using the previous remark, we obtain

$$\overline{F}(z) = \begin{cases} +\infty & \text{if } z \leq 0 \\ \dfrac{1+2^{-12}}{z^{12}} k_1 - \dfrac{1+2^{-6}}{z^6} k_2 & \text{if } 0 < z \leq z_{\min} \\ -\dfrac{k_2^2}{4k_1}(1+2^{-12})(1+2^{-6})^2 =: m_{\overline{F}} & \text{if } z > z_{\min}, \end{cases}$$

where

$$z_{\min} = \Big(2\frac{(1+2^{-12})k_1}{(1+2^{-6})k_2}\Big)^{1/6}$$

is the minimum point for $\tilde{\psi} = \tilde{\psi}_n$ given by the previous remark. The two values $z_{\min}$ and $m_{\overline{F}}$ can be compared to the corresponding minimum point and minimum value for $\tilde{\psi}$, that are $(2k_1/k_2)^{1/6}$ and $-k_2^2/4k_1$, respectively. It is also interesting to note that

$$\lim_{w \to +\infty} \tilde{\psi}(w) = -\frac{k_2^2}{8k_1} = \frac{1}{2} \min \psi.$$

References

[1] R. Alicandro and M. Cicalese, Representation result for continuum limits of discrete energies with superlinear growth. Preprint, 2003.

[2] R. Alicandro and M. Cicalese, Representation result for continuum limits of discrete energies with linear growth, paper in preparation.

[3] R. Alicandro, M. Focardi and M.S. Gelli, Finite-difference approximation of energies in fracture mechanics. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **29** (2000), 671–709

[4] L. Ambrosio, Existence theory for a new class of variational problems, *Arch. Rational Mech. Anal.* **111** (1990), 291-322.

[5] L. Ambrosio and A. Braides, Energies in SBV and variational models in fracture mechanics. In *Homogenization and Applications to Material Sciences*, (D. Cioranescu, A. Damlamian, P. Donato eds.), GAKUTO, Gakkōtosho, Tokio, Japan, 1997, 1-22.

[6] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, Oxford, 2000.

[7] A. Blake and A. Zisserman, *Visual Reconstruction*, MIT Press, Cambridge, Massachussets, 1987.

[8] X. Blanc, C. Le Bris and P.-L. Lions, From molecular models to continuum mechanics. *Arch. Rational Mech. Anal.* **164** (2002), 341–381.

[9] A. Braides, *Approximation of Free-Discontinuity Problems*, Lecture Notes in Mathematics **1694**, Springer Verlag, Berlin, 1998.

[10] A. Braides, Non-local variational limits of discrete systems. *Commun. Contemp. Math.* **2** (2000), 285–297.

[11] A. Braides, *Γ-convergence for Beginners*, Oxford University Press, Oxford, 2002.

[12] A. Braides and M. Cicalese, Surface energies in nonconvex discrete systems, in preparation.

[13] A. Braides, G. Dal Maso and A. Garroni, Variational formulation of softening phenomena in fracture mechanics: the one-dimensional case, *Arch. Rational Mech. Anal.* **146** (1999), 23-58.

[14] A. Braides and A. Defranceschi, *Homogenization of Multiple Integrals*, Oxford University Press, Oxford, 1998.

[15] A. Braides and M.S. Gelli, Limits of discrete systems with long-range interactions. *J. Convex Anal.* **9** (2002), 363–399.

[16] A. Braides and M.S. Gelli, Continuum limits of discrete systems without convexity hypotheses, *Math. Mech. Solids* **7** (2002), 41–66.

[17] A. Braides, M.S. Gelli and M. Sigalotti, The passage from non-convex discrete systems to variational problems in Sobolev spaces: the one-dimensional case, *Proc. Steklov Inst. Math.* **236** (2002), 408–427.

[18] A. Chambolle, Un theoreme de Γ-convergence pour la segmentation des signaux, *C. R. Acad. Sci., Paris, Ser. I* **314** (1992), 191–196.

[19] A. Chambolle, Image segmentation by variational methods: Mumford and Shah functional and the discrete approximations. *SIAM J. Appl. Math.* **55** (1995), 827–863.

[20] A. Chambolle, Finite-differences discretizations of the Mumford-Shah functional. *M2AN Math. Model. Numer. Anal.* **33** (1999), 261–288.

[21] M. Charlotte and L. Truskinovsky, Linear elastic chain with a hyper-prestress. *J. Mech. Phys. Solids* **50** (2002), 217–251.

[22] G. Dal Maso, *An Introduction to Γ-convergence*, Birkhäuser, Boston, 1993.

[23] E. De Giorgi and L. Ambrosio, Un nuovo funzionale del calcolo delle variazioni, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* **82** (1988), 199-210.

[24] G. Del Piero and L. Truskinovsky, A one-dimensional model for localized and distributed failure, *Journal de Physique IV France 8*, **8** (1998), 95-102.

[25] M. Focardi and M.S. Gelli, Approximation results by difference schemes of fracture energies: the vectorial case. *Nonlinear Diff. Eq. Appl.*, to appear.

[26] G. Friesecke and F. Theil, Validity and failure of the Cauchy-Born hypothesis in a two-dimensional mass-spring lattice. *J. Nonlinear Sci.* **12** (2002), 445–478.

[27] M.S. Gelli and G. Royer-Carfagni, Separation of scales in Fracture Mechanics. From molecular to continuum theory via G-convergence. Preprint, Pisa, 2001.

[28] B. Houchmandzadeh, J. Lajzerowicz and E. Salje, Relaxation near surfaces and interfaces for first-, second- and third-neighbour interactions: theory and applications to polytypism, *J. Phys.: Condens. Matter* **4** (1992), 9779-9794.

[29] D. Mumford and J. Shah, Optimal approximation by piecewise smooth functions and associated variational problems, *Comm. Pure Appl. Math.* **17** (1989), 577-685.

[30] S. Pagano and R. Paroni, A simple model for phase transitions: from the discrete to the continuum problem. *Quart. Appl. Math.* **61** (2003), 89–109.

[31] J. Pouget, Dynamics of patterns in ferroelastic-martensitic transformations, *Phys. Rev. B* **43** (1991), 3575-3581.

[32] G. Puglisi and L. Truskinovsky, Mechanics of a discrete chain with bi-stable elements. *J. Mech. Phys. Solids* **48** (2000), 1–27.

[33] L. Truskinovsky, Fracture as a phase transition, *Contemporary research in the mechanics and mathematics of materials* (R.C. Batra and M.F. Beatty eds.) CIMNE, Barcelona, 1996, 322-332.